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Crystals and Mirror Constructions for Quotients

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by

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Abstract

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This thesis develops a new approach to computing the quantum cohomology of symplectic reductions of partial flag varieties X; such symplectic reductions are known as weight varieties. Motivated by a conjecture of Teleman [129], we use a mirror family Landau-Ginzburg model (M_P, f_P) of X introduced by Rietsch [119] to give a conjectural explicit description of the quantum cohomology of weight varieties. We specialise to the class of polygon spaces $\mathcal{P}_{r,n}$, these are symplectic reductions of the complex Grassmannian of 2-planes $\mathrm{Gr}_{\mathbb{C}}(2, n)$ by the maximal torus action. Polygon spaces in low rank have been classified and the quantum cohomology of these varieties is known. As a result, we are able to verify our conjectural description explicitly.

In addition, we investigate the appearance of representation-theoretic combinatorial structures in the mirror symmetry of complete flag varieties. We show that, on the B-model side, the extended string cone $\underline{C}_{\mathbf{i}}$ introduced by Caldero [24] to define toric degenerations on the A-model can be recovered via a discretisation process known as tropicalisation. Specifically, using a non-standard parameterisation of M_B, tropicalisation recovers the precise inequalities defining $\underline{C}_{\mathbf{i}}$. This provides an explicit approach to results previously obtained by Berenstein-Kazhdan [13]. We conclude with a description of a conjectural program relating these combinatorial structures on the B-model with hierarchies of integrable systems on the A-model.

For my Mum.

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Chapter 1

Introduction

1.1 Background

The phenomenon of mirror symmetry was first observed in the Hodge numbers of pairs of Calabi-Yau manifolds in the late 1980s by Greene-Plesser [56] and Candelas-Lynker-Schimmrigk [25]. In taking a Calabi-Yau resolution of the quotient of the smooth quintic three-fold $X \subseteq \mathbb{P}^4$ by the natural action of $\mathbb{Z}_5^5$, Greene-Plesser provided one of the first mirror constructions, constructing a family of mirrors $\{M_\omega\}$ to X. The subsequent calculation, by Candelas-de la Ossa-Greene-Parkes [26], of enumerative invariants of X via period calculations on the mirror family $\{M_\omega\}$ stunned the algebraic geometry community and hinted at a remarkable connection between mirror pairs.

In this section we recall the development of mirror symmetry since this original contribution.

Mirror constructions, mirror conjectures and quantum cohomology

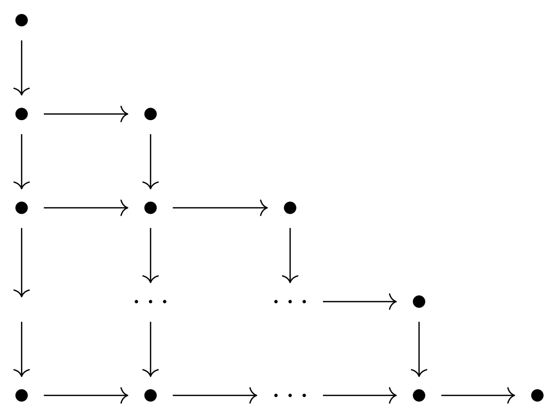
Batyrev [6] generalised the Greene-Plesser construction, providing a general framework to construct mirror candidates for Calabi-Yau hypersurfaces in toric varieties. These methods were later extended to Calabi-Yau complete intersections in toric varieties [8] . To construct a mirror candidate of Calabi-Yau hypersurface in a (Gorenstein Fano) toric variety, Batyrev used the moment polytope of the toric variety. He showed that the toric variety is Fano if and only if its moment polytope is reflexive, In this case, the dual polytope is also reflexive and corresponds to a Fano toric variety. The mirror candidates are then constructed from data attached to the dual reflexive polytope.

Givental [48] proposed an extension of mirror symmetry to Fano manifolds, conjecturing that the mirror to a Fano manifold X (A-model) is a Landau-Ginzburg model (M, f) (B-model), where $M \to T$ is a smooth family of varieties with quasi-affine total space and $f : M \to \mathbb{C}$ is a nonconstant holomorphic function called the *superpotential*. Givental's *mirror conjecture* states an equivalence between the quantum cohomology D-module of X and the D-module generated by certain oscillatory integrals $\int_{\Gamma_t \subseteq M_t} \exp(f_t/\hbar)\omega_t$ associated with a family $(M_t, f_t, \omega_t)_{t\in T}$. Here $(M_t, f_t, \omega_t)_{t\in T}$ is the data of non-vanishing top forms ω_t on the fibres of the family $M \to T$, f_t is the restriction of f to each fibre, and Γ_t is an appropriate family of Morse-theoretic middle dimensional cycles of $Re(f_t)$. In this setting, mirror symmetry for the pair $(X, (M, f))$ predicts an isomorphism

$$qH^*(X) \cong \mathrm{Jac}(f) := \mathbb{C}[M]/(\partial f)$$

between the (small) quantum cohomology algebra $qH^*(X)$ of X and the Jacobian ring of f. The quantum structure is given by variation in the family. In particular, homogeneous spaces for compact, connected Lie groups should exhibit mirror-symmetric phenomena.

In the case of complete flag manifolds $\mathrm{SL}_{n+1}(\mathbb{C})/B$, Givental verified the mirror conjecture by considering a "2-dimensional Toda lattice" [49]. Starting from a *(complete) Gelfand-Tsetlin quiver* having $(n+1)(n+2)/2$ vertices,



Givental constructs a trivial family Y_t, $t \in (\mathbb{C}^\times)^n$, with each Y_t isomorphic to an $n(n+1)/2$-dimensional complex algebraic torus. The superpotential and volume forms are constructed from the combinatorial data of the quiver. The relation with the Toda lattice was later exploited to provide presentations of the quantum cohomology for complete flag manifolds G/B ([50] in type A; [86] in general type).

Givental's construction and mirror conjectures are generalised by Batyrev-Ciocan-Fontanine-Kim-van Straten (BCKS) in [10] (see also [9]) to provide a conjectural mirror family to complete intersections in partial flag manifolds $\mathrm{SL}_{n+1}(\mathbb{C})/P$. The initial input is now a (degenerate) Gelfand-Tsetlin quiver corresponding to the stabiliser P of a fixed partial flag in $\mathbb{C}^{n+1}$. From this data is constructed a toric degeneration of $\mathrm{SL}_{n+1}(\mathbb{C})/P$ to a (in general, singular) Gorenstein Fano toric variety V_P (see also [128], [54]). The conjectural mirror family to $\mathrm{SL}_{n+1}(\mathbb{C})/P$ is a small toric desingularisation $\hat{V}_P$ of V_P^* [7], where V_P^* is the toric variety whose moment polytope is dual to that of V_P. In certain cases, the generalised GKZ hypergeometric series ([46]) of the toric variety $\hat{V}_P$ is shown to give a solution to the quantum D-module of $\mathrm{SL}_{n+1}(\mathbb{C})/P$ [10, Section 5].

The theory of *standard monomial bases*, due to Lakshmibai-Musili-Seshadri [95], provides a monomial basis for spaces of sections of projective embeddings of partial flag varieties G/P, for G a connected semisimple complex algebraic group, $P \subseteq G$ a parabolic subgroup. The explicit nature of these bases has led to consequences in geometry and representation theory: for example, effective determinations of the singular locus of Schubert varieties, and generalisations of the Littlewood-Richardson rule [95, Chapter 13]. In addition, Gonciulea-Lakshmibai [54] use the standard monomial basis to construct toric degenerations of G/B, where $B \subseteq G$ is a Borel subgroup, and Schubert varieties in miniscule G/P.

For G an arbitrary connected semisimple, simply-connected complex algebraic group, a generalisation was given by Caldero [24] who, for every reduced expression $\mathbf{i}$ of the longest element w_0 of the Weyl group of G, obtained toric degenerations for all Schubert varieties in G/B. The key tool used by Caldero is the specialisation at $q = 1$ of (the dual of) *Lusztig's canonical basis* [102] for the upper/lower part of the quantised universal enveloping algebra $U_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ of G. A key feature of his work is the construction of a lattice-semigroup whose points parameterise bases of representations of G, the *string cone lattice semigroup*, and a lattice-semigroup parameterising a weight basis of the coordinate ring of the base affine space known as the *extended string cone*. Alexeev-Brion [1] later determine conditions for the central toric fibre of Caldero's degeneration to be Gorenstein Fano, with a view to obtaining a mirror family construction similar to BCKS.

Rietsch [119] describes a Lie-theoretic construction of a mirror family (M_P^t, f_P^t, ω_t) to the flag variety G/P. Here G is connected semisimple, simply-connected complex algebraic group, $P \subseteq G$ a parabolic subgroup; fix a maximal torus $T \subseteq G$. The remarkable feature is that the family M_P is a subvariety of a Borel containing the dual torus ${}^LT$ inside the Langlands dual ${}^LG$, with base $Z(L_{{}^LP})$ being the centre of a Levi subgroup $L_{{}^LP} \subseteq {}^LP$ inside the dual ${}^LP$ of P. Building on the unpublished work of Dale Peterson, Rietsch gives an isomorphism

$$qH^*(G/P)_{(q)} \cong \mathrm{Jac}(f_P),$$

and an extension to the T-equivariant setting. Specific (T-equivariant) mirror conjectures for G/P are stated, extending the previously formulated conjectures of Givental and BCKS in type A.

The T-equivariant Rietsch mirror conjectures are verified for complete flag varieties G/B by Lam [96] (see earlier work of Rietsch [120] for the non-equivariant case) and, recently, for miniscule flag varieties G/P by Lam-Templier [97]. An essential feature of these works is Berenstein-Kazhdan's notion of *geometric crystal* [12], [13]. Incredibly, the mirror family (M_P, f_P, ω_t) proposed by Rietsch are part of the decorated geometric crystals studied by Berenstein-Kazhdan; moreover, it appears that the introduction of this central object was not known to either author(s). Originally introduced as a tool to understand W-invariant γ-functions appearing in the local Langlands program [20], geometric crystals associated to G are birational models of the *Kashiwara crystals* [80] associated to the Langlands dual ${}^LG$. Kashiwara crystals are combinatorial models of Kashiwara's crystal bases [79], which are specialisations of Lusztig's canonical base at $q = 0$. The recovery of the Kashiwara crystal from the geometric crystal is via the process of *tropicalisation*.

Homological mirror symmetry

There have been proposed two *intrinsic* approaches to mirror symmetry: Kontsevich's program of *homological mirror symmetry* and the geometric approach proposed by Strominger-Yau-Zaslow (known as the *SYZ conjecture*).

At his 1994 ICM address, Kontsevich proposed the following conjecture:

Conjecture (Homological Mirror Symmetry Conjecture [91])**.** For a mirror pair of Calabi-Yau manifolds (X, M), (some enhanced version of) the Fukaya category $\mathfrak{F}(X)$ of X [40, 41] is equivalent to the derived category of coherent sheaves on M. The same statement holds with the roles of X and M swapped.

Kontsevich's program of homological mirror symmetry (HMS) highlights a profound connection between the symplectic geometry of a Calabi-Yau manifold X and the complex geometry of its mirror M. A consequence of the homological mirror symmetry conjecture would be the Givental-BCKS-Rietsch mirror conjectures relating quantum cohomology D-modules with oscillatory integrals, and identifications of quantum cohomology with Jacobian rings of superpotentials. In particular, obtaining an identification of quantum cohomology with the Jacobian ring of the superpotential for the mirror provides a first order approximation to conjectures in homological mirror symmetry program.

In his 2014 ICM address, Teleman [129] described a conjectural mirror construction for symplectic reductions $M /\!\!/ G$, with G a compact, connected Lie group and M a compact Hamiltonian G-space. This construction is a consequence of a new program of topological actions of G on Fukaya categories arising from Hamiltonian G-spaces and gauging topological quantum field theories. When $M = G/L$ is a coadjoint orbit considered as a Hamiltonian T-space, for $T \subseteq G$ a maximal torus, Teleman conjectured the following:

Conjecture A (Teleman, [129])**.** Let ν be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian T-action. Let $t \in Z({}^L L_{\mathbb{C}})$ denote the symplectic structure on G/L. Then, the Fukaya category of the symplectic reduction $(G/L) /\!\!/ T(\nu)$ can be computed as the category $\mathrm{Hom}(S_\nu, \Lambda(t))$, where S_ν is the cotangent fibre over $\exp(\nu)$ considered as an element in ${}^L T$ by duality.

Identify G/L with $G_{\mathbb{C}}/P$, for some parabolic subgroup of the complexification $G_{\mathbb{C}}$ of G. Let (M_P, f_P) be the mirror family introduced by Rietsch. Then, M_P is a subgroup of a Borel containing the dual torus ${}^L T_{\mathbb{C}}$ inside the Langlands dual ${}^L G_{\mathbb{C}}$. A first approximation to the confirmation of Teleman's conjecture is the following consequence for quantum cohomology:

Conjecture B (Teleman, [129])**.** Let ν be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian T-action. Let $t \in Z({}^L L_{\mathbb{C}})$ denote the symplectic structure on G/L. Then, the quantum cohomology of the symplectic reduction $(G/L) /\!\!/ T(\nu)$ can be computed as the Jacobian ring of the restriction of the T-equivariant superpotential to the fibre of the canonical quotient homomorphism $e : M_P \to {}^L T_{\mathbb{C}}$ over $\exp(\nu)$. The quantum structure comes from the variation of $t \in Z({}^L L_{\mathbb{C}})$.

Moreover, if G has nontrivial (finite) centre Z, then the number of critical points appears with multiplicity $|Z|$.

Let G be a compact, connected Lie group, $T \subseteq G$ a maximal torus. The symplectic reductions of coadjoint orbits (with respect to the Hamiltonian T-action) are known as *weight varieties* and were initially studied in [89]. Weight varieties are geometric analogues of weight spaces of irreducible representations. Indeed, let λ be a strictly dominant weight and $(G/T, \mathcal{L}_\lambda)$ be the complete flag variety together with polarisation $\mathcal{L}_\lambda$ such that, by the Borel-Weil theorem, $H^0(G/T, \mathcal{L}_\lambda) \cong V(\lambda)^*$, the unique irreducible representation of G having lowest weight $-\lambda$. Then, the weight variety $(G/T) /\!\!/ T(\nu)$ inherits a polarisation $\mathcal{L}_{\lambda,\nu}$ and $\dim H^0((G/T) /\!\!/ T(\nu), \mathcal{L}_{\lambda,\nu}) = \dim V(\lambda)^*_\nu$, where $V(\lambda)^*_\nu$ is the ν-weight space of $V(\lambda)^*$.

The (co)homology of weight varieties has been investigated and computed by several authors [67], [37], [51], [53], [52]. For certain weight varieties that can be explicitly identified, the quantum cohomology has been computed (see, for example, [31]). However, a general framework for computations of the quantum cohomology of weight varieties (in the spirit of Rietsch, say) has yet to be obtained. One aim of this thesis is to develop an approach to address this problem.

An important class of weight varieties are quotients of $\mathrm{Gr}_{\mathbb{C}}(2, n)$. These symplectic reductions have a moduli intepretation as the moduli of spatial n-gons with fixed side-lengths $r \in \mathbb{R}^n_{>0}$, called *polygon spaces* $\mathcal{P}_{r,n}$. Polygon spaces are related to the moduli space $\overline{M}_{0,n}$ of stable n-pointed rational curves ([84], [37], [88]) and the moduli space of flat connections on a punctured sphere [89]. Examples of polygon spaces include $\mathbb{P}^{n-3}_{\mathbb{C}}$, $(\mathbb{P}^1_{\mathbb{C}})^{n-3}$ and blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ at $0, 1, 2, 3, 4$ points [38].

SYZ conjecture

In [127], Strominger-Yau-Zaslow interpret mirror symmetry in terms of T-*duality* in string theory.

Conjecture (SYZ Conjecture [127])**.** If X and M are mirror pairs of Calabi-Yau n-folds, then there exist fibrations $g : X \to B$ and $g' : M \to B$, whose fibres are special Lagrangian, with general fibre an n-torus. Furthermore, these fibrations are dual in the sense that, canonically, $X_b \cong H^1(M_b, \mathbb{R}/\mathbb{Z})$ and $M_b \cong H^1(X_b, \mathbb{R}/\mathbb{Z})$, whenever the fibres X_b and M_b are non-singular tori.

The SYZ conjecture proposes an approach for a geometric construction of the mirror of a Calabi-Yau manifold X: once a Lagrangian torus fibration $X \to B$ of X has been obtained, attempt to build M by dualising the toric fibres [58], [70]. In the Fano setting, Auroux [4] extended the SYZ-conjecture:

Conjecture ([4, Conjecture 1.1])**.** Let X be a compact Kähler manifold, $D \subseteq X$ an anticanonical divisor, Ω a holomorphic volume form defined over $X \setminus D$. Then, the mirror LG-model (M, f) can be constructed as a moduli space of special Lagrangian tori in $X \setminus D$ equipped with flat U(1)-connections, with superpotential $f : M \to \mathbb{C}$ given by Fukaya-Oh-Ohta-Ono's m_0 obstruction to Floer homology [39].

One method of constructing Lagrangian torus fibration of a variety X is via integrable systems on (a dense subset of) X. The notion of a toric degeneration of an integrable system on a projective manifold was introduced by Nishinou-Nohara-Ueda [116] (see also [65]): informally, this is a toric degeneration of X such that the integrable can be transported to an integrable system on the toric limit. If the toric limit is Fano and admits a small resolution then the authors compute Floer-theoretic potential functions for X, using deep results of Fukaya-Oh-Ohta-Ono [39].

By degenerating the Gelfand-Tsetlin integrable system [64] on the complete flag variety $\mathrm{SL}_{n+1}(\mathbb{C})/B$ and making explicit computations of holomorphic disks, Nishinou-Nohara-Ueda compute the Floer-theoretic potential function. In this way, they recover the superpotential introduced by Givental using the Gelfand-Tsetlin quiver.

In later work, Nohara-Ueda [117] construct a family of integrable systems Ψ_Γ on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ parameterised by triangulations Γ of a fixed convex planar n-gon Π (the *reference polygon*). For any triangulation Γ of Π, the integrable system Ψ_Γ admits a toric degeneration (originally determined in [125]) and, if the Kahler structure on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ represents the first Chern class, then the toric limit of Ψ_Γ is Gorenstein Fano and admits a small resolution. This allows Nohara-Ueda to compute the Floer-theoretic potential function associated to $\mathrm{Gr}_{\mathbb{C}}(2, n)$. Moreover, for a certain triangulation they recover the superpotential constructed by physical considerations in [35].

The integrable system Ψ_Γ is invariant with respect to the natural torus action on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ and induces an integrable system Φ_Γ on the polygon spaces $\mathcal{P}_{r,n}$. Moreover, the toric degenerations of Ψ_Γ induce toric degenerations of Φ_Γ. The moment polytope of the central limit of Φ_Γ is realised as the intersection of the moment polytope of Ψ_Γ with an affine subspace. In Section 4.5 we describe a conjectural relationship between this hierarchy of integrable systems for the A-model and the appearance of Kashiwara crystal structures in the mirror symmetry of $\mathrm{Gr}_{\mathbb{C}}(2, n)$.

Thesis results

This thesis is comprised of two parts.

In the first part, we develop a new approach to computing the quantum cohomology rings of symplectic reductions of partial flag varieties X, also known as *weight varieties*. Motivated by a conjecture of Teleman [129], we use a mirror family (M_p, f_P) of X introduced by Rietsch [119] to give a conjectural explicit presentation of the quantum cohomology of weight varieties. We determine explicit expressions for the superpotential f_P with respect to a family of parameterisations of M_P, originally studied by Lusztig, Fomin-Zelevinsky in the context of total positivity of reductive groups.

In order to test our conjectural description, we specialise our focus to the class of *polygon spaces* $\mathcal{P}_{r,n}$ in type A. The polygon space $\mathcal{P}_{r,n}$ is a symplectic quotient of the Grassmannian of 2-planes in $\mathbb{C}^n$ and has a modul interpretation as the moduli space of spatial n-gons with fixed consecutive side length given by $r \in \mathbb{R}^n_{>0}$.

Polygon spaces of low dimension have been classified: the moduli space of 4-gons $\mathcal{P}_{r,4}$ is diffeomorphic to $\mathbb{P}^1_{\mathbb{C}}$ (independent of r); the moduli space of 5-gons $\mathcal{P}_{r,5}$ is a rational surface diffeomorphic to either $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, $\mathbb{P}^2_{\mathbb{C}}$ or the Del Pezzo surface obtained by blowing up $\mathbb{P}^2_{\mathbb{C}}$ at $1, 2, 3, 4$ points.

We obtain the following results in Section 3.4.

Theorem 3.4.15. Let $X = \mathrm{Gr}_{\mathbb{C}}(2,4) = \mathrm{SL}_4(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, (M_P, F_P) the Rietsch mirror family. Let $e : M_P \to {}^L T$ be the equivariant structure map. Let $\mathcal{P}_{r,4}$, $r \in \mathbb{R}^4_{>0}$, be the space of 4-gons realised as the symplectic reduction of X. Then, the quantum cohomology of $\mathcal{P}_{r,4}$ can be computed as the Jacobian ring of the restriction of f_P to a generic fibre of e.

Theorem 3.4.16. Let $X = \mathrm{Gr}_{\mathbb{C}}(2,5) = \mathrm{SL}_5(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, (M_P, F_P) the Rietsch mirror family. Let $e : M_P \to {}^L T$ be the equivariant structure map. Let $\mathcal{P}_{r,5}$, $r \in \mathbb{R}^5_{>0}$, be the space of 5-gons realised as the symplectic reduction of X. Let $r = (1,1,1,1,3)$. Then, for the quantum cohomology of $\mathcal{P}_{r,5}$ can be computed as the Jacobian ring of the restriction of f_P to a fibre of e.

In the second part of this thesis, we investigate the appearance of representation-theoretic combinatorial structures, known as *Kashiwara crystals*, in the mirror symmetry of complete flag varieties. We show that, on the B-model side of mirror symmetry for the complete flag variety, the extended string cone introduced by Caldero to define a family of toric degenerations on the A-model side, and later used by Alexeev-Brion [1] in the context of mirror symmetry, can be recovered via a discretisation process known as *tropicalisation*. Specifically, using a non-standard parameterisation of M_P we explicitly recover the extended string cone via tropicalisation. Our description of tropicalisation and the tropicalisation functor Trop is given in Section 4.3. The following result is described in Section 4.4.

Theorem 4.4.5. Let G be a reductive complex algebraic group, $B \subseteq G$ a Borel subgroup. Let (M_B, f_B) be the Rietsch mirror family to ${}^L G/{}^L B$. Then, for every $\mathbf{i}$, a reduced expression of the longest element w_0 of the Weyl group of G, there exists a parameterisation $\underline{\theta}_{\mathbf{i}}$ of a dense open subset of M_P with respect to which the tropical locus $\{\mathrm{Trop}(f_B) \geq 0\}$ is precisely the extended string cone $\underline{C}_{\mathbf{i}}$. Moreover, the λ-inequalities defining $\underline{C}_{\mathbf{i}}$ are explicitly recovered.

We conclude with an observation on how the crystal structure on the B-model side controls aspects of integrable systems appearing on the A-model side.

1.2 Outline

The structure of this thesis is as follows: in Chapter 2 we introduce the background required from symplectic geometry. In Section 2.1 we discuss the general setting of Hamiltonian G-spaces, for G a compact, connected Lie group, and introduce the moment map. For Hamiltonian T-spaces, with T a compact torus, we see that the moment polytope admits internal structure, decomposing into chambers and walls. In Section 2.2 we introduce, following Marsden-Weinstein-Meyer, the symplectic reduction of a Hamiltonian G-space. In this section we describe the important example of polygon spaces $\mathcal{P}_{r,n}$. In Section 2.3 we discuss the symplectic geometry of coadjoint orbits and show that they are Hamiltion T-spaces for the coadjoint action of T. We describe the wall structure on their moment polytopes in terms of root-theoretic data. In Section 2.4 we provide some examples of the symplectic reduction of coadjoint orbits (so-called weight varieties). We finish the chapter with a brief discussion on the well-known connection between the symplectic reduction and GIT quotients.

In Chapter 3 we develop a new approach to computing the quantum cohomology of weight varieties. In Section 3.1 we introduce the Landau-Ginzburg model (M_P, f_P) first proposed by Rietsch, and discuss its connection to computing (T-equivariant) quantum cohomology of partial flag varieties. In Section 3.2 we present Teleman's conjectural mirror construction for the quantum cohomology of weight varieties. Our conjectural description of the quantum cohomology of weight varieties is given in Conjecture 3.2.7. In Section 3.3 we obtain explicit expressions for the superpotential f_P, which will be essential in verifying Conjecture 3.2.7. Section 3.4 specialises to the type A setting and we make new quantum cohomology computations for polygon spaces $\mathcal{P}_{r,n}$ of low rank, thereby verifying Conjecture 3.2.7 in this setting. We conclude this chapter with an outline of future directions of research.

Chapter 4 is an investigation into the appearance of representation-theoretic structures in the mirror symmetry for complete flag varieties. In Section 4.1, we recall background from the theory of quantised universal enveloping algebras. Section 4.2 introduces Lusztig's canonical basis $\mathcal{B}$ and its consequences for representation theory. In particular, we give a brief account of the role of $\mathcal{B}$ in determining combinatorial tensor product multiplicity formulae. We define several parameterisations of $\mathcal{B}$ including the the family of string parameterisations due to Littelmann. We conclude this section by introducing the extended string cone $\underline{C}_{\mathbf{i}}$ and the λ-inequalities that define it. In Section 4.3, we give a brief account of Kashiwara's theory of crystals and their geometric counterparts developed by Berenstein-Kazhdan. In this section we develop the tool of tropicalisation, realised as a functor Trop from a certain class of varieties to Set. Section 4.4 introduces a non-standard parameterisation of the Rietsch mirror (M_B, f_B) and we state and prove our main result Theorem 4.4.5. We conclude with a discussion illuminating intriguing similarities between the hierarchy of a family of integrable systems on the A-model side (introduced in [117]) and the crystal structure obtained in Theorem 4.4.5 (on the B-model side); this will be focus of future work.

1.3 Notation

In this preliminary section we introduce the conventions and definitions we adopt throughout this thesis.

Let $\mathbb{P}$ be a monoid, A some nonempty set. We write $\mathbb{P}A$ for the $\mathbb{P}$-span of A (i.e. the free $\mathbb{P}$-module generated by A if A is not a subset of some $\mathbb{P}$-module). If G is a group then $Z(G)$ will denote the centre of G.

We introduce our conventions for Lie theoretic objects, for further details see [126]. Let G be a complex reductive algebraic group; unless otherwise stated G will be assumed connected. We fix a choice of maximal torus $T \subseteq G$, a Borel subgroup $B_+ \subseteq G$ containing T, and opposite Borel subgroup B_- so that $B_- \cap B_+ = T$. We write $N_\pm$ for the unipotent radical of $B_\pm$. A parabolic subgroup $P \subseteq G$ admits a Levi decomposition $P = L_P N_P$, where N_P is the unipotent radical, L_P is reductive and $N_P \cap L_P = \{e\}$. We write $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}_\pm, \mathfrak{n}_\pm, \mathfrak{p}$ for the corresponding Lie algebras. We denote the Weyl group $W = N_G(T)/T$. For $w \in W$, $t \in T$, we will sometimes write $t^w = wtw^{-1}$.

The above choices are uniquely determined (up to isomorphism) by the root datum $\Psi(G) = (X, R, X^\vee, R^\vee)$ associated to the pair (G, T). Here

$$X \coloneqq \mathrm{Hom}(T, \mathbb{G}_m), \quad X^\vee \coloneqq \mathrm{Hom}(\mathbb{G}_m, T)$$

and $R \subseteq X$ is the set of roots relative to T; $R^\vee \subseteq X^\vee$ is the corresponding set of coroots. We will interchangeably refer to elements of X (resp. $X^\vee$) as weights or characters (resp. coweights or cocharacters). There is a canonical pairing between X and $X^\vee$

$$\begin{array}{rccc} \langle , \rangle : & X \times X^\vee & \longrightarrow & \mathbb{Z} \\ & (\lambda, \mu^\vee) & \longmapsto & \langle \lambda, \mu^\vee \rangle \end{array}$$

defined by $(\lambda \circ \mu^\vee)(z) = z^{\langle \lambda, \mu^\vee \rangle}$. With respect to this pairing there are canonical identifications

$$X \cong \mathrm{Hom}(X^\vee, \mathbb{Z}), \quad X^\vee \cong \mathrm{Hom}(X, \mathbb{Z}).$$

Denote the root lattice $Q \coloneqq \mathbb{Z}R$, and the coroot lattice $Q^\vee \coloneqq \mathbb{Z}R^\vee$. We define the lattice of integral weights $\Pi \subseteq X_\mathbb{Q} \coloneqq \mathbb{Q} \otimes X$ to be the lattice

$$\Pi = \{\lambda \in X_\mathbb{Q} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ \alpha^\vee \in R^\vee\}.$$

We write $w(\lambda)$, or simply $w\lambda$, (resp. $w(\lambda^\vee)$) for the action of $w \in W$ on $\lambda \in X$ (resp. $\lambda^\vee \in X^\vee$); this descends to an action on Q (resp. $Q^\vee$) preserving R (resp. $R^\vee$).

The choice of Borel B_+ induces a choice of positive roots $R^+ \subseteq R$ and simple roots $S \subseteq R^+$. We write $S^\vee \subseteq X^\vee$ for the simple coroots. We write $Q_{\geq 0} \coloneqq \mathbb{Z}_{\geq 0} R^+ = \mathbb{Z}_{\geq 0} S$, and $Q_{\leq} \coloneqq \mathbb{Z}_{\geq 0} R^- = -\mathbb{Z}_{\geq 0} S$, with analogous definitions for $Q^\vee_{\geq 0}$, $Q^\vee_{\leq 0}$. The monoid of dominant weights is

$$X_+ \coloneqq \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0,\ \alpha^\vee \in S^\vee\},$$

with an analogous definition for the monoid of dominant coweights $X_+^\vee$. A weight $\lambda \in X$ is antidominant if $w_0(\lambda) \in X_+$ (see below for the definition of w_0); there is an analogous definition of antidominant coweight. We denote the monoid of dominant weights (resp. dominant coweights) X_- (resp. $X_-^\vee$).

If G is a reductive complex algebraic group with root datum $(X, R, X^\vee, R^\vee)$, then we call $(\Pi, S, \Pi^\vee, S^\vee)$ the associated Cartan datum.

There is a partial ordering on X (resp. $X^\vee$) defined as follows:

$$\lambda \geq \mu \text{ (resp. } \lambda^\vee \geq \mu^\vee) \quad \iff \quad \lambda - \mu \in Q_+ \text{ (resp. } \lambda^\vee - \mu^\vee \in Q_+^\vee).$$

There is a unique identification

$$\begin{array}{ccc} S & \longleftrightarrow & S^\vee \\ \alpha & \longleftrightarrow & \alpha^\vee \end{array}$$

such that $\langle \alpha, \alpha^\vee \rangle = 2$. Using this identification we index both S and $S^\vee$ by the same set I, so that $S = \{\alpha_i\}_{i \in I}$ and $S^\vee = \{\alpha_i^\vee\}_{i \in I}$, where $\alpha_i^\vee = (\alpha_i)^\vee$. Define the fundamental weights $\varpi_i \in \Pi$, $i \in I$, to be the weights such that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$, for $i, j \in I$.

There is an involution $i \mapsto i^*$ on I, where $-w_0(\alpha_i) = \alpha_{i^*}$, for $i \in I$ (see below for definition of w_0). This is equivalent to $w_0 s_i w_0 = s_{i^*}$.

For $\alpha \in R^+$, we make a choice of corresponding root subgroup homomorphism

$$x_\alpha : \ \mathbb{A}^1 \longrightarrow N_+$$

satisfying

$$t x_\alpha(c) = x_\alpha(\alpha(t)c)t, \quad t \in T.$$

We write $x_i \coloneqq x_{\alpha_i}$ and $y_i \coloneqq x_{-\alpha_i}$, for $i \in I$. If we totally order $R^+ = \{\beta_1, \ldots, \beta_m\}$ then there is an isomorphism of varieties $\prod_{j=1}^m x_{\beta_i} : \mathbb{A}^m \to N_+$. It is well-known that G is generated by T and $\operatorname{im} x_\alpha$, $\alpha \in S \cup -S$.

If $P \supseteq B_+$ then there is a unique subset $J = J(P) \subseteq I$ such that P is generated by B_+ and $\operatorname{im} y_j$, $j \in J$. We write $P = P_J$ if we want to make J explicit; P is called a standard parabolic subgroup. The Levi subgroup L_P is generated by T and $\operatorname{im} x_j$, $\operatorname{im} y_j$, $j \in J$. We write W_P for the Weyl group of the pair (L_P, T). If $P = P_J$ then W_P is identified with the subgroup of W generated by s_j, $j \in J$. Write $W^P \subseteq W$ for the set of minimal length coset representatives of W/W_P. The centre $Z(L_P)$ is a subgroup of T equal to T^{W_P}, the elements in T fixed by W_P

The Weyl group W is generated by reflections s_i, $i \in I$, subject to the standard Coxeter relations

$$s_i^2 = 1, \quad (s_i s_j)^{m_{ij}} = 1,$$

where $m_{ij} = 2, 3, 4$ or 6 whenever, respectively, $\langle \alpha_i, \alpha_j^\vee \rangle = 0, 1, 2, 3$. The latter relations are called the braid relations. If $w = s_{i_1} \cdots s_{i_r}$, with r minimal, then we define $\ell(w) = r$, the length of w. For such a presentation of w we call the sequence $(i_1, \ldots, i_r)$ a reduced expression of w. The set of all reduced expressions of w will be denoted $R(w)$. There is a unique element $w_0 \in W$, with $w_0^2 = e \in W$, having maximal length $\ell(w_0) = \dim N_+ = |R^+|$. We write W_P for the Weyl group of the pair (L_P, T). If $P = P_J$ then W_P is identified with the subgroup of W generated by s_j, $j \in J$. Let $w_0^P \in W_P$ be the longest element. Define $w_P^{-1} \in W$ to be the longest element of W^P.

A result of Matsumoto, Tits (see [18]) shows that any two reduced expressions are related by braid relations. Define

$$\overline{s}_i \coloneqq x_i(-1)y_i(1)x_i(-1), \quad i \in I.$$

Then, $\overline{s}_i \in N_G(T)$ and is a representative of $s_i \in W$. The $\overline{s}_i$, $i \in I$, satisfy the braid relations so that the element

$$\overline{w} = \overline{s}_{i_1} \cdots \overline{s}_{i_r} \in N_G(T),$$

where $(i_1, \dots, i_r) \in R(w)$, is a well-defined representative of $w \in W$. In particular, if $u, v \in W$ and $w = uv$, with $\ell(w) = \ell(u) + \ell(v)$, then $\overline{w} = \overline{uv}$. In general, the $\overline{s}_i$ do not satisfy $\overline{s}_i^2 = 1 \in G$, although we have $\overline{s}_i^2 = \alpha_i^\vee(-1)$. For the longest element $w_0 \in W$, $\overline{w}_0 B_\pm \overline{w}_0^{-1} = B_\mp$: in particular, $\overline{w}_0 N_\pm \overline{w}_0^{-1} = N_\mp$.

We will use the following involutive antiautomorphisms of G

(i) the transpose $g \mapsto g^T$, determined by

$$x_i(a)^T = y_i(a), \quad y_i(a)^T = x_i(a), \quad t^T = t, \quad i \in I, t \in T; \tag{1.3.1}$$

(ii) the positive inverse $g \mapsto g^\iota$, determined by

$$x_i(a)^\iota = x_i(a), \quad y_i(a)^\iota = y_i(a), \quad t^\iota = t^{-1}, \quad i \in I, t \in T. \tag{1.3.2}$$

These antiautomorphisms commute with each other and with the involutive antiautomorphism $g \mapsto g^{-1}$ of G. We have

$$\overline{w}^T = \overline{w}^{-1}, \quad \text{and} \quad \overline{w}^\iota = \overline{w^{-1}}.$$

For any $g = utv \in G_0 = N_- T N_+$ admitting Gauss decomposition, we define

$$\pi^-(g) = u, \quad \pi^0(g) = t, \quad \pi^+(g) = v, \quad \pi^{\leq 0}(g) = ut, \quad \pi^{\geq 0}(g) = tv. \tag{1.3.3}$$

Following [15, Section 6], we define the generalised minors $\Delta_{u\mu, v\mu}$, $\mu \in X_+$, $u, v \in W$, to be the regular functions on G whose restriction to $\overline{u} G_0 \overline{v}^{-1}$ is given by

$$\Delta_{u\mu, v\mu}(g) \coloneqq \mu(\pi^0(\overline{u}^{-1} g \overline{v})).$$

When G is type A, so that W is identified with a group of permutations, the generalised minor $\Delta_{u\varpi_i, v\varpi_i}$ is the matrix minor with row set $I = \{u(1), \dots, u(i)\}$ and column set $J = \{v(1), \dots, v(i)\}$.

If G is a reductive complex algebraic group with root datum $(X, R, X^\vee, R^\vee)$ then the Langlands dual group ${}^L G$ is the reductive complex algebraic group with dual root datum $(X^\vee, R^\vee, X, R)$. When referring to subgroups of the Langlands dual we will write ${}^L T$, ${}^L B_\pm$, ${}^L N_\pm$ etc. We will also write $X({}^L T)$ when referring to the weight lattice of the pair $({}^L G, {}^L T)$, with similar notation for the other objects defined above.

Chapter 2

Symplectic geometry of coadjoint orbits

In this chapter we introduce the necessary background from symplectic geometry and Hamiltonian actions of compact Lie groups. In Section 2.1 we introduce Hamiltonian G-spaces, where G is a compact, connected Lie group. We introduce the additional data of the moment map and describe how the moment polytope admits a chamber structure. In Section 2.2 we recall the notion of symplectic reduction and indicate the construction of the modul space of spatial polygons. We also present the construction of the complex Grassmannian of 2-planes $\mathrm{Gr}_{\mathbb{C}}(2,n)$ via symplectic reduction. In Section 2.3 we consider in more detail a special case of Hamiltonian G-spaces, namely, the coadjoint orbits of G. We show that the chamber structure of the moment polytope can be obtained from the structure of the Weyl group and root system in $\mathfrak{g}$. Section 2.4 introduces the class of weight varieties: these are those symplectic manifolds that can be realised as reductions of coadjoint orbits. We close this section with an analysis of the chamber structure for the $\mathrm{Gr}_{\mathbb{C}}(2,n)$. Finally, in Section 2.5 we briefly discuss the relationship between symplectic reduction and GIT quotients in algebraic geometry, focusing mainly on the case of coadjoint orbits.

Most of the material in this chapter is standard and can be found in any graduate textbook on symplectic geometry, for example [3],[111]. The material concerning reduction of coadjoint orbits and the wall structure of moment polytopes can be found in [89], or [60].

2.1 Hamiltonian G-spaces

Let G be a compact, connected Lie group, (M,ω) a symplectic manifold. We are interested in symplectic (left) actions of G on M,

$$\begin{aligned} a: \quad G &\longrightarrow \mathrm{Symp}(M,\omega) \\ g &\longmapsto a_g \end{aligned}$$

where $\mathrm{Symp}(M,\omega)$ is the group of symplectomorphisms of (M,ω).

Remark 2.1.1. We will write $g \cdot m := a_g(m)$, $g \in G$, $m \in M$, whenever a group G acts on a set M.

For each $X \in \mathfrak{g}$, we let $\underline{X}$ denote the infinitesimal action of X on M induced by a. This is the (unique) vector field on M with flow $\{a_{\exp(-tX)}\}_{t\in\mathbb{R}}$. Explicitly, for each $m \in M$, we consider the orbit map

$$\begin{aligned} \sigma_m : \quad G &\longrightarrow M \\ g &\longmapsto g \cdot m \end{aligned}$$

Then, $\underline{X}_m := (d\sigma_m)_e(-X) \in T_m M$.

Remark 2.1.2.

1) The sign appearing in the definition of $\underline{X}$ ensures that $[\underline{X}, \underline{Y}] = \underline{[X, Y]}$.

2) We will refer to the vector field $\underline{X}$, $X \in \mathfrak{g}$, as a *fundamental vector field.*

For any $m \in M$, the tangent space at m to the orbit $G \cdot m$ is spanned by the fundamental vector fields

$$T_m(G \cdot m) = \{\underline{X}_m \mid X \in \mathfrak{g}\}$$

Definition 2.1.3. The action $a : G \to \mathrm{Symp}(M, \omega)$ is *Hamiltonian* if, for every $X \in \mathfrak{g}$, there exists a function

$$\mu : M \longrightarrow \mathfrak{g}^*,$$

such that

1) for each $X \in \mathfrak{g}$, the function

$$\begin{aligned} \mu^X : \quad M &\longrightarrow \mathbb{R} \\ m &\longmapsto \langle \mu(m), X \rangle \end{aligned}$$

is a Hamiltonian function for the fundamental vector field $\underline{X}$, so that

$$d\mu^X = i_{\underline{X}}\omega.$$

2) μ is equivariant: for every $g \in G$, we have

$$\mu \circ a_g = \mathrm{Ad}^*(g) \circ \mu$$

Here $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ is the coadjoint action of G on $\mathfrak{g}^*$.

We call the datum (M, ω, a, μ) a *Hamiltonian G-space*, and μ is the *moment map.*

Remark 2.1.4. We will refer to a Hamiltonian G-space (M, ω, a, μ) as (M, ω), the extra data of the action and choice of a moment map being implicit.

If μ is the moment map for a Hamiltonian G-space then, for $X \in \mathfrak{g}$, we have

$$\mu^X = H^X \circ \mu,$$

where $H^X : \mathfrak{g}^* \to \mathbb{R}$ is the linear 'evaluation at X' map. Infinitesimally, we obtain

$$i_{\underline{X}}\omega = d\mu^X = H^X \circ d\mu \implies \omega_m(\underline{X}_m, V) = \langle d\mu_m(V), X\rangle, \quad V \in T_mM. \tag{2.1.1}$$

Hence, $\ker d\mu_m$ is the ω-complement of $T_m(G \cdot m)$, and the annihilator of $\operatorname{im} d\mu_m \subseteq \mathfrak{g}^*$ is

$$(\operatorname{im} d\mu_m)^\circ = \{X \in \mathfrak{g} \mid \underline{X}_m = 0\},$$

which can be identified with the Lie algebra of the stabiliser $G_m = \{g \in G \mid a_g(m) = m\}$.

Lemma 2.1.5. *$d\mu_m$ is surjective if and only if the stabiliser G_m is discrete (hence, finite). In particular, $m \in M$ is a critical point of μ if and only if* $\dim G_m \geq 1$.

Let $H \subseteq G$ be a subgroup, and denote (H) be the *type of* H: (H) is the set of subgroups of G that are conjugate to H. The *orbit-type stratification* of M is the partition of M into subsets

$$M_{(H)} = \{m \in M \mid G_m \in (H)\},$$

where $H \subseteq G$ is a subgroup. By the equivariant Darboux theorem [63], each subset $M_{(H)}$ is a union of G-invariant symplectic submanifolds of M (not necessarily of the same dimension). Moreover, $M_{(H)}$ is a Hamiltonian G-space with moment map $\mu_{|M_{(H)}}$.

Remark 2.1.6. An example when the connected components have different dimensions is easily seen: consider the action of S^1 on $M = \mathbb{C}P^2$, where $e^{it} \cdot [t \cdot z_0 : z_1 : z_2]$, $e^{it} \in S^1$, then the fixed point set is $M_{(S^1)}$ consists of the point $[1 : 0 : 0]$ and the line at infinity $[0 : z_1 : z_2]$.

If G is commutative then $(H) = \{H\}$, and M partitions into a union of G-invariant submanifolds

$$M = \bigcup_{H \subseteq G} \operatorname{Fix}(H), \tag{2.1.2}$$

where

$$\operatorname{Fix}(H) = \{m \in M \mid h \cdot m = m, \text{ for every } h \in H\}.$$

When M is compact the union in (2.1.2) is finite [3, Ch. 2]. Hence, by Lemma 2.1.5 we obtain the following result.

Proposition 2.1.7. *Let G be a commutative compact, connected Lie group, and (M, ω) be a Hamiltonian G-space, with μ the corresponding moment map. The collection of critical points of μ is a union of Hamiltonian G-spaces of the form* $\operatorname{Fix}(H)$, *for $H \subseteq G$ a positive dimensional stabiliser of some point: if M is compact then this union is finite. The critical values of μ are the images of these submanifolds.*

Remark 2.1.8. For the remainder of this section we assume that $G = T$ is a torus.

Let (M, ω) be a Hamiltonian T-space with moment map μ.

Theorem 2.1.9 (Atiyah, Guillemin-Sternberg [2, 62])**.** *Let (M, ω) be a Hamiltonian T-space with moment map $\mu : M \to \mathfrak{t}^*$. Assume that M is compact. Then, the set of fixed points of the action is a finite union of connected symplectic submanifolds $C_1, \dots, C_N$. Moreover, μ is constant on each of these components, $\mu(C_i) = \xi_i \in \mathfrak{t}^*$, and $\mu(M)$ is the convex hull of $\xi_1, \dots, \xi_N$,*

$$\mu(M) = \left\{ \sum_{i=1}^{N} c_i \xi_i \;\middle|\; \sum_{i=1}^{N} c_i = 1, c_j \geq 0 \right\} \subseteq \mathfrak{t}^*$$

Definition 2.1.10. The *moment polytope* associated to a Hamiltonian T-space (M, ω) with moment map μ is the polytope $\mu(M) \subseteq \mathfrak{t}^*$.

Remark 2.1.11. For a Hamiltonian T-space (M, ω) with moment map μ we will write $\Delta_M \coloneqq \mu(M)$ for its moment polytope, or simply Δ if there is no risk of confusion.

Let $F_1, \dots, F_r$ be the closures of the connected components $X_1, \dots, X_r$ of the orbit-type strata from Proposition 2.1.7 with corresponding stabiliser subgroups $T_1, \dots, T_r \subseteq T$. Each F_j is a connected component of $\mathrm{Fix}(T_j)$, where T_j is the stabiliser of a generic point in F_j, and F_j is a Hamiltonian T-space with moment map $\mu_{|F_j}$.

Set $H_j = T/T_j$. The T-action has kernel T_j and F_j inherits an effective Hamiltonian H_j-action. The moment map $\mu_j : F_j \to \mathfrak{h}_j^*$ for this action is unique up to a constant, which we now specify.

For any $m \in F_j$, (2.1.1) shows that $\mathrm{im}\left(d\mu_{|F_j}\right)_m$ is the annihilator of $\mathfrak{t}_j$ inside $\mathfrak{t}^*$, so that $\mathrm{im}\left(d\mu_{|F_j}\right)_m = \mathfrak{h}_j^*$. In particular, $\mu(F_j)$ is some translate of $\mathfrak{h}_j^*$ inside $\mathfrak{t}^*$. For example, if $m_j \in F_j$ as a fixed point for the T-action then $\mu(F_j) \subseteq \mu(m_j) + \mathfrak{h}_j^*$. Hence, a moment map μ_j for the H_j-action on F_j can be specified by requiring that $\mu(F_j)$ lands in $\mathfrak{h}_j^*$.

Remark 2.1.12. Applying Theorem 2.1.9 we obtain $\mu(F_j)$ is a convex polytope, for each j. By the above discussion, this convex polytope is a subset of the intersection $\Delta_M \cap (\xi_j + \mathfrak{h}_j^*)$, where ξ_j is the image of some fixed point in F_j. Furthermore, if $F_i \subseteq F_j$ then $\mu(F_i) \subseteq \mu(F_j)$.

Definition 2.1.13. A *codimension-k wall in Δ_M* (or simply, a *wall in Δ_M*) is the image of some F_j in Δ_M, where $\dim T_j = k$. We denote the set of all walls by $\mathcal{F}$.

A wall is *proper* if it has positive codimension.

An *internal wall* is a proper wall containing a point in the interior of Δ_M. An *external wall* is any proper wall that is not internal.

Definition 2.1.14. A *chamber* is a connected component of the set

$$\Delta_M^\circ \coloneqq \Delta_M \setminus \bigcup_{\substack{d \in \mathcal{F} \\ d \text{ proper}}} d \tag{2.1.3}$$

Remark 2.1.15. By Proposition 2.1.7, Δ_M° is precisely the set of regular values of μ. Moreover, Δ_M decomposes into a union of polytopes: the interior of each polytope corresponds to a unique chamber.

2.2 Symplectic reduction

Let G be a compact Lie group (not necessarily a torus) and (M, ω) a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Let $\lambda \in \Delta_M$. By equivariance of the moment map, the level set $\mu^{-1}(\lambda)$ is G_λ-invariant, where G_λ is the stabiliser of λ under the coadjoint action of G on $\mathfrak{g}^*$. If λ is a regular value then the action of G_λ is locally free and the quotient $\mu^{-1}(\lambda)/G_\lambda$ admits, at worst, orbifold singularities.

Consider the diagram:

$$\begin{array}{ccc} \mu^{-1}(\lambda) & \xhookrightarrow{i_\lambda} & M \\ {\scriptstyle p_\lambda}\downarrow & & \\ \mu^{-1}(\lambda)/G_\lambda & & \end{array}$$

Whenever the action on $\mu^{-1}(\lambda)$ is free, the quotient M_λ admits a canonical manifold structure, and the quotient map p_λ is a principal G_λ-bundle.

Theorem 2.2.1 (Marsden-Weinstein [108], Meyer [113])**.** *Let G be a compact Lie group and (M, ω) be a Hamiltonian G-space with moment map μ. Assume that $\lambda \in \mathfrak{g}^*$ is a regular value of μ. Then, the topological quotient $\mu^{-1}(\lambda)/G_\lambda$ is a symplectic orbifold of dimension $\dim M - 2\dim G_\lambda$, and there exists a unique symplectic form ω^{red} on $\mu^{-1}(\lambda)/G_\lambda$ such that*

$$p_\lambda^*\omega^{\mathrm{red}} = i_\lambda^*\omega$$

In particular, whenever G acts freely on $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/G$ inherits the structure of a symplectic manifold.

Definition 2.2.2. The symplectic orbifold $(\mu^{-1}(\lambda), \omega^{\mathrm{red}})$ is the *symplectic reduction of M by G_λ at λ*, and will be denoted $M /\!\!/ G_\lambda(\lambda)$.

We will need the following result.

Proposition 2.2.3 (Reduction in stages)**.** *Let $G = H \times K$ be a compact, connected Lie group and (M, ω) be a Hamiltonian G-space with moment map μ. Let μ_H and μ_K be the moment maps of the induced actions of H and K on M. Then, μ can be canonically identified with $\mu_H \times \mu_K$. For any regular value $\nu = (\alpha, \beta) \in \mathfrak{g}^* = \mathfrak{h}^* \times \mathfrak{k}^*$ so that α is a regular value of μ_H and β is a regular value of μ_K, the symplectic reduction $\mu^{-1}(\nu)/G$ is symplectomorphic to the symplectic reduction of the Hamiltonian K-space $\mu_H^{-1}(\alpha)/H$ at β. An analogous statement holds with the roles of H and K reversed.*

We finish this section with some examples of symplectic reduction that we will return to in Section 2.5.

Example 2.2.4. Let (S^2, ω) be the 2-sphere with its standard $SO(3)$-invariant symplectic form ω so that $\int_{S^2} \omega = 4\pi$. Let $n \geq 3$ and fix $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, a sequence of positive real numbers. The product manifold $(S^2)^n$ is given the symplectic form $\Omega_r = \sum_{i=1}^n r_i \omega_i$, where ω_i is the pull-back of ω along the j^{th} projection. Points in $(S^2)^n$ can be identified with polygonal paths in $\mathbb{R}^3$ having consecutive edge-lengths $r_1, \ldots, r_n$. The natural action of $SO(3)$ provides a diagonal action on $((S^2)^n, \Omega_r)$ with moment map

$$\begin{array}{rccc} \mu_{r,n}: & (S^2)^n & \longrightarrow & \mathfrak{so}(3)^* \cong \mathbb{R}^3 \\ & (a_1, \ldots, a_n) & \longmapsto & \sum_{i=1}^n r_i a_i \end{array}$$

The set $\mu_{r,n}^{-1}(0)$ can be identified with polygons in $\mathbb{R}^3$ having consecutive side-lengths $r_1, \ldots, r_n$. The critical points of $\mu_{r,n}$ are the degenerate polygons: these are those polygons P lying completely in a line. In particular, whenever n is odd, there are no degenerate polygons in $\mu_{r,n}^{-1}(0)$ and Theorem 2.2.1 implies that the symplectic reduction is a smooth $(2n-6)$-dimensional symplectic manifold.

Definition 2.2.5. The symplectic reduction of $(S^2)^n$ by SO(3) at 0 is called the *moduli space of spatial n-gons $\mathcal{P}_{r,n}$* or, simply, a *polygon space*.

Polygon spaces have been studied intensively over the past couple of decades and have connections with the moduli space $\overline{M}_{0,n}$ of stable n-pointed rational curves ([84], [37], [88]) and the moduli space of flat connections on a punctured sphere [89].

We record the following examples of polygon spaces (see [66], [38]).

Example 2.2.6. (i) The simplest case when $n = 3$ is trivial as there is exactly one 3-gon in $\mathbb{R}^3$ with prescribed side-lengths, up to SO(3)-invariance. Hence, $\mathcal{P}_{r,n}$ is a point.

(ii) Let $n = 4$. In this case $\mathcal{P}_{r,n}$ is a 2-dimensional symplectic manifold diffeomorphic to S^2 (in particular, independent of r).

(iii) Let $n = 5$. Then, $\mathcal{P}_{r,n}$ is either $S^2 \times S^2$ or a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at $0, 1, 2, 3, 4$ points [37].

(iv) When $r = (1, 1 \ldots, 1, n, n, n)$, $\mathcal{P}_{r,n}$ is identified with $(\mathbb{P}^1_{\mathbb{C}})^{n-3}$. This can be seen by the following argument: consider $\mathcal{P}_{r,n}$ to be the moduli space of weighted configurations of points in $\mathbb{P}^1_{\mathbb{C}}$ [33]. Let $z_1, \ldots, z_n$ be such a configuration all lying in the same affine chart, which we take to be $\mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$. Consider the cross-ratios

$$w_i = \frac{z_{n-2} - z_n}{z_n - z_{n-1}} \cdot \frac{z_i - z_{n-1}}{z_n - z_i}, \quad i = 1, \ldots, n-3.$$

The points z_{n-2}, z_{n-1}, z_n never collide on the subset of semi-stable configurations implying the existence of a map

$$\mathcal{P}_{r,n} \longrightarrow (\mathbb{P}^1_{\mathbb{C}})^{n-3}$$

This map is an isomorphism (see [37], [38]).

We recall the well-known construction of Grassmannians of k-planes in $\mathbb{C}^n$ via symplectic reduction.

Example 2.2.7. Consider $G = U(k)$ acting on the space of $n \times k$ complex matrices $\mathrm{Mat}_{n\times k}(\mathbb{C}) \cong \mathbb{C}^{kn}$: $k \cdot A = Ak^{-1}$. Consider $\mathrm{Mat}_{n\times k}(\mathbb{C})$ equipped with the standard symplectic form on complex affine space. Then, we have

$$\begin{array}{rccl} \mu: & \mathrm{Mat}_{n\times k}(\mathbb{C}) & \longrightarrow & \mathfrak{u}(k)^* \\ & A & \longmapsto & \mu(A) : x \mapsto \frac{i}{2}\operatorname{tr}(xAx^*) \end{array}$$

Using a $U(k)$-equivariant identification (via the Killing form, say), we identify $\mathfrak{u}(k)^* \cong \mathfrak{u}(k)$ and the moment map is

$$\mu(A) = \frac{i}{2}A^*A.$$

The point $y = \frac{i}{2}\mathbb{I}_k \in \mathfrak{u}(k)$ is fixed by the coadjoint action of $U(k)$ on $\mathfrak{u}(k)$ and

$$\mu^{-1}(y) = \{A \mid \mathrm{Mat}_{n\times k} \mid A^*A = \mathbb{I}_k\}.$$

This is the set of unitary k-frames in $\mathbb{C}^n$. Hence, the symplectic reduction $\mu^{-1}(y)/U(k)$ is the complex Grassmannian of k-planes in $\mathbb{C}^n$.

2.3 Coadjoint orbits

Let G be a compact, connected Lie group, $T \subseteq G$ a maximal torus, W the Weyl group for the pair (G, T). Denote the Lie algebra of G (resp. T) by $\mathfrak{g}$ (resp. $\mathfrak{t}$), and let $\mathfrak{g}^*$ (resp. $\mathfrak{t}^*$) by the dual vector space.

In this section we will consider the (co)adjoint actions of G on $\mathfrak{g}$ and $\mathfrak{g}^*$. For each $X \in \mathfrak{g}$, we define the following function on $\mathfrak{g}^*$:

$$\begin{array}{rccl} H^X: & \mathfrak{g}^* & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & H^X(\xi) = \langle \xi, X\rangle \end{array}$$

Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit, and $\xi \in \mathcal{O}$. Identify $\mathcal{O}$ with G/G_ξ via the orbit map

$$\begin{array}{rccl} \sigma_\xi: & G & \longrightarrow & \mathcal{O} \\ & g & \longmapsto & g \cdot \xi \end{array}$$

where $G_\xi = \{g \in G \mid g \cdot \xi = \xi\}$ is the stabiliser of ξ in G. With this identification, the tangent space to $\mathcal{O}$ at ξ is $\mathfrak{g}/\mathfrak{g}_\xi$, where $\mathfrak{g}_\xi$ is the Lie algebra of G_ξ.

For the coadjoint action of G on $\mathfrak{g}^*$, the fundamental vector field $\underline{X}$ generated by $X \in \mathfrak{g}$ satisfies

$$\langle \underline{X}_\xi, Y \rangle = \langle \xi, [X,Y] \rangle, \qquad \xi \in \mathfrak{g}^*, Y \in \mathfrak{g}. \tag{2.3.1}$$

In particular, for any $X, Y \in \mathfrak{g}$, we have

$$\underline{X}(H_Y) = H^{[X,Y]} = \langle dH^Y, \underline{X} \rangle, \tag{2.3.2}$$

For each $\xi \in \mathfrak{g}^*$, there is defined on $\mathfrak{g}$ a skew-symmetric bilinear form ω_ξ,

$$\omega_\xi(Y, X) := \langle \xi, [X,Y] \rangle, \qquad X, Y \in \mathfrak{g}. \tag{2.3.3}$$

The form ω_ξ descends to a nondegenerate skew-symmetric form on the quotient $\mathfrak{g}/\mathfrak{g}_\xi$: the kernel of ω_ξ is precisely $\mathfrak{g}_\xi$. Hence, we obtain a nondegenerate skew-symmetric bilinear form on the tangent space $T_\xi\mathcal{O}$, which we also denote ω_ξ. In this way, we obtain a nondegenerate 2-form $\omega_\mathcal{O}$ on $\mathcal{O}$, known as the *Kostant-Kirillov-Souriau (KKS) form* [94, 124].

For $X, Y \in \mathfrak{g}$, (2.3.3) implies that $\omega_\mathcal{O}(\underline{X}, \underline{Y}) = H^{[Y,X]}$. Fixing $X \in \mathfrak{g}$, and using (2.3.1), we obtain, for all $Y \in \mathfrak{g}$,

$$\langle i_{\underline{X}}\omega_\mathcal{O}, \underline{Y} \rangle = \omega_\mathcal{O}(\underline{X}, \underline{Y}) = \langle dH^X, \underline{Y} \rangle.$$

As the vector fields $\underline{Y}$, for $Y \in \mathfrak{g}$, span the tangent spaces to $\mathcal{O}$ at each point, we have

$$i_{\underline{X}}\omega_\mathcal{O} = dH^X. \tag{2.3.4}$$

Applying the Lie derivative $\mathcal{L}_{\underline{X}}$ to dH^Y, for $X \in \mathfrak{g}$, this shows, together with (2.3.2),

$$\mathcal{L}_{\underline{X}} dH^Y = dH^{[X,Y]} = i_{\underline{[X,Y]}}\omega_\mathcal{O}. \tag{2.3.5}$$

Using the formula $[\mathcal{L}_{\underline{X}}, i_{\underline{Y}}] = i_{[\underline{X},\underline{Y}]}$, and the fact that $[\underline{X}, \underline{Y}] = \underline{[X,Y]}$ (Remark 2.1.2),

$$\mathcal{L}_{\underline{X}} i_{\underline{Y}} \omega_\mathcal{O} = i_{\underline{[X,Y]}}\omega_\mathcal{O} + i_{\underline{Y}} \mathcal{L}_{\underline{X}} \omega_\mathcal{O}$$

By (2.3.4) and (2.3.5), we obtain $i_{\underline{Y}} \mathcal{L}_{\underline{X}} \omega_\mathcal{O} = 0$, for all $X, Y \in \mathfrak{g}$. Hence, $\mathcal{L}_{\underline{X}} \omega_\mathcal{O} = 0$, for all $X \in \mathfrak{g}$ and $\omega_\mathcal{O}$ is G-invariant.

Using Cartan's formula, and (2.3.4), we find that, for every $X \in \mathfrak{g}$,

$$0 = \mathcal{L}_{\underline{X}} \omega_\mathcal{O} = d i_{\underline{X}} \omega_\mathcal{O} + i_{\underline{X}} d\omega_\mathcal{O} = i_{\underline{X}} d\omega_\mathcal{O}$$

Since the fundamental vector fields $\underline{X}$, $X \in \mathfrak{g}$, span the tangent spaces to $\mathcal{O}$ at every point, $\omega_\mathcal{O}$ is closed.

In summary,

Proposition 2.3.1 ([94, 124])**.** *Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit. Then, there exists a G-invariant symplectic form $\omega_\mathcal{O}$ on $\mathcal{O}$, the* Kirillov-Kostant-Siourau form*. The coadjoint action is Hamiltonian with moment map being the canonical inclusion*

$$\mu_\mathcal{O} : \mathcal{O} \hookrightarrow \mathfrak{g}^*$$

so that the fundamental vector fields $\underline{X}$, for $X \in \mathfrak{g}$, admit Hamiltonian functions H^X.

Restricting the action of G to T, a coadjoint orbit (equipped with its KKS symplectic structure) is a Hamiltonian T-space. A moment map for the action is the composition

$$\mathcal{O} \xhookrightarrow{\mu_{\mathcal{O}}} \mathfrak{g}^* \longrightarrow \mathfrak{t}^*, \tag{2.3.6}$$

where the second map is the canonical projection, which we will also denote $\mu_{\mathcal{O}} : \mathcal{O} \to \mathfrak{t}^*$.

As a compact Hamiltonian T-space with moment map μ, the image

$$\Delta_{\mathcal{O}} \coloneqq \mu_{\mathcal{O}}(\mathcal{O}) \subseteq \mathfrak{t}^*$$

of the moment map is a convex polytope (Theorem 2.1.9) with additional internal chamber/wall structure, which we now describe.

Choosing a G-invariant positive-definite inner product (for example, the Killing form) on $\mathfrak{g}$ induces a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, setting up a correspondence between adjoint and coadjoint orbits. Using this isomorphism, we consider $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$ (by identifying $\mathfrak{t}^* \subseteq \mathfrak{g}^*$ with $\mathfrak{t} \subseteq \mathfrak{g}$). An adjoint orbit admits the structure of a Hamiltonian T-space by pulling back the symplectic form on the corresponding coadjoint orbit. The moment map $\mu_{\mathcal{O}}$ for the T-action becomes orthogonal projection on to the subspace $\mathfrak{t}$.

Remark 2.3.2. In the proceeding discussion, we will fix such an identification and will refer to elements of $\mathfrak{g}^*$ as elements of $\mathfrak{g}$, without reference to the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$.

Suppose that $\mathcal{O} = \mathcal{O}_X \subseteq \mathfrak{g}$ is the adjoint orbit through $X \in \mathfrak{t}$. Then, $\mathcal{O} \cap \mathfrak{t} = W \cdot X$ (see [34, Ch.3]), and each point in the intersection is a fixed point of the T-action. Conversely, if $Y \in \mathcal{O}$ is a fixed point of the T-action then, for any $Z \in \mathfrak{t}$, we have $[Z, Y] = 0$, and Z is an element of the centraliser of $\mathfrak{t}$ in $\mathfrak{g}$. But T is a maximal torus so that its centraliser is itself. Hence, $Z \in \mathcal{O} \cap \mathfrak{t}$ and $Z = w \cdot X$, for some $w \in W$.

Combining the previous discussion with Theorem 2.1.9 proves the following:

Theorem 2.3.3 (Kostant [93])**.** *Let $\xi \in \mathfrak{g}^*$, and $\mathcal{O} = G \cdot \xi$ be the coadjoint orbit through ξ, considered as a Hamiltonian T-space. Then, the moment polytope is realised as*

$$\Delta_{\mathcal{O}} = \mathrm{Conv}(W \cdot \xi). \tag{2.3.7}$$

Moreover, each $w \cdot \xi \in \Delta_{\mathcal{O}}$ is a vertex.

Remark 2.3.4. Theorem 2.3.3 appeared first as the Schur-Horn theorem: let A be an $n \times n$ matrix with diagonal entries $a_1, \ldots, a_n$ and spectrum $\lambda_1 \geq \ldots \geq \lambda_n$. Then, $(a_1, \ldots, a_n)$ lies in the convex hull of $w \cdot (\lambda_1, \ldots, \lambda_n)$. This result was later generalised by Kostant to Theorem 2.3.3.

Let $\lambda_1, \ldots, \lambda_N \in \mathfrak{t}^*$ denote the fundamental weights and their W-conjugates, and let $H_i = H_{\lambda_i}$ be the stabiliser for the (co)adjoint action of G. Let $\mathfrak{h}_i$ be the Lie algebra of H_i. Then, T is a maximal torus in H_i, for each i, so we can consider W_i, the Weyl group of the pair (H_i, T). The Weyl group W_i is a parabolic subgroup of W (the Weyl group of (G, T)) and is generated by the reflections corresponding to those roots that are orthogonal to λ_i. The weight λ_i generates a 1-dimensional torus $S_i \subseteq H_i$. In particular, by Proposition 2.1.7, any point in $\mathcal{O} = \mathcal{O}_X$ that is fixed by S_i is a critical point of the moment map $\mu_{\mathcal{O}}$.

Lemma 2.3.5. *1) The fixed point set of S_i is $\mathcal{O} \cap \mathfrak{h}_i$.*

2) $\mu_{\mathcal{O}}(\mathcal{O} \cap \mathfrak{h}_i) = \bigcup_{w \in W} \mathrm{Conv}(W_i \cdot wX)$.

Proof. (a) $Z \in \mathcal{O}$ is fixed by S_i if and only if $[\lambda_i, Z] = 0$ if and only if $Z \in \mathfrak{h}_i$.

(b) Suppose that $\mathcal{O} = \mathcal{O}_X$, for $X \in \mathfrak{t}$. The intersection $\mathcal{O} \cap \mathfrak{h}_i \supseteq \mathcal{O} \cap \mathfrak{t} = W \cdot X$ is H_i-invariant, so it consists of the H_i-orbits passing through $W \cdot X$. Each orbit $H_i w \cdot X$, for $w \in W$, is a symplectic submanifold and becomes a Hamiltonian T-space with moment map being the restriction of $\mu_{\mathcal{O}}$ to $H_i w \cdot X$. Hence, by Theorem 2.3.3

$$\mu_{\mathcal{O}}(H_i w \cdot X) = \mathrm{Conv}(W_i w \cdot X) \tag{2.3.8}$$

and the result follows.

□

Theorem 2.3.6 ([68, 60])**.** *Let $\mathcal{O} = \mathcal{O}_\xi$ be a coadjoint orbit, $\xi \in \mathfrak{g}^*$, consider as a Hamiltonian T-space with moment map $\mu_{\mathcal{O}} : \mathcal{O} \to \mathfrak{t}^*$. Let $\lambda_1, \dots, \lambda_N \in \mathfrak{t}^*$ be collections of fundamental weights and all their W-conjugates, and denote the stabiliser of λ_i in G by H_i. Write W_i for the Weyl group of the pair (H_i, T). Then, the critical points of the moment map are the symplectic submanifolds*

$$H_i w \cdot \xi, \qquad w \in W, i = 1, \dots, N. \tag{2.3.9}$$

The codimension-1 walls in the moment polytope $\Delta_{\mathcal{O}}$ are the convex polytopes

$$\mathrm{Conv}(W_i w \cdot \xi), \qquad w \in W, i = 1, \dots, N. \tag{2.3.10}$$

Proof. Identify $\mathcal{O}$ with an adjoint orbit, so that $\mathcal{O} = \mathcal{O}_X$, for $X \in \mathfrak{t}$. A critical point $Y \in \mathcal{O}$ must be fixed by some positive dimensional subtorus $T' \subseteq T$ (Proposition 2.1.7). Hence, for any $Z \in \mathfrak{t}'$, where $\mathfrak{t}'$ is the Lie algebra of T', we have $[Y, Z] = 0$. Thus, Y lies in the centraliser of $\mathfrak{t}'$. The wall structure on $\Delta_{\mathcal{O}}$ implies that the centraliser of any point in $\mathcal{O}$ must be a subalgebra of one of the maximal centralisers $\mathfrak{h}_i$ ([60, Ch. 5]). Hence, $Y \in \mathfrak{h}_i$ □

Example 2.3.7. Let $G = \mathrm{SU}(2)$, $T \subseteq G$ the diagonal matrices. We identify $\mathfrak{g}^*$ with the set of traceless 2×2 Hermitian matrices $\mathcal{H}_2$

$$\begin{aligned} \mathrm{tr} : \ \mathcal{H}_2 &\longrightarrow \mathfrak{g}^* \\ A &\longmapsto (X \mapsto i\,\mathrm{tr}(AX)) \end{aligned} \tag{2.3.11}$$

This map is G-equivariant and the moment map for the resulting Hamiltonian T-space $\mathcal{H}_2$ is projection onto the diagonal.

Any G-orbit is uniquely determined by a non-negative real number $\lambda \in \mathbb{R}_{\geq 0}$. Let $\mathcal{O} = \mathcal{O}_\lambda$ be the corresponding orbit. Thus, $A \in \mathcal{O}$ if and only if its eigenvalues are $\pm\lambda$. Theorem 2.3.3 implies that the top-left diagonal entry of A must lie in the interval $[-\lambda, \lambda]$. By Theorem 2.3.6, the walls of the interval are $\{\pm\lambda\}$, and the chamber (equal to the set of regular values of the moment map) is the open interval $(-\lambda, \lambda)$.

We check this directly: consider a traceless Hermitian matrix

$$A = \begin{bmatrix} a & b+ic \\ b-ic & -a \end{bmatrix}, \qquad a, b, c \in \mathbb{R}.$$

such that A has eigenvalues $\pm\lambda$. Then, we must have

$$\lambda^2 = -\det A = a^2 + b^2 + c^2 \geq a^2 \implies a \in [-\lambda, \lambda].$$

Moreover, if $a \in [-\lambda, \lambda]$ then $a^2 \leq \lambda^2$ and we can choose $z \in \mathbb{C}$ such that $|z|^2 = \lambda^2 - a^2$. Then, the matrix

$$A = \begin{bmatrix} a & z \\ \overline{z} & -a \end{bmatrix}$$

lies in $\mathcal{O}$. That the set of regular values for the moment map is $(-\lambda, \lambda)$ follows immediately.

Example 2.3.8. Let $G = \mathrm{SU}(3)$. Then, the moment polytope with chamber stucture for a generic (six dimensional) orbit is given in Figure 2.1.

2.4 Weight varieties

Let G be a compact, semisimple Lie group, $T \subseteq G$ a maximal torus in G. Let $\mathcal{O} = \mathcal{O}_\xi \subseteq \mathfrak{g}^*$ be the coadjoint orbit through $\xi \in \mathfrak{g}^*$, considered as a Hamiltonian T-space with moment map $\mu : \mathcal{O} \to \mathfrak{t}^*$. Let $\Delta = \mu(\mathcal{O}) \subseteq \mathfrak{t}^*$ be the (convex) moment polytope, Δ° the union of chambers in Δ (Definition 2.1.14). Recall that Δ° is precisely the set of regular values of μ.

The coadjoint action of T on $\mathfrak{t}^*$ is trivial so that, for any $\xi \in \mathfrak{t}^*$, the stabiliser of ξ in T is T itself. The equivariance of the moment map implies that the level set $\mu^{-1}(\xi)$ carries a (proper) T-action.

Definition 2.4.1 (Knutson [89])**.** Let $\nu \in \mathfrak{t}^*$. The *ν-weight variety of $\mathcal{O}_\xi$*, denoted $\mathcal{O}_\xi(\nu)$, is the symplectic reduction $\mathcal{O}_\xi /\!\!/ T(\nu)$ of $\mathcal{O}$ by T at ν.

Remark 2.4.2. Theorem 2.1.9 and Theorem 2.2.1 imply that weight varieties are defined (nonempty) whenever $\nu \in \Delta^\circ$.

Remark 2.4.3. By Theorem 2.2.1 we know that weight varieties are orbifolds. However, in type A it is a fact (see [89, Ch. 1]) that weight varieties are always *manifolds*.

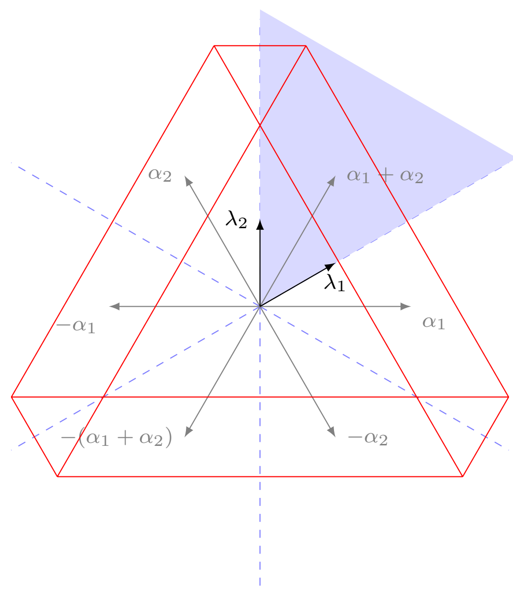


Figure 2.1: Generic hexagonal $SU(3)$ moment polytope. The chambers are the connected regions bounded by the interior lines.

For the remainder of this section we describe some specific weight varieties in type A.

Example 2.4.4. Let $G = \mathrm{SU}(2)$ with maximal torus $T \cong S^1$ consisting of the diagonal matrices in G. Identify $\mathfrak{g}^*$ with the traceless 2×2 Hermitian matrices $\mathcal{H}_2$. A 2-dimensional coadjoint orbit $\mathcal{O}$ consists of those $A \in \mathcal{H}_2$ with distinct nonzero eigenvalues $\pm\lambda$. In Example 2.3.7 we saw that the a level set of the moment map, at a regular value $a \in (-\lambda, \lambda)$, could be identified with the circle $\mu^{-1}(a) = \{z \in \mathbb{C} \mid |z|^2 = \lambda^2 - a^2\}$. The T-action on the level set is is $t \cdot z = t^2 z$, $z \in \mu^{-1}(a)$, $t \in T$. In particular, the quotient is a point (which is to be expected).

Example 2.4.5. Let $G = \mathrm{SU}(3)$. Then, the coadjoint orbits have (real) dimension 2 or 6. A generic coadjoint orbit has dimension 6 and is diffeomorphic to the variety of complete flags in $\mathbb{C}^3$. Any weight variety $\mathcal{O}(\nu)$ of a generic coadjoint orbit $\mathcal{O}$ must be a compact, symplectic manifold having dimension 2. Using the Kirwan surjectivity theorem [87], there is a surjection from $H^*(\mu^{-1}(0))$ on to $H^*(\mathcal{O}(\nu))$. The level set $\mu^{-1}(0)$ is identified with a complex submanifold of the variety of complete flags in $\mathbb{C}^3$. In particular, it has cohomology in even degrees only. Thus, The Euler characteristic of $\mathcal{O}(\nu)$ is 2, so that $\mathcal{O}(\nu)$ is diffeomorphic to S^2.

Example 2.4.6. Let $G = U(1)^n \times U(2)$. Then, G acts on $\mathrm{Mat}_{n\times 2}(\mathbb{C})$ by conjugation. By Proposition 2.2.3 and Example 2.2.7, the symplectic reduction of $\mathrm{Mat}_{n\times 2}(\mathbb{C})$ by G is the symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2, n)$ by $U(1)^n$. Hence, this symplectic reduction is a weight variety for a degenerate coadjoint orbit $\mathcal{O}$ of $U(n)$ diffeomorphic to $\mathrm{Gr}_{\mathbb{C}}(2,n)$. The Hamiltonian action of the torus $T = U(1)^n$ on $\mathrm{Gr}_{\mathbb{C}}(2,n)$ has associated moment map

$$\begin{array}{rccc} \mu_T: & \mathrm{Gr}_{\mathbb{C}}(2,n) & \longrightarrow & \mathfrak{t}^* \cong \mathbb{R}^n \\ & \mathrm{span}\{u,v\} & \longmapsto & \frac{1}{2}(|u_1|^2+|v_1|^2,\dots,|u_n|^2+|v_n|^2) \end{array}$$

Here (u,v) is a unitary 2-frame in $\mathbb{C}^n$. Hence, the image of the moment map is

$$\Xi \coloneqq \left\{ (r_1,\dots,r_n) \in \mathbb{R}^n \mid \sum_{i=1}^n r_i = 1,\ 0 \le r_i \le 1/2 \right\}$$

The critical values of moment polytope consists of those $r \in \Xi$ such that either

(a) $r_i = 0$, for some i, or

(b) $r_i = 1/2$, for some i, or

(c) there exists a subset $I \subseteq \{1,\dots,n\}$ with $|I|$ and $|I^c|$ at least two, and $\sum_{i\in I} r_i = \sum_{i\notin I} r_i$.

Points in the moment polytope satisfying one of the first two conditions are points of external walls: the interior walls are described by those points in the moment polytope satisfying the last condition. Therefore, the walls can be described as the subsets

$$\Xi_I \coloneqq \left\{ (r_1,\dots,r_n) \in \Xi \mid \sum_{i\in I} r_i = \sum_{i\notin I} r_i \right\}, \quad I \subseteq \{1,\dots,n\}.$$

Observe that $\Xi_I = \Xi_{I^c}$, for every $I \subseteq \{1,\dots,n\}$. For each $i \in I$, we write Ξ_i instead of $\Xi_{\{i\}}$.

Hence, the set of regular values of μ_T, Ξ°, is the set of points such that $\sum_{i\in I} r_i \neq \sum_{i\notin I} r_i$, for any subset $I \subseteq \{1,\dots,n\}$. We call a subset $I \subseteq \{1,\dots,n\}$ *r-short* if $\sum_{i\in I} r_i < \sum_{i\notin I} r_i$.

If $\gamma \subseteq \Xi$ is a chamber then γ consists of points such that $\sum_{i\in I} r_i \neq \sum_{i\notin I} r_i$, for every $I \subseteq \{1,\dots,n\}$. If $r \in \gamma$ then, for any $I \subseteq \{1,\dots,n\}$, I or I^c is r-short. By definition of a chamber, if I is r-short, for some $r \in \gamma$, then I is r-short, for any $r \in \gamma$. Therefore, we will say that I is γ-short if I is r-short, for some $r \in \gamma$. Thus, γ is characterised by

$$\mathcal{S}(\gamma) \coloneqq \{ I \subseteq \{1,\dots,n\} \mid\ I \text{ is } \gamma\text{-short}\}.$$

Observe that $\mathcal{S}(\gamma)$ is a poset via inclusion of subsets. Given $I \in \mathcal{S}(\gamma)$, Ξ_I is in the closure $\overline{\gamma}$ if and only if $I \in \mathcal{S}(\gamma)$ is maximal with respect to inclusion (see [66]). In particular, the chamber γ is adjacent to an external wall Ξ_i if and only if $\{i\} \in \mathcal{S}(\gamma)$ is maximal. Remarkably, the poset $\mathcal{S}(\gamma)$ characterises the diffeomorphism type of $\mathcal{P}_{r,n}$

Proposition 2.4.7 (Knutson-Hausmann, [66]). *Let $r, r' \in \Xi^\circ$, so that $r \in \gamma$ and $r \in \gamma'$ are the respective chambers. If there is an isomorphism of posets $\mathcal{S}(\gamma) \cong \mathcal{S}(\gamma')$ then $\mathcal{P}_{r,n} \cong \mathcal{P}_{r',n}$.*

For r and r' in distinct chambers γ and γ', the diffeomorphism type of the polygon spaces $\mathcal{P}_{r,n}$ and $\mathcal{P}_{r',n}$ are related by blowing up and blowing down submanifolds [61]. If γ and γ' are adjacent and separated by a wall Ξ_I such that I^c is γ-short and I is γ'-short, Mandini [107] obtained the following explicit description of the birational map.

Theorem 2.4.8 (Mandini, [107]). *Let $|I| = p$. There is a birational map between $\mathcal{P}_{r,n}$ and $\mathcal{P}_{r',n}$ obtained by blowing up along a subvariety in $\mathcal{P}_{r,n}$ diffeomorphic to $\mathbb{P}^{p-2}_{\mathbb{C}}$ and blowing down along a subvariety in $\mathcal{P}_{r',n}$ diffeomorphic to $\mathbb{P}^{n-p-2}_{\mathbb{C}}$.*

As a consequence, we obtain the following result.

Corollary 2.4.9. *Let $\gamma \subseteq \Xi$ be a chamber containing an external wall in its closure. Then, for any $r \in \gamma$, $\mathcal{P}_{r,n}$ is diffeomorphic to $\mathbb{P}^{n-3}_{\mathbb{C}}$.*

Remark 2.4.10. If we scale the symplectic form on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ by $\lambda > 0$ then the moment polytope Ξ in Example 2.4.6 gets 'inflated' to

$$\Xi_\lambda = \left\{ (r_1, \dots, r_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} r_i = \lambda,\ 0 \leq r_i \leq \lambda/2 \right\}$$

As such, we should consider the following cone over Ξ

$$C(\Xi) := \left\{ (r_1, \dots, r_n) \in \mathbb{R}^n_{>0} \mid \sum_i r_i \neq 0,\ \frac{(r_1, \dots, r_n)}{\sum_i r_i} \in \Xi \right\}.$$

By abuse of notation we will write $r \in \Xi$ when we really mean $r \in C(\Xi)$.

2.5 Algebraic viewpoint

In this section we briefly outline the relation between symplectic reductions and GIT quotients. First we recall the notion of the GIT quotient and then we apply the Kempf-Ness theorem (Theorem 2.5.5) in the setting of coadjoint orbits. For more details see [114].

Let G be a complex connected reductive group with associated root datum $(X, R, X^\vee, R^\vee)$ and fix a choice of Borel subgroup $B \subseteq G$ and maximal torus $T \subseteq B$. Let $S = \{\alpha_i\}_{i \in I} \subseteq X$ be simple roots corresponding to this choice of B and $P = P_J \supseteq B$ a standard parabolic subgroup corresponding to a subset $J \subseteq I$. The choice of a dominant weight $\lambda \in X^*(T)$ satisfying $\langle \lambda, \alpha^\vee \rangle = 0$, whenever $\alpha \in I$, and $\langle \lambda, \alpha^\vee \rangle > 0$, whenever $\alpha \in S \setminus I$, determines an embedding of the (partial) flag variety $i_\lambda : G/P \to \mathbb{P}(V_\lambda)$, where V_λ is the finite dimensional irreducible representation of G with highest weight λ (see [73]). Let $\mathcal{L}_\lambda := i_\lambda^* \mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$ be the (ample) invertible sheaf of hyperplane sections associated to this embedding, and let L_λ be the total space of $\mathcal{L}_\lambda$.

Notation 2.5.1. We write $G/_\lambda P$ to denote that we are considering the (partial) flag variety G/P together with the projective embedding associated to $\mathcal{L}_\lambda$.

The Borel-Weil-Bott theorem [73] identifies the space of global sections $H^0(G/P, \mathcal{L}_\lambda) \cong V_{\lambda^*}$ as the irreducible representation of G with highest weight $\lambda^* = -w_0\lambda$, where $w_0 \in W$ is the longest element. The homogeneous coordinate ring R_λ associated to this embedding is

$$R_\lambda = \bigoplus_{n\geq 0} H^0(G/P, \mathcal{L}_\lambda^{\otimes n}) \cong \bigoplus_{n\geq 0} V_{n\lambda^*}. \tag{2.5.1}$$

Remark 2.5.2. For any dominant weight $\xi \in X_{\geq 0}$, there exists, up to a non-zero scalar, a unique G-invariant ring structure on the (graded) G-module $\bigoplus_{n\geq 0} V_{n\xi}$, known as the *Cartan product*, defined as follows: the irreducible representation $V_{(m+n)\xi}$ appears with multiplicity one in the tensor product $V_{n\xi} \otimes V_{m\xi}$. Hence, up to a non-zero scalar, there is a unique G-invariant surjection

$$V_{n\xi} \otimes V_{m\xi} \twoheadrightarrow V_{(n+m)\xi}.$$

This is how multiplication is defined in the ring $\bigoplus_{n\geq 0} V_{n\xi}$.

The line bundle L_λ admits a G-linearisation. By restriction, L_λ can also be considered as a T-linearised line bundle.

Now, let $\Delta(\lambda) := \mathrm{Conv}(W\lambda^*) \subseteq \mathfrak{t}^*$ be the convex hull of the W-orbit through λ^*, and choose $\mu \in X^*(T) \cap \Delta(\lambda)$. We can twist L_λ by μ to obtain a T-linearised line bundle $L_\lambda(-\mu)$: as a line bundle, $L_\lambda(-\mu) = L_\lambda$, and we twist the action of T in the fibres of $L_\lambda(-\mu)$ by $-\mu$. We let $\mathcal{L}_\lambda(-\mu)$ denote the sheaf of sections of $L_\lambda(-\mu)$.

It is straightforward to see the following:

Lemma 2.5.3. *The space of T-invariants $H^0\left(G/P, \mathcal{L}_\lambda(-\mu)^{\otimes n}\right)^T$ is the μ-weight space in the (irreducible) representation $H^0(G/P, \mathcal{L}_\lambda^{\otimes n}) \cong V_{n\lambda^*}$.*

Definition 2.5.4 ([89])**.** The *ν-weight variety* of $G/\!_\lambda P$ is the G.I.T. quotient defined by the T-linearised line bundle $L_\lambda(-\mu)$,

$$T_\mu \backslash\!\backslash G/\!_\lambda P := \mathrm{Proj} \bigoplus_{n\geq 0} H^0\left(G/P, \mathcal{L}(-\mu)^n\right)^T = \mathrm{Proj} \bigoplus_{n\geq 0} (V_{n\lambda^*})^{n\mu}$$

Let $K \subseteq G$ be a maximal compact subgroup, which we will assume is acting unitarily on V_λ. Let $H \subseteq T$ be a maximal compact torus in T with $H \subseteq K$. Then, the quotient $X = G/\!_\lambda P$ is a Hamiltonian H-space with moment map $\mu : X \to \mathfrak{h}^*$.

Applying the Kempf-Ness theorem [114] we have the following result.

Theorem 2.5.5. *There is an inclusion $\mu^{-1}(0) \subseteq X^{ss}$ inducing a homeomorphism between the symplectic reduction and the GIT quotient*

$$\mu^{-1}(0)/H \cong T_\mu \backslash\!\backslash G/\!_\lambda P$$

In particular, the symplectic reduction is a projective variety.

We end this section with an elaboration of Example 2.2.4. We will come back to this example in Section 3.4.

Recall the construction of the moduli space of spatial n-gons $\mathcal{P}_{r,n}$, $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$, from Example 2.2.4. Suppose that $r \in \mathbb{Z}^n_{>0}$. An application of the Kempf-Ness Theorem [114] implies that there is an identification

$$\mathcal{P}_{r,n} \cong (\mathbb{P}^1_{\mathbb{C}})^n /\!\!/ \mathrm{PGL}_2(\mathbb{C})$$

The Gelfand-Macpherson correspondence [43] provides the following isomorphism of G.I.T. quotients

$$T_\mu \backslash\!\!\backslash \mathrm{SL}_n(\mathbb{C}) /\!\!/_\lambda P \cong (\mathbb{P}^1_{\mathbb{C}})^n /\!\!/ \mathrm{PGL}_2(\mathbb{C})$$

where $P \subseteq \mathrm{SL}_n(\mathbb{C})$ is a maximal parabolic such that $G/\!\!/_\lambda P \cong \mathrm{Gr}(2, n)$, $T \subseteq \mathrm{SL}_n(\mathbb{C})$ is a maximal torus. The linearisation defined by μ corresponds to the action of T on $\mathbb{C}^n$ given by

$$\mathrm{diag}(t_1, \dots, t_n) \longmapsto \mathrm{diag}(t^r t_1, \dots, t^r t_n)$$

where $t^r = t_1^{r_1} \cdots t_n^{r_n}$ is the character defined by r.

Hence, we have the following result (recall Example 2.4.6).

Theorem 2.5.6 (Hausmann-Knutson, [66])**.** *Let $r \in \mathbb{Z}^n_{>0}$. Then, the polygon space $\mathcal{P}_{r,n}$ admits the structure of a projective variety and can be identified with a weight variety. Specifially, the polygon space $\mathcal{P}_{r,n}$ is a symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2, n)$, the Grassmannian of 2-planes in $\mathbb{C}^n$, by the compact torus $H \subseteq T$ at $r \in \Xi^\circ$.*

Remark 2.5.7. To ensure compatibility of symplectic forms coming from our constructions of $\mathcal{P}_{r,n}$ (Example 2.2.4) and the symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2, n)$, we scale the symplectic form on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ by $|r| > 0$ (cf. Remark 2.4.10). In particular, given $r \in \mathbb{R}^n_{>0}$, the moment polytope of the torus action on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ used to define $\mathcal{P}_{r,n}$ as a symplectic quotient is

$$\Xi_{|r|} = \{s \in C(\Xi) \mid |s| = |r|\}.$$

Chapter 3

Mirror constructions

In this chapter we develop a new approach to computing quantum cohomology of weight varieties motivated by a conjecture of Teleman (Conjecture 3.2.6). Let G be a semisimple complex algebraic group, $P \subseteq G$ a parabolic subgroup containing a maximal torus T. Using the mirror Landau-Ginzburg model (M_P, f_P) of a partial flag variety $X = G/P$ introduced by Rietsch [119], the quantum cohomology of a symplectic reduction of X by a compact torus $H \subseteq T$ at $\nu \in \mathfrak{h}^*$ is conjectured to be obtained by restricting the *superpotential* f_P to a certain subvariety Y_ν of M_P and computing the Jacobian ring. In fact, the mirror family M_P is a subvariety of a Borel subgroup ${}^L B_-$ of the Langlands dual ${}^L G$ and the subvariety Y_ν is the fibre of the canonical homomorphism ${}^L B_- \to {}^L T$ over $\exp(2\pi i \nu)$. Here we canonically identify $\mathfrak{h}^*$ with the subalgebra ${}^L\mathfrak{h} \subseteq {}^L\,\mathfrak{t}$.

In Section 3.1 we recall the construction of the Rietsch mirror family Landau-Ginzburg model (M_P, f_P) and the definition of the (T-equivariant) superpotential. We briefly discuss the work of Rietsch relating the (T-equivariant) quantum cohomology to the Jacbobian ring of f_P and related mirror conjectures. In Section 3.2, we describe recent work of Teleman on topological actions of compact, connected Lie groups on Fukaya categories. We introduce Teleman's conjecture on the construction of Fukaya categories of weight varieties and the consequences for quantum cohomology of weight varieties (Conjectures 3.2.6, 3.2.7). Section 3.3 presents several formulae for computing the superpotential f_P and provides an explicit expression for f_P with respect to a family of parameterisations of M_P (Proposition 3.3.9). In Section 3.4 we provide a new conjectural description of the quantum cohomology of weight varieties in type A. We focus on polygon spaces $\mathcal{P}_{r,n}$ (see Examples 2.2.4, 2.2.6 and Section 2.5, verifying our description for polygon spaces of low rank. The verification makes use of computations of the quantum cohomology rings of del Pezzo surfaces [31], [55]. Finally, in Section 3.5 we discuss further avenues of research.

Let G be a complex reductive algebraic group with root datum $(X, R, X^\vee, R^\vee)$. We will use the notation and conventions from Section 1.3. Let $T \subseteq B_\pm \subseteq G$ be a choice of maximal torus and opposite Borel subgroups, $N_\pm \subseteq B_\pm$ the unipotent radicals. Let $P = P_J \supseteq B_+$ be a standard parabolic subgroup. Let ${}^L G$ be the Langlands dual group, ${}^L T, {}^L B_\pm, {}^L N_\pm$ the corresponding subgroups (associated to the choice of simple coroots), and ${}^L P$ the dual standard parabolic subgroup (corresponding to the subset $J(P) \subseteq I$). At certain points in this chapter we will restrict to the case when G is semisimple.

3.1 Rietsch mirror construction

In this section we introduce a candidate for the (B-model) equivariant Landau-Ginzburg model (M_P, f_P) of the (A-model) generalised flag variety G/P proposed by Rietsch [119]. The mirror family M_P is realised as a subvariety of the opposite Langlands dual subgroup ${}^LB_- \subseteq {}^LG$ and parameterised by $Z(L_{{}^LP})$, the centre of the unique Levi $L_{{}^LP} \subseteq {}^LP$ containing ${}^LT$. The superpotential f_P is considered as a family of holomorphic functions f_P^t, $t \in Z(L_P)$, on the fibres of the mirror family M_P.

The verification that M_P is the 'correct' mirror family takes the form of an identification of the equivariant quantum cohomology ring $qH^*(G/P)$ with the Jacobian ring of the superpotential $\mathrm{Jac}(f_P)$. This result is originally due to Rietsch [119].

Rietsch proposed a stronger statement of the mirror symmetry between G/P and M_P in the form of a *mirror conjecture*. We briefly discuss this conjecture and some partial verifications due to Lam, Lam-Templier [96], [97].

For ease of notation we swap the roles of G and ${}^LG$. In particular, we are going to describe the mirror family $M_{{}^LP}$ to the flag variety ${}^LG/{}^LP$. By abuse of notation we will write M_P instead of $M_{{}^LP}$.

Let $w_P^{-1} \in W^P$ be the minimal length coset representative having maximal length. We record the following elementary result (see [17]).

Lemma 3.1.1. *$w_P^{-1} = w_0 w_0^P$ and $\ell(w_P^{-1}) = \ell(w_P) = \ell(w_0) - \ell(w_0^P) = \dim G/P$.*

By Lemma 3.1.1 we have $w_P^{-1} w_0^P = w_0 = w_0^P w_P$ and

$$\overline{w_P^{-1}} \cdot \overline{w_0^P} = \overline{w}_0, \quad \overline{w_0^P} \cdot \overline{w}_P = \overline{w}_0.$$

Let P^* be the standard parabolic subgroup containing B_+ and $\overline{w}_0 L_P \overline{w}_0^{-1}$. Thus, $J(P^*) = J(P)^*$, $w_0^P w_0 = w_0 w_0^{P^*}$ and

$$\overline{w_0^{P^*}} \cdot \overline{w_{P^*}^{-1}}, \quad \overline{w}_{P^*} \cdot \overline{w_0^{P^*}} = \overline{w}_0.$$

In particular, $w_P^{-1} = w_{P^*}$.

The Landau-Ginzburg model (M_P, f_P)

Consider the incidence variety

$$Z_P := \{(t, b) \in Z(L_P) \times B_- \mid b \in N_+ Z(L_P) \overline{w}_P N_+\},$$

where $Z(L_P)$ is the centre of the Levi subgroup L_P. Projection onto the second factor $\mathrm{pr}_2 : Z_P \to B_-$ is an isomorphism

$$Z_P \cong M_P := B_- \cap N_+ Z(L_P)\overline{w}_P N_+ \subseteq B_-. \tag{3.1.1}$$

Let $\mathrm{pr}_1 : Z_P \to Z(L_P)$ be the projection onto the first factor. For $t \in Z(L_P)$, we can identify the fibre over t as

$$\mathrm{pr}_1^{-1}(t) \cong M_P^t := B_- \cap N_+ t\overline{w}_P N_+ \tag{3.1.2}$$

We will require a unique decomposition of elements in M_P. Recall the Levi decomposition $P = L_P N_P$. Here L_P is the reductive subgroup generated by T and $\mathrm{im}\, x_j, \mathrm{im}\, y_j$, $j \in J(P)$, and $N_P = N_+(w_P)$, where, for $w \in W$, we define

$$N_+(w) := \prod_{\substack{\alpha \in R^+ \\ w^{-1}(\alpha) \in R^-}} \mathrm{im}\, x_\alpha = N_+ \cap \overline{w} N_- \overline{w}^{-1}$$

Therefore, $\overline{w} N_+(w) \overline{w}^{-1} \subseteq N_-$, for any $w \in W$.

The following result follows from the standard description of Bruhat cells in G (see [126]).

Lemma 3.1.2. *Let $w \in W$.*

(a) Any $x \in B_+ w B_+$ can be written uniquely as $x = zt\overline{w}u$, where $z \in N_+(w), t \in T, u \in N_+$.

(b) Any $x \in B_+ w B_+$ can be written uniquely as $x = v\overline{w}sy$, where $v \in N_+, s \in T, y \in N_+(w^{-1})$.

Applying this result to $M_P = B_- \cap N_+ Z(L_P)\overline{w}_P N_+$, we obtain the unique decompositions

$$M_P = B_- \cap N_+(w_P) Z(L_P) \overline{w}_P N_+$$

and

$$M_P = B_- \cap N_+ \overline{w}_P Z(L_{P^*}) N_+(w_{P^*}).$$

Definition 3.1.3. (i) Define the *quantum structure map* to be the projection

$$\begin{array}{rccc} q: & M_P & \longrightarrow & Z(L_P) \\ & x = zt\overline{w}_P u & \longmapsto & t \end{array}$$

This map is well-defined. The fibres of q are the subvarieties

$$M_P^t = B_- \cap N_+ t \overline{w}_P N_+$$

from (3.1.2).

(ii) Define the *equivariant structure map* to be the projection

$$\begin{array}{rrcl} e: & M_P \subseteq B_- = N_-T & \longrightarrow & T \\ & x = vs & \longmapsto & s \end{array}$$

As $B_-/N_- \cong T$, the map e can be identified with the canonical quotient homomorphism.

Definition 3.1.4. Let ${}^LG/{}^LP$ be a generalised flag variety. Define the *mirror of* ${}^LG/{}^LP$ to be the subvariety

$$B_-^{w_P} := B_- \cap N_+ \overline{w}_P N_+ \subseteq B_-. \tag{3.1.3}$$

We define the *mirror family* to be the subvariety M_P defined in (3.1.1) considered as a (trivial) family over $Z(L_P)$ via the quantum structure map q. We will write (M_P, q) to denote this family.

Remark 3.1.5. The mirror family M_P (resp. mirror $B_-^{w_P}$) of ${}^LG/{}^LP$ is an example of a *double Bruhat cell* (resp. *reduced double Bruhat cell*). These are subvarieties of a reductive group G of the form

$$B_- v B_- \cap B_+ u B_+ \quad \text{(resp. } B_- v B_- \cap N_+ \overline{u} N_+)$$

where $u, v \in W$ [36].

We record some straightforward properties of the fibres M_P^t.

Proposition 3.1.6.

(a) Let $t \in Z(L_P)$. Then, M_P^t is smooth variety of dimension $\dim M_P^t = \dim \ell(w_P) = \dim {}^LG/{}^LP$, *isomorphic to $B_-^{w_P}$.*

(b) Multiplication in G induces an isomorphism of varieties

$$\begin{array}{rrcl} m: & Z(L_P) \times B_-^{w_P} & \xrightarrow{\sim} & M_P \\ & (t, z\overline{w}_P u) & \longmapsto & tz\overline{w}_P u \end{array} \tag{3.1.4}$$

Proof. (a) That M_P^t is isomorphic to $B_-^{w_P}$ is immediate. The map

$$\begin{array}{rcl} M_P^t & \longrightarrow & G/B_- \\ x & \longmapsto & b\overline{w}_0 B_- \end{array} \tag{3.1.5}$$

identifies $B_-^{w_P}$ with the *(open) Richardson variety*

$$\mathcal{R}^-_{w_0^P, w_0} := \left(B_+ \overline{w_0^P} B_- \cap B_- \overline{w}_0 B_- \right) / B_- \subseteq G/B_-.$$

The varieties $\mathcal{R}_{w_0^P,w_0}$ are known to be smooth of dimension $\ell(w_0) - \ell(w_0^P) = \ell(w_P)$, see [21]. We also have

$$\begin{aligned}\ell(w_0) - \ell(w_0^P) &= (\dim {}^L G - \dim {}^L B_+) - (\dim {}^L P - \dim {}^L B_+) \\ &= \dim {}^L G - \dim {}^L P \\ &= \dim {}^L G / {}^L P.\end{aligned}$$

(b) This is obvious.

□

Remark 3.1.7. For any $t \in Z(L_P)$, we can embed M_P^t in G/B_+ using the map

$$\begin{array}{ccc} M_P^t & \longrightarrow & G/B_+ \\ x & \longmapsto & x^{-1}\overline{w_0^P}B_+ \end{array}$$

and then use the canonical projection $p : G/B_+ \to G/P$ to embed M_P^t in G/P. In this way, G/P can be considered to be a compactification of the fibres M_P^t, $t \in Z(L_P)$, of the mirror family (M_P, q). The image is a (projected) open Richardson variety [90] and M_P^t can be considered as an open subvariety of G/P.

As an open (projected) Richardson variety, the image of the embedding $M_P^t \to G/P$ is the complement of an anticanonical divisor $\partial_{G/P}$ in G/P [90, Lemma 5.4], where $\partial_{G/P}$ is the multiplicity-free union of the divisors D^i, $i \in I$, and D_i, $i \notin J(P)$,

$$D^i := \overline{p(\mathcal{R}_{w_0^P}^{w_0 s_i})}, \quad \text{and} \quad D_i := \overline{p(\mathcal{R}_{s_i w_0^P}^{w_0})}.$$

Here $p : G/B_+ \to G/P$ is the canonical projection and the Richardson variety $\mathcal{R}_u^v \subseteq G/B_+$ is the intersection of the Schubert cell $B_-\overline{u}B_+/B_+$ with the opposite Schubert cell $B_+\overline{v}B_+/B_+$.

We now proceed to define the (equivariant) superpotential f_P associated with M_P.

Definition 3.1.8. For $i \in I$, define the *elementary characters* $\chi_i : N_+ \to \mathbb{A}^1$ uniquely determined by $\chi_i(x_j(a)) = \delta_{ij} \cdot a$. The *standard regular character* is the character

$$\chi := \sum_{i \in I} \chi_i. \tag{3.1.6}$$

Definition 3.1.9. Define the *superpotential function* f_P to be the holomorphic function

$$\begin{array}{rccc} f_P : & M_P = B_- \cap N_+ Z(L_P)\overline{w}_P N_+ & \longrightarrow & \mathbb{C} \\ & zt\overline{w}_P u & \longmapsto & \chi(z) + \chi(u) \end{array} \tag{3.1.7}$$

For $t \in Z(L_P)$, define $f_P^t : M_P^t \to \mathbb{C}$ to be the restriction of f_P to a fibre of q.

Remark 3.1.10. Writing $x = zt\overline{w}_P u \in B_- \cap N_+(w_P)t\overline{w}_P N_+$, with $z \in N_+(w_P)$, $t \in Z(L_P)$, $u \in N_+$ uniquely determined by x, we have

$$f_P(x) = \chi(z) + \chi(u) = \sum_{i \notin J(P)} \chi_i(z) + \chi(u).$$

For the remainder of this section we assume that G is semisimple. We will also require the ${}^L T$*-equivariant superpotential*. This is a holomorphic function defined on $M_P \times {}^L\mathfrak{t}$ and is (informally) the map

$$(x, h) \longmapsto f_P(x) + \exp(\langle h, \log \pi^0(x) \rangle).$$

Here we have made the canonical identification ${}^L\mathfrak{t} \cong \mathfrak{t}^*$.

Define the variety $\tilde{M}_P$ by the fibre diagram

$$\begin{array}{ccc} \tilde{M}_P & \longrightarrow & \mathfrak{t} \\ \downarrow & & \downarrow \exp \\ M_P & \xrightarrow[e]{} & T \end{array}$$

Hence, $\tilde{M}_P$ may be identified with $\{(b; y) \in M_P \times \mathfrak{t} \mid b \exp(-y) \in N_-\}$. For $t \in Z(L_P)$, consider the correspondence

$$\tilde{M}_P^t := \{(b, y) \in M_P^t \times \mathfrak{t} \mid b \exp(-y) \in N_-\}.$$

The projection

$$\begin{array}{rccc} c_P : & \tilde{M}_P & \longrightarrow & M_P \\ & (b; y) & \longmapsto & b \end{array}$$

is a covering map and there is a commutative diagram

$$\begin{array}{ccc} \tilde{M}_P & \xrightarrow{c_P} & M_P \\ \mathrm{pr}_1 \downarrow & & \downarrow \mathrm{pr}_1 \\ Z(L_P) & = & Z(L_P) \end{array} \tag{3.1.8}$$

inducing a covering on fibres

$$\begin{array}{ccc} \tilde{M}_P^t & \longrightarrow & M_P \\ (b, y) & \longmapsto & b \end{array}$$

In (3.1.8) both projections pr_1 corrspond to projections on to the first factor. Define the holomorphic function

$$\begin{array}{rccc} \tilde{\phi} : & \tilde{M}_P \times {}^L\mathfrak{t} & \longrightarrow & \mathbb{C} \\ & (b, y; h) & \mapsto & \exp(\langle h, y \rangle) \end{array} \tag{3.1.9}$$

Here we canonically identify ${}^L\mathfrak{t}$ with $\mathfrak{t}^*$.

Definition 3.1.11. Define the *${}^LT$-equivariant superpotential function* to be the (multi-valued) holomorphic function

$$f_{P,{}^LT} \coloneqq f_P + \log\tilde{\phi} : \quad M_P \times {}^L\mathfrak{t} \longrightarrow \mathbb{C}$$

Remark 3.1.12. It is immediate from the definition of $\tilde{\phi}$ that the logarithmic derivative in the direction of $\tilde{M}_P$ is independent of y; that is, the logarithmic deriviative of $\tilde{\phi}$ depends only on the M_P directions. In particular, fixing $h \in {}^L\mathfrak{t}$, we can talk about the (logarithmic) critical points of $\tilde{\phi}(\ ;h)$ in a fibre M_P^t in the original mirror family M_P. As such, we can define

$$M_{P,{}^LT}^{\mathrm{crit}} \coloneqq \{(b;h) \in M_P \times {}^L\mathfrak{t} \mid b \text{ is a critical point of } (f_P + \log\tilde{\phi}(\ ;h))_{|M_P^t},\, t \in Z(L_P)\}$$

to be the set of (logarithmic) critical points of $\tilde{\phi}$ in a fibre M_P^t of M_P.

Remark 3.1.13. For our purposes, we will be interested in the critical points of the ${}^LT$-equivariant superpotential function $f_{P,{}^LT}$ restricted to $M_P^t \times \{h\}$, for fixed $t \in Z(L_P)$ and $h \in {}^L\mathfrak{t}$. In particular, the equivariant part will not come into consideration when determining the critical points of $f^t_{P,{}^LT}$.

Quantum cohomology and mirror conjectures

We briefly indicate why the mirror family M_P together with equivariant superpotential $f_{P,{}^LT}$ are the 'correct' Landau-Ginzburg B-model to be considered as the (equivariant) mirror to ${}^LG/{}^LP$. We will describe Rietsch's construction of (a localisation of) the ${}^LT$-equivariant (small) quantum cohomology $qH_T^*({}^LG/{}^LP)$ and recent results of Lam and Lam-Templier on a *mirror conjecture* formulated by Rietsch in [119, Conjecture 8.2]. In this section we assume that G is semisimple.

The *(small) quantum cohomology ring of ${}^LG/{}^LP$*, $qH^*({}^LG/{}^LP)$, is a deformation of the usual cohomology ring $H^*({}^LG/{}^LP) \equiv H^*({}^LG/{}^LP, \mathbb{C})$ with $k = \dim H^2({}^LG/{}^LP)$ parameters, admitting the structure of a $\mathbb{C}[q_1,\ldots,q_k]$-module. As a $\mathbb{C}[q_1,\ldots,q_k]$-module we have

$$qH^*({}^LG/{}^LP) \cong H^*({}^LG/{}^LP) \otimes \mathbb{C}[q_1,\ldots,q_k].$$

The ring structure is defined by deforming the usual cup product, with new (deformed) structure constants defined in terms of genus 0, 3-point Gromov-Witten invariants. The *${}^LT$-equivariant quantum cohomology*, $qH^*_{{}^LT}({}^LG/{}^LP)$, is defined in terms of equivariant genus 0, 3-point Gromov-Witten invariants, and is a module over both $\mathbb{C}[q_1,\ldots,q_k]$ and $H^*(B{}^LT) \cong \mathbb{C}[{}^L\mathfrak{t}]$. $qH^*_{{}^LT}({}^LG/{}^LP)$ is a deformation of the usual ${}^LT$-equivariant cohomology $H^*_{{}^LT}({}^LG/{}^LP)$. See [30], [42] and [11] for further details about (equivariant) Gromov-Witten invariants in general, and [85] for the (equivariant) Gromov-Witten invariants of (partial) flag varieties.

The small (equivariant) quantum cohomology of full flag varieties ${}^LG/{}^LB_+$ has seen significant progress over the past two decades. Presentations of $qH^*({}^LG/{}^LB_+)$ were given by Givental-Kim [50], Ciocan-Fontanine [29] and Kim [86] and identified with the regular functions of the nilpotent leaf of the Toda lattice of the Langlands dual G. In [49], Givental proved a mirror conjecture relating oscillatory integrals on the mirror manifold with solutions to his quantum D-module.

The corresponding results for partial flag varieties have proved more elusive. However, building on the (unpublished) work of D. Peterson, Rietsch obtained the following result for all partial flag varieties.

Theorem 3.1.14 (Rietsch, [119, Theorem 4.1])**.** *There exists an isomorphism*

$$qH^*_{{}^LT}({}^LG/{}^LP)[q_1^{-1},\dots,q_k^{-1}] \cong \mathbb{C}[M^{\mathrm{crit}}_{P,{}^LT}] \tag{3.1.10}$$

between (a localisation of) the ${}^LT$-equivariant quantum cohomology of ${}^LG/{}^LP$ and the coordinate ring of the (possibly non-reduced) variety $M^{\mathrm{crit}}_{P,{}^LT}$ (i.e. the Jacobian ring *of $f_{P,{}^LT}$). The quantum parameters on the right hand side of the isomorphism arise from the quantum structure map $q : M^{\mathrm{crit}}_{P,{}^LT} \to Z(L_P)$, and the equivariant structure is given by projection $M^{\mathrm{crit}}_{P,{}^LT} \to {}^L\mathfrak{t}$ onto the second factor.*

Specialising the equivariant parameters to 0 gives the following identification of the non-equivariant quantum cohomology with the Jacobian ring of the superpotential

Corollary 3.1.15. *There is an isomorphism*

$$qH^*({}^LG/{}^LP) \cong \mathbb{C}[M_P^{\mathrm{crit}}] \tag{3.1.11}$$

where the right hand side is the (possibly non-reduced) variety

$$M_P^{\mathrm{crit}} := \{b \in M_P \mid b \textit{ is a critical point of } (f_P)_{|M_P^t},\ t \in Z(L_P)\} \tag{3.1.12}$$

In [119] Rietsch proposed the following (${}^LT$-equivariant) mirror conjecture:

Conjecture 3.1.16 (Rietsch, [119, Conjecture 8.2])**.** A full set of solutions to the ${}^LT$-equivariant quantum differential equations of ${}^LG/{}^LP$ (defined in [47], [30], for example) is given by the period integrals

$$S_\Gamma(t,h) = \int_{\Gamma_t} \exp(f_P/\hbar)\tilde{\phi}(\ ,h)\omega_t \tag{3.1.13}$$

where $\Gamma = \{\Gamma_t\}_{t\in Z(L_P)}$ is a continuous family of cycles in the fibres M_P^t, and ω_t is a family of non-vanishing to forms on the fibres.

Conjecture 3.1.16 can be considered as a strengthening of Theorem 3.1.14. Conjecture 3.1.16 has been shown to hold in several cases.

Theorem 3.1.17 (Lam, [96])**.** *Let ${}^LP = {}^LB_+$, so that ${}^LG/{}^LB_+$ is a full flag variety. Then, Conjecture 3.1.16 holds.*

Recall that a Dynkin node $i \in I$ is *miniscule* if the set of weights of $V(\varpi_i)$, where ϖ_i is a fundamental weight, are extremal. A parabolic subgroup P is *miniscule* if $J(P) = I \setminus \{i\}$, where i is miniscule.

If ${}^LP \subseteq {}^LG$ is miniscule then the partial flag varieties ${}^LG/{}^LP$ includes Grassmannians, orthogonal Grassmannians and even dimensional quadrics as examples.

Theorem 3.1.18 (Lam-Templier, [97])**.** *Let ${}^LP \subseteq {}^LG$ be a miniscule parabolic subgroup. Then, Conjecture 3.1.16 holds.*

3.2 A conjectural mirror construction for weight varieties

In his 2014 ICM address, Teleman [129] described a conjectural mirror construction for symplectic reductions $M /\!\!/ G$, with G a compact, connected Lie group and M a compact Hamiltonian G-space. This construction is a consequence of a proposed general framework focusing on topological actions of G on Fukaya categories arising from Hamiltonian G-spaces and gauging topological quantum field theories (TQFTs). We will briefly describe this conjecture when M is a flag variety of G, omitting the majority of the (conjectural) details and definitions. For the general story we refer to [129] and the references therein.

Let G be a compact, connected Lie group and $T \subseteq G$ be a maximal torus. Suppose that $M = G/L \cong \mathcal{O}_q \subseteq \mathfrak{g}^*$ is a coadjoint orbit for G with its Kirillov-Kostant-Souriau symplectic structure given by q. Then, M is a Hamiltonian G-space and therefore, upon restriction to T, is a Hamiltonian T-space.

Definition 3.2.1 ([16])**.** The *Bezrukavnikov-Mirkovic-Finkelberg space* $\mathrm{BFM}(G)$ is the holomorphic symplectic reduction of $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ by conjugation under $G_{\mathbb{C}}$ (the complexification of G), where $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ denotes the (open) submanifold of elements that are regular in the cotangent fibre.

Remark 3.2.2 ([16], [129, Theorem 5.1])**.**

1) If $G = T$ then $\mathrm{BFM}(T) = T^*T_{\mathbb{C}}$.

2) The zero fibre of the moment map for the Hamiltonian $G_{\mathbb{C}}$-space $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ is the *universal centraliser*
$$Z_{\mathrm{reg}} = \{(g, \nu) \in G_{\mathbb{C}} \times (\mathfrak{g}^*_{\mathbb{C}})_{\mathrm{reg}} \mid g \cdot \nu = \nu,\ \nu \text{ regular}\} \tag{3.2.1}$$
The space Z_{reg} is smooth with stabilisers of constant dimension. As such, the symplectic manifold structure is evident.

In [129] Teleman considers the (conjectural) 2-category $\sqrt{\mathfrak{Coh}}(\mathrm{BFM}({}^LG))$, the *Kapustin-Rozansky-Saulina (KRS) 2-category* of $\mathrm{BFM}({}^LG)$ (see [76], [77] for further details). This 2-category (conjecturally) contains, for example, (G-equivariant) Fukaya categories of coadjoint orbits $\mathfrak{F}(G/L, q)$ and the dg-category $\mathfrak{Coh}(L)$ of coherent sheaves on smooth holomorphic Lagrangians $L \subseteq \mathrm{BFM}(^L G)$ as objects. To be more precise, $\sqrt{\mathfrak{Coh}}(\mathrm{BFM}(^L G)$ is the sheaf of global sections of a sheaf of $\mathcal{O}_{\mathrm{BFM}(^L G)}$-linear 2-categories, over the space $\mathrm{BFM}(^L G)$.

Remark 3.2.3. Given holomorphic Lagrangians $L, L' \subseteq \mathrm{BFM}(^L G)$, which we consider as objects in $\sqrt{\mathfrak{Coh}}(\mathrm{BFM}(^L G))$, their Hom-category $\mathrm{Hom}(L, L')$ will be a sheaf of categories supported on the intersection $L \cap L'$ and equivalent to the *matrix factorization category* $\mathrm{MF}(L, \Psi)$, for some holomorphic function $\Psi : L \to \mathbb{C}$. For more details see [129, Section 3].

'Theorem' 3.2.4 ([129])**.** *The space* $\mathrm{BFM}(^L G)$ *admits a smooth Lagrangian foliation, parameterised by pairs* (L, q)*, where* $T \subseteq L \subseteq G$ *is a Levi subgroup and* $q \in Z(^L L_{\mathbb{C}})$*. Moreover, the leaves of this foliation (conjecturally) arise as the support of the* G*-equivariant Fukaya categories* $\mathfrak{F}(G/L, q)$*.*

Remark 3.2.5.

1) The quote marks appearing in 'Theorem' 3.2.4 should be interpreted as follows: the existence of a smooth foliation of $\mathrm{BFM}(^L G)$ of the type indicated is proved in [129, Theorem 6.8]. However, the statement concerning Fukaya categories relies on (yet unproven) equivalences of categories predicted by *homological mirror symmetry*, and on the (conjectural) construction of the KRS 2-category.

2) The story here is formally analogous to the Borel-Weil construction of irreducible representions of G. The appearance of the Fukaya category $\mathfrak{F}(G/L, q)$ arises from *symplectic induction* of the category of vector spaces admitting actions of L (passing through $q \in Z(^L L_{\mathbb{C}})$).

The flag variety $(G/L, q)$ admits the structure of a Hamiltonian T-space. Therefore, the T-equivariant Fukaya category $\mathfrak{F}(G/L, q)$ is an object in $\mathrm{BFM}(^L T) = T^* {}^L T_{\mathbb{C}}$. We denote its holomorphic Lagrangian support $\Lambda(q) \subseteq \mathrm{BFM}(^L T)$.

Conjecture 3.2.6 (Teleman, [129])**.** Let ν be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian T-action. Let $t \in Z(^L L_{\mathbb{C}})$ denote the (exponential of the) symplectic structure on G/L. Then, the Fukaya category of the symplectic reduction $(G/L)/\!\!/T(\nu)$ can be computed as the category $\mathrm{Hom}(S_\nu, \Lambda(t))$, where S_ν is the cotangent fibre over $\exp(\nu) \in T^\vee$.

At the level of quantum cohomology, the conjecture is reformulated as follows:

Conjecture 3.2.7. Let ν be a regular value of the moment map $\mu : G/L \to \mathfrak{t}^*$ for the Hamiltonian T-action. Let $t \in Z(^L L_{\mathbb{C}})$ denote the symplectic structure on G/L. Then, whenever the symplectic reduction $(G/L) /\!\!/ T(\nu)$ is Fano, its quantum cohomology can be computed as the Jacobian ring of the restriction of the T-equivariant superpotential to the fibre of the equivariant structure map $e : M_P \to {}^L T$ lying over $\exp(2\pi i \nu)$. Here we canonically identify $\mathfrak{t}^* \cong {}^L\,\mathfrak{t}$. The quantum structure comes from the variation of $t \in Z(^L L_{\mathbb{C}})$.

Moreover, if G has nontrivial (finite) centre Z, then the number of critical points appears with multiplicity $|Z|$.

3.3 Formulae for the superpotential

In this section we describe formulae that will allow us to compute f_P. This will be essential in our approach to computing the quantum cohomology of weight varieties.

To each $\mathbf{i} \in R(w_P^{-1})$ we will describe an explicit expression for the restriction of the *superpotential* f_P to a dense open subset $U \subseteq M_P$, where $U \cong (\mathbb{C}^\times)^{\ell(w_P)}$. We will see that, with respect to this parameterisation, f_P is a rational function (in fact, a Laurent polynomial) whose polar divisors correspond to the divisor constituents of an anticanonical divisor in G/P. This is also observed in [97].

First, we have the following formula for f_P in terms of $x \in M_P$.

Lemma 3.3.1. *For any $x \in N_+ Z(L_P)\overline{w}_P N_+$ we have*

$$f_P(x) = \chi\left(\pi^+(\overline{w}_P^{-1}x)\right) + \sum_{i \notin J(P)} \chi_i\left(\pi^+(\overline{w_P^{-1}}^{-1} x^\iota)\right)$$

Here $g \mapsto g^\iota$ is the positive inverse defined in Section 1.3.

Proof. Let $x = zt\overline{w}_P u$, with $z \in N_+(\overline{w}_P), u \in N_+$. Then, $\pi^+(\overline{w}_P^{-1}x) = u$. Let P^* be the standard parabolic containing B_+ and $\overline{w}_0 L_P \overline{w}_0$. Write $u = u_L v$, where $u_L \in N_+ \cap L_{P^*}$, and $v \in N_+(w_P^{-1})$. Define $v' := \overline{w}_P u_L \overline{w}_P^{-1} \in N_+ \cap L_P$; in particular, $tv' = v't$. Hence,

$$x = zt\overline{w}_P u = zt\overline{w}_P u_L v = zv't\overline{w}_P v$$

and

$$\pi^+(\overline{w_P^{-1}}^{-1} x^\iota) = \pi^+(\overline{w_P^{-1}}^{-1} v^\iota \overline{w_P^{-1}} t^{-1} (zv')^\iota) = (zv')^\iota$$

Here we have used that $v \in N_+(w_p^{-1})$.

For any $i \notin J(P)$, $\chi_i((zv)^\iota) = \chi_i(z)$: the fact that $\chi_i(n^\iota) = \chi_i(n)$, for all $i \in I$, $n \in N_+$, follows from the definition of the map $n \mapsto n^\iota$. Hence, by Remark 3.1.10, for $x = zt\overline{w}_P u \in N_+(w_P)t\overline{w}_P N_+$,

$$f_P(x) = \sum_{i \notin J(P)} \chi_i(z) + \chi(u) = \sum_{i \notin J(P)} \chi_i(\pi^+(\overline{w_P^{-1}}^{-1} x^\iota)) + \chi(\pi^+(\overline{w}_P^{-1}x))$$

□

In the case that G is simply connected we can use the identity

$$\chi_i(\pi^+(g)) = \frac{\Delta_{\varpi_i, s_i\varpi_i}(g)}{\Delta_{\varpi_i,\varpi_i}(g)}$$

to obtain a formula for f_P using generalised minors (recall Section 1.3).

Lemma 3.3.2. *Let G be simply-connected. For any $g \in N_+ Z(L_P)\overline{w}_P N_+$,*

$$f_P(g) = \sum_{i \in I} \frac{\Delta_{w_P\omega_i, s_i\omega_i}(g)}{\Delta_{w_P\omega_i,\omega_i}(g)} + \sum_{i \notin J(P)} \frac{\Delta_{w_0 s_i\omega_i, w_i}(g)}{\Delta_{w_0\omega_i,\omega_i}(g)}$$

Remark 3.3.3. Observe that the terms in the above descriptions of f_P correspond to the divisors D^i, $i \in I$, D_i, $i \notin J(P)$, defined in Remark 3.1.7. This is analogous to the situation for mirror symmetry of Fano toric varieties. Further discussion can be found in [97], [83, Chapter 2].

We will also make use of the following description of f_P due to Lam-Templier [97].
For any $w \in W$, define the variety

$$N_+^w \coloneqq B_- \overline{w} B_- \cap N_+ \subseteq N_+ \tag{3.3.1}$$

This variety is a reduced Bruhat cell (Remark 3.1.5).

Lemma 3.3.4. *(a) There is an isomorphism*

$$\begin{array}{cccc} \eta: & B_-^{w_P} & \longrightarrow & N_+^{w_P^{-1}} \\ & x & \longmapsto & \pi^+(\overline{w}_P^{-1} x) \end{array}$$

(b) There is an injection

$$\begin{array}{cccc} \tau: & N_+^{w_P^{-1}} & \longrightarrow & N_+(w_P) \\ & u & \longmapsto & \pi^+((\overline{w}_P u)^{-1})) \end{array}$$

Proof. (a) If $x = z\overline{w}_P u \in B_- \cap N_+(\overline{w}_P)\overline{w}_P N_+$ then $\pi^+(\overline{w}_P^{-1} x) = u$. Now, observe that

$$u^{-1} = x^{-1} z \overline{w}_P = x^{-1}\overline{w}_P(\overline{w}_P^{-1} z \overline{w}_P) \in B_- \overline{w}_P B_-$$

so that $u \in B_- w_P^{-1} B_- \cap N_+$. Hence, η is well-defined. Conversely, if $u \in N_+^{w_P^{-1}}$ then $u^{-1} \in N_+^{w_P}$ and $u\overline{w}_P^{-1} \in B_- N_+(w_P)$. Then, the inverse to η is seen to be

$$u \longmapsto \pi^-(u^{-1}\overline{w}_P^{-1})^{-1}.$$

(b) In the course of the proof above we saw that $\pi^+(u^{-1}\overline{w}_P^{-1}) \in N_+(w_P)$. In fact, we find

$$\tau(u) = \eta(u)(\overline{w}_P u)^{-1}$$

so that τ is injective.

□

Lemma 3.3.4 implies that the restriction of the superpotential f_P to a fibre of q can be defined as a map on $N_+^{w_P^{-1}}$: we can trivialise the mirror family

$$\begin{array}{ccc} Z(L_P) \times N_+^{w_P^{-1}} & \xrightarrow{m\circ(\mathrm{id}\times\eta^{-1})} & M_P \\ (t,u) & \longmapsto & t\eta^{-1}(u) = t\tau(u)\overline{w}_P u \end{array} \tag{3.3.2}$$

Hence, if $x = zt\overline{w}_P u \in M_P^t$ then, as function of $(t,u) \in Z(L_P) \times N_+^{w_P^{-1}}$,

$$f_P^t(t,u) = \chi(t\tau(u)t^{-1}) + \chi(u). \tag{3.3.3}$$

Lemma 3.3.5. *Let $u \in N_+^{w_P^{-1}}$. Then,*

$$\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}}) = \overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0,$$

where P^* is the standard parabolic subgroup containing B_+ and $\overline{w}_0 L_P \overline{w}_0^{-1}$.

Proof. Let $u \in N_+^{w_P^{-1}}$. Then, $x = \tau(u)\overline{w}_P u \in B_-^{w_P}$ and $u = \overline{w}_P^{-1}\tau(u)^{-1}x$. Hence,

$$\overline{w}_0^{-1}u^T = \overline{w}_0^{-1}x^T\tau(u)^{-T}\overline{w}_P = \overline{w}_0^{-1}x^T\overline{w}_0\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0\overline{w_0^{P^*}}^{-1} \tag{3.3.4}$$

Here we use that $w_P w_0^{P^*} = w_0$, with $\ell(w_P)+\ell(w_0^{P^*}) = \ell(w_0)$, so that $\overline{w}_P = \overline{w}_0\overline{w_0^{P^*}}^{-1}$. Finally, $x \in B_-$ so that $\overline{w}_0^{-1}x^T\xi\overline{w}_0 \in B_-$ and the result follows. □

Since, for any $i \in I$,

$$\overline{w}_0^{-1}y_i(a)\overline{w}_0 = x_{i^*}(-a),$$

we see that

$$\chi_i(\tau(u)) = \chi_{i^*}(\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0)$$

and

$$\sum_{i\notin J(P)} \chi_i(\tau(u)) = \sum_{i\notin J(P)} \chi_{i^*}(\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0) = \sum_{i\notin J(P^*)} \chi_i(\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}}))$$

Lemma 3.3.6. *For $u \in N_+^{w_P^{-1}}$, $i \in I$,*

$$\chi_i(\tau(u)) = \chi_{i^*}(\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}})).$$

Hence, if $x = zt\overline{w}_P u \in B_- \cap N_+(w_P)Z(L_P)\overline{w}_P N_+$ then

$$f_P(x) = \chi(u) + \sum_{i\notin J(P^*)} \alpha_{i^*}(t)\chi_i(\pi^+(\overline{w}_0^{-1}u^T\overline{w_0^{P^*}}))$$

In particular, when G is simply-connected we have

$$f_P^t(zt\overline{w}_P u) = \chi(u) + \sum_{i\notin J(P^*)} \alpha_{i^*}(t)\frac{\Delta_{w_0^{P^*}s_i\varpi_i, w_0\varpi_i}(u)}{\Delta_{\varpi_i, w_0\varpi_i}(u)}$$

Proof. Noting that $\chi_i(tnt^{-1}) = \alpha_i(t)\chi_i(n)$, for any $n \in N_+$, the result follows from (3.3.3) and Lemma 3.3.5. For the last formula recall the definition of the generalised minors in Section 1.3 and note that, for any $i \notin J(P^*)$, $w_0^{P^*}\varpi_i = \varpi_i$. □

We will now describe a formula for the restriction of f_P to a family of open subsets $U_{\mathbf{i}} \subseteq N_+^{w_P^{-1}}$, $\mathbf{i} \in R(w_P^{-1})$, each of which is isomorphic to a complex algebraic torus.

Let $\mathbf{i} = (i_1, \ldots, i_r) \in R(w)$, $w \in W$. Define the map

$$\begin{array}{rccc} x_{\mathbf{i}}: & (\mathbb{C}^\times)^r & \longrightarrow & N_+^w \\ & (a_1, \ldots, a_r) & \longmapsto & x_{i_1}(a_1)\cdots x_{i_r}(a_r) \end{array}$$

where x_i, $i \in I$, is the root subgroup corresponding to $\alpha_i \in S$ (see Section 1.3. An essential property of the maps $x_{\mathbf{i}}$ is the following result.

Lemma 3.3.7 (Fomin-Zelevinsky, [36, Theorem 1.2])**.** *Let* $\mathbf{i} \in R(w)$*,* $w \in W$*. Then,* $x_{\mathbf{i}}$ *is an open embedding.*

Definition 3.3.8. Let $\mathbf{i} \in R(w_P^{-1})$. Define the open embedding

$$\begin{array}{rccc} j_{\mathbf{i}}: & Z(L_P) \times (\mathbb{C}^\times)^{\ell(w_P)} & \longrightarrow & M_P \\ & (t, a) & \longmapsto & t\tau(x_{\mathbf{i}}(a))\overline{w}_P x_{\mathbf{i}}(a) \end{array} \tag{3.3.5}$$

We call $j_{\mathbf{i}}$ the *FZ-parameterisation in the direction* $\mathbf{i}$.

Proposition 3.3.9 ([97])**.** *Assume G is simply-connected. Let* $\mathbf{i} = (i_1, \ldots, i_r) \in R(w_P^{-1})$*. Then,*

$$f_P(j_{\mathbf{i}}(t, a)) = a_1 + \ldots + a_r + \sum_{i \notin J(P^*)} \alpha_{i^*}(t) F_i(a)$$

where $F_i(a) \in \mathbb{Z}_{\geq 0}[a_1^{\pm}, \ldots a_r^{\pm}]$ *is a Laurent polynomial with nonnegative integer coefficients.*

Proof. We use Lemma 3.3.6. Let $\mathbf{i} = (i_1, \ldots, i_r) \in R(w_P^{-1})$. First, we observe that if $u = x_{\mathbf{i}}(a)$ then

$$\chi_i(u) = \sum_{i_j = i} \chi_{i_j}(x_{i_j}(a_j)) = \sum_{i_j = i} a_{i_j}.$$

Hence, $\chi(u) = a_1 + \ldots + a_r$. Finally, [14, Theorem 5.8] shows that $\Delta_{w_0^{P^*} s_i \varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))$ is a polynomial with nonegative integer coefficients, and [14, Corollary 9.4] shows that $\Delta_{w_0^{P^*}\varpi_i, w_0\varpi_i}(x_{\mathbf{i}}(a))$ is a monomial. □

3.4 Computing the quantum cohomology of polygon spaces

In this section we describe a new approach to computing the quantum cohomology of a class of weight varieties in type A: the *polygon spaces* $\mathcal{P}_{r,n}$ (see Examples 2.2.4, 2.2.6 and Section 2.5). First, we set up our notation specific to this setting.

Let $G = \mathrm{SL}_{n+1}(\mathbb{C})$, and write $I = \{1, \dots, n\}$. Choose T to be the maximal torus consisting of diagonal matrices and write $t = \mathrm{diag}(t_1, \dots, t_{n+1})$ for elements of T. Let $H \subseteq T$ be a maximal compact torus in T, identified with $(S^1)^n$. We set B_+ to be the subgroup of upper triangular matrices with unipotent radical N_+ being the subgroup of upper triangular unipotent matrices. The opposite Borel B_- consists of lower triangular matrices and its unipotent radical is N_-, the subgroup of lower triangular unipotent matrices. If $P \supseteq B_+$ is a standard parabolic, $J(P) = \{j_1, \dots, j_l\} \subseteq I$, so that $\{k_1, \dots, k_m\} = I \setminus J(P)$, then P consists of those upper block-triangular matrices having blocks of the form

$$\begin{bmatrix} A_1 & * & * & * & * \\ 0 & A_2 & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & * \\ 0 & 0 & \cdots & 0 & A_{m+1} \end{bmatrix} \in G$$

where A_1 is $k_1 \times k_1$, A_i is $(k_i - k_{i-1}) \times (k_2 - k_1)$, for $i = 2, \dots, m$, and A_{m+1} is $(n+1-k_m) \times (n+1-k_m)$. The Levi subgroup is the subgroup of block-diagonal matrices in P_J. In particular, $Z(L_P)$ is identified with an algebraic torus of rank m.

The Langlands dual group is ${}^LG = \mathrm{PGL}_{n+1}(\mathbb{C})$ which we identify with G/Z, $Z \subseteq G$ is the (finite, cyclic) centre of G. The dual torus ${}^LT$ is identified with T/Z. The corresponding subgroups ${}^LB_\pm$ and ${}^LN_\pm^\vee$ are the images of corresponding subgroups of G. We identify ${}^LN_\pm$ with $N_\pm$ (there is a unique lift under the canonical quotient homomorphism). For standard parabolic $P \subseteq G$ we identify ${}^LP$ with the image of P in ${}^LG$.

Let $(X, R, X^\vee, R^\vee)$ be the root datum of G. The weight lattice $X = \mathrm{Hom}(T, \mathbb{G}_m)$ admits a basis of fundamental weights $\varpi_1, \dots, \varpi_n$,

$$\begin{aligned} \varpi_i : \qquad T \quad &\longrightarrow \quad \mathbb{C}^\times \\ \mathrm{diag}(t_1, \dots, t_{n+1}) \quad &\longmapsto \quad t_1 \cdots t_i \end{aligned}$$

The positive roots corresponding to B_+ are $R_+ = \{\alpha_{ij} \mid 1 \le i < j \le n+1\}$, where

$$\begin{aligned} \alpha_{ij} : \qquad T \quad &\longrightarrow \quad \mathbb{C}^\times \\ \mathrm{diag}(t_1, \dots, t_{n+1}) \quad &\longmapsto \quad t_i t_j^{-1}, \end{aligned}$$

and corresponding simple roots $\alpha_i \coloneqq \alpha_{i,i+1}$, $i = 1, \dots, n$. In particular,

$$\alpha_{ij} = \alpha_i + \ldots + \alpha_j, \quad i < j.$$

The simple coroots are $S^\vee = \{\alpha_i^\vee \mid i = 1, \dots, n\}$, where

$$\begin{array}{rccl} \alpha_i^\vee : & \mathbb{C}^\times & \longrightarrow & T \\ & c & \longmapsto & \mathrm{diag}(1, \dots, \underbrace{c, c^{-1}}_{i,i+1}, \dots, 1) \end{array}$$

The Weyl group is $W = S_{n+1}$, the permutation group on $n+1$ letters. Define s_i, $i = 1, \dots, n$, to be the standard adjacent transpositions. The longest element is the permutation

$$w_0 = (1\ n+1)(2\ n)(3\ n-1)\cdots . \tag{3.4.1}$$

For a standard parabolic subgroup P_J, $J = \{j_1, \dots, j_l\}$, the Weyl group of the pair (L_P, T) is the subgroup generated by $s_{j_1}, \dots, s_{j_l}$ and can be identified with a product of permutation groups. The longest element w_0^P is the product of the longest elements for each of these permutation groups.

For each $i \in I$, we have the root subgroups

$$\begin{array}{rccl} x_i : & \mathbb{C} & \longrightarrow & N_+ \\ & c & \longmapsto & \mathbb{I} + cE_{i,i+1} \end{array}$$

and

$$\begin{array}{rccl} y_i : & \mathbb{C} & \longrightarrow & N_+ \\ & c & \longmapsto & \mathbb{I} + cE_{i+1,i} \end{array}$$

Here E_{ij} is the matrix with 1 in the ij-entry and 0s elsewhere, $\mathbb{I}$ is the identity matrix.

The monomial matrix representative $\overline{s}_i = \in N_G(T)$, $i \in I$, is the the image of the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in the SL_2-triple generated by $\mathrm{im}\, x_i$ and $\mathrm{im}\, y_i$.

The quantum cohomology of a point

As a sanity check, we describe the simplest case of a (proper) parabolic subgroup $P \subseteq G$ of maximal dimension. This case will also provide highlights of the methods we use in the next section when we consider polygon spaces. In fact, the following calculation can be considered as a verification of Conjecture 3.2.7 for the moduli of triangles (the $n = 3$ polygon space).

Let $J = \{2, \dots, n\}$ and $P = P_J$ be the parabolic subgroup

$$P = \left\{ \begin{bmatrix} a & b^t \\ 0 & C \end{bmatrix} \in G \right\}.$$

Then, ${}^LG/{}^LP \cong \mathbb{P}^n_{\mathbb{C}}$. As $\mathbb{P}^n_{\mathbb{C}}$ is a toric variety under the (diagonal) action of ${}^LT$, any symplectic reduction by H will be a point. Hence, Conjecture 3.2.7 predicts a single critical point for the superpotential f_P when restricted to a generic fibre of the equivariant structure map $e : M_P \to T$. Let's verify that this does indeed hold. Our computation will provide a template for calculations in the remaining sections of this Chapter.

The Levi subgroup $L_P \subseteq P$ is

$$L_P = \left\{ \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix} \in G \right\}.$$

The parabolic subgroup is $W_P = \langle s_2, \ldots, s_n \rangle \subseteq W$ and the longest element in W_P is w_0^P. Using the reduced expressions

$$w_0^P = s_2 s_3 s_2 \cdots s_n \cdots s_2, \quad \text{and} \quad w_0 = s_n s_{n-1} s_n s_{n-2} s_{n-1} s_n \cdots s_n,$$

we find that

$$w_P^{-1} = w_0 w_0^P = s_n \cdots s_1.$$

As any two reduced expressions are related by a sequence of braid relations, this last expression implies that $R(w_P^{-1}) = \{(n, \ldots, 1)\}$. Let $\mathbf{i} = (n, \ldots, 1) \in R(w_P^{-1})$ be this unique reduced expression.

The parabolic P^* containing B_+ and $\overline{w}_0 L_P \overline{w}_0^{-1}$ is such that $J(P^*) = \{1, \ldots, n-1\}$. Hence, $w_0^{P^*}$ is the longest element in the permutation group $\langle s_1, \ldots s_{n-1} \rangle \cong S_{n-1}$.

Using Lemma 3.3.6 we compute f_P^t, $t \in Z(L_P)$, with respect to the FZ-parameterisation in the direction $\mathbf{i}$. We have

$$\begin{aligned} f_P(j_{\mathbf{i}}(t,a)) &= a_1 + \ldots + a_n + \sum_{i \notin J(P^*)} \alpha_{i^*}(t) \frac{\Delta_{w_0^{P^*} s_i \varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(a))} \\ &= a_1 + \ldots + a_n + \alpha_1(t) \left(\frac{\Delta_{w_0^{P^*} s_n \varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a))} \right) \end{aligned}$$

Identifying generalised minors with matrix minors (see Section 1.3) we find

$$\Delta_{w_0^{P^*} s_n \varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a)) = \Delta_{12 \cdots (n-1)(n+1), 23 \cdots (n+1)}(x_{\mathbf{i}}(a))$$

and

$$\Delta_{\varpi_n, w_0 \varpi_n}(x_{\mathbf{i}}(a)) = \Delta_{12 \cdots n, 23 \cdots (n+1)}(x_{\mathbf{i}}(a)).$$

Using induction on n, we find

$$x_{\mathbf{i}}(a_1, \ldots, a_n) = \begin{bmatrix} 1 & a_n & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Hence, $\Delta_{12\cdots n,23\cdots(n+1)}(x_{\mathbf{i}}(a)) = 1$ and, since $\Delta_{12\cdots n,23\cdots(n+1)}$ is the determinant of the top right $n \times n$ matrix,

$$\Delta_{12\cdots n,23\cdots(n+1)}(x_{\mathbf{i}}(a)) = a_1 \cdots a_n$$

Hence,

$$f_P(j_{\mathbf{i}}(t,a)) = a_1 + \ldots + a_n + \frac{\alpha_1(t)}{a_1 \cdots a_n}. \tag{3.4.2}$$

Remark 3.4.1. The superpotential f_P computed in (3.4.2) is precisely the well-known superpotential associated to projective space [30].

By Conjecture 3.2.7, the quantum cohomology of $\mathbb{P}^n_{\mathbb{C}} /\!\!/ {}^L T$ (which is a point) is obtained by restricting (3.4.2) to a fibre of $e : M_P \to B_-/N_- = T$ and computing the Jacobian ring. We now determine e with respect to the FZ-parameterisation $j_{\mathbf{i}}$.

Let $u \in N_+^{w_P^{-1}}$, $t \in Z(L_P)$. Then, $j_{\mathbf{i}}(t,u) = t\eta^{-1}(u)$ and the image of $j_{\mathbf{i}}(t,u)$ under e is $te(\eta^{-1}(u))$. Observe that $e(\eta^{-1}(u)) = \pi^0(\eta^{-1}(u))$.

Let $\eta(x) = u$ so that $x = \tau(u)\overline{w}_P u$. We saw in the proof of Lemma 3.3.5 that

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0.$$

As $\tau(u) \in N_+(w_P)$ we obtain $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1})$. Therefore,

$$\pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}) = \pi^0(\overline{w}_0^{-1} x^T \overline{w}_0) = \overline{w}_0^{-1}\pi^0(x)\overline{w}_0$$

In particular, we can compute

$$e(\eta^{-1}(u)) = \overline{w}_0 \pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}})\overline{w}_0^{-1}.$$

We will require to use the fact that we are in a special situation, namely that P is very large.

Proposition 3.4.2. *Let $u \in N_+^{w_P^{-1}}$. Then, $e(\eta^{-1}(u))$ is uniquely determined by the diagonal entries of the matrix $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$.*

Proof. We recall the notation immediately preceding the statement of the Proposition. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0$$

and $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1}) = N_+(w_{P^*})$. The subgroup

$$N_+(w_{P^*}) = \prod_{\substack{\alpha \in R^+ \text{ s.t.} \\ w_{P^*}^{-1}(\alpha) \in R^-}} \operatorname{im} x_\alpha = \prod_{i=1}^{n} \operatorname{im} x_{\alpha_{in}}$$

consists of unipotent matrices $n \in N_+$ of the form

$$n = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & * \\ 0 & 1 & 0 & \cdots & \cdots & * \\ 0 & 0 & 1 & \cdots & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & * \\ 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \tag{3.4.3}$$

Write $x = vs \in N_-T$, with $v \in N_-$, $s \in T$. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \left(\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0\right) \overline{w}_0^{-1} s \overline{w}_0 \left(\overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0\right)$$

As $\overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0$ is a matrix of the form (3.4.3) and $\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0 \in N_-$, the element $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$ is a matrix of the form

$$\begin{bmatrix} d_1(u) & 0 & 0 & \cdots & * \\ * & d_2(u) & 0 & \cdots & * \\ * & * & d_3(u) & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}$$

Projecting onto the first n diagonal entries allows us to define a morphism of varieties

$$\begin{aligned} g: \quad N_+^{w_P^{-1}} & \longrightarrow T \\ u & \longmapsto g(u) \coloneqq \operatorname{diag}(d_1(u), \ldots, d_n(u), (d_1(u) \cdots d_n(u))^{-1}) \end{aligned}$$

Then, by construction, we have

$$e(\eta^{-1}(u)) = \overline{w}_0 g(u) \overline{w}_0^{-1}.$$

□

We illustrate the proof of the above Proposition with an example.

Example 3.4.3. Consider $G = \mathrm{SL}_4(\mathbb{C})$, $\mathbf{i} = (3, 2, 1)$. Then,

$$\overline{w}_0^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \overline{w_0^{P^*}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then, for

$$u = x_{\mathbf{i}}(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_3 & 0 & 0 \\ 0 & 1 & a_2 & 0 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have

$$\overline{w_0}^{-1} u^T \overline{w_0^{P*}} = \begin{bmatrix} -a_1 & 0 & 0 & 1 \\ 1 & -a_2 & 0 & 0 \\ 0 & 1 & -a_3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute

$$\eta^{-1}(u) = \begin{bmatrix} a_1^{-1}a_2^{-1}a_3^{-1} & 0 & 0 & 0 \\ 1 & a_3 & 0 & 0 \\ 0 & 1 & a_2 & 0 \\ 0 & 0 & 1 & a_1 \end{bmatrix}$$

If we let $y = \operatorname{diag}(-a_1, -a_2, -a_3, -(a_1a_2a_3)^{-1})$, then we have

$$e(\eta^{-1}(u)) = \overline{w}_0 y \overline{w}_0^{-1}.$$

A straightforward generalisation of Example 3.4.3 provides the following computation of e restricted to the FZ-parameterisation.

Proposition 3.4.4. *Let* $\mathbf{i}$ *be the unique reduced expression for* w_P^{-1}*, where* P *is the standard parabolic subgroup with* $J(P) = \{2, \dots, n\}$*. Let* $j_{\mathbf{i}}$ *be the FZ-parameterisation in the direction* $\mathbf{i}$*,* $e : M_P \to T$ *the equivariant structure map. Then,*

$$e(j_{\mathbf{i}}(t, a_1, \dots, a_n)) = t \operatorname{diag}((a_1 \cdots a_n)^{-1}, a_n, a_{n-1}, \dots, a_1) \in T$$

Hence, we are comforted to see that the intersection of the fibres of the quantum structure map q and the equivariant structure map e is a single point.

Theorem 3.4.5. *Conjecture 3.2.7 holds for symplectic reductions of* ${}^LG/{}^LP \cong \mathbb{P}^n_{\mathbb{C}}$*.*

Remark 3.4.6. If we swap the role of G and ${}^LG$, so that ${}^LG = \mathrm{SL}_{n+1}(\mathbb{C})$, then Conjecture 3.2.7 states that we expect $|Z({}^LG)|$ critical points of f_P. Indeed, the computation can proceed as above, and f_P (in the FZ-parameterisation) is equal to the expression (3.4.2).

In this situation, determining the equivariant structure map with respect to the FZ-parameterisation is similar to Proposition 3.4.2: for any $u \in N_+^{w_P^{-1}}$, $e(\eta^{-1}(u)) \in T \subseteq \mathrm{PGL}_{n+1}(\mathbb{C})$ is determined by the diagonal entries of $\overline{w_0}^{-1} u^T \overline{w_0^{P*}}$. We compute, for $t \in Z(L_P)$, $u \in N_+^{w_P^{-1}}$,

$$e(j_{\mathbf{i}}(t, u)) = t \operatorname{diag}((a_1^2 a_2 \cdots a_n)^{-1}, a_n a_1^{-1}, \dots, a_2 a_1^{-1}, 1) \in T$$

Here we are choosing the unique representative of elements in $T = {}^LT/Z({}^LG)$ whose last diagonal entry is 1.

Then, the intersection of the fibres of $q^{-1}(t)$ and $e^{-1}(s)$, where $t = \operatorname{diag}(c, 1, \dots, 1)Z \in Z(L_P)$, $s = \operatorname{diag}(c_1, \dots, c_n, 1)Z \in T$, can be identified with the set

$$\left\{ a \in \mathbb{C}^\times \mid a^{n+1} = \frac{c}{c_1 \cdots c_n} \right\}.$$

Hence, we have $n + 1 = |Z({}^LT)|$ critical points of the restriction of f_P^t to a fibre of the equivariant structure map.

The quantum cohomology of polygon spaces

In this section we will outline a new approach to computing the quantum cohomology of the class of weight varieties realised as symplectic reductions of the complex Grassmannian of 2-planes $\mathrm{Gr}_{\mathbb{C}}(2, n+1)$: these are the polygon spaces $\mathcal{P}_{r,n+1}$ (Examples 2.2.4, 2.2.6 and Section 2.5).

Let $P \subseteq G$ be a standard parabolic subgroup such that $J(P) = \{1, 3, \ldots, n+1\}$ or $J(P) = \{1, 2, \ldots, n-2, n\}$. Recall that, in the former case ${}^L G/{}^L P \cong \mathrm{Gr}_{\mathbb{C}}(2, n+1)$ and, in the latter case ${}^L G/{}^L P$ is isomorphic to the Grassmannian of $(n-1)$-planes in $\mathbb{C}^{n+1}$ (by duality, this is the same as $\mathrm{Gr}_2((\mathbb{C}^{n+1})^*)$).

Let $P \subseteq G$ be the parabolic subgroup

$$P = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in G \mid A \text{ is } 2 \times 2 \right\}$$

with Levi subgroup

$$L_P = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in P \right\}$$

and $Z(L_P) \cong \mathbb{G}_m$. Therefore, we are considering the case $J(P) = \{1, 3, \ldots, n\}$. The parabolic subgroup P^* is the standard parabolic subgroup such that $J(P^*) = \{1, 2, \ldots, n-2, n\}$. The parabolic subgroup $W_P = \langle s_1, s_3, \ldots, s_n \rangle \subseteq W$ and is isomorphic to a product of permutation groups $S_2 \times S_{n-1}$. The longest element in W_P is w_0^P.

By Theorem 2.5.6, a symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2, n+1) \cong {}^L G/{}^L P$ is a polygon space $\mathcal{P}_{r,n+1}$. Here $r \in \mathbb{R}^{n+1}_{>0}$ is such that $|r|$ corresponds to the Kahler form on $\mathrm{Gr}_{\mathbb{C}}(2, n)$ defined via its realisation as a symplectic reduction of complex affine space (Example 2.2.7). In this setting, we are considering $\mathbb{R}^{n+1} \cong \mathfrak{h}'$, where $H \subseteq H' \subseteq U(n+1)$ is the maximal diagonal torus. Since $G = \mathrm{SL}_{n+1}(\mathbb{C})$, we project $\mathfrak{u}(n+1)$ along $\mathbb{R}(1, \ldots, 1)$ onto $\mathfrak{su}(n+1)$ and write $\hat{r} \in \mathfrak{h}$ for the image of r under this projection. In this way, we can associate to r the element $t(r) \in T$ where, if $\hat{r} = (\hat{r}_1, \ldots, \hat{r}_{n+1})$, we define

$$t(r) \coloneqq \mathrm{diag}(\exp(2\pi i \hat{r}_1), \ldots, \exp(2\pi i \hat{r}_{n+1})) \in T \subseteq \mathrm{SL}_{n+1}(\mathbb{C}).$$

We now proceed to describe our main conjecture: an explicit description of the quantum cohomology of $\mathcal{P}_{r,n}$. First, we require the following elementary result.

Lemma 3.4.7. $w_{P^*}^{-1} = w_0 w_0^{P^*} = s_2 \cdots s_n s_1 \cdots s_{n-1}$.

Proof. Consider the following reduced expressions

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_n \cdots s_1, \quad \text{and} \quad w_0^{P^*} = s_1 s_2 s_1 \cdots s_{n-2} \cdots s_1 s_n$$

For $i = 1, \ldots, n$, write $w_i = s_i s_{i-1} \cdots s_2 s_1$, so that $w_0 = w_1 w_2 \cdots w_n$. First we note that, for any n,

$$w_n w_1 w_2 \cdots w_{n-2} = w_1 \cdots w_{n-3} s_n s_{n-1}. \tag{3.4.4}$$

Indeed, proceeding by induction we find

$$\begin{aligned} s_n(w_{n-1}w_1\cdots w_{n-3})w_{n-2} &= s_n(w_1\cdots w_{n-4})s_{n-1}s_{n-2}w_{n-2} \\ &= s_n w_1\cdots w_{n-4}s_{n-1}w_{n-3} \\ & w_1\cdots w_{n-3}s_n s_{n-1} \end{aligned}$$

The last equality follows because $s_j w_i = w_i s_j$, whenever $j > i+1$.

Thus, assuming the result holds for $n-1$ we have,

$$\begin{aligned} w_0 w_0^{P^*} &= w_1 w_2\cdots w_n w_1 w_2\cdots w_{n-2}s_n \\ &= w_1\cdots w_{n-1}w_1\cdots w_{n-3}s_n s_{n-1}s_n, \quad \text{by (3.4.4)}, \\ &= w_1\cdots w_{n-1}w_1\cdots w_{n-3}s_{n-1}s_n s_{n-1} \\ &= s_2\cdots s_{n-1}s_1\cdots s_{n-2}s_n s_{n-1}, \quad \text{by induction}, \\ &= s_2\cdots s_n s_1\cdots s_{n-1}. \end{aligned}$$

□

Hence, $w_P^{-1} = s_{n-1}\cdots s_1 s_n\cdots s_2$.

Let $\mathbf{i} = (n-1, n-2, \ldots, 1, n, n-1, \ldots, 2) \in R(w_P^{-1})$. Using Lemma 3.3.6 we compute f_P^t, $t \in Z(L_P)$, with respect to the FZ-parameterisation in the direction $\mathbf{i}$. By Lemma 3.3.6, we have

$$\begin{aligned} f_P(j_{\mathbf{i}}(t,a)) &= a_1 + \ldots + a_n + \sum_{i\notin J(P^*)} \alpha_{i^*}(t)\frac{\Delta_{w_0^{P^*}s_i\varpi_i, w_0\varpi_i}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_i, w_0\varpi_i}(x_{\mathbf{i}}(a))} \\ &= a_1 + \ldots + a_n + \alpha_2(t)\left(\frac{\Delta_{w_0^{P^*}s_{n-1}\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a))}{\Delta_{\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a))}\right) \end{aligned}$$

Identifying generalised minors with matrix minors (see Section 1.3), and using

$$w_0^{P^*} = s_1 s_2 s_1 s_3 s_2 s_1\cdots s_{n-2}\cdots s_1 s_n,$$

we find

$$\Delta_{w_0^{P^*}s_{n-1}\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a)) = \Delta_{2\cdots(n-1)(n+1), 3\cdots(n+1)}(x_{\mathbf{i}}(a))$$

and

$$\Delta_{\varpi_{n-1}, w_0\varpi_{n-1}}(x_{\mathbf{i}}(a)) = \Delta_{1\cdots(n-1), 3\cdots n(n+1)}(x_{\mathbf{i}}(a)).$$

Now,

$$x_{\mathbf{i}}(a_1,\ldots,a_{n-1},b_1,\ldots,b_{n-1}) = x_{n-1}(a_1)\cdots x_1(a_{n-1})x_n(b_1)\cdots x_2(b_{n-1})$$

Using

$$x_{n-1}(a_1)\cdots x_1(a_{n-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & 0 & \cdots & 0 \\ 0 & 0 & 1 & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$x_n(b_1)\cdots x_2(b_{n-1}) = \begin{bmatrix} 1 & b_{n-1} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & b_{n-2} & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

we find

$$x_{n-1}(a_1)\cdots x_1(a_{n-1})x_n(b_1)\cdots x_2(b_{n-1}) = \begin{bmatrix} 1 & a_{n-1} & a_{n-1}b_{n-1} & 0 & \cdots & 0 \\ 0 & 1 & a_{n-2}+b_{n-1} & a_{n-2}b_{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1b_1 \\ 0 & 0 & 0 & 0 & \cdots & b_1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

In particular,

$$\Delta_{1\cdots(n-1),3\cdots(n+1)}(x_{\mathbf{i}}(a,b)) = a_1\cdots a_{n-1}b_1\cdots b_{n-1}.$$

An induction argument gives

$$\Delta_{2\cdots(n-1)(n+1),3\cdots(n+1)}(x_{\mathbf{i}}(a,b)) = a_1\cdots a_{n-2} + a_1\cdots a_{n-3}b_{n-1} + \cdots + a_1b_3\cdots b_{n-1} + b_2\cdots b_{n-1}$$

Hence,

$$\begin{aligned} f_P(j_{\mathbf{i}}(t,a,b)) &= a_1 + \ldots + a_{n-1} + b_1 + \ldots + b_{n-1} \\ &\quad + \alpha_2(t)\frac{a_1\cdots a_{n-2} + a_1\cdots a_{n-3}b_{n-1} + \cdots + b_2\cdots b_{n-1}}{a_1\cdots a_{n-1}b_1\cdots b_{n-1}} \end{aligned} \tag{3.4.5}$$

This expression for the superpotential is related to previous mirror constructions of [49] and [10] in the following way.

First we introduce the definition of a Gelfand-Tsetlin quiver.

Definition 3.4.8. Let $P \subseteq G$ be a standard parabolic subgroup. A *Gelfand-Tsetlin quiver of shape* P is a quiver GT_P with underlying set of vertices $\mathcal{V}_P = \{\alpha \in R^+ \mid w_P^{-1}(\alpha) \in -R^+\}$, and arrows defined by

$$\alpha \longrightarrow \beta \qquad \text{if and only if} \qquad \beta = \alpha + \alpha_i, \quad \text{for some } i \in I.$$

We include two additional vertices v_h, v_t and two additional arrows

$$\alpha_{13} \longrightarrow v_h \qquad \text{and} \qquad v_h \longrightarrow \alpha_{2,n+1}.$$

For an arrow $a \in \mathrm{GT}_P$, we define $h(a)$ to be the head of a, $t(a)$ to be the tail of a.

For P such that $J(P) = \{1, 3 \ldots, n\}$, the Gelfand-Tsetlin quivers take the form

$$\begin{array}{ccccccccc}
v_h & \longleftarrow & \alpha_{13} & \longleftarrow & \alpha_{14} & \longleftarrow & \cdots & \longleftarrow & \alpha_{1,n+1} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & \alpha_{23} & \longleftarrow & \alpha_{24} & \longleftarrow & \cdots & \longleftarrow & \alpha_{2,n+1} \\
 & & & & & & & & \uparrow \\
 & & & & & & & & v_t
\end{array}$$

Given GT_P, a Gelfand-Tsetlin quiver of shape P, where $J(P) = \{1, 3, \ldots, n\}$, we define a family of monomial transformations of $(\mathbb{C}^\times)^{2(n-1)}$. Let $t \in Z(L_P)$ and $q = \alpha_2(t) \in \mathbb{C}^\times$. Let $(a_1, \ldots, a_{n-1}, b_1, b_{n-1})$ denote the standard coordinates on $(\mathbb{C}^\times)^{2(n-1)}$. Associate the variables z_{1i}, z_{2i}, $i = 1, \ldots, n-1$, to the vertices of Γ as follows

$$\begin{array}{ccccccccc}
q & \longleftarrow & z_{1,n-1} & \longleftarrow & z_{1,n-2} & \longleftarrow & \cdots & \longleftarrow & z_{11} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & z_{2,n-1} & \longleftarrow & z_{2,n-2} & \longleftarrow & \cdots & \longleftarrow & z_{21} \\
 & & & & & & & & \uparrow \\
 & & & & & & & & 1
\end{array}$$

and define the following monomial transformation of $(\mathbb{C}^\times)^{2(n-1)}$:

$$\begin{aligned}
z_{1i} &:= \frac{q}{a_{n+2-i} \cdots a_{n-1}}, & i &= 3, \ldots, n-1, \\
z_{2i} &:= b_1 \cdots b_i, & i &= 3, \ldots .n-2, \\
z_{2,n-1} &:= \frac{q}{a_{n-1} b_{n-1}}.
\end{aligned} \tag{3.4.6}$$

The following result is a generalisation of Givental's mirror construction for the complete flag variety [49], and is similar to an observation of Marsh-Rietsch [109]. An analogous construction of the superpotential given a Gelfand-Tsetlin quiver of type P was given in [10] (see also [35] for a physical derivation).

Proposition 3.4.9. *Let $P \subseteq \mathrm{SL}_{n+1}(\mathbb{C})$ be a standard parabolic subgroup such that $J(P) = \{1, 3, \ldots, n\}$ with Levi subgroup P containing the maximal torus T of diagonal matrices. Let $\mathbf{i} = (n-1, \ldots, 1, n, \ldots, 2) \in R(w_P^{-1})$. Composing the FZ-parameterisation in the direction $\mathbf{i}$ with the inverse of the monomial transformation (3.4.6), the superpotential f_P takes the form*

$$f_p = \sum_{a \in \mathrm{GT}_P} \frac{z_{h(a)}}{z_{t(a)}}$$

Remark 3.4.10. Given a Gelfand-Tsetlin quiver of an arbitrary standard parabolic subgroup P, we conjecture that there exists a monomial transformation on $(\mathbb{C}^\times)^{\ell(w_P)}$ giving rise to a similar description of the superpotential.

We now compute the equivariant structure map $e : M_P \to T$ with respect to the FZ-parameterisation in the direction $\mathbf{i}$.

Let $u \in N_+^{w_P^{-1}}$, $t \in Z(L_P)$. Then, $j_\mathbf{i}(t, u) = t\eta^{-1}(u)$ and the image of $j_\mathbf{i}(t, u)$ under e is $te(\eta^{-1}(u))$. Observe that $e(\eta^{-1}(u)) = \pi^0(\eta^{-1}(u))$.

Let $\eta(x) = u$ so that $x = \tau(u)\overline{w}_P u$. We saw in the proof of Lemma 3.3.5 that

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0.$$

As $\tau(u) \in N_+(w_P)$ we obtain $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1})$. Therefore,

$$\pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}) = \pi^0(\overline{w}_0^{-1} x^T \overline{w}_0) = \overline{w}_0^{-1} \pi^0(x) \overline{w}_0$$

In particular, we can compute

$$e(\eta^{-1}(u)) = \overline{w}_0 \pi^0(\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}) \overline{w}_0^{-1}.$$

Proposition 3.4.11. *Let $u \in N_+^{w_P^{-1}}$. Then, $e(\eta^{-1}(u))$ is uniquely determined by the diagonal entries of the matrix $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$ and $u^{-1}\overline{w}_P^{-1}$.*

Proof. We recall the notation immediately preceding the statement of the Proposition. The proof is similar to the proof of Proposition 3.4.2. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \overline{w}_0^{-1} x^T \overline{w}_0 \overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0$$

and $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0 \in N_+(w_P^{-1}) = N_+(w_{P^*})$. The subgroup

$$N_+(w_{P^*}) = \prod_{\substack{\alpha \in R^+ \text{ s.t.} \\ w_{P^*}^{-1}(\alpha) \in R^-}} \operatorname{im} x_\alpha = \prod_{i=1}^{n-1} \operatorname{im} x_{\alpha_{i,n}} \times \prod_{i=1}^{n-1} \operatorname{im} x_{\alpha_{i,n-1}}$$

consists of unipotent matrices $n \in N_+$ of the form

$$n = \begin{bmatrix} 1 & 0 & 0 & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{3.4.7}$$

Write $x = vs \in N_-T$, with $v \in N_-$, $s \in T$. In particular, $s = e(\eta^{-1}(u))$. We have

$$\overline{w}_0^{-1} u^T \overline{w_0^{P^*}} = \left(\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0\right) \overline{w}_0^{-1} s \overline{w}_0 \left(\overline{w}_0^{-1} \tau(u)^{-T} \overline{w}_0\right)$$

As $\overline{w}_0^{-1}\tau(u)^{-T}\overline{w}_0$ is a matrix of the form (3.4.7) and $\overline{w}_0^{-1} s v^T s^{-1} \overline{w}_0 \in N_-$, the element $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$ is a matrix of the form

$$\begin{bmatrix} d_1(u) & 0 & 0 & \cdots & 0 & * & * \\ * & d_2(u) & 0 & \cdots & 0 & * & * \\ * & * & d_3(u) & \cdots & 0 & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \end{bmatrix}$$

Consider the map that projects onto the first $n-1$ diagonal entries of $\overline{w}_0^{-1} u^T \overline{w_0^{P^*}}$

$$\begin{aligned} h: \quad N_+^{w_P^{-1}} &\longrightarrow (\mathbb{C}^\times)^{n-1} \\ u &\longmapsto h(u) := (d_1(u), \ldots, d_{n-1}(u)) \end{aligned}$$

Then, by construction, we have

$$\overline{w}_0 \operatorname{diag}(h(u), 1, 1) \overline{w}_0^{-1} = (1, 1, s_3, \ldots, s_{n+1}),$$

where $s = \operatorname{diag}(s_1, \ldots, s_{n+1}) \in T$.

The diagonal entries s_1, s_2 can be determined by the following argument. For $u \in N_+^{w_P^{-1}}$, and $x = vs$ such that $\eta(x) = u$, we have

$$u^{-1}\overline{w}_P^{-1} = \eta(u)^{-1}\tau(u) = (s^{-1}v^{-1}s)s^{-1}\tau(u) \in N_-TN_+(w_P).$$

By a similar analysis as above, we have $u^{-1}\overline{w}_P^{-1}$ is a matrix whose top left 2×2 block is of the form

$$\begin{bmatrix} s_1^{-1} & 0 \\ * & s_2^{-1} \end{bmatrix}$$

Hence, the map that projects onto the first two diagonal entries of $u^{-1}\overline{w}_P^{-1}$

$$g = (g_1, g_2) : \quad N_+^{w_P^{-1}} \longrightarrow (\mathbb{C}^\times)^2$$

determines the remaining diagonal entries of $s = e(\eta^{-1}(u))$.

Finally, define

$$\begin{array}{rccl} f: & N_+^{w_P^{-1}} & \longrightarrow & T \\ & u & \longmapsto & \operatorname{diag}(g_1(u)^{-1}, g_2(u)^{-1}, 1 \ldots, 1)\overline{w}_0 \operatorname{diag}(h(u), 1, 1)\overline{w}_0^{-1} \end{array}$$

By construction, $f(u) = e(\eta^{-1}(u))$. □

We highlight the proof of Proposition 3.4.11 with an example.

Example 3.4.12. Let $G = \mathrm{SL}_5(\mathbb{C})$ and $\mathbf{i} = (3, 2, 1, 4, 3, 2) \in R(w_P^{-1})$. Then,

$$\overline{w}_0^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \overline{w_0^{P*}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \overline{w}_P^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$u = x_{\mathbf{i}}(a_1, a_2, a_3, b_1, b_2, b_3) = \begin{bmatrix} 1 & a_3 & a_3b_3 & 0 & 0 \\ 0 & 1 & a_2 + b_3 & a_2b_2 & 0 \\ 0 & 0 & 1 & a_1 + b_2 & a_1b_1 \\ 0 & 0 & 0 & 1 & b_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\overline{w}_0^{-1} u^T \overline{w_0^{P*}} = \begin{bmatrix} a_1b_1 & 0 & 0 & 1 & -b_1 \\ -(a_1 + b_2) & a_2b_2 & 0 & 0 & 1 \\ 1 & -(a_2 + b_3) & a_3b_3 & 0 & 0 \\ 0 & 1 & -a_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Projecting onto the first three diagonal entries gives $h(u) = (a_1b_1, a_2b_2, a_3b_3)$.

Also,

$$u^{-1}\overline{w}_P^{-1} = \begin{bmatrix} a_1a_2a_3 & 0 & 1 & -a_3 & a_2a_3 \\ -(a_1a_2 + a_1b_3 + b_2b_3) & b_1b_2b_3 & 0 & 1 & -(a_2 + b_3) \\ a_1 + b_2 & -b_1b_2 & 0 & 0 & 1 \\ -1 & b_1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Projecting onto the first two diagonal entries gives $g(u) = (a_1a_2a_3, b_1b_2b_3)$.

Then, the Proposition shows that $e(\eta^{-1}(u))$ is the matrix

$$\begin{aligned}&\operatorname{diag}((a_1a_2a_3)^{-1},(b_1b_2b_3)^{-1},1,1,1)\cdot\overline{w}_0\operatorname{diag}(a_1b_1,a_2b_2,a_3b_3,1,1)\overline{w}_0^{-1}\\=&\operatorname{diag}((a_1a_2a_3)^{-1},(b_1b_2b_3)^{-1},a_3b_3,a_2b_2,a_1b_1)\in T,\end{aligned}$$

in agreement with

$$\eta^{-1}(u)=\begin{bmatrix}(a_1a_2a_3)^{-1} & 0 & 0 & 0 & 0\\ \frac{a_1a_2+a_1b_3+b_2b_3}{a_1a_2a_3b_1b_2b_3} & (b_1b_2b_3)^{-1} & 0 & 0 & 0\\ 1 & a_3 & a_3b_3 & 0 & 0\\ 0 & 1 & a_2+b_3 & a_2b_2 & 0\\ 0 & 0 & 1 & a_1+b_2 & a_1b_1\end{bmatrix}$$

Proposition 3.4.11 allows us to determine an explicit formula for e in the FZ-parameterisation.

Theorem 3.4.13. *Let* $\mathbf{i}=(n-1,n-2,\dots,1,n,n-1,\dots,2)\in R(w_P^{-1})$, *where* $P\subseteq G$ *is the standard parabolic with* $J(P)=\{1,3\dots,n\}$. *Let* $j_{\mathbf{i}}$ *be the FZ-parameterisation in the direction* $\mathbf{i}$, $e:M_P\to T$ *the equivariant structure map. Then,*

$$\begin{aligned}&e(j_{\mathbf{i}}(t,a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}))\\=&t\cdot\prod_{j=3}^{n+1}\left(\alpha_{1j}^\vee(a_{n+2-j})\alpha_{2j}^\vee(b_{n+2-j})\right)\in T.\end{aligned}$$

Proof. This follows from a calculation similar to Example 3.4.12. □

Corollary 3.4.14. *Fix* $t\in Z(L_P)$.

(a) For any $j=1,\dots,n$,

$$\varpi_j\left(e(j_{\mathbf{i}}(t,a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}))\right)=\begin{cases}\varpi_1(t)(a_1\cdots a_{n-1})^{-1}, & \text{if } j=1,\\ \varpi_2(t)(a_1\cdots a_{n-1}b_1\cdots b_{n-1})^{-1}, & \text{if } j=2,\\ \varpi_j(t)\prod_{i=1}^{j}a_{n+1-i}^{-1}b_{n+1-i}^{-1}, & \text{if } j=3,\dots,n.\end{cases}$$

(b) Let $c_1,\dots,c_n\in\mathbb{C}^\times$ *be generic. Then,*

$$\frac{\mathbb{C}[a_1^\pm,\dots,a_{n-1}^\pm,b_1^\pm,\dots,b_{n-1}^\pm]}{\langle c_i-\varpi_i(e(j_{\mathbf{i}}(t,\underline{a},\underline{b})))\rangle_{i\in I}}\cong\frac{\mathbb{C}[b_1^\pm,\dots,b_{n-1}^\pm]}{\langle\varpi_1(t)c_2b_1\cdots b_{n-1}-\varpi_2(t)c_1\rangle}$$

Quantum cohomology of polygon spaces in low rank

In this section we will verify Conjecture 3.2.7 for polygon spaces $\mathcal{P}_{r,n}$ with $n=4,5$.

Quantum cohomology of the moduli space of 4-gons $\mathcal{P}_{r,4}$

Let $n+1=4$. Let $P \subseteq \mathrm{SL}_4(\mathbb{C})$ be the standard parabolic subgroup of upper block-triangular matrices such that $J(P)=\{1,3\}$; then $P=P^*$. Therefore, $Z(L_P)=\{\mathrm{diag}(a,a,a^{-1},a^{-1}) \mid a \in \mathbb{C}^\times\}$. Let $\mathbf{i}=(2,1,3,2) \in R(w_P^{-1})$. With respect to the FZ-parameterisation in the direction $\mathbf{i}$, (3.4.5) implies that the superpotential takes the form

$$f_P(t,a_1,a_2,b_1,b_2)=a_1+a_2+b_1+b_2+\alpha_2(t)\frac{a_1+b_2}{a_1a_2b_1b_2}, \quad t \in Z(L_P), a_1,a_2,b_1,b_2 \in \mathbb{C}^\times.$$

Theorem 3.4.13 shows that, with respect to the FZ-parameterisation, the equivariant structure map e takes the form

$$e(t,a_1,a_2,b_1,b_2)=t\,\mathrm{diag}((a_1a_2)^{-1},(b_1b_2)^{-1},a_2b_2,a_1b_1).$$

We trivialise T as follows

$$\begin{array}{ccc} T & \longrightarrow & (\mathbb{C}^\times)^3 \\ t & \longmapsto & (\varpi_1(t),\varpi_2(t),\varpi_3(t)) \end{array}$$

Then, for $c=(c_1,c_2,c_3) \in (\mathbb{C}^\times)^3$, and $t=\mathrm{diag}(a,a,a^{-1},a^{-1}) \in Z(L_P)$, the intersection of the fibres $e^{-1}(c) \cap q^{-1}(t)$ is described by the equations

$$\frac{a}{a_1a_2}=c_1, \quad \frac{a^2}{a_1a_2b_1b_2}=c_2, \quad \frac{a}{a_1b_1}=c_3$$

Hence, we can eliminate a_1, a_2 and b_2 and the restriction of the superpotential f_P^t to the fibre $e^{-1}(c)$ is

$$f_P^t(b_1)=\frac{a}{c_3b_1}+\frac{c_3b_1}{c_1}+b_1+\frac{c_1a}{c_2b_1}+c_2\left(\frac{a}{c_3b_1}+\frac{c_1a}{c_2b_1}\right)$$

Here we have used $\alpha_2(t)=a^2$. Hence, for generic c, we obtain

$$\mathrm{Jac}(f_P^t) \cong \frac{\mathbb{C}[b]}{(b^2-a)}.$$

Recall from Example 2.2.4 that a weight variety constructed as the symplectic reduction of $\mathrm{Gr}(2,4)$ is diffeomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Hence, our construction recovers the well-known quantum cohomology ring of $\mathbb{P}^1_{\mathbb{C}}$ (see [30]).

Theorem 3.4.15. *Let $X=\mathrm{Gr}_{\mathbb{C}}(2,4)=\mathrm{SL}_4(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, (M_P,F_P) the Rietsch mirror family. Let $e: M_P \to {}^LT$ be the equivariant structure map. Let $\mathcal{P}_{r,4}$, $r \in \mathbb{Z}^4_{>0}$, be the space of 4-gons realised as the symplectic reduction of X. Then, the quantum cohomology of $\mathcal{P}_{r,4}$ can be computed as the Jacobian ring of the restriction of f_P to a generic fibre of e. In particular, Conjecture 3.2.7 is verified.*

Quantum cohomology of the moduli space of 5-gons $\mathcal{P}_{r,5}$

Let $n+1=5$. Let $P \subseteq \mathrm{SL}_5(\mathbb{C})$ be the standard parabolic subgroup of upper block-triangular matrices such that $J(P)=\{1,3,4\}$; then P^* is the parabolic subgroup such that $J(P^*)=\{1,2,4\}$. Let $\mathbf{i}=(3,2,1,4,3,2)\in R(w_P^{-1})$. With respect to the FZ-parameterisation in the direction $\mathbf{i}$, (3.4.5) implies that the superpotential takes the form

$$f_P(t,a_1,a_2,a_3,b_1,b_2,b_3)=a_1+a_2+a_3+b_1+b_2+b_3+\alpha_2(t)\frac{a_1a_2+a_1b_3+b_2b_3}{a_1a_2a_3b_1b_2b_3}$$

for $t\in Z(L_P), a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{C}^\times$.

As above, we trivialise T using the fundamental weights

$$\begin{array}{ccc} T & \longrightarrow & (\mathbb{C}^\times)^4 \\ t & \longmapsto & (\varpi_1(t),\varpi_2(t),\varpi_3(t),\varpi_4(t)) \end{array}$$

and Corollary 3.4.14 implies that fibre of e over $(c_1,c_2,c_3,c_4)\in(\mathbb{C}^\times)^4$ are defined by the equations

$$c_1=\varpi_1(t)(a_1a_2a_3)^{-1},\ c_2=\varpi_2(t)(a_1a_2a_3b_1b_2b_3)^{-1},\ c_3=\varpi_3(t)(a_1a_2b_1b_2)^{-1},\ c_4=\varpi_4(t)(a_1b_1)^{-1}. \tag{3.4.8}$$

Recall Example 2.2.6. Since the diffeomorphism type of $\mathcal{P}_{r,5}$ is constant through chambers of the moment polytope Ξ of the Hamiltonian H-space $\mathrm{Gr}_{\mathbb{C}}(2,5)$, we may perturb r so that $r\in\Xi\cap\mathbb{Q}^5$. As such we assume that $r=(r_1,\dots,r_5)$ has rational entries. In [37] it is shown that $\mathcal{P}_{r,5}$ is diffeomorphic to one of the following complex projective varieties

$$\mathbb{P}^2_{\mathbb{C}},\quad \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}},\quad \text{or}\quad X_j,\quad j=1,2,3,4,$$

where X_j is a blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at j distinct points. In particular, being a del Pezzo surface, a moduli space of pentagons is always Fano.

By Corollary 2.4.9, if r lies in a chamber bounded by an external wall of Ξ then $\mathcal{P}_{r,5}\cong\mathbb{P}^2_{\mathbb{C}}$. The discussion in Example 2.2.6 shows that for such $r=(r_1,r_2,r_3,r_4,r_5)$ there exists some $i\in\{1,\dots,5\}$ such that

$$r_i<\sum_{j\neq i}r_j$$

while

$$r_i+r_j>\sum_{k\neq i,j}r_k,\quad \text{for all } j\neq i.$$

For example, $r=\frac{1}{7}(1,1,3,1,1)\in\Xi$ lies in a chamber bordering the external wall Ξ_3. Hence, letting $\xi=e^{4\pi i/35}$, the equations defining the fibre of e over $t(r)=(\xi^{-1},\xi^{-1},\xi^4,\xi^{-1},\xi^{-1})\in T$ become

$$a_1a_2a_3=\varpi_1(t)\xi,\ a_1a_2a_3b_1b_2b_3=\varpi_2(t)\xi^2,\ a_1a_2b_1b_2=\varpi_3(t)\xi^{-2},\ a_1b_1=\varpi_4(t)\xi^{-1}.$$

Eliminating a_1, a_2, a_3, b_3, and using the monomial transformation $p_1 = b_1$, $p_2 = b_1 b_2$, the restriction of the superpotential to this fibre becomes

$$f_P^1 = \frac{1}{\xi p_1} + \frac{p_1}{\xi p_2} + \xi^3 p_2 + p_1 + \frac{p_2}{p_1} + \frac{\xi}{p_2} + \frac{1}{\xi^4 p_2} + \frac{1}{\xi^2 p_1 p_2} + \frac{1}{\xi p_1}.$$

The Jacobian ring of f_P^1 is described by the equations

$$p_1^2 = p_2 + \frac{1}{\xi}, \quad \text{and} \quad 0 = -\frac{p_1^2}{\xi} + \xi^3 p_1 p_2^2 + p_2^2 - \frac{1}{\xi^2}$$

Substituting the first equation into the second, and recalling that $p_1 \neq 0$, we have

$$\mathrm{Jac}(f_P^1) \cong \frac{\mathbb{C}[x]}{\langle p(x) \rangle},$$

where $p(x)$ is a degree 3 polynomial with distinct roots.

Theorem 3.4.16. *Let $X = \mathrm{Gr}_{\mathbb{C}}(2,5) = \mathrm{SL}_5(\mathbb{C})/P$ be the complex Grassmannian of 2-planes, (M_P, F_P) the Rietsch mirror family. Let $e : M_P \to {}^L T$ be the equivariant structure map. Let $\mathcal{P}_{r,5}$, $r \in \mathbb{Z}^5_{>0}$, be the space of 5-gons realised as the symplectic reduction of X. Let $r = (1,1,1,1,3)$. Then, for the quantum cohomology of $\mathcal{P}_{r,5}$ can be computed as the Jacobian ring of the restriction of f_P to a generic fibre of e.*

3.5 Future directions

It would be interesting to extend the methods developed in this chapter to further classes of weight varieties in type A. One particular class where Conjecture 3.2.7 could be verified is for certain symplectic reductions of complete flag varieties ${}^L G / {}^L B_+$.

Let $\lambda \in \mathfrak{t}_+^*$ be generic and $\mathcal{O}_\lambda \subseteq \mathfrak{sl}_n$ the corresponding coadjoint orbit. By mapping a complete flag to its 1-dimensional constituent, we obtain a bundle

$$\mathcal{O}_\lambda \longrightarrow \mathbb{P}_{\mathbb{C}}^{n-1}$$

This bundle is a $\mathrm{SL}_n(\mathbb{C})$-equivariant symplectic fibration (see [60]) with fibre being a complete flag variety for $\mathrm{SL}_{n+1}(\mathbb{C})$. The Minimal Coupling Theorem (see [60, Chapter 4]) states that if the fibres F of an equivariant symplectic fibration $X \to B$ are small enough then the symplectic reduction of X at μ, for an open subset of μ's, is a bundle over the symplectic reduction of B with fibre F. In this situation, 'small enough' means that λ is in a certain open neighbourhood of the line through the first fundamental weight. Hence, as the reduction of $\mathbb{P}_{\mathbb{C}}^{n-1}$ is a point, this implies that the reduction of $\mathcal{O}_\lambda$ is a complete flag variety for $\mathrm{SL}_{n-1}(\mathbb{C})$.

Therefore, for λ close enough to ϖ_1 we expect that the approach developed in this thesis to compute the quantum cohomology of weight varieties of $\mathrm{SL}_n(\mathbb{C})$ will recover the quantum cohomology of a complete flag variety for $\mathrm{SL}_{n-1}(\mathbb{C})$. The quantum cohomology rings of complete flag varieties are known by [50], allowing us to verify our computation.

Chapter 4

Crystals

This Chapter investigates the appearance of combinatorial structures from representation theory in the mirror symmetry for flag varieties. Let G be a reductive complex algebraic group with Lie algebra $\mathfrak{g}$. Associated to $\mathfrak{g}$ is the Drinfeld-Jimbo quantised universal enveloping algbra $U_q(\mathfrak{g})$, a Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$. The representation theory of $U_q(\mathfrak{g})$ is similar to the representation theory of $\mathfrak{g}$ and admits sufficient extra structure to be able to better understand the combinatorial representation theory of $\mathfrak{g}$. The key is Lusztig's *canonical basis* $\mathcal{B}$ of the positive part of $U_q(\mathfrak{g})$ [102] (independently discovered by Kashiwara [79]). $\mathcal{B}$ is canonical in the sense that it gives rise to canonical weight bases of all finite dimensional simple $U_q(\mathfrak{g})$-modules via the standard (co)Verma module construction.

Determining elements of the basis $\mathcal{B}$ itself is difficult. However, there exist several useful parameterisations of $\mathcal{B}$. For $\mathbf{i}$, a reduced expression for the longest element w_0 in the Weyl group of $\mathfrak{g}$, the *string parameterisation* $c_{\mathbf{i}}$ of Littelmann [98] provides a parameterisation of $\mathcal{B}$ by the lattice points in a rational polyhedral cone $C_{\mathbf{i}} \subseteq \mathbb{R}^{\ell(w_0)}$, the *string cone (in the direction* $\mathbf{i}$*)*. The *extended string cone* $\underline{C}_{\mathbf{i}} \subseteq \mathbb{R}^{\mathrm{rk}(\mathfrak{g})+\ell(w_0)}$ is a modification of $C_{\mathbf{i}}$ that 'remembers' the way in which $\mathcal{B}$ interacts with the finite dimensional simple $U_q(\mathfrak{g})$-modules.

The canonical basis $\mathcal{B}$ gives rise to a rich combinatorial structure known as a Kashiwara crystal [80]. These combinatorial objects model the representation theory of $\mathfrak{g}$ and provide effective approaches to studying tensor product multiplicities. Birational analogues of Kashiwara crystals, known as *geometric crystals*, were introduced by Berenstein-Kazhdan [12]. From a geometric crystal, one can construct a Kashiwara crystal via *tropicalisation*. For us, tropicalisation is a functor Trop from the class $\mathcal{V}$ of *positive varieties* (birational to algebraic tori) to Set such that, if $f : X \to \mathbb{A}^1$ is a rational function, $X \in \mathcal{V}$ birational to the algebraic torus S, then $\mathrm{Trop}(f) : X^\vee(S) \to \mathbb{Z}$ is a piecewise linear function. Details can be found in Section 4.3.

A remarkable fact is that the Rietsch mirror family (M_P, f_P) introduced in Section 3.1 is part of a geometric crystal. This has been observed in [96], [97], and used to prove mirror conjectures of Rietsch [119]. Our main result, Theorem 4.4.5, uses a family of nonstandard parameterisation $j_{\mathbf{i}}$ of M_P, $\mathbf{i}$ a reduced expression of w_0, to explicitly show that the tropical locus $\{\mathrm{Trop}(f_P) \geq 0\}$ can be identified with the lattice points in the extended string cone $\underline{C}_{\mathbf{i}}(\mathbb{Z})$. Specifically, with respect to the parameterisation $j_{\mathbf{i}}$, we recover precisely the inequalities defining $\underline{C}_{\mathbf{i}}$.

In Section 4.1 we recall background from the theory of quantised universal enveloping algebras. Section 4.2 introduces Lusztig's canonical basis $\mathcal{B}$, its consequences for representation theory and a brief account of the role of $\mathcal{B}$ in determining combinatorial tensor product multiplicity formulae. We define several parameterisations of $\mathcal{B}$ including the the family of string parameterisations due to Littelmann. We conclude this section by introducing the extended string cone $\underline{C}_{\mathbf{i}}$ and the λ-inequalities that define it. In Section 4.3, we give a brief account of Kashiwara's theory of crystals and their geometric counterparts developed by Berenstein-Kazhdan. In this section we develop the tool of tropicalisation, realised as a functor from a certain class of varieties to Set. Section 4.4 introduces a non-standard parameterisation of the Rietsch mirror (M_B, f_B), and we state and prove our main result Theorem 4.4.5. We conclude with a discussion illuminating intriguing similarities between the hierarchy of a family of toric degenerations on the A-model side (introduced in [117]) and the crystal structure obtained in Theorem 4.4.5.

4.1 Some quantum algebra

In this section we recall some of the structure theory of quantised universal enveloping algebras associated to the Lie algebra $\mathfrak{g}$ of a reductive complex algebraic group G and their representation theory.

Convention 4.1.1. Throughout this section G will be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$ and Lie algebra $\mathfrak{g}$. We adopt the conventions and notation from Section 1.3. We will assume that $X = \Pi$ is the lattice of integral weights.

The Cartan matrix $[c_{ij}]_{i,j\in I}$, where $c_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$, $i, j \in I$, is symmetrisable. Let $d_i \in \mathbb{Z}_{>0}$, $i \in I$, be such that $d_i c_{ij} = d_j c_{ji}$. We assume that the integers $\{d_i\}_{i\in I}$ are pairwise relatively prime.

Let $\mathbb{C}(q)$ be the field of rational functions. Given $n \in \mathbb{Z}$ we define

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-3)} + q^{-(n-1)} \in \mathbb{C}(q).$$

Set $[0]_q! := 1$, $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$, for $n > 0$, and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q!}{[n]_q![m-n]_q!}, \quad 0 \leq n \leq m.$$

Quantised universal enveloping algebras

Definition 4.1.2. The *(Drinfeld-Jimbo) quantised universal enveloping algebra associated to* $\mathfrak{g}$, $U_q(\mathfrak{g})$, or simply U when there is no risk of confusion, is the associative $\mathbb{C}(q)$-algebra with unit generated by the elements E_i, F_i, $i \in I$, and K_h, $h \in X^\vee$, such that the function

$$\begin{array}{ccc} \mathbb{C}(q)[X^\vee] & \longrightarrow & U_q(\Psi(G)) \\ e^h & \longmapsto & K_h \end{array}$$

is a homomorphism of (commutative, unital) $\mathbb{C}(q)$-algebras (here $\mathbb{C}(q)[X^\vee]$ is the group algebra of $X^\vee$ over $\mathbb{C}(q)$), and such that the following relations hold:

(1) $K_h E_i K_{-h} = q^{\langle \alpha_i, h\rangle} E_i$, for $i \in I, h \in X^\vee$,

(2) $K_h F_i K_{-h} = q^{-\langle \alpha_i, h\rangle} F_i$, for $i \in I, h \in X^\vee$,

(3) $E_i F_j - F_j E_i = \delta_{ij} \frac{K_{d_i \alpha_i^\vee} - K_{-d_i \alpha_i^\vee}}{q^{d_i} - q^{-d_i}}$, for $i, j \in I$,

(4) for every $i \neq j$,

$$\sum_{r+s=1-c_{ij}} (-1)^r E_i^{(r)} E_j E_i^{(s)} = \sum_{r+s=1-c_{ij}} (-1)^r F_i^{(r)} F_j F_i^{(s)} = 0.$$

Here $E_i^{(r)} = E_i/[r]_{q^{d_i}}!$ and $F_i^{(r)} = F_i/[r]_{q^{d_i}}!$ are the *q-divided powers*.

We write $U^0(\mathfrak{g})$, or simply U^0 when there is no risk of confusion, for the image of the homomorphism $\mathbb{C}(q)[X^\vee] \to U_q(\mathfrak{g})$ described above. A standard argument shows that $U^0 \cong \mathbb{C}(q)[X^\vee]$.

Denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the $\mathbb{C}(q)$-subalgebra generated by E_i, $i \in I$, (resp. F_i, $i \in I$). When there is no risk of confusion we write U^+ (resp. U^-).

We write $U_q(\mathfrak{g})^{\geq 0}$ (resp. $U_q(\mathfrak{g})^{\leq 0}$) to be the subalgebra generated by E_i, $i \in I$, and K_h, $h \in X^\vee$, (resp. F_i, $i \in I$, and K_h, $h \in X^\vee$). When there is no risk of confusion we write $U^{\geq 0}$ (resp. $U^{\leq 0}$).

For any $i \in I$, we define $U_q(\mathfrak{g})_i$ to be the $\mathbb{C}(q^{d_i})$-subalgebra generated by $E_i, F_i, K_{\pm d_i \alpha_i^\vee}$. $U_q(\mathfrak{g})_i$ is a subalgebra isomorphic to $U_q(\mathfrak{sl}_2)$ (this follows from Proposition 4.1.5 below). When there is no risk of confusion we write U_i.

Define, for every $i \in I$, $h \in X^\vee$,

$$\deg(K_h) = 0, \text{ and } \deg(E_i) = -\deg(F_i) = \alpha_i.$$

The defining relations of $U_q(\mathfrak{g})$ are homogeneous and $U_q(\mathfrak{g})$ is a Q-graded algebra. Furthermore, there is the root space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha,$$

where

$$U_q(\mathfrak{g})_\alpha = \{u \in U_q(\mathfrak{g}) \mid K_h u K_{-h} = q^{\langle h, \alpha\rangle} u, \text{ for all } h \in X^\vee\}.$$

When there is no risk of confusion we write U_α in place $U_q(\mathfrak{g})_\alpha$.

We will also make use of the following involutions:

(i) the *bar involution*, $u \mapsto \overline{u}$, is the $\mathbb{C}$-algebra automorphism of $U_q(\mathfrak{g})$ such that

$$\overline{q} = q^{-1}, \quad \overline{E}_i = E_i, \quad \overline{F}_i = F_i, \quad \overline{K}_h = K_{-h}, \quad i \in I, h \in X^\vee. \tag{4.1.1}$$

An element $u \in U_q(\mathfrak{g})$ is *bar-invariant* if $u = \overline{u}$.

(ii) the $\mathbb{C}(q)$-algebra automorphism $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, uniquely determined by

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_h) = K_{-h}, \quad i \in I, h \in X^\vee. \tag{4.1.2}$$

(iii) The $\mathbb{C}(q)$-algebra antiautomorphism $\iota : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, uniquely determined by

$$\iota(E_i) = E_i, \quad \iota(F_i) = F_i, \quad \iota(K_h) = K_{-h}, \quad i \in I, h \in X^\vee. \tag{4.1.3}$$

Observe that ι is Q-graded.

Remark 4.1.3. The quantised universal enveloping algebra $U = U_q(\mathfrak{g})$ can be given the structure of a non-commutative, non-cocommutative Hopf algebra $(U, \Delta, S, \varepsilon)$, where

$$\Delta(E_i) = E_i \otimes 1 + K_{d_i\alpha_i^\vee} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-d_i\alpha_i^\vee} + 1 \otimes F_i, \quad \Delta(K_h) = K_h \otimes K_h, \tag{4.1.4}$$

$$S(E_i) = -K_{-d_i\alpha_i^\vee} E_i, \quad S(F_i) = -F_i K_{d_i\alpha_i^\vee}, \quad S(K_h) = K_{-h}, \tag{4.1.5}$$

$$\varepsilon(K_h) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \tag{4.1.6}$$

for $i \in I$, $h \in X^\vee$. With this structure of a Hopf algebra $U_q(\mathfrak{g})$ is a *Hopf algebra deformation* of $U(\mathfrak{g})$, the universal enveloping algebra associated to $\mathfrak{g}$ (equipped with its usual (noncommutative, cocommutative) Hopf algebra structure).

As a Hopf algebra $U_q(\mathfrak{g})$ is a deformation of $U(\mathfrak{g})$ as follows: in the specialisation $q \to 1$, the Hopf algebra structure given to $U_q(\mathfrak{g})$ specialises to the Hopf algebra $U(\mathfrak{g})$. Details can be found in [69, Chapter 3], [32, Chapter 6], or [72, Chapter 4].

Remark 4.1.4. The antipode map S given in Remark 4.1.3 is an antiautomorphism of $\mathbb{C}(q)$-algebras.

A standard application of the Hopf algebra structure on $U_q(\mathfrak{g})$ is the existence of a *triangular decomposition*, originally due to Rosso:

Proposition 4.1.5 (Rosso, [122])**.** *Let $U = U_q(\mathfrak{g})$. Then, $U = U^- \otimes U^0 \otimes U^+$.*

Corollary 4.1.6. *The subalgebras $U^\pm$ are completely determined by the relations in Definition 4.1.2 (4). In particular, the automorphism ω induces isomorphisms of $\mathbb{C}(q)$-algebras $U^\pm \cong \mathrm{U}^\mp$.*

Also, $U^{\geq 0} \cong U^0 \otimes U^+$, $U^{\leq 0} \cong U^- \otimes U^0$.

Remark 4.1.7. Using the involution ω and Corollary 4.1.6, we also have $U = U^+ \otimes U^0 \otimes U^-$.

Proposition 4.1.5 and Corollary 4.1.6 imply that a basis for $U_q(\mathfrak{g})$ is determined once we specify a basis for $U_q^+(\mathfrak{g})$ (equivalently $U_q^-(\mathfrak{g})$). In Section 4.2 we will construct several bases for $U^\pm$: the *PBW-type bases* and Lusztig's *canonical basis*.

Remark 4.1.8. Given any symmetrisable Kac-Moody algebra $\mathfrak{g}$ with associated Cartan datum $(X, S, X^\vee, S^\vee)$ one can define an associated quantised universal enveloping algebra $U_q(\mathfrak{g})$. For further details see [69].

Representation theory

In light of Remark 4.1.3, aspects of the representation theory of $U_q(\mathfrak{g})$ should closely resemble the representation theory of G. Some precise statements on how the representation theory of $U_q(\mathfrak{g})$ is a 'deformation' of the representation theory of G are given in [27, Chapter 6].

We record the basic definitions and results from the representation theory of $U_q(\mathfrak{g})$ that we need. Let $U = U_q(\mathfrak{g})$ and denote the category of U-modules by U-mod. A U-module V is a *weight module* if there is a *weight space decomposition*

$$V = \bigoplus_{\lambda \in X} V_\lambda, \quad \text{where } V_\lambda = \{v \in V \mid K_h v = q^{\langle \lambda, h\rangle} v, \ \text{ for all } h \in X^\vee\}.$$

A weight $\lambda \in X$ is called a *weight of* V if $V_\lambda \neq 0$, in which case V_λ is called a *weight space*. Denote by $\mathrm{wt}(V) \subseteq X$ the set of weights of V. If $\lambda \in \mathrm{wt}(V)$ then any nonzero $x \in V_\lambda$ is called a *weight vector*. Observe that, for a U-module V, the generators of U permute weight spaces:

$$E_i V_\lambda \subseteq V_{\lambda+\alpha_i}, \quad \text{and} \quad F_i V_\lambda \subseteq V_{\lambda-\alpha_i}. \tag{4.1.7}$$

A vector $v \in V$ is called a *highest weight vector of weight* λ (resp. *lowest weight vector of weight* λ) if there exists $\lambda \in \mathrm{wt}(V)$ such that $v \in V_\lambda$ and

$$U^+ v = 0, \quad \text{and} \quad V = Uv, \quad (\text{resp.} \quad U^- v = 0, \quad \text{and} \quad V = Uv).$$

By Proposition 4.1.5, weight modules V admitting a highest weight vector (resp. lowest weight vectors) are cyclic U^--modules (resp. cyclic U^+-modules). A weight module V is a *highest weight module with highest weight* λ (resp. *lowest weight module with lowest weight* λ) if there exists a highest weight vector $v \in V_\lambda$ (resp. if there exists a lowest weight vector $v \in V_\lambda$). If V is a highest/lowest weight module of highest/lowest λ then $\dim V_\lambda = 1$.

Let V be a weight module such that $\dim V_\lambda < \infty$, for all $\lambda \in X$, and such that there exists $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l \in X$ so that

$$\mathrm{wt}(V) \subseteq \{\leq \lambda_1\} \cup \cdots \cup \{\leq \lambda_k\} \cup \{\geq \mu_1\} \cup \cdots \cup \{\geq \mu_l\}.$$

Here $\{\leq \lambda\} := \{\nu \in X \mid \nu \leq \lambda\}$ and $\{\geq \mu\} := \{\nu \in X \mid \nu \geq \mu\}$. The *character of* V is

$$\mathrm{ch}\, V := \sum_{\lambda \in X} \dim V_\lambda e^\lambda \in \mathbb{Z}[[X]],$$

where $\mathbb{Z}[[X]]$ is the formal group ring of X.

Let $\mathcal{O}^q \subseteq U$-mod be the full subcategory of finitely-generated weight modules V with finite dimensional weight spaces. Define $\mathcal{O}^q_+ \subseteq \mathcal{O}^q$ to be the full subcategory of modules for which E_i, $i \in I$, acts locally nilpotently: for any $v \in V$, and any $i \in I$, there exists r such that $E_i^r v = 0$. Analogously, define $\mathcal{O}^q_- \subseteq \mathcal{O}^q$ to be the full subcategory of modules for which F_i, $i \in I$, acts locally nilpotently. Define $\mathcal{O}^q_{int} = \mathcal{O}^q_+ \cap \mathcal{O}^q_-$ to be the full subcategory consisting of those U-modules $V \in \mathcal{O}^q$ for which E_i, F_i, $i \in I$, act locally nilpotently. Objects in $\mathcal{O}^q_{int}$ are called *integrable U-modules.*

Remark 4.1.9. The category of weight modules considered above is often referred to as the category of Type **1** U-modules. See [72, Chapter 5] for further details.

Now we introduce some endofunctors on $\mathcal{O}^q$. They will restrict to give endofunctors on $\mathcal{O}^q_{int}$.

The automorphism ω defined in (4.1.2) induces an autoequivalence on $\mathcal{O}^q$, $V \mapsto {}^\omega V$: as a vector space ${}^\omega V = V$ and we define the *twisted* U-action $*$ on V

$$u * v := \omega(u)v, \quad u \in U, v \in V. \tag{4.1.8}$$

Then, $({}^\omega V)_\lambda = V_{-\lambda}$ and twisting induces an equivalence $\mathcal{O}^q_\pm \xrightarrow{\sim} \mathcal{O}^q_\mp$. In particular, we obtain an autoequivalence of $\mathcal{O}^q_{int}$.

For any $V \in \mathcal{O}^q$, with $V = \bigoplus_{\lambda \in X} V_\lambda$ and $\dim V_\lambda < \infty$, define the *graded dual of* V to be

$$V_* := \bigoplus_\lambda V_\lambda^*, \quad \text{where } V_\lambda^* = \mathrm{Hom}_{\mathbb{C}(q)}(V_\lambda, \mathbb{C}(q)). \tag{4.1.9}$$

We provide V_* with the structure of a U-module as follows: for $x \in U$, $f \in V_*$, define $xf \in V_*$ by

$$(xf)(u) = f(S(x)u), \quad u \in U. \tag{4.1.10}$$

Observe that, if $V \in \mathcal{O}^q$, $\lambda, \mu \in \mathrm{wt}(V)$, and $f \in V_\lambda^*$, $v \in V_\mu$, then

$$(K_h f)(v) = q^{-\langle \mu, h\rangle} f(v), \quad h \in X^\vee.$$

Hence, $(V_*)_\lambda = V^*_{-\lambda}$. Therefore, we have an endofunctor on $\mathcal{O}^q$, $V \mapsto V_*$. If $V \in \mathcal{O}^q_+$ (resp. $V \in \mathcal{O}^q_-$) then $V_* \in \mathcal{O}^q_-$ (resp. $V_* \in \mathcal{O}^q_+$): indeed, for $i \in I$, we have

$$F_i^r (V_*)_\lambda \subseteq (V_*)_{\lambda - r\alpha_i} = V^*_{-\lambda + r\alpha_i}$$

so that, if $V \in \mathcal{O}^q_+$ then $V_{-\lambda + r\alpha_i} \equiv 0$, whenever $-\lambda \in \mathrm{wt}(V)$ and r is sufficiently large. Hence, $V \mapsto V_*$ restricts to give an endofunctor on $\mathcal{O}^q_{int}$.

Proposition 4.1.10. *(a) The functor $V \mapsto V_*$ is exact, and,*

(b) $(\mathcal{O}^q_\pm)_* \subseteq \mathcal{O}^q_\mp$. In particular, $(\mathcal{O}^q_{int})_* \subseteq \mathcal{O}^q_{int}$.

We introduce some examples of U-modules that we use throughout this thesis. They are the quantum analogues of (co)Verma modules from the representation theory of $U(\mathfrak{g})$.

Example 4.1.11. (a) Let $\lambda \in X$. Define

$$\Delta(\lambda) := U / \left(\sum_{i \in I} U E_i + \sum_{h \in X^\vee} U(K_h - q^{\langle \lambda, h \rangle} 1) \right) \in \mathcal{O}^q_- \tag{4.1.11}$$

and

$$\nabla(\lambda) := U / \left(\sum_{i \in I} U F_i + \sum_{h \in X^\vee} U(K_h - q^{\langle \lambda, h \rangle} 1) \right) \in \mathcal{O}^q_+. \tag{4.1.12}$$

Denote the left ideal in (4.1.11) (resp. (4.1.12)) by J_λ^- (resp. J_λ^+). The triangular decomposition (Proposition 4.1.5) implies that $\Delta(\lambda)$ is a highest weight module having highest weight λ, with highest weight vector $1 + J_\lambda^-$, and that $\nabla(\lambda)$ is a lowest weight module having lowest weight λ, with lowest weight vector $1 + J_\lambda^+$.

As a cyclic U^--module, $\Delta(\lambda) \cong U^-$; as a cyclic U^+-module, $\nabla(\lambda) \cong U^+$. Moreover, ${}^\omega\Delta(\lambda) = \nabla(-\lambda)$ and ${}^\omega\nabla(\lambda) = \Delta(-\lambda)$.

(b) Let $a = (a_i)_{i \in I}, b = (b_i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$. Define the left ideal

$$J_{a,b,\lambda} := \sum_{i \in I} U E_i^{a_i+1} + \sum_{i \in I} U F_i^{b_i+1} + \sum_{h \in X^\vee} U(K_h - q^{\langle \lambda, h \rangle} 1) \subseteq U. \tag{4.1.13}$$

A straightforward but lengthy calculation shows that the quotient $U/J_{a,b,\lambda} \in \mathcal{O}^q_{int}$ (see [72, Lemma 5.7]).

Definition 4.1.12. (a) Let $\lambda \in X_+$ be a dominant weight and $b = (b_i) \in \mathbb{Z}^I_{\geq 0}$ be defined by $b_i = \langle \lambda, \alpha_i^\vee \rangle$. Define $V^q(\lambda) \in \mathcal{O}^q_{int}$ to be the U-module

$$V^q(\lambda) := U/J_{0,b,\lambda}.$$

(b) Let $\lambda \in X_-$ be an antidominant weight and $a = (a_i) \in \mathbb{Z}^I_{\geq 0}$ be defined by $a_i = -\langle \lambda, \alpha_i^\vee \rangle$. Define $V_q(\lambda) \in \mathcal{O}^q_{int}$ to be the U-module

$$V_q(\lambda) := U/J_{a,0,\lambda}.$$

The following result shows that the combinatorics of the representation theory of U is 'the same as' the corresponding representation theory of G. For details see [69, Chapter 3], [72, Chapter 5].

Proposition 4.1.13. *(a) The category $\mathcal{O}^q_{int}$ is semisimple and closed under taking tensor products and graded duals. The objects of $\mathcal{O}^q_{int}$ are precisely the finite-dimensional U-modules.*

(b) *(i)* *Let $\lambda \in X_+$ be dominant. Then, $V^q(\lambda)$ is a finite-dimensional irreducible highest weight module with highest weight λ.*

(ii) *Let $\lambda \in X_-$ be antidominant. Then, $V_q(\lambda)$ is a finite-dimensional irreducible lowest weight module with lowest weight λ.*

(c) *If $V \in \mathcal{O}^q_{int}$ is irreducible then V is a highest weight module.*

(d) *If $V \in \mathcal{O}^q_{int}$ is a highest weight module with highest weight $\lambda \in X$ then $\lambda \in X_+$ and $V \cong V^q(\lambda)$. Similarly, if $V \in \mathcal{O}^q_{int}$ is a lowest weight module having lowest weight $\lambda \in X$ then $\lambda \in X_-$ and $V \cong V_q(\lambda)$.*

(e) *Let $\lambda \in X_+$ be dominant. let $V(\lambda)$ be the finite-dimensional irreducible G-module with highest weight λ. Then, $\operatorname{ch} V(\lambda) = \operatorname{ch} V^q(\lambda)$.*

Remark 4.1.14. Using the twisted action (4.1.8), Proposition 4.1.13 implies that the irreducible highest weight module ${}^{\omega}V^q(\lambda)$ is a lowest weight module having lowest weight $-\lambda$, so that ${}^{\omega}V^q(\lambda) \cong V_q(-\lambda)$. Similarly, we have $V^q(\lambda)_* \cong V_q(-\lambda)$.

In fact, as in the classical setting, we have, for $\lambda \in X_+$,

$$V_q(-\lambda) \cong V^q(-w_0(\lambda)).$$

Remark 4.1.15. The subalgebras $U^{\geq 0}$ and $U^{\leq 0}$ are Hopf subalgebras and most of the above constructions can be defined by restricting the Hopf algebra structure from U.

The subalgebras $U^{\pm}$ are not Hopf subalgebras (they are not closed under Δ, for example); however, we can define a (twisted) Hopf algebra structure. We describe this Hopf algebra structure for U^+, the structure on U^- is obtained via ω. Define a (twisted) multiplication (on homogeneous elements)

$$\begin{array}{ccc} U^+ \otimes U^+ \times U^+ \otimes U^+ & \longrightarrow & U^+ \otimes U^+ \\ (a \otimes b, c \otimes d) & \longmapsto & q^{\langle \deg b, (\deg c)^\vee \rangle} ac \otimes bd. \end{array}$$

Define

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes 1, \quad S(E_i) = -E_i, \quad \varepsilon(E_i) = 0, \quad (i \in I) \tag{4.1.14}$$

Then, $(U^+, \Delta, S, \varepsilon)$ is a (twisted) Hopf algebra. For further discussion see [102, Chapter 1].

4.2 Bases and parameterisations

In this section we give several constructions of bases for the quantised universal enveloping algebra $U = U_q(\mathfrak{g})$. Using Proposition 4.1.5, it suffices to obtain a basis for either U^+ or U^-.

PBW-type bases

For $\mathfrak{g}$ of simply-laced, finite type, Lusztig introduces in [103] an action on U of the braid group covering W. In subsequent joint work with M. Dyer [104, Appendix], and again still for simply-laced, finite type $\mathfrak{g}$, Lusztig uses the braid group action to determine a collection of bases $\mathcal{B}_{\mathbf{i}} \subseteq U^+$ depending on a reduced expression $\mathbf{i}$ of the longest element $w_0 \in W$. This construction is a quantum analogue of the existence of the well-known Poincaré-Birkhoff-Witt (PBW) basis for $U(\mathfrak{g})$. The bases $\mathcal{B}_{\mathbf{i}} \subseteq U^+$ are known as PBW-type bases. Saito later extended this construction to $\mathfrak{g}$ of arbitrary finite type and obtained PBW-type bases of U^+ (see [123]).

We will briefly outline this construction. Following [104], define an algebra automorphism T_i of U, $i \in I$, defined by

$$T_i(E_i) = -F_i K_{d_i\alpha_i^\vee}, \quad T_i(F_i) = -K_{-d_i\alpha_i^\vee} E_i, \tag{4.2.1}$$

$$T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r q^{-d_i s} E_i^{(r)} E_j E_i^{(s)}, \quad T_i(F_j) = \sum_{r+s=-c_{ij}} (-1)^r q^{d_i s} F_i^{(s)} F_j F_i^{(r)} \quad j \in I, j \neq i,$$

$$T_i(K_h) = K_{s_i(h)}, \quad h \in X^\vee.$$

It can be seen that $T_i^{-1} = \iota \circ T_i \circ \iota$.

The following result is fundamental (see [72, Chapter 8]).

Proposition 4.2.1 (Lusztig [103], Saito [123])**.** *The collection $\{T_i \mid i \in I\}$ satisfy the braid relations associated to the Weyl group of G.*

Hence, there is an action of the braid group covering W on U. If $w = s_{i_1} \cdots s_{i_k} \in W$, with $\ell(w) = k$, then the automorphism $T_w := T_{i_1} \cdots T_{i_k}$ is well-defined. Observe that, for each $i \in I$,

$$T_i(U_\alpha) \subseteq U_{s_i(\alpha)}, \quad \alpha \in Q.$$

The following Lemma can be found in [72, Proposition 8.20]).

Lemma 4.2.2. *Let $w = s_{i_1} \cdots s_{i_k} \in W$, $\ell(w) = k$, such that*

$$s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_j \in S, \quad \textit{for some } j \in I.$$

Then, $T_{s_{i_1} \cdots s_{i_{k-1}}}(E_{i_k}) = E_j \in U^+$.

Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ be a reduced expression for the longest element $w_0 \in W$, and $t = (t_1, \ldots, t_m) \in \mathbb{Z}_{\geq 0}^m$. Define

$$p_{\mathbf{i}}(t) := E_{i_1}^{(t_1)} T_{i_1}\left(E_{i_2}^{(t_2)}\right) \cdots (T_{i_1} \cdots T_{i_{m-1}})\left(E_{i_m}^{(t_m)}\right). \tag{4.2.2}$$

Then,

$$\deg p_{\mathbf{i}}(t) = \sum_{j=1}^m t_j \beta_j, \tag{4.2.3}$$

where $\beta_j \coloneqq s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. Define

$$\mathcal{B}_{\mathbf{i}} \coloneqq \{p_{\mathbf{i}}(t) \mid t \in \mathbb{Z}^m_{\geq 0}\}. \tag{4.2.4}$$

Proposition 4.2.3 (Lusztig, [104, Appendix])**.** *For every reduced expression* $\mathbf{i} \in R(w_0)$ *the set* $\mathcal{B}_{\mathbf{i}}$ *is a (homogeneous)* $\mathbb{C}(q)$*-basis of* U^+*, called a* PBW-type basis of U^+*.*

Remark 4.2.4. Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. Then, $\mathbf{i}$ induces a total ordering on the set of positive roots determined by the simple roots $S \subseteq R$ as follows: define $\beta_1 = \alpha_{i_1}$ and, for each $j > 1$, define

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}).$$

Then, the sequence $\beta_1, \ldots, \beta_m \in R_+$ consists of distinct positive roots and provides an enumeration of R_+. We call the sequence $\beta_1, \ldots, \beta_m$ the *root sequence* associated to $\mathbf{i}$.

Recall the PBW theorem (see [71, Chapter 17]) for the universal enveloping algebra $U(\mathfrak{n}_+)$: for any ordered basis $x_1, \ldots, x_m$ of $\mathfrak{n}_+$, the set of monomials

$$\{x_1^{a_1} \cdots x_m^{a_m} \mid (a_1, \ldots, a_m) \in \mathbb{Z}^m_{\geq 0}\} \subseteq U(\mathfrak{n}_+)$$

is a $\mathbb{C}$-basis of $U(\mathfrak{n}_+)$. In particular, if $\mathbf{i} \in R(w_0)$ then the corresponding root sequence $\beta_1, \ldots, \beta_m$ and root vectors $x_i \coloneqq e_{\beta_i} \in \mathfrak{n}_+(\beta_i)$, $i = 1, \ldots, m$, induces a $\mathbb{C}$-basis $B_{\mathbf{i}}$ of $U(\mathfrak{n}_+)$. Here $e_\alpha \in \mathfrak{n}_+$ is a root vector of weight α.

The bases $\mathcal{B}_{\mathbf{i}}$ are q-analogues of the PBW-bases defined for $U(\mathfrak{n}_+)$. In fact, in the $q \to 1$ limit, $\mathcal{B}_{\mathbf{i}}$ specialises to $B_{\mathbf{i}}$ (see [104], Appendix).

Remark 4.2.5. Recall from (4.1.2) the isomorphism of algebras $\omega : U^+ \xrightarrow{\sim} U^-$. Using this isomorphism, a PBW-type basis $\mathcal{B}_{\mathbf{i}}$ gives rise to a basis of U^-. Using (4.2.1) we have

$$T_i(\omega(E_j)) = (-q^{-d_i})^{\langle \alpha_j, \alpha_i^\vee \rangle} \omega(T_i(E_j)), \quad i, j \in I.$$

Hence, if $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ the set

$$\left\{ F_{i_1}^{(t_1)} T_{i_1} \left(F_{i_2}^{(t_2)} \right) \cdots (T_{i_1} \cdots T_{i_{m-1}}) \left(F_{i_m}^{(t_m)} \right) \mid (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0} \right\} \subseteq U^- \tag{4.2.5}$$

is a basis of U^-, called a *PBW-type basis of* U^-.

Recall the antiautomorphism ι of U given in (4.1.3) and the involution $i \mapsto i^*$ from Section 1.3. We will see that ι induces a permutation on the set $\{\mathcal{B}_{\mathbf{i}} \mid \mathbf{i} \in R(w_0)\}$ of PBW-type bases. First, we require some notation.

Definition 4.2.6. Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be a sequence. Define

$$\mathbf{i}^* \coloneqq (i_1^*, \ldots, i_r^*), \quad \mathbf{i}^{op} = (i_r, \ldots, i_1). \tag{4.2.6}$$

The operations $\mathbf{i} \mapsto \mathbf{i}^*$ and $\mathbf{i} \mapsto \mathbf{i}^{op}$ commute with each other,

$$(\mathbf{i}^{op})^* = (\mathbf{i}^*)^{op}, \quad \text{for } \mathbf{i} \in I^r.$$

If $\mathbf{i} \in R(w_0)$ then $\mathbf{i} \mapsto \mathbf{i}^*$, $\mathbf{i} \mapsto \mathbf{i}^{op}$, are (commuting) permutations of $R(w_0)$.

Proposition 4.2.7. *Let $\mathbf{i} \in R(w_0)$ and let $\mathbf{i}' = (\mathbf{i}^{op})^*$. Then, $\iota(p_{\mathbf{i}}(t)) = p_{\mathbf{i}'}(t^{op})$, for any $t = (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0}$, and where $t^{op} = (t_m, \ldots, t_1)$.*

Proof. Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ and denote $\mathbf{i}' = (\mathbf{i}^{op})^* = (i'_1, \ldots, i'_m) \in R(w_0)$. For any $k \in \{1, \ldots, m\}$, we have

$$s_{i_{k-1}} \cdots s_{i_1} s_{i'_1} \cdots s_{i'_{m-k+1}} = w_0,$$

and

$$s_{i_{k-1}} \cdots s_{i_1} s_{i'_1} \cdots s_{i'_{m-k}} (\alpha_{i'_{m-k+1}}) = \alpha_{i_k}.$$

By Lemma 4.2.2, we obtain

$$T_{w_0 s_{i'_{m-k+1}}} (E_{i_{m-k+1}}) = E_{i_k}.$$

Hence,

$$T_{i'_1} \cdots T_{i'_{m-k}} (E_{i'_{m-k+1}}) = T^{-1}_{i_1} \cdots T^{-1}_{i_{k-1}} (E_{i_k}) = \iota(T_{i_1} \cdots T_{i_{k-1}} (E_k)).$$

Applying the antiautomorphism ι to $p_{\mathbf{i}}(t) \in \mathcal{B}_i$, $t = (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0}$, we obtain

$$\begin{aligned}
\iota(p_{\mathbf{i}}(t)) &= \iota\left(E^{(t_1)}_{i_1} T_{i_1}\left(E^{(t_2)}_{i_2}\right) \cdots (T_{i_1} \cdots T_{i_{m-1}})\left(E^{(t_m)}_{i_m}\right)\right) \\
&= \iota\left(T_{i_1} \cdots T_{i_{m-1}}\left(E^{(t_m)}_{i_m}\right)\right) \cdots \iota\left(T_{i_1}\left(E^{(t_2)}_{i_2}\right)\right) \iota(E^{(t_1)}_{i_1}) \\
&= E^{(t_m)}_{i'_1} T_{i'_1}\left(E^{(t_{m-1})}_{i'_2}\right) \cdots (T_{i'_1} \cdots T_{i'_{m-1}})\left(E^{(t_1)}_{i'_m}\right) \\
&= p_{\mathbf{i}'}(t^{op}) \in \mathcal{B}_{\mathbf{i}'}.
\end{aligned}$$

□

An immediate consequence is the following:

Corollary 4.2.8. *Let $\mathbf{i} \in R(w_0)$ and let $\mathbf{i}' = (\mathbf{i}^{op})^*$. Then, $\iota(\mathcal{B}_{\mathbf{i}}) = \mathcal{B}_{\mathbf{i}'}$.*

Remark 4.2.9. There is an analogous result for the PBW-type bases of U^- (see Remark 4.2.5): the antiautomorphism ι is an involutive permutation on the set of PBW-type bases of U^-.

Canonical bases and Lusztig parameterisation

In [100] Lusztig introduces a simultaneous modification of the PBW-type bases, called the *canonical basis*. Each of the PBW-type bases is related to the canonical basis by a unitriangular change of basis (with respect to some order). Originally the construction of the canonical basis was restricted to simply-laced, finite type $\mathfrak{g}$ as it relied on results of Ringel relating ADE quantised enveloping algebras U^+ with the Hall algebra of the type ADE quiver. Moreover, Lusztig's construction made essential use of deep results in algebraic geometry and topology coming from the theory of perverse sheaves and intersection cohomology. A favourable feature of Lusztig's canonical basis is that it provides a construction of a *canonical basis* for each finite dimensional irreducible U-module admitting remarkable consequences (for example, Theorem 4.2.15).

Independently and simultaneously, Kashiwara provided an elementary algebraic construction of a *global basis* of U^- admitting similar consequences for the representation theory of U. Kashiwara's construction had the advantage of working for an arbitrary symmetrisable Kac-Moody algebra and the only 'geometry' required was a basic result on the triviality of vector bundles on $\mathbb{P}^1$ (which is essentially an algebraic problem). We will discuss Kashiwara's result further in Section 4.3.

Lusztig [105] extended his construction of the canonical basis of U^+ to the (positive part of) quantised universal enveloping algebras associated to an arbitrary symmetric Kac-Moody algebras, and outlined a construction for the non-symmetric setting. Again, his construction relied on deep results from the theory of perverse sheaves and intersection cohomology. Complete details of Lusztig's topological construction of the canonical basis can be found in [102, Part II].

In this section we recall the essential features of Lusztig's results and indicate some of the consequences for the determination of tensor product multiplicities in representation theory.

Theorem 4.2.10 (Lusztig, [100])**.** *Let $U = U_q(\mathfrak{g})$.*

(a) The $\mathbb{Z}[q^{-1}]$-submodule $\mathcal{L} = \mathrm{span}_{\mathbb{Z}[q^{-1}]}\mathcal{B}_{\mathbf{i}} \subseteq U^+$ is independent of $\mathbf{i}$.

(b) The $\mathbb{Z}$-basis $B = \mathcal{B}_{\mathbf{i}} + \mathcal{L} \subseteq \mathcal{L}/q^{-1}\mathcal{L}$ is independent of $\mathbf{i}$.

(c) The projection $\mathcal{L} \to \mathcal{L}/q^{-1}\mathcal{L}$ induces a Q-graded isomorphism $f : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/q^{-1}\mathcal{L}$ of $\mathbb{Z}$-modules. The $\mathbb{Z}$-basis $\mathcal{B} := f^{-1}(B)$ is a $\mathbb{Z}[q^{-1}]$-basis of $\mathcal{L}$ and consists of bar-invariant, homogeneous elements.

Corollary 4.2.11. *$\iota(\mathcal{B}) = \mathcal{B}$.*

Proof. By Corollary 4.2.8 and Theorem 4.2.10(a), the antiautomorphism ι induces an action on $\mathcal{L}/q^{-1}\mathcal{L}$ and a permutation of $B \subseteq \mathcal{L}/q^{-1}\mathcal{L}$. Moreover, ι commutes with $^- : U \to U$ and therefore preserves $\mathcal{L} \cap \overline{\mathcal{L}}$. Also, ι commutes with the natural projection $\pi : \mathcal{L} \to \mathcal{L}/q^{-1}\mathcal{L}$, so that $f \circ \iota = \iota \circ f$. The result follows. □

The $\mathbb{C}(q)$-basis $\mathcal{B} \subseteq U^+$ is Lusztig's *canonical basis*. By Theorem 4.2.10, $\mathcal{B}$ is the unique homogeneous basis of U^+ such that

(i) for every $b \in \mathcal{B}$, $b = \overline{b}$,

(ii) for every $\mathbf{i} \in R(w_0)$, $t \in \mathbb{Z}^m_{\geq 0}$, there is a unique $b = b_{\mathbf{i}}(t) \in \mathcal{B}$ such that $b - p_{\mathbf{i}}(t)$ is a linear combination of elements in $\mathcal{B}_{\mathbf{i}}$ with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$.

Define $\mathcal{B}_\alpha \coloneqq \mathcal{B} \cap U_\alpha$, $\alpha \in Q$. Observe that $\mathcal{B}_0 = \{1\}$.

Definition 4.2.12. Let $\mathbf{i} \in R(w_0)$. The map

$$\begin{array}{rccc} b_{\mathbf{i}}: & \mathbb{Z}^m_{\geq 0} & \longrightarrow & \mathcal{B} \\ & t & \longmapsto & b_{\mathbf{i}}(t) \end{array}$$

is called the *Lusztig parameterisation of* $\mathcal{B}$ *(in the direction* $\mathbf{i}$*)*. If $b = b_{\mathbf{i}}(t)$ then we call t the *Lusztig data of* b *(in the direction* $\mathbf{i}$*)*.

Remark 4.2.13. Using the isomorphism $\omega : U^+ \xrightarrow{\sim} U^-$, the image of $\mathcal{B}$ in U^- can be shown to be a basis possessing analogous properties as those described in Theorem 4.2.10.

The canonical basis $\mathcal{B} \subseteq U_q^+(\mathfrak{g})$ admits remarkable consequences for the representation theory of $U_q(\mathfrak{g})$ and, by Proposition 4.1.13, for the representation theory of G itself. The following technical result is essential.

Lemma 4.2.14. *Let* $i \in I$, $r \geq 0$.

(a) $\mathcal{B} \cap U^+E_i^r$ *spans* $U^+E_i^r$.

(b) $\mathcal{B} \cap E_i^rU^+$ *spans* $E_i^rU^+$.

Proof. (a) Let $b \in \mathcal{B}_\nu$, where $\nu = \sum_{i\in I} \nu_i\omega_i$, and fix $i \in I$. Define $s_i(b) \in \mathbb{Z}_{\geq 0}$ to be the largest integer r such that $0 \leq r \leq \nu_i$ and satisfying the condition:

(A) there exists $z' \in U^+$ such that b appears with nonzero coefficient in $z'E_i^r$

Observe that $b \mapsto s_i(b)$ is well-defined: (A) is always satisfied when $r = 0$. Using [105, Section 11.6], we have $b \in U^+E_i^{s_i(b)}$.

Let $z \in U^+E_i^r$. By Theorem 4.2.10, we can write $z = \sum_{b\in\mathcal{B}} a_b b$, $a_b \in \mathbb{C}(q)$. Suppose $a_b \neq 0$. Then, degree considerations give $r \leq \nu_i$ and $r \leq s_i(b)$ by definition of $s_i(b)$. Hence, $b \in U^+E_i^{s_i(b)} \subseteq U^+E_i^r$.

(b) Applying the antiautomorphism ι to U^+, the result follows from (a) and Corollary 4.2.11.

□

Theorem 4.2.15. *Let $\lambda \in X_-$ be an antidominant weight, $V_q(\lambda)$ the corresponding irreducible lowest weight module (see Definition 4.1.12). Then, if $V_q(\lambda) \cong U^+/J_\lambda$ as a cyclic U^+-module then $\mathcal{B} \cap J_\lambda$ spans J_λ. Equivalently, $\{b + J_\lambda \mid b \notin J_\lambda\}$ spans $V_q(\lambda)$.*

Proof. Let $\lambda = -\sum_{i\in I} c_i\omega_i$, with $c_i \in \mathbb{Z}_{\geq 0}$. As a U^+-module we have $V_q(\lambda) \cong U^+/\sum_{i\in I} U^+E_i^{c_i+1}$ and it suffices to show that $\mathcal{B} \cap U^+E_i^n$ spans $U^+E_i^n$, for every $i \in I$ and every $n \geq 0$. This follws from Lemma 4.2.14(a).

□

Definition 4.2.16. Let $\lambda \in X_+$, $v_\lambda \in V_q(w_0(\lambda))$ be a lowest weight vector. Define

$$\mathcal{B}(\lambda) := \{b \in \mathcal{B} \mid bv_\lambda \neq 0\}.$$

Therefore,

$$\mathcal{B}(\lambda) = \{b \in \mathcal{B} \mid b \notin U^+E_i^{-\langle w_0(\lambda),\alpha_i^\vee\rangle+1},\ i \in I\}.$$

Proposition 4.2.17. *Let $\mathbf{i} \in R(w_0)$. Then, for any $\lambda \in X_+$, $\mathcal{B}(\lambda)$ is parameterised by a subset of Lusztig data via the Lusztig parameterisation of $\mathcal{B}$ in the direction $\mathbf{i}$.*

In the remainder of this section we illustrate an application of the canonical basis to determining combinatorial tensor product multiplicity formulae, following Lusztig [101]. First we recall the basic problem.

For $\lambda \in X_+$, we write $V(\lambda)$ for the irreducible $\mathfrak{g}$-module, and let $\mathrm{wt}(\lambda) := \mathrm{wt}\, V(\lambda)$ be the set of weights of $V(\lambda)$.

Let $\lambda, \mu, \nu \in X_+$. Then, the tensor product $V(\lambda) \otimes V(\mu)$ decomposes as a direct sum of irreducibles

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu\in X_+} V(\nu)^{c^\nu_{\lambda,\mu}}$$

We would like to determine manifestly positive combinatorial models that compute the nonnegative integers $c^\nu_{\lambda,\mu}$. Such combinatorial models are called *Littlewood-Richardson rules* in reference to the type A model involving skew-tableaux.

Gelfand-Zelevinsky proposed in [44], [45], an approach to determining the tensor product multiplicities $c^\nu_{\lambda,\mu}$ by counting lattice points in polytopes (see also [5]). Their argument relied on the notion of a *good basis* in $V(\lambda)$ that we will now describe.

Definition 4.2.18. Let $\lambda, \gamma \in X_+$, $\beta \in \mathrm{wt}(\lambda)$. Define the *$\gamma$-primitive β-weight vectors in* $V(\lambda)$ to be the nonzero elements in the following subspace

$$V(\lambda; \beta, \gamma) := \{v \in V(\lambda)_\beta \mid e_i^{\langle\gamma,\alpha_i^\vee\rangle+1}(v) = 0,\ i \in I\}.$$

The relation between $V(\lambda; \beta, \gamma)$ and $c^\nu_{\lambda,\mu}$ is given by the following result due to Kostant [92, Lemma 4.1].

Proposition 4.2.19. *Let $\lambda, \mu, \nu \in X_+$, Then,*

$$c^\nu_{\lambda,\mu} = \dim V(\lambda; \nu - \mu, \mu).$$

Proof. Let $\lambda, \mu, \nu \in X_+$. Then,

$$\begin{aligned} c^{\nu}_{\lambda,\mu} &= \dim \mathrm{Hom}_{\mathfrak{g}}(V(\nu), V(\lambda) \otimes V(\mu)) \\ &= \dim \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu), V(\lambda) \otimes V(\mu)) \\ &= \dim \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu) \otimes V(\mu)^*, V(\lambda)) \end{aligned}$$

Any $f \in \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu) \otimes V(\mu)^*, V(\lambda))$ is determined by $f(1_\nu \otimes v_{-\mu})$, where $v_{-\mu} \in V(\mu)^*$ is a lowest weight vector. Then, we must have

$$f(1_\nu \otimes v_{-\mu}) \in V(\lambda)_{\nu-\mu}.$$

Moreover, we must have $e_i^{\langle \mu, \alpha_i^\vee \rangle + 1}(v_{-\mu}) = 0$, for each $i \in I$. Therefore, there is an isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}(\nu) \otimes V(\mu)^*, V(\lambda)) &\longrightarrow V(\lambda; \nu - \mu, \mu) \\ f &\longmapsto f(1_\nu \otimes v_{-\mu}). \end{aligned}$$

The result follows. ∎

A *good basis for* $V(\lambda)$, $\lambda \in X_+$, is a weight basis $B \subseteq V(\lambda)$ such that $B \cap V(\lambda; \beta, \gamma)$ spans, for all $\beta \in \mathrm{wt}(\lambda)$, $\gamma \in X_+$. In particular, by Proposition 4.2.19, a subset of a good basis of $V(\lambda)$ computes $c^{\nu}_{\lambda,\mu}$.

Mathieu showed proved the existence of good bases in [110] using Frobenius splitting methods (in particular, his proof is restricted to finite type). Lusztig provided a proof using the canonical basis; his approach extends to arbitrary symmetrisable type (upon equating Lusztig's canonical basis with Kashiwara's global basis).

Proposition 4.2.20 (Mathieu [110], Lusztig [101, Section 4]). *Let $\lambda \in X_+$. Then, there exists a good basis of $V(\lambda)$.*

Proof. Let $\lambda \in X_+$ and consider the $U_q(\mathfrak{g})$-module $V^q(\lambda)$. There is an analogous definition of the space of ν-primitive μ-weight vectors in $V^q(\lambda)$. For $\beta, \gamma \in X_+$, define

$$I_\beta := \sum_{i \in I} U^+ E_i^{\langle \beta, \alpha_i^\vee \rangle + 1}, \quad \text{and} \quad J_\gamma := \sum_{i \in I} E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1} U_+.$$

By Lemma 4.2.14, $\mathcal{B} \cap I_\beta$ spans I_β, $\mathcal{B} \cap J_\gamma$ spans J_γ, and $\mathcal{B} \cap I_\beta \cap J_\gamma$ spans $I_\beta \cap J_\gamma$, for all $\beta, \gamma \in X_+$.

Recall from Remark 4.1.14 that $V^q(\lambda) = V_q(w_0(\lambda)) \cong U^+/I_{-w_0(\lambda)}$. Hence, for any $\gamma \in X_+$ there is an isomorphism (of vector spaces)

$$J_\gamma/(I_{-w_0(\lambda)} \cap J_\gamma) \cong J_\gamma V^q(\lambda)$$

and $(\mathcal{B} \cap J_\gamma) \setminus (\mathcal{B} \cap I_{-w_0(\lambda)} \cap J_\gamma)$ maps to a basis of $J_\gamma V^q(\lambda) = \sum_{i \in I} E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1} V^q(\lambda)$. We have just shown that $\mathcal{B}(\lambda) \cap J_\gamma$ maps to a weight basis $B_{\lambda,\gamma}$ of $\sum_{i \in I} E_i^{\langle \gamma, \alpha_i^\vee \rangle + 1} V^q(\lambda)$, for any $\gamma \in X_+$.

If we consider the dual space $V^q(\lambda)^*$ as a U-module then the annihilator of $J_\gamma V^q(\lambda)$ in $V^q(\lambda)^*$ is seen to be the subspace

$$(J_\gamma V^q(\lambda))^\perp = \{\xi \in V^q(\lambda)^* \mid E_i^{\langle\gamma,\alpha_i^\vee\rangle+1}(\xi) = 0,\ i \in I\}.$$

Hence, $(J_\gamma V^q(\lambda))^\perp$ is spanned by part of the dual basis $B^*_{\lambda,\gamma}$. Hence, at the specialisation $q = 1$ we obtain a good basis for $V(\lambda)^* \cong V(-w_0(\lambda))$ and the result follows. □

Let $\mathbf{i} \in R(w_0)$, with corresponding root sequence $\beta_1, \dots, \beta_{\ell(w_0)}$, and $b_\mathbf{i}$ the corresponding Lusztig parameterisation in direction $\mathbf{i}$. Let $b \in \mathcal{B}$ be such that $b = b_\mathbf{i}(t)$. Then, (recall (4.2.3))

$$\deg b = \deg b_\mathbf{i}(t) = \sum_{j=1}^{\ell(w_0)} t_j\beta_j.$$

As $V(\lambda; \nu-\mu, \mu) \subseteq V(\lambda)_{\nu-\mu}$, Proposition 4.2.19 implies that the tensor product multiplicity $c^\nu_{\lambda,\mu}$ is equal to the cardinality of a subset of lattice points in the following polytope sitting inside the *Kostant partition space* $\mathbb{R}^{R^+}$:

$$\{t \in \mathbb{R}^{R^+}_{\geq 0} \mid \lambda - \nu + \mu = \sum_{j=1}^{\ell(w_0)} t_j\beta_j\} \subseteq \mathbb{R}^{R^+}. \tag{4.2.7}$$

Remark 4.2.21. Berenstein-Zelevinsky show that the subset whose lattice points count $c^\nu_{\lambda,\mu}$ is a polytope embedded in the space (4.2.7) and explicitly determine a set of defining inequalities [14, Theorem 2.3]. A remarkable feature of their description is that the inequalities they obtain are determined by the representation theory of the Langlands dual group ${}^LG$.

Canonical bases and string parameterisations

There is another, quite different, parameterisation of the canonical basis $\mathcal{B}$ called the *string parameterisation*. The string parameterisation was introduced by Kashiwara in his work on Littelmann's generalised Demazure character formulae [80]. Similar parameterisations were considered by Littelmann [98], and Berenstein-Zelevinsky [14]. In this section we introduce a string parameterisation that is different, but equivalent to, the string parameterisation defined in [14, Section 3]. We relate our string parameterisation to Berenstein-Zelevinsky's in Remark 4.2.25.

Let V be a U^--module satisfying the following property: for any nonzero $v \in V$, $i \in I$, $F_i^r v = 0$, for sufficiently large r. We will call a U^--module with this property *locally nilpotent*. For locally nilpotent U^--module V, the function

$$\begin{array}{rccl} c_i : & V \setminus \{0\} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ & v & \longmapsto & \max\{r \in \mathbb{Z}_{\geq 0} \mid F_i^r v \neq 0\}. \end{array} \tag{4.2.8}$$

is well-defined.

Definition 4.2.22. Let V be a locally nilpotent U^--module, $v \in V$ nonzero. Given any sequence $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ define the *string of* v *in the direction* $\mathbf{i}$ to be

$$c_{\mathbf{i}}(v) = (t_1, \ldots, t_r) \in \mathbb{Z}^r_{\geq 0},$$

where we define recursively

$$t_1 = c_{i_1}(v),\ t_2 = c_{i_2}\left(F_{i_1}^{t_1}(v)\right), \ldots,\ t_r = c_{i_r}\left(F_{i_{r-1}}^{t_{r-1}} \cdots F_{i_1}^{t_1}(v)\right).$$

The *string map of* V *in the direction* $\mathbf{i}$ is the function

$$\begin{array}{rccc} c_{\mathbf{i}}: & V \setminus \{0\} & \longrightarrow & \mathbb{Z}^r_{\geq 0} \\ & v & \longmapsto & c_{\mathbf{i}}(v). \end{array}$$

We define string maps of U^+ in the direction $\mathbf{i} \in R(w_0)$ that give rise to a family of parameterisations of the canonical basis $\mathcal{B}$. These *string parameterisations* are different to the Lusztig parameterisations in Definition 4.2.12. First, we need to specify a locally nilpotent action of U^- on U^+.

In [72, Chapter 6], there is described a graded perfect pairing

$$(,): \quad U^- \times U^+ \longrightarrow \mathbb{C}(q)$$

satisfying

$$(\omega(x), \omega(y)) = (y, x) = (\iota(y), \iota(x)), \quad y \in U^-, x \in U^+,$$

and such that, if $x \in U^+(\alpha)$, $y \in U^-(\beta)$, with $\alpha + \beta \neq 0$, then $(y, x) = 0$. Composing with ω we obtain a nondegenerate symmetric bilinear form on U^+

$$\begin{array}{rccc} (,)': & U^+ \times U^+ & \longrightarrow & \mathbb{C}(q) \\ & (u, v) & \longmapsto & (\omega(u), v). \end{array} \tag{4.2.9}$$

For $i \in I$, let L_i be the linear operator on U^+ adjoint to left multiplication by E_i: L_i is uniquely specified by the condition that

$$(L_i(y), x)' = (y, E_i x)', \quad y, x \in U^+. \tag{4.2.10}$$

Then, for all $\mu \in Q_+$,

$$L_i(U^+_\mu) \subseteq U^+_{\mu - \alpha_i}. \tag{4.2.11}$$

Since U^- and U^+ are isomorphic algebras we obtain an action of $U^{-,op}$, the opposite algebra of U^-, on U^+, uniquely determined by

$$F_i \bullet y := L_i(y), \quad i \in I, y \in U^+.$$

Twisting by ι we obtain an action of U^- on U^+

$$x \cdot y := \iota(x) \bullet y, \quad x \in U^-, y \in U^+.$$

Specifically, for $F \in U^-$, $u \in U^+$, $F \bullet u \in U^+$ is the unique element such that

$$(F \bullet u, v)' = (u, \omega(\iota(F))v)', \quad v \in U^+.$$

By (4.2.11), U^+ is a locally nilpotent U^--module. Hence, for any sequence $\mathbf{i} \in I^r$ we can consider the string map $c_{\mathbf{i}}$ associated to U^+ with respect to this locally nilpotent U^--structure.

Theorem 4.2.23 (Littlemann [98], Berenstein-Zelevinsky [14, Proposition 3.5])**.** *Let* $\mathbf{i} \in R(w_0)$. *Then, the string map* $c_{\mathbf{i}}$ *associated to* U^+ *defines a bijection from* $\mathcal{B}$ *onto the set of all lattice points* $C_{\mathbf{i}}(\mathbb{Z})$ *of some rational polyhedral cone* $C_{\mathbf{i}} \subseteq \mathbb{R}^{\ell(w_0)}$.

Definition 4.2.24. Let $\mathbf{i} \in R(w_0)$. We define the *string parameterisation of* $\mathcal{B}$ *in the direction* $\mathbf{i}$ to be the function

$$\begin{aligned} c_{\mathbf{i}} : \quad \mathcal{B} &\longrightarrow C_{\mathbf{i}}(\mathbb{Z}) \subseteq \mathbb{Z}_{\geq 0}^{\ell(w_0)} \\ b &\longmapsto c_{\mathbf{i}}(b). \end{aligned} \tag{4.2.12}$$

The cone $C_{\mathbf{i}} \subseteq \mathbb{R}_{\geq 0}^{\ell(w_0)}$ spanned by the image of $c_{\mathbf{i}}$ is called the *string cone in the direction* $\mathbf{i}$.

Remark 4.2.25. In [98] and [14], the authors define string maps for locally nilpotent U^+-modules and obtain string parameterisations for the *dual canonical basis* $\mathcal{B}^*$. We briefly describe the construction of string maps from [14] and explain why it is equivalent to the definition given above.

Certainly, for any locally nilpotent U^+-module V and $\mathbf{i} \in I^r$, there is an analogous notion of string maps $c_{\mathbf{i}}$ (replace F_i by E_i in the construction). Let U_*^+ be the graded dual of U^+ (recall (4.1.9)),

$$U_*^+ := \bigoplus_{\alpha \geq 0} U_\alpha^*, \qquad \text{where } U_\alpha^* = \mathrm{Hom}_{\mathbb{C}(q)}(U_\alpha, \mathbb{C}(q)).$$

By Proposition 4.2.3 and Remark 4.2.4, for any $\alpha > 0$, we have $\dim U_\alpha = \mathcal{P}(\alpha) < \infty$, where $\mathcal{P}$ is Kostant's partition function ([71, Section 24]), so that $\dim U_\alpha^+ < \infty$, for all α. Recall that the grading on U^+ is given by the U^0-action $u \mapsto K_h u K_{-h}$, $u \in U^+$, $h \in X^\vee$. Therefore, U_*^+ is a Q_--graded vector space and $U_{*,\alpha}^+ = U_{-\alpha}^*$.

The *dual canonical basis* $\mathcal{B}^*$ is the basis of U_*^+ dual to $\mathcal{B} \subseteq U^+$: for $b \in \mathcal{B}$, we define $b^* \in \mathcal{B}^*$ by

$$b^*(b') = \delta_{b,b'}, \quad b' \in \mathcal{B}.$$

There is an action of U^+ on U_*^+: for $E \in U^+$, $f \in U_*^+$, we have $E \cdot f \in U_*^+$ determined by

$$(E \cdot f)(u) = f(\iota(E)u), \quad u \in U^+. \tag{4.2.13}$$

With this definition U^+ acts locally nilpotently on U_*^+. Hence, for any sequence $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$, we can consider the string map $c_{\mathbf{i}}$ associated to U_*^+. It is this string map that appears in [14]: the authors show that $c_{\mathbf{i}}$ is a bijection between $\mathcal{B}^*$ and $C_{\mathbf{i}}(\mathbb{Z})$.

Twisting the $U^{\geq 0}$-module U_*^+ by ω we obtain a Q_+-graded $U^{\leq 0}$-module. The form (4.2.9) identifies the $U^{\leq 0}$-modules $U^+ \cong {}^{\omega}U_*^+$. In particular, the string cones defined via either construction are equal.

In [14], Berenstein-Zelevinsky gave an explicit description of $C_{\mathbf{i}}$ by describing a set of defining inequalities. A remarkable feature of this description is that the defining inequalities are defined in terms of the representation theory of the Langlands dual ${}^L\mathfrak{g}$. We recall their result.

Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. An $\mathbf{i}$-trail from γ to δ in $V(\omega_i^\vee)$, where $V(\omega_i^\vee)$ is the irreducible representation of ${}^LG$ having highest weight $\omega_i^\vee$, is a sequence of weights $\pi = (\gamma = \gamma_0, \gamma_1 \ldots, \gamma_m = \delta)$, $\gamma_i \in \mathrm{wt}(V(\omega_i^\vee)) \subseteq X^\vee$, such that

(i) for $k = 1, \ldots, m$ we have $\gamma_{k-1} - \gamma_k = c_k \alpha_{i_k}^\vee$, for some $c_k \in \mathbb{Z}_{\geq 0}$, and

(ii) $e_{i_1}^{c_1} \cdots e_{i_m}^{c_m}$ is a nonzero linear map from $V(\omega_i^\vee)_\delta$ to $V(\omega_i^\vee)_\gamma$.

Given an $\mathbf{i}$-trail $\pi = (\gamma_0, \ldots, \gamma_m)$ in $V(\omega_i^\vee)$, define (recall that $\gamma_i \in X^\vee$)

$$d_k^{(i)}(\pi) := \frac{1}{2} \langle \alpha_{i_k}, \gamma_{k-1} + \gamma_k \rangle, \quad k = 1, \ldots, m. \tag{4.2.14}$$

Condition (i) implies that $d_k(\pi) \in \mathbb{Z}$, for all k.

Theorem 4.2.26 ([14, Theorem 3.10])**.** *Let $\mathbf{i} \in R(w_0)$, $m = \ell(w_0)$. Then, the string cone $C_{\mathbf{i}}$ is the cone in $\mathbb{R}^m$ consisting of all $(t_1, \ldots, t_m)$ such that $\sum_k d_k^{(i)}(\pi) t_k \geq 0$, for any $i \in I$ and any $\mathbf{i}$-trail from $\omega_i^\vee$ to $w_0 s_i \omega_i^\vee$ in $V(\omega_i^\vee)$.*

We describe the relationship between string cones and the representation theory of U. Let $\lambda \in X_-$ be an antidominant weight. There is an exact sequence of U-modules

$$0 \longrightarrow I_\lambda \longrightarrow \nabla(\lambda) \longrightarrow V_q(\lambda) \longrightarrow 0 \tag{4.2.15}$$

where $I_\lambda = \sum_{i \in I} U(E_i^{-\langle \lambda, \alpha_i^\vee \rangle + 1} + J_\lambda^+) \subseteq \Delta(\lambda)$ and $V_q(\lambda)$ is the finite-dimensional irreducible U-module having lowest weight λ (see (4.1.11) and Definition 4.1.12).

In Theorem 4.2.15 we saw that the canonical basis $\mathcal{B} \subseteq U^+$ gives rise to a canonical basis of $V_q(\lambda) = V^q(w_0(\lambda))$. Recall the corresponding subset $\mathcal{B}(w_0(\lambda)) \subseteq \mathcal{B}$ (Definition 4.2.16). Therefore, we have the following result.

Proposition 4.2.27. *Let $\mathbf{i} \in R(w_0)$. Then, for any $\lambda \in X_+$, $\mathcal{B}(\lambda)$ is parameterised by a subset of $C_{\mathbf{i}}(\mathbb{Z})$ via the string map in the direction $\mathbf{i}$.*

Definition 4.2.28. Let $\mathbf{i} \in R(w_0)$. Define the *extended string cone* $\underline{C}_{\mathbf{i}} \subseteq \mathbb{R}X \times \mathbb{R}^{\ell(w_0)}$ to be the $\mathbb{R}_{\geq 0}$-span of

$$\underline{C}_{\mathbf{i}}(\mathbb{Z}) \coloneqq \{(\lambda, t) \in X \times \mathbb{Z}^{\ell(w_0)} \mid t = c_{\mathbf{i}}(b), \text{ for some } b \in \mathcal{B}(\lambda)\}$$

Observe that, for any $\lambda \in X_+$, $1 \in \mathcal{B}(\lambda) \subseteq U^+$: by construction, 1 corresponds to a lowest weight vector in $V^q(\lambda)$. Hence, for any $\lambda, \lambda' \in X_+$, $\mathcal{B}(\lambda) \cap \mathcal{B}(\lambda') \neq \varnothing$. The extended string cone 'separates' the subsets $\mathcal{B}(\lambda)$: consider the projection onto the first factor

$$\begin{array}{rccc} p: & \underline{C}_{\mathbf{i}} & \longrightarrow & \mathbb{R}X \\ & (\lambda, t) & \longmapsto & \lambda \end{array} \tag{4.2.16}$$

Then, for any $\lambda \in X_+$, we can identify $p^{-1}(\lambda) \cap \underline{C}_{\mathbf{i}}(\mathbb{Z}) = \mathcal{B}(\lambda)$, and we obtain an identification

$$\underline{C}_{\mathbf{i}}(\mathbb{Z}) = \bigsqcup_{\lambda \in X_+} \mathcal{B}(\lambda). \tag{4.2.17}$$

We make the following definitions.

Definition 4.2.29. Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. Define the *highest weight map*

$$\begin{array}{rccc} \mathrm{hw}: & \underline{C}_{\mathbf{i}} & \longrightarrow & \mathbb{R}X \\ & (\lambda, t) & \longmapsto & \lambda \end{array} \tag{4.2.18}$$

and the *weight map*

$$\begin{array}{rccc} \mathrm{wt}: & \underline{C}_{\mathbf{i}} & \longrightarrow & \mathbb{R}X \\ & (\lambda, t) & \longmapsto & w_0(\lambda) + \sum_{j=1}^{m} a_j \alpha_{i_j} \end{array} \tag{4.2.19}$$

Remark 4.2.30. The weight map (4.2.19) also appears in [1], albeit in slightly different form. Observe that, for $\mu \in X$, $\lambda \in X_+$, the intersection of fibres $\mathrm{wt}^{-1}(\mu) \cap \mathrm{hw}^{-1}(\lambda)$ can be identified with the μ-weight space in $V(\lambda)$.

A description of the inequalities defining $\underline{C}_{\mathbf{i}} \subseteq \mathbb{R}X \times \mathbb{R}^{\ell(w_0)}$ first appeared in [98]. An implicit description can be found in later work of Berenstein-Zelevinsky [14] for arbitrary $\mathbf{i} \in R(w_0)$, and was explicitly given in [1, Theorem 1.1]).

Theorem 4.2.31. *Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. The extended string cone $\underline{C}_{\mathbf{i}}$ is the intersection of $\mathbb{R}X \times C_{\mathbf{i}}$ with the $\ell(w_0)$ half spaces defined by the inequalities*

$$t_k + \sum_{l=k+1}^{m} \langle \alpha_{i_l}, \alpha_{i_k}^{\vee} \rangle t_l \leq \langle \lambda^*, \alpha_{i_k}^{\vee} \rangle, \quad k = 1, \ldots, m. \tag{4.2.20}$$

Here $\lambda^* = -w_0(\lambda)$. The inequalities in (4.2.20) are called λ-inequalities.

We provide some the motivation for the above definitions. The definition of the weight map and the appearance of the λ-inequalities can be understood in terms of the combinatorics of the representation theory of U. By [102, Theorem 14.3.2(c)], we have

$$L_i^{(c_i(b))}(b) \in \mathcal{B}, \quad \text{for any } b \in \mathcal{B}.$$

Here $L_i^{(r)} := \frac{L_i^r}{[r]_q!}$ is the associated q-divided power.

Hence, if $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$ and $c_{\mathbf{i}}(b) = (a_1, \ldots, a_m)$, $b \in \mathcal{B}$, then

$$b_j := L_{i_j}^{(a_j)} \cdots L_{i_1}^{(a_1)}(b) \in \mathcal{B}, \quad j = 1, \ldots, m.$$

Littelmann [98, Section 1] has shown that, for any $b \in \mathcal{B}$,

$$L_{i_m}^{(a_m)} \cdots L_{i_1}^{(a_1)}(b) = 1 \in \mathcal{B}.$$

Let $\lambda \in X_+$ and choose $v_\lambda \in V^q(\lambda)$ a lowest weight vector. Then, $\{bv_\lambda \mid b \in \mathcal{B}(\lambda)\}$ is a basis of $V^q(\lambda)$. In particular, if $b \in \mathcal{B}(\lambda)$ with $c_{\mathbf{i}}(b) = (a_1, \ldots, a_m)$, then bv_λ is a weight vector having weight

$$w_0(\lambda) + \sum_{j=1}^m a_j \alpha_{i_j} = \mathrm{wt}(\lambda, a_1, \ldots, a_m).$$

This is where the definition for the weight map comes from.

Identify $b_j \in \mathcal{B}$ with the basis element $b_j v_\lambda$ so that b_j has weight $\mathrm{wt}(\lambda, a)$. Recall from (4.2.8) the definition of c_i, $i \in I$, the string in the direction i. Consider the i_m-string in $V^q(\lambda)$ through 1. Then, $a_m = c_{i_m}(b_{m-1})$ implies

$$a_m \leq -\langle w_0(\lambda), \alpha_{i_m}^\vee \rangle.$$

Next, consider the i_{m-1}-string in $V^q(\lambda)$ through b_{j-1}. Since $a_{m-1} = c_{i_{m-1}}(b_{j-2})$ we must have

$$a_{m-1} \leq -\langle w_0(\lambda) + a_m \alpha_{i_m}, \alpha_{i_{m-1}}^\vee \rangle \quad \Longrightarrow \quad a_{m-1} + a_m \langle \alpha_{i_m}, \alpha_{m-1}^\vee \rangle \leq \langle -w_0(\lambda), \alpha_{i_{m-1}}^\vee \rangle.$$

Continuing in this fashion we recover the λ-inequalities (4.2.20): consider the i_k-string in $V^q(\lambda)$ through b_k. Then, $a_k = c_{i_k}(b_{k-1})$, implies

$$a_k \leq -\langle w_0(\lambda) + \sum_{j=k+1}^m a_j \alpha_{i_j}, \alpha_{i_k}^\vee \rangle \quad \Longrightarrow \quad a_k + \sum_{j=k+1}^m a_j \langle \alpha_{i_j}, \alpha_{i_k}^\vee \rangle \leq \langle -w_0(\lambda), \alpha_{i_k}^\vee \rangle.$$

The content of the Littelmann, Berenstein-Zelevinsky results cited aboved is that these necessary conditions are sufficient to describe the extended string cone $\underline{C}_{\mathbf{i}}$ in the direction $\mathbf{i}$.

Remark 4.2.32. As Lusztig's canonical basis $\mathcal{B}$ is determined by $U_q(\mathfrak{g})$, it does not depend on the isogeny class of a semisimple complex algebraic group. Therefore, the string cone is independent of the isogeny class of a semisimple complex algebraic group. The explicit description of the string cone given by Berenstein-Zelevinsky (see Theorem 4.2.26) requires G to be simply-connected in order to define the appropriate **i**-trails. However, this is not a problem when we are considering parameterisations of bases of irreducible representations as parameterisations of bases for representations extend across isogeny classes. If G is reductive then G is an extension of a semisimple algebraic group G^{ss} by a central torus and a similar argument allows us to consider parameterisations of bases of irreducible representations of G via G^{ss}. Of course, if we want to keep track of weights then we must remember how the central torus acts.

4.3 Combinatorial and geometric crystals

'*In some sense the* $q \to 0$ *limit strips a module of its linear structure, so we are reduced to combinatorics.*' A. Joseph, [74, p.26]

Let $U = U_q(\mathfrak{g})$ be the quantised universal enveloping algebra associated to the Lie algebra $\mathfrak{g}$ of a reductive complex algebraic group G. In this section we will describe discrete combinatorial models of the representation theory of U introduced by Kashiwara, called *crystals*. The theory of crystals developed from Kashiwara's investigations into bases of U^- at the specialisation $q = 0$. A purely combinatorial construction of crystals, not relying on U^-, and using an arbitrary (not necesarily symmetrisable) Cartan datum $(\Pi, S, \Pi^\vee, S^\vee)$, was given by Littelmann soon thereafter in [99] using his path model.

Since this early development, crystal structures have been discovered throughout mathematics: using the geometry of the affine Grassmannian of ${}^L\mathfrak{g}$ [19], [75]; using the symplectic geometry of quiver varieties [82]; in the study of the generalisation of the Casselmann-Shalika formula to the metaplectic group and associated Eisenstein series[22].

For a reductive complex algebraic group G, Berenstein-Kazhdan described a general geometric framework to obtain crystal structures [12], [13]. Using only the geometry and representation theory of G they recovered the crystal structures obtained by Kashiwara and Littelmann. The tool that they used to construct Kashiwara crystals was the tropicalisation functor Trop. In thi section we provide a construction of Trop on the category of algebraic tori and extend its domain to the category of positive varieties (Definition 4.3.22).

Kashiwara crystals

In this section we briefly recall Kashiwara's notion of an *(abstract) crystal*. Crystals provide combinatorial models of the crystal bases of (specialisations of) integrable U-modules: essentially, the crystal base is the $q \to 0$ limit of the canonical basis $\mathcal{B}$. Hence, crystals admit consequences for the combinatorial representation theory of G. For further details on crystal bases see [79]; for further details on the category of abstract crystals see [80], [78].

A Kashiwara crystal encodes the combinatorial data of the crystal base of an integrable U-module V. As a first approximation, and sufficient for our considerations, a crystal base is a basis B of V *at* $q = 0$ satisfying the following property: for any $i \in I$, the $\mathbb{C}(q^{d_i})$-subalgebra U_i of U generated by $E_i, F_i, K_{\pm d_i \alpha_i^\vee}$ is isomorphic to $U_q(\mathfrak{sl}_2)$ and V decomposes as a U_i-module

$$V \cong \bigoplus_j V(l_j^{(i)}).$$

Here $V(l_j^{(i)})$ is the $(l_j^{(i)}+1)$-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module. Then, B is a basis of the specialisation at $q = 0$ of V such that, for any $i \in I$, B induces an isomorphism $V_0 \cong \bigoplus_j V(l_j^{(i)})_0$. Here W_0 is the specialisation at $q = 0$ of a U_i-module W.

In [79] Kashiwara showed the existence of a crystal basis for any integrable U-module V (more generally, he proved the existence of a crystal basis for the negative part U^- of the quantised universal enveloping algebra associated to any symmetrisable Kac-Moody algebra). Moreover, Kashiwara shows that a crystal basis could be 'melted' (i.e. lifted from the specialisation at $q = 0$) to provide a basis of the U-module V, called a *global basis*, and that the global basis for the irreducible U-modules $V^q(\lambda)$, $\lambda \in X_+$, can be obtained from a *global basis* of U^-. Grojnowski-Lusztig later showed in [57] that Kashiwara's global basis was equal to Lusztig's canonical basis in U^- (in the symmetrisable Kac-Moody setting): the composition of ω with the bar involution $^-$ takes $\mathcal{B}$ to Kashiwara's (lower) global basis.

We now define the abstract notion of a crystal and restrict ourselves to those crystals associated to the Lie algebra $\mathfrak{g}$. Essentially all of the definitions and constructions extend to symmetrisable Kac-Moody type. For the further details on crystals coming from symmetrisable Kac-Moody algebras see [69]; for details on the theory of abstract crystals associated to arbitrary Cartan datum (without requiring recourse to quantised universal enveloping algebras) see [99], [74], [23].

Extend the standard order on $\mathbb{Z}$ to a linear ordering on $\mathbb{Z}_{-\infty} \coloneqq \mathbb{Z} \cup \{-\infty\}$ so that $-\infty$ is the smallest element. Define

$$-\infty + x = -\infty, \quad \text{for any } x \in \mathbb{Z}_{-\infty}.$$

Let $(X, R, X^\vee, R^\vee)$ be the associated root datum of G and assume that $X = \Pi$ is the lattice of integral weights.

Definition 4.3.1. An *abstract (Kashiwara) crystal of type* (R, X) is a (nonempty) set B together with maps

$$\mathrm{wt} : B \to X, \quad \varepsilon_i, \varphi_i : B \to \mathbb{Z}_{-\infty}, \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\}, \quad i \in I. \tag{4.3.1}$$

Here 0 is a ghost element not contained in B. We call the maps $\mathrm{wt}, \varepsilon_i, \varphi_i$, $i \in I$, the structure maps. The collection $(B, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)_{i \in I}$ is subject to the following axioms:

(C1) $\varphi_i(b) - \epsilon_i(b) = \langle wt(b), \alpha_i^\vee \rangle$, for each $i \in I$;

(C2) if $b \in B$ satisfies $\tilde{e}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i$, $\epsilon(\tilde{e}_i(b)) = \epsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$;

(C2)' if $b \in B$ satisfies $\tilde{f}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i$, $\epsilon(\tilde{f}_i(b)) = \epsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1$;

(C3) for $b, b' \in B$, $b' = \tilde{f}_i(b)$ if and only if $\tilde{e}_i(b') = b$;

(C4) if $\varphi_i(b) = -\infty$ then $\tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

For $\mu \in X$, define $B_\mu := \{b \in B \mid \mathrm{wt}(b) = \mu\}$. The functions $\tilde{e}_i, \tilde{f}_i$, $i \in I$, are called *crystal operators*. Given a crystal B one may associate an I-coloured directed graph called the *crystal graph*: the set of vertices is B and there exists a directed arrow $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i(b) = b'$. We say that B is *connected* if its crystal graph is connected. A union of connected components of the crystal graph determines a subcrystal $B' \subseteq B$. B is a *highest weight crystal* if there exists unique $b \in B$ such that $\tilde{e}_i b = 0$, for all $i \in I$; B is a *lowest weight crystal* if there exists unique $b \in B$ such that $\tilde{f}_i b = 0$, for all $i \in I$.

Let B_1 and B_2 be crystals of the same type. A morphism of crystals is a function $\psi : B \sqcup \{0\} \to B' \sqcup \{0\}$ such that

(CM1) $\psi(0) = 0$;

(CM2) if $\psi(b) \neq 0$ then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\epsilon_i(\psi(b)) = \epsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$, for all $i \in I$;

(CM3) for $b \in B$ such that $\psi(b) \neq 0$ and $\psi(\tilde{e}_i(b)) \neq 0$, we have $\psi(\tilde{e}_i(b)) = \tilde{e}_i(\psi(b))$;

(CM3)' for $b \in B$ such that $\psi(b) \neq 0$ and $\psi(\tilde{f}_i(b)) \neq 0$, we have $\psi(\tilde{f}_i(b)) = \tilde{f}_i(\psi(b))$.

Crystals together with crystal morphisms define a category $\mathcal{K}$. If B_1 and B_2 are crystals of the same type then their disjoint union $B_1 \sqcup B_2$ is a crystal in the obvious way: this provides $\mathcal{K}$ with a coproduct. Moreover, $\mathcal{K}$ can be equipped with a tensor structure, defined as follows: if $B_1, B_2 \in \mathcal{K}$ are crystals of the same type then define the crystal $B_1 \otimes B_2$, where

(i) $B_1 \otimes B_2 = B_1 \times B_2$ as a set. We write $b_1 \otimes b_2$ for the pair (b_1, b_2).

(ii) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$.

(iii)
$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1), \\ b_1 \otimes \tilde{f}_i(b_2), & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1). \end{cases}$$

(iv)
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \\ b_1 \otimes \tilde{e}_i(b_2), & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1). \end{cases}$$

We understand $b \otimes 0 = 0 \otimes b = 0$.

(v) $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1), \varphi_i(b_2) + \langle \mathrm{wt}(b_1), \alpha_i^\vee \rangle)$, and

(vi) $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_2), \varepsilon_i(b_1)\langle \mathrm{wt}(b_2), \alpha_i^\vee \rangle)$.

Remark 4.3.2. If $B_1, \ldots, , B_k$ are crystals such that

$$\varepsilon_i(b) = \max\{r \mid \tilde{e}_i^r(b) \neq 0\}, \quad \varphi_i(b) = \max\{r \mid \tilde{f}_i(b) \neq 0\}, \quad b \in B_1 \cup \ldots \cup B_k, i \in I, \tag{4.3.2}$$

then the action of the crystal operators $\tilde{e}_i, \tilde{f}_i$ on a tensor $b_1 \otimes \cdots \otimes b_k$ can be computed using the *signature rule*: decorate each tensorand b_j with $\varphi_i(b_j)$ '$-$' signs followed by $\varepsilon_i(b_j)$ '$+$' signs. This gives rise to a sequence in the alphabet $\{-,+\}$. Successively cancel all adjacent pairs $+-$ to obtain a sequence having a '$-$' signs followed by b '$+$' signs. Then,

$$\varphi_i(b_1 \otimes \cdots \otimes b_k) = a, \quad \text{and} \quad \varepsilon_i(b_1 \otimes \cdots \otimes b_k) = b,$$

and $\tilde{f}_i$ acts on the tensor factor associated to the rightmost remaining $-$, and $\tilde{e}_i$ acts on the tensor fact associated to the leftmost $+$.

Example 4.3.3. (1) Let G be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$, $\mathfrak{g}$ its Lie algebra. Let $U = U_q(\mathfrak{g})$ be the associated quantised universal enveloping algebra. Let $\lambda \in X_-$ be antidominant and $V_q(\lambda) = U^+/I_{-\lambda}$, where $I_{-\lambda} = \sum_i U^+ E_i^{-\langle \lambda, \alpha_i^\vee \rangle + 1}$. Let $\lambda' = w_0(\lambda)$. Hence, $V_q(\lambda)$ is the irreducible finite dimensional U-module having lowest weight λ and highest weight λ'. Let $\mathcal{B}(\lambda') \subseteq \mathcal{B}$ be the subset from Definition 4.2.16. Then, $\mathcal{B}(\lambda')$ maps onto a basis of $V_q(\lambda)$. Let $\mathcal{V}(\lambda')$ be the $\mathbb{Z}[q^{-1}]$-span of this base. By Theorem 4.2.10, we obtain a homogenous basis $B(\lambda')$ of the $\mathbb{Z}$-module $\mathcal{V}(\lambda')/q^{-1}\mathcal{V}(\lambda')$. The pair $(B(\lambda'), \mathcal{V}(\lambda'))$ is an example of a crystal base (see [79]). Recall the $\mathbb{Z}[q^{-1}]$-submodule $\mathcal{L}$ from Theorem 4.2.10. If $b \in B(\lambda')$, let $\mathbf{b} \in \mathcal{B}(\lambda')$ be the unique element in $\mathcal{B}$ that maps to b. Let $\pi_\lambda : \mathcal{L} \to \mathcal{V}(\lambda')/q^{-1}\mathcal{V}(\lambda')$ be the composition of the above projections. We define a crystal structure of type (R, X) on $B(\lambda')$:

(i) if $b \in B(\lambda')_\mu$ then $\mathrm{wt}(b) = \mu$;

(ii) if $b \in B(\lambda')$ then $\tilde{e}_i b \coloneqq \pi_\lambda(E_i \mathbf{b})$, $\tilde{f}_i b \coloneqq \pi_\lambda(L_i(\mathbf{b}))$, $i \in I$;

(iii) $\varepsilon_i(b) = \max\{r \mid \tilde{e}_i^r(b) \neq 0\}$, $\varphi_i(b) = \max\{r \mid \tilde{f}_i^r(b) \neq 0\}$.

For details regarding the well-definedness of these definitions see [102, Part III].

(2) More generally, given any integrable U-module V, one can define a crystal $B(V)$ in an analogous manner.

(3) Let $B(-\infty) \subseteq \mathcal{L}/q^{-1}\mathcal{L}$ be the image of the canonical basis $\mathcal{B}$ (recall the lattice $\mathcal{L}$ from Theorem 4.2.10). Define the weight map wt, the crystal operators $\tilde{e}_i$, $\tilde{f}_i$ and φ_i, $i \in I$, as in the previous example. Define $\varepsilon_i(b) = \varphi_i(b) + \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$. Ths endows $B(-\infty)$ with a crystal structure.

(4) If B is a crystal then we define $B^\vee$ as follows: as a set $B^\vee = \{b^\vee \mid b \in B\}$ and

$$\mathrm{wt}(b^\vee) = -\mathrm{wt}(b),\ \varepsilon_i(b^\vee) = \varphi_i(b),\ \varphi_i(b^\vee) = \varepsilon_i(b),\ \tilde{e}_i(b^\vee) = (\tilde{f}_i(b))^\vee,\ \tilde{f}_i(b^\vee) = (\tilde{e}_i(b))^\vee.$$

Here $0^\vee = 0$. $B^\vee$ is called the dual of B and $B^{\vee\vee} = B$.

(5) $B(\infty) := B(-\infty)^\vee$. This is the crystal associated to the crystal basis of U^- constructed by Kashiwara [79].

(6) Let $\lambda \in X$. Then, there is a crystal $T_\lambda = \{t_\lambda\}$ with $\mathrm{wt}(t_\lambda) = \lambda$ and $\varphi_i(t_\lambda) = -\infty$, for all $i \in I$.

(7) Let $\lambda \in X_-$ be antidominant. There is an isomorphism of crystals $B(\lambda)^\vee \cong B(-w_0(\lambda))$.

Remark 4.3.4. Let B_1, B_2 be crystals associated to crystal bases of integrable U-modules V_1, V_2. The crystal $B_1 \cup B_2$ is the crystal associated to the integrable U-module $V_1 \oplus V_2$, and the crystal $B_1 \otimes B_2$ is the crystal associated to the integrable U-module $V_1 \otimes V_2$. It is shown in [78] that there are isomorphisms $B \otimes T_0 \cong B$, $T_0 \otimes B \cong B$, where T_0 is the crystal from Example 4.3.3, and that $\mathcal{K}$ is a monoidal category.

The following result indicates the relationship between crystals and representation theory.

Proposition 4.3.5. *Let $B = B(V)$ be a finite crystal of type (R, X) associated to an integrable $U_q(\mathfrak{g})$-module V. Then, if $V \cong \bigoplus_{\lambda \in X_+} V^q(\lambda)^{c_\lambda}$ then B decomposes as a disjoint union*

$$B = \bigsqcup_{\lambda \in X_+} B(\lambda)^{\sqcup c_\lambda}.$$

Therefore, crystals provide a tool to approach tensor product decomposition computations: if $\lambda_1, \ldots, \lambda_k \in X_-$ are antidominant then the connected components of the crystal $B(\lambda_1) \otimes \cdots \otimes B(\lambda_k)$ correspond precisely to the irreducible summands of $V^q(\lambda_1) \otimes \cdots \otimes V^q(\lambda_k)$. Upon specialisation at $q = 1$ this provides an effective computational model for computing tensor product multiplicities for $\mathfrak{g}$.

An important problem is to determine combinatorially accessible examples of crystals. We provide a construction for all crystals $B(\lambda)$, $\lambda \in X_+$, in type A.

Let $\mathfrak{g} = \mathfrak{gl}_n$ be the Lie algebra of the general linear group $\mathrm{GL}_n(\mathbb{C})$. Let $I = \{1, \ldots, n-1\}$, $X = \mathbb{Z}^n$ with standard basis $\epsilon_1, \ldots, \epsilon_n$, $S = \{\alpha_i\}_{i=1}^{n-1}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, \ldots, n-1$. Let $\varpi_i = \sum_{j=1}^{i} \epsilon_j$, for $i = 1, \ldots, n$. Recall that, for $\mathfrak{gl}_n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in X_+$ if and only if $\lambda_1 \geq \cdots \geq \lambda_n$.

Define the crystal $\mathbb{B} = \{\mathbf{1}, \ldots, \mathbf{n}\}$ as follows: its crystal graph is

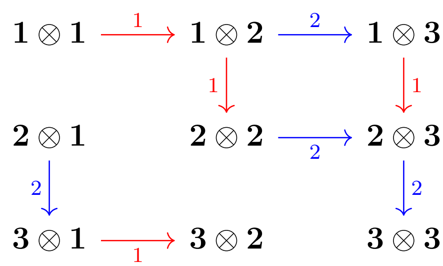
$$\mathbf{1} \xrightarrow{1} \mathbf{2} \xrightarrow{2} \mathbf{3} \xrightarrow{3} \cdots \xrightarrow{n-2} \mathbf{n-1} \xrightarrow{n-1} \mathbf{n}$$

This specifies how the crystal operators act. We set

$$\varphi_i(\mathbf{j}) = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i. \end{cases} \quad \text{and} \quad \varepsilon_i(\mathbf{j}) = \begin{cases} 0, & \text{if } j \neq i+1, \\ 1, & \text{if } j = i+1, \end{cases}$$

and $\mathrm{wt}(\mathbf{1}) = \varpi_1$. Observe that $\mathbb{B}$ satisfies the condition in (4.3.2). Axiom (C4)' and the crystal graph imply that $\mathrm{wt}(\mathbf{k}) = \varpi_1 - \sum_{j=1}^{k-1} \alpha_j$. In particular, $\mathrm{wt}(\mathbf{n}) = -\varpi_{n-1}$. $\mathbb{B} = B(\varpi_1)$ is the crystal graph associated to the irreducible $U_q(\mathfrak{gl}_n)$-module $V^q(\varpi_1)$. In the specialisation $q = 1$ this is the defining representation $\mathbb{C}^n$ of $\mathfrak{gl}_n$.

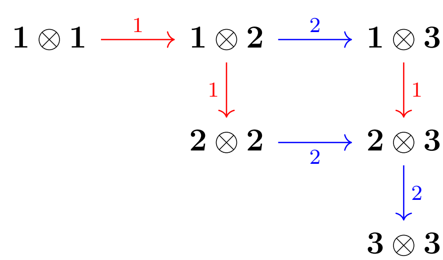
Fix $n = 3$. Using the signature rule (Remark 4.3.2), we can determine the crystal graph of $\mathbb{B} \otimes \mathbb{B}$:



The connected component

$$\mathbf{2} \otimes \mathbf{1} \xrightarrow{2} \mathbf{3} \otimes \mathbf{1} \xrightarrow{1} \mathbf{3} \otimes \mathbf{2}$$

is isomorphic to the dual crystal $\mathbb{B}^\vee = B(\varpi_2)$ and the large connected component



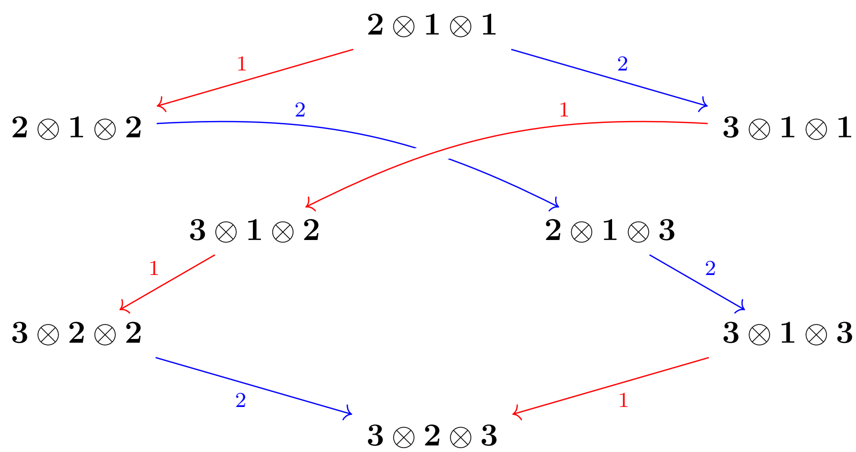
is isomorphic to the crystal $B(2\varpi_1)$. We find $\mathbb{B} \otimes \mathbb{B} = B(\varpi_2) \sqcup B(2\varpi_1)$, corresponding to the decomposition of $\mathfrak{gl}_3$-modules $V \otimes V \cong V(\varpi_2) \oplus V(2\varpi_2)$. In particular, using the crystal $\mathbb{B} = B(\varpi_1)$ we've obtained $B(2\varpi_1)$. More generally, we can obtain the crystal $B(k\varpi_1)$ as a subcrystal of $\mathbb{B}^{\otimes k}$. This is a special instance of the following result (see [69, Chapter 7]).

Proposition 4.3.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in X_+$. Assume $\lambda_n \geq 0$. Then, $B(\lambda)$ is isomorphic to the connected component of $\mathbb{B}^{\otimes|\lambda|}$ containing the highest weight element*

$$\underbrace{\mathbf{n} \otimes \cdots \otimes \mathbf{n}}_{\lambda_n} \otimes \cdots \otimes \underbrace{\mathbf{2} \otimes \cdots \otimes \mathbf{2}}_{\lambda_2} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{\lambda_1}$$

When (R, X) is of classical type, Kashiwara-Nakashima [81] obtained models of the crystals $B(\lambda)$, $\lambda \in X_+$, using Young tableaux. See [69] for further details and examples.

Example 4.3.7. The crystal graph for the crystal $B(\varpi_1 + \varpi_2)$ is given below.



Tropicalisation and positivity

In this section we will describe a 'geometrisation' of crystals, following Berenstein-Kazhdan [12], [13]. We will describe a birational model of Kashiwara crystals called a *geometric crystal*. Geometric crystals are varieties birational to algebraic tori, together with a collection of birational maps that 'model' the combinatorial data of a Kashiwara crystal. Through the process of *tropicalisation* (or *ultra-discretization* [115]), the geometry of the geometric crystal is stripped away revealing the data of a Kashiwara crystal (Definition 4.3.1).

Let G be a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$. Let $T \subseteq G$ be a maximal torus.

Definition 4.3.8. A decorated geometric crystal is the data $(X, \gamma, \varphi_i, \varepsilon_i, e_i^\cdot, f \mid i \in I)$, where X is an irreducible variety, γ is a rational morphism $X \to T$, called the *weight map*, $\varphi_i, \varepsilon_i : X \to \mathbb{A}^1$ are rational functions, and each $e_i^\cdot : \mathbb{G}_m \times X \to X$ is a unital rational action (denoted $(c, x) \mapsto e_i^c(x)$) such that, for each $i \in I$, one has either

(i) $\varphi_i = \varepsilon_i = 0$ and the action is trivial, or,

(ii) $\varphi_i \neq 0$ and $\varepsilon_i \neq 0$, and

$$\gamma(e_i^c(x)) = \alpha_i^\vee(c)\gamma(x), \quad \varepsilon_i(x) = \alpha_i(\gamma(x))\varphi_i(x),$$

$$\varepsilon_i(e_i^c(x)) = c\varepsilon_i(x), \quad \varphi_i(e_i^c(x)) = c^{-1}\varphi_i(x).$$

Moreover, $f : X \to \mathbb{A}^1$ is a rational function, called the *decoration*, on X such that

$$f(e_i^c(x)) = f(x) + \frac{c-1}{\varphi_i(x)} + \frac{c^{-1}-1}{\varepsilon_i(x)}.$$

Remark 4.3.9. Observe the analogy between the conditions defining a geometric crystal and a Kashiwara crystal (Definition 4.3.1). For our purposes we will only be interested in the data (X, γ, f). For general results and examples see [12], [13].

Our interest in geometric crystals is the process by which we can recover Kashiwara crystals. This is the process of tropicalisation, which we now describe.

First, given a real vector space E, we construct the *semi-field of polytopes in E*. This construction will be the foundation of our notion of *tropicalisation*. Further details and proofs can be found in [112], [13, Section 4].

Let E be a finite-dimensional real vector space, E^* the dual vector space. Define $\mathcal{P}_E$ to be the set of all convex polytopes in E, and define the Minkowski sum

$$P + Q := \{p + q \mid p \in P, q \in Q\}, \quad P, Q \in \mathcal{P}_E.$$

Then, $(\mathcal{P}_E, +)$ is a monoid with unit $\{0\}$. For $P \in \mathcal{P}_E$, define the *support function of P*, to be

$$\begin{array}{rccc} \chi_P : & E^* & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \min\{\xi(p) \mid p \in P\}. \end{array}$$

If $\mathrm{vert}(P)$ is the set of vertices of P then we have $\chi_P(\xi) = \min\{\xi(p) \mid p \in \mathrm{vert}(P)\}$. We have the following elementary result.

Lemma 4.3.10. *The assignment*

$$\begin{array}{rccc} \chi : & \mathcal{P}_E & \longrightarrow & \mathrm{Fun}(E^*, \mathbb{R}) \\ & P & \longmapsto & \chi_P \end{array}$$

is an injective homomorphism of monoids. Here $\mathrm{Fun}(E^*, \mathbb{R}) = \{f : E^* \to \mathbb{R}\}$ *is the set of* $\mathbb{R}$*-valued functions on* E^**, considered as a monoid under pointwise addition.*

Corollary 4.3.11. *Let $P, Q, R \in \mathcal{P}_E$. Then, $\mathcal{P}_E$ admits the canellation property: $P + R = Q + R$ if and only if $P = Q$.*

For $P, Q \in \mathcal{P}_E$ we define the *join of P and Q* to be

$$P \vee W := \mathrm{conv}(P \cup Q),$$

the convex hull of $P \subseteq Q$. Hence, $\chi_{P \vee Q} = \min(\chi_P, \chi_Q)$ and, since

$$\min(\chi_P, \chi_Q) + \chi_R = \min(\chi_P + \chi_R, \chi_Q + \chi_R),$$

we obtain the following identity in $\mathcal{P}_E$,

$$(P \vee Q) + R = (P + R) \vee (Q + R).$$

This shows that $(\mathcal{P}_E, +, \vee)$ is a semi-ring with addition $\vee$, and multiplication $+$.

Define $\mathcal{P}_E^+$ to be the Grothendieck group of the monoid $(\mathcal{P}_E, +)$, with generators $[P]$, $P \in \mathcal{P}_E$, subject to the relation $[P+Q] = [P] + [Q]$. Then, the join operation can be uniquely extended to $\mathcal{P}_E^+$ using the 'quotient' rule

$$([P] - [Q]) \vee ([P'] - [Q']) \coloneqq [(P + Q') \vee (P' + Q)] - [Q + Q'].$$

Hence, $\mathcal{P}_E^+$ is a semi-field. Moreover, the homomorphism $\chi : (\mathcal{P}_E, +) \to (\mathrm{Fun}(E^*, \mathbb{R}), +)$ extends uniquely to an injective homomorphism of semi-fields

$$\begin{aligned} \tilde{\chi} : \ \mathcal{P}_E^+ &\longrightarrow \mathrm{Fun}(E^*, \mathbb{R}) \\ P &\longmapsto \chi_P \end{aligned}$$

Here $\mathrm{Fun}(E^*, \mathbb{R})$ is a semi-field with the operation of 'addition' $(f, g) \mapsto \min(f, g)$ and 'multiplication' $(f, g) \mapsto f + g$. By abuse of notation we will simply write χ instead of $\tilde{\chi}$.

We apply the above polytope algebra construction in the category of rational tori. Let S be an algebraic torus split over $\mathbb{Q}$, and define

$$X(S) \coloneqq \mathrm{Hom}(S, \mathbb{G}_m), \quad X^\vee(S) \coloneqq \mathrm{Hom}(\mathbb{G}_m, S),$$

with canonical pairing

$$\langle , \rangle : \ X(S) \times X^\vee(S) \longrightarrow \mathbb{Z}$$

Denote the group algebra of $X(S)$ over $\mathbb{Q}$ by $\mathbb{Q}[X(S)]$. The elements in $\mathbb{Q}[X(S)]$ can be canonically identified with the algebra of regular functions on S. For $f \in \mathbb{Q}[X(S)]$, say $f = \sum_{\mu \in X(S)} a_\mu e^\mu$, define the *Newton polytope of* f

$$N(f) \coloneqq \mathrm{conv}\{\mu \mid a_\mu \neq 0\} \subseteq \mathbb{R}X(S).$$

The *support of* f is the set

$$\mathrm{supp}(f) \coloneqq \{\mu \mid f = \sum_{\mu \in X(S)} a_\mu e^\mu, \ a_\mu \neq 0\} \subseteq \mathrm{vert}(N(f)).$$

We have the following consequence of the definition.

Lemma 4.3.12. *Let $f, g : S \to \mathbb{A}^1$ be regular functions, both nonzero. Then,*

(a) $N(fg) = N(f) + N(g)$, *and*

(b) $N(f + g) \subseteq N(f) \vee N(g)$.

Lemma 4.3.12 gives the following result.

Proposition 4.3.13. *The assignment*

$$\begin{array}{rccc} N: & \mathbb{Q}[X(S)]^\times & \longrightarrow & \mathcal{P}_E \\ & f & \longmapsto & N(f) \end{array}$$

is a homomorphism of monoids. Moreover, N extends to a well-defined homomorphism of abelian groups

$$\begin{array}{rccc} \tilde{N}: & \mathrm{Frac}(S)^\times & \longrightarrow & \mathcal{P}_E^+ \\ & \frac{f}{g} & \longmapsto & [N(f)] - [N(g)] \end{array}$$

By abuse of notation we will simply write N instead of $\tilde{N}$.

We will now define the notion of *tropicalisation* for algebraic tori defined over $\mathbb{Q}$. Let S be an algebraic torus defined over $\mathbb{Q}$, and set $E = \mathbb{R}X(S)$. We canonically identify $E^* = \mathbb{R}X^\vee(S)$. Consider the following modification of the homomorphism $\chi : \mathcal{P}_E^+ \to \mathrm{Fun}(E^*, \mathbb{R})$:

$$\begin{array}{rccc} \chi^0: & \mathcal{P}_E^+ & \longrightarrow & \mathrm{Fun}(X^\vee(S), \mathbb{Z}) \\ & P & \longmapsto & \chi_P^0 \end{array} \tag{4.3.3}$$

where

$$\chi_P^0(\xi) := \min\{\langle p, \xi\rangle \mid p \in P \cap X(S)\}, \quad \xi \in X^\vee(S).$$

Definition 4.3.14. Define *tropicalisation (with respect to S)* to be the composition

$$\mathrm{Trop}_S := \chi^0 \circ N: \quad \mathrm{Frac}(S)^\times \longrightarrow \mathrm{Fun}(X^\vee(S), \mathbb{Z}).$$

If $S = \mathbb{G}_m^k$ is the standard torus then we simply write Trop_k.

If $f : S \to S'$ is a rational morphism of tori, define the *tropicalisation of f*

$$\mathrm{Trop}(f): \quad X^\vee(S) \longrightarrow X^\vee(S')$$

to be the unique function such that the following diagram commutes

$$\begin{array}{ccc} X(S) \subseteq \mathrm{Frac}(S)^\times & \xrightarrow{\mathrm{Trop}_S} & \mathrm{Fun}(X^\vee(S), \mathbb{Z}) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \mathrm{Trop}(f)^*} \\ X(S') \subseteq \mathrm{Frac}(S')^\times & \xrightarrow[\mathrm{Trop}_{S'}]{} & \mathrm{Fun}(X^\vee(S'), \mathbb{Z}) \end{array}$$

That is, $\mathrm{Trop}(f) : X^\vee(S) \to X^\vee(S')$ is the unique function such that, for every $\lambda' \in X(S')$, we have an equality of functions

$$\mathrm{Trop}_S(\lambda' \circ f) = \mathrm{Trop}_{S'} \circ \mathrm{Trop}(f) : X^\vee(S) \to \mathbb{Z}.$$

Equivalently, $\mathrm{Trop}(f) : X^\vee(S) \to X^\vee(S')$ is the unique function such that

$$\mathrm{Trop}_S(\lambda' \circ f)(\mu) = \langle \lambda', \mathrm{Trop}(f)(\mu)\rangle, \quad \text{for every } \mu \in X^\vee(S),\ \lambda' \in X(S').$$

Example 4.3.15. Consider the rational morphism

$$\begin{array}{rccc} f: & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 \\ & (x,y) & \longmapsto & \left(\frac{x}{x^2+y}, xy^{-1}\right) \end{array}$$

Write $f = (f_1, f_2)$, and let $e_1, e_2 \in \mathbb{Z}^2$ be the standard basis with dual basis e_1^*, e_2^*. Hence,

$$\mathrm{Trop}(f)(a_1e_1^* + a_2e_2^*) = b_1e_1^* + b_2e_2^*$$

must satisfy

$$\begin{aligned} b_1 &= \langle e_1, \mathrm{Trop}(f)(a_1e_1^* + a_2e_2^*)\rangle \\ &= \mathrm{Trop}_{\mathbb{G}_m^2}(f_1)(a_1e_1^* + a_2e_2^*) \\ &= \chi_{N(f_1)}(a_1e_1^* + a_2e_2^*) \\ &= \langle e_1, a_1e_1^* + a_2e_2^*\rangle - \min\left(\langle 2e_1, a_1e_1^* + a_2e_2^*\rangle, \langle e_2, a_1e_1^* + a_2e_2^*\rangle\right) \\ &= a_1 - \min(2a_1, a_2). \end{aligned}$$

Similarly, we compute

$$b_2 = a_1 - a_2.$$

Hence,

$$\begin{array}{rccc} \mathrm{Trop}(f): & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ & (a_1,a_2) & \longmapsto & (a_1 - \min(2a_1,a_2), a_1 - a_2). \end{array}$$

Remark 4.3.16. Observe that if we define

$$\begin{array}{rccc} g: & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 \\ & (x,y) & \longmapsto & \left(\frac{x}{x^2-y}, xy^{-1}\right) \end{array}$$

then $\mathrm{Trop}(g) = \mathrm{Trop}(f)$, where f is from Example 4.3.15.

Example 4.3.15 indicates that the notion of tropicalisation we have defined is the same as the standard notion of tropicalisation appearing in the field of tropical geometry (see [106]). We verify this observation with the following result.

Proposition 4.3.17. *Let $f = (f_1, \dots, f_l) : \mathbb{G}_m^k \to \mathbb{G}_m^l$ be a rational morphism. Then,*

$$\begin{array}{rccc} \mathrm{Trop}(f): & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}^l \\ & (a_1,\dots,a_k) & \longmapsto & (\mathrm{Trop}_k(f_1)(a_1,\dots,a_k), \dots, \mathrm{Trop}_k(f_l)(a_1,\dots,a_k)) \end{array}$$

where $\mathrm{Trop}_k(f_j)(a_1,\ldots,a_k)$ *is considered to be a tropical rational function (see [106]). Namely,* $\mathrm{Trop}_k(f_j)(a_1,\ldots,a_k)$ *is the piecewise linear function obtained from* $f_j(x_1,\ldots,x_k)$ *by replacing*

$$\begin{array}{ccc} x_i & \longleftrightarrow & a_i \\ + & \longleftrightarrow & \min \\ \times & \longleftrightarrow & + \\ \div & \longleftrightarrow & - \end{array}$$

Proof. It suffices to show that $\mathrm{Trop}_k(f)$ is the claimed piecewise linear function, for $f \in \mathrm{Frac}(\mathbb{G}_m^k) = \mathbb{Q}(x_1,\ldots,x_k)$.

Let $f = \frac{g}{h}$ with $g,h \in \mathbb{Q}[x_1,\ldots,x_k]$. By definition, we have $\mathrm{Trop}_k(f) \in \mathrm{Fun}(\mathbb{Z}^k,\mathbb{Z})$ and $\mathrm{Trop}_k(f) = \chi^0 \circ N(f)$, where χ^0 is given in (4.3.3) and $N(f) = [N(g)] - [N(h)]$ is the (virtual) Newton polytope of f. Since χ^0 is a morphism of semi-fields, it suffices to consider the case when $f \in \mathbb{Q}[x_1,\ldots,x_k]$.

Let $f \in \mathbb{Q}[x_1,\ldots,x_k]$. Then, by definition

$$\mathrm{Trop}_k(f)(a_1,\ldots,a_l) = \chi^0(N(f)) = \min\{(b_1,\ldots,b_k)\cdot(a_1,\ldots,a_k) \mid (b_1,\ldots,b_k) \in \mathrm{supp}(f)\}$$

and this expression is what we are looking for. Here $\cdot$ is the standard dot product on $\mathbb{Z}^k$. □

The tropicalisation of homomorphisms between algebraic tori will be of most interest to us. We record some elementary results.

Proposition 4.3.18. *Let S be an algebraic torus defined over $\mathbb{Q}$.*

(a) Let $\lambda \in X(S)$, considered as a rational function $\lambda : S \to \mathbb{G}_m$. Recall the canonical identification $X(S) \cong \mathrm{Hom}_{\mathbb{Z}}(X^\vee(S),\mathbb{Z})$. Then,

$$\mathrm{Trop}_S(\lambda) = \lambda \in \mathrm{Hom}(X^\vee(S),\mathbb{Z}) \subseteq \mathrm{Fun}(X^\vee,\mathbb{Z}).$$

(b) Let $\xi \in X^\vee(S)$, considered as a rational function $\xi : \mathbb{G}_m \to S$. Then, for any $\mu \in X^\vee(\mathbb{G}_m)$,

$$\mathrm{Trop}(\xi)(\mu) = \xi \circ \mu \in X^\vee(S).$$

Identifying $X^\vee(\mathbb{G}_m) \cong \mathbb{Z}$, $\mathrm{id}_{\mathbb{G}_m} \mapsto 1 \in \mathbb{Z}$, we have

$$\mathrm{Trop}(\xi)(n) = n\xi \in X^\vee(S), \quad n \in \mathbb{Z}.$$

(c) More generally, if $f : S \to T$ is an homomorphism of algebraic tori then, for any $\mu \in X^\vee(S)$,

$$\mathrm{Trop}(f)(\mu) = f \circ \mu \in X^\vee(T)$$

(d) *Let $f : S_1 \to T$ and $g : S_2 \to T$ be rational morphisms, and*

$$\begin{array}{rccc} fg: & S_1 \times S_2 & \longrightarrow & T \\ & (x, y) & \longmapsto & f(x)g(y) \end{array}$$

Then, $\mathrm{Trop}(fg) = \mathrm{Trop}(f) + \mathrm{Trop}(g)$.

Proof. (a) Let $\lambda \in X(S)$. By definition, for any $\mu \in X^\vee(S)$,

$$\begin{aligned} \mathrm{Trop}_S(\lambda)(\mu) &= \chi^0_{N(\lambda)}(\mu) \\ &= \min\{\langle p, \mu\rangle \mid p \in N(\lambda) \cap X(S)\} \\ &= \langle \lambda, \mu\rangle, \quad \text{since } N(\lambda) = \{\lambda\}. \end{aligned}$$

(b) Let $\xi \in X^\vee(S)$. Then, $\mathrm{Trop}(\xi) : X^\vee(\mathbb{G}_m) \to X^\vee(S)$ is the unique function such that, for every $\mu \in X^\vee(\mathbb{G}_m)$, $\lambda \in X(S)$,

$$\mathrm{Trop}_1(\lambda \circ \xi)(\mu) = \langle \lambda, \mathrm{Trop}(\xi)(\mu)\rangle.$$

Then,

$$\begin{aligned} \mathrm{Trop}_1(\lambda \circ \xi)(\mu) &= \langle \lambda \circ \xi, \mu\rangle, \quad \text{by (a)}, \\ &= \langle \lambda, \xi \circ \mu\rangle, \quad \text{by definition of the pairing } \langle , \rangle. \end{aligned}$$

Now, let $\mu_n \in X^\vee(\mathbb{G}_m)$, $\mu_n(z) = z^n$. Then,

$$\mathrm{Trop}(\xi)(\mu_n) = \xi \circ \mu_n = \xi^n,$$

because ξ is a homomorphism. Writing the group structure additively the result follows.

(c) The argument is similar.

(d) Let $f : S_1 \to T$, $g : S_2 \to T$. There is a canonical identification $X^\vee(S_1 \times S_2) = X^\vee(S_1) \oplus X^\vee(S_2)$. For any $\mu = (\mu_1, \mu_2) \in X^\vee(S_1) \oplus X^\vee(S_2)$, $\lambda \in X(T)$, we must have

$$\mathrm{Trop}_{S_1 \times S_2}(\lambda \circ fg)(\mu) = \langle \lambda, \mathrm{Trop}(fg)(\mu)\rangle.$$

Then,

$$\begin{aligned} \mathrm{Trop}_{S_1 \times S_2}(\lambda \circ fg)(\mu) &= \langle \lambda \circ fg, \mu\rangle, \quad \text{by (a)}, \\ &= \langle (\lambda \circ f)(\lambda \circ g), \mu\rangle, \quad \text{since } \lambda \text{ is a homomorphism}, \\ &= \langle (\lambda \circ f), \mu\rangle + \langle (\lambda \circ g), \mu_2\rangle, \quad \text{by definition of the pairing } \langle , \rangle, \\ &= \langle \lambda, \mathrm{Trop}(f)(\mu_1)\rangle + \langle \lambda, \mathrm{Trop}(g)(\mu_2)\rangle, \\ &= \langle \lambda, \mathrm{Trop}(f)(\mu_1) + \mathrm{Trop}(g)(\mu_2)\rangle. \end{aligned}$$

Hence, $\mathrm{Trop}(fg) = \mathrm{Trop}(f) + \mathrm{Trop}(g)$.

□

Consider the rational morphism

$$\begin{array}{rccc} h: & \mathbb{G}_m^2 & \longrightarrow & \mathbb{G}_m^2 \\ & (x,y) & \longmapsto & (x, x+y) \end{array}$$

This map is a birational isomorphism with rational inverse $(x,y) \mapsto (x, y-x)$. However,

$$\begin{array}{rccc} \mathrm{Trop}(h): & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ & (a,b) & \longmapsto & (a, \min(a,b)) \end{array}$$

is not a bijection. In light of this example, if we want to define a *tropicalisation functor* on some category of split algebraic tori (defined over $\mathbb{Q}$) we need to restrict morphisms.

In [12], Berenstein-Kazhdan determined an appropriate category on which to define a tropicalisation functor. We recall their results.

Definition 4.3.19. Let S be an algebraic torus, split over $\mathbb{Q}$.

(a) Let $f \in \mathrm{Frac}(S)^\times$. We say f is *positive* if it can be written as a quotient $f = g/h$, where $g, h \in \mathbb{Z}_{>0}[X(S)]$. Denote the semi-field of positive rational functions on S by $\mathrm{Frac}_+(S)$.

(b) A rational morphism $f : S \to S'$ is *positive* if the pullback map

$$f^* : \quad X(S') \subseteq \mathrm{Frac}(S') \longrightarrow \mathrm{Frac}_+(S)$$

is well-defined. Denote the set of positive rational morphisms $S \to S'$ by $\mathrm{Mor}_+(S, S')$.

Remark 4.3.20. It can be shown that $f \in \mathrm{Mor}_+(S, S')$ if and only if $f : S(\mathbb{Q}_{>0}) \to S'(\mathbb{Q}_{>0})$ is well-defined (see [13, Section 4]).

Theorem 4.3.21 (Berenstein-Kazhdan, [12, Section 2.4])**.** *Let $\mathcal{T}_+$ be the monoidal category of algebraic tori split over $\mathbb{Q}$ with positive rational morphisms. Then,*

$$\begin{array}{rccc} \mathrm{Trop}: & \mathcal{T}_+ & \longrightarrow & \mathrm{Set} \\ & S & \longmapsto & X^\vee(S) \\ & f & \longmapsto & \mathrm{Trop}(f) \end{array}$$

is a (covariant) functor. If we equip $\mathcal{T}_+$ and Set *with their standard monoidal structure then* Trop *is monoidal.*

Tropicalisation, as we have defined it, applies to the positive rational functions on a split algebraic torus. Therefore, we might expect to be able to apply tropicalisation to those varieties birational to an algebraic torus. As we will see, this will be possible, but will require our introducing a notion of *positivity* for varieties.

Definition 4.3.22. Let X be a variety defined over $\mathbb{Q}$.

(a) A *toric chart on* X is a birational isomorphism $\theta : S \to X$, for some algebraic torus S split over $\mathbb{Q}$.

(b) We say that two toric charts

$$\theta : S \to X, \quad \theta' : S' \to X$$

are *positively equivalent* if $\theta^{-1} \circ \theta'$ is an isomorphism in $\mathcal{T}_+$. In particular,

$$\theta^{-1} \circ \theta' \in \mathrm{Mor}_+(S', S), \quad \text{and} \quad (\theta')^{-1} \circ \theta \in \mathrm{Mor}_+(S, S').$$

(c) A *positive atlas on* X is a positive equivalence class of toric charts on X. We will also call a positive atlas a *positive structure on* X.

(d) A *positive variety* (X, Θ_X) is a variety together with a choice of positive atlas. A *morphism of positive varieties* (X, Θ_X), (Y, Θ_Y) is a rational morphism

$$f : \ X \longrightarrow Y$$

such that

$$f : \ X(\mathbb{Q}_{>0}) \longrightarrow Y(\mathbb{Q}_{>0})$$

is a well-defined function, and $\mathrm{range}(f) \cap \mathrm{dom}(\theta^{-1}) \neq \varnothing$, for any $\theta \in \Theta_Y$.

With these objects and morphisms, we define the *category of positive varieties* $\mathcal{V}_+$.

Example 4.3.23. (1) Let S be a split algebraic torus, defined over $\mathbb{Q}$. The *standard positive structure on* S is the positive structure containing $\mathrm{id}_S : S \to S$, which we will denote Θ_S. The *standard positive structure on* $\mathbb{A}^k$ is the positive structure containing the canonical open embedding $\mathbb{G}_m^k \to \mathbb{A}^k$, which we will denote Θ_k. It's straightforward to see that, as positive varieties, $\mathbb{G}_m^k \cong \mathbb{A}^k$.

(2) More generally, let (X, Θ_X) be a positive variety, $\theta : S \to X \in \Theta_X$. Then, θ defines an isomorphism of positive varieties $(S, \Theta_S) \xrightarrow{\sim} (X, \Theta_X)$.

(3) Any homomorphism of algebraic tori is positive.

Remark 4.3.24. We will always consider an algebraic torus as a positive variety equipped with the standard positive structure.

The importance of the notion of positivity is that it gives the correct condition to construct a tropicalisation functor. As we are motivated to reconstruct Kashiwara crystals via tropicalisation, we introduce the following definition.

Definition 4.3.25. A *positive decorated geometric crystal* is a decorated geometric crystal $(X, \gamma, \varepsilon_i, \varphi_i, e_i^{\cdot}, f \mid i \in I)$ such that X is equipped with a positive structure Θ, with respect to which all of the maps appearing in the definition are positive (with the respect to the appropriate standard positive structures). We will write simply (X, Θ, f) for the data of a positive decorated geometric crystal.

The following result is an imediate consequence of the definition of a positive structure.

Lemma 4.3.26. *Let (X, Θ_X) and (Y, Θ_Y) be positive varieties. Then, $(X \times Y, \Theta_X \times \Theta_Y)$, where*

$$\Theta_X \times \Theta_Y \coloneqq \{\theta \times \theta' \mid \theta \in \Theta_X, \theta' \in \Theta_Y\},$$

is a positive variety. Therefore, $\mathcal{V}_+$ is a monoidal category.

Consider the natural inclusion functor

$$\begin{array}{ccc} \mathcal{T}_+ & \longrightarrow & \mathcal{V}_+ \\ S & \longmapsto & (S, \Theta_S) \end{array} \tag{4.3.4}$$

Clearly, this functor is fully faithful and monoidal. Moreover, by Example 4.3.23, we have the following result.

Proposition 4.3.27. *The inclusion functor (4.3.4) is an equivalence of monoidal categories.*

Consider the category $\mathcal{V}_{++}$ with objects (X, Θ_X, θ), where $(X, \Theta_X) \in \mathcal{V}_+$ and $\theta \in \Theta_X$, and morphisms being morphisms of the underlying positive varieties. By definition, the forgetful functor

$$\begin{array}{ccc} \mathcal{V}_{++} & \longrightarrow & \mathcal{V}_+ \\ (X, \Theta_X, \theta) & \longmapsto & (X, \Theta_X) \end{array} \tag{4.3.5}$$

is an equivalence of monoidal categories. Any adjoint to this functor corresponds to a simultaneous choice of toric chart $\theta : S \xrightarrow{\sim} X \in \Theta_X$, for every (X, θ_X). All such adjoints are isomorphic.

Define the functor

$$\begin{array}{cccc} \tau : & \mathcal{V}_{++} & \longrightarrow & \mathcal{T}_+ \\ & (X, \Theta_X, \theta) & \longmapsto & \operatorname{dom}(\theta) \\ & (X, \Theta_X, \theta) \xrightarrow{f} (Y, \Theta_Y, \theta') & \longmapsto & (\theta')^{-1} \circ f \circ \theta \end{array} \tag{4.3.6}$$

Then, τ is an equivalence of monoidal categories.

We will now extend the tropicalisation functor to the category of positive varieties. Let $\mathcal{G} : \mathcal{V}_+ \to \mathcal{V}_{++}$ be an adjoint to the forgetful functor. By the discussion above, all such adjoints are isomorphic to each other.

Definition 4.3.28. The composition

$$\mathrm{Trop}_{\mathcal{G}} := \mathrm{Trop} \circ \tau \circ \mathcal{G} : \ \mathcal{V}_+ \longrightarrow \mathrm{Set} \tag{4.3.7}$$

will be called a *tropicalisation functor*.

Remark 4.3.29. All tropicalisation functors $\mathrm{Trop}_{\mathcal{G}} : \mathcal{V}_+ \to \mathrm{Set}$ are isomorphic to each other. For the remainder of this thesis we assume that we have fixed a choice of tropicalisation functor and write $\mathrm{Trop} : \mathcal{V}_+ \to \mathrm{Set}$ (by abuse of notation).

Moreover, if $f : X \to \mathbb{A}^1$ is a rational function then $\mathrm{Trop}(f) : X^\vee(S) \to \mathbb{Z}$ depends only only the positive equivalence class of f. Here, rational functions $f, f' \in \mathrm{Frac}_+(X)$ are *positively equivalent* if there is an isomorphism of positive varieties $h : (X, \Theta_X) \xrightarrow{\sim} (X, \Theta_X)$ such that $f' = f \circ h$ (see [13, Section 6.1]).

We finish this section with the main result in [13], which states that the tropicalisation of a positive decorated geometric crystal is a Kashiwara crystal. For more details see [13].

Theorem 4.3.30 (Berenstein-Kazhdan, [13])**.** *Let* (X, Θ, f) *be a positive decorated geometric crystal. Define the* tropical locus of f (with respect to Θ) *to be*

$$B_{f,\Theta} := \{x \in \mathrm{Trop}(X) \mid \mathrm{Trop}(f)(x) \geq 0\}.$$

Then, the data $(B_{f,\Theta}, \mathrm{Trop}(\gamma), \mathrm{Trop}(\varepsilon_i), \mathrm{Trop}(\varphi_i), \mathrm{Trop}(e_i^\cdot) \mid i \in I)$ *is a Kashiwara crystal, where we consider the tropicalised data as being restricted to* $B_{f,\Theta}$*. The map* $\mathrm{Trop}(\gamma) = \mathrm{wt}$ *for the crystal.*

4.4 Crystal structures in mirror symmetry

In this section we will demonstrate the appearance of Kashiwara crystal structures in the Rietsch mirror family (M_B, f_B) to the complete flag variety ${}^LG/{}^LB_+$. For each reduced expression $\mathbf{i}$ of the longest element w_0, we define a toric chart $\underline{\theta}_{\mathbf{i}}$ of M_B. The collection of toric charts $\{\underline{\theta}_{\mathbf{i}} \mid \mathbf{i} \in R(w_0)\}$ are positively equivalent and define a positive structure $\underline{\Theta}_0$ on M_B. With respect to this positive structure the superpotential f_B is a positive morphism. In Theorem 4.4.5 we identify the tropical locus $\{\mathrm{Trop}(f_B \circ \underline{\theta}_{\mathbf{i}}) \geq 0\}$ of the Rietsch mirror family (M_B, f_B) with the extended string cone $\underline{C}_{\mathbf{i}}(\mathbb{Z})$ (Definition 4.2.28), explicitly recovering the inequalities defining the string cone and the λ-inequalities. We will see that the quantum structure map q and the equivariant structure map e from Definition 3.1.3 play an essential role in these structures: the tropicalisation $\mathrm{Trop}(q)$ is the highest weight map hw (4.2.18) and $\mathrm{Trop}(e)$ (4.2.19) is the weight map hw. These results are inspired by, and similar to, [13].

As usual, G is a reductive complex algebraic group with associated root datum $(X, R, X^\vee, R^\vee)$, and we use the notation and conventions from Section 1.3.

For $w \in W$, define the varieties

$$B_-^w := B_- \cap N_+\overline{w}N_+, \quad N_+^w := B_-\overline{w}B_- \cap N_+. \tag{4.4.1}$$

Let $M := B_- \cap N_+T\overline{w}_0N_+$ be the mirror family of ${}^LG/{}^LB_+$. Recall the quantum and equivariant structure maps (Definition 3.1.3),

$$\begin{array}{ccc} T & \xleftarrow{q} & M \\ & & \downarrow e \\ & & T \end{array} \tag{4.4.2}$$

where

$$\begin{aligned} q: \quad M = B_- \cap N_+T\overline{w}_0N_+ &\longrightarrow T \\ b = zt\overline{w}_Pu &\longmapsto t \end{aligned} \tag{4.4.3}$$

and

$$\begin{aligned} e: \quad M \subseteq B- = N_-T &\longrightarrow T \\ b = vs &\longmapsto s \end{aligned} \tag{4.4.4}$$

By Section 3.1, q is a smooth trivial fibration with fibre $B_-^{w_0} := B_- \cap N_+\overline{w}_0N_+$. We fix the following trivialisation

$$\begin{aligned} j: \quad T \times B_-^{w_0} &\longrightarrow M \\ (t,x) &\longmapsto \pi^+\left(\left(\overline{w}_0x^T\right)^{-1}\right)\overline{w}_0x^T\pi^0\left(x^{-T}\right)t^{w_0} \end{aligned} \tag{4.4.5}$$

Lemma 4.4.1. *The trivialisation j is well-defined.*

Proof. We must show that

(i) if $j_t(x) = j(t,x)$ then $j_t(x) \in B_- \cap N_+t\overline{w}_0N_+$, and

(ii) j is an isomorphism.

Write $x = z\overline{w}_0u$. Then,

$$\begin{aligned} (\overline{w}_0x^T)^{-1} &= x^{-T}\overline{w}_0^{-1} \\ &= z^{-T}\overline{w}_0u^{-T}\overline{w}_0^{-1} \end{aligned}$$

Hence, $\pi^+((\overline{w}_0x^T)^{-1}) = \overline{w}_0u^{-T}\overline{w}_0^{-1}$ and we have

$$j_t(z\overline{w}_0u) = z^T\pi^0(z^T\overline{w}_0u^T)t^{\overline{w}_0} \in B_-.$$

Also, we have $x = vs \in B_- = N_-T$ and $\pi^0(x^{-T}) = s^{-1}$, Hence,

$$j_t(x) = j_t(vs) = \pi^+\left(\left(\overline{w}_0x^T\right)^{-1}\right)\overline{w}_0sv^Ts^{-1}t^{\overline{w}_0} \in N_+\overline{w}_0N_+t^{\overline{w}_0} = N_+t\overline{w}_0N_+.$$

The inverse to j is seen to be

$$\begin{array}{ccc} M & \longrightarrow & T \times B_-^{w_0} \\ b = zt\overline{w}_0 u & \longmapsto & \left(\pi^0(\overline{w}_0^{-1}b), \pi^{\geq 0}\left(\overline{w}_0^{-1}\pi^-(b)\right)^T\right) \end{array} \tag{4.4.6}$$

□

Remark 4.4.2. Observe that the trivialisation j is not the obvious choice of trivialisation induced by multiplication. A similar trivialisation appears in [28].

For any $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$, define

$$\begin{array}{cccc} x_{-\mathbf{i}}: & \mathbb{G}_m^m & \longrightarrow & B_-^{w_0} \\ & (a_1, \ldots, a_m) & \longmapsto & x_{-i_1}(a_1) \cdots x_{-i_m}(a_m), \end{array} \tag{4.4.7}$$

where, for $i \in I$,

$$\begin{array}{cccc} x_{-i}: & \mathbb{G}_m & \longrightarrow & B_-^{w_0} \\ & c & \longmapsto & y_i(c)\alpha_i^\vee(c^{-1}). \end{array}$$

Here $y_i : \mathbb{A}^1 \to N_-$ is the root subgroup corresponding to the (simple) negative root $-\alpha_i$, and $\alpha_i^\vee \in X^\vee(T)$.

Definition 4.4.3. For $\mathbf{i} \in R(w_0)$, define

$$\begin{array}{cccc} \underline{\theta}_\mathbf{i}: & T \times \mathbb{G}_m^{\ell(w_0)} & \longrightarrow & M \\ & (t, a) & \longmapsto & j(t, x_{-\mathbf{i}}(a)) \end{array}$$

By Fomin-Zelevinsky [36], we have the following result.

Proposition 4.4.4. *(a) For any* $\mathbf{i} \in R(w_0)$, $\underline{\theta}_\mathbf{i}$ *is a toric chart.*

(b) For $\mathbf{i}, \mathbf{i}' \in R(w_0)$, *the birational isomorphism*

$$\underline{\theta}_\mathbf{i}^{\mathbf{i}'} := \underline{\theta}_{\mathbf{i}'}^{-1} \circ \underline{\theta}_\mathbf{i} : \; T \times \mathbb{G}_m^{\ell(w_0)} \longrightarrow T \times \mathbb{G}_m^{\ell(w_0)}$$

is a positive morphism. In other words, $\underline{\theta}_\mathbf{i}$ *and* $\underline{\theta}_{\mathbf{i}'}$ *are positively equivalent.*

Proposition 4.4.4 implies that we can equip M with the structure of a positive variety $(M, \underline{\Theta}_0)$, where we define $\underline{\Theta}_0$ to be the positive equivalence class of toric charts on M containing $\{\underline{\theta}_\mathbf{i} \mid \mathbf{i} \in R(w_0)\}$.

Recall from Definition 3.1.9 the *superpotential* f_B

$$\begin{array}{cccc} f_B: & M & \longrightarrow & \mathbb{C} \\ & b = zt\overline{w}_0 u & \longmapsto & \chi(z) + \chi(u) \end{array}$$

where $\chi = \sum_{i \in I} \chi_i \in \mathrm{Hom}(N_+, \mathbb{A}^1)$ and χ_i, $i \in I$, is the character of N_+ uniquely determined by

$$\chi_i(x_j(a)) = \delta_{ij} a, \quad a \in \mathbb{A}^1.$$

Define

$$\begin{array}{rccc} f_B^{(1)} : & M & \longrightarrow & \mathbb{C} \\ & b = zt\overline{w}_0 u & \longmapsto & \chi(z) \end{array} \tag{4.4.8}$$

and

$$\begin{array}{rccc} f_B^{(2)} : & M & \longrightarrow & \mathbb{C} \\ & b = zt\overline{w}_0 u & \longmapsto & \chi(u) \end{array} \tag{4.4.9}$$

For $\mathbf{i} \in R(w_0)$, we define

$$f_{B,\mathbf{i}} := f_B \circ \underline{\theta}_{\mathbf{i}}, \quad f_{B,\mathbf{i}}^{(1)} := f_B^{(1)} \circ \underline{\theta}_{\mathbf{i}}, \quad f_{B,\mathbf{i}}^{(2)} := f_B^{(2)} \circ \underline{\theta}_{\mathbf{i}}, \tag{4.4.10}$$

We now state the main result of this section.

Theorem 4.4.5. *Let* $\mathbf{i} \in R(w_0)$*. Then, the subset*

$$\{(\lambda, a) \in X^\vee \times \mathbb{Z}^{\ell(w_0)} \mid \mathrm{Trop}(f_{B,\mathbf{i}})(\lambda, a) \geq 0\} \subseteq X^\vee(T \times \mathbb{G}_m^{\ell(w_0)}) = X(^LT) \times \mathbb{Z}^{\ell(w_0)} \tag{4.4.11}$$

equals the set of lattice points in the weighted string cone $\underline{C}_{\mathbf{i}}(\mathbb{Z})$ *of the Langlands dual group* ^LG*. More precisely,*

(a) $f_{B,\mathbf{i}}^{(1)}$ *is a positive morphism and* $\mathrm{Trop}(f_{B,\mathbf{i}}^{(1)})(\lambda, a) \geq 0$ *are the inequalites defining* $\mathbb{R}X(^LT) \times C_{\mathbf{i}}$ *in* $\mathbb{R}X(^LT) \times \mathbb{R}^{\ell(w_0)}$*, and*

(b) $f_{B,\mathbf{i}}^{(2)}$ *is a positive morphism and* $\mathrm{Trop}(f_{B,\mathbf{i}}^{(2)})(\lambda, a) \geq 0$ *are the* λ*-inequalities (see (4.2.20)).*

Moreover, the maps q *and* e *are positive morphisms and*

$$\mathrm{Trop}(q) = \mathrm{hw}, \quad \mathrm{Trop}(e) = \mathrm{wt}\,. \tag{4.4.12}$$

Proof. First, we show the statements on the description of the weighted string cone. Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_0)$. Then, there exists $z, u \in N_+$ such that

$$j(t, x_{-\mathbf{i}}(a)) = zt\overline{w}_0 u \in N_+ t\overline{w}_0 N_+ \cap B_-.$$

Observe that

$$u = \pi^+\left(\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))\right).$$

Hence, we have

$$f_{B,\mathbf{i}}^{(2)}(t, a) = \chi\left(\pi^+\left(\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a))\right)\right).$$

Using the definition of j,

$$\overline{w}_0^{-1} j(t, x_{-\mathbf{i}}(a)) = \overline{w}_0^{-1}\pi^+\left(\left(\overline{w}_0 x_{-\mathbf{i}}(a)^T\right)^{-1}\right)\overline{w}_0 x_{-\mathbf{i}}(a)^T \pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right) t^{w_0}$$

so that

$$\pi^+\left(\overline{w}_0^{-1}j(t,x_{-\mathbf{i}}(a))\right) = \pi^+\left(x_{-\mathbf{i}}(a)^T\pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right)t^{w_0}\right).$$

By definition of the transpose map we obtain

$$\begin{aligned} x_{-\mathbf{i}}(a)^T &= x_{-i_m}(a_m)^T\cdots x_{-i_1}(a_1)^T \\ &= \alpha_{i_m}^\vee(a_m^{-1})x_{i_m}(a_m)\cdots\alpha_{i_1}^\vee(a_1^{-1})x_{i_1}(a_1) \\ &= \left(\prod_{j=1}^m \alpha_{i_j}^\vee(a_j^{-1})\right)x_{i_m}(b_m)\cdots x_{i_1}(b_1), \end{aligned}$$

where

$$b_j = a_j\prod_{l<j} a_l^{\langle\alpha_{i_j},\alpha_{i_l}^\vee\rangle}, \quad j=1,\dots,m.$$

Hence,

$$\pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right) = \prod_{j=1}^m \alpha_{i_j}^\vee(a_j).$$

Therefore, we obtain

$$x_{-\mathbf{i}}(a)^T\pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right)t^{w_0} = t^{w_0}x_{\mathbf{i}^{op}}(c^{op}),$$

where $c=(c_1,\dots,c_m)\in\mathbb{G}_m^m$ is defined by

$$c_j = (-w_0\alpha_{i_j})(t)a_j^{-1}\prod_{l>j} a_l^{-\langle\alpha_{i_j},\alpha_{i_l}^\vee\rangle}, \quad j=1,\dots,m,$$

and

$$\pi^+\left(x_{-\mathbf{i}}(a)^T\pi^0\left(x_{-\mathbf{i}}(a)^{-T}\right)t^{w_0}\right) = x_{\mathbf{i}^{op}}(c^{op})$$

Putting this all together, and recalling the involution $i\mapsto i^*$ on I (see Definition 4.2.6), we have

$$\begin{aligned} f_{B,\mathbf{i}}^{(2)}(t,a_1,\dots,a_m) &= \chi(u) \\ &= \chi(x_{\mathbf{i}^{op}}(c^{op})) \\ &= \sum_{j=1}^m \alpha_{i_j^*}(t)a_j^{-1}\prod_{l>j} a_l^{-\langle\alpha_{i_j},\alpha_{i_l}^\vee\rangle}. \end{aligned}$$

Hence, $f_{B,\mathbf{i}}^{(2)}$ is a positive morphism (Definition 4.3.22). By Proposition 4.3.18, we have, for any $(\lambda,a)\in X^\vee\times\mathbb{Z}^m$,

$$\mathrm{Trop}(f_{B,\mathbf{i}}^{(2)})(\lambda,a) = \min\{\langle\alpha_{i_j^*},\lambda\rangle - a_j - \sum_{l=j+1}^m \langle\alpha_{i_j},\alpha_{il}^\vee\rangle a_l \mid j=1,\dots,m\}$$

Recalling that the root datum of the Langlands dual group ${}^LG$ is the dual root datum for G, we see that the locus $\mathrm{Trop}(f^{(2)}_{B,\mathbf{i}}) \geq 0$ is precisely the locus defined by the λ-inequalities for ${}^LG$ (Theorem 4.2.31).

Now we obtain the inequalities defining the string cone $C_{\mathbf{i}}$. Note that, if $b = zt\overline{w}_0u \in M$ then

$$z = \pi^+\left(\overline{w}_0^{-1}b^\iota\right)^\iota$$

Hence, we have

$$f^{(1)}_{B,\mathbf{i}}(t,a) = \chi\left(\pi^+(\overline{w}_0^{-1}j(t,x_{-\mathbf{i}}(a))^\iota)^\iota\right).$$

Now,

$$\begin{aligned}&\overline{w}_0^{-1}j(t,x_{-\mathbf{i}}(a))^\iota\\ =&\overline{w}_0^{-1}t^{-w_0}\pi^0(x_{-\mathbf{i}}(a)^T)x_{-\mathbf{i}}(a)^{\iota T}\overline{w}_0\pi^+((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})^\iota\end{aligned}$$

so that

$$\pi^+(\overline{w}_0^{-1}j(t,x_{-\mathbf{i}}(a))^\iota) = \pi^+((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})^\iota,$$

and we have

$$f^{(1)}_{B,\mathbf{i}}(t,a) = \chi\left(\pi^+((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})\right)$$

We can assume that G is semisimple and simply-connected (see Remark 4.2.32)). In this situation, for any $g \in N_-TN_+$,

$$\chi_i(\pi^+(g)) = \frac{\Delta_{\omega_i,s_i\omega_i}(g)}{\Delta_{\omega_i,\omega_i}(g)}, \quad i \in I,$$

where $\Delta_{u\omega_i,v\omega_i}(g) = \omega_i(\pi^0(\overline{u}^{-1}g\overline{v}))$ is a generalised minor (see [36, Proposition 2.6]). Hence,

$$f^{(1)}_{B,\mathbf{i}}(t,a) = \sum_{i\in I}\frac{\Delta_{\omega_i,s_i\omega_i}((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})}{\Delta_{\omega_i,\omega_i}((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})}. \tag{4.4.13}$$

Define the involutive antiautomorphism

$$\begin{aligned}\tau_{w_0}:\ G &\longrightarrow G\\ g &\longmapsto \overline{w}_0g^{-\iota T}\overline{w}_0^{-1}.\end{aligned}$$

Then,

$$(\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1} = x_{-\mathbf{i}}(a)^{-T}\overline{w}_0^{-1} = \overline{w}_0^{-1}\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota)$$

and (4.4.13) becomes

$$\begin{aligned}&\sum_{i\in I}\frac{\Delta_{\omega_i,s_i\omega_i}((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})}{\Delta_{\omega_i,\omega_i}((\overline{w}_0x_{-\mathbf{i}}(a)^T)^{-1})}\\ =&\sum_{i\in I}\frac{\Delta_{w_0\omega_i,s_i\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))}{\Delta_{w_0\omega_i,\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))}.\end{aligned}$$

We have the following formulae from [14, (4.6)],

$$\Delta_{u\omega_i,v\omega_i}(x) = \Delta_{-v\omega_i,-u\omega_i}(x^\iota) = \Delta_{w_0v\omega_i,w_0u\omega_i}(\tau_{w_0}(x)), \quad u,v\in W, i\in I.$$

Using these formulae we obtain

$$\sum_{i\in I}\frac{\Delta_{w_0\omega_i,s_i\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))}{\Delta_{w_0\omega_i,\omega_i}(\tau_{w_0}(x_{-\mathbf{i}}(a)^\iota))} = \sum_{i\in I}\frac{\Delta_{-\omega_i,-w_0s_i\omega_i}(x_{-\mathbf{i}}(a))}{\Delta_{w_0\omega_{i^*},\omega_{i^*}}(x_{-\mathbf{i}}(a))}.$$

Observe that, for any $b = z\overline{w}_0u \in B_-^{w_0}$, $\overline{w}_0^{-1}b \in N_-N_+$, so that

$$\Delta_{w_0\omega_i,\omega_i}(b) = 1, \quad i\in I.$$

Hence,

$$f^{(1)}_{B,\mathbf{i}}(t,a) = \sum_{i\in I}\Delta_{-\omega_i,-w_0s_i\omega_i}(x_{-\mathbf{i}}(a)).$$

Finally, using [14, Corollary 5.9], we can compute the tropicalisation of a generalised minor: we have

$$\mathrm{Trop}(\Delta_{u\omega_i,v\omega_i}(x_{-\mathbf{i}}(a_1,\ldots,a_m)) = \min\left\{\sum_{k=1}^m d_k^{(i)}(\pi)a_k \mid \pi \text{ is } \mathbf{i}\text{-trail from } -u\omega_i \text{ to } -v\omega_i \text{ in } V_{\omega_i}\right\}.$$

Thus,

$$\begin{aligned}\mathrm{Trop}(f^{(1)}_{B,\mathbf{i}})(\lambda,a) &= \min\{\mathrm{Trop}(\Delta_{-\omega_i,-w_0s_i\omega_i}(x_{-\mathbf{i}}(a))) \mid i\in I\}\\ &= \min\left\{\sum_{k=1}^m d_k^{(i)}(\pi)a_k \mid \pi \text{ is } \mathbf{i}\text{-trail from } \omega_i \text{ to } w_0s_i\omega_i \text{ in } V_{\omega_i},\ i\in I\right\}\end{aligned}$$

and the locus defined by $\mathrm{Trop}(f^{(1)}_{B,\mathbf{i}}) \geq 0$ is precisely the string cone $C_{\mathbf{i}}$ for ${}^LG$ (see Theorem 4.2.26).

We will now show that q and e are positive morphisms and

$$\mathrm{Trop}(q) = \mathrm{hw}, \quad \mathrm{Trop}(e) = \mathrm{wt}\,.$$

To show that q is positive it suffices to show that the rational morphism $q\circ\underline{\theta}_{\mathbf{i}}$ is positive, for some $\mathbf{i}\in R(w_0)$. The result follows immediately since

$$q\circ\underline{\theta}_{\mathbf{i}}(t,a) = t \tag{4.4.14}$$

is a homomorphism. Then, $\mathrm{Trop}(q) = \mathrm{hw}$ follows from (4.4.14).

The argument for e is bit more involved. Let $\mathbf{i}\in R(w_0)$. Using (4.4.6), we see that

$$\begin{aligned}\pi^0(x_{-\mathbf{i}}(a)) &= \pi^0\left(\pi^{\geq 0}(\overline{w}_0^{-1}\pi^-(j(t,x_{-\mathbf{i}}(a))))^T\right)\\ &= \pi^0(\overline{w}_0^{-1}\pi^-(j(t,x_{-\mathbf{i}}(a))))\\ &= \pi^0(\overline{w}_0^{-1}j(t,x_{-\mathbf{i}}(a))\pi^0(j(t,x_{-\mathbf{i}}(a)))^{-1})\\ &= t^{w_0}\pi^0(j(t,x_{-\mathbf{i}}(a)))^{-1}\end{aligned}$$

Hence, we find

$$e(j(t, x_{-\mathbf{i}}(a))) = \pi^0(j(t, x_{-\mathbf{i}}(a))) = t^{w_0}\pi^0(x_{-\mathbf{i}}(a))^{-1} = t^{w_0}\left(\prod_{j=1}^{m} \alpha_{i_j}^\vee(a_j)\right)$$

Since homomorphisms are positive we see that e is positive. Denote the conjugation map

$$\begin{aligned} c_{w_0} : \ T &\longrightarrow T \\ t &\longmapsto \overline{w}_0 t \overline{w}_0^{-1} \end{aligned}$$

Then, by Proposition 4.3.18, we have

$$\mathrm{Trop}(e) = \mathrm{Trop}(c_{w_0}) + \sum_{j=1}^{m} \mathrm{Trop}(\alpha_{i_j}^\vee)$$

and, for $(\lambda, a) \in X^\vee(T) \times X^\vee(\mathbb{G}_m^{\ell(w_0)})$,

$$\mathrm{Trop}(e)(\lambda, a) = \mathrm{Trop}(c_{w_0})(\lambda) + \sum_{j=1}^{\ell(w_0)} \mathrm{Trop}(\alpha_{i_j}^\vee)(a).$$

Making the canonical identifications $X^\vee(T) \cong X({}^L T)$, where ${}^L T \subseteq {}^L G$ is the torus dual to T, and $X^\vee(\mathbb{G}_m^{\ell(w_0)}) \cong \mathbb{Z}^{\ell(w_0)}$, we obtain $\mathrm{Trop}(c_{w_0})(\lambda) = w_0(\lambda)$, $\lambda \in X({}^L T)$, and

$$\mathrm{Trop}(e)(\lambda, a_1, \ldots, a_m) = w_0(\lambda) + \sum_{j=1}^{m} a_j \alpha_{i_j}^\vee.$$

The result follows. □

Remark 4.4.6. A similar result to Theorem 4.4.5 is obtained by Berenstein-Kazhdan [13, Theorem 6.15]. Their result is a consequence of the fact that $(M_B, \underline{\Theta}_0, f_B)$ can be given the structure of a *positive decorated geometric crystal*; we briefly outline their argument. By Theorem 4.3.30, the tropical locus $B_{f,\underline{\Theta}_0}$ is a Kashiwara crystal. The fibre of $\mathrm{Trop}(q)$ over $\lambda \in X^\vee(T)$ is shown to be a highest weight crystal isomorphic having highest weight λ. They then use the following theorem of Joseph [118]: a family $\{\mathcal{C}_\lambda \mid \lambda \in X_+^\vee\}$ of highest weight crystals, so that $c_\lambda \in \mathcal{C}_\lambda$ is a unique highest weight element, is *closed* if, for any $\lambda, \mu \in X_+^\vee$, the correspondence $c_{\lambda+\mu} \mapsto (c_\lambda, c_\mu) \in C_\lambda \otimes C_\mu$ extends to an injective morphism of crystals $C_{\lambda+\mu} \to C_\lambda \otimes C_\mu$.

Theorem 4.4.7 (Joseph, [118])**.** *If $\{\mathcal{C}_\lambda \mid \lambda \in X_+^\vee\}$ is a closed family of crystals then each C_λ is isomorphic to $B(\lambda)$.*

4.5 Future directions

In this final section we describe how the crystal structure appearing on the B-model side of mirror symmetry of complete flag varieties $X = {}^LG/{}^LB$ plays a conjectural organisational role with regards to certain integrable systems appearing on the A-model side of mirror symmetry for symplectic reductions of X. This will be the focus of future work. We focus on the case of polygon spaces $\mathcal{P}_{r,n}$ to be explicit. For background on completely integrable systems see [63].

Recall Examples 2.2.4, 2.4.6 and the construction of the polygon space $\mathcal{P}_{r,n}$ by symplectic reduction of $\mathrm{Gr}_{\mathbb{C}}(2, n)$. Assume that $R \coloneqq |r| \in \mathbb{Z}_{>0}$, and suppose $\mathrm{Gr}_{\mathbb{C}}(2, n) = \mathrm{PGL}_n(\mathbb{C})/{}^LP$ admits Kahler form corresponding to $R\alpha_2^\vee$.

In [117, Section 3], Noharu-Ueda construct a family of completely integrable systems $\Psi_\Gamma : \mathrm{Gr}_{\mathbb{C}}(2, n) \to \mathbb{R}^{2(n-2)}$ parameterised by triangulations Γ of some fixed n-gon Π. The functions in Ψ_Γ are in bijection with $n-1$ consecutive edges of Π and the $n-3$ diagonals defining Γ. Moreover, the integrable system Ψ_Γ descends to an integrable system $\Phi_\Gamma : \mathcal{P}_{r,n} \to \mathbb{R}^{n-3}$ of the symplectic reduction $\mathcal{P}_{r,n}$ (these are the *bending systems* in [66]).

Let $\Delta_\Gamma \subseteq \mathbb{R}^{2(n-2)}$ be the moment polytope of the integrable system Ψ_Γ. Let $(u_1, \ldots, u_{n-1})$ be the coordinates on $\mathbb{R}^{2(n-2)}$ corresponding to $(n-1)$ consecutive edges, and $(v_1, \ldots, v_{n-3})$ the coordinates corresponding to the diagonals in Γ. Then, the moment polytope of Φ_Γ is shown to be the following subset of Δ_Γ

$$\Delta_\Gamma(r) \coloneqq \{(u_1, \ldots, u_{n-1}, v_1, \ldots, v_{n-3}) \mid (u_1, \ldots, u_{n-1}, |r| - \sum_{i=1}^{n-1} u_i) = r\}$$

Recall the moment polytope Ξ_R from Example 2.4.6. Thus, $r \in \Xi_R$. The above discussion is summarised in Figure 4.1.

For a particular triangulation Γ_0 of Π, Noharu-Ueda show ([117, Example 4.1]) that Ψ_{Γ_0} is equivalent to the Gelfand-Tsetlin system [64] and that the moment polytope Δ_{Γ_0} is equivalent to a Gelfand-Tsetlin polytope $\mathrm{GT}_P^{(R)}$ consisting of all Gelfand-Tsetlin patterns

$$\begin{array}{cccccccc}
R & & \lambda_2^{(n-1)} & & & & & \\
& \lambda_1^{(n-1)} & & \lambda_2^{(n-2)} & & & & \\
& & \ddots & & \ddots & & & \\
& & & \lambda_1^{(3)} & & \lambda_2^{(2)} & & 0 \\
& & & & \lambda_1^{(2)} & & \lambda_2^{(1)} & \\
& & & & & \lambda_1^{(1)} & &
\end{array} \tag{4.5.1}$$

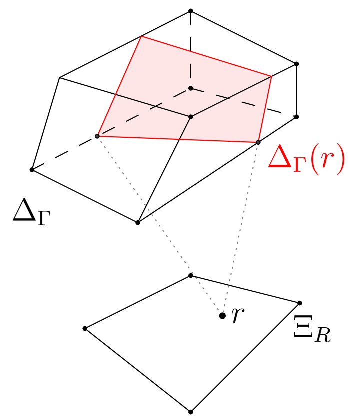


Figure 4.1: Relation between the moment polytopes Ξ, Δ_Γ and $\Delta_\Gamma(r)$.

Recall that each subtriangular array

$$\begin{matrix} \lambda_1^{(j)} & & \lambda_2^{(j-1)} \\ & \lambda_1^{(j-1)} & \end{matrix}$$

corresponds to the relation $\lambda_1^{(j)} \geq \lambda_1^{(j-1)} \geq \lambda_2^{(j-1)}$.

Recall the mirror family (M_P, f_P) and explicit formula for the superpotential f_P from Section 3.4. By Proposition 3.4.9, there is a monomial transformation of $(\mathbb{C}^\times)^{2(n-2)}$ such that the superpotential takes the form

$$f_P = \sum_{a \in \mathrm{GT}_P} \frac{z_{h(a)}}{z_{t(a)}}. \tag{4.5.2}$$

Here GT_P is the Gelfand-Tsetlin quiver of shape P. The tropical locus of f_P with respect to the z-coordinates is now seen to be precisely the space of Gelfand-Tsetlin patterns of shape

P. Namely, $\mathrm{Trop}(f_P)(\lambda, Z_i^{(j)}) \geq 0$, where $(\lambda, Z_i^{(j)}) \in X^\vee(Z(L_P)) \times \mathbb{Z}^{2(n-1)}$, if and only if

$$\begin{array}{cccccccc}
\lambda & & Z_2^{(n-1)} & & & & & \\
 & Z_1^{(n-1)} & & Z_2^{(n-2)} & & & & \\
 & & \ddots & & \ddots & & & \\
 & & & Z_1^{(3)} & & Z_2^{(2)} & & 0 \\
 & & & & Z_1^{(2)} & & Z_2^{(1)} & \\
 & & & & & Z_1^{(1)} & &
\end{array}$$

is a Gelfand-Tsetlin pattern. In particular, fixing $\lambda = R\alpha_2^\vee$ we obtain the Gelfand-Tsetlin patterns in (4.5.1).

We now describe a project for further research.

Recall the quantum structure map q and the equivariant structure map e for (M_B, L_B) (Definition 3.1.3). Let $\lambda = R\alpha_2^\vee \in X^\vee(L_P) \cap X_+^\vee$. Then, by Theorem 4.4.5, $\mathrm{Trop}(q)$ is the highest weight map for the extended string cone, and $\mathrm{Trop}(e)$ is the weight map. By [1, Section 5], $\Delta_\lambda := \mathrm{Trop}^{-1}(\lambda)$ is equivalent to the polytope $\mathrm{GT}_P^{(R)}$. Hence, Δ_λ can be identified with Δ_{Γ_0}. Now, $\mathrm{Trop}(e)(\Delta_\lambda)$ is the convex hull of $W \cdot \lambda \subseteq \mathbb{R}X$. By Theorem 2.3.3, this is precisely the moment polytope Ξ_R. This situation (occuring on the B-model side) is similar to that described in Figure 4.1 (on the A-model side). This motivates the following conjecture.

Conjecture 4.5.1. Under the identification $\Delta_\lambda = \Delta_{\Gamma_0}$, the moment polytope $\Delta_{\Gamma_0}(r) \subseteq \Delta_{\Gamma_0}$ is equal to $\mathrm{Trop}(e)^{-1}(\hat{r}) \cap \Delta_\lambda$

Recall the definition of $\hat{r}$ from Example 2.2.6: $\hat{r}$ is the orthogonal projection of r onto the hyperplane $\{r_1 + \ldots + r_n = 0\}$. Conjecture 4.5.1 is similar to work of Rietsch-Williams [121].

If $\hat{r}$ is integral then we have the following heuristic interpretation of Conjecture 4.5.1: the lattice points of the moment polytope $\Delta_{\Gamma_0}(r)$ of the weight variety $\mathcal{P}_{r,n}$ is equal to the dimension of the $\hat{r}$-weight space in the irreducible representation $V(R\alpha_2^\vee)$ of PGL_n. Similar results have been obtained via different methods in [65]. It would be interesting to determine if this observation can be extended to more general weight varieties: this is expected to be related to recent work of Gross-Hacking-Keel-Kontsevich [59].

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