Lawrence Berkeley National Laboratory

Title
INVARIANT SURFACES OF ORDINARY DIFFERENTIAL EQUATIONS WITH AND WITHOUT TIME LAG

Permalink
https://escholarship.org/uc/item/1xk9p6t6

Author
Schaeffer, Anthony J.

Publication Date
1967-04-01

University of California

Ernest O. Lawrence Radiation Laboratory

INVARIANT SURFACES OF ORDINARY DIFFERENTIAL EQUATIONS
WITH AND WITHOUT TIME LAG

Berkeley, California

DISCLAIMER

INVARIANT SURFACES OF ORDINARY DIFFERENTIAL EQUATIONS
WITH AND WITHOUT TIME LAG

Anthony J. Schaeffer

Lawrence Radiation Laboratory
University of California
Berkeley, California

April 7, 1967

ABSTRACT

Consider the system of differential equations

$$(1) \qquad \dot{\theta}(t) = \Theta[\theta(t - r), x(t - r), \varepsilon]$$

$$(2) \qquad \dot{x}(t) = A[\theta(t - r), \varepsilon] \, x(t - r) + X[\theta(t - r), x(t - r), \varepsilon],$$

where $r = 0$ when there is no time lag. A mapping $S : R^m \to R^n$ is an Invariant Surface of (1, 2) if the pair $\{\psi(t), S[\psi(t)]\}$ is a solution of (1, 2) where $\psi(t)$ is a solution of the equation

$$(3) \qquad \dot{\theta}(t) = \Theta\{\theta(t - r), S[\theta(t - r)], \varepsilon\},$$

which must be defined for all time if $r > 0$. In this paper, for $\varepsilon = 0$, it is assumed that $S_0(\theta) \equiv 0$ is an invariant surface of (1, 2), and it is shown that under suitable conditions on Θ, A, and X there exists a unique invariant surface of (1, 2) near S_0 for ε small. The essential conditions on Θ, A, and X are that:

(a) All be differentiable with respect to x and θ;

(b) No characteristic roots of A ever be purely imaginary, and A not depend on θ if there is time lag;

(c) X have a small linear term, plus higher order terms, in x.

More general time lag terms are allowed in the paper and a systematic treatment of differential equations with time lag is given.

INTRODUCTION

This paper sharpens results of Sacker [1], Marica [3], Hale [13], and Diliberto [14] on the existence of Invariant Surfaces of a perturbed system of differential equations without time lag. It also extends these results to systems with time lag in a much stronger form than was done previously (Halany [9, pp 501-509]). The equations are considered in normal (or polar) form - that is, after a change of variables has been made that maps a known invariant surface into the zero vector. (The existence of such changes of variables is assumed here, as in previous works.) The explicit equations considered here are:

(1) $$\dot{\theta} = \Theta(\theta, x, y, z, \varepsilon)$$

(2) $$\dot{x} = A(\theta) x + X(\theta, x, y, z, \varepsilon)$$

(3) $$\dot{y} = \varepsilon \{B(\theta) y + Y(\theta, x, y, z, \varepsilon)\}$$

(4) $$\dot{z} = (\frac{1}{\varepsilon}) \{C(\theta) z + Z(\theta, x, y, z, \varepsilon)\},$$

and the known surface when $\varepsilon = 0$ is given by $\{x, y, z\} = 0$.

The work is divided into two parts; the first - Chapters I and II - is devoted to equations without time lag, and the second - Chapters III and IV - to equations with time lag. Lemmas and theorems are numbered by chapter and result so that 1.3 refers to Result 3 of Chapter 1. Corollaries are numbered with the main result number and a corollary number so that 1.3.2 is the second corollary to Result 1.3.

Chapter I is a collection of known results stated or derived in notation consistent with the rest of the paper. Theorem 1.1 gives rate of growth estimates for solutions of linear homogeneous time dependent equations when time dependent estimates are assumed on the eigenvalues. The estimates are

given both for time increasing and time decreasing. These estimates are used to derive Lemma 1.2 which asserts the existence of a unique bounded solution to a linear non-homogeneous time dependent equation with a bounded non-homogeneous term when the eigenvalues are never purely imaginary. This lemma is fundamental to the paper.

Chapter II contains the existence theorems for the first part. The main results, Theorem 2.2, asserts the existence of unique invariant surfaces of the system (1-4) for small $\varepsilon \neq 0$. We assume that A, B, C, Θ, X, Y, and Z are continuously differentiable in θ, x, y, z; $A(\theta)$, $B(\theta)$, and $C(\theta)$ never have purely imaginary eigenvalues; X, Y, and Z are quadratic in x, y, and z except for a small linear term; Θ is bounded; and the partial derivatives of A, B, C, Θ, X, Y, and Z must satisfy certain bounds.

In order to see clearly what our results say and to compare them with previous results, it is convenient to rewrite the equations (1 - 4) in more detail. We let w be the vector obtained by grouping x, y, and z as one; for example, $w = \{x, y\ z\}$. Then we have

$$(5) \qquad \dot{\theta} = \omega + \varepsilon[c(\theta) + \Theta_R(\theta, x, y, z, \varepsilon)]$$

$$(6) \qquad \dot{x} = A(\theta) x + [d_1(\theta) + D_1(\theta, x, y, z, \varepsilon) w + e_1(\theta, \varepsilon)]$$

$$(7) \qquad \dot{y} = \varepsilon \{B(\theta) y + [d_2(\theta) + D_2(\theta, x, y, z, \varepsilon) w + e_2(\theta, \varepsilon)]\}$$

$$(8) \qquad \dot{z} = (\frac{1}{\varepsilon}) \{C(\theta) z + [d_3(\theta) + D_3(\theta, x, y, z, \varepsilon) w + e_3(\theta, \varepsilon)]\}.$$

Our assumptions - in Theorem 2.2 - imply that $c(\theta)$ is small, $\Theta_R(\theta, 0, 0, 0, 0) = 0$; $d_i(\theta) \equiv 0$, $D_i(\theta, x, y, z, \varepsilon)$ is small, $\|e_i(\theta, \varepsilon)\| \to 0$ as $\varepsilon \to 0$, for $i = 1, 2, 3$.

Previous authors, except Sacker [1], have assumed that $\overline{c}(\theta) \equiv 0$ and $\overline{d}_i(\theta) \equiv 0$, where the bar indicates a mean value given by

$$\overline{c}(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T c(\theta_1 + \omega_1\tau, \ \theta_2 + \omega_2\tau, \ \ldots, \ \theta_n + \omega_n\tau) \ d\tau.$$

Marica [3] and Hale [13] exhibit a change of variables that transforms equations in their form to our form. Sacker imposes no condition on $d_i(\theta)$, but the surfaces he obtains are not continuous in ε - the perturbation parameter - at $\varepsilon = 0$. Simple examples demonstrate that this loss of continuity cannot be avoided if $\overline{d}_i(\theta) \neq 0$. (See Diliberto [14].) Previous results have required $D_i(\theta, 0, 0, 0, 0) \equiv 0$, while we only require that they be small. Diliberto and Marica also allow a cross linear term of "$\varepsilon E(\theta) x$" in the equation for $\dot{y}$. In Corollary 2.2.1, we allow more general cross linear terms. The essential requirement is that the extra linear terms do not destroy the estimates on the eigenvalues of $A(\theta)$, $B(\theta)$, and $C(\theta)$. In this paper, we assume that the equations are continuously differentiable and obtain Lipschitz continuous surfaces. Diliberto and Sacker require at least twice continuous differentiability to get continuously differentiable surfaces, while Hale and Marica assume Liptschitz equations and obtain Lipschitz surfaces. We also include a singular perturbation - $1/\varepsilon$ - term, which other authors have not.

Differential equations with time lag are introduced in Chapter III. These equations are treated as functionals from a Banach space to a Euclidian space. Theorem 3.1, Lemma 3.2, and Theorem 3.3 show existence, uniqueness - Gronwall's Lemma - and continuous dependency, respectively. Lemma 3.5 gives rate of growth estimates for autonomous linear homogeneous equations, similar to Theorem 1.1; and Theorem 3.6 generalizes Lemma 1.2 to equations with time lag. Theorem 3.14 treats Fréchét differentiation of solutions with respect to initial conditions. The results of Chapter III through Theorem 3.14 - with the exception of Theorem 3.6 - are extensions of well-known results to the time lag case. They are derived rigorously here, in consistent notation, because of a lack of a collection of such results in any one place.

The remainder of Chapter III is devoted to the initial value problem with time lag. Usually, in the time lag case, an initial function problem is treated, and then only for time increasing. Theorem 3.16 demonstrates the existence and uniqueness of a solution to the initial value problem in the class of functions defined for all time and having a certain maximum exponential rate of growth. The assertion is made under the assumption that the equation is uniformly Lipschitz, and that the Lipschitz constant times the time lag is less than $1/e$. Corollary 3.16.2 shows that this solution is the unique solution defined for all time if the right hand side of the equation is bounded. Corollary 3.16.3 and Theorem 3.17 show continuity and differentiability of this solution with respect to the initial value.

Chapter IV extends the results of Chapter II to equations with time lag. Here it is required that the linear part of $(2 - 4)$ - A, B, and C - be independent of θ, but the other assumptions are essentially the same. Our result is much stronger than the few previous results; for example, Halany [9, pp 501-509]. Halany considers only a scaler normal - x - perturbation term, does not allow time lag in the θ variable, allows lag of one fixed amount, and requires uniform asymptotic stability for the linear portion of the $\dot{x}$ equation. We, as before, allow normal, degenerate, and singular perturbation terms, and all may be vectors. We also allow time lag in θ, and the lag depends on the solution over an entire interval. We assume that the characteristic roots of the linear portions - A, B, and C - never be purely imaginary. Halany's surfaces are defined from the real line into a Banach space of continuous functions, while ours are defined from a Euclidian space to a Euclidian space. We have to require that the surfaces be defined for all time. This is not a severe restriction, in particular if the equations are periodic. In applications, the surfaces one is usually interested in are defined naturally for all time.

I. PRELIMINARIES IN ORDINARY DIFFERENTIAL EQUATIONS

This chapter is devoted to deriving and stating results that will be needed in the second chapter. These results are collected here to avoid interrupting later proofs.

Notation: If x and y are complex n-vectors and A is a complex n x n matrix, then we will use the notation: $x, y \in C^n$; $\langle x, y \rangle = \sum\limits_{i=1}^{n} x_i \bar{y}_i$, for the standard inner product; $|x| = \langle x, x \rangle^{\frac{1}{2}}$, for the norm of x; and $\|A\| = \sup\limits_{\|x\| \neq 0} \frac{|Ax|}{|x|}$, for the matrix norm.

The condition in the following lemma is used by Sacker [1], which stimulated this lemma. Part of the lemma is stated and proven by Hartman [2, pp. 56, 56].

Lemma 1.1: Let A(t) be a complex n x n matrix, continuous for $t \in \ell = [\alpha \ \beta] \subset R^1$. Assume that $\lambda_1(t)$ is continuous and

$$(1) \qquad \text{Re}\{\langle A(t) \ x, \ x \rangle\} \leq \lambda_1(t) \ |x|^2$$
$$t \in \ell, \ x \in C^n.$$
$$(2) \qquad \text{Re}\{\langle A(t) \ x, \ x \rangle\} \geq \lambda_2(t) \ |x|^2$$

Let $J(t, t_o)$ be the fundamental matrix associated with A(t). Then

$$(3) \qquad \|J(t, t_o)\| \leq \exp[\int_{t_o}^{t} \lambda_1(\tau) \ d\tau] \qquad , \quad \alpha \leq t_o \leq t \leq \beta$$

$$(4) \qquad \|J(t, t_o)\| \geq \exp[\int_{t_o}^{t} \lambda_1(\tau) \ d\tau] \qquad , \quad \alpha \leq t \leq t_o \leq \beta$$

$$(5) \qquad \|J(t, t_o)\| \leq \exp[\int_{t_o}^{t} \lambda_2(\tau) \ d\tau] \qquad , \quad \alpha \leq t \leq t_o \leq \beta$$

$$(6) \qquad \|J(t, t_o)\| \geq \exp[\int_{t_o}^{t} \lambda_2(\tau) \ d\tau] \qquad , \quad \alpha \leq t_o \leq t \leq \beta .$$

Remark: By the fundamental matrix associated with A(t), we mean the matrix solution $J(t, t_o)$ of

$$\dot{x}(t) = A(t) \ x(t)$$

that satisfies $J(t_o, t_o) = I$, where I is the identity matrix.

Proof: Estimates (3) and (5) follow directly from Hartman [2, p. 55], while (4) and (6) follow from (3) and (5) by the argument below given only for (4). We start by noting that

$$J(t, \tau) = J(t, t_o) \circ J(t_o, \tau) \qquad \text{for } t, \tau, t_o \in \ell,$$

and $\|B\| \geq \|B^{-1}\|^{-1}$ if B is an invertable matrix. Thus we have $J^{-1}(t_o, t) = J(t, t_o)$, and hence,

$$\|J^{-1}(t_o, t)\| = \|J(t, t_o)\| \leq \exp[\int_{t_o}^{t} \lambda_1(\tau) \, d\tau], \quad t_o \leq t \leq \beta.$$

Thus we have

$$\|J(t_o, t)\| \geq \|J^{-1}(t_o, t)\|^{-1} = \|J(t, t_o)\|^{-1}$$
$$\geq \exp[-\int_{t_o}^{t} \lambda_1(\tau) \, d\tau] \quad , \quad \alpha \leq t_o \leq t \leq \beta.$$

Interchanging the role of t and t_o gives (4).

Q.E.D.

The following lemma is essentially due to Marica [3, p. 8].

Lemma 1.2: Let $A(t)$ be a continuous complex $n \times n$ matrix defined for all $t \in R$ and satisfy

(7) $\text{Re}\{\langle A(t) \, x, \, x \rangle\} \leq \lambda \, |x|^2$ for $t \in R$, $x \in C^n$,

for some $\lambda < 0$. Assume that $f(t)$ is defined for all $t \in R$ into R^n and $|f(t)| \leq K$. Then there exists a unique bounded solution, $x^o(t)$, of

(8) $\dot{x}(t) = A(t) \, x(t) + f(t)$,

and $x^o(t)$ satisfies the equation

$$(9) \qquad x^O(t) = \int_{-\infty}^{O} J(t, \; t + \tau) \; f(t + \tau) \; d\tau,$$

where $J(t, \; t_o)$ is the fundamental matrix associated with $A(t)$.

Proof: The general solution of (8) is

$$(10) \qquad x(t) = J(t, \; 0) \{x_o + \int_{0}^{t} J^{-1}(\tau, \; 0) \; f(\tau) \; d\tau\}.$$

The integrals

$$(11) \qquad \int_{-\infty}^{O} J^{-1}(\tau, \; 0) \; f(\tau) \; d\tau,$$

$$(12) \qquad J(t, \; 0) \int_{-\infty}^{t} J^{-1}(\tau, \; 0) \; f(\tau) \; d\tau = \int_{-\infty}^{t} J(t, \; \tau) \; f(\tau) \; d\tau$$

exist and are uniformly bounded since

$$|f(t)| \leq K; \quad \|J^{-1}(\tau, \; 0)\| = \|J(0, \; \tau)\| \leq e^{-\lambda\tau}, \; \tau \leq 0;$$

and $\qquad \|J(t, \; 0) \circ J^{-1}(\tau, \; 0)\| = \|J(t, \; \tau)\| \leq e^{\lambda(t-\tau)}, \; \tau \leq t.$

Where the last two estimates follow from (7) and Lemma 1.1. Thus (10) can be written as

$$x(t) = J(t, \; 0) \{x_o - \int_{-\infty}^{O} J^{-1}(\tau, \; 0) \; f(\tau) \; d\tau\} + \int_{-\infty}^{t} J(t, \; \tau) \; f(\tau) \; d\tau.$$

The term inside the brackets is a constant and the second term is bounded, so for $x(t)$ to be bounded, we must have

$$x_o' = \int_{-\infty}^{O} J^{-1}(\tau, \; 0) \; f(\tau) \; d\tau,$$

since $\|J(t, \; 0)\| \to \infty$ as $t \to -\infty$. Thus x_o, and hence $x^O(t)$, are uniquely determined and $x^O(t)$ satisfies (9).

Q.E.D.

Corollary 1.2.1: If in Lemma 1.2 the condition (7) is replaced by

$$\text{Re}\{\langle A(t) \; x, \; x \rangle\} \geq \lambda \; |x|^2 \qquad \text{for} \quad t \in R, \; x \in C^n$$

for some $\lambda > 0$, then the same conclusion holds, but with $x^o(t)$ now given by

$$x^o(t) = -\int_0^\infty J(t, \; t + \tau) \; f(t + \tau) \; d\tau.$$

Proof: In (11) and (12), integrate to $+ \infty$ rather than $- \infty$ and proceed as before.

<div style="text-align:center">Q.E.D.</div>

Corollary 1.2.2: Let $A(t)$ be a continuous complex $n \times n$ matrix defined on R. Assume that

$$A(t) = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix},$$

where $A_1(t)$ and $A_2(t)$ satisfy

$$\text{Re}\{\langle A_1(t) \; \xi, \; \xi \rangle\} \leq -\lambda \; |\xi|^2 \; ,$$

$$\text{Re}\{\langle A_2(t) \; \xi, \; \xi \rangle\} \geq \lambda \; |\xi|^2$$

for all $t \in R$ and some $\lambda > 0$. Assume also that $X [t, x]$ is defined, bounded, continuous in t, and Lipshitz in x for $t \in R$, $x \in C^n$, $|x| \leq \rho$, into C^n. Then any bounded solution, $x^o(t)$, of the equation

$$\dot{x}(t) = A(t) \; x(t) + X [t, x(t)]$$

satisfies

$$x^o(t) = \begin{cases} \int_{-\infty}^0 J_1(t, \; t + \tau) \; X_1 [t + \tau, \; x^o(t + \tau)] \; d\tau \\ -\int_0^\infty J_2(t, \; t + \tau) \; X_2 [t + \tau, \; x^o(t + \tau)] \; d\tau \end{cases}$$

where J_i is the fundamental matrix association with A_i and $X = \{X_1, X_2\}$.

Lemma 1.3: (Gronwall's lemma) Let $\lambda(t)$ be a real continuous function and $\mu(t)$ a non-negative continuous function on the interval $[a, b]$. If a continuous function $y(t)$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s) \, y(s) \, ds$$

for $a \leq t \leq b$, then on the same interval

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s) \, \mu(s) \, \exp[\int_s^t \mu(\tau) \, d\tau] \, ds.$$

In particular, if $\lambda(t) \equiv \lambda$ is a constant,

$$y(t) \leq \lambda \, \exp[\int_a^t \mu(s) \, ds].$$

See Coppel [4, p. 19] for a proof.

Notation: Let $g(x)$ be a differentiable function from R^n into R^m. Then $\mathcal{D}_x \, g(x_o)$ will denote the Jacobian matrix $[\partial g/\partial x]$ evaluated at x_o.

Lemma 1.4: (Differentiability of Solutions) Let x be an n-vector, y an m-vector, and assume that $f(t, x, y)$, $\mathcal{D}_x f(t, x, y)$, and $\mathcal{D}_y f(t, x, y)$ are defined and continuous on all of $R \times R^n \times R^m$. Let $X(t, t_o, \phi, y)$ denote the solution of

$$\dot{x}(t) = f[t, x(t), y]$$

such that $X(t_o, t_o, \phi, y) = \phi$. Then the derivatives $\mathcal{D}_\phi \, X(t, t_o, \phi, y)$ and $\mathcal{D}_y \, X(t, t_o, \phi, y)$ exist for all t for which $X(t, t_o, \phi, y)$ exists. Also, $\mathcal{D}_\phi \, X(t, t_o, \phi, y)$ satisfies the matrix equation.

$$\dot{Z} = \mathcal{D}_x \, f \, [t, X(t, t_o, \phi, y), y] \circ Z,$$

where $\mathcal{D}_\phi \, X(t_o, t_o, \phi, y) = I$; and $\mathcal{D}_y \, X(t, t_o, \phi, y)$ satisfies

$$\dot{Z} = \mathcal{D}_x \, f[t, \, X(t, \, t_o, \, \phi, \, y), \, y] \circ Z \; + \mathcal{D}_y \, f[t, \, X(t, \, t_o, \, \phi, \, y), \, y]$$

where $\mathcal{D}_y \, X(t_o, \, t_o, \, \phi, \, y) = 0$.

The proof follows from Hartman [2, pp. 95-98].

II. INVARIANT SURFACES OF ORDINARY DIFFERENTIAL EQUATIONS

We now investigate systems of differential equations of the form

(1) $$\dot{\theta}(t) = \Theta[\theta(t), x(t), \varepsilon]$$

(2) $$\dot{x}(t) = A[\theta(t), \varepsilon] \circ x(t) + X[\theta(t), x(t), \varepsilon],$$

where θ and Θ are m-vectors; x and X are n-vectors; and Θ, A, X are defined for all $\theta \in R^m$, $x \in R^n$, $|x| \le \alpha$, $|\varepsilon| \le \beta$. We will assume that Θ, A, and X are at least continuously differentiable in θ and x, and continuous in ε.

Definition: A Lipschitz continuous function $S(\theta)$ from R^m into R^n is called an Invariant Surface of the system (1, 2) if for fixed ε,

$$\theta(t) = \psi(t, \theta^o, S)$$

$$x(t) = S[\psi(t, \theta^o, S)]$$

is a solution of (1, 2), where $\psi(t, \theta^o, S)$ is the solution of

(3) $$\dot{\theta}(t) = \Theta\{\theta(t), S[\theta(t)], \varepsilon\},$$

(4) $$\psi(0, \theta^o, S) = \theta^o.$$

We shall show that under suitable conditions on Θ, A, and X, there exists an invariant surface of (1, 2), unique in some class of functions for $|\varepsilon|$ small but non-zero. In proving this result, we will:

1. Derive an integral equation that an invariant surface must satisfy.

2. Use this integral equation to define a mapping whose fixed points are invariant surfaces.

3. Show that the above mapping is a contraction.

The method originated with Marica [3]. Step 1 is done in Lemma 2.1, and Steps 2 and 3 are done in Theorem 2.2.

Lemma 2.1: Given the system (1, 2), assume $A(\theta, \varepsilon)$ can be written

$$A(\theta, \varepsilon) = \begin{bmatrix} A_1(\theta, \varepsilon) & 0 \\ 0 & A_2(\theta, \varepsilon) \end{bmatrix},$$

and that there exists a $\lambda > 0$ such that

(5) $$\langle A_1(\theta, \varepsilon) x_1, x_1 \rangle \leq -\varepsilon \lambda |x_1|^2$$

(6) $$\langle A_2(\theta, \varepsilon) x_2, x_2 \rangle \geq \varepsilon \lambda |x_2|^2 .$$

If for some $\varepsilon \neq 0$, there exists an invariant surface $S(\theta)$ of (1, 2), $S(\theta) = \{S_1(\theta), S_2(\theta)\}$, then

$$S_1(\theta) = \int_{-\infty}^{0} J_1^{-1}(\tau, \theta, \psi) \circ X_1 \{\psi(\tau, \theta, S), S [\psi(\tau, \theta, S)], \varepsilon\} d\tau$$

$$S_2(\theta) = -\int_{0}^{\infty} J_2^{-1}(\tau, \theta, \psi) \circ X_2 \{\psi(\tau, \theta, S), S [\psi(\tau, \theta, S)], \varepsilon\} d\tau,$$

where $\psi(\tau, \theta^o, S)$ satisfies (3, 4), and $J_i(\tau, \theta, \psi)$ is the fundamental matrix associated with $A_i [\psi(t, \theta, S), \varepsilon]$ such that $J(0, \theta, \psi) = I$.

Remark: A splitting of x, X, and S into two components correspondent to the assumed splitting of A is used implicitly in this lemma.

Proof: The proof will be carried out only in the case when $A = A_2$. This involves no loss of generality, and the extension to the general case will be clear. To simplify the notation, the subscript "2" will be omitted. With the simplification, (5) and (6) become

(7) $$\langle A(\theta, \varepsilon) x, x \rangle \geq \varepsilon \lambda |x|^2 .$$

Since $S(\theta)$ is assumed to exist, $\psi(t, \theta^0, S)$ is known, and $x(t) = S[\psi(t, \theta^0, S)]$ satisfies the equation (2) with $\theta(t) = \psi(t, \theta^0, S)$. By Corollary 1.2.2, it follows that

$$S[\psi(t, \theta^0, S)] = -\int_0^\infty J(t, t + \tau) \circ X \{\psi(t + \tau, \theta^0, S), S[\psi(t + \tau, \theta^0, S)], \varepsilon\} \, d\tau,$$

where $J(t, t_0)$ is the fundamental matrix associated with $A[\psi(t, \theta^0, S), \varepsilon]$. Fix t and let $\theta = \psi(t, \theta^0, S)$; then

$$\psi(t + \tau, \theta^0, S) = \psi[\tau, \psi(t, \theta^0, S), S] = \psi(\tau, \theta, S),$$

since (3) is autonomous. Thus

$$S(\theta) = -\int_0^\infty J(t, t + \tau) \circ X \{\psi(\tau, \theta, S), S[\psi(\tau, \theta, S)], \varepsilon\} \, d\tau.$$

We know that

$$(8) \qquad J(t + \sigma, t + \tau) = I + \int_{t+\tau}^{t+\sigma} A[\psi(\mu, \theta^0, S), \varepsilon] \circ J(\mu, t + \tau) \, d\mu$$

$$= I + \int_\tau^\sigma A[\psi(\mu, \theta, S), \varepsilon] \circ J(t + \mu, t + \tau) \, d\mu.$$

In this form, the dependence of J on σ, τ, θ, and ψ is apparent, so to emphasize this dependency, we write

$$J(t + \sigma, t + \tau) = J(\sigma, \tau, \theta, \psi),$$

and specifically,

$$J(t, t + \tau) = J(0, \tau, \theta, \psi) = J^{-1}(\tau, 0, \theta, \psi) = J^{-1}(\tau, \theta, \psi).$$

It is now clear that $J(\tau, \theta, \psi)$ is the fundamental matrix associated with $A[\psi(\tau, \theta, \psi), \varepsilon]$ and that $J(0, \theta, \psi) = I$. Thus we obtain

$$S(\theta) = -\int_0^\infty J^{-1}(\tau, \theta, \psi) \circ X \{\psi(\tau, \theta, S), S[\psi(\tau, \theta, S)], \varepsilon\} \, d\tau.$$

This expression makes sense for all θ, independently of any choice of θ^o or t.

Q.E.D.

Notation: Let $f(\theta)$ be a scaler, vector, or matrix valued function defined for all $\theta \in R^m$. We will use the norm

$$\|f\| = \sup_{\theta \in R^m} |f(\theta)|,$$

where $|f(\theta)|$ is the absolute value of a scaler, or the Euclidian norm of a matrix or vector. Also, we will use the Banach spaces

$$\Omega(\rho_1) = \{S: R^m \to R^n \quad : \quad \|S\| \leq \rho_1\},$$

and the subspaces of $\Omega(\rho_1)$, $\Omega(\rho_1, \rho_2) = \{S \in \Omega(\rho_1) : \mathcal{D}_\theta S(\theta)$ is continuous and $\|\mathcal{D}_\theta S\| \leq \rho_2\}$.

Theorem 2.2: Consider the real system of equations

(9) $$\dot{\theta}(t) = \Theta[\theta(t), x(t), \varepsilon]$$

(10) $$\dot{x}(t) = E(\varepsilon) \circ \{A[\theta(t)] \circ x(t) + X[\theta(t), x(t), \varepsilon]\},$$

where θ, Θ are m-vectors, x and X are n-vectors, and E and A are $n \times n$ matrices. Assume that Θ, A, X are defined and bounded for all $\theta \in R^m$, $x \in R^n$, $|x| \leq \alpha$, $|\varepsilon| \leq \beta$. Assume also that:

(a) $E(\varepsilon)$ is the matrix

$$E(\varepsilon) = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & \varepsilon I_2 & 0 \\ 0 & 0 & (1/\varepsilon)I_3 \end{bmatrix}.$$

(b) $A(\theta)$ is the matrix

$$A(\theta) \;=\; \begin{bmatrix} A_1(\theta) & 0 & 0 \\ 0 & A_2(\theta) & 0 \\ 0 & 0 & A_3(\theta) \end{bmatrix},$$

and there exists a $\lambda > 0$ and matrices $A_{ij}(\theta)$ ($i = 1, 2, 3$; $j = 1, 2$) such that

$$A_i(\theta) \;=\; \begin{bmatrix} A_{i1}(\theta) & 0 \\ 0 & A_{i2}(\theta) \end{bmatrix} \quad i = 1, 2, 3;$$

and the inequalities

(11)
$$\langle A_{i1}(\theta)\, \xi_1, \; \xi_1 \rangle \leq -\lambda\, |\xi_1|^2$$
$$\langle A_{i2}(\theta)\, \xi_2, \; \xi_2 \rangle \geq \lambda\, |\xi_2|^2$$
$$i = 1, 2, 3$$

are satisfied for all $\theta \in R^m$. Also, there exists a constant $c_1 \geq 0$ such that

$$\| \mathcal{D}_\theta\, A \| \leq c_1.$$

(c) $\| \mathcal{D}_\theta\, \Theta(\cdot, \; x, \; \varepsilon) \| \leq \varepsilon \lambda / 2$

$\| \mathcal{D}_\theta\, \Theta(\cdot, \; x, \; \varepsilon) \| \leq \varepsilon c_2$

for some constant $c_2 \geq 0$.

(d) $\| X(\cdot, \; x, \; \varepsilon) \| \leq c_3\, |x| + c_4(\varepsilon)$

$\| \mathcal{D}_\theta\, X(\cdot, \; x, \; \varepsilon) \| \leq c_3\, |x| + c_4(\varepsilon)$

$\| \mathcal{D}_\theta\, X(\cdot, \; x, \; \varepsilon) \| \leq c_3,$

where $0 \leq c_3 \leq \lambda / 18$ and $c_4(\varepsilon) \to 0$ as $|\varepsilon| \to 0$.

Then there exists a $\rho_2{}^0$, $\rho_1{}^0$ ($\rho_2{}^0$), ε^0 ($\rho_1{}^0$, $\rho_2{}^0$) such that for

$$0 < |\varepsilon| \leqq \varepsilon^0$$

there exists a unique invariant surface $S(\theta)$ of (9, 10) in $\Omega(\rho_1{}^0)$ that is

Lipschitz continuous with constant $\rho_2{}^0$, and $\|S\| \to 0$ as $\|\varepsilon\| \to 0$.

Proof: We will, as in Lemma 2.1, assume that the A_{i1} terms are not

present; thus (11) becomes

(12) $$\langle A_i(\theta)\, \xi,\, \xi \rangle \geqq \lambda \,|\xi|^2 \quad (i = 1, 2, 3).$$

Assume that for some $\varepsilon \neq 0$, $\rho_1 > 0$, and $\rho_2 > 0$, there exists an

$S(\theta) \in \Omega(\rho_1, \rho_2)$ that is an invariant surface of the system (9, 10). Then,

by Lemma 2.1, $S(\theta)$ satisfies

(13) $$S(\theta) = -\int_0^\infty E(\varepsilon) \circ J_E^{-1}(\tau,\, \theta,\, \psi) \circ X\, \{\psi(\tau),\, S[\psi(\tau)],\, \varepsilon\}\, d\tau,$$

where $\psi(\tau) = \psi(\tau,\, \theta,\, S)$ satisfies

(14) $$\dot\theta(t) = \Theta\{\theta(t),\, S[\theta(t)],\, \varepsilon\}, \qquad \psi(0,\, \theta,\, S) = \theta,$$

and $J_E(\tau,\, \theta,\, \psi)$ is the fundamental matrix associated with $E(\varepsilon) \circ A[\psi(\tau,\, \theta,\, \psi)]$

such that $J_E(0,\, \theta,\, \psi) = I$. Equation (13) has three terms, corresponding to

the "1", "ε", and "$1/\varepsilon$" in $E(\varepsilon)$; which will be called the "normal", "degenerate",

and "singular" terms, respectively, when it is necessary to distinguish them.

If, in the degenerate term, we make the change of integration variable $\tau \to {}^\tau/\varepsilon$

and in the singular term the change $\tau \to \varepsilon\tau$, then (13) becomes

(15) $$S(\theta) = -\int_0^\infty J^{-1}(\tau,\, \theta,\, \psi,\, \varepsilon) \circ X\, \{\psi(\tau,\, \varepsilon),\, S[\psi(\tau,\, \varepsilon)],\, \varepsilon\}\, d\tau,$$

where

$$\psi(\tau, \varepsilon) = \begin{cases} \psi(\tau, \theta, S) & \text{normal} \\ \psi(^{\tau}/\varepsilon, \theta, S) & \text{in the} \quad \text{degenerate} \quad \text{term} \\ \psi(\varepsilon\tau, \theta, S) & \text{singular} \end{cases},$$

and $J(\tau, \theta, \psi, \varepsilon) = J(\tau, 0, \theta, \psi, \varepsilon)$ is the fundamental matrix associated with

$$(16) \qquad A[\psi(\tau, \varepsilon)] = \begin{cases} A_1[\psi(\tau)] & 0 & 0 \\ 0 & A_2[\psi(^{\tau}/\varepsilon)] & 0 \\ 0 & 0 & A_3[\psi(\varepsilon\tau)] \end{cases}.$$

By (12), we have

$$(17) \qquad \langle A[\psi(\tau, \theta, S, \varepsilon)] \, x, x \rangle \geqq \lambda \, |x|^2$$

for all $\theta \in R^m$, $\tau \in R$, $x \in R^n$; thus, by Lemma 1.1, we know that

$$(18) \qquad \|J^{-1}(\tau, \theta, \psi, \varepsilon)\| = \|J(0, \tau, \theta, \psi, \varepsilon)\| \leqq e^{-\lambda \tau}, \quad \tau \geqq 0.$$

The sup norm is valid here, since (17) holds uniformly for $\theta \in R^m$.

Now, we define a mapping T_ε on $\Omega(\infty, \infty)$ from equation (15) by

$$(19) \qquad [T_\varepsilon S](\theta) = -\int_0^\infty J^{-1}(\tau, \theta, \psi, \varepsilon) \circ X \{\psi(\tau, \varepsilon), S[\psi(\tau, \varepsilon)], \varepsilon\} \, d\tau.$$

Our procedure is to show for suitable ε, ρ_1 and ρ_2 that T_ε maps $\Omega(\rho_1, \rho_2)$ into itself and is a contraction. Thus T_ε will have a unique fixed point in $\Omega(\rho_1)$ which clearly is an invariant surface.

We start by showing $\|T_\varepsilon S\| \leqq \rho_1$ if $\|S\| \leqq \rho_1$ for suitable ρ_1 and ε.

$$\|T_\varepsilon S\| = \sup_\theta \left| \int_0^\infty J^{-1}(\tau, \theta, \psi, \varepsilon) \circ X \{\psi(\tau, \varepsilon), S[\psi(\tau, \varepsilon)], \varepsilon\} \, d\tau \right|$$

$$\leqq \int_0^\infty \|J^{-1}(\tau, \cdot, \psi, \varepsilon)\| \, \|X\| \, d\tau.$$

From assumption (d), we have

$$\|X\| = \|X \{\cdot, \ S(\psi), \ \varepsilon\}\| \leq c_3 \|S\| + c_4(\varepsilon).$$

Using this and (18) yields

(20) $\quad \|T_\varepsilon S\| \leq [c_3 \rho_1 + c_4(\varepsilon)] \int_0^\infty e^{-\lambda \tau} d\tau \leq [c_3 \rho_1 + c_4(\varepsilon)]/\lambda,$

since $\|S\| \leq \rho_1$. But $c_3/\lambda \leq {}^1\!/18$ by assumption and $c_4(\varepsilon) \to 0$ as $|\varepsilon| \to 0$, so $\|T_\varepsilon S\| \leq \rho_1$ if $\|S\| \leq \rho_1$ and ε_0 is small enough that

(21) $\qquad\qquad\qquad c_4(\varepsilon) \leq 17 \lambda \rho_1/18, \ \text{for} \ |\varepsilon| \leq \varepsilon_0.$

Now we show that $\|D_\theta T_\varepsilon\| \leq \rho_2$ if $\|S\| \leq \rho_1$, $\|D_\theta S\| \leq \rho_2$ for a suitable choice of ρ_1, ρ_2, and ε. Differentiating (19) yields

(22) $\quad [D_\theta T_\varepsilon S] (\theta) =$

$$= - \int_0^\infty \left\{ \begin{array}{l} D_\theta J^{-1}(\tau, \theta, \psi, \varepsilon) \circ X \{\psi, S[\psi], \varepsilon\} \\[6pt] + J^{-1}(\tau, \theta, \psi, \varepsilon) \circ D_\theta X \{\psi, S[\psi], \varepsilon\} \circ D_\theta \psi \\[6pt] + J^{-1}(\tau, \theta, \psi, \varepsilon) \circ D_x X \{\psi, S[\psi], \varepsilon\} \circ D_\theta S \circ D_\theta \psi \end{array} \right\} d\tau$$

by the Chain Rule, where $\psi = \psi(\tau, \varepsilon)$. To proceed we will need estimates on $\|D_\theta \psi(\tau, \varepsilon)\|$ and $\|D_\theta J^{-1}(\tau, \theta, \psi, \varepsilon)\|$. We will compute an estimate on $\|D_\theta \psi(\tau)\| = \|D_\theta \psi(\tau, \theta, S)\|$ (that is, before the change of integration variable) first. From Lemma 1.4 and (14), we obtain

$$D_\theta \psi(\tau, \theta, S) =$$

$$= I + \int_0^\tau \left\{ \begin{array}{l} D_\theta \Theta \{\psi(\mu), S[\psi(\mu)], \varepsilon\} \circ D_\theta \psi(\mu) \\[6pt] + D_x \Theta \{\psi(\mu), S[\psi(\mu)], \varepsilon\} \circ D_\theta S[\psi(\mu)] \circ D_\theta \psi(\mu) \end{array} \right\} d\mu.$$

Thus, taking norms yields

$$\|D_\theta \psi(\tau)\| \leq 1 + \int_0^\tau (\|D_\theta \Theta\| + \|D_x \Theta\| \|D_\theta S\|) \|D_\theta \psi(\mu)\| \, d\mu,$$

to which we apply Gronwall's lemma to obtain

$$\|\mathcal{D}_\theta \, \psi(\tau)\| \leq \exp \{[\|\mathcal{D}_\theta \, \Theta\| + \|\mathcal{D}_x \, \Theta\| \, \|\mathcal{D}_\theta \, S\|] \, \tau\}.$$

Putting the assumed estimates (c) into this yields

$$\|\mathcal{D}_\theta \, \psi(\tau)\| \leq \exp \{\varepsilon[\lambda/2 + c_2 \, \rho_2] \, \tau\}, \qquad \tau \geq 0,$$

since $\|\mathcal{D}_\theta \, S\| \leq \rho_2$. Let

$$\gamma = \lambda/2 + c_2 \, \rho_2.$$

Later we will need $\gamma < \lambda$, so we require

(23) $$\rho_2 \leq \lambda/(3c_2),$$

and then we have

(24) $$\gamma \leq 5\lambda/6 < \lambda,$$

and the estimate

(25) $$\|\mathcal{D}_\theta \, \psi(\tau)\| \leq \exp(\varepsilon \, \gamma \, \tau) \leq \exp(5 \, \varepsilon \, \lambda \, \tau/6).$$

After taking norms, the change of integrations variables affects only $\mathcal{D}_\theta \, \psi$ in (22) and most severely in the degenerate term. For this term, $\tau \to \tau/\varepsilon$, so (25) must be replaced by

(26) $$\|\mathcal{D}_\theta \, \psi(\tau, \, \varepsilon)\| \leq \exp(\gamma \, \tau) \leq \exp(5 \, \lambda \, \tau/6), \qquad \tau \geq 0.$$

This estimate will be used in all terms for notational ease.

Now we need to estimate $\|\mathcal{D}_\theta \, J^{-1}(\tau, \, \theta, \, \psi)\|$. Recall from (16) that $J^{-1}(\tau, \, \theta, \, \psi, \, \varepsilon) = J(0, \, \tau, \, \theta, \, \psi, \, \varepsilon)$, and is the matrix solution of

$$\dot{x}(t) = A[\psi(\tau, \, \varepsilon)] \, x(t)$$

such that $J(\tau, \tau, \theta, \psi, \varepsilon) = I$. Applying Lemma 1.4 to the columns of J, one sees that $\mathcal{D}_\theta$ J must satisfy

$$(27) \quad \frac{d}{dt} [\mathcal{D}_\theta J(t, \tau, \theta, \psi, \varepsilon)] = A[\psi(t, \theta, S, \varepsilon)] \circ \mathcal{D}_\theta J(t, \tau, \theta, \psi, \varepsilon)$$

$$+ \{\mathcal{D}_\theta A[\psi(t, \theta, S, \varepsilon)] \circ \mathcal{D}_\theta \psi(t, \theta, S, \varepsilon)\} \circ J(t, \tau, \theta, \psi, \varepsilon),$$

and that $\mathcal{D}_\theta J(\tau, \tau, \theta, \psi, \varepsilon) = 0$.

The notation used here requires caution because many subscripts are concealed in it. $\mathcal{D}_\theta A(\cdot)$ and $\mathcal{D}_\theta J(\cdot)$ are linear maps from R^m into the m x m matrices. Thus $\mathcal{D}_\theta A(\cdot) \circ \mathcal{D}_\theta \psi(\cdot)$ is still a linear map from R^m into the n x n matrices. The resulting matrix is then multiplied by $J(\cdot)$.

Equation (27) is a linear non-homogenous differential equation and its solution is

$$D_\theta J(t, \tau, \theta, \psi, \varepsilon) =$$

$$\int_\tau^t J(t, \sigma, \theta, \psi, \varepsilon) \circ \mathcal{D}_\theta A[\psi(\sigma, \varepsilon)] \circ \mathcal{D}_\theta \psi(\sigma, \varepsilon) \circ J(\sigma, \tau, \theta, \psi, \varepsilon) \, d\sigma$$

by the standard method of solving such equations. Taking norms (sup over θ), we obtain

$$\|\mathcal{D}_\theta J(t, \tau, \cdot, \psi, \varepsilon)\| \leq$$

$$\leq \left| \int_\tau^t \|J(t, \sigma, \cdot, \psi, \varepsilon)\| \, \|\mathcal{D}_\theta A\| \, \|\mathcal{D}_\theta \psi(\sigma, \varepsilon)\| \, \|J(\sigma, \tau, \cdot, \psi, \varepsilon)\| \, d\sigma \right|,$$

and when $t = 0$ and $\tau \geq 0$,

$$(28) \quad \|\mathcal{D}_\theta J^{-1}(\tau, \cdot, \psi, \varepsilon)\| = \|\mathcal{D}_\theta J(0, \tau, \cdot, \psi, \varepsilon)\| \leq$$

$$\leq \|\mathcal{D}_\theta A\| \int_0^\tau \|J(0, \sigma, \cdot, \psi, \varepsilon)\| \, \|\mathcal{D}_\theta \psi(\sigma, \varepsilon)\| \, \|J(\sigma, \tau, \cdot, \psi, \varepsilon)\| \, d\sigma.$$

By Lemma 1.1 and (17) (see the remarks around (18)), we have that

(29) $$\|J(\sigma, \tau, \cdot, \psi, \varepsilon)\| \le \exp \lambda(\sigma - \tau), \quad \text{for } \sigma \le \tau.$$

Putting this, (26), and the estimate $\|\mathcal{D}_\theta A\| \le c_1$ assumed in (b) into (28) yields

(30) $$\|\mathcal{D}_\theta J^{-1}(\tau, \cdot, \psi, \varepsilon)\| \le c_1 \int_0^\tau e^{-\lambda\sigma} e^{\gamma\tau} e^{\lambda(\sigma-\tau)} d\sigma \le c_1 e^{-\lambda\tau}(e^{\gamma\tau}-1)/\gamma.$$

We now take norms in (22) and then use estimates (18), (26), and (30) to obtain

$$\|\mathcal{D}_\theta T_\varepsilon S\| \le \int_{0-}^\infty \left\{ \begin{array}{l} c_1 e^{-\lambda\tau}(e^{\gamma\tau} - 1) \|x\|/\gamma \\[2mm] + e^{-\lambda\tau} \|\mathcal{D}_\theta x\| e^{\gamma\tau} \\[2mm] e^{-\lambda\tau} \|\mathcal{D}_x x\| \|\mathcal{D}_\theta S\| e^{\gamma\tau} \end{array} \right\} d\tau.$$

This integral converges since $\gamma \le 5\lambda/6 < \lambda$ (see 24). Thus

$$\|\mathcal{D}_\theta T_\varepsilon S\| \le$$

$$\le [\frac{c_1}{\lambda} \|x\| + \|\mathcal{D}_\theta x\| + \|\mathcal{D}_\theta S\| \|\mathcal{D}_x x\|]/[\lambda - \gamma]$$

$$\le \{\frac{c_1}{\lambda} [\rho_1 c_3 + c_4(\varepsilon)] + \rho_1 c_3 + c_4(\varepsilon) + \rho_2 c_3\}/\{\lambda - \gamma\}$$

$$\le \{(1 + c_1/\lambda) [\rho_1 c_3 + c_4(\varepsilon)] + \rho_2 c_3\} \{6/\lambda\},$$

by (24) and assumptions (d). But in (d) we also assumed that $c_3/\lambda \le {}^1/18$, so that $6 \rho_2 c_3/\lambda \le \frac{\rho_2}{3}$. Also, if

(31) $$\rho_1 \le \lambda^2 \rho_2/[18 c_3(\lambda + c_1)]$$

and ε_0 is small enough that

(32) $$c_4(\varepsilon) \le \lambda^2 \rho_2/[18(\lambda + c_1)] \text{ for } |\varepsilon| \le \varepsilon_0,$$

then for $\|S\| \le \rho_1 \|\mathcal{D}_\theta S\| \le \rho_2 \le \lambda/(3 c_2)$, we have

$$\|\mathcal{D}_\theta \, \dot{T}_\varepsilon \, S\| \leq \frac{\rho_2}{3} + \frac{\rho_2}{3} + \frac{\rho_2}{3} = \rho_2.$$

Thus we have shown that, if (31) and (23) hold and $|\varepsilon|$ is small enough, then T_ε maps $\Omega(\rho_1, \rho_2)$ into itself.

Continuing on our outlined procedure, we now have to show for suitable ρ_1, ρ_2, ε that T_ε is a contraction. To this end, let ρ_1 and ρ_2 satisfy (31) and (23) and let S^1, $S^2 \in \Omega(\rho_1, \rho_2)$. Let $\psi^1(t, \varepsilon) = \psi^1(t, \theta, S^1, \varepsilon)$ be the solution of (14) with $S = S^1$, and $J_i(t, \varepsilon) = J_i(t, \theta, \psi^1, \varepsilon)$ be the fundamental matrix associated with $A[\psi^1(t, \varepsilon)]$. We wish to estimate $\|T_\varepsilon \, S^1 - T_\varepsilon \, S^2\|$. From (19), it follows that

$$[T_\varepsilon \, S^1](\theta) - [T_\varepsilon \, S^2](\theta) =$$

$$= -\int_0^\infty \left\{ \begin{array}{l} J_1^{-1}(\tau, \varepsilon) \circ X \{\psi^1(\tau, \varepsilon), S^1[\psi^1(\tau, \varepsilon)], \varepsilon\} \\[2mm] -J_2^{-1}(\tau, \varepsilon) \circ X \{\psi^2(\tau, \varepsilon), S^2[\psi^2(\tau, \varepsilon)], \varepsilon\} \end{array} \right\} d\tau$$

$$= -\int_0^\infty \left\{ \begin{array}{l} \{J_1^{-1}(\tau, \varepsilon) - J_2^{-1}(\tau, \varepsilon)\} \circ X[\psi^1, S^1(\psi^1), \varepsilon] \\[1mm] + J_2^{-1}(\tau, \varepsilon) \circ \{X[\psi^1, S^1(\psi^1), \varepsilon] - X[\psi^2, S^1(\psi^1), \varepsilon]\} \\[1mm] + J_2^{-1}(\tau, \varepsilon) \circ \{X[\psi^2, S^1(\psi^1), \varepsilon] - X[\psi^2, S^1(\psi^2), \varepsilon]\} \\[1mm] + J_2^{-1}(\tau, \varepsilon) \circ \{X[\psi^2, S^1(\psi^2), \varepsilon] - X[\psi^2, S^2(\psi^2), \varepsilon]\} \end{array} \right\} d\tau.$$

Thus,

(33) $$\|T_\varepsilon \, S^1 - T_\varepsilon \, S^2\| \leq$$

$$\leq \int_0^\infty \left\{ \begin{array}{l} \|J_1^{-1}(\tau, \varepsilon) - J_2^{-1}(\tau, \varepsilon)\| \, \|X\| \\[1mm] + \|J_2^{-1}(\tau, \varepsilon)\| \, \|\mathcal{D}_\theta X\| \, \|\psi^1(\tau, \varepsilon) - \psi^2(\tau, \varepsilon)\| \\[1mm] + \|J_2^{-1}(\tau, \varepsilon)\| \, \|\mathcal{D}_x X\| \, \|\mathcal{D}_\theta S^1\| \, \|\psi^1(\tau, \varepsilon) - \psi^2(\tau, \varepsilon)\| \\[1mm] + \|J_2^{-1}(\tau, \varepsilon)\| \, \|\mathcal{D}_x X\| \, \|S^1 - S^2\| \end{array} \right\} d\tau.$$

Now we have to estimate $\|\psi^1(\tau, \varepsilon) - \psi^2(\tau, \varepsilon)\|$ and $\|J_1^{-1}(\tau, \varepsilon) - J_2^{-1}(\tau, \varepsilon)\|$. As before, we will make the estimate on ψ before the change of variables, then make the change and use the worst case estimate. By (14)

$$\|\psi^1(t, \theta, s^1) - \psi^2(t, \theta, s^2)\| \leq$$

$$\leq \int_o^t \|\Theta \{\psi^1(\tau), s^1[\psi^1(\tau)], \varepsilon\} - \Theta \{\psi^2(\tau), s^2[\psi^2(\tau)], \varepsilon\}\| \, d\tau$$

$$\leq \int_o^t \{\|D_\theta \Theta\| + \|D_x \Theta\| \|D_\theta s^1\|\} \|\psi^1(\tau) - \psi^2(\tau)\| + \|D_x \Theta\| \|s^1 - s^2\| \, d\tau$$

$$\leq \|D_x \Theta\| \|s^1 - s^2\| t + \{\|D_\theta \Theta\| + \|D_x \Theta\| \|D_\theta s^1\|\} \int_o^t \|\psi^1(\tau) - \psi^2(\tau)\| \, d\tau$$

$$\leq \varepsilon c_2 \|s^1 - s^2\| t + \{\varepsilon(\lambda/2 + \rho_2 c_2)\} \int_o^t \|\psi^1(\tau) - \psi^2(\tau)\| \, d\tau.$$

Applying Gronwall's lemma and recalling the definition of γ (before (23)), we obtain

$$\|\psi^1(t) - \psi^2(t)\| \leq c_2 \|s^1 - s^2\| [e^{\varepsilon\gamma t} - 1]/\gamma, \quad t \geq 0.$$

After making the change of variables, this becomes

$$(34) \qquad \|\psi^1(t, \varepsilon) - \psi^2(t, \varepsilon)\| \leq c_2 \|s^1 - s^2\| [e^{\gamma t} - 1]/\gamma, \quad t \geq 0.$$

To estimate $\|J_1^{-1}(\tau, \varepsilon) - J_2^{-1}(\tau, \varepsilon)\|$, let

$$\Delta J(t, \tau) = J_1(t, \tau, \theta, \psi^1, \varepsilon) - J_2(t, \tau, \theta, \psi^2, \varepsilon).$$

Then $\Delta J(t, \tau)$ satisfies the linear non-homogenous matrix differential equation

$$\frac{d}{dt} \Delta J(t, \tau) = A[\psi^1(t, \varepsilon)] \circ J_1(t, \tau, \theta, \psi^1, \varepsilon)$$

$$- A[\psi^2(t, \varepsilon)] \circ J_2(t, \tau, \theta, \psi^2, \varepsilon)$$

$$= A[\psi^1(t, \varepsilon)] \circ \Delta J(t, \tau) + \Delta A(t) \circ J_2(t, \tau, \theta, \psi^2, \varepsilon),$$

with the initial condition $\Delta J(\tau, \tau) = 0$, where

$$\Delta A(t) = A[\psi^1(t, \varepsilon)] - A[\psi^2(t, \varepsilon)].$$

The solution of this equation is

$$\Delta J(t, \tau) = \int_{\tau}^{t} J_1(t, \sigma, \theta, \psi^1, \varepsilon) \circ \Delta A(\sigma) \circ J_2(\sigma, \tau, \theta, \psi^2, \varepsilon) \, d\sigma.$$

Thus, by taking norms and setting $t = 0$, we obtain

$$\|\Delta J(0, \tau)\| = \|J_1(0, \tau, \cdot, \psi^1, \varepsilon) - J_2(0, \tau, \cdot, \psi^2, \varepsilon)\| \leqq$$

$$\leqq \int_{0}^{\tau} \|J_1(0, \sigma, \cdot, \psi^1, \varepsilon)\| \ \|\Delta A(\sigma)\| \ \|J_2(\sigma, \tau, \cdot, \psi^2, \varepsilon)\| \, d\sigma.$$

We have by the mean value theorem that

$$\|\Delta A(\sigma)\| \leqq \|D_\theta A\| \ \|\psi^1(\sigma, \varepsilon) - \psi^2(\sigma, \varepsilon)\|,$$

and thus

(35) $\qquad \|\Delta J(0, \tau)\| \leqq$

$$\leqq \frac{c_1 \, c_2 \, \|s^1 - s^2\|}{\gamma} \int_{0}^{\tau} e^{-\lambda\sigma} \, [e^{\gamma\sigma} - 1] \, e^{\lambda(\sigma-\tau)} \, d\sigma$$

$$= \frac{c_1 \, c_2 \, \|s^1 - s^2\|}{\gamma} \ e^{-\lambda\tau} \left[\frac{e^{\gamma\tau} - 1 - \gamma\tau}{\gamma} \right], \ \tau \geqq 0,$$

where (34) and (b) have been used. Clearly we have

$$\|J_1^{-1}(\tau, \varepsilon) - J_2^{-1}(\tau, \varepsilon)\| = \|\Delta J(0, \tau)\|.$$

Finally, we take (34) and (35), factor out $\|s^1 - s^2\|$ and $e^{-\lambda\tau}$ to obtain

$$\|T_\varepsilon s^1 - T_\varepsilon s^2\| \leqq \|s^1 - s^2\| \int_{0}^{\infty} e^{-\lambda\tau} \ F(\tau) \, d\tau$$

in (33), where

$$F(\tau) = \left\{ \begin{array}{l} c_1\, c_2\, \|X\|\ [e^{\gamma\tau} - 1 - \gamma\tau]/\gamma^2 \\ +\ c_2[\|\mathcal{D}_\theta\, X\| + \|\mathcal{D}_x\, S\|\ \|\mathcal{D}_\theta\, X\|]\ [e^{\gamma\tau} - 1]/\gamma \\ +\ \|\mathcal{D}_x\, X\| \end{array} \right\}.$$

All we have to show now is $\Gamma = \int_0^\infty e^{-\lambda\tau}\, F(\tau)\, d\tau\ < 1.$

$$\Gamma = \int_0^\infty e^{-\lambda\tau}\, F(\tau)\, d\tau =$$

$$= \left\{ \begin{array}{l} c_1\, c_2\, \|X\|\ [\frac{1}{\lambda-\gamma} - \frac{1}{\lambda} - \frac{\gamma}{\lambda^2}]/\gamma^2 + \|\mathcal{D}_x\, X\|/\lambda \\ +\ c_2\ [\|\mathcal{D}_\theta\, X\| + \|\mathcal{D}_x\, S\|\ \|\mathcal{D}_\theta\, X\|]\ [\frac{1}{\lambda-\gamma} - \frac{1}{\lambda}]/\gamma \end{array} \right\}$$

$$\leq \left\{ \begin{array}{l} c_1\, c_2\, [\rho_1\, c_3 + c_4(\varepsilon)]/[\lambda^2(\lambda - \gamma)] + c_3/\lambda \\ +\ c_2\, [\rho_1\, c_3 + c_4(\varepsilon) + \rho_2\, c_3]/[\lambda(\lambda - \gamma)] \end{array} \right\}.$$

Now select ρ_1, ρ_2, and ε_0 such that

$$(36) \qquad \rho_1 \leq \frac{\lambda^2(\lambda - \gamma)}{4\, c_2\, c_3\, (c_1 + \lambda)},$$

$$(37) \qquad \rho_2 \leq \frac{\lambda(\lambda - \gamma)}{4\, c_2\, c_3},$$

$$(38) \qquad c_4(\varepsilon) \leq \frac{\lambda^2(\lambda - \gamma)}{4\, c_2(c_1 + \lambda)}\ \text{for}\ |\varepsilon| \leq \varepsilon_0.$$

Then, $\Gamma \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{18} < 1.$

It becomes apparent by looking at (21), (23), (31), (32), (36), (37), and (38) that there exists a $\rho_2{}^0$, $\rho_1{}^0(\rho_2{}^0)$ and $\varepsilon^0\,(\rho_1{}^0,\ \rho_2{}^0)$ such that for $0 < |\varepsilon| \leq \varepsilon^0(\rho_1{}^0,\ \rho_2{}^0)$ there exists a unique fixed point $S(\theta)$ of T_ε in $\Omega(\rho_1{}^0)$, which in turn is an invariant surface of (9, 10). It is clear that $S(\theta)$ is Lipschitz continuous with constant $\rho_2{}^0$. Also, by iterating (20), we see that

$$\|S\| \leq \frac{18\, c_4(\varepsilon)}{17\, \lambda},$$

so that $\|S\| \to 0$ as $|\varepsilon| \to 0$.

$$\text{Q.E.D.}$$

Corollary 2.2.1: Consider the system

(39) $\qquad \dot{\theta}(t) = \Theta[\theta(t), x(t), \varepsilon]$

(40) $\qquad \dot{x}(t) = E(\varepsilon) \cdot \{(A[\theta(t)] + B[\theta(t)]) x(t) + X[\theta(t), x(t), \varepsilon]\}$

where Θ, E, A, and X are as in Theorem 2.2. Let P be a non-singular matrix that commutes with $A(\theta)$ for all θ. Assume that $B(\theta)$ is continuously differentiable in B^m and

(41) $\qquad \|P^{-1} B(\cdot) P\|, \|P^{-1} D_\theta B(\cdot) P\| \leq \alpha,$

(42) $\qquad \|P^{-1}\| \leq \beta, \qquad \|P\| \leq \gamma.$
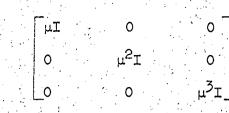
Assume also that c_3 satisfies

(43) $\qquad \alpha + \beta \gamma c_3 \leq \lambda/18,$

where c_3 is the constant in assumption (d) of Theorem 2.2. Then the conclusions of Theorem 2.2 hold for the system (39, 40).

Remark: The condition of this corollary is best motivated by considering an example. Let $B(\theta)$ be the matrix

$$B(\theta) = \begin{bmatrix} 0 & B_1(\theta) & B_2(\theta) \\ 0 & 0 & B_3(\theta) \\ 0 & 0 & 0 \end{bmatrix},$$

and P the matrix

$$P = \begin{bmatrix} \mu I & 0 & 0 \\ 0 & \mu^2 I & 0 \\ 0 & 0 & \mu^3 I \end{bmatrix},$$

where the blocking is the same as in A. It is clear that P commutes with $A(\theta)$ for all θ. Then $\|P^{-1} B(\cdot) P\| \leq \mu b = \alpha$, $\|P\| \leq \mu = \gamma$, $\|P^{-1}\| \leq \mu^{-3} = \beta$, for some constant b. Condition (43) becomes

$$\mu b + \mu^{-2} c_3 \leq \lambda/18,$$

which can be satisfied by some μ if $c_3 \leq \lambda^3 / [54 (27 b)^2]$.

Proof: If an invariant surface $S(\theta)$ exists, then as in (15) it satisfies

$$S(\theta) = -\int_0^\infty J^{-1}(\tau, \theta, \psi, \varepsilon) \circ W\{\psi(\tau, \varepsilon), S[\psi(\tau, \varepsilon)], \varepsilon\} d\tau,$$

where $W(\theta, x, \varepsilon) = B(\theta) x + X(\theta, x, \varepsilon)$, and J and ψ are as before. Let $Q(\theta) = P^{-1} S(\theta)$. Then, since by assumption P commutes with A, P must also commute with J, so that

$$Q(\theta) = -\int_0^\infty J^{-1}(\tau, \theta, \psi, \varepsilon) \circ P^{-1} \circ W\{\psi(\tau, \varepsilon), P Q[\psi(\tau, \varepsilon)], \varepsilon\} d\tau.$$

Thus, if we can show that $P^{-1} \circ W$ satisfies condition (d) on X imposed in Theorem 2.2, then the existence of an invariant surface $Q(\theta)$ will follow. But $S(\theta) = P Q(\theta)$, so that the existence of Q implies the existence of S since P is assumed non-singular. We show only the first part of (d); the rest follows in the same manner. Let $x = P w$, then

$$|P^{-1} \circ W[\theta, w, \varepsilon]|$$
$$\leq \|P^{-1} B(\cdot) P\| |w| + \|P^{-1}\| \|X(\cdot, Pw, \varepsilon)\|$$
$$\leq \alpha |w| + \beta [c_3 |Pw| + c_4(\varepsilon)]$$
$$\leq [\alpha + \beta \gamma c_3] |w| + \beta c_4(\varepsilon)$$
$$\leq (\lambda/18) |w| + \beta c_4(\varepsilon).$$

<div align="center">Q.E.D.</div>

In Theorem 2.2, it was assumed that X contained a small, but otherwise arbitrary, linear term. Corollary 2.2.1 treats the case where X has not necessarily small cross linear terms that do not change the eigenvalues of $A(\theta)$ by much. X may still contain a small additional linear term. The following corollary treats the case when the degenerate term is absent.

Corollary 2.2.2: If in Theorem 2.2,

$$E(\varepsilon) = \begin{bmatrix} I_1 & 0 \\ 0 & (1/\varepsilon)I_3 \end{bmatrix},$$

$$A(\theta) = \begin{bmatrix} A_1(\theta) & 0 \\ 0 & A_3(\theta) \end{bmatrix},$$

then condition (c) may be weakened to

c´) $\qquad \|D_\theta \Theta(\cdot,\ x,\ \varepsilon)\| \leq \lambda/2$

$\qquad\qquad \|D_x \Theta(\cdot,\ x,\ \varepsilon)\| \leq c_2$

with the same conclusion holding.

Proof: After the change of integration variable made in going from (13) to (15), the only effect of the degenerate term is in $\psi(\tau,\ \varepsilon)$ and this effect is noticed only in the estimates on $\|D_\theta \psi(\tau,\ \varepsilon)\|$ and $\|\psi^1(\tau,\ \varepsilon) - \psi^2(\tau,\ \varepsilon)\|$. If the degenerate term is absent, then the "worst case" term becomes the normal term (that is, in $\psi(\tau)$). With the new assumptions, these are

$$\|D_\theta \psi(\tau)\| \leq \exp(\gamma t),$$

$$\|\psi^1(\tau) - \psi^2(\tau)\| \leq c_2 \|s^1 - s^2\| [e^{\gamma\tau} - 1]/\gamma,$$

which are precisely the estimates (26) and (34) that were actually used in the proof of Theorem 2.2. The proof of this corollary then proceeds as in Theorem 2.2.

Q.E.D.

III. PRELIMINARIES IN EQUATIONS WITH TIME LAG

This chapter will be devoted to a fairly exhaustive study of ordinary differential equations with time lag. The exhaustiveness is necessary because of the relative lack of general basic theorems proven rigorously. Although most of the material presented is known -- well known in the case without time lag -- some of the material appears to be new.

We will treat these equations as functionals from a Banach space to a Euclidian space. The approach appears to have originated with Krasovshii [5], although this work has been influenced more by Hale [6, 7], Hale and Perelló [8], Halany [9], and Oguztöreli [10]. The functional analysis necessary for this work has been taken from Dieudonné [11].

Definition: A function $f(\lambda)$ from $[-r, 0]$ into R^n is called a Regulated Function if the upper limit, $f(\lambda+) = \lim_{\mu \to \lambda+} f(\mu)$, exists for $\lambda \in [-r, 0)$ and the lower limit, $f(\lambda-) = \lim_{\mu \to \lambda-} f(\mu)$, exists for $\lambda \in (-r, 0]$. (This is true if and only if f is the uniform limit of step functions.) The space of all bounded regulated functions from $[-r, 0]$ into R^n is a Banach space under the norm, $\|f\| = \sup_{-r \le \lambda \le 0} |f(\lambda)|$, and will be denoted by B_r^n. (Dieudonné [11, p. 139]).

We will denote an element of B_r^n with a subscript "r" (that is, ϕ_r), and use λ as the argument in $[-r, 0]$. Let $x(\tau)$ be defined at least on $[t - r, t]$ into R^n; then we will define an element $x_r(t) \in B_r^n$ by

$$x_r(t) = \{x_r(t, \lambda) = x(t + \lambda), \qquad -r \le \lambda \le 0\}.$$

The function $x_r(t)$ is the segment of $x(\tau)$ obtained by letting τ range from $t - r$ to t. Let $X(t, \phi_r)$ be a mapping from a subset of $R \times B_r^n$ into R^n. Thus, for fixed t, $X(t, \phi_r)$ is a functional. A differential equation with time lag will then be written

(1) $$\dot{x}(t) = X[t, x_r(t)],$$

where $\dot{x}(t) = \lim_{h \to 0+} \frac{1}{h}[x(t + h) - x(t)]$ denotes the right-hand derivative of x at t. Initial conditions for (1) will be imposed as

(2) $\quad x_r(0) = \phi_r^0;$ that is, $\quad x(\lambda) = \phi_r^0(\lambda), -r \leq \lambda \leq 0.$

A function $X(t, \phi_r)$ will be called continuous (in both arguments) at (t^0, ϕ_r^0) if, given $\epsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that

$$|X(t, \phi_r) - X(t^0, \phi_r^0)| < \epsilon$$

whenever $|t - t^0| < \delta_1$ and $\|\phi_r - \phi_r^0\| < \delta_2$. $X(t, \phi_r)$ will be called Lipschitz in ϕ_r if there exists a constant L such that

$$|X(t, \phi_r^1) - X(t, \phi_r^2)| \leq L \|\phi_r^1 - \phi_r^2\|.$$

Existence, Uniqueness, and Continuous Dependence Theorems

Theorem 3.1: (Existence and Uniqueness)

Let $H = [t_0, t_0 + a] \times \{\phi_r : \|\phi_r - \phi_r^0\| \leq b\}$ be a set in $R \times B_r^n$, where $\phi_r^0(\lambda)$ is a constant function. Let $X(t, \phi_r)$ be a continuous function from H into R^n. Assume that $X(t, \phi_r)$ is uniformly Lipschitz in ϕ_r, with constant L, and uniformly bounded by M in H. Then for any ψ_r^0 such that $\|\psi_r^0 - \phi_r^0\| \leq \frac{b}{2}$, there exists an $\alpha(\psi_r^0)$, $0 < \alpha \leq a$, such that

(3) $$\dot{x}(t) = X[t, x_r(t)], \qquad x_r(t^0) = \psi_r^0$$

has the unique regulated solution in $[t_0 - r, t_0 + \alpha]$, which is continuous for $t \geq t_0$.

Proof: We assume without loss that $t_0 = 0$. Define the Banach spaces C^α by

$$C^\alpha = \left\{ u(t): \begin{array}{l} u(t) \text{ regulated for } t \in [t_0 - r, \ t_0 + \alpha] \\ u_r(0) = \psi_r^o, \ \|u_r(t) - \phi_r^o\| \le b, \ 0 \le t \le \alpha \end{array} \right\},$$

with the sup norm; and define a map T from C^α into the space of regulated functions by

$$[Tu](t) = \left\{ \begin{array}{ll} \psi_r^o(t) & -r \le t \le 0 \\ \psi_r^o(0) + \int_0^t X[\sigma, u_r(\sigma)] \, d\sigma & 0 \le t \le \alpha \end{array} \right.$$

The integral here is well defined because $X[t, u_r(t)]$ is regulated if $X(t, \phi_r)$ is continuous and $u(t)$ is regulated. (See Dieudonné [11, Section 8.7] for a discussion of integrals of regulated functions.) It is clear that $[Tu](t)$ is defined and regulated for $-r \le t \le \alpha$, and continuous for $0 \le t \le \alpha$. We show that for $0 < \alpha \le \min \{a, \ ^b/2M\}$, $T : C^\alpha \to C^\alpha$. Clearly $[Tu]_r(0) = \psi_r^o$, so all we need to show is that $\|[Tu]_r(t) - \phi_r^o\| \le b$ for $0 \le t \le \alpha$. But

$$\|[Tu]_r(t) - \phi_r^o\| = \sup_{-r \le \lambda \le 0} |[Tu](t + \lambda) - \phi_r^o(\lambda)|$$

$$\le \sup_\lambda \left\{ \begin{array}{ll} |\psi_r^o(t + \lambda) - \phi_r^o(\lambda)| & t + \lambda \le 0 \\ |\psi_r^o(0) - \phi_r^o(\lambda)| + \int_0^{t+\lambda} |X[\sigma, u_r(\sigma)]| d\sigma & t + \lambda \ge 0 \end{array} \right.$$

$$\le \sup_\lambda \left\{ \begin{array}{ll} \|\psi_r^o - \phi_r^o\| & t + \lambda \le 0 \\ \|\psi_r^o - \phi_r^o\| + \int_0^t M \, d\sigma & t + \lambda \ge 0 \end{array} \right.$$

since $\phi_r^o(\lambda)$ is a constant function and $|X| \le M$. Since $\alpha \le (^b/2M, a)$, we see that

$$\|[Tu]_r(t) - \phi_r^o\| \le {}^b/2 + {}^b/2 = b$$

for $0 \le t \le \alpha$. We complete the proof by showing that for some α, $T : C^\alpha \to C^\alpha$ is a contraction. Thus T has a fixed point in C^α which clearly must be a

solution to the differential equation (3). For $u \in C^\alpha$, $\|u\|_\alpha = \sup_{-r \leq t \leq \alpha} |u(t)|$; thus

$$\|[Tu^1] - [Tu^2]\|_\alpha = \sup_{0 \leq t \leq \alpha} |[Tu^1](t) - [Tu^2](t)|$$

$$\leq \sup_{0 \leq t \leq \alpha} \int_0^t |X[\sigma, u_r^1(\sigma)] - X[\sigma, u_r^2(\sigma)]| \, d\sigma$$

$$\leq \int_0^\alpha L \, \|u_r^1(\sigma) - u_r^2(\sigma)\|_B \, d\sigma.$$

The sup is only taken for $0 \leq t \leq \alpha$ since Tu^1 and Tu^2 agree on $-r \leq t \leq 0$. The norm in the last expression above is the B_r^n norm, but

$$\|u_r^1(\sigma) - u_r^2(\sigma)\|_B = \sup_{-r \leq \lambda \leq 0} |u^1(\sigma + \lambda) - u^2(\sigma + \lambda)|$$

$$\leq \sup_{-r \leq t \leq \alpha} |u^1(t) - u^2(t)| = \|u^1 - u^2\|_\alpha, \text{ for } \sigma \leq \alpha.$$

Thus we have

$$\|[Tu^1] - [Tu^2]\|_\alpha \leq \alpha L \|u^1 - u^2\|_\alpha,$$

so that if $\alpha L < 1$, T is a contraction on C^α.

$$Q.E.D.$$

Lemma 3.2: (Gronwall's) Let $u(t)$, $v(t)$, $w(t)$ be non-negative continuous functions, u and v on $[a - r, b]$, and w on $[a, b]$. Assume that

$$u(t) \leq \left\{ \begin{array}{ll} v(t) & , \quad a - r \leq t \leq a \\ v(t) + \int_a^t w(\tau) \|u_r(\tau)\| \, d\tau & , \quad a \leq t \leq b \end{array} \right\}.$$

Then $u(t) \leq \|u_r(t)\| \leq \|v_r(t)\| + \int_a^t \|v_r(s)\| \, w(s) \exp[\int_s^t w(\tau) \, d\tau] \, ds.$

Proof: Clearly $u(t) \leq \|u_r(t)\|$. But we have

$$\|u_r(t)\| = \sup_{-r \le \lambda \le 0} |u(t + \lambda)|$$

$$\le \sup_{\lambda} \left\{ \begin{array}{ll} v(t + \lambda) & , \quad t + \lambda \le a \\[2ex] v(t + \lambda) + \int_a^{t+\lambda} w(\tau) \|u_r(\tau)\| \, d\tau & , \quad t + \lambda \ge a \end{array} \right\}$$

$$\le \|v_r(t)\| + \int_a^t w(\tau) \|u_r(\tau)\| \, d\tau, \quad a \le t \le b.$$

The lemma now follows from Lemma 1.3 applied to $\|u_r(t)\|$.

<center>Q.E.D.</center>

Gronwall's lemma in the time lag case does not appear to have been stated or proven previously.

<u>Theorem 3.3</u>: (Continuous Dependence) Let $H = [t_o, t_o + a] \times \{\phi_r: \|\phi_r - \xi_r^o\| \le b\} \times \{\psi_r: \|\psi_r - \psi_r^o\| \le C\}$ be a set in $R \times R^n \times R^m$, where $\xi_r(\lambda)$ is a constant function. Let $X[t, \phi_r, \psi_r]$ be a function from H to R^n. Assume that X is continuous in t and ϕ_r, and uniformly Lipschitz in ϕ_r, with constant L, in H. Let $\psi_r^1, \psi_r^2 \in B_r^m$ be such $\|\psi_r^i - \psi_r^o\| \le C$, and $\phi_r^1, \phi_r^2 \in B_r^n$ be such that $\|\phi_r^i - \xi_r^o\| \le b/2$. Then there exists an $\alpha > 0$ such that the equations

$$\dot{x}^i(t) = X[t, x_r^i(t), \psi_r^i], \qquad\qquad x_r^i(0) = \phi_r^i (i = 1, 2)$$

have unique solutions for $t_o - r \le t \le t_o + \alpha$. Also, if $|X(t, \phi_r, \psi_r^1) - X(t, \phi_r, \psi_r^2)| \le \varepsilon$ for all t, $t_o \le t \le t_o + \alpha$, $\|\phi_r - \xi_r^o\| \le b$, then

$$|x^1(t) - x^2(t)| \le \{\|\phi_r^1 - \phi_r^2\| + \varepsilon(t - t_o)\} e^{L(t - t_o)}, \ t_o \le t \le t_o + \alpha.$$

<u>Proof</u>: The existence of α follows immediately from Theorem 3.1 All we have to prove is the estimate on the difference of the solutions. Let $\Delta(t) = x^1(t) - x^2(t)$; then

$$\Delta(t) = \begin{cases} \phi_r^1(t) - \phi_r^2(t) & , \ t \leq t_o \\ \phi_r^1(0) - \phi_r^1(0) + \int_{t_o}^t \{X[\sigma, \ x_r^1(\sigma), \ \psi_r^1] \\ \qquad\qquad\qquad\qquad -X[\sigma, \ x_r^2(\sigma), \ \psi_r^2]\} \ d\sigma, \ t \geq t_o \end{cases}.$$

Thus we have

$$|\Delta(t)| \leq \begin{cases} \|\phi_r^1 - \phi_r^2\| & , \ t \leq t_o \\ \|\phi_r^1 - \phi_r^2\| + \int_{t_o}^t \{L \|\Delta_r(\tau)\| + \varepsilon\} \ d\tau, & , \ t \geq t_o \end{cases}$$

$$\leq \begin{cases} \|\phi_r^1 - \phi_r^2\| & , \ t \leq t_o \\ \|\phi_r^1 - \phi_r^2\| + \varepsilon(t - t_o) + L \int_{t_o}^t \|\Delta_r(\tau)\| \ d\tau, & t \geq t_o \end{cases}.$$

By applying Gronwall's lemma with

$$v(t) = \begin{cases} \|\phi_r^1 - \phi_r^2\| & , \quad t \leq t_o \\ \|\phi_r^1 - \phi_r^2\| + \varepsilon(t - t_o) & ; \quad t \geq t_o \end{cases}$$

we see that the result follows:

<div align="center">Q.E.D.</div>

Linear Equations

Consider the linear differential equation with time lag,

$$\dot{x}(t) = A(t) \circ x_r(t),$$

where $A(t)$ is a continuous linear functional from B_r^n into R^n for fixed t. In what we will be doing here, $A(t)$ will be assumed continuous in t for all $t \in R$.

Lemma 3.4: Let $A(t)$ be a continuous linear mapping from B_r^n into R^n for fixed $t \in R$. Assume that $A(t)$ is continuous in t for all $t \in R$. Then there exists a continuous linear mapping $J(t, t_o)$, called the fundamental operator associated with $A(t)$, from B_r^n into R^n, defined for $t \geq t_o - r$, that maps an initial condition ϕ_r into a solution of

$$(4) \qquad \dot{x}(t) = A(t) \circ x_r(t).$$

That is, if $\phi_r \in B_r^n$, then $x(t) = J(t, t_o) \circ \phi_r$ satisfies (4) and $x_r(t_o) = \phi_r$.
Also, the mapping $J_r(t, t_o)$ from B_r^n into B_r^n, defined by

$$J_r(t, t_o, \lambda) \circ \phi_r = J(t + \lambda, t_o) \circ \phi_r \qquad , \qquad -r \le \lambda \le 0,$$

is continuous and linear for fixed t, defined for $t \ge t_o$, and the equality

$$J(t, t_o) = J(t, t_1) \circ J_r(t_1, t_o)$$

holds for $t_1 \ge t_o$, $t \ge t_1 - r$.

    Proof: Existence on an interval $[t_o - r, t_o + h]$ follows immediately
from Theorem 3.1. Since $A(t)$ is a continuous mapping for each $t \in R$, h can
be selected independently of t_o for t_o in any compact set, so that a solution
on $[t_o - r, t_o + h]$ can be continued to $[t_o - r, t_o + 2h]$, and so on for all
$t \ge t_o - r$. Uniqueness of solution follows from Gronwall's inequality in the
same manner as in equations without time lag. Thus, $J(t, t_o) \circ \phi_r$, the
solution of (4), can be defined uniquely for all $t \ge t_o - r$. Solutions to (4)
were shown to depend continuously on ϕ_r in Theorem 3.3, so that $J(t, t_o)$ is a
continuous mapping. It is also linear, since if

$$x(t) = J(t, t_o) \circ \alpha \phi_r^1 + J(t, t_o) \circ \beta \phi_r^2,$$

where α and $\beta \in R$, then

$$x_r(t_o) = J_r(t_o, t_o) \circ \alpha \phi_r^1 + J(t_o, t_o) \circ \beta \phi_r^2 = \alpha \phi_r^1 + \beta \phi_r^2.$$

Thus, by uniqueness,

$$x(t) = J(t, t_o) \circ [\alpha \phi_r^1 + \beta \phi_r^2].$$

Let $t_1 \ge t_o$, and let $\phi_r^2 = J_r(t_1, t_o) \circ \phi_r^1$. Then clearly $J(t, t_1)$ is defined,

continuous and linear. Also, $J(t, t_1) \circ \phi_r^2$ and $J(t, t_o) \circ \phi_r^1$ agree on $[t_1 - r, t_1]$, and thus by uniqueness must agree for all $t \geq t_1 - r$. Thus

$$J(t, t_1) \circ J_r(t_1, t_o) \circ \phi_r^1 = J(t, t_o) \circ \phi_r^1 \quad \text{for } t \geq t_1 - r, \; t_1 \geq t_o.$$

Q.E.D.

We now wish to obtain an expression for the solution of the non-homogenous equation

(5) $$\dot{x}(t) = A(t) \circ x_r(t) + f(t) \quad ; \quad x_r(t_o) = \phi_r.$$

Halany [9, Section 4.3] derives an expression for the solution of (5), which in our notation is

(6) $$x(t) = J(t, t_o) \circ \phi_r^o + \int_{t_o}^{t} [J(t, \tau) \circ I_r] f(\tau) \, d\tau,$$

where I_r is the matrix function

(7) $$I_r = \left\{ \; I_r(\lambda) = \begin{bmatrix} 0, & -r \leq \lambda < 0 \\ I, & \lambda = 0 \end{bmatrix} \right\}.$$

The discontinuity of I_r is the reason for using regulated functions rather than continuous functions.

For the rest of the considerations in this section, we will restrict ourselves to the autonomous case; that is, A does not depend on t. We define the function $\exp_r(\mu)$ as an element of $B_{\overline{r}}^1$ by

$$\exp_r(\mu) = \{\exp_r(\mu; \lambda) = e^{\mu\lambda} \quad , \quad -r \leq \lambda \leq 0\}.$$

Definition: A number μ will be called a <u>characteristic root</u> of the linear functional A or of the differential equation

(8) $$\dot{x}(t) = A \circ x_r(t)$$

if there exists a non-zero complex n-vector w for which

$$\mu w = A \circ (\exp_r(\mu) \, I) \circ w,$$

where I is the identity matrix, or equivalently if

$$\det [A \circ (\exp_r(\mu) \, I) - \lambda I] = 0.$$

This definition can be motivated by "guessing" a solution to (6) of the form $e^{\mu t} w$. It is known that in any half plane, Re $z \geq \gamma$, there are only a finite number of characteristic roots of (6), and that these have only finite multiplicity. Hale [6] proves this when B_r^n is a space of continuous functions. The result also applies in our case since $\exp_r(\mu)$ is continuous. The following lemma is proven by Hale and Perelló [8] when B_r^n is a space of continuous functions.

Lemma 3.5: Let A be a continuous linear mapping from B_r^n into R^n. Assume that no characteristic roots of A are purely imaginary. Then there exist subspaces B_+^n and B_-^n of B_r^n and constants μ, K, K' > 0 such that

(a) $B_r^n = B_+^n \oplus B_-^n$

(b) $J_r(t, t_o) \circ \phi_r^+$ is defined and contained in B_+^n for all t if $\phi_r^+ \in B_+^n$;

(9) $\qquad |J(t, t_o) \circ \phi_r^+| \leq K \, \|\phi_r^+\| \; e^{\mu(t-t_o)}$, $\qquad\qquad t \leq t_o$;

(10) $\qquad |J(t, t_o) \circ \phi_r^+| \geq K' \, \|\phi_r^+\| \; e^{\mu(t-t_o)}$, $\qquad\qquad t \geq t_o$.

(c) $J_r(t, t_o) \circ \phi_r^-$ is defined and contained in B_-^n for all $t \geq t_o$ if $\phi_r^- \in B_-^n$;

(11) $\qquad |J(t, t_o) \circ \phi_r^-| \leq K \, \|\phi_r^-\| \; e^{-\mu(t-t_o)}$, $\qquad\qquad t \geq t_o$.

Remark: In our case, where B_r^n is a space of regulated functions, parts (a) and (b) in the above lemma follow as in Hale and Perelló since B_+^n is the finite dimensional space of "characteristic" functions of A. The only part of (c) that is not immediately clear is the estimate (11), but since $J(t, t_0)$ is a continuous operator and $\psi_r^- = J_r(t_0 + r, t_0) \circ \phi_r^-$ is a continuous function (of λ), (11) holds with perhaps a different K.

From this lemma, it follows that $x(t)$, the solution of (5) given in (6), can be written

$$(12) \qquad\qquad x(t) = x^+(t) + x^-(t)$$

where

$$(13) \qquad x^+(t) = J(t, t_0) \circ \phi_r^+ + \int_{t_0}^{t} [J(t, \sigma) \circ I_r^+] \; f(\sigma) \, d\sigma,$$

$$(14) \qquad x^-(t) = J(t, t_0) \circ \phi_r^- + \int_{t_0}^{t} [J(t, \sigma) \circ I_r^-] \; f(\sigma) \, d\sigma,$$

with $\phi_r = \phi_r^+ + \phi_r^-$ and $I_r = I_r^+ + I_r^-$. We are now in a position to prove an analogy of Lemma 1.2 and Corollary 1.2.1.

Theorem 3.6: Let A be a continuous linear mapping from B_r^n into R^n and $f(t)$ a continuous bounded function from R to R^n. If A has no purely imaginary characteristic roots, then any bounded solution of

$$(15) \qquad\qquad \dot{x}(t) = A \circ x_r(t) + f(t)$$

that is defined for all $t \geq t_0$ can be written

$$(16) \qquad\qquad x(t) = x^-(t, t_0) - \int_0^{\infty} [J(t, t + \sigma) \circ I_r^+] f(t + \sigma) \, d\sigma$$

where $x^-(t, t_0)$ is given by

$$(17) \qquad\qquad x^-(t, t_0) = J(t, t_0) \circ \phi_r^- + \int_{t_0}^{t} [J(t, \sigma) \circ I_r^-] f(\sigma) \, d\sigma.$$

Also there exists a unique bounded solution $x^0(t)$ of (15) that is defined for all t, given by

$$(18) \qquad x^0(t) = \int_{-\infty}^{0} [J(t, \ t + \sigma) \circ I_r^-] \ f(t + \sigma) \ d\sigma$$

$$- \int_{0}^{\infty} [J(t, \ t + \sigma) \circ I_r^+] \ f(t + \sigma) \ d\sigma.$$

Proof: We know that the general solution of (15) is given by (12), (13), and (14). For $t \geq t_0$,

$$|x^-(t, \ t_0)| \leq |J(t, \ t_0) \circ \phi_r^-| + \int_{t_0}^{t} |J(t, \ \sigma) \circ I_r^-| \ |f(\sigma)| \ d\sigma$$

$$\leq K \ \|\phi_r^-\| \ e^{-\mu(t-t_0)} + \int_{t_0}^{t} K \ e^{-\mu(t-\sigma)} \ M \ d\sigma$$

$$\leq K \ \|\phi^-\| \ e^{-\mu(t-t_0)} + M \ K \ e^{-\mu(t-t_0)}/\mu < \infty,$$

where μ and K are constants guaranteed by Lemma 3.5, and M is a bound on $|f(\sigma)|$. Thus, $x^-(t, \ t_0)$ is bounded for all $t \geq t_0$. Also,

$$x^+(t, \ t_0) = J(t, \ t_0) \circ \phi_r^+ + \int_{t_0}^{t} [J(t, \ \sigma) \circ I_r^+] \ f(\sigma) \ d\sigma$$

$$= J(t, \ t_0) \circ \{\phi_r^+ + \int_{t_0}^{\infty} [J_r(t_0, \ \sigma) \circ I_r^+] \ f(\sigma) \ d\sigma\}$$

$$- \int_{t}^{\infty} [J(t, \ \sigma) \circ I_r^+] \ f(\sigma) \ d\sigma,$$

if the integrals make sense and converge. But, $J(t, \ \sigma) \circ I_r^+ = J(t, \ t_0) \circ J_r(t_0, \ \sigma) \circ I_r^+$ for all t, t_0, σ since $I_r^+ \in B_+^n$. The integrals converge, as in Lemma 1.2, because the estimate (9) is valid and $f(\sigma)$ is bounded. The only way for $x^+(t)$ to be bounded, since $|J(t, \ t_0) \circ \phi_r^+| \to \infty$ as $t \to \infty$ by (10), is for its multiplier to vanish; that is,

$$\phi_r^+ = -\int_{t_0}^{\infty} [J_r(t_0, \ \sigma) \circ I_r^+] \ f(\sigma) \ d\sigma.$$

Thus, we are left with the expression

$$x^+(t, t_o) = -\int_0^\infty [J(t, t + \sigma) \circ I_r^+] f(t + \sigma) \ d\sigma,$$

and (16) and (17) follow. Notice that $x^+(t, t_o)$ is uniquely defined for all t and t_o, and thus the second integral in (18) is valid; but $x^-(t, t_o)$ need not be defined for $t \leq t_o$ and is not unique. To obtain an $x^-(t, t_o)$ that is defined for all t, we follow the procedure used on x^+. Thus,

$$(19) \qquad x^-(t, t_o) = J(t, t_o) \circ \{\phi_r^- - \int_{-\infty}^{t_o} [J_r(t_o, \tau) \circ I_r^-] f(\tau) \ d\tau\}$$

$$+ \int_{-\infty}^t [J(t, \sigma) \circ I_r^-] f(\sigma) \ d\sigma,$$

if the integrals make sense and converge. But $J(t, t_o) \circ J_r(t_o, \tau) \circ I_r^- = J(t, \tau) \circ I_r^-$ is valid since $\tau \leq t_o \leq t$. We can assume $t_o \leq t$ because t_o will drop out shortly. The convergence follows as before. Thus, if we assume that

$$\phi_r^- = \int_{-\infty}^{t_o} [J_r(t_o, \tau) \circ I_r^-] f(\tau) \ d\tau,$$

then $x^-(t, t_o)$ is defined and bounded for all t, and

$$(20) \qquad x^-(t, t_o) = \int_{-\infty}^0 [J(t, t + \sigma) \circ I_r^-] f(t + \sigma) \ d\sigma,$$

as stated in (18) independently of t_o. To show the uniqueness $x^-(t, t_o)$, let $y^-(t)$ be defined and bounded for all t, and

$$(21) \qquad y^-(t) = J(t, \tau_o) \circ \psi_r^-(\tau_o) + \int_{\tau_o}^t [J(t, \sigma) \circ I_r^-] f(\sigma) \ d\sigma$$

for some τ_o. Thus, as in (19),

$$y^-(t) = J(t, \tau_o) \circ \{\psi_r^-(\tau_o) - \int_{-\infty}^{\tau_o} [J_r(\tau_o, \sigma) \circ I_r^-] f(\sigma) \ d\sigma\}$$

$$+ \int_{-\infty}^t [J(t, \sigma) \circ I_r^-] f(\sigma) \ d\sigma$$

for $t \geq \tau_o$. Notice that $y^-(t)$ is actually independent of τ_o, the term "inside the { }'s" is uniformly bounded in τ_o, and the second integral is actually

$x^-(t, t_o)$ defined in (20). Thus, since $|J(t, \tau_o) \circ \{\cdot\}| \to 0$ as $\tau_o \to -\infty$ by (11), we see that

$$y^-(t) \to x^-(t, t_o) \quad \text{as } \tau_o \to -\infty.$$

But, as noted before, $y^-(t)$ is independent of τ_o; this implies equality, and uniqueness follows.

$$Q.E.D.$$

Calculus in a Banach Space

Before we can proceed to the differentiability properties of solutions, we collect some facts on differentiation in a Banach space. We have taken this material almost verbatim from Chapter VIII of Dieudonné's book [9], and will state it without proof.

Definition: Let E and F be Banach spaces; let f be a continuous mapping of A, an open subset of E, into F. We say that f is (Fréchét) differentiable at $x_o \in A$ if there is a linear map u of E into F such that

$$\lim_{x \to x_o, x \neq x_o} \frac{\|f(x) - f(x_o) - u(x - x_o)\|}{\|x - x_o\|} = 0.$$

We will denote the derivative of f at x_o by $\mathcal{D}_x f(x_o)$.

Lemma 3.7: If the continuous map f from $A \subset E$ into F is differentiable at the point x_o, then $\mathcal{D}_x f(x_o)$ is a uniquely determined continuous linear mapping of E into F.

Examples: If f is a constant function, then $\mathcal{D}_x f(x_o) = 0$, since $\|f(x) - f(x_o)\| = 0$.

If f is a continuous linear mapping of E into F, then $\mathcal{D}_x f(x_o) = f$ for all $x_o \in E$. Since $f(x) - f(x_o) = f(x - x_o)$, thus

$$\|f(x) - f(x_o) - f(x - x_o)\| = 0.$$

Theorem 3.8: (Chain Rule) Let E, F, G be three Banach spaces; A an open neighborhood of $x_0 \in$ E; f a continuous mapping of A into F; $y_0 = f(x_0)$; B an open neighborhood of $y_0 \in$ F; and g a continuous mapping of B into G. Then, if f is differentiable at x_0 and g is differentiable at y_0, the mapping h = g [f] (which is defined and continuous in a neighborhood of x_0 into G) is differentiable at x_0, and we have

$$\mathcal{D}_x h(x_0) = \mathcal{D}_y g [f(x_0)] \circ \mathcal{D}_x f(x_0).$$

Application: Let f, g be two continuous mappings of the open subset A of E into F. If f and g are differentiable at x_0, so are f + g and αf (α a scalar), and

$$\mathcal{D}_x [f + g] (x_0) = \mathcal{D}_x f(x_0) + \mathcal{D}_x g(x_0),$$

$$\mathcal{D}_x [\alpha f] (x_0) = \alpha \mathcal{D}_x f(x_0).$$

Definition: Let E be a Banach space. A Segment joining two points a, b $\in$ E is the set of points $\{a + \xi(b - a): \quad 0 \leq \xi \leq 1\}$.

Theorem 3.9: (Mean Value) Let E, F be two Banach spaces; f a continuous mapping into F of a neighborhood of a segment S joining two points x_0, $x_0 + \Delta x$ of E. If f is differentiable at every point of S, then

$$\|f(x_0 + \Delta x) - f(x_0)\| \leq \|\Delta x\| \sup_{0 \leq \xi \leq 1} \|\mathcal{D}_x f(x_0 + \xi \Delta x\|.$$

Theorem 3.10: Let E, F be two Banach spaces; f a differentiable mapping into F of an open neighborhood A of a segment S joining two points x, x + Δx of E. Then, for each $x_0 \in$ A, we have

$$\|f(x + \Delta x) - f(x) - \mathcal{D}_x f(x_0) \circ \Delta x\|$$

$$\leq \|\Delta x\| \sup_{\tau \in S} \|\mathcal{D}_x f(\tau) - \mathcal{D}_x f(x_0)\|.$$

Notice that these last theorems are not as strong as the ones in ordinary Calculus because a point where equality holds is not guaranteed.

__Theorem 3.11__: Let A be an open connected subset in a Banach space E, $\{f_n\}$ a sequence of differentiable mappings of A into a Banach space F. Suppose that: (1°) there exists one point $x_0 \in A$ such that the sequence $\{f_n(x_0)\}$ converges in F, (2°) for every point $a \in A$, there is a ball $B(a)$ of center a contained in A such that in $B(a)$ the sequence $\{D_x f_n\}$ converges uniformly. Then for each $a \in A$, the sequence $\{f_n\}$ converges uniformly in $B(a)$; moreover, if, for each $x \in A$, $f(x) = \lim_{n \to \infty} f_n(x)$ and $g(x) = \lim_{n \to \infty} D_x f_n(x)$, then $g(x) = D_x f(x)$ for each $x \in A$.

__Lemma 3.12__: Let $I = [\alpha, \beta] \in R$ be a compact interval, E and F real Banach spaces, f a continuous mapping of $I \times A$ (A an open subset of E) into F. Suppose also that $D_x f(t, x)$ exists and is continuous on $I \times A$. Then

$$g(x) = \int_\alpha^\beta f(t, x)\, dt$$

is continuously differentiable in A, and

$$D_x g(x) = \int_\alpha^\beta D_x f(t, x)\, dt.$$

__Theorem 3.13__: Let $I = [0, \infty]$ and $f(t, x)$ be a function from $I \times A$ (A an open subset of a Banach space E) into a Banach space F. Suppose that the integral $\int_0^\infty f(t, x)\, dt$ converges to $g(x)$ on A. If $D_x f(t, x)$ exists and is continuous in $I \times A$, and if the integral

$$\int_0^\infty D_x f(t, x)\, dt$$

converges uniformly on A, then $D_x g(x)$ exists on A and

$$D_x g(x) = \int_0^\infty D_x f(t, x)\, dt.$$

Proof: Dieudonné does not state this theorem, but it follows from Theorem 3.11 and Lemma 3.12 as in the standard real variable analogy. See Apostol [12, p. 443] for a proof in this case.

Differentiation with Respect to Initial Conditions

Theorem 3.14: Let

$$H = [t_o, \; t_o + a] \times \{\phi_r : \|\phi_r - \xi_r\| \le b\} \subset R \times B_r^n,$$

where $\xi_r(\lambda)$ is a constant function. Let $X(t, \phi_r)$ be defined and continuous from H into R^n, and continuously differentiable in ϕ_r for $\|\phi_r - \xi_r\| < b$. Let $x(t, t_o, \phi_r)$ denote the solution of

$$(22) \qquad \dot{x}(t) = X[t, \; x_r(t)]$$

that satisfies

$$(23) \qquad x_r(t_o, \; t_o, \; \phi_r) = \phi_r.$$

Then, if $\|\phi_r - \xi_r\| \le b/2$, there exists an $\alpha > 0$ such that $x(t, t_o, \phi_r)$ is defined for $t_o - r \le t \le t_o + \alpha$; also if $\|\phi_r^o - \xi_r\| \le b/2$, then the derivative $\mathcal{D}_{\phi_r} x(t, t_o, \phi_r^o)$ of $x(t, t_o, \phi_r)$ at ϕ_r^o exists for $t_o - r \le t \le t_o + \alpha$, and

$$(24) \qquad \mathcal{D}_{\phi_r} x(t, \; t_o, \; \phi_r) = J(t, \; t_o)$$

where $J(t, t_o)$ is the fundamental operator associated with $\mathcal{D}_{\phi_r} X[t, \; x_r(t, \; t_o, \; \phi_r^o)]$.

Proof: The existence of $x(t, t_o, \phi_r)$ and α follows from Theorem 3.1. By Theorem 3.4, $J(t, t_o)$ exists as long as $x(t, t_o, \phi_r^o)$ is defined; that is, for $t_o - r \le t \le t_o + \alpha$. All we have to show is the equality (24). This reduces to showing that

$$\lim_{\|\Delta\phi_r\| \to 0} \left\{ \frac{|x(t, t_o, \phi_r^o + \Delta\phi_r) - x(t, t_o, \phi_r^o) - J(t, t_o) \circ \Delta\phi_r|}{\|\Delta\phi_r\|} \right\} = 0.$$

We assume that $t_o = 0$, without loss, and will supress it. Then $J(t) \circ \Delta\phi_r$ is the solution of

$$\dot{\xi}(t) = \mathcal{D}_{\phi_r} X[t, x_r(t, \phi_r^o)] \circ \xi_r(t)$$

that satisfies $\xi_r(0) = \Delta\phi_r$. Thus,

$$J(t) \circ \Delta\phi_r = \begin{cases} \Delta\phi_r(t) & , \ t \leq 0 \\ \Delta\phi_r(0) + \int_0^t \mathcal{D}_{\phi_r} X[\tau, x(\tau, \phi_r^o)] \circ J_r(\tau) \circ \Delta\phi_r \ d\tau, & t \geq 0 \end{cases}.$$

Let $\xi(t) = J(t) \circ \Delta\phi_r$, $x(t) = x(t, \phi_r^o + \Delta\phi_r)$, $x^o(t) = x(t, \phi_r^o)$, $D(t) = \mathcal{D}_{\phi_r} X[t, x_r^o(t)]$. Then

(25)
$$x(t, \phi_r^o + \Delta\phi_r) - x(t, \phi_r^o) - J(t) \circ \Delta\phi_r$$
$$= x(t) - x^o(t) - \xi(t)$$
$$= \begin{cases} 0 & , \ t \leq 0 \\ \int_0^t \{X[\tau, x_r(\tau)] - X[\tau, x_r^o(\tau)] - D(\tau) \circ \xi_r(\tau)\} \ d\tau, & \\ & , \ t \geq 0 \end{cases}.$$

But we have

(26)
$$|X[\tau, x_r(\tau)] - X[\tau, x_r^o(\tau)] - D(\tau) \circ \xi_r(\tau)|$$
$$\leq |X(\tau, x_r(\tau) - X[\tau, x_r^o(\tau)] - D(\tau) \circ [x_r(\tau) - x_r^o(\tau)]|$$
$$+ |D(\tau) \circ [x_r(\tau) - x_r^o(\tau) - \xi_r(\tau)]|$$
$$\leq \|x_r(\tau) - x_r^o(\tau)\| \sup_{\psi_r} \|\mathcal{D}_{\phi_r} X[\tau, \psi_r(\tau)] - \mathcal{D}_{\phi_r} X[\tau, x_r^o(\tau)]\|$$
$$+ \|D(\tau)\| \ \|x_r(\tau) - x_r^o(\tau) - \xi_r(\tau)\|,$$

by Theorem 3.10, where $\psi_r(\tau) \to x_r^o(\tau)$ as $\|\Delta\phi_r\| \to 0$. By Theorem 3.3, we have

$\|x_r(\tau) - x_r^o(\tau)\| \leq \|\Delta\phi_r\| \, e^{L\tau}$ for $\tau \geq 0$, where $L = \sup \{\|\mathcal{D}_{\phi_r} X[\tau, \psi_r]\|$: $(\tau, \psi_r) \in H\}$. Thus

$$(27) \qquad \|x_r(\tau) - x_r^o(\tau)\| \leq \|\Delta\phi_r\| \, e^{L\alpha} \leq K \|\Delta\phi_r\|$$

for $0 \leq \tau \leq \alpha$. Let

$$\Delta X(\Delta\phi_r) =$$

$$= \sup \{\|\mathcal{D}_{\phi_r} X(\tau, \psi_r) - \mathcal{D}_{\phi_r} X[\tau, x_r^o(\tau)]\| : (\tau, \psi_r) \in H\};$$

then $\Delta X(\Delta\phi_r) \to 0$ as $\|\Delta\phi_r\| \to 0$ by (27) and the assumption that $\mathcal{D}_{\phi_r} X(t, \phi_r)$ is continuous. Thus, upon putting (26) and (27) into (25), we obtain

$$|x(t) - x^o(t) - \xi(t)|$$

$$\leq \begin{cases} 0 & , \ t \leq 0 \\ K \|\Delta\phi_r\| \, \Delta X(\Delta\phi_r) \, t + L \int_0^t \|x_r(\tau) - x_r^o(\tau) - \xi_r(\tau)\| \, d\tau, & t \geq 0 \end{cases},$$

to which we apply Gronwall's inequality, yielding

$$|x(t) - x^o(t) - \xi(t)|$$

$$\leq \begin{cases} 0 & , \quad t \leq 0 \\ K \|\Delta\phi_r\| \, \Delta X(\Delta\phi_r) \, t \, e^{Lt}, & t \geq 0 \end{cases}$$

$$\leq K' \|\Delta\phi_r\| \, \Delta X(\Delta\phi_r) \quad \text{for} \qquad -r \leq t \leq \alpha.$$

Going back to the difference quotient, we see that

$$\lim_{\|\Delta\phi_r\| \to 0} \frac{|x(t, t_o, \phi_r^o + \Delta\phi_r) - x(t, t_o, \phi_r) - J(t, t_o) \circ \Delta\phi_r|}{\|\Delta\phi_r\|}$$

$$\leq \lim_{\|\Delta\phi_r\| \to 0} K' \, \Delta X(\Delta\phi_r) = 0.$$

Q.E.D.

Equations with Small Lipschitz Constant

In this section, we discuss solutions of differential equations with time lag or advance that are defined for all time. In defining a solution for all time, it is easier to start from an initial value rather than an initial function as was done in the earlier sections. We show that if the Lipschitz constant is small, then there is a unique solution that has minimum rate of growth for the initial value problem.

We deal with equations with time advance, so that the difficult part of the problem occurs for positive time; but the results are equally valid in the time lag case, as a simple reversal of time shows. We use the notation

$$(28) \qquad\qquad x(t) = X[t, x^r(t)]$$

for an equation with advance, where

$$x^r(t) = \{x^r(t, \lambda) = x(t + \lambda)\} \qquad \text{for } 0 \le \lambda \le r.$$

With a slight stretch of the notation, we will say $x^r(t) \in B_r^n$.

Since the results we are after appear to be previously unknown (and are somewhat surprising), it is worthwhile treating the simplest non-trivial case before starting formal proofs. The simplest case of (28) is

$$(29) \qquad\qquad \dot{x}(t) = L\, x(t + r)$$

where x is a scalar and $L > 0$. If a solution (29) is defined for all $t \ge 0$, then it can be extended uniquely for all $t \le 0$, since for $t \le 0$ (29) is an equation with time lag. Thus we are looking for a solution $u(t)$ of (29) that is defined for all $t \ge 0$ such that $u(0) = 1$. If such a solution exists, it must satisfy

$$(30) \qquad\qquad u(t) = 1 + L \int_0^t u(\sigma + r)\, d\sigma.$$

We now show that a Piccard iteration scheme converges to

$$(31) \qquad f(t, r, L) = \{ \sum_{n=0}^{\infty} L^n t(t + nr)^{n-1}/n! \}$$

uniformly in any compact interval $[0, T]$, if $L r e < 1$.

Let $u^0(t) = 1$ for $t \geq 0$. Assume $u^N(t)$ is defined for $t \geq 0$, and define

$$u^{N+1}(t) = 1 + L \int_0^t u^N(\sigma + r) \, d\sigma.$$

Our induction hypothesis is that

$$u^N(t) = \sum_{n=0}^{N} L^n t(t + nr)^{n-1}/n! \, ,$$

which is clearly true for $N = 0$. Thus

$$u^{N+1}(t) = 1 + L \int_0^t u^N(\sigma + r) \, d\sigma$$

$$= 1 + \sum_{n=0}^{N} (L^{n+1}/n!) \int_0^t (\sigma + r) [\sigma + (n + 1)r]^{n-1} \, d\sigma$$

$$= \sum_{n=0}^{N+1} L^n t(t + nr)^{n-1}/n! \, ,$$

as can easily be checked. It is clear that $u^N(t)$ is defined for all $t \geq 0$ and all integers $N \geq 0$ by the induction step. To show convergence of (31), fix $T > 0$ and let $\delta > 0$ satisfy $L(r + \delta) e = 1$, which is possible since $L r e < 1$. Select n large enough that Sterling's estimate for $n!$ is valid and that $n\delta \geq T$. Then for $0 \leq t \leq T \leq n\delta$, we have

$$\frac{L^n t(t + nr)^{n-1}}{n!} \qquad \leq \qquad \frac{L^n T[n(r + \delta)]^{n-1} e^n}{(2\pi n)^{1/2} n^n} \qquad =$$

$$= \frac{T L e [L(r + \delta) e]^{n-1}}{(2\pi)^{1/2} n^{3/2}} \qquad = \qquad \frac{T L e}{(2\pi)^{1/2}} n^{-3/2} \, .$$

Thus, the sum $\sum_{n=0}^{\infty} L^n t(t + nr)^{n-1}/n!$ converges uniformly and absolutely for

$0 \leq t \leq T$. The standard interchange of limit and integration then shows that (31) is a solution of (30) and hence (29).

We now show that $f(t, r, L)$ is the unique solution of (29) with minimum rate of growth that satisfies the initial condition $x(0) = 1$. Let $x(t)$ be any solution of (29) that satisfies $x(0) = 1$ and

$$|x(t)| \leq K' f(t, r, L) \text{ for } t \geq 0.$$

Then, if $u(t) = f(t, r, L)$ and $\delta(t) = u(t) - x(t)$, we have

$$|\delta(t)| \leq (K' + 1) f(t, r, L) = K f(t, r, L), \qquad t \geq 0$$

$$|\delta(t)| \leq L \int_0^t |\delta(\sigma + r)| \, d\sigma \leq K L \int_0^t f(\sigma + r, r, L) \, d\sigma$$

$$= K [f(t, r, L) - 1] = K \sum_{n=1}^\infty L^n t(t + nr)^{n-1}/n! \; .$$

Take the new estimate $\left(|\delta(t)| \leq K \sum_{n=1}^\infty L^n t(t + nr)^{n-1}/n! \right)$ and put it back in $|\delta(t)| \leq L \int_0^t |\delta(\sigma + r)| \, d\sigma$. This yields $|\delta(t)| \leq K \sum_{n=2}^\infty L^n t(t + nr)^{n-1}/n!$, $t \geq 0$. Iterate the procedure; thus, after the Nth step,

$$|\delta(t)| \leq K \sum_{n=N}^\infty L^n t(t + nr)^{n-1}/n! \; , \qquad t \geq 0.$$

But, since the series (31) converges uniformly and absolutely, the partial sums from N to ∞ must go to zero as N approaches ∞. Thus, $|\delta(t)| \equiv 0$, and hence

$$x(t) = u(t) \quad \text{for} \quad t \geq 0.$$

We now show that $f(t, r, L) = e^{\mu_0 t}$ where $\mu_0 = L f(r, r, L)$, and μ_0 is the characteristic root of (29) with minimum real part. The characteristic equation of (29) is (see (8))

$$(32) \qquad \qquad \mu = L e^{\mu r}.$$

Since this cannot be solved simply for μ, we study an inverted problem and treat L as a function of μ; that is

$$L(\mu) = \mu \, e^{\mu r},$$

for $\mu \geq 0$. $L(\mu)$ has one and only one maximum for $\mu \geq 0$ (at $\mu = {}^1/r$), and $L(0) = \lim\limits_{\mu \to \infty} L(\mu) = 0$. Hence

$$0 \leq L(\mu) \leq (r \, e)^{-1}.$$

Going back to (32), we see that if $L \, r \, e < 1$, then (32) has two real positive solutions. Let μ_o be the smaller of these; then $\mu_o < {}^1/r$, and $e^{\mu_o t}$ is a solution of (29). But $f(t, r, L) \leq K \, e^{\mu_o t}$, since $f(t, r, L)$ is the unique solution of (29) with minimum rate of growth. Since $f(0, r, L) = e^{\mu_o t}\big|_{t=0} = 1$, the following Gronwallian lemma shows $f(t, r, L) = e^{\mu_o t}$, since there can only be one solution of (29) with growth rate less than $e^{t/r}$.

Lemma 3.15: Let $u(t)$ be defined and positive for $t \geq 0$. Assume that $u(t)$ satisfies

(33) $$u(t) \leq K \, e^{t/r},$$

(34) $$u(t) \leq c + L \int_0^t \|u^r(\sigma)\| \, d\sigma,$$

where $L \, r \, e = \rho < 1$ and $c \geq 0$. Then

$$u(t) \leq c \, f(t, r, L) \leq c \, e^{t/r}.$$

Note: $\|u^r(\sigma)\| = \sup\limits_{0 \leq \lambda \leq r} |u(\sigma + r)|$.

Proof: $|u^r(t)| \leq K \, e^{(t + r)/r} = K \, e \, e^{t/r}$. Thus
$$u(t) \leq c + K \, L \, e \int_0^t e^{\sigma/r} \, d\sigma \leq c + K \, L \, e \, r \, e^{t/r} = c + K \, \rho \, e^{t/r}.$$

We claim that for $N \geq 0$,

$$(35) \qquad u(t) \leq c \sum_{n=0}^{N} L^n t(t + nr)^{n-1}/n! + K \rho^{N+1} e^{t/r},$$

which has just been shown for $N = 0$. Assume (35) true at N. Then

$$\|u^r(t)\| \leq c \sum_{n=0}^{N} L^n(t + r) [t + (n + 1)r]^{n-1}/n! + K e \rho^{N+1} e^{t/r},$$

and by (33),

$$u(t) \leq c [1 + \sum_{n=0}^{N} (L^{n+1}/n!) \int_{0}^{t} (\sigma + r) [\sigma + (n + 1) r]^{n-1} d\sigma$$

$$+ K L e \rho^{N+1} \int_{0}^{t} e^{\sigma/r} d\sigma$$

$$\leq c \sum_{n=0}^{N+1} L^n t(t + nr)^{n-1}/n! + K \rho^{N+2} e^{t/r},$$

and induction is valid. Let $N \to \infty$ in (35), then

$$u(t) \leq c f(t, r, L) \leq c e^{t/r}.$$

$$Q.E.D.$$

Finally, we need to show that $\mu_0 = L f(r, r, L)$. But, from (32),

$$\mu_0 = L e^{\mu_0 r} = L f(r, r, L)$$

since $e^{\mu_0 t} = f(t, r, L)$.

Summarizing the above, if $L r e < 1$, then

$$f(t, r, L) = \sum_{n=0}^{\infty} L^n t(t + nr)^{n-1}/n! = e^{\mu_0 t}$$

is the unique solution of

$$\dot{x}(t) = L x(t + r)$$

that is defined for all t, satisfies

$$x(0) = 1,$$

and has minimum rate of growth. Also

$$(36) \qquad \mu_o = L \ f(r, \ r, \ L) < {}^1/r.$$

The condition $L \ r \ e \leq 1$ becomes more reasonable if one notes that its negation implies that the characteristic equation (32) has no real roots, and thus a unique solution with minimum rate of growth cannot be defined since complex roots appear in conjugate pairs. It may be possible to extend this result by constructing a real solution with minimum rate of growth. The double root case when $L \ r \ e = 1$ is ignored here.

Later we will need stronger Gronwallian type results; in particular, the case where

$$u(t) \leq v(t) + L \int_o^t \ \|u^r(\sigma)\| \ d\sigma.$$

No general results are available for this problem yet, but the following corollaries are strong enough for our needs here and give some hint in the general case.

Corollary 3.15.1: Let $u(t)$ be as in the previous lemma, but with (34) replaced by

$$(37) \qquad u(t) \leq c \ t + L \int_o^t \ \|u^r(\sigma)\| \ d\sigma.$$

Then it follows that

$$u(t) \leq c \ e \ t \ f(t, \ r, \ L) \leq c \ e \ t \ e^{t/r}.$$

Proof: Putting (33) in (37) yields

$$u(t) \leq c \ t + K \ \rho \ e^{t/r}.$$

Iterating through (37) yields

$$u(t) \leq c \sum_{n=1}^{N} L^{n-1} t(t + nr)^{n-1}/n! + K \rho^N e^{t/r},$$

for all $N \geq 1$ and $t \geq 0$. Let $N \to \infty$, then

$$u(t) \leq c \sum_{n=1}^{\infty} t [L(t + nr)]^{n-1}/n! = c[f(t, r, L) - 1]/L$$

$$= c(e^{\mu_0 t} - 1)/L = c \{\exp[L f(r, r, L) t] - 1\}/L$$

$$\leq c t f(r, r, L) \exp[L f(r, r, L) t]$$

$$\leq c e t f(t, r, L).$$

$$\text{Q.E.D.}$$

Corollary 3.15.2: Let $u(t)$ be as before, but with (33) replaced by

(38) $$u(t) \leq c e^{t/r} + L \int_0^t \|u^r(\sigma)\| d\sigma,$$

where $c \geq 0$. Then if $\rho = L r e < 1$,

$$u(t) \leq c e^{t/r}/(1 - \rho).$$

Proof: Starting the iteration procedure as before, we see that

$$u(t) \leq c e^{t/r} + K \rho e^{t/r}.$$

The first iteration through (38) yields

$$u(t) \leq c e^{t/r} + c L \int_0^t e e^{\sigma/r} d\sigma + K \rho L \int_0^t e e^{\sigma/r} d\sigma$$

$$\leq c[1 + L r e] e^{t/r} + K \rho^2 e^{t/r}$$

$$= c[1 + \rho] e^{t/r} + K \rho^2 e^{t/r}.$$

Subsequent iterations yield

$$u(t) \leq c e^{t/r} (\sum_{n=0}^{N} \rho^n) + K \rho^{N+1} e^{t/r}$$

for all integers $N \geq 0$ and $t \geq 0$. Since $\rho < 1$, we have $\rho^{N+1} \to 0$ and $\sum_{n=0}^{M} \rho^n \to 1/(1 - \rho)$. Thus the result follows.

<div align="center">Q.E.D.</div>

Theorem 3.16: Let $X[t, \phi^r]$ be defined into R^m, continuous in t, and Lipschitz in ϕ^r with constant L for all t in R and all continuous ϕ^r in B_r^m. Let $\phi_0^r(\lambda)$ be a constant function and assume that $|X(t, \phi_0^r)| \leq M$ for all t. If $L r e < 1$, then there exists a solution $u(t)$ to the initial value problem

$$(39) \qquad \dot{x}(t) = X[t, x^r(t)],$$

$$(40) \qquad x(t_0) = x_0.$$

Further, $u(t)$ is the unique solution of (39, 40) that is

(a) defined for all t, $-\infty < t < \infty$, and

(b) satisfies $|u(t)| \leq H e^{t/r}$ for $t \geq 0$ and some $H > 0$.

Also, $u(t)$ satisfies the estimates

$$(41) \qquad |u(t)| \leq |x_0| + (K/L) \{f(t - t_0, r, L) - 1\}, \qquad t \geq t_0$$

$$(42) \qquad |u(t)| \leq |x_0| + (K/L) [f(r, r, L) - 1] e^{L(t_0 - t)}, \qquad t \leq t_0$$

where $K = M + L |x_0 - \phi_0^r(0)|$.

Proof: Assume without loss that $t_0 = 0$. Let $u_0(t) = x_0$ for $t \geq 0$. Assume that $u_n(t)$ is defined and continuous for $t \geq 0$. Then

$$u_n^r(t) = u_n^r(t, \lambda) = u_n(t + \lambda) \qquad , \qquad 0 \leq \lambda \leq r$$

is defined, continuous in λ, and in B_r^m for all $t \geq 0$; and thus $X[t, u_n^r(t)]$ is defined, and

$$u_{n+1}(t) = x_0 + \int_0^t X[\sigma, u_n^r(\sigma)] d\sigma$$

is defined and continuous for all $t \geq 0$. Thus the sequence $\{u_n(t)\}$ is defined for $t \geq 0$, $n \geq 0$.

We now show that on any compact interval $[0, T]$, the sequence $\{u_n(t)\}$ converges uniformly to a solution of (39, 40).

$$|u_1(t) - u_0(t)| \leq \int_0^t |X[\sigma, x_0^r]| \, d\sigma$$

$$\leq \int_0^t \{|X[\sigma, x_0^r] - X[\sigma, \phi_0^r]| + |X[\sigma, \phi_0^r]|\} \, d\sigma$$

$$\leq L \|x_0^r - \phi_0^r\| \, t + Mt = Kt,$$

for $t \geq 0$, where $x_0^r(x) = x_0$, $0 \leq \lambda \leq r$. Thus,

$$\|u_1^r(t) - u_0^r(t)\| \leq K(t + r).$$

Claim that

$$(43) \qquad |u_n(t) - u_{n-1}(t)| \leq Kt \, \{L(t + nr)\}^{n-1}/n!.$$

By the above, (43) is true for $n = 1$. Assume it true for $n = N$, then

$$\|u_N^r(t) - u_{N-1}^r(t)\| \leq K(t + r) \, \{L[t + (n + 1) \, r]\}^{N-1}/N! \, .$$

Thus

$$|u_{N+1}(t) - u_N(t)| \leq \int_0^t |X[\sigma, x_n^r(\sigma)] - X[\sigma, x_{N-1}^r(\sigma)]| \, d\sigma$$

$$\leq L \int_0^t \|u_N^r(\sigma) - u_{N-1}^r(\sigma)\|$$

$$\leq \frac{K \, L^N}{N!} \int_0^t (\sigma + r) \, [\sigma + (n + 1) \, r]^{n-1} \, d\sigma$$

$$\leq K \, t \, \{L[t + (N + 1) \, r]\}^N/(N + 1)!,$$

and the induction is valid so that (43) holds for all $n \geq 0$ and all $t \geq 0$.

Let

$$(44) \qquad u(t) = x^0 + \sum_{n=1}^{\infty} \; u_n(t) - u_{n-1}(t),$$

then

$$|u(t)| \leq |x^0| + \sum_{n=1}^{\infty} |u_n(t) - u_{n-1}(t)|$$

$$\leq |x^0| + (K/L) \sum_{n=1}^{\infty} \; L^n \, t(t + nr)^{n-1}/n!$$

$$= |x^0| + (K/L) \, \{f(t, \, r, \, L) - 1\},$$

for $t \geq 0$ and (41) holds. It has already been seen that the series for $f(t, \, r, \, L)$ converges uniformly on $[0, \, T]$ for $L \, r \, e < 1$, so it follows that (44) converges uniformly on $[0, \, T]$. The fact that $u(t)$ is a solution of (39) follows by the same interchange of limit and integration used in the standard Piccard iteration proof that is valid by the uniform convergence. Thus we have existence for $t \geq 0$.

It follows from the Gronwallian lemma (3.15) that $u(t)$ is the unique solution of (39, 40) that is defined for $t \geq 0$ and satisfies (b). For $t \leq 0$, existence and uniqueness follow from Theorem 3.1, because (39) is an equation with time delay for $t \leq 0$. The estimate (42) follows from Gronwall's lemma for equations with time delay (Lemma 3.2) and

$$\|u^r(0) - x_0^r\| \leq (K/L) \, \{f(r, \, r, \, L) - 1\},$$

since

$$|u(t) - x_0| \leq \left\{ \begin{array}{ll} \|u^r(0) - x_0^r\| & , \; 0 \leq t \leq r \\ \|u^r(t) - x_0^r\| + L \int_0^t \|u^r(\sigma) - x_0^r\| \, d\sigma, & t \leq 0 \end{array} \right\}.$$

Q.E.D.

Corollary 3.16.1: Let $X[t, \phi^r]$ be as in the previous theorem, but assume that $\phi_o^r = 0$ and $M = 0$ (that is, $X[t, 0] = 0$). Then the same conclusion holds with $K = L |x_o|$.

Corollary 3.16.2: Let $X(t, \phi^r)$ be as in the previous theorem, but assume that $|X(t, \phi^r)| \leq M$ for all t and all continuous $\phi^r \in B_r^m$. Then there exists a unique solution of (39, 40) that is defined for all t, $-\infty < t < \infty$.

Proof: The conditions of Theorem 3.16 are satisfied so that there exists a solution $u(t)$ of (39, 40) that is defined for all t and is unique in the class $\{\xi(t) : |\xi(t)| \leq H e^{t/r}, t \leq 0, H > 0\}$. But any solution $v(t)$ of (39, 40) that is defined for all t satisfies $|v(t)| \leq |x^o| + M(t - t_o)$, $t \geq t_o$. Thus $u(t)$ is the unique solution defined for all t.

Q.E.D.

Corollary 3.16.3: Let $X(t, \phi^r)$ be as in the previous theorem. Let $u(t, \xi)$ be the solution of (39) guaranteed by Theorem 3.16 that satisfies $u(t_o, \xi) = \xi$. Then

$$(45) \qquad |u(t, \xi_1) - u(t, \xi_2)| \leq |\xi_1 - \xi_2| f(t - t_o, r, L), \qquad t \geq t_o;$$

$$(46) \qquad |u(t, \xi_1) - u(t, \xi_2)| \leq |\xi_1 - \xi_2| f(r, r, L) e^{L(t_o - t)}, \quad t \leq t_o.$$

Proof: Let $\delta(t) = |u(t, \xi_1) - u(t, \xi_2)|$ and $\delta_o = |\xi_1 - \xi_2|$. Then $\delta(t) \leq K e^{t/r}$ for $t \geq 0$, and

$$\delta(t) \leq \delta_o + L \int_o^t \|\delta^r(\sigma)\| \, d\sigma, \qquad\qquad t \geq 0,$$

and (45) follows from the Gronwallian lemma. For $t \leq 0$,

$$\delta(t) \leq \begin{cases} \delta_o \, f(r, r, L) & 0 \leq t \leq r \\ \delta_o \, f(r, r, L) + L \int_o^t \|\delta^r(\sigma)\| \, d\sigma, & t \leq 0 \end{cases}$$

Thus (46) follows from Theorem 3.2.

Q.E.D.

Theorem 3.17: Let $X[t, \phi^r]$ map $R \times B_r^m$ into R^m. Assume X is continuous in t, continuously differentiable in ϕ^r, and $\|D_\phi r \, X[t, \phi^r]\| \leq L$ for all $t \in R$, $\phi^r \in B_r^m$. Assume also that $L \, r \, e = \rho < 1$ and $|X[t, \phi_o^r]| \leq M$, where $\phi_o^r(\lambda)$ is a constant function. Let $u(t, \xi)$ be the unique solution of

$$(47) \qquad \dot{x}(t) = X[t, x^r(t)]$$

that is defined for all $t \geq 0$, has minimum rate of growth, and $u(0, \xi) = \xi$. Then $u(t, \xi)$ is differentiable with respect to ξ for all $t \geq 0$, and $D_\xi \, u(t, \xi)$ is the unique solution of the equation

$$(48) \qquad \dot{U}(t) = D_\phi r \, X[t, u^r(t, \xi)] \circ U^r(t)$$

that is defined for all $t \geq 0$, has minimum rate of growth, and $U(0) = D_\xi \, \dot{u}(0, \xi) = I$.

Proof: Since $X[t, \phi^r]$ is differentiable for all $\phi^r \in B_r^m$, Theorem 3.9 applies and

$$|X[t, \phi_1^r] - X[t, \phi_2^r]|$$

$$\leq \|\phi_1^r - \phi_2^r\| \, \sup_{\psi^r} \|D_\phi r \, X[t, \psi^r]\|$$

$$\leq L \, \|\phi_1^r - \phi_2^r\|.$$

Thus the existence and uniqueness results apply to equations (47) and (48) so that all that is needed is to show $D_\xi \, u(t, \xi) = U(t)$. This requires showing that $\lim_{|\delta| \to 0} \frac{1}{|\delta|} |u(t, \xi + \delta) - u(t, \xi) - U(t) \circ \delta| = 0$. Let

$$\Delta(t) = |u(t, \xi + \delta) - u(t, \xi) - U(t) \circ \delta|$$

$$\leq \int_0^t \left| \begin{array}{l} X[\sigma, u^r(\sigma, \xi + \delta)] - X[\sigma, u^r(\sigma, \xi)] \\[6pt] - \mathcal{D}_\phi^{\;r} X[\sigma, u^r(\sigma, \xi)] \circ U^r(\sigma) \circ \delta \end{array} \right| \, d\sigma$$

$$\leq \int_0^t \left\{ \left| \begin{array}{l} X[\sigma, u^r(\sigma, \xi + \delta)] - X[\sigma, u^r(\sigma, \xi)] \\[6pt] - \mathcal{D}_\phi^{\;r} X[\sigma, u^r(\sigma, \xi)] \circ [u^r(\sigma, \xi + \delta) - u^r(\sigma, \xi)] \end{array} \right| \\[20pt] + |\mathcal{D}_\phi^{\;r} X[\sigma, u^r(\sigma, \xi)] \circ \Delta^r(\sigma)| \end{array} \right\} \, d\sigma$$

$$\leq \int_0^t \{ \varepsilon(\sigma, \delta) \| u^r(\sigma, \xi + \delta) - u^r(\sigma, \xi)\| + L \|\Delta^r(\sigma)\| \} \, d\sigma,$$

where by Theorem 3.10 (Mean Value)

$$\varepsilon(\sigma, \delta) = \sup \| \mathcal{D}_\phi^{\;r} X[\sigma, v^r(\sigma)] - \mathcal{D}_\phi^{\;r} X[\sigma, u^r(\sigma, \xi)]\|,$$

and the sup is over all $v^r(\sigma)$ such that

$$\{ v^r(\sigma) = \alpha \, u^r(\sigma, \xi + \delta) + (1 - \alpha) \, u^r(\sigma, \xi) \, , \qquad 0 \leq \alpha \leq 1 \}.$$

Without loss, we will assume that $\varepsilon(\sigma, \delta)$ is non-decreasing in either argument alone. By continuous dependence (Corollary 3.16.3), we have

$$\| u^r(\sigma, \xi + \delta) - u^r(\sigma, \xi)\| \leq |\delta| \, f(\sigma + r, r, L).$$

Thus, combining, we have

$$(49) \qquad \Delta(t) \leq |\delta| \int_0^t f(\sigma + r, r, L) \, \varepsilon(\sigma, \delta) \, d\sigma + L \int_0^t \|\Delta^r(\sigma)\| \, d\sigma.$$

We are going to apply Corollary 3.15.2 to (49), and it is here that a stronger form of Gronwall's lemma would be helpful. Fix $T > 0$, and choose $\varepsilon_0 > 0$. Select δ small enough that

(50) $$\varepsilon(\sigma, \delta) \leq \varepsilon_o \qquad\qquad 0 \leq \sigma \leq T;$$

(51) $$f(\sigma + r, r, L) \ \varepsilon(\sigma, \delta) \leq \varepsilon_o \ e^{\sigma/r}, \qquad \sigma \geq T.$$

Estimate (50) can be satisfied because $\mathcal{D}_\phi{}^r \ X[t, \phi^r]$ is continuous and (51) is possible because $f(\sigma + r, r, L) \ e^{-\sigma/r} \to 0$ as $\sigma \to \infty$ (see (36)). Thus (49) becomes

$$\Delta(t) \leq |\delta| \int_o^t \varepsilon_o \ e^{\sigma/r} \ d\sigma + L \int_o^t \|\Delta^r(\sigma)\| \ d\sigma$$

$$\leq \varepsilon_o \ r \ |\delta| \ e^{t/r} \ + \ L \int_o^t \|\Delta^r(\sigma)\| \ d\sigma.$$

Finally, by Corollary 3.15.2,

$$\Delta(t) \leq [\varepsilon_o \ |\delta| \ r \ e^{t/r}]/[1 - \rho].$$

Since ε_o was arbitrary, we see that

$$\lim_{|\delta| \to 0} \frac{1}{|\delta|} \left| u(t, \xi + \delta) - u(t, \xi) - U(t) \circ \delta \right| = \lim_{|\delta| \to 0} \frac{\Delta(t)}{|\delta|} = 0.$$

Thus $U(t)$ is the derivative $\mathcal{D}_\xi \ u(t, \xi)$ of $u(t, \xi)$.

$$\text{Q.E.D.}$$

IV. INVARIANT SURFACES OF ORDINARY DIFFERENTIAL EQUATIONS WITH TIME LAG

In this chapter, the results of Chapter II will be generalized to equations with time lag. The method will be the same as before with the details only slightly different.

Definition: A bounded, Lipschitz continuous function $S(\theta)$ from R^m into R^n will be called an Invariant Surface of the system

$$(1) \qquad \dot{\theta}(t) = \Theta[\theta_r(t), x_r(t), \varepsilon]$$

$$(2) \qquad \dot{x}(t) = A(\varepsilon) \circ x_r(t) + X[\theta_r(t), x_r(t), \varepsilon]$$

if the pair $\{\psi(t, \theta^o, S), S[\psi(t, \theta^o, S)]\}$ is a solution of (1, 2), where $\psi(t, \theta^o, S)$ is a solution of

$$(3) \qquad \dot{\theta}(t) = \Theta\{\theta_r(t), S[\theta_r(t)], \varepsilon\}$$

that is defined for all time and such that

$$(4) \qquad \psi(0, \theta^o, S) = \theta^o.$$

Lemma 4.1: Given the system (1, 2), assume that there exists a $\mu_o > 0$ such that all characteristic roots μ of the continuous linear operator $A(\varepsilon)$ satisfy

$$|\text{Re}(\mu)| > \varepsilon \mu_o.$$

Assume also that for some $\varepsilon \neq 0$, there exists an invariant surface $S(\theta)$ of (1, 2), such that

$$(5) \qquad \Psi[\theta_r(t), \varepsilon] = \Theta\{\theta_r(t), S[\theta_r(t)], \varepsilon\}$$

is bounded and uniformly Lipschitz in θ_r with constant L and that L r e < 1.

Then

$$S(\theta) = \int_{-\infty}^{0} [J(-\sigma) \circ I_r^-] \, X \, \{\psi_r(\sigma, \theta, S), \ S[\psi_r(\sigma, \theta, S)], \ \varepsilon\} \, d\sigma$$

$$-\int_{0}^{\infty} [J(-\sigma) \circ I_r^+] \, X \, \{\psi_r(\sigma, \theta, S), \ S[\psi_r(\sigma, \theta, S)], \ \varepsilon\} \, d\sigma$$

where $J(\sigma)$ is the fundamental operator associated with $A(\varepsilon)$, and $\psi(t, \theta, S)$ is the unique solution of (3), (4) that is defined for all time.

Proof: Equation (3) in the notation of (5) is $\dot{\theta}(t) = \Psi[\theta_r(t), \varepsilon]$. Since Ψ is Lipschitz and $L \, r \, e < 1$, Theorem 3.16 applies, and $\psi(t, \theta^\circ, S)$ is the unique solution of (3) satisfying (4) that is defined for all t. Then by the definition of an invariant surface,

$$x(t) = S[\psi(t, \theta^\circ, S)]$$

is a bounded solution of (2) that is defined for all time, so that Theorem 3.6 applies and we must have

$$S[\psi(t, \theta^\circ, S)] =$$

$$= \int_{-\infty}^{0} [J(-\sigma) \circ I_r^-] \, X \, \{\psi_r(t + \sigma, \theta^\circ, S), \ S[\psi_r(t + \sigma, \theta^\circ, S)], \ \varepsilon\} \, d\sigma$$

$$- \int_{0}^{\infty} [J(-\sigma) \circ I_r^+] \, X \, \{\psi_r(t + \sigma, \theta^\circ, S), \ S[\psi_r(t + \sigma, \theta^\circ, S)], \ \varepsilon\} \, d\sigma.$$

Since $\psi(t, \theta^\circ, S)$ is uniquely determined by θ°, we have

$$\psi(t + \sigma, \theta^\circ, S) = \psi[\sigma, \psi(t, \theta^\circ, S) \, S].$$

Let $\theta = \psi(t, \theta^\circ, S)$, then

$$S(\theta) = \int_{-\infty}^{0} [J(-\sigma) \circ I_r^-] \, X \, \{\psi_r(\sigma, \theta, S), \ S[\psi_r(\sigma, \theta, S)], \ \varepsilon\} \, d\sigma$$

$$- \int_{0}^{\infty} [J(-\sigma) \circ I_r^-] \, X \, \{\psi_r(\sigma, \theta, S), \ S[\psi_r(\sigma, \theta, S)], \ \varepsilon\} \, d\sigma.$$

This expression is independent of t or θ° and is defined for all θ.

Q.E.D.

Notation: In this chapter, B_r^n will denote the space of continuous function in R^n. I_r is the only non-continuous function we need here and $[J(-\sigma) \circ I_r^{\pm}]$ has already been discussed. Let $f(\cdot)$ be a function defined on R^n or B_r^n. We will use the norm

$$\|\!\| f \|\!\| \ = \sup_{\theta \, \in \, R^n} \ \|f(\theta)\|,$$

$$\|\!\| f \|\!\| \ = \sup_{\theta_r \, \in \, B_r^n} \ \|f(\theta_r)\|,$$

where $\|f(\cdot)\|$ is an appropriate norm. As before, we will work in the Banach spaces

$$\Omega(\rho_1) = \{S : R^m \to R^n \ : \ \|\!\| S \|\!\| \ \leq \rho_1\}, \text{ and the subspaces of } \Omega(\rho_1)$$

$$\Omega(\rho_1 \ \rho_2) = \{S \in \Omega(\rho_1) : \mathcal{D}_\theta \, S(\theta) \text{ is continuous, and } \|\!\| \mathcal{D}_\theta \, S \|\!\| \ \leq \rho_2\}.$$

Theorem 4.2: Consider the system of equations

(6) $$\dot{\theta}(t) = \Theta[\theta_r(t), x_r(t), \varepsilon]$$

(7) $$\dot{x}(t) = E(\varepsilon) \circ \{A \circ x_r(t) + X[\theta_r(t), x_r(t), \varepsilon]\}$$

where θ and Θ are m-vectors, x and X are n-vectors, E is an $n \times n$ matrix, and A is a continuous linear mapping from B_r^n into R^n. Assume that Θ and X are defined and bounded for all $\theta_r \in B_r^m$, $x_r \in B_r^n$, $\|x_r\| \leq \alpha$, and $|\varepsilon| \leq \beta$. Assume also that:

(a) $\quad$ $E(\varepsilon)$ is the matrix

$$E(\varepsilon) = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & \varepsilon I_2 & 0 \\ 0 & 0 & (^1/\varepsilon)I_3 \end{bmatrix},$$

(b)   A is the block diagonal operator

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix},$$

no characteristic root is purely imaginary, and μ and K are the constants guaranteed by Lemma 3.5.

(c) $\||\mathcal{D}_{\theta_r} \Theta(\cdot, x_r, \varepsilon)\|| \leq \varepsilon \mu/(2\,e)$

$\||\mathcal{D}_{x_r} \Theta(\cdot, x_r, \varepsilon\|| \leq \varepsilon\, c_1,$

for some constant $c_1 \geq 0$.

(d) $\|| X(\cdot, x_r, \varepsilon)\|| \leq c_2 \|x_r\| + c_3(\varepsilon),$

$\||\mathcal{D}_{\theta_r} X(\cdot, x_r, \varepsilon)\|| \leq c_2 \|x_r\| + c_3(\varepsilon),$

$\||\mathcal{D}_{x_r} X(\cdot, x_r, \varepsilon)\|| \leq c_2,$

where $c_3(\varepsilon) \to 0$ as $|\varepsilon| \to 0$, and $0 \leq c_2 \leq \mu/(18\,K\,e)$.

Then there exists a ρ_2^0, $\rho_1^0(\rho_2^0)$, $\varepsilon^0(\rho_1^0, \rho_2^0)$ such that for $0 < |\varepsilon| \leq \varepsilon^0$ there exists a unique invariant surface $S(\theta)$ of (6, 7) in $\Omega(\rho_1^0)$ that is Lipschitz continuous with constant ρ_2^0, and $\|| S \|| \to 0$ as $|\varepsilon| \to 0$.

Proof: Assume that for some ε, ρ_1, and ρ_2 an invariant surface $S(\theta)$ exists. For this $S(\theta)$, equation (6) becomes

(8) $\dot{\theta}(t) = \Psi_S[\theta_r(t), \varepsilon] = \Theta \{\theta_r(t), S[\theta_r(t)], \varepsilon\}.$

$\Psi_S(\theta_r, \varepsilon)$ is uniformly Lipschitz in θ_r since

$$\||| \mathcal{D}_{\theta_r} \Psi_S(\cdot, \varepsilon) \|||$$

$$\leq \||| \mathcal{D}_{\theta_r} \Theta[\cdot, S(\cdot), \varepsilon] \||| + \||| \mathcal{D}_{x_r} \Theta[\cdot, S(\cdot), \varepsilon] \||| \, \||| \mathcal{D}_\theta S \|||$$

$$\leq |\varepsilon| \, [\mu/2 + c_1 \rho_2] = L_\varepsilon.$$

It should be noted here that, although $S(\theta)$ is a mapping from R^m into R^n, it induces a mapping $\overline{S}$ from B_r^m into B_r^n by

$$[\overline{S}(\phi_r)](\lambda) = S[\phi_r(\lambda)],$$

and upon differentiation,

$$[\mathcal{D}_{\theta_r} \overline{S}(\phi_r)](\lambda) = \mathcal{D}_\theta S[\phi_r(\lambda)].$$

Thus taking norms we have

$$\||| \overline{S} \||| = \||| S \|||, \text{ and } \||| \mathcal{D}_{\theta_r} S \||| = \||| \mathcal{D}_\theta S \|||.$$

This last fact has been tacitly used above and will be used later without comment.

Continuing on Ψ_S, we see that if $|\varepsilon|$ is small enough, then $L_\varepsilon r e < 1$ so that Lemma 4.1 applies, and

$$S(\theta) = \int_{-\infty}^{0} [J_E(-\sigma) \circ I_r^-] \, X\{\psi_r(\sigma, \theta, S), s[\psi_r(\sigma, \theta, S)], \varepsilon\} \, d\sigma$$

$$- \int_{0}^{\infty} [J_E(-\sigma) \circ I_r^+] \, X\{\psi_r(\sigma, \theta, S), s[\psi_r(\sigma, \theta, S)], \varepsilon\} \, d\sigma,$$

where $J_E(\sigma)$ is the fundamental operator associated with $E(\varepsilon) \circ A$. As before, we will notationally suppress the first integral. This will require some caution since the estimates on $\psi(t, \theta, S)$ are different for $t \leq 0$ and $t \geq 0$. The poorer estimates occur for $t \leq 0$, and we will have to use them. Then $S(\theta)$ must satisfy

$$S(\theta) = -\int_{0}^{\infty} [J_E(-\sigma) \circ I_r^+] \, X\{\psi_r(\sigma, \theta, S), s[\psi_r(\sigma, \theta, S)], \varepsilon\} \, d\sigma,$$

where $\psi(\sigma, \theta, S)$ is the unique solution of (8) that is defined for all time and satisfies $\psi(0, \theta, S) = \theta$.

We now make the change of integration variable used before; that is, $\tau \to \tau/\varepsilon$ in the degenerate term and $\tau \to \varepsilon\tau$ in the singular term. Then

$$S(\theta) = -\int_0^\infty j(\sigma)\, X\{\psi_r(\sigma, \varepsilon), S[\psi_r(\sigma, \varepsilon)], \varepsilon\}\, d\sigma,$$

where

$$\psi(\sigma, \varepsilon) = \begin{cases} \psi(\sigma, \theta, S) & \text{normal} \\ \psi(\sigma/\varepsilon, \theta, S) & \text{in the degenerate term} \\ \psi(\varepsilon\sigma, \theta, S) & \text{singular} \end{cases},$$

and $j(\sigma) = J(-\sigma) \circ I_r^+$, where now $J(\sigma)$ is the fundamental operator associated with A. Then by Lemma 3.5 and assumption (b), we have

$$|J(\sigma)| \le K\, e^{-\mu\sigma}, \qquad \sigma \ge 0.$$

We define T_ε on $\Omega(\infty, \infty)$ by

$$(9) \qquad [T_\varepsilon S](\theta) = -\int_0^\infty j(\sigma)\, X\{\psi_r(\sigma, \varepsilon), S[\psi_r(\sigma, \varepsilon)], \varepsilon\}\, d\sigma.$$

Then following the procedure used before

$$\|T_\varepsilon S\| \le \int_0^\infty |j(\sigma)|\, \|\|X[\cdot,\, S(\cdot),\, \varepsilon]\|\|\, d\sigma$$

$$\le K[c_2\, \rho_1 + c_3(\varepsilon)] \int_0^\infty e^{-\mu\sigma}\, d\sigma$$

$$\le K[c_2\, \rho_1 + c_3(\varepsilon)]/\mu.$$

But $K\, c_2/\mu \le {}^1/18$ and $c_3(\varepsilon) \to 0$ as $|\varepsilon| \to 0$, so that $\|T_\varepsilon S\| \le \rho_1$ if $\|\|S\|\| \le \rho_1$ and $|\varepsilon|$ is small enough.

Before we can estimate $\||\mathcal{D}_\theta\, T_\varepsilon\, S\||$, we will need an estimate on $\||\mathcal{D}_\theta\, \psi_r(\sigma,\,\theta,\,S,\,\varepsilon)\||$. By Lemma 3.17 we know that $\mathcal{D}_\theta\, \psi(\sigma,\,\theta,\,S)$ satisfies

$$\dot{\Phi}(t) = \mathcal{D}_{\theta_r}\Psi_S[\psi_r(t,\,\theta,\,S),\,\varepsilon] \circ \Phi(t), \qquad \Phi(0) = I.$$

For Lemma 3.17 to apply, we have to assume that $|\varepsilon|$ is small enough that $L_\varepsilon\, r\, e < 1$. Then by Corollary 3.16.1, we have that

$$|\Phi(t)| \le e^{L_\varepsilon e|t|} \quad \text{for all } t.$$

$(f(t,\,r,\,L) \le e^{Let}, \quad t \ge 0.)$ Thus

$$\||\mathcal{D}_\theta\, \psi_r(\sigma,\,\cdot,\,S)\|| \le e\, e^{L_\varepsilon e\,|\sigma|}$$

and, after the change of integration variable,

$$\||\mathcal{D}_\theta\, \psi_r(\sigma,\,\cdot,\,S,\,\varepsilon)\|| \le e\,\exp\{L_\varepsilon\, e\sigma)/\varepsilon\}$$

$$\le e\,\exp\{(\mu/2 + c_1\,\rho_2)\, e\sigma\}, \qquad \sigma \ge 0.$$

Let $\gamma = \mu/2 + c_1\,\rho_2\, e$ and require

$$(10) \qquad\qquad \rho_2 \le \mu/(3\, c_1\, e).$$

Then $\gamma \le 5\mu/6 < \mu$,

$$(11) \qquad \||\mathcal{D}_\theta\, \psi_r(\sigma,\,\cdot,\,S,\,\varepsilon)\|| \le e\, e^{\gamma\sigma}, \qquad \sigma \ge 0.$$

Coming back to $\||\mathcal{D}_\theta\, T_\varepsilon\, S\||$, we have

$$\||\mathcal{D}_\theta\, T_\varepsilon\, S\|| \le \int_0^\infty j(\sigma)\,\||\mathcal{D}_\theta\, X\,\{\cdot\}\||\, d\sigma,$$

where

$$\mathcal{D}_\theta\, X\{\psi_r(\sigma,\,\varepsilon),\,S[\psi_r(\sigma,\,\varepsilon)],\,\varepsilon\} =$$

$$= \mathcal{D}_{\theta_r} X\{\cdot\} \circ \mathcal{D}_\theta \psi_r(\cdot) + \mathcal{D}_{x_r} X\{\cdot\} \circ \mathcal{D}_\theta S(\cdot) \circ \mathcal{D}_\theta \psi_r(\cdot).$$

Thus

$$\|\!|\!| \mathcal{D}_\theta T_\varepsilon S |\!|\!|\!$$

$$\leq \int_0^\infty |j(\sigma)| \{\|\!|\!| \mathcal{D}_{\theta_r} X(\cdot) |\!|\!| + \|\!|\!| \mathcal{D}_{x_r} X(\cdot) |\!|\!| \ \|\!|\!| \mathcal{D}_\theta S |\!|\!| \} \ \|\!|\!| \mathcal{D}_\theta \psi_r(\sigma) |\!|\!| \ d\sigma$$

$$\leq K\,e[c_2\,\rho_1 + c_3(\varepsilon) + c_2\,\rho_2] \int_0^\infty e^{\gamma\sigma}\,e^{-\mu\sigma}\,d\sigma$$

$$\leq K\,e[c_2(\rho_1 + \rho_2) + c_3(\varepsilon)]/(\mu - \gamma)$$

$$\leq 6\,K\,e[c_2(\rho_1 + \rho_2) + c_3(\varepsilon)]/\mu.$$

Estimate (11), assumption (d), and the definition of γ have been used. But we have assumed in (d) that $K\,e\,c_2/\mu \leq {}^1\!/18$, so that if

$$(12) \qquad\qquad \rho_1 \leq \mu\,\rho_2/(18\,K\,e\,c_2)$$

and ε^0 is small enough that for $0 < |\varepsilon| \leq \varepsilon^0$,

$$6\,K\,e\,c_3(\varepsilon)/\mu \leq {}^1\!/3.$$

Then for $\|\!|\!| S |\!|\!| \leq \rho_1$, $\|\!|\!| \mathcal{D}_\theta S |\!|\!| \leq \rho_2 \leq \mu/(3\,c_1\,e)$, we have $\|\!|\!| \mathcal{D}_\theta T_\varepsilon S |\!|\!| \leq \rho_2$. Thus we have shown that if (10) and (12) hold, and if $|\varepsilon|$ is small enough, then T_ε maps $\Omega(\rho_1, \rho_2)$ into itself.

To show that T_ε is a contraction, let $S^1, S^2 \in \Omega(\rho_1 \ \rho_2)$. Then

$$(13) \qquad \|\!|\!| T_\varepsilon S^1 - T_\varepsilon S^2 |\!|\!|$$

$$\leq \int_0^\infty |j(\sigma)| \ \|\!|\!| X[\psi_r^1, S^1(\psi_r^1), \varepsilon] - X[\psi_r^2, S^2(\psi_r^2), \varepsilon] |\!|\!| \ d\sigma$$

$$\leq \int_0^\infty |j(\sigma)| \left\{ \begin{array}{l} \|\!|\!| X[\psi_r^1, S^1(\psi_r^1), \varepsilon] - X[\psi_r^2, S^1(\psi_r^1), \varepsilon] |\!|\!| \\[4pt] + \|\!|\!| X[\psi_r^2, S^1(\psi_r^1), \varepsilon] - X[\psi_r^2, S^1(\psi_r^2), \varepsilon] |\!|\!| \\[4pt] + \|\!|\!| X[\psi_r^2, S^1(\psi_r^2), \varepsilon] - X[\psi_r^2, S^2(\psi_r^2), \varepsilon] |\!|\!| \end{array} \right\} \ d\sigma$$

$$\leq \int_0^\infty |j(\sigma)| \left\{ \begin{array}{l} \|\mathcal{D}_{\theta_r} X(\cdot)\| \ \|\psi_r^1 - \psi_r^2\| \\[2mm] + \|\mathcal{D}_{X_r} X(\cdot)\| \ \|\mathcal{D}_\theta S^1\| \ \|\psi_r^1 - \psi_r^2\| \\[2mm] + \|\mathcal{D}_{X_r} X(\cdot)\| \ \|S^1 - S^2\| \end{array} \right\} d\sigma.$$

Before we can proceed, we need to obtain an estimate on $\|\psi_r^1(\sigma, \cdot, S; \varepsilon) - \psi_r^2(\sigma, \cdot, S, \varepsilon)\|$. Before the change of variables we have

$$|\psi^1(t, \theta, S^1) - \psi^2(t, \theta, S^2)|$$

$$\leq \left| \int_0^t \Theta\{\psi_r^1(\sigma), S^1[\psi_r^1(\sigma)]\} - \Theta\{\psi_r^2(\sigma), S^2[\psi_r^2(\sigma)]\} \, d\sigma \right|$$

$$\leq \left| \int_0^t \left\{ \begin{array}{l} \|\mathcal{D}_{\theta_r} \Theta(\cdot)\| \ \|\psi_r^1(\sigma, S^1) - \psi_r^2(\sigma, S^2)\| \\[2mm] + \|\mathcal{D}_{X_r} \Theta(\cdot)\| \ \|\mathcal{D}_\theta S\| \ \|\psi^1(\sigma, S^1) - \psi_r^2(\sigma, S^2)\| \\[2mm] + \|\mathcal{D}_{X_r} \Theta(\cdot)\| \ \|S^1 - S^2\| \end{array} \right\} d\sigma \right|$$

$$\leq \varepsilon \, c_1 \ \|S^2 - S^2\| \ |t| + L_\varepsilon \left| \int_0^t \|\psi_r^1(\sigma, S^1) - \psi_r^1(\sigma, S^2) \, d\sigma \right|.$$

Thus the proof of Corollary 3.15.1 implies that

$$|\psi^1(t, \theta, S^1) - \psi^2(t, \theta, S^2)|$$

$$\leq \ \varepsilon \, c_1 \ \|S^1 - S^2\| \ (e^{L_\varepsilon \, e \, |t|} - 1)/L_\varepsilon$$

$$\leq \ c_1 \ \|S^1 - S^2\| \ (e^{\gamma \, |t|/\varepsilon} - 1)/\gamma.$$

This is uniform in θ, and is valid for all t. After the change of variables, we have

$$\|\psi_r^1(\sigma, S^1) - \psi_r^2(\sigma, S^2)\|$$

$$\leq \ c_1 \, e \, \|S^1 - S^2\| \ (e^{\gamma t} - 1)/\gamma, \qquad \text{for} \qquad t \geq 0.$$

Taking this back to (13) yields

$$\||T_\varepsilon S^1 - T_\varepsilon S^2\|| \le \||S^1 - S^2\|| \int_0^\infty |j(\sigma)| \, F(\sigma) \, d\sigma,$$

where

$$F(\sigma) = [\||\mathcal{D}_{\theta_r} X(\cdot)\|| + \||\mathcal{D}_{x_r} X(\cdot)\|| \; \||\mathcal{D}_\theta S\||] .$$
$$\||\psi_r^1(\sigma, S^1) - \psi_r^2(\sigma, S^2)\|| + \||\mathcal{D}_{\theta_r} X(\cdot)\||$$

$$\le c_1 \, e[c_2 \, \rho_1 + c_3(\varepsilon) + c_2 \, \rho_2] \, (e^{\gamma\sigma} - 1)/\gamma + c_2 .$$

$$\Gamma = \int_0^\infty |j(\sigma)| \, F(\sigma) \, d\sigma$$

$$\le K \, c_1 \, e[c_2 \, \rho_1 + c_3(\varepsilon) + c_2 \, \rho_2] . \int_0^\infty e^{-\mu\sigma} (e^{\gamma\sigma} - 1)/\gamma \, d\sigma$$
$$+ K \, c_2 \int_0^\infty e^{-\mu\sigma} \, d\sigma$$

$$\le K \, c_2/\mu + K \, c_1 \, e[c_2 \, \rho_1 + c_3(\varepsilon) + c_2 \, \rho_2]/[\mu(\mu + \gamma)].$$

Select ρ_1, ρ_2, and ε_0 such that

$$\rho_1 \le \mu(\mu + \gamma)/(4 \, K \, c_1 \, c_2 \, e)$$

$$\rho_2 \le \mu(\mu + \gamma)/(4 \, K \, c_1 \, c_2 \, e)$$

$$c_3(\varepsilon) \le \mu(\mu + \gamma)/(4 \, K \, c_1 \, c_2 \, e) \qquad \text{for } |\varepsilon| \le \varepsilon_0.$$

Then $\Gamma \le 1/18 + 1/4 + 1/4 + 1/4 < 1$ and T_ε is a contraction on $\Omega(\rho_1, \rho_2)$.
The proof now follows as the proof of Theorem 2.2.

Q.E.D.

Corollary 4.2.1: If in Theorem 4.2, A is an arbitrary continuous
operator with no purely imaginary characteristic roots, the theorem remains
true.

Proof: Constants μ and K are still guaranteed to exist by Lemma 3.5, and the change of integration variable is still valid, so that equation (9) is valid. The proof follows identically.

Q.E.D.

Corollary 4.2.2: If in Theorem 4.2 the degenerate term is absent; that is

$$E(\varepsilon) = \begin{bmatrix} I_1 & 0 \\ 0 & (^1/\varepsilon)I_3 \end{bmatrix},$$

then condition (c) can be replaced by

(c′) $\||D_{\theta_r} \Theta(\cdot, x_r, \varepsilon)\|| \leq \min[\mu/(2\ e), (2\ r\ e)^{-1}]$

$\||D_{x_r} \Theta(\cdot, x_r, \varepsilon)\|| \leq c_1,$

and the conclusion remains valid.

Proof: See Corollary 2.2.1.

Remark: The invariant surfaces obtained in this work are entities in Euclidian spaces, while previous authors (in particular Halany [9, pp. 501-509] have had their invariant surfaces be entities in Banach spaces of continuous functions. Also, there the "Angular" variables, θ, enter with time delay, which Halany [9, p. 509] describes as an open question. The price paid for these extensions is that the surfaces obtained are proven to be invariant only in relation to solutions defined for all time. Thus if $\psi(t)$ is a solution of (6) that is only defined for $t \geq t_o$, it is not known under what conditions $S[\psi(t)]$ is a solution of (7) for $t \geq t_o$. It is conjectured that $S[\psi(t)]$ is not necessarily a solution of (7) for $t \geq t_o$ and that the behavior of $S[\psi(t)]$ for $t \geq t_o$ will have to be deduced from stability results.

REFERENCES

1. Sacker, R. J., A New Approach to the Perturbation Theory of Invariant Surfaces, Comm. Pure & App. Math., Vol. XVIII (4) 1965, pp. 717-732.

2. Hartman, Philip, Ordinary Differential Equations, John Wiley and Sons, Inc., New York, 1964.

3. Marica, J. G., Almost Periodic Surfaces (Thesis), University of California, Berkeley, 1961 (unpublished).

4. Coppell, W. A., Stability and Asymptotic Behavior of Differential Equations, D. C. Heath and Company, Boston, 1965.

5. Krasovskii, N., Some Problems in the Theory of Stability of Motion, Moscow, 1959. Translated by Stanford Univ. Press, Palo Alto, California, 1962.

6. Hale, J. K., Linear Functional-Differential Equations with Constant Coefficients, Contrib. Differential Equations, Vol. 2 (1963), pp. 291-317.

7. Hale, J. K., Averaging Methods for Differential Equations with Retarded Arguments and a Small Parameter, J. Diff. Equations, Vol. 2 (1966), pp. 57-73.

8. Hale, J. K. and Perelló, C., The Neighborhood of a Singular Point of Functional Differential Equations, Contrib. Diff. Equations, Vol. 3 (1964), pp. 351-375.

9. Halany, A., Differential Equations, Academic Press, New York, 1966.

10. Oguztöreli, N, Time Lag Control Processes, Academic Press, New York, 1966.

11. Dieudonné, J., Foundations of Modern Analysis, Academic Press, New York, 1960.

12.   Apostol, T. M., _Mathematical Analysis_, Addison-Wesley Publishing
       Company, Inc., Reading, Mass., 1957.

13.   Hale, J. K., _Oscillations in Nonlinear Systems_, McGraw-Hill Book
       Company, Inc., New York, N. Y., 1963.

14.   Diliberto, S. P., Perturbation Theorems for Periodic Surfaces, Circ.
       Mat. Palermo, Vol. II (9) 1960, pp. 265-299, and Vol. II (10) 1961,
       pp. 111-112.