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Lifshitz Holography

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Abstract

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In this dissertation, we examine the holographic description of strongly-coupled quantum field theories with Lifshitz fixed points. After reviewing the standard dictionary of Lifshitz holography, we carry out the holographic renormalization procedure for two different bulk gravitational theories that support asymptotically Lifshitz spacetimes. The first bulk theory is relativistic gravity with a massive vector and the second is an anisotropic theory of gravity.

In the bulk theory of relativistic gravity with a massive vector, we find that the holographic counterterms induced near anisotropic infinity take the form of the action for Hořava-Lifshitz (HL) gravity, with the appropriate value of the dynamical critical exponent z. In the particular case of $3+1$ bulk dimensions and $z=2$ asymptotic scaling near infinity, we find a logarithmic counterterm, related to anisotropic Weyl anomaly of the dual CFT, and show that this counterterm reproduces precisely the action of conformal gravity at a $z=2$ Lifshitz point in $2+1$ dimensions, which enjoys anisotropic local Weyl invariance. We find, however, that only one of two independent central charges appears in the anomaly.

We next argue that bulk HL gravity provides the minimal holographic dual for Lifshitz-type field theories with anisotropic scaling and dynamical exponent z. First we show that Lifshitz spacetimes are vacuum solutions of HL gravity, without the need for additional matter. Then we show that it reproduces the full structure of the $z=2$ anisotropic Weyl anomaly in dual field theories in $2+1$ dimensions, while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly.

Finally, we search for static asymptotically Lifshitz black hole solutions in HL gravity. In contrast to general relativity, we find that these static solutions do not have black hole horizons and instead contain naked singularities. In general, we argue that it is necessary to search for stationary (but non-static) black holes with universal horizons.

Contents

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Chapter 1

Introduction to Holography

Quantum field theory (QFT) provides the foundation for describing phenomena in particle physics, statistical mechanics and condensed matter physics. While many perturbative techniques have been developed to study quantum field theories at weak coupling, analyzing strongly-coupled field theories has been notoriously difficult. Given the ubiquity of QFTs in describing our physical world, the development of a tool that provides us with any information about the behavior of field theories at strong coupling is of great potential value.

Gauge-gravity duality is the surprising conjecture that, in some cases, a QFT can mathematically be described by a (quantum) theory of gravity. At first, this may not seem to be much of a simplification since quantum gravity is perhaps even more difficult to formulate and understood. But gauge-gravity duality is a strong-weak duality, which means that a strongly-coupled QFT corresponds to a weakly-coupled theory of gravity and vice-versa. In this way, the two descriptions are complementary to each other. Furthermore, in some specific limits, the dual gravitational theory can even be treated classically, making the theory very tractable. This, amazingly, means that properties of strongly-coupled QFTs can be studied using a weakly-coupled classical theory of gravity.

The first example of a gauge-gravity duality was provided by Maldacena [1], which provided strong evidence that the $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills on $\mathbf{R}^{3+1}$ is dual to Type IIB string theory on $AdS_5 \times S^5$. In this example, the classical gravity limit of the string theory describes the strongly coupled Yang-Mills QFT in the large N limit. Further examples of the duality soon followed and it became known as the *AdS/CFT correspondence*. Note that a key feature of this duality is that the QFT exists in a smaller number of spacetime dimensions than the gravity dual. For this reason, the duality has also become to be known as *holography*. We will provide a brief heuristic overview of the AdS/CFT correspondence, as well as its extension to nonrelativistic systems, in Section 1.2 below (for more complete reviews see, *e.g.*, [2, 3, 4, 5, 6]). Before proceeding, let us briefly outline the content and structure of this dissertation.

1.1 Outline

Following the introduction to holography presented in this chapter, Chapter 2 provides a brief review of the theory of Hořava-Lifshitz (HL) gravity. An overview of the systematics of holographic renormalization in Lifshitz spacetimes is then given in Chapter 3. An illustrative example of the holographic renormalization procedure is worked through in Chapter 4 for the simple example of pure Einstein gravity with cosmological constant. Chapters 1-4 serve as background material needed for the remainder of the dissertation. All of the content in Chapters 1-4 has appeared previously in the literature.

Chapters 5 and 6 feature original research conducted by the author in conjunction with Petr Hořava and Charles Melby-Thompson, as published in [7]. It involves the holographic renormalization of Lifshitz space with a relativistic bulk theory and the subsequent calculation of the $z = 2$ Weyl anomaly. Chapter 5 builds on work by [8, 9] and in addition writes the results using an ADM decomposition on the boundary and generalizes it to any number of spatial dimensions. Chapter 6 presents the first holographic calculation of a $z = 2$ gravitational Weyl anomaly and classifies such anomalies. It also confirms the natural appearance of conformal HL gravity in the Weyl anomaly.

Chapter 7 features original research conducted by the author in conjunction with Petr Hořava and Charles Melby-Thompson, as published in [10]. It presents a novel Lifshitz spacetime solution of a low-energy HL gravity action and applies the procedure of holographic renormalization to once again calculate the $z = 2$ Weyl anomaly.

Chapter 8 involves original research conducted by the author in conjunction with Petr Hořava and Omid Saremi. It involves a search for static Lifshitz black hole solutions of the low-energy HL gravity action of Chapter 7 that had not yet been attempted in the literature.

Finally, Chapter 9 summarizes the main findings of this dissertation.

1.2 The AdS/CFT Correspondence

We begin by identifying the gravitational theories that are dual to a class of QFTs known as conformal field theories (CFTs). A CFT in d spacetime dimensions is invariant under the symmetries of the $SO(d, 2)$ conformal symmetry group. The conformal group includes Poincaré transformations as well as the scale symmetry

$$x^\mu \to b x^\mu, \tag{1.1}$$

and the special conformal transformations:

$$x^\mu \to \frac{x^\mu - B^\mu x^2}{1 - 2B \cdot x + B^2 x^2}, \tag{1.2}$$

where b and B^μ parametrize the symmetry transformation and $\mu = 1, \ldots d$. Note that (1.1) scales the time and space coordinates isotropically.

In looking for a gravitational dual, we need to look for a space-time metric which has $SO(d,2)$ as its isometry group, that is, the conformal symmetry is realized geometrically. The metric we are looking for is that of $(d+1)$-dimensional Anti de Sitter (AdS) spacetime[1]:

$$ds^2 = \left(\frac{r}{\ell}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \left(\frac{\ell}{r}\right)^2 dr^2. \tag{1.3}$$

(From now on, we will set its radius of curvature $\ell = 1$ for convenience.) This AdS metric is a solution of pure Einstein gravity with negative cosmological constant. Recall that the AdS metric in (1.3) has a *boundary* at $r = \infty$ and a *horizon* at $r = 0$. We denote the region between the boundary and the horizon as the *bulk*.

It is easy to check that (1.12) is invariant under the scaling symmetry:

$$x^\mu \to b x^\mu, \qquad r \to \frac{r}{b}, \tag{1.4}$$

and the special conformal symmetry:

$$x^\mu \to \frac{x^\mu - B^\mu(x^2 + r^{-2})}{1 - 2B\cdot x + B^2(x^2 + r^{-2})}, \qquad r \to r\left(1 - 2B\cdot x + B^2(x^2 + r^{-2})\right). \tag{1.5}$$

We also see that in addition to the original d spacetime coordinates of the CFT, the metric (1.12) has an addition dimension, r, known as the radial coordinate. How is this radial direction to be interpreted in the original CFT? The key to this interpretation comes from the fact that r is rescaled under the scaling symmetry (1.4). We already have an interpretation for what this scaling means in a field theory: it represents flow under the renormalization group. So, heuristically, we can think of the radial direction as representing the renormalization group energy scale of the dual field theory, with the region near $r = \infty$ (the boundary) representing the field theory at high (UV) energies and the region near $r = 0$ (the horizon) representing low (IR) energies. Of course, a CFT by definition does not change with energy scale and, correspondingly, AdS spacetime has the isometry (1.4) under changes of r.

The next challenge is to describe not just a CFT, but also a CFT that is deformed by relevant operators so that it flows away from conformality in the IR. Clearly the dual gravitational theory can no longer be purely AdS spacetime, since this is invariant under the scaling symmetry. What we need is a space-time that asymptotically looks like AdS as $r \to \infty$ but deviates away from AdS in the rest of the spacetime. In this way, we see we can describe a QFT near a conformal fixed point by a gravitational dual in asymptotically AdS spacetime[2].

This gives a suitable physical interpretation of the AdS background of the bulk gravity theory. The next question to ask is what do bulk fields in the gravitational field theory represent in the dual QFT? In fact, a bulk field corresponds to a local operator in the CFT.

[1]The AdS metric is written here in the Poincaré patch.

[2]Asymptotically AdS spacetimes will be defined more precisely in Section 3.1.

For example, the stress-energy tensor of the CFT is dual to gravitational modes in the bulk[3]. To provide some more concrete details of the correspondence in terms of partition functions, consider the example of a free scalar field in AdS spacetime:

$$S = -\frac{1}{2}\int d^dx dr\sqrt{-G}\left(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \mu^2\Phi^2\right) + S_{ct}, \tag{1.6}$$

where S_{ct} contains the boundary counterterms necessary to any remove divergences[4]. The equation of motion for the scalar has two solutions, which behave asymptotically near the boundary ($r \to \infty$) as $\Phi \sim r^{-\Delta_\pm}$, where $\Delta_\pm = \frac{d}{2} \pm \sqrt{(\frac{d}{2})^2 + m^2}$. That is:

$$\Phi \to \alpha r^{-\Delta_-} + \beta r^{-\Delta_+} \qquad \text{as } r \to \infty. \tag{1.7}$$

Φ will be dual to some local scalar operator $\mathcal{O}$ in the dual CFT. However, the dual CFT depends on the boundary condition that is chosen for Φ. If a Dirichlet (standard) boundary condition is chosen for the scalar ($\Phi \to \phi_0 r^{-\Delta_-}$ as $r \to \infty$), then the dual operator $\mathcal{O}_+$ will have scaling dimension Δ_+. Furthermore, we can now use the AdS/CFT correspondence to calculate the generating function for the operator $\mathcal{O}_+$, with source ϕ_0:

$$\left\langle e^{i\int\phi_0\mathcal{O}_+}\right\rangle_{CFT} = \int D\Phi e^{iS} \approx e^{iS}|_{Eq.of Motion} \tag{1.8}$$

where the last equality is satisfied when the gravitational theory can be treated classically. (1.8) allows us to calculate all CFT correlation functions involving $\mathcal{O}_+$ by functional differentiation. With this choice of boundary conditions, α in (1.7) represents the source ϕ_0 for $\mathcal{O}_+$ and β contributes to $\langle\mathcal{O}_+\rangle$.

We can instead use a Neumann (alternative) boundary condition, setting the renormalized radial momentum[5] on the boundary to $\Pi_\Phi \to J_0 r^{-\Delta_+}$ as $r \to \infty$. The dual operator $\mathcal{O}_-$ now has scaling dimension Δ_- and (1.8) becomes:

$$\left\langle e^{i\int J_0\mathcal{O}_-}\right\rangle_{CFT'} = \int D\Phi e^{iS+i\int_{boundary}\sqrt{-g}\Phi(J_0 r^{-\Delta_+})}. \tag{1.9}$$

With these alternative boundary conditions, β in (1.7) now acts as the source J_0 for $\mathcal{O}_-$ and α contributes to $\langle\mathcal{O}_-\rangle$. In fact, the theory with alternative boundary conditions is related to the theory with standard boundary conditions by a Legendre transformation:

$$\left\langle e^{i\int J_0\mathcal{O}_-}\right\rangle_{CFT'} = \int D\phi_0 e^{\int\phi_0 J_0}\left\langle e^{i\int\phi_0\mathcal{O}_+}\right\rangle_{CFT}. \tag{1.10}$$

[3]CFTs always have a stress-energy operator and this explains why we always require dynamical gravity in the bulk.

[4]The first counterterm takes the form $S_{ct} = -\int_{boundary} d^dx\sqrt{-g}(\frac{\Delta_-}{2}\Phi^2 + ...)$, where $g_{\alpha\beta} = r^2\eta_{\alpha\beta}$ is the induced boundary metric [11].

[5]This is defined as $\Pi_\Phi \equiv -r\partial_r\Phi + \frac{1}{\sqrt{-g}}\frac{\delta S_{ct}}{\delta\Phi} \approx -r\partial_r\Phi - \Delta_-\Phi + ...$

Furthermore, the addition of a double-trace deformation to the boundary field theories induces a flow from the CFT' in the alternative quantization to the CFT in the standard quantization. More details on the choice of boundary conditions and their interpretation can be found in [12, 13].

The well-known BF-bound [14, 15] states that the allowed boundary conditions for a scalar in AdS spacetime depends upon the scalar's mass. The standard boundary condition is only allowed for $m^2 \geq -(d/2)^2$ whereas the alternative boundary condition is only allowed for $1-(d/2)^2 \geq m^2 \geq -(d/2)^2$. This implies that the operator $\mathcal{O}_+$ has scaling dimension $\Delta_+ \geq d/2$ and the operator $\mathcal{O}_-$ has scaling dimension $(d-2)/2 \leq \Delta_- \leq d/2$. Recall that the scaling dimension of any scalar operator in a unitary CFT satisfies the unitarity bound $\Delta \geq (d-2)/2$ and thus every such operator can be related to a scalar in the bulk with an appropriate choice of boundary condition.

1.3 Lifshitz Holography

The AdS/CFT correspondence provides a dual description of conformal fixed points. But there is another class of nonrelativistic fixed points that occurs in quantum field theories known as Lifshitz fixed points. Such Lifshitz QFTs in $D+1$ spacetime dimensions are invariant under the Lifshitz symmetry group, which contains Euclidean symmetry in the spatial dimensions, time translation symmetry and the following scaling symmetry:

$$t \to b^z t, \qquad x^i \to b x^i, \qquad i = 1, \ldots D. \tag{1.11}$$

Note that there is an anisotropy between time and space, with the degree of anisotropy measured by the dynamical critical exponent z. Systems with such Lifshitz scaling appear frequently in quantum and statistical field theory of condensed matter systems [16], especially in the context of Lifshitz multicritical points, and in nonequilibrium statistical mechanics. More recently, in a seemingly unrelated development, anisotropic Lifshitz-type scaling (1.11) has played a central role in the new approach to quantum gravity initiated in [17, 18]. This anisotropic theory of gravity will be reviewed in Chapter 2.

Once again, we look for dual gravitational theory with a metric that has the Lifshitz symmetry group as its isometry group. The result is the metric of Lifshitz spacetime in $D+2$ dimensions,

$$ds^2 = -r^{2z} dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2}. \tag{1.12}$$

The holographic gravity duals of Lifshitz-type QFTs should therefore have (1.12) as their solution. This geometry appears as a solution in several effective theories, such as the theory considered in [19] in which bulk Einstein gravity is coupled to a massive vector, and more recently also in a variety of constructions obtained from string theory [20, 21, 22, 23]. It will sometimes be useful to rewrite (1.12) in the coordinate $u = \frac{1}{r}$:

$$ds^2 = -\frac{1}{u^{2z}} dt^2 + \frac{1}{u^2} d\mathbf{x}^2 + \frac{du^2}{u^2}. \tag{1.13}$$

Using the action of (1.6), we can again analyze the behavior of a bulk scalar field, now for Lifshitz spacetime. Once again, the equation of motion for the scalar has two solutions, which behave asymptotically near the boundary ($r \to \infty$) as $\Phi \sim r^{-\Delta_\pm}$, where now:

$$\Delta_\pm = \frac{D+z}{2} \pm \sqrt{\left(\frac{D+z}{2}\right)^2 + m^2}. \tag{1.14}$$

In Lifshitz spacetime, the allowed boundary conditions still depend upon the mass of the scalar but needs to be modified from that of AdS spacetime [24]. The standard boundary condition is now allowed for $m^2 \geq -\frac{1}{4}(D+z)^2$ whereas the alternative boundary condition is allowed for $-\frac{1}{4}(D+z)^2 \leq m^2 \leq z^2 - \frac{1}{4}(D+z)^2$. This implies that the operator $\mathcal{O}_+$ has scaling dimension $\Delta_+ \geq \frac{1}{2}(D+z)$ and the operator $\mathcal{O}_-$ has scaling dimension $\frac{1}{2}(D-z) \leq \Delta_- \leq \frac{1}{2}(D+z)$. As before, this range covers all scalar operators above the unitarity bound of $\Delta \geq \frac{1}{2}(D-z)$.

1.4 Weyl Symmetry and Weyl Anomalies

So far we have examined field theories in flat space but one can also extend the holographic duality to field theories on a curved spacetime background. Let us take a CFT and add a curved spacetime background $g_{\mu\nu}(x)$. This has a gravitational dual formulated on an asymptotically *locally* AdS spacetime background:

$$ds^2 = \frac{dr^2}{r^2} + r^2(\bar{g}_{\mu\nu}(r,x)dx^\mu dx^\nu), \tag{1.15}$$

where $\bar{g}_{\mu\nu}(r,x) \to g_{\mu\nu}(x)$ as $r \to \infty$. Spacetime diffeomorphism transformations in the bulk gravitation theory translate asymptotically into a Weyl transformation of $g_{\mu\nu}(x)$:

$$g_{\mu\nu}(x) \to e^{2\Omega(x)} g_{\mu\nu}(x). \tag{1.16}$$

So we would expect the field theory to be invariant under the Weyl transformation (1.16). This is indeed true classically but there is one subtlety we must take into account. In the statement of the AdS/CFT correspondence (1.8), both sides are infinite (at least initially) and need to be carefully regulated in order for this equality to make sense. On the field theory side, standard UV divergences appear and require conventional renormalization. On the gravity side, the divergences are IR effects that come from the fact the volume of AdS spacetime becomes infinite as $r \to \infty$. These divergences can be dealt with systematically using the procedure of holographic renormalization (see Section 3.2). But in the renormalization procedure of a QFT it is not always possible to maintain the classical symmetries; quantum anomalies can appear [25]. Thus the dual field theory is Weyl invariant except for the possible presence of a Weyl anomaly.

Similarly, in Lifshitz holography, the dual Lifshitz field theories will be invariant under an anisotropic version of the Weyl symmetry, up to a possible anomaly. The calculation of these Weyl anomalies, using the technique of holographic renormalization, is one of the main results of the dissertation.

Chapter 2

Review of Anisotropic Gravity

Before continuing with the formulation of Lifshitz holography, we take a detour to introduce anisotropic gravity which will prove to be very useful in later chapters. An anisotropic theory of gravity with Lifshitz-like anisotropic scaling at short distances was first introduced by Hořava [17, 18] and we shall refer to this theory as Hořava-Lifshitz (HL) gravity (see, *e.g.*, [26, 27, 28] for some reviews). In this chapter, we briefly review some features of HL gravity, concentrating on aspects relevant for the main points of this dissertation.

2.1 Main Features of HL Gravity

The theory can be formulated in the general number of $d = D+1$ spacetime dimensions. Since the spacetime manifold $\mathcal{M}$ is assumed to carry a preferred foliation structure $\mathcal{F}$, consisting of codimension-one leaves Σ of constant absolute time, it is natural to use nonrelativistic coordinates t and $\mathbf{x} \equiv \{x^i, i = 1, \dots D\}$, adapted to the foliation. In such coordinates, the theory is then described by specifying its fields and its symmetries. The gravity field metric multiplet consists of fields

$$N, \quad N_i, \quad g_{ij}, \tag{2.1}$$

familiar from the ADM decomposition of the relativistic metric on spacetime: N is the lapse function[1], N_i the shift vector, and g_{ij} the spatial metric on the leaves Σ.

In the simplest version of the theory, the gauge symmetries are given by those spacetime diffeomorphisms that preserve the preferred foliation $\mathcal{F}$. Such foliation-preserving diffeomorphisms Diff$(\mathcal{M}, \mathcal{F})$, generated by

$$\delta t = f(t), \qquad \delta x^i = \xi^i(\mathbf{x}, t), \tag{2.2}$$

contain one fewer gauge symmetry per spacetime point than the symmetries of general relativity. Theories of gravity with anisotropic scaling whose symmetries are as large as those of general relativity can be constructed [29], but we will not deal with them here.

[1] The lapse function is taken to be a function of both space and time, which is known in the literature as the *non-projectable* case.

The action respecting the symmetries of Diff$(\mathcal{M}, \mathcal{F})$ consists of a kinetic term,

$$S_K = \frac{1}{2\kappa^2} \int_{\mathcal{M}} dt\, d^D\mathbf{x} \sqrt{g}\, N \left(K_{ij}K^{ij} - \lambda K^2 \right), \tag{2.3}$$

where

$$K_{ij} = \frac{1}{2N} \left(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i \right) \tag{2.4}$$

is the extrinsic curvature of Σ, $K \equiv K^i_i$, and λ is dimensionless coupling constant; and a potential term

$$S_{\mathcal{V}} = \frac{1}{\kappa^2} \int_{\mathcal{M}} dt\, d^D\mathbf{x} \sqrt{g}\, N\, \mathcal{V}, \tag{2.5}$$

with $\mathcal{V}$ a scalar function independent of the time derivatives of all fields. Specifically, the potential term $\mathcal{V}$ is a local function of the Riemann tensor of the spatial metric g_{ij}, its covariant derivatives and the spatial vector field $\nabla_i N/N$. At low energies, only the most relevant operators will contribute to the potential, resulting in the low-energy potential:

$$S_{\mathcal{V}} \;=\; \frac{1}{2\kappa^2} \int_{\mathcal{M}} dt\, d^{D+1}x \sqrt{g} N \left[\beta(R - 2\Lambda) + \frac{\alpha^2}{2} \frac{\nabla_i N \nabla^i N}{N^2} \right]. \tag{2.6}$$

The novelty compared to General Relativity is in the three couplings β, λ and α, which in GR are fixed to $\lambda = \beta = 1$ and $\alpha = 0$. Note that turning on the α coupling is important for the consistency of anisotropic gravity [30, 26]: Taking the naive $\alpha \to 0$ limit in (2.6) would lead to a non-closure of the constraint algebra.

When $\Lambda = 0$, the flat spacetime $\mathbf{R}^{D+2}$ is a solution of anisotropic gravity with potential (2.6). The propagating graviton modes consist of the transverse-traceless tensor polarizations with dispersion relation $\omega^2 = \beta k^2$ (here $k \equiv \sqrt{k_i k_i}$ is the magnitude of the spatial momentum), plus an extra scalar graviton polarization, with dispersion

$$\omega^2 = \frac{\beta(1-\lambda)}{[1-(D+1)\lambda]} \left[1 + D \left(\frac{2\beta}{\alpha^2} - 1 \right) \right] k^2. \tag{2.7}$$

The requirement of stability and perturbative unitarity around flat spacetime constrains the couplings to be in the range $\beta > 0$,

$$\alpha^2 \leq \frac{2\beta D}{D-1}, \tag{2.8}$$

and

$$\lambda \geq 1 \quad \text{or} \quad \lambda \leq 1/(D+1). \tag{2.9}$$

As one flows to higher energies, higher derivative operators will become important in the potential $\mathcal{V}$. Anisotropic gravity is of interest because it allows one to flow to a Lifshitz fixed point at higher energies. When z equals the number of spatial dimensions D, several interesting things happen: First, the theory becomes power-counting renormalizable, when we allow all terms compatible with the gauge symmetries in the action. In addition, the effective spectral dimension of spacetime flows to two at short distances, in accord with the lattice results first obtained in the causal dynamical triangulations approach to quantum gravity in [31, 32, 33], and independently confirmed recently in [34]. Moreover, when $z = D$, one can further restrict the classical action by requiring its invariance under the local version of the rigid anisotropic scaling, which acts on the spacetime metric via anisotropic Weyl transformations. This leads to an anisotropic version of conformal gravity [17], which will be described in Section 2.2.

In higher dimensions, and for higher values of z, the number of available relevant and marginal terms that can appear in $\mathcal{V}$ proliferates quickly. One can further limit the independent terms by imposing an additional symmetry. For example, one can impose the *detailed balance condition* [17, 18]. This condition means that $\mathcal{V}$ is constructed from an auxiliary local action $\mathcal{W}$ in D Euclidean dimensions, as the sum of squares of the $\mathcal{W}$ equations of motion:

$$\mathcal{V} = \mathcal{G}_{ijk\ell}\frac{\delta\mathcal{W}}{\delta g_{ij}}\frac{\delta\mathcal{W}}{\delta g_{k\ell}}, \tag{2.10}$$

with an appropriately chosen non-derivative DeWitt metric tensor $\mathcal{G}_{ijk\ell}$. This condition – inspired by the theory of non-equilibrium systems – has a straightforward generalization in the presence of matter. When the theory is in detailed balance, the number of independent couplings in $\mathcal{V}$ reduces to the number of independent couplings in $\mathcal{W}$.

2.2 Conformal HL Gravity

Under certain circumstances, we can impose additional gauge symmetries to further constrain the classical action of gravity with anisotropic scaling. When $z = D$, one can require invariance under a local version of the anisotropic scaling (1.11), which acts on the metric multiplet by anisotropic Weyl transformations

$$N \to e^{z\omega}N \qquad N_i \to e^{2\omega}N_i \qquad g_{ij} \to e^{2\omega}g_{ij}, \tag{2.11}$$

with an arbitrary local function $\omega(t, \mathbf{x})$. We denote the group of anisotropic Weyl transformations (2.11) with dynamical exponent z by $\mathrm{Weyl}_z(\mathcal{M}, \mathcal{F})$. Crucially, this group extends the group of foliation preserving diffeomorphisms into a semi-direct product [17, 35]

$$\mathrm{Weyl}_z(\mathcal{M}, \mathcal{F}) \rtimes \mathrm{Diff}(\mathcal{M}, \mathcal{F}). \tag{2.12}$$

Indeed, the commutator between an infinitesimal foliation-preserving diffeomorphism $\delta_{(f,\ \xi^i)}$ of (2.2) and an infinitesimal generator δ_ω of the anisotropic Weyl transformation (2.11) yields another infinitesimal anisotropic Weyl transformation,

$$[\delta_\omega, \delta_{(f,\ \xi^i)}] = \delta_{f\partial_t\omega + \xi^i\partial_i\omega}, \tag{2.13}$$

with the same fixed – but otherwise arbitrary – value of z. On the other hand, had we tried to extend $\mathrm{Diff}(\mathcal{M}, \mathcal{F})$ into the full spacetime diffeomorphism group, the closure of the symmetries would have forced the relativistic scaling with $z = 1$. Thus, anisotropic Weyl symmetry is only possible when we relax the spacetime diffeomorphism symmetry to the symmetries of the preferred foliation $\mathcal{F}$. Insisting on the additional symmetries (2.11) implies that the coupling constant λ must take a fixed value, $\lambda = 1/D$. We will refer to it as the "conformal value" of λ.

Conformal anisotropic gravity (with $z = D$) can be formulated for any number of spatial dimensions, but in Chapter 6 we will make use of the $D = 2$ theory in detail and we therefore focus on this case. For the case of $D = 2$, which requires $z = 2$, the unique kinetic term is

$$S_K = \frac{1}{2\kappa^2}\int_{\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{g}\,N\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right). \tag{2.14}$$

One can easily check that this term is indeed invariant under (2.11) and, moreover, satisfies the detailed balance condition.

The potential term $\mathcal{V}$ is also strongly constrained by the condition of anisotropic Weyl invariance (2.11). In $D = 2$, where the Riemann tensor of the spatial metric reduces to the Ricci scalar R, there is only one term that can appear in $\mathcal{V}$:

$$S_{\mathcal{V}} = \frac{1}{2\kappa_{\mathcal{V}}^2}\int_{\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{g}\,N\left\{R + \frac{\nabla^i\nabla_i N}{N} - \frac{\nabla^i N\nabla_i N}{N^2}\right\}^2. \tag{2.15}$$

This term is also invariant under (2.11), but it does not satisfy the detailed balance condition: There is no *local* action that would yield this term as the sum of squares of its equations of motion.[2] Thus, pure $z = 2$ conformal gravity in $2+1$ dimensions with detailed balance has no potential term.

This conformal $z = 2$ gravity in $2+1$ dimensions can be coupled to scalars $X^a(t, \mathbf{x})$. Anisotropic Weyl invariance of the classical action will be preserved when we assign scaling dimension zero to X^a. The kinetic term becomes

$$S_K = \frac{1}{2\kappa^2}\int_{\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{g}\,N\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) + \frac{1}{2}\int_{\mathcal{M}} dt\, d^2\mathbf{x}\,\frac{\sqrt{g}}{N}\left(\partial_t X^a - N^i\nabla_i X^a\right)^2. \tag{2.16}$$

Even under the condition of detailed balance, this coupled theory develops a nontrivial potential. There is a unique potential term compatible both with anisotropic conformal invariance and the detailed balance condition,

$$S_{\mathcal{V}} = \int_{\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{g}\,N\left\{(\nabla^i\nabla_i X^a)^2 + \kappa^2\left(\nabla_i X^a\nabla_j X^a - \frac{1}{2}g_{ij}\nabla^k X^a\nabla_k X^a\right)^2\right\}. \tag{2.17}$$

This theory, of $z = 2$ conformal gravity coupled to scalars and satisfying the detailed balance condition, first appeared in [17] as the worldvolume action of "membranes at quantum criticality", whose ground-state wavefunction on Riemann surface Σ reproduces the partition

[2] However, as was discussed in [17], one *can* get $\mathcal{V} \sim R^2$ by squaring the equation of motion of a nonlocal action: the Polyakov conformal anomaly action $\int d^2\mathbf{x}\sqrt{g}R\frac{1}{\nabla^2}R$.

function of the critical bosonic string on Σ. The Euclidean action in $D = 2$ dimensions which yields (2.17) via the detailed balance construction is simply given by the action familiar from the critical string,

$$\mathcal{W} = \frac{1}{2}\int d^2\mathbf{x}\,\sqrt{g}\,g^{ij}\nabla_i X^a \nabla_j X^a. \tag{2.18}$$

We recognize the first term in (2.17) as the square of the X^a equation of motion, and the second term as the square of the energy-momentum tensor obtained from the g_{ij} variation of (2.18).

Chapter 3

Holography in Asymptotically Lifshitz Spacetimes

In this chapter, we discuss some general features of Lifshitz holography that are universal and independent of the precise model. This chapter also serves as an overview of the the technique of holographic renormalization, which will be described qualitatively here. First we must precisely define the notion of asymptotically Lifshitz spacetime.

3.1 Anisotropic Conformal Infinity and Asymptotically Lifshitz Spacetimes

The notion of conformal infinity plays a central role in general relativity [36, 37]. It is constructed by mapping the original metric $G_{\mu\nu}$ on a manifold $\mathcal{M}$ via a smooth conformal Weyl transformation to

$$\tilde{G}_{\mu\nu} = \Omega^2(x) G_{\mu\nu}, \tag{3.1}$$

such that the rescaled metric $\tilde{G}_{\mu\nu}$ is extendible to a larger manifold $\tilde{\mathcal{M}}$, which contains the closure $\bar{\mathcal{M}}$ of $\mathcal{M}$ as a closed submanifold. The idea is to define the conformal infinity of $\mathcal{M}$ to be the set $\bar{\mathcal{M}} \setminus \mathcal{M}$. The scaling factor Ω must extend to $\tilde{\mathcal{M}}$ and satisfy certain regularity conditions at $\bar{\mathcal{M}} \setminus \mathcal{M}$ (the most essential being that it should have a single zero there and that its exterior derivative should be nonzero), but is otherwise arbitrary. A change from one permissible scaling factor to another is interpreted as a conformal transformation at $\bar{\mathcal{M}} \setminus \mathcal{M}$: Hence, conformal infinity carries a naturally defined preferred conformal structure.

This notion of conformal infinity allows one to define precisely, and in a coordinate-independent way, the notion of an event horizon (and hence the notion of black holes), as the boundary of the causal past of the future infinity. Moreover, it allows us to define precisely the concept of spacetimes which "asymptotically approach" a chosen vacuum solution "at infinity." In the example of AdS, this picture is naturally compatible with the physical ideas of holography: The conformal infinity of AdS is of codimension one, and carries the natural conformal structure induced from the asymptotic isometries of the bulk, just as predicted by the holographic dictionary.

As pointed out in previous work [35], the intuition of holographic renormalization in Lifshitz spacetime clashes with this classic notion of conformal infinity as defined by Penrose. This tension has been remedied [35], for spacetimes carrying the additional structure of an asymptotic foliation, by generalizing Penrose's notion of conformal infinity to reflect the asymptotic anisotropy permitted by the foliation. The basic idea is simple: When $\mathcal{M}$ carries a preferred foliation at least near infinity, we can use the anisotropic Weyl transformation (2.11), instead of the relativistic rescaling (3.1), to map $\mathcal{M}$ inside a larger manifold $\tilde{\mathcal{M}}$ such that $\bar{\mathcal{M}} \subset \tilde{\mathcal{M}}$. Even in this case, the rescaling factor $\Omega = e^{\omega}$ must satisfy regularity conditions at $\bar{\mathcal{M}} \setminus \mathcal{M}$. In particular, Ω must have a simple zero there. With a judiciously chosen value of z, the *anisotropic conformal infinity* $\bar{\mathcal{M}} \setminus \mathcal{M}$ can be of codimension one. Moreover, it naturally inherits a preferred "anisotropic conformal structure," with conformal transformations given by those foliation-preserving diffeomorphisms that preserve the boundary metric up to an anisotropic Weyl rescaling.

The resulting notion of anisotropic conformal infinity for Lifshitz spacetime matches the intuitive expectations from holography [35]. In the case of the Lifshitz spacetime (1.12), we start with the metric as given in (1.13). We interpret this geometry as carrying a natural codimension-one foliation by leaves of constant t, at least near $u \to 0$. This additional structure of an asymptotic foliation gives us the additional freedom to use anisotropic Weyl transformations (2.11) without violating the symmetries. Choosing the rescaling factor

$$\Omega = u \tag{3.2}$$

and applying the anisotropic Weyl transformation (2.11) maps the Lifshitz metric in the asymptotic regime of $u \to 0$ to the flat metric,

$$\tilde{ds}^2 = -dt^2 + d\mathbf{x}^2 + du^2. \tag{3.3}$$

u can now be analytically extended from $u > 0$ to all real values. The anisotropic conformal infinity of the $(D+2)$-dimensional Lifshitz spacetime is at $u = 0$. Topologically, it is $\mathbf{R}^{D+1}$, and very similar to the conformal infinity of the Poincaré patch of AdS_{D+2}. However, even though the induced metric on anisotropic conformal infinity at $u = 0$ in (3.3) looks naively relativistic, one must remember that its natural symmetries are not relativistic: This conformal infinity carries a preferred foliation by leaves of constant t, and a natural anisotropic conformal structure characterized by dynamical exponent z. The natural symmetries are given by those foliation-preserving diffeomorphisms that preserve the metric up to an anisotropic Weyl transformation [35]. In addition to the spatial rotations and spacetime translations of $\mathbf{R}^{D+1}$, one can easily check that this symmetry group contains also the anisotropic scaling transformations (1.11). Thus, the conformal structure of anisotropic conformal infinity nicely matches the expected conformal symmetries of the dual field theory.

Equipped with the notion of anisotropic conformal infinity of spacetime, we can now give a precise definition of spacetime geometries that are "asymptotically Lifshitz". Simply put, given a value of z, a spacetime is said to be asymptotically Lifshitz if it exhibits the same anisotropic conformal infinity as the Lifshitz spacetime for that value of dynamical exponent z. This definition follows the logic that leads to the notions of asymptotic flatness and asymptotic AdS [36, 37], and extends such notions naturally to the case of anisotropic scaling.

As a part of their definition, the spacetimes which are asymptotically Lifshitz must carry an asymptotic foliation structure near their anisotropic conformal infinity. In the context of holographic renormalization, this condition translates into an important restriction on the form of the vielbein fall-off,

$$\frac{e_i^0}{r^z} \to 0, \qquad \text{as } r \to \infty. \tag{3.4}$$

This provides an answer to a question discussed in [8]: Our definition of asymptotically Lifshitz spacetimes using the notion of anisotropic conformal infinity requires that the sources for the energy flux vanish.[1]

With the definition of "asymptotically Lifshitz" at hand, it is now possible to define precisely black holes and their event horizons in Lifshitz spacetimes, by referring to the properties of the anisotropic conformal infinity of spacetime just as in the more traditional spacetimes which have codimension-one isotropic conformal infinity.

3.2 Holographic Renormalization in Asymptotically Lifshitz Spacetimes

Holographic duality in asymptotically AdS spacetimes – or, by logical extension, in asymptotically Lifshitz spacetimes – relates the partition function of a bulk gravity system with Dirichlet boundary conditions at conformal infinity to the generating function of correlators in the appropriate dual quantum field theory. At low energies and to leading order, this correspondence gives the connected generating functional W with sources $f^{(0)}$ on the field theory side, in terms of the on-shell bulk gravity action evaluated with Dirichlet boundary conditions given by $f^{(0)}$:

$$W[f^{(0)}] = -S_{on-shell}[f^{(0)}]. \tag{3.5}$$

Both sides of this correspondence are divergent: Standard ultraviolet divergences appear on the field theory side, and they require conventional renormalization. This behavior is matched on the gravity side, where the divergences are infrared effects, due to the scales that diverge as we approach the spacetime boundary. Holographic renormalization [38, 39, 40, 41, 42, 43] (for reviews, see [44, 11, 45]) is the technology designed to perform the subtraction of infinities on the gravity side, in the form of divergent boundary terms in the on-shell action, and to make precise sense of (3.5).

[1] More precisely, it would be sufficient to impose $(\partial_i e_j^0 - \partial_j e_i^0)/r^z \to 0$ at infinity, a constraint which also emerges naturally in the vielbein formulation of gravity with anisotropic scaling. In this dissertation, we impose the stronger condition (3.4).

Recent papers [8, 46, 47] have performed various steps of holographic renormalization in Lifshitz spacetime at the non-linear level, and this dissertation builds on the results established there. Since we choose for our analysis the Hamiltonian approach to holographic renormalization [48, 49], our treatment is closest to that of [8].

Hamiltonian approach to holographic renormalization

The original analysis of holographic renormalization relied on properties of asymptotic expansions near the conformal infinity of spacetime [50, 51, 52]. The Hamiltonian approach of [48, 49] aspires to give a somewhat more covariant picture, and the results of the earlier asymptotic expansion approach can be reproduced from it [48]. Either way, we start by choosing a radial coordinate, r, in some neighborhood of the anisotropic conformal infinity of the Lifshitz spacetime $\mathcal{M}$, such that the hypersurfaces of constant r are diffeomorphic to the boundary $\partial\mathcal{M}$, and they equip $\mathcal{M}$ near $\partial\mathcal{M}$ with a codimension-one foliation structure.[2] This foliation should not be confused with the preferred foliation of the anisotropic conformal boundary by leaves of constant t – the asymptotic regime of our spacetime carries a nested foliation structure, with leaves of constant radial coordinate r further foliated by leaves of constant t.

Our task is to evaluate the on-shell action as a functional of the boundary fields, and perform the corresponding renormalization. Because of the infinite volume of Lifshitz space, the on-shell action diverges, and must be regularized by inserting a cutoff at finite volume and identifying terms that diverge in the asymptotic expansion in the cutoff, and then renormalized by introducing appropriate counterterms to eliminate the divergences. The on-shell action is regulated by cutting the bulk spacetime off at some value $r < \infty$ of the radial coordinate. If $\mathcal{M}_r$ is the cut-off manifold, its boundary $\partial\mathcal{M}_r$ represents a regulated boundary of spacetime. The on-shell action is a function of the regulator r, and the boundary fields which include the metric multiplet, N, N_i and g_{ij}, plus all sources associated with the bulk matter Φ, which we collectively denote by ϕ. From now on, we simply denote the on-shell action $S_{on-shell}$ – viewed as a functional of the boundary values of the fields – by S, and parametrize it as

$$S \sim \int_{\partial\mathcal{M}_r} dt\, d^D\mathbf{x}\, \sqrt{g}\, N\mathcal{L}. \tag{3.6}$$

Since the on-shell action S is a function of r and the boundary values of the fields, we can naturally interpret it as a solution to the Hamilton-Jacobi equation, regarding r as the evolution parameter. This is the starting point for the Hamiltonian approach to holographic renormalization. The Hamilton-Jacobi theory implies that the first variation of the on-shell action with respect to the boundary fields gives the conjugate momenta. In the holographic dictionary, the boundary fields serve as sources in the field theory, and their conjugate

[2] In our conventions, $\partial\mathcal{M}$ is at $r = \infty$. The choice of $u = 1/r$ instead of r as a coordinate near $\partial\mathcal{M}$ would be more appropriate, since u is finite through $\partial\mathcal{M}$. In this section, we leave this more rigorous coordinate choice implicit.

momenta are thus directly related to the one-point functions of the operators conjugate to the sources.

A convenient way of computing the divergent part of $\mathcal{L}$ is to organize the terms with respect to their scaling with r. More precisely, we define the dilatation operator by

$$\delta_{\mathcal{D}} = \int_{\partial\mathcal{M}_r} dt\, d^D\mathbf{x} \left(zN\frac{\delta}{\delta N} + 2N_i\frac{\delta}{\delta N_i} + 2g_{ij}\frac{\delta}{\delta g_{ij}} - \sum_{\phi} \Delta_\phi \phi \frac{\delta}{\delta\phi}, \right), \tag{3.7}$$

where Δ_ϕ collectively denotes the asymptotic decay exponents of the bulk matter fields Φ. Quantities of interest can then be decomposed into a sum of terms with definite scaling dimension under $\delta_{\mathcal{D}}$. For example, the object of our central interest, $\mathcal{L}$, can be expanded as

$$\mathcal{L} = \sum_{\Delta} \mathcal{L}^{(\Delta)} + \tilde{\mathcal{L}}^{(z+D)} \log r. \tag{3.8}$$

Throughout this dissertation, superscripts in parentheses on any object $\mathcal{O}$ always denote the scaling dimension in the decomposition of $\mathcal{O}$ as a sum of terms of definite engineering scaling dimensions. For example, $T^{(0)}_{AB}$ is the constant part of the stress tensor, and $R^{(2)}$ is the dimension-two part of the scalar curvature.

The individual terms of the expansion (3.8) satisfy

$$\delta_{\mathcal{D}}\mathcal{L}^{(\Delta)} = -\Delta\, \mathcal{L}^{(\Delta)} \qquad \text{for} \quad \Delta \neq z + D. \tag{3.9}$$

When $\Delta = z + D$, the scaling behavior is anomalous,

$$\delta_{\mathcal{D}}\mathcal{L}^{(z+D)} = -(z+D)\mathcal{L}^{(z+D)} + \tilde{\mathcal{L}}^{(z+D)}, \tag{3.10}$$

with the inhomogeneous term satisfying

$$\delta_{\mathcal{D}}\tilde{\mathcal{L}}^{(z+D)} = -(z+D)\tilde{\mathcal{L}}^{(z+D)}. \tag{3.11}$$

This logarithmic term in (3.8) reflects the possibility of an anisotropic Weyl anomaly.

The dynamical equations for the divergent part of $\mathcal{L}$ are determined as follows. Since the on-shell action satisfies the Hamilton-Jacobi equation, its radial derivative is determined in terms of the Hamiltonian. Because the fields have fixed asymptotic behavior, in the asymptotic region the radial derivative is equivalent to the anisotropic scaling operator,

$$r\frac{d}{dr} \approx \delta_{\mathcal{D}}. \tag{3.12}$$

The Hamilton-Jacobi equation then relates the action of $\delta_{\mathcal{D}}$ on the on-shell action to the Hamiltonian of the system. Using the bulk equations of motion, one obtains a first-order differential equation for $\mathcal{L}$ in terms of the boundary values of the fields that can be solved iteratively in the expansion in eigenmodes of $\delta_{\mathcal{D}}$.

Equivalently, one can expand the Hamiltonian constraint in eigenmodes of $\delta_{\mathcal{D}}$. The structure of these equations allows for the momentum modes to be obtained recursively in terms of the boundary data. In this method, the dilatation operator acting on the on-shell action gives an expression linear in the canonical momenta, so that the values for the momenta obtained recursively from the Hamiltonian constraint give rise directly to the desired expression on-shell action. The resulting on-shell action will have divergent pieces that can be expressed as local functionals of the boundary data. These pieces can be subtracted, leading to the finite renormalized on-shell action.

Chapter 4

Illustrative Example: Holographic Renormalization in AdS

Let us begin with a simple example that illustrates many of the key calculations of holographic renormalization. We will examine the theory of general relativity with cosmological constant but no other matter fields, which has AdS spacetime as a background solution. The holographic renormalization of this theory has been examined extensively in the literature (e.g., [38, 41, 48]) but it will be useful to repeat it here before proceeding to the more complicated cases in Lifshitz holography.

4.1 The Bulk Action and Notation

The bulk spacetime relativistic action is:

$$S = \frac{1}{16\pi G_{d+1}} \int_{\mathcal{M}} d^d x\, dr \sqrt{-G}\,(\mathcal{R} - 2\Lambda) + \frac{1}{8\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-g}\, K. \tag{4.1}$$

Note that in order for AdS spacetime (1.3) to be a classical solution, we set

$$\Lambda = -\frac{1}{2} d(d-1). \tag{4.2}$$

We denote the metric in the bulk by:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta} dx^\alpha dx^\beta + \frac{dr^2}{r^2}. \tag{4.3}$$

The boundary is at $r = \infty$. d is the number of spacetime dimensions on the boundary and so there are $d+1$ spacetime dimensions in the bulk. For coordinate indices, α, β are used for the d spacetime boundary indices (x^α) and μ, ν are used for the $d+1$ bulk dimensions (x^α, r). ∇_α represents the covariant derivative for the metric $g_{\alpha\beta}$. Note that in

(4.3), the bulk diffeomorphisms have been gauge fixed by setting the the radial shift vector[1] $\mathcal{N}_\alpha = 0$, and the radial lapse function $\mathcal{N} = 1/r$. This *radial gauge* is adopted throughout this chapter. Note that in this radial gauge, the extrinsic curvature on the boundary is $K_{\alpha\beta} \equiv \frac{1}{2\mathcal{N}}(\partial_r g_{\alpha\beta} - \nabla_\alpha \mathcal{N}_\beta - \nabla_\beta \mathcal{N}_\alpha) = r\partial_r g_{\alpha\beta}/2$.

It is often convenient to work in terms of vielbeins, which we define via

$$ds^2 = \eta_{MN} E^M_\mu E^N_\nu dx^\mu dx^\nu = \eta_{AB} e^A_\alpha e^B_\beta dx^\alpha dx^\beta + \frac{dr^2}{r^2}. \tag{4.4}$$

For the internal frame indices, $M, N = 0, 1, ..., d$ are used for the $d+1$ bulk dimensions, $A, B = 0, 1, ..., d-1$ are used for the d spacetime boundary indices. The vielbeins allow coordinate indices to be changed to frame indices and vice versa, for example $T^{AB} = e^A_\alpha e^B_\beta T^{\alpha\beta}$.

In order to distinguish the Riemann tensor of the two different metrics $G_{\mu\nu}$ and $g_{\alpha\beta}$, we use the notation wherein $(d+1)$-dimensional quantities are written in curly letters (for example, $\mathcal{R}$ for the Ricci scalar) and the d-dimensional quantities are written in standard italics.

4.2 Radial Decomposition

When written in radial ADM variables $\mathcal{N}$, $\mathcal{N}_\alpha$ and $g_{\alpha\beta}$, the bulk action (4.1) becomes (when ignoring surface terms)

$$S = \frac{1}{16\pi G_{d+1}} \int_{\mathcal{M}} d^d x\, dr\, \sqrt{-g}\mathcal{N} \left(K^2 - K_{\alpha\beta}K^{\alpha\beta} + R - 2\Lambda\right). \tag{4.5}$$

Varying this action with respect to the radial lapse function $\mathcal{N}$ gives the Hamiltonian constraint:

$$K^2 - K_{\alpha\beta}K^{\alpha\beta} = R - 2\Lambda. \tag{4.6}$$

This equation will be useful in later sections. Furthermore, we define[2] the (radial) momenta corresponding to the metric $g_{\alpha\beta}$ by:

$$\pi^{\alpha\beta} \equiv -\frac{16\pi G_{d+1}}{\sqrt{-g}} \frac{\delta S}{\delta \partial_r g_{\alpha\beta}} = K^{\alpha\beta} - g^{\alpha\beta} K. \tag{4.7}$$

4.3 Functional Derivatives and the Stress Tensor

We will be applying the Hamilton-Jacobi formalism in the radial direction to perform the holographic renormalization. We form the on-shell action by taking the bulk action (4.1)

[1]Recall that $\mathcal{N}$ and $\mathcal{N}_\alpha$ are defined via $G_{r\alpha} = \mathcal{N}_\alpha$ and $G_{rr} = \mathcal{N}^2 + g^{\alpha\beta}\mathcal{N}_\alpha\mathcal{N}_\beta$

[2]This differs from the usual canonical momenta by a factor of $\sqrt{-g}(-16\pi G_{d+1})^{-1}$ in order to simplify some of the subsequent equations.

and evaluating it as a function of the boundary fields:

$$S = \frac{1}{16\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-g}\mathcal{L}. \tag{4.8}$$

As in the standard Hamilton-Jacobi theory, the radial momenta (4.7) can be obtained by functional differentiation of this on-shell action:

$$\pi^{\alpha\beta} = -\frac{16\pi G_{d+1}}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}}. \tag{4.9}$$

Equivalently, the variation of the on-shell action is:

$$\delta S = -\frac{1}{16\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-g} \left[\pi^{\alpha\beta}\delta g_{\alpha\beta}\right] = -\frac{1}{16\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-g} \left[2\pi^{\alpha}{}_{\beta} e^{\beta}_{B} \delta e^{B}_{\alpha}\right]. \tag{4.10}$$

The boundary stress-energy tensor $T^{\alpha}{}_{B}$ is defined by functional differentiation of the on-shell action with respect to the vielbeins e^{B}_{α}. Therefore, the variation of the on-shell action can also be written as:

$$\delta S = -\frac{1}{16\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-g} \left[T^{\alpha}{}_{B} \delta e^{B}_{\alpha}\right], \tag{4.11}$$

where

$$T^{A}{}_{B} = -\frac{16\pi G_{d+1}}{\sqrt{-g}} e^{A}_{\alpha} \frac{\delta S}{\delta e^{B}_{\alpha}}. \tag{4.12}$$

By comparing (4.10) and (4.11) we get the following relation:

$$T_{\alpha\beta} = 2\pi_{\alpha\beta}. \tag{4.13}$$

We will use the results of this section to determine the stress-energy tensor from the on-shell action.

4.4 Analysis of Linearized Constant Modes

The bulk fields in general relativity will correspond to operators in the dual quantum field theory. To elucidate these operators and their scaling dimensions it is useful to conduct an analysis of the constant linearized modes around the AdS background. By constant, we mean that the modes are independent of x^{α} (but can depend on r).

Before doing this, recall that the bulk theory has a gauge group of diffeomorphisms ($\delta G_{\mu\nu} = \partial_\mu \zeta^\sigma G_{\nu\sigma} + \partial_\nu \zeta^\sigma G_{\mu\sigma} + \zeta^\sigma \partial_\sigma G_{\mu\nu}$), so some modes will be pure gauge. Since we will only be looking at the linearized modes, we can substitute in our background solution of $G_{rr} = \frac{1}{r^2}$, $G_{r\alpha} = 0$, $G_{\alpha\beta} = g_{\alpha\beta} = r^2\eta_{\alpha\beta}$:

$$\delta G_{rr} = \frac{2}{r^2}\partial_r \zeta^r - \frac{2}{r^3}\zeta^r,$$

$$\begin{aligned}\delta G_{r\alpha} &= r^2\partial_r\zeta^\beta\eta_{\alpha\beta} + \frac{1}{r^2}\partial_\alpha\zeta^r, \\ \delta g_{\alpha\beta} &= \partial_\alpha\zeta^\gamma r^2\eta_{\beta\gamma} + \partial_\beta\zeta^\gamma r^2\eta_{\alpha\gamma} + 2r\zeta^r\eta_{\alpha\beta}.\end{aligned} \tag{4.14}$$

When choosing the radial gauge of (4.3), we must maintain $\delta G_{rr} = \delta G_{r\alpha} = 0$, which requires $\zeta^r = r\zeta$ and $\zeta^\alpha = \frac{1}{2r^2}\eta^{\alpha\beta}\partial_\beta\zeta$, where ζ is independent of r. This leaves the following diffeomorphisms unfixed:

$$\delta g_{\alpha\beta} = \partial_\alpha\partial_\beta\zeta + 2r^2\zeta\eta_{\alpha\beta}. \tag{4.15}$$

In addition, we are just looking at constant modes so we can take $\delta g_{\alpha\beta}$ to be independent of x^α (but it can depend on r). This means that ζ must be independent of both x^α and r and so (4.15) becomes

$$\delta g_{\alpha\beta} = 2r^2\zeta\eta_{\alpha\beta}. \tag{4.16}$$

This is a pure gauge mode, even after the radial gauge has been chosen. We can now analyze the linearized modes. We only consider the modes that are constant in x^α. By the above discussion, we take

$$g_{\alpha\beta} = r^2[\eta_{\alpha\beta}(1+t) + t_{\alpha\beta}], \tag{4.17}$$

where $\eta^{\alpha\beta}t_{\alpha\beta} = 0$ and t, $t_{\alpha\beta}$ are functions of r only. We now substitute this into the equations of motion ($\mathcal{R}_{\mu\nu} = \frac{1}{2}(\mathcal{R} - 2\Lambda)G_{\mu\nu}$), keeping only terms linear in $t_{\alpha\beta}$ and t to get:

$$(d+1)t'_{\alpha\beta} + rt''_{\alpha\beta} = 0, \qquad t' = 0, \tag{4.18}$$

where a prime denotes differentiation with respect to r. Other than the pure gauge mode of (4.16), this equation has general solution $t_{\alpha\beta} = c_{1\alpha\beta} + c_{2\alpha\beta}r^{-d}$ for traceless constants $c_{1\alpha\beta}$ and $c_{2\alpha\beta}$. $c_{1\alpha\beta}$ is the asymptotic value of $g_{\alpha\beta}$ and thus it represents a source for the stress-energy tensor $T^{\alpha\beta}$. These linearized modes give a contribution to the bulk stress-energy tensor of:

$$T_{AB} \sim -dc_{2AB}r^{-d}. \tag{4.19}$$

Thus c_{2AB} contributes to the expectation value of T_{AB} and, from the exponent of r, we can deduce that the CFT stress-energy tensor operator has scaling dimension d as expected.

4.5 Boundary Source Fields and Asymptotic Scaling

The boundary conditions are specified by fixing the sources for the various field theory operators on the boundary. We can use the results of Section 4.4 to determine the asymptotic behavior of the fields. The correct boundary condition involves the following finite fixed source as $r \to \infty$ (denoting the source by a bar):

$$\bar{e}^A_\alpha = \frac{e^A_\alpha}{r}. \tag{4.20}$$

The above scaling behavior allows us to determine the scaling behavior of other quantities near the boundary. Any boundary quantity can be written in terms of the source fields and then the scaling behavior can be read off from the resulting exponents of r. Consider a general object $\mathcal{O}$. When written in terms of the boundary source fields, we say that the term in $\mathcal{O}$ scaling as $r^{-\Delta}$ is of "order Δ" and denote it by $\mathcal{O}^{(\Delta)}$. For example, e^A_α has order -1, $\gamma_{\alpha\beta}$ has order -2, $\gamma^{\alpha\beta}$ has order 2 and R has order 2.

We can also use the leading order behavior of the fields (4.20) to determine the leading order behavior of the momenta:

$$K^{(0)}_{AB} = \eta_{AB}, \tag{4.21}$$

$$T^{(0)}_{AB} = 2\pi^{(0)}_{AB} = -2(d-1)\eta_{AB}. \tag{4.22}$$

Note also that (4.12) implies that the term $\mathcal{L}^{(\Delta)}$ in the on-shell action determines ${T^A}_B^{(\Delta)}$.

4.6 Holographic Renormalization Equations

When the action (4.1) is evaluated on shell as a function of the boundary fields we write it as:

$$S = \frac{1}{16\pi G_{d+1}} \int_{\partial\mathcal{M}} d^dx\,\sqrt{-g}\mathcal{L}. \tag{4.23}$$

A convenient way of computing the divergent part of $\mathcal{L}$ is to organize the terms with respect to how they scale with r. More precisely, we define the dilatation operator by:

$$\delta_{\mathcal{D}} = \int_{\partial\mathcal{M}} d^dx \left(e^A_\mu \frac{\delta}{\delta e^A_\mu} \right). \tag{4.24}$$

This operator asymptotically represents $r\dfrac{\partial}{\partial r}$.

$\mathcal{L}$ can then be decomposed into a sum of terms as follows:

$$\mathcal{L} = \sum_{\Delta\geq 0} \mathcal{L}^{(\Delta)} + \tilde{\mathcal{L}}^{(d)} \log r. \tag{4.25}$$

Note that from the form of (4.23), $\mathcal{L}^{(\Delta)}$ only results in a divergent term in the on-shell action if $\Delta < d$. Furthermore, we include a logarithmic term at order d due to the possibility of a Weyl scaling anomaly. The individual terms of the expansion (4.25) satisfy

$$\delta_{\mathcal{D}}\mathcal{L}^{(\Delta)} = -\Delta\,\mathcal{L}^{(\Delta)} \qquad \text{for} \quad \Delta \neq d, \tag{4.26}$$

$$\delta_{\mathcal{D}}\mathcal{L}^{(d)} = -(d)\mathcal{L}^{(d)} + \tilde{\mathcal{L}}^{(d)}, \tag{4.27}$$

$$\delta_{\mathcal{D}}\tilde{\mathcal{L}}^{(d)} = -(d)\tilde{\mathcal{L}}^{(d)}. \tag{4.28}$$

Applying $\delta_{\mathcal{D}}$ to the on-shell action (4.23) and using (4.12) yields:

$$(d + \delta_{\mathcal{D}})\mathcal{L} = -T^{A}{}_{A}. \tag{4.29}$$

Expanding this at each order then results in:

$$(d - \Delta)\mathcal{L}^{(\Delta)} = -T^{A}{}_{A}{}^{(\Delta)}, \tag{4.30}$$

except for $\Delta = d$, when this becomes

$$\tilde{\mathcal{L}}^{(\Delta)} = -T^{A}{}_{A}{}^{(\Delta)}. \tag{4.31}$$

This allows us to solve for the anomaly. The above equations imply that the anomaly term can also be found by:

$$\tilde{\mathcal{L}}^{(\Delta)} = \lim_{\Delta \to d}\Big((d - \Delta)\mathcal{L}^{(\Delta)}\Big). \tag{4.32}$$

Note that the value of $\mathcal{L}^{(d)}$ cannot be found by this asymptotic analysis.

We now move on to finding an explicit expression for these divergent terms in the on-shell action $\mathcal{L}^{(\Delta)}$. Recall from (4.6) that the variation of the bulk action (4.1) with respect to $\mathcal{N}$ produces the constraint equation:

$$K^2 - K_{AB}K^{AB} = R - 2\Lambda. \tag{4.33}$$

We will now expand this equation in its dilatation eigenvalues and then substitute it into (4.30) to yield an expression for $\mathcal{L}^{(\Delta)}$.

Explicitly, noting that $K_{AB}K^{AB} - K^2 = K_{AB}\pi^{AB}$, the left hand side of (4.33) contains a term at order Δ equal to $-K_{AB}^{(\Delta/2)}\pi^{AB(\Delta/2)} - \sum_{s<\Delta/2} 2K_{AB}^{(s)}\pi^{AB(\Delta-s)}$. Using (4.21), the $s = 0$ term of this sum is then equal to the right hand side of (4.30). Therefore, combining these results, the terms in the on-shell action are given for $\Delta \neq 0$ by:

$$(d - \Delta)\mathcal{L}^{(\Delta)} = \mathcal{Q}^{(\Delta)} + \mathcal{S}^{(\Delta)}, \tag{4.34}$$

where the quadratic term $\mathcal{Q}^{(\Delta)}$ is given by

$$\mathcal{Q}^{(\Delta)} = \Big[\sum_{0<s<\Delta/2} 2K_{AB}^{(s)}\pi^{AB(\Delta-s)}\Big] + K_{AB}^{(\Delta/2)}\pi^{AB(\Delta/2)}, \tag{4.35}$$

and the source $\mathcal{S}$ is

$$\mathcal{S} = R - 2\Lambda. \tag{4.36}$$

When $\Delta = 0$, (4.34) becomes:

$$(d - 0)\mathcal{L}^{(0)} = 2\mathcal{S}^{(0)}. \tag{4.37}$$

$\mathcal{S}$ can also be expanded in its dilatation eigenvalues and the only contributing terms are:

$$\begin{aligned} \mathcal{S}^{(0)} &= -2\Lambda = d(d-1), \\ \mathcal{S}^{(2)} &= R. \end{aligned} \tag{4.38}$$

4.7 Calculation of Divergences

We now proceed to use the holographic renormalization equations of Section 4.6 to calculate the divergent terms in the on-shell action at each order. From the form of $\mathcal{S}$ in (4.38), it is clear that there will only be divergent terms at order $\Delta = 2k$ for $k \in \mathbb{N}$. Once these divergent terms have been calculated, counterterms must be added to the action in order to subtract these divergences. With a boundary cutoff at $r = \frac{1}{\epsilon}$, the counterterms are

$$S_{ct} = -\frac{1}{16\pi G_{d+1}} \int_{\partial\mathcal{M}} d^d x \sqrt{-g} \left[\sum_{0\leq\Delta<d} \mathcal{L}^{(\Delta)} - \tilde{\mathcal{L}}^{(d)} \log(\epsilon) \right]. \tag{4.39}$$

Non-derivative counterterms

At order 0, we have:

$$\mathcal{L}^{(0)} = \frac{2\mathcal{S}^{(0)}}{d} = 2(d-1). \tag{4.40}$$

This divergent term in the on-shell action results in the following divergence in the stress tensor (see Section 4.3):

$$T^{(0)}_{AB} = -2(d-1)\eta_{AB}. \tag{4.41}$$

Note that this agrees with the previous result in (4.22).

Two-derivative counterterms

Up to total derivatives, the divergent term in the on-shell action of order 2 is:

$$(d-2)\mathcal{L}^{(2)} = \mathcal{S}^{(2)} = R. \tag{4.42}$$

This gives the following contribution to the stress tensor (see Section 4.3):

$$(d-2)T^{(2)}_{AB} = 2R_{AB} - \eta_{AB}R. \tag{4.43}$$

In addition, this divergent term leads to the well known Weyl anomaly in $d = 2$ (using (4.32)):

$$\tilde{\mathcal{L}}^{(d=2)} = \lim_{\Delta\to 2} \left((2-\Delta)\mathcal{L}^{(\Delta)} \right) = R. \tag{4.44}$$

Four-derivative counterterms

At fourth order we have:

$$(d-4)\mathcal{L}^{(4)} = K^{(2)}_{AB}\pi^{AB(2)}. \tag{4.45}$$

Note that $\pi_{\alpha\beta} = K_{\alpha\beta} - g_{\alpha\beta}K$ implies that $\pi^{\alpha}{}_{\alpha} = -(d-1)K$ and so $K_{\alpha\beta} = \pi_{\alpha\beta} - \dfrac{\pi^{\gamma}{}_{\gamma} g_{\alpha\beta}}{d-1}$. Therefore:

$$\begin{aligned}
(d-4)\mathcal{L}^{(4)} &= K^{(2)}_{AB}\pi^{AB(2)} \\
&= \pi^{(2)}_{AB}\pi^{AB(2)} - \frac{(\pi^{A}{}_{A}{}^{(2)})^2}{d-1} \\
&= \frac{1}{4}\left(T^{(2)}_{AB}T^{AB(2)} - \frac{(T^{A}{}_{A}{}^{(2)})^2}{d-1}\right) \\
&= \frac{1}{4(d-2)^2}\left[(2R_{AB} - \delta_{AB}R)(2R^{AB} - \delta^{AB}R)\right] - \frac{R^2}{4(d-1)} \\
&= \frac{1}{(d-2)^2}\left(R_{AB}R^{AB} - \frac{d}{4(d-1)}R^2\right).
\end{aligned} \tag{4.46}$$

In particular, this result leads to the well-known expression for the Weyl anomaly in $d = 4$ (using (4.32)):

$$\tilde{\mathcal{L}}^{(d=4)} = \lim_{\Delta\to 4}\left((4-\Delta)\mathcal{L}^{(\Delta)}\right) = \frac{1}{4}\left(R_{AB}R^{AB} - \frac{1}{3}R^2\right). \tag{4.47}$$

Higher-derivative counterterms

The divergent terms calculated above are all the divergent terms for $d < 6$. For higher dimensions there will be further divergences, which can be calculated systematically using this method.

Chapter 5

Lifshitz Holography 1: GR with a massive vector

Now that we have had some practice with the holographic renormalization of AdS spacetime, we move on to Lifshitz holography. The first theory we examine is the simplest one in the literature, first found in [19], in which bulk Einstein gravity is coupled to a massive vector. In this chapter we carry out the holographic renormalization of this theory and in the next chapter we analyze some of the implications of the calculation.

5.1 The Bulk Action and Notation

The bulk spacetime relativistic action is:

$$\begin{aligned} S &= \frac{1}{16\pi G_{D+2}} \int_{\mathcal{M}} dt\, d^D\mathbf{x}\, dr\, \sqrt{-G} \left(\mathcal{R} - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 \mathcal{A}_\mu \mathcal{A}^\mu \right) \\ &\quad + \frac{1}{8\pi G_{D+2}} \int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\, \sqrt{-g}\, K. \end{aligned} \tag{5.1}$$

Note that in order for the Lifshitz spacetime (1.12) to be a classical solution, we set

$$m^2 = Dz \quad \mathit{and} \quad \Lambda = -\frac{1}{2}\left(z^2 + (D-1)z + D^2\right). \tag{5.2}$$

The Lifshitz metric is sourced by a non-zero condensate of the vector field, with background solution:

$$\mathcal{A}_A = \alpha \delta^0_A, \tag{5.3}$$

with

$$\alpha^2 = \frac{2(z-1)}{z}. \tag{5.4}$$

Note that the non-zero condensate of this massive vector field in the time direction causes the dual field theory to be non-relativistic, with anisotropic scaling between space and time.

Therefore, in addition to distinguishing boundary and bulk coordinates (as was done in Chapter 4), we now also need to distinguish boundary spacetime coordinates from boundary spatial coordinates. So let us clarify our notation for this chapter. The bulk metric is written as follows, with the boundary at $r = \infty$:

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta} dx^\alpha dx^\beta + \frac{dr^2}{r^2} \\ &= -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) + \frac{dr^2}{r^2}. \end{aligned} \tag{5.5}$$

We take D to be the number of spatial dimensions on the boundary and so there are $D+2$ spacetime dimensions in the bulk and $d \equiv D+1$ spacetime dimensions on the boundary. For coordinate indices, i, j are used for the D spatial boundary indices (x^i), whereas α, β are used for the $D+1$ spacetime boundary indices (t, x^i) and μ, ν are used for the $D+2$ bulk dimensions (t, x^i, r). $\tilde{\nabla}_\alpha$ represents the covariant derivative for the metric $g_{\alpha\beta}$ and ∇_i represents the covariant derivative for the metric γ_{ij}. Note that in (5.5), the bulk diffeomorphisms have been gauge fixed by setting the the bulk shift vector $\mathcal{N}_\alpha = 0$, and the bulk lapse function $\mathcal{N} = 1/r$. This *radial gauge* is adopted throughout this chapter. Moreover, in order to distinguish the lapse and shift variables in the bulk from those of the ADM decomposition on the boundary, we refer to the bulk variables $\mathcal{N}$ and $\mathcal{N}_i$ as the "radial lapse" and "radial shift". In this radial gauge, the extrinsic curvature on the boundary is $K_{\alpha\beta} \equiv \frac{1}{2\mathcal{N}}(\partial_r g_{\alpha\beta} - \tilde{\nabla}_\alpha \mathcal{N}_\beta - \tilde{\nabla}_\beta \mathcal{N}_\alpha) = r\partial_r g_{\alpha\beta}/2$.

It is often convenient to work in terms of vielbeins, which we define via

$$\begin{aligned} ds^2 &= \eta_{MN} E^M_\mu E^N_\nu dx^\mu dx^\nu = \eta_{AB} e^A_\alpha e^B_\beta dx^\alpha dx^\beta + \frac{dr^2}{r^2} \\ &= -N^2 dt^2 + \delta_{IJ} \hat{e}^I_i \hat{e}^J_j (dx^i + N^i dt)(dx^j + N^j dt) + \frac{dr^2}{r^2}. \end{aligned} \tag{5.6}$$

For the internal frame indices, $M, N = 0, 1, ..., D+1$ are used for the $D+2$ bulk dimensions, $A, B = 0, 1, ..., D$ are used for the $D+1$ spacetime boundary indices and $I, J = 1, ..., D$ are used for the D spatial boundary indices.

In order to distinguish the Riemann tensor and the extrinsic curvature tensor of the three different metrics $G_{\mu\nu}$, $g_{\alpha\beta}$ and γ_{ij}, we use the notation wherein $(D+2)$-dimensional quantities are written in curly letters (for example, $\mathcal{R}$ for the Ricci scalar), $(D+1)$-dimensional quantities are written in standard italics and D-dimensional quantities are written with hats.

5.2 Radial Decomposition

When written in radial ADM variables $\mathcal{N}$, $\mathcal{N}_\alpha$, $g_{\alpha\beta}$, the bulk action (5.1) becomes (when ignoring surface terms)

$$S = \frac{1}{16\pi G_{D+2}} \int_{\mathcal{M}} d^d x\, dr\, \sqrt{-g} \mathcal{N} \Big(K^2 - K_{\alpha\beta} K^{\alpha\beta} + R - 2\Lambda$$

$$-\frac{1}{2\mathcal{N}^2}F_{r\alpha}F_r{}^\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2\mathcal{N}^2}m^2\mathcal{A}_r^2 - \frac{1}{2}m^2\mathcal{A}_\alpha\mathcal{A}^\alpha\Big). \quad (5.7)$$

We define[1] the radial momenta corresponding to the metric $g_{\alpha\beta}$ and the vector field $\mathcal{A}_\alpha$ by:

$$\begin{aligned}
\pi^{\alpha\beta} &\equiv -\frac{16\pi G_{D+2}}{\sqrt{-g}}\frac{\delta S}{\delta\partial_r g_{\alpha\beta}} = K^{\alpha\beta} - g^{\alpha\beta}K, \\
\pi^{\alpha} &\equiv -\frac{16\pi G_{D+2}}{\sqrt{-g}}\frac{\delta S}{\delta\partial_r \mathcal{A}_{\alpha}} = \frac{F_r{}^\alpha}{\mathcal{N}}.
\end{aligned} \quad (5.8)$$

Varying the action (5.7) with respect to the radial lapse function $\mathcal{N}$ gives the Hamiltonian constraint:

$$K^2 - K_{\alpha\beta}K^{\alpha\beta} - \frac{1}{2}\pi_\alpha\pi^\alpha - \frac{1}{2\mathcal{N}^2}m^2\mathcal{A}_r^2 = R - 2\Lambda - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}m^2\mathcal{A}_\alpha\mathcal{A}^\alpha, \quad (5.9)$$

which will be useful in later sections.

5.3 The ADM decomposition

We next need to perform an ADM decomposition of $g_{\alpha\beta}$, separating out the time from the space coordinates on the boundary.

ADM decomposition in the metric formalism

In our calculations, we decompose the metric $g_{\alpha\beta}$ on the $(D+1)$-dimensional boundary of spacetime into the ADM decomposition

$$\begin{aligned}
g_{tt} = -N^2 + N^i N_i, \qquad g_{ij} = \gamma_{ij}, \qquad g_{ti} = N_i, \\
g^{tt} = -\frac{1}{N^2}, \qquad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2}, \qquad g^{ti} = \frac{N^i}{N^2}.
\end{aligned}$$

This metric leads to the following Christoffel symbols:

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{\partial_t N}{N} + \frac{N^j\nabla_j N}{N} + \frac{N^i N^j \hat{K}_{ij}}{N}, \\
\Gamma^i_{tt} &= \gamma^{ij}N\nabla_j N + N\gamma^{ij}\partial_t\left(\frac{N_j}{N}\right) - \frac{N^i N^j \nabla_j N}{N} - \gamma^{ij}N^k\nabla_j N_k - \frac{N^i N^j N^k \hat{K}_{jk}}{N}, \\
\Gamma^t_{ti} &= \frac{\nabla_i N}{N} + \frac{N^j \hat{K}_{ij}}{N},
\end{aligned}$$

[1] This differs from the usual canonical momenta by a factor of $\sqrt{-g}(-16\pi G_{D+2})^{-1}$ in order to simplify some of the subsequent equations.

$$\begin{aligned}\Gamma^j_{ti} &= N\gamma^{jk}\hat{K}_{ik} + N\nabla_i\left(\frac{N^j}{N}\right) - \frac{N^jN^k\hat{K}_{ik}}{N},\\ \Gamma^t_{ij} &= \frac{\hat{K}_{ij}}{N},\\ \Gamma^k_{ij} &= \hat{\Gamma}^k_{ij} - \frac{\hat{K}_{ij}N^k}{N},\end{aligned}$$

where $\hat{K}_{ij} = \frac{1}{2N}(\partial_t\gamma_{ij} - \nabla_iN_j - \nabla_jN_i)$ is the D-dimensional extrinsic curvature.

These result in the following $(D+1)$-dimensional Ricci scalar R for the metric $g_{\alpha\beta}$ in terms of $\hat{R}$, the D-dimensional Ricci scalar for the metric γ_{ij}:

$$R = \hat{R} - \frac{2\nabla^i\nabla_iN}{N} + \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \frac{\partial_tZ}{N\sqrt{\gamma}} + \frac{\nabla^iY_i}{N}, \tag{5.10}$$

where:

$$Z \equiv \gamma^{ij}\sqrt{\gamma}\nabla_i(\frac{N_j}{N}) + 2\hat{K}\sqrt{\gamma}, \tag{5.11}$$

$$Y_i \equiv -\partial_t(\frac{N_i}{N}) + \frac{N^j\nabla_iN_j}{N} + 2N^j\hat{K}_{ij} - 3N_i\hat{K} + \frac{N^j\nabla_jN_i}{N} - \frac{N_i\nabla_jN^j}{N}. \tag{5.12}$$

ADM decomposition in the vielbein formalism

The vielbeins are defined by $g_{\alpha\beta} = e^A_\alpha e^B_\beta\eta_{AB}$ and $\gamma_{ij} = \hat{e}^I_i\hat{e}^J_j\delta_{IJ}$. The $(D+1)$ dimensional boundary has vielbeins e^A given by:

$$e^0 = Ndt, \qquad e^I = \hat{e}^I_i(N^idt + dx^i) = N^Idt + \hat{e}^I. \tag{5.13}$$

The Ricci rotation coefficients are defined by $de^C = \Omega_{AB}{}^Ce^A\wedge e^B$:

$$de^0 = \nabla_iNdx^i\wedge dt = \frac{\nabla_IN}{N}e^I\wedge e^0, \tag{5.14}$$

$$de^I = (\frac{\nabla_JN^I}{N} - \frac{\hat{e}^j_J\partial_t\hat{e}^I_j}{N})e^J\wedge e^0 + \hat{\Omega}_{JK}{}^Ie^J\wedge e^K. \tag{5.15}$$

This means that:

$$\Omega_{0I}{}^0 = \frac{\nabla_IN}{2N}, \tag{5.16}$$

$$\Omega_{IJ}{}^0 = 0, \tag{5.17}$$

$$\Omega_{0J}{}^I = -(\frac{\nabla_JN^I}{2N} - \frac{\hat{e}^j_J\partial_t\hat{e}^I_j}{2N}), \tag{5.18}$$

$$\Omega_{JK}{}^I = \hat{\Omega}_{JK}{}^I, \tag{5.19}$$

$$\Omega_{0I}{}^I = -(\frac{\nabla_IN^I}{2N} - \frac{\hat{e}^j_I\partial_t\hat{e}^I_j}{2N}) = -(\frac{\nabla_IN^I}{2N} - \frac{\gamma^{ij}\partial_t\gamma_{ij}}{4N}) = \frac{\hat{K}}{2}. \tag{5.20}$$

Note that by definition ${\Omega_{AB}}^C = -{\Omega_{BA}}^C$. The covariant derivative is then given by:

$$\tilde{\nabla}_\alpha V_B = \partial_\alpha V_B - {\omega_{\alpha B}}^C V_C, \tag{5.21}$$

where $\omega_{ABC} = -\Omega_{ABC} + \Omega_{ACB} + \Omega_{BCA}$. Note that $\omega_{ABC} = -\omega_{ACB}$. Also ${\omega_{[AB]}}^C = -{\Omega_{AB}}^C$ and ${\omega_{CD}}^C = 2{\Omega_{DC}}^C$.

5.4 The massive vector

By a choice of frame, we take the massive vector to be:

$$\mathcal{A}_A = (\alpha + \psi)\delta^0_A. \tag{5.22}$$

Also, the massive vector has a non-zero component in the r direction, which the equation of motion for $\mathcal{A}_r$ gives as:

$$\mathcal{A}_r = -\frac{\tilde{\nabla}^\alpha F_{r\alpha}}{m^2} = -\frac{\tilde{\nabla}^A(\mathcal{N}\pi_A)}{m^2}. \tag{5.23}$$

Then:

$$\mathcal{A}_\alpha \;=\; e^A_\alpha \mathcal{A}_A = e^0_\alpha(\alpha+\psi) = N(\alpha+\psi)\delta^t_\alpha. \tag{5.24}$$

The only non-zero component of $F_{\alpha\beta}$ is:

$$F_{it} = -F_{ti} = \partial_i \mathcal{A}_t = \alpha\nabla_i N + \nabla_i(N\psi). \tag{5.25}$$

The non-zero components of $F^{\alpha\beta}$ are:

$$F^{jt} = -F^{tj} = -\frac{\gamma^{ij}F_{it}}{N^2}, \qquad F^{jk} = \frac{(\gamma^{ij}N^k - \gamma^{ik}N^j)F_{it}}{N^2}. \tag{5.26}$$

Therefore we have that:

$$\begin{aligned} F_{AB}F^{AB} = F_{\alpha\beta}F^{\alpha\beta} &= -\frac{2(\alpha\nabla_i N + \nabla_i(N\psi))(\alpha\nabla^i N + \nabla^i(N\psi))}{N^2} \\ &= -\frac{2\alpha^2\nabla_i N\nabla^i N}{N^2} - \frac{4\alpha\nabla_i(N\psi)\nabla^i N}{N^2} - \frac{2\nabla_i(N\psi)\nabla^i(N\psi)}{N^2}. \end{aligned} \tag{5.27}$$

5.5 Functional derivatives and the stress tensor

As before, we will be applying the Hamilton-Jacobi formalism in the radial direction to perform the holographic renormalization. We form the on-shell action by taking the bulk action (5.1) and evaluating it as a function of the boundary fields:

$$S = \frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\mathcal{L}. \tag{5.28}$$

As in the standard Hamilton-Jacobi theory, the radial momenta (5.8) can be obtained by functional differentiation of the on-shell action:

$$\pi^{\alpha\beta} = -\frac{16\pi G_{D+2}}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}, \qquad \pi^{\alpha} = -\frac{16\pi G_{D+2}}{\sqrt{-g}}\frac{\delta S}{\delta \mathcal{A}_{\alpha}}. \tag{5.29}$$

Equivalently, the variation of the on-shell action is:

$$\begin{aligned}\delta S &= -\frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} d^d x\sqrt{-g}\left[\pi^{\alpha\beta}\delta g_{\alpha\beta} + \pi^{\alpha}\delta\mathcal{A}_{\alpha}\right] &(5.30)\\ &= -\frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} d^d x\sqrt{-g}\left[(2\pi^{\alpha}{}_{\beta} + \pi^{\alpha}\mathcal{A}_{\beta})e^{\beta}_{B}\delta e^{B}_{\alpha} + \pi^{A}\delta\mathcal{A}_{A}\right]. &(5.31)\end{aligned}$$

The boundary stress tensor $T^{\alpha}{}_{B}$, however, is defined by functional differentiation of the on-shell action with respect to the vielbeins e^{B}_{α}, while holding the vector field with *frame indices* ($\mathcal{A}_A$) fixed. Note that $\mathcal{A}_0 = \alpha + \psi$ is the only non-zero component of $\mathcal{A}_A$. Therefore, the variation of the on-shell action can also be written as:

$$\delta S = -\frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} d^d x\sqrt{\gamma}N\left[T^{\alpha}{}_{B}\delta e^{B}_{\alpha} + \pi_{\psi}\delta\psi\right], \tag{5.32}$$

where

$$T^{A}{}_{B} = -\frac{16\pi G_{D+2}}{\sqrt{\gamma}N}e^{A}_{\alpha}\frac{\delta S}{\delta e^{B}_{\alpha}}, \qquad \pi_{\psi} = -\frac{16\pi G_{D+2}}{\sqrt{\gamma}N}\frac{\delta S}{\delta\psi}. \tag{5.33}$$

One fact to remember is that a non-relativistic stress tensor is not necessarily symmetric. $T^{t}{}_{0}$ is the vacuum expectation value (vev) of the energy density, $T^{i}{}_{0}$ is the vev of the energy flux, $T^{0}{}_{I}$ is the vev of the momentum density and $T^{I}{}_{j}$ is the vev of the stress tensor.

By comparing (5.31) and (5.32) we get the following relations:

$$T_{\alpha B} = (2\pi_{\alpha\beta} + \pi_{\alpha}\mathcal{A}_{\beta})e^{\beta}_{B}, \qquad \pi_{\psi} = \pi^{0}. \tag{5.34}$$

Rearranging these expressions we have the following:

$$\pi_{AB} = \frac{1}{2}(T_{AB} - \pi_{A}\mathcal{A}_{B}), \qquad \pi_{I}\mathcal{A}_{0} = T_{I0} - T_{0I}. \tag{5.35}$$

Finally, by using the expressions for the vielbeins derived in Section 5.3, we can write the stress tensor as:

$$\begin{aligned}T^{0}{}_{0} &= -\frac{16\pi G_{D+2}}{\sqrt{\gamma}}\frac{\delta S}{\delta N}, &(5.36)\\ T^{0}{}_{I} &= -\frac{16\pi G_{D+2}}{\sqrt{\gamma}}\frac{\delta S}{\delta N^{I}}, &(5.37)\\ T^{I}{}_{J} &= -16\pi G_{D+2}\left[\frac{N^{I}}{\sqrt{\gamma}N}\frac{\delta S}{\delta N^{J}} + \frac{1}{\sqrt{\gamma}N}\hat{e}^{I}_{i}\frac{\delta S}{\delta\hat{e}^{J}_{i}}\right]\\ &= -16\pi G_{D+2}\left[\frac{N^{I}}{\sqrt{\gamma}N}\frac{\delta S}{\delta N^{J}} + \frac{2}{\sqrt{\gamma}N}\hat{e}^{I}_{i}\hat{e}_{Jj}\frac{\delta S}{\delta\gamma_{ij}}\right]. &(5.38)\end{aligned}$$

We will use these expressions to determine the stress tensor and vector momentum from the on-shell action.

5.6 Analysis of Linearized Constant Modes

As in Section 4.4, the bulk fields will correspond to operators in the dual quantum field theory and it is useful to conduct an analysis of the constant linearized modes around the Lifshitz background. We parametrize the linearized fields by:

$$e^0 = r^z(1+\frac{1}{2}f)dt + rv_{1i}dx^i, \quad e^I = r^z v_2^I dt + r(\delta^I{}_i(1+\frac{1}{2}k)+\frac{1}{2}k^I{}_i)dx^i, \quad \psi = \alpha j, \tag{5.39}$$

where $k_{ij}\delta^{ij} = 0$ and where $f, j, v_{1i}, v_{2i}, k_{ij}$ are functions of r only. At the linearized level, this is equivalent to the following parametrization of the metric and massive vector:

$$\begin{aligned} g_{tt} = -r^{2z}(1+f), \qquad g_{ti} = r^{z+1}(-v_{1i}+v_{2i}), \qquad g_{ij} = r^2(\delta_{ij}(1+k)+k_{ij}), \\ \mathcal{A}_t = \alpha r^z(1+\frac{1}{2}f+j), \qquad \mathcal{A}_i = \alpha r v_{1i}. \end{aligned} \tag{5.40}$$

Since the equations and background solution have D-dimensional rotational symmetry, the linearized equations of motion decouple into scalar, vector and tensor equations and we can analyze each separately. But first we examine the pure gauge modes that appear.

Pure gauge modes

Before doing this, recall that the bulk theory has a gauge group of diffeomorphisms ($\delta G_{\mu\nu} = \partial_\mu\zeta^\sigma G_{\nu\sigma} + \partial_\nu\zeta^\sigma G_{\mu\sigma} + \zeta^\sigma\partial_\sigma G_{\mu\nu}$ and $\delta\mathcal{A}_\mu = \partial_\mu\zeta^\sigma\mathcal{A}_\sigma + \zeta^\sigma\partial_\sigma\mathcal{A}_\mu$), so some modes will be pure gauge. Since we will only be looking at the linearized modes, we can substitute in our background solution of $G_{rr} = \frac{1}{r^2}$, $G_{r\alpha} = 0$, $G_{ij} = g_{ij} = r^2\delta_{ij}$, $G_{tt} = g_{tt} = -r^{2z}$, $G_{it} = g_{it} = 0$, $\mathcal{A}_\mu = \alpha r^z\delta^t_\mu$:

$$\begin{aligned} \delta G_{rr} &= \frac{2}{r^2}\partial_r\zeta^r - \frac{2}{r^3}\zeta^r, \\ \delta G_{ri} &= r^2\partial_r\zeta^j\delta_{ij} + \frac{1}{r^2}\partial_i\zeta^r, \\ \delta G_{rt} &= -r^{2z}\partial_r\zeta^t + \frac{1}{r^2}\partial_t\zeta^r, \\ \delta g_{tt} &= -2r^{2z}\partial_t\zeta^t - 2zr^{2z-1}\zeta^r, \\ \delta g_{it} &= -r^{2z}\partial_i\zeta^t + r^2\partial_t\zeta^j\delta_{ij}, \\ \delta g_{ij} &= r^2\partial_i\zeta^k\delta_{jk} + r^2\partial_j\zeta^k\delta_{ik} + 2r\zeta^r\delta_{ij}, \\ \delta\mathcal{A}_t &= \partial_\mu\zeta^t\alpha r^z + \alpha z r^{z-1}\zeta^r, \\ \delta\mathcal{A}_r &= \partial_r\zeta^t\alpha r^z, \\ \delta\mathcal{A}_i &= 0. \end{aligned} \tag{5.41}$$

When choosing the radial gauge of (5.5), we must maintain $\delta G_{rr} = \delta G_{rt} = \delta G_{ri} = 0$, which requires $\zeta^r = r\zeta$, $\zeta^i = \frac{1}{2r^2}\delta^{ij}\partial_j\zeta$ and $\zeta^t = -\frac{1}{2zr^{2z}}\partial_t\zeta$, where ζ is independent of r. This leaves the following diffeomorphisms unfixed:

$$\delta g_{tt} = \frac{1}{z}\partial_t^2\zeta - 2zr^{2z}\zeta,$$

$$\begin{aligned}
\delta g_{it} &= \frac{z+1}{2z}\partial_i\partial_t\zeta, \\
\delta g_{ij} &= \partial_i\partial_j\zeta + 2r^2\zeta\delta_{ij}, \\
\delta\mathcal{A}_t &= -\frac{\alpha}{2r^z}\partial_\mu\partial_t\zeta + \alpha z r^z\zeta, \\
\delta\mathcal{A}_r &= -\frac{\alpha}{2r^z}\partial_r\partial_t\zeta, \\
\delta\mathcal{A}_i &= 0.
\end{aligned} \tag{5.42}$$

In addition, we are just looking at constant modes so we can take $\delta g_{tt}, \delta g_{it}, \delta g_{ij}$ to be independent of x^i, t (but they can depend on r). This means that ζ must be independent of x^i, t and r and so (5.42) becomes

$$\begin{aligned}
\delta g_{tt} &= -2zr^{2z}\zeta, \\
\delta g_{it} &= 0, \\
\delta g_{ij} &= 2r^2\zeta\delta_{ij}, \\
\delta\mathcal{A}_t &= \alpha z r^z\zeta, \\
\delta\mathcal{A}_r &= 0, \\
\delta\mathcal{A}_i &= 0.
\end{aligned} \tag{5.43}$$

This is a pure gauge mode, even after the radial gauge and massive vector frame (5.22) have been chosen.

Scalar modes

We now substitute our linearized scalar modes (f, k, j) into the equation of motion:

$$\mathcal{R}_{\mu\nu} = \frac{1}{D}(2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu})G_{\mu\nu} + \frac{1}{2}F_{\sigma\mu}F^{\sigma}{}_{\nu} + \frac{1}{2}m^2\mathcal{A}_\mu\mathcal{A}_\nu, \tag{5.44}$$

keeping only linear terms. The resulting equations for $(\mu,\nu) = (r,r)$, $(\mu,\nu) = (t,t)$ and $(\mu,\nu) = (i,j)$ respectively are:

$$\begin{aligned}
0 &= D^2r^2k'' + 3D^2rk' + Dr^2f'' + (3D+2z-2)rf' - 4(z-1)(D-1)(zj+rj'), \\
0 &= Dr^2f'' + (D^2+3D+2z-2)rf' + zD^2k' - 4(z-1)[D^2j + (D-1)(zj+rj')], \\
0 &= D(z+2D+1)rk' + Dr^2k'' + (D+2z-2)rf' + 4(z-1)(zj+rj'),
\end{aligned} \tag{5.45}$$

where a prime denotes differentiation with respect to r. The equation of motion for the massive vector ($\nabla^\mu F_{\mu\nu} = m^2\mathcal{A}_\nu$) gives a further equation when $\mu = t$:

$$0 = r^2f'' + 2j'' + Dzrk' + (D+1)rf' + 2(D+z+1)rj'. \tag{5.46}$$

These linear equations have the following general solution (for $z \neq D$):

$$j = -\frac{(D+z-1)}{(z-1)}\frac{c_1}{r^{D+z}} - \frac{(D+z-1)}{(z-1)}\frac{c_2}{r^{\Delta_+}} + \frac{(D+z-1)}{(z-1)}\frac{c_3}{r^{\Delta_-}},$$

$$
\begin{aligned}
k &= \frac{2}{(D+z)}\frac{c_1}{r^{D+z}} - \frac{2(3z-D-2-\beta_z)}{(D+z+\beta_z)}\frac{c_2}{r^{\Delta_+}} + \frac{2(3z-D-2+\beta_z)}{(D+z-\beta_z)}\frac{c_3}{r^{\Delta_-}} + 2\zeta, \\
f &= \frac{2D}{(D+z)}\frac{c_1}{r^{D+z}} + \frac{2((3D-1)z-D(D-1)-(D-1)\beta_z)}{(D+z+\beta_z)}\frac{c_2}{r^{\Delta_+}} \\
&\quad - \frac{2((3D-1)z-D(D-1)+(D-1)\beta_z)}{(D+z-\beta_z)}\frac{c_3}{r^{\Delta_-}} + c_4 - 2z\zeta,
\end{aligned} \tag{5.47}
$$

where $c_1, c_2, c_3, c_4, \zeta$ are arbitrary constants and

$$
\Delta_\pm = \frac{1}{2}(z+D\pm\beta_z), \qquad \beta_z = \sqrt{(z+D)^2+8(z-1)(z-D)}. \tag{5.48}
$$

When $z=D$, the general solution is instead:

$$
\begin{aligned}
j &= -\frac{(2D-1)}{(D-1)}\frac{c_1+c_2\ln r}{r^{2D}} + c_3, \\
k &= \frac{1}{D}\frac{c_1+c_2\ln r}{r^{2D}} + \frac{(3D-1)}{2D^2(D-1)}\frac{c_2}{r^{2D}} - 2(D-1)c_3\ln r + 2\zeta, \\
f &= \frac{c_1+c_2\ln r}{r^{2D}} - \frac{(3D-1)}{2D(D-1)}\frac{c_2}{r^{2D}} + 2D(D-1)c_3\ln r + c_4 - 2z\zeta.
\end{aligned} \tag{5.49}
$$

As explained previously, ζ is a pure gauge mode.

Vector modes

We now substitute our linearized vector modes (v_{1i}, v_{2i}) into the equation of motion (5.44) and only keep linear terms. The resulting equation for $(\mu,\nu)=(t,i)$ is:

$$
0 = (z-1)(D+1)(v_{2i}+v_{1i}) + (z+D+1)rv'_{2i} - (D-z+3)rv'_{1i} + r^2(v''_{2i}-v''_{1i}). \tag{5.50}
$$

The equation of motion for the massive vector ($\nabla^\mu F_{\mu\nu} = m^2\mathcal{A}_\nu$) gives a further equation when $\mu = i$:

$$
0 = r^2v''_{1i} + (D+1)rv'_{1i} - (D+z-1)(z-1)v_{1i} + zrv'_{2i} + z(z-1)v_{2i}. \tag{5.51}
$$

These two linear equations have the following general solution:

$$
\begin{aligned}
v_{1i} &= c_{1i}r^{z-1} + c_{2i}r^{-(D+1)} + c_{3i}r^{-(2z+D-1)}, \\
v_{2i} &= c_{4i}r^{-(z-1)} + \frac{(z-2)(D+z)}{z(z-2-D)}c_{2i}r^{-(D+1)} + \frac{(D+3z-2)}{(D+z)}c_{3i}r^{-(2z+D-1)}.
\end{aligned} \tag{5.52}
$$

where $c_{1i}, c_{2i}, c_{3i}, c_{4i}$ are arbitrary constants.

Tensor modes

Finally, we substitute our linearized tensor mode (k_{ij}) into the equation of motion (5.44) and only keep linear terms. The resulting equation for $(\mu, \nu) = (i, j)$ is:

$$0 \;=\; r^2 k''_{ij} + (z + D + 1) r k'_{ij}. \tag{5.53}$$

This has general solution $k_{ij} = c_{1ij} + c_{2ij} r^{-(D+z)}$ for traceless constants c_{1ij} and c_{2ij}.

Analysis of the modes

We can identify each pair of linearized modes with an operator in the dual field theory. In the dual field theory, the nonrelativistic stress-energy tensor is made up of the energy density $\mathcal{E}$ and the spatial stress tensor Π^i_j, which have dimension $z + D$, the momentum density $\mathcal{P}_i$, which has dimension $D + 1$ and the energy flux, $\mathcal{E}_i$, which has dimension $2z + D - 1$. As shown in [8, 9], we can identify c_4, c_{4i}, c_{1i} and c_{1ij} as the sources for $\mathcal{E}$, $\mathcal{P}_i$, $\mathcal{E}_i$ and Π^i_j respectively. Furthermore, c_1, c_{2i}, c_{3i} and c_{2ij} contribute to expectation values of $\mathcal{E}$, $\mathcal{P}_i$, $\mathcal{E}_i$ and Π^i_j respectively. Note that there is not a separate source for the trace of Π^i_j since the anisotropic scaling implies that $z\mathcal{E} + \Pi^i_i = 0$. We now also have scalar modes (c_2 and c_3), which correspond to a new dual operator $\mathcal{O}_\psi$ of dimension $\Delta_\pm$. The $\pm$ sign here depends on whether the standard or alternative boundary condition is chosen for this mode (see Section 1.2). If the standard boundary condition is chosen, then c_2 is the source and the $\mathcal{O}_\psi$ has dimension Δ_+. If $\frac{8+D}{5} \leq z \leq D$ for $D > 1$ (or $1 \leq z \leq \frac{9}{5}$ for $D = 1$), then the alternative boundary condition can be chosen instead, which results in c_3 being the source and $\mathcal{O}_\psi$ having dimension Δ_-.

5.7 Boundary Source Fields and Asymptotic Scaling

The boundary conditions are specified by fixing the sources for the various field theory operators on the boundary. Using the results of the previous section, our boundary conditions involve the following finite fixed sources as $r \to \infty$ (denoting each source with a bar):

$$\bar{e}^0_\alpha = \frac{e^0_\alpha}{r^z}, \qquad \bar{e}^I_\alpha = \frac{e^I_\alpha}{r}, \qquad \bar{\psi} = \frac{\psi}{r^{-\Delta_-}}. \tag{5.54}$$

In order to have a foliation on the boundary, it is necessary to set $\bar{e}^0_i$ (the source for the energy flux $\mathcal{E}^i$) equal to zero [8]. For all of this chapter, we have set $\bar{e}^0_i = 0$.

Note that $T^\alpha{}_A$ is the vacuum expectation value of the operator sourced by $\bar{e}^A_\alpha$. In other words, $\bar{e}^0_\alpha$ is the source for the energy density $\mathcal{E}$ and the energy flux $\mathcal{E}^i$, whereas $\bar{e}^I_\alpha$ is the source for the momentum density $\mathcal{P}_i$ and the stress tensor $\Pi^i{}_j$. $\bar{\psi}$ is the source for $\mathcal{O}_\psi$, the operator dual to the massive vector ψ. The operator $\mathcal{O}_\psi$ is relevant for $z < D$ and irrelevant for $z > D$. Therefore, for $z > D$, we must take $\bar{\psi} = 0$ in order to preserve the asymptotic boundary conditions above. In the case $z = D$, the operator is marginal and there is some evidence suggesting that it becomes marginally relevant in the case of $D = 2$ [53]. We have assumed in the boundary conditions (5.54) that the standard boundary condition has been chosen for ψ. If the alternative boundary condition is chosen instead, one simply replaces Δ_- by Δ_+ in (5.54) and identical results are obtained in what follows. Note that in this case the operator $\mathcal{O}_\psi$ is always relevant.

The scaling dimensions discussed here are the classical scaling dimensions, consistent with the fact that we perform our analysis near the ultraviolet fixed point with fixed z. In the bulk, this corresponds to the asymptotic analysis in the vicinity of the space-time boundary at conformal infinity. Hence, in our analysis we systematically ignore most of the possible nontrivial infrared dynamics, such as the flow - generically expected in Lifshitz-type theories - towards lower values of z under the influence of relevant operators.

The above scaling behavior allows us to determine the scaling behavior of other quantities near the boundary. Any boundary quantity can be written in terms of the source fields $\bar{e}^A_\alpha$ and $\bar{\psi}$ and then the scaling behavior can be read off from the resulting exponents of r. Consider a general object $\mathcal{O}$. When written in terms of the boundary source fields, we say that the term in $\mathcal{O}$ scaling as $r^{-\Delta}$ is of "order Δ" and denote it by $\mathcal{O}^{(\Delta)}$. For example, e^0_α has order $-z$, e^I_α has order -1 and ψ has order Δ_-. This means that N has order $-z$, N_i has order -2, γ_{ij} has order -2 and γ^{ij} has order 2.

From (5.10), R has components of order 2 and $2z$ given by:

$$R^{(2)} = \hat{R} - \frac{2\nabla^i\nabla_i N}{N}, \tag{5.55}$$

$$R^{(2z)} = \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \frac{\partial_t Z}{N\sqrt{\gamma}} + \frac{\nabla^i Y_i}{N}. \tag{5.56}$$

From (5.27), $F_{AB}F^{AB}$ has components of order 2, $2+\Delta_-$ and $2+2\Delta_-$ given by:

$$(F_{AB}F^{AB})^{(2)} = -\frac{2\alpha^2\nabla_i N\nabla^i N}{N^2}, \tag{5.57}$$

$$(F_{AB}F^{AB})^{(2+\Delta_-)} = -\frac{4\alpha\nabla_i(N\psi)\nabla^i N}{N^2}, \tag{5.58}$$

$$(F_{AB}F^{AB})^{(2+2\Delta_-)} = -\frac{2\nabla_i(N\psi)\nabla^i(N\psi)}{N^2}. \tag{5.59}$$

Also, $\mathcal{A}_A\mathcal{A}^A = -(\alpha+\psi)^2$ has components of dimension $0, \Delta_-, 2\Delta_-$:

$$(\mathcal{A}_A\mathcal{A}^A)^{(0)} = -\alpha^2, \tag{5.60}$$

$$(\mathcal{A}_A\mathcal{A}^A)^{(\Delta_-)} = -2\alpha\psi, \tag{5.61}$$

$$(\mathcal{A}_A\mathcal{A}^A)^{(2\Delta_-)} = -\psi^2. \tag{5.62}$$

Note that (5.33) implies that the term $\mathcal{L}^{(\Delta)}$ in the on-shell action determine ${T^0}_0{}^{(\Delta)}$, ${T^0}_I{}^{(\Delta+1-z)}$, ${T^I}_0{}^{(\Delta+z-1)}$, ${T^I}_J{}^{(\Delta)}$ and ${\pi_\psi}^{(\Delta-\Delta_-)}$.

We can also determine the leading order divergences of the momenta from the asymptotic boundary conditions (5.54):

$$K^0{}_0{}^{(0)} = z, \qquad K^I{}_J{}^{(0)} = \delta^I{}_J. \tag{5.63}$$

Also, the zero-component of the vector momentum is given by:

$$\pi_0 = rF_{r0} = r\partial_r \mathcal{A}_0 + \mathcal{A}_0 K^0{}_0 - r\partial_0 \mathcal{A}_r. \tag{5.64}$$

This gives:

$$\pi_0^{(0)} = \alpha K^0{}_0{}^{(0)} = \alpha z, \tag{5.65}$$

$$\pi_0^{(\Delta_-)} = r\partial_r \psi + \alpha K^0{}_0{}^{(\Delta_-)} + \psi K^0{}_0{}^{(0)} = \alpha K^0{}_0{}^{(\Delta_-)} + (z - \Delta_-)\psi. \tag{5.66}$$

Remember that $\pi_\psi \equiv \pi^0$.

5.8 Holographic Renormalization Equations

When the action (5.1) is evaluated on shell as a function of the boundary fields we write it as:

$$S = \frac{1}{16\pi G_{D+2}} \int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\, \sqrt{\gamma}\, N\mathcal{L}. \tag{5.67}$$

A convenient way of computing the divergent part of $\mathcal{L}$ is to organize the terms with respect to how they scale with r. More precisely, we define the dilatation operator by:

$$\delta_{\mathcal{D}} = \int_{\partial\mathcal{M}} dt\, d^D\mathbf{x} \left(z e^0_\mu \frac{\delta}{\delta e^0_\mu} + e^I_\mu \frac{\delta}{\delta e^I_\mu} - \Delta_- \psi \frac{\delta}{\delta \psi} \right). \tag{5.68}$$

This operator asymptotically represents $r\dfrac{\partial}{\partial r}$.

$\mathcal{L}$ can then be decomposed into a sum of terms as follows:

$$\mathcal{L} = \sum_{\Delta \geq 0} \mathcal{L}^{(\Delta)} + \tilde{\mathcal{L}}^{(z+D)} \log r. \tag{5.69}$$

Note that we include a logarithmic term at order $z + D$ due to the possibility of a Weyl scaling anomaly. The individual terms of the expansion (5.69) satisfy

$$\delta_{\mathcal{D}} \mathcal{L}^{(\Delta)} = -\Delta\, \mathcal{L}^{(\Delta)} \qquad \text{for} \;\; \Delta \neq z + D, \tag{5.70}$$

$$\delta_{\mathcal{D}} \mathcal{L}^{(z+D)} = -(z+D)\mathcal{L}^{(z+D)} + \tilde{\mathcal{L}}^{(z+D)}, \tag{5.71}$$

$$\delta_{\mathcal{D}} \tilde{\mathcal{L}}^{(z+D)} = -(z+D)\tilde{\mathcal{L}}^{(z+D)}. \tag{5.72}$$

Applying $\delta_{\mathcal{D}}$ to the on-shell action (5.67) and using (5.33) then yields:

$$(z + D + \delta_{\mathcal{D}})\mathcal{L} = -zT^0{}_0 - T^I{}_I + \Delta_-\psi\pi_\psi. \tag{5.73}$$

Expanding this at each order then results in:

$$(z + D - \Delta)\mathcal{L}^{(\Delta)} = -zT^0{}_0{}^{(\Delta)} - T^I{}_I{}^{(\Delta)} + \Delta_-\psi\pi_\psi^{(\Delta-\Delta_-)}, \tag{5.74}$$

except for $\Delta = z + D$, when this becomes

$$\tilde{\mathcal{L}}^{(\Delta)} = -zT^0{}_0{}^{(\Delta)} - T^I{}_I{}^{(\Delta)} + \Delta_-\psi\pi_\psi^{(\Delta-\Delta_-)}. \tag{5.75}$$

This allows us to solve for the anomaly. The above equations imply that the anomaly term can also be found by:

$$\tilde{\mathcal{L}}^{(\Delta)} = \lim_{\Delta\to z+D}\Big[(z + D - \Delta)\mathcal{L}^{(\Delta)}\Big]. \tag{5.76}$$

Note that the value of $\mathcal{L}^{(z+D)}$ cannot be found by this asymptotic analysis.

We now move on to finding an explicit expression for these divergent terms in the on-shell action $\mathcal{L}^{(\Delta)}$. Recall from (4.6) that the variation of the bulk action (4.1) with respect to $\mathcal{N}$ produces the constraint equation:

$$K^2 - K_{AB}K^{AB} - \frac{1}{2}\pi_A\pi^A - \frac{1}{2m^2}(\tilde{\nabla}^A\pi_A)^2 = R - 2\Lambda - \frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}m^2\mathcal{A}_A\mathcal{A}^A, \tag{5.77}$$

where we have also used (5.23) and are working in the radial gauge. Expanding this equation in its dilatation eigenvalues (utilizing (5.35), (5.63), (5.65), (5.66)) and then substituting it into (5.74) yields an expression for $\mathcal{L}^{(\Delta)}$ (see [8] for more details). Explicitly, the terms in the on-shell action are given for $\Delta \neq 0, \Delta_-$ and $2\Delta_-$ by:

$$(z + D - \Delta)\mathcal{L}^{(\Delta)} = \mathcal{Q}^{(\Delta)} + \mathcal{S}^{(\Delta)}, \tag{5.78}$$

where the quadratic term $\mathcal{Q}^{(\Delta)}$ is given by

$$\begin{aligned}\mathcal{Q}^{(\Delta)} \;=\; &\sum_{0<s<\Delta/2;s\neq\Delta_-}\left[2K_{AB}^{(s)}\pi^{AB(\Delta-s)} + \pi_A^{(s)}\pi^{A(\Delta-s)} + \frac{1}{m^2}(\tilde{\nabla}_A\pi^A)^{(s)}(\tilde{\nabla}_A\pi^A)^{(\Delta-s)}\right]\\ &+\left[K_{AB}^{(\Delta_-)}T^{AB(\Delta-\Delta_-)} + K_{00}^{(\Delta_-)}\pi^{0(\Delta-2\Delta_-)}\psi + \pi_I^{(\Delta_-)}\pi^{I(\Delta-\Delta_-)}\right]\\ &+\left[K_{AB}^{(\Delta/2)}\pi^{AB(\Delta/2)} + \frac{1}{2}\pi_A^{(\Delta/2)}\pi^{A(\Delta/2)} + \frac{1}{2m^2}(\tilde{\nabla}_A\pi^A)^{(\Delta/2)2}\right],\end{aligned} \tag{5.79}$$

and the source $\mathcal{S}$ is

$$\mathcal{S} = R - 2\Lambda - \frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}m^2\mathcal{A}_A\mathcal{A}^A. \tag{5.80}$$

We also have the following exceptions to the above formula:

$$(z+D)\mathcal{L}^{(0)} = 2\mathcal{S}^{(0)}, \tag{5.81}$$

$$(z+D-\Delta_-)\mathcal{L}^{(\Delta_-)} = (\Delta_- - z)\psi\pi_\psi^{(0)} + \mathcal{S}^{(\Delta_-)}, \tag{5.82}$$

$$\begin{aligned}(z+D-2\Delta_-)\mathcal{L}^{(2\Delta_-)} &= (\Delta_- - z)\psi\pi_\psi^{(\Delta_-)} + K_{AB}^{(\Delta_-)}\pi^{AB(\Delta_-)} \\ &\quad + \frac{1}{2}\pi_A^{(\Delta_-)}\pi^{A(\Delta_-)} + \mathcal{S}^{(2\Delta_-)}.\end{aligned} \tag{5.83}$$

$\mathcal{S}$ needs to be calculated at each order. The calculation in Section 5.7 shows that R has components of order 2 and $2z$, $F_{AB}F^{AB}$ has components of order $2, 2+\Delta_-, 2+2\Delta_-$ and $\mathcal{A}_A\mathcal{A}^A$ has components of order $0, \Delta_-, 2\Delta_-$, resulting in:

$$\mathcal{S}^{(0)} = -2\Lambda + \frac{1}{2}m^2\alpha^2 = (z+D)(z+D-1), \tag{5.84}$$

$$\mathcal{S}^{(\Delta_-)} = m^2\alpha\psi = Dz\alpha\psi, \tag{5.85}$$

$$\mathcal{S}^{(2\Delta_-)} = \frac{1}{2}m^2\psi^2 = \frac{Dz}{2}\psi^2, \tag{5.86}$$

$$\mathcal{S}^{(2)} = R^{(2)} - \frac{1}{4}(F_{AB}F^{AB})^{(2)} = \hat{R} - \frac{2\nabla^i\nabla_i N}{N} + \frac{\alpha^2}{2}\frac{\nabla^i N\nabla_i N}{N^2}, \tag{5.87}$$

$$\mathcal{S}^{(2+\Delta_-)} = -\frac{1}{4}(F_{AB}F^{AB})^{(2+\Delta_-)} = \frac{\alpha\nabla_i N\nabla^i(N\psi)}{N^2}, \tag{5.88}$$

$$\mathcal{S}^{(2+2\Delta_-)} = -\frac{1}{4}(F_{AB}F^{AB})^{(2+2\Delta_-)} = \frac{\nabla^i(N\psi)\nabla_i(N\psi)}{2N^2}, \tag{5.89}$$

$$\mathcal{S}^{(2z)} = R^{(2z)} = \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \text{total derivatives}. \tag{5.90}$$

5.9 Calculation of Divergences

We now proceed to use these formulae to calculate the divergent terms in the on-shell action at each order. Once these divergent terms have been calculated, counterterms must be added to the action in order to subtract these divergences. With a boundary cutoff at $r = \frac{1}{\epsilon}$, the counterterms are

$$S_{ct} = -\frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\,\sqrt{\gamma}\, N\left[\sum_{0\le\Delta<z+D}\mathcal{L}^{(\Delta)} - \tilde{\mathcal{L}}^{(z+D)}\log(\epsilon)\right]. \tag{5.91}$$

We shall determine $\mathcal{L}^{(\Delta)}$ and $\tilde{\mathcal{L}}^{(z+D)}$ only up to total derivatives of the form $\frac{\partial_t}{\sqrt{\gamma}N}$ or $\frac{\nabla_i}{N}$. This will not allow us to determine anomalies in $\tilde{\mathcal{L}}^{(z+D)}$ that are total derivatives (such as was found for $D=z=1$ in (4.44)), but we shall see that this suffices for our calculation of anomalies in $D=z=2$. Therefore, we can drop the total derivative terms in (5.90). We now proceed to calculate the divergences at each order.

Non-derivative counterterms with $\psi = 0$

At order 0, we have:

$$\mathcal{L}^{(0)} = \frac{2\mathcal{S}^{(0)}}{z+D} = 2(z+D-1). \tag{5.92}$$

This yields ${T^A}_B^{(0)} = -2(z+D-1)\delta^A{}_B$.

Non-derivative counterterms involving ψ

First we evaluate the order Δ_- and $2\Delta_-$ counterterms. Using (5.63, 5.65, 5.66) and inserting this into (5.78) we have

$$(z+D-\Delta_-)\mathcal{L}^{(\Delta_-)} = -(z-\Delta_-)\psi\pi_\psi^{(0)} + \mathcal{S}^{(\Delta_-)} = (z-\Delta_-)\psi\alpha z + Dz\alpha\psi \tag{5.93}$$

$$\mathcal{L}^{(\Delta_-)} = z\alpha\psi, \tag{5.94}$$

which yields ${T^A}_B^{(\Delta_-)} = -z\alpha\psi\delta^A{}_B$.

Note that $\pi^A{}_B = \frac{1}{2}(T^A{}_B - \pi^A\mathcal{A}_B) = K^A{}_B - K\delta^A{}_B$ and this means that:

$$\pi^0{}_0{}^{(\Delta_-)} = \frac{1}{2}(T^0{}_0{}^{(\Delta_-)} - \alpha\pi_\psi^{(\Delta_-)} - \psi\pi_\psi^{(0)}) = -\frac{1}{2}\alpha\pi_\psi^{(\Delta_-)}, \tag{5.95}$$

$$\pi^I{}_J{}^{(\Delta_-)} = \frac{1}{2}T^I{}_J{}^{(\Delta_-)} = -\frac{z\alpha\psi}{2}\delta^I{}_J, \tag{5.96}$$

$$K^{(\Delta_-)} = -\frac{\pi^A{}_A{}^{(\Delta_-)}}{D} = \frac{\alpha\pi_\psi^{(\Delta_-)}}{2D} + \frac{z\alpha\psi}{2}, \tag{5.97}$$

$$K^0{}_0{}^{(\Delta_-)} = \pi^0{}_0{}^{(\Delta_-)} + K^{(\Delta_-)} = -\frac{\alpha\pi_\psi^{(\Delta_-)}(D-1)}{2D} + \frac{z\alpha\psi}{2}, \tag{5.98}$$

$$K^I{}_J{}^{(\Delta_-)} = \pi^I{}_J{}^{(\Delta_-)} + K^{(\Delta_-)}\delta^I{}_J = \frac{\alpha\pi_\psi^{(\Delta_-)}}{2D}\delta^I{}_J. \tag{5.99}$$

Substituting this into the expression $\pi_0^{(\Delta_-)} = \alpha K^0{}_0{}^{(\Delta_-)} + (z-\Delta_-)\psi$ derived above gives:

$$\pi_0^{(\Delta_-)} = \alpha\left(-\frac{\alpha\pi_\psi^{(\Delta_-)}(D-1)}{2D} + \frac{z\alpha\psi}{2}\right) + (z-\Delta_-)\psi, \tag{5.100}$$

$$\pi_0^{(\Delta_-)} = \frac{2D(2z-1-\Delta_-)}{2D-\alpha^2(D-1)}\psi = \frac{Dz(2z-1-\Delta_-)}{z+D-1}\psi. \tag{5.101}$$

Therefore, using this result for $\pi_0^{(\Delta_-)}$:

$$K^{(\Delta_-)} = -\frac{\alpha z(2z-1-\Delta_-)}{2(z+D-1)}\psi + \frac{z\alpha\psi}{2} = -\frac{\alpha z(z-D-\Delta_-)}{2(z+D-1)}\psi, \tag{5.102}$$

$$K^0{}_0{}^{(\Delta_-)} = \frac{\alpha z(D-1)(2z-1-\Delta_-)}{2(z+D-1)}\psi + \frac{z\alpha\psi}{2} = \frac{\alpha z((2D-1)z-(D-1)\Delta_-)}{2(z+D-1)}\psi, \tag{5.103}$$

$$K^I{}_J{}^{(\Delta_-)} = -\frac{\alpha z(2z-1-\Delta_-)}{2(z+D-1)}\psi\delta^I{}_J. \tag{5.104}$$

Then:

$$\begin{aligned}
(z+D-2\Delta_-)\mathcal{L}^{(2\Delta_-)} &= -(z-\Delta_-)\psi\pi_\psi^{(\Delta_-)} + K_{AB}^{(\Delta_-)}\pi^{AB(\Delta_-)} + \frac{1}{2}\pi_A^{(\Delta_-)}\pi^{A(\Delta_-)} + \mathcal{S}^{(2\Delta_-)} \\
&= -(z-\Delta_-)\psi\pi_\psi^{(\Delta_-)} + (-\frac{\alpha\pi_\psi^{(\Delta_-)}(D-1)}{2D} + \frac{z\alpha\psi}{2})(-\frac{1}{2}\alpha\pi_\psi^{(\Delta_-)}) \\
&\quad +(\frac{\alpha\pi_\psi^{(\Delta_-)}}{2})(-\frac{z\alpha\psi}{2}) + \frac{1}{2}(\pi_\psi^{(\Delta_-)})^2 - \frac{Dz}{2}\psi^2 \\
&= \frac{Dz\psi^2(4z^2-4z-4z\Delta_-+1+2\Delta_-+\Delta_-^2+z+D-1)}{2(z+D-1)} \\
&= \frac{Dz\psi^2(z+D-2\Delta_-)(2z-1-\Delta_-)}{2(z+D-1)}, \qquad (5.105)\\
\mathcal{L}^{(2\Delta_-)} &= \frac{Dz\psi^2(2z-1-\Delta_-)}{2(z+D-1)}, \qquad (5.106)
\end{aligned}$$

where $\Delta_- = \frac{1}{2}(z+D-\beta_z)$ and $\beta_z = \sqrt{(z+D)^2+8(z-1)(z-D)}$ has been used.

This result yields ${T^A}_B^{(2\Delta_-)} = -\frac{Dz\psi^2(2z-1-\Delta_-)}{2(z+D-1)}{\delta^A}_B$. Next we can calculate:

$$\begin{aligned}
(z+D-3\Delta_-)\mathcal{L}^{(3\Delta_-)} &= K_{AB}^{(\Delta_-)}T^{AB(2\Delta_-)} + K_{00}^{(\Delta_-)}\pi^{0(0)}\psi \\
&= +\frac{D\alpha z^2(2z-1-\Delta_-)(z-D-\Delta_-)}{4(z+D-1)^2}\psi^3 \\
&\quad +\frac{D\alpha z^2((2D-1)z-(D-1)\Delta_-)(2z-1-\Delta_-)}{2(z+D-1)^2}\psi^3 \\
&= \frac{D\alpha z^2(2z-1-\Delta_-)(-D+(4D-1)z-(2D-1)\Delta_-)}{4(z+D-1)^2}\psi^3,
\end{aligned}$$

which yields $\pi_\psi^{(2\Delta_-)} = -\frac{3D\alpha z^2(2z-1-\Delta_-)(-D+(4D-1)z-(2D-1)\Delta_-)}{4(z+D-3\Delta_-)(z+D-1)^2}\psi^2$.

This allows us to calculate $K_{AB}^{(2\Delta_-)}$:

$$\begin{aligned}
{\pi^0}_0^{(2\Delta_-)} &= \frac{1}{2}({T^0}_0^{(2\Delta_-)} - \alpha\pi_\psi^{(2\Delta_-)} - \psi\pi_\psi^{(\Delta_-)}) = -\frac{1}{2}\alpha\pi_\psi^{(2\Delta_-)} - \frac{1}{4}\psi\pi_\psi^{(\Delta_-)}, \qquad (5.107)\\
{\pi^I}_J^{(2\Delta_-)} &= \frac{1}{2}{T^I}_J^{(2\Delta_-)} = \frac{1}{4}\psi\pi_\psi^{(\Delta_-)}{\delta^I}_J, \qquad (5.108)\\
K^{(2\Delta_-)} &= -\frac{{\pi^A}_A^{(\Delta_-)}}{D} = \frac{1}{2D}\alpha\pi_\psi^{(2\Delta_-)} - \frac{(D-1)}{4D}\psi\pi_\psi^{(\Delta_-)}, \qquad (5.109)\\
{K^0}_0^{(2\Delta_-)} &= {\pi^0}_0^{(2\Delta_-)} + K^{(2\Delta_-)} = -\frac{\alpha\pi_\psi^{(2\Delta_-)}(D-1)}{2D} - \frac{(2D-1)}{4D}\psi\pi_\psi^{(\Delta_-)}, \qquad (5.110)\\
{K^I}_J^{(2\Delta_-)} &= {\pi^I}_J^{(2\Delta_-)} + K^{(2\Delta_-)}{\delta^I}_J = (\frac{1}{2D}\alpha\pi_\psi^{(2\Delta_-)} + \frac{1}{4D}\psi\pi_\psi^{(\Delta_-)}){\delta^I}_J. \qquad (5.111)
\end{aligned}$$

Higher order non-derivative terms can be calculated in a similar manner.

Two-derivative counterterms with $\psi = 0$

The divergent term in the on-shell action of order 2 is:

$$(z+D-2)\mathcal{L}^{(2)} \;=\; \mathcal{S}^{(2)} = \hat{R} - \frac{2\nabla^i\nabla_i N}{N} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}. \tag{5.112}$$

Up to a total derivative, this becomes:

$$(z+D-2)\mathcal{L}^{(2)} \;=\; \hat{R} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}. \tag{5.113}$$

This gives the following contribution to the stress tensor (see Section 5.5):

$$\begin{aligned}
(z+D-2)T_{00}^{(2)} &= \hat{R} - \frac{\alpha^2\nabla^i\nabla_i N}{N} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2},\\
(z+D-2)T_{0I}^{(3-z)} &= 0,\\
(z+D-2)T_{I0}^{(1+z)} &= \frac{N_I}{N}(-\hat{R} + \frac{\alpha^2\nabla^i\nabla_i N}{N} - \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}),\\
(z+D-2)T_{IJ}^{(2)} &= 2\hat{R}_{IJ} - \frac{2\nabla_I\nabla_J N}{N} + \frac{\alpha^2\nabla_I N\nabla_J N}{N^2}\\
&\quad + \delta_{IJ}(-\hat{R} + \frac{2\nabla^i\nabla_i N}{N} - \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}),\\
(z+D-2){T^I}_I^{(2)} &= -(D-2)\hat{R} + \frac{2(D-1)\nabla^i\nabla_i N}{N} - \frac{\alpha^2(D-2)\nabla^i N\nabla_i N}{2N^2}.
\end{aligned}$$

At order $2z$ there is a contribution from the quadratic term $\frac{1}{2m^2}\left[(\nabla_A\pi^A)^{(z)}\right]^2$. Note that:

$$\begin{aligned}
(\nabla_A\pi^A)^{(z)} &= (\partial_A\pi^A - {\omega_A}^{AB}\pi_B)^{(z)} = (\partial_A\pi^A - 2{\Omega_A}^{BA}\pi_B)^{(z)}\\
&= (\partial_0(\pi^{0(0)}) - 2{\Omega_I}^{0I}\pi_0^{(0)}) = (\partial_0(-z\alpha) - 2{\Omega_I}^{0I}z\alpha)\\
&= -\alpha z\hat{K},
\end{aligned} \tag{5.114}$$

where expressions from Section 5.3 have been used. Therefore, up to total derivatives:

$$\begin{aligned}
(z+D-2z)\mathcal{L}^{(2z)} &= \mathcal{S}^{(2z)} + \frac{1}{2m^2}\left[(\tilde{\nabla}_A\pi^A)^{(z)}\right]^2\\
&= \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \frac{1}{2m^2}(-\alpha z\hat{K})^2\\
&= \hat{K}_{ij}\hat{K}^{ij} - \frac{(D+1-z)}{D}\hat{K}^2.
\end{aligned} \tag{5.115}$$

Two-derivative counterterms involving ψ

We can also calculate various divergent terms involving ψ, for example:

$$(z+D-2-\Delta_-)\mathcal{L}^{(2+\Delta_-)} = K_{AB}^{(\Delta_-)}T^{AB(2)} + \frac{\alpha\nabla_i N\nabla^i(N\psi)}{N^2}$$

$$
\begin{aligned}
&= -\frac{\alpha z \psi}{2(z+D-1)}\Big[((2D-1)z-(D-1)\Delta_-)T^{00(2)}+(2z-1-\Delta_-){T^I{}_I}^{(2)}\Big] \\
&\qquad -\frac{\alpha\nabla^i\nabla_i N\psi}{N}+\frac{\alpha\nabla_i N\nabla^i N\psi}{N^2} \\
&= -\frac{\alpha z\psi}{2(z+D-1)(z+D-2)}\cdot \\
&\qquad \Big[((2D-1)z-(D-1)\Delta_-)(\hat{R}-\frac{\alpha^2\nabla^i\nabla_i N}{N}+\frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}) \\
&\qquad +(2z-1-\Delta_-)(-(D-2)\hat{R}+\frac{2(D-1)\nabla^i\nabla_i N}{N}-\frac{\alpha^2(D-2)\nabla^i N\nabla_i N}{2N^2})\Big] \\
&\quad -\frac{\alpha\nabla^i\nabla_i N\psi}{N}+\frac{\alpha\nabla_i N\nabla^i N\psi}{N^2}. \qquad (5.116)
\end{aligned}
$$

Or, by defining some constants:

$$
\mathcal{L}^{(2+\Delta_-)} = -\psi\left(c_1\hat{R}+c_2\frac{\nabla^i\nabla_i N}{N}+c_3\frac{\nabla_i N\nabla^i N}{N^2}\right), \qquad (5.117)
$$

where:

$$
\begin{aligned}
c_1 &= \frac{\alpha z(-2+D-\Delta_-+3z)}{2(D-2+z)(D-1+z)(z+D-2-\Delta_-)}, \\
c_2 &= \frac{\alpha(4+2D^2-(4+(\alpha^2-2)\Delta_-)z+(1-2D)(\alpha^2-2)z^2+D((2+(\alpha^2-2)\Delta_-)z-6))}{2(D-2+z)(D-1+z)(z+D-2-\Delta_-)}, \\
c_3 &= \frac{\alpha(-8-4D^2-(-12+\alpha^2(2+\Delta_-))z+(3\alpha^2-4)z^2+D(12+(\alpha^2-8)z))}{4(D-2+z)(-1+D+z)(-2+D-\Delta_-+z)}.
\end{aligned}
$$

(Note that for $z=D=2$ we have $c_1=\frac{1}{2}$, $c_2=\frac{1}{2}$ and $c_3=-\frac{1}{4}$.)

This results in:

$$
\pi_\psi^{(2)} = c_1\hat{R}+c_2\frac{\nabla^i\nabla_i N}{N}+c_3\frac{\nabla_i N\nabla^i N}{N^2} \qquad (5.118)
$$

and also:

$$
\begin{aligned}
T_{00}^{(2+\Delta_-)} &= -c_1\psi\hat{R}-c_2\nabla^i\nabla_i\psi-c_3\frac{\nabla_i N\nabla^i N\psi}{N^2}+2c_3\frac{\nabla_i\nabla^i N\psi}{N}+2c_3\frac{\nabla^i N\nabla_i\psi}{N}, \qquad (5.119) \\
T_{0I}^{(3-z+\Delta_-)} &= 0, \qquad (5.120) \\
T_{I0}^{(1+z+\Delta_-)} &= \frac{N_I}{N}(c_1\psi\hat{R}+c_2\nabla^i\nabla_i\psi+c_3\frac{\nabla_i N\nabla^i N\psi}{N^2}-2c_3\frac{\nabla_i N\nabla^i\psi}{N}), \\
T_{IJ}^{(2+\Delta_-)} &= \delta_{IJ}(c_1\psi\hat{R}-c_2\frac{\nabla_i N\nabla^i\psi}{N}+c_3\frac{\nabla_i N\nabla^i N\psi}{N^2}) \\
&\quad -2c_1\psi\hat{R}_{IJ}+c_2\frac{\nabla_I N\nabla_J\psi}{N}+c_2\frac{\nabla_J N\nabla_I\psi}{N}-2c_3\frac{\nabla_I N\nabla_J N\psi}{N^2}
\end{aligned}
$$

$$-2\delta_{IJ}c_1\frac{\nabla^i\nabla_i(N\psi)}{N}+2c_1\frac{\nabla_I\nabla_J(N\psi)}{N}, \tag{5.121}$$

$$T^I{}_I{}^{(2+\Delta_-)} = (D-2)(c_1\psi\hat{R}-c_2\frac{\nabla_iN\nabla^i\psi}{N}+c_3\frac{\nabla_iN\nabla^iN\psi}{N^2})-2c_1\frac{\nabla^i\nabla_i(N\psi)}{N}. \tag{5.122}$$

There are many more two-derivative terms involving ψ. For example:

$$\begin{aligned}(z+D-2-2\Delta_-)\mathcal{L}^{(2+2\Delta_-)} &= 2K_{AB}^{(2\Delta_-)}\pi^{AB(2)}+\pi_A^{(2\Delta_-)}\pi^{A(2)}\\ &+ K_{AB}^{(\Delta_-)}T^{AB(2+\Delta_-)}+K_{00}^{(\Delta_-)}\pi^{0(2)}\psi+\mathcal{S}^{(2+2\Delta_-)}.\end{aligned} \tag{5.123}$$

This has been calculated explicitly in the case $D=z=2$:

$$\mathcal{L}^{(2+2\Delta_-)} = \psi^2(\frac{\nabla^i\nabla_iN}{8N}-\frac{\nabla^iN\nabla_iN}{2N^2})+\frac{3}{4}\psi\nabla^i\nabla_i\psi. \tag{5.124}$$

Four-derivative counterterms with $\psi = 0$

At fourth order we have:

$$\begin{aligned}&(z+D-4)\mathcal{L}^{(4)}\\ &\quad= K_{AB}^{(2)}\pi^{AB(2)}+\frac{1}{2}\pi_A^{(2)}\pi^{A(2)}\\ &\quad=\frac{1}{a_0}\Big[a_1\Big(\frac{\nabla_iN\nabla^iN}{N^2}\Big)^2+a_2\frac{\nabla_iN\nabla^iN}{N^2}\hat{R}+a_3\frac{\nabla^i\nabla_iN}{N}\frac{\nabla_jN\nabla^jN}{N^2}\\ &\qquad+a_4\frac{\nabla^iN\nabla^jN}{N^2}\hat{R}_{ij}+a_5\frac{\nabla^i\nabla_iN}{N}\hat{R}+a_6\Big(\frac{\nabla^i\nabla_iN}{N}\Big)^2+a_7\hat{R}_{ij}^2+a_8\hat{R}^2\Big].\end{aligned} \tag{5.125}$$

where:

$$\begin{aligned}a_0 &= -2Dz^2(-2+D+z)^2(-1+D+z)(-4+\beta+D+z)^2,\\ a_1 &= 32(z-1)^3+D^4(-11+z(6+z))\\ &\quad+D^3(52-3\beta_z-z(77+2\beta_z-(34+\beta_z)z+z^2))\\ &\quad+D^2(16(-8+\beta_z)+z(2(116+\beta_z)+z(-145-8\beta_z+2(9+\beta_z)z+3z^2)))\\ &\quad+D(z-1)(16(-8+\beta_z)+z(184+z(-68-5\beta_z+z(-13+\beta_z+5z)))),\\ a_2 &= 2z(z-1)(D^4+D^3(\beta_z-z)+16(z-1)z+D^2(8-4\beta_z+z(-16+2\beta_z+3z))\\ &\quad+Dz(-4(-8+\beta_z)+z(-24+\beta_z+5z))),\\ a_3 &= -2z(D^4(z-4)+32(z-1)^2+D^3(-2(7+2\beta_z)+(21+\beta_z-z)z)\\ &\quad+D^2(32+18\beta_z+z(-60-11\beta_z+z(10+2\beta_z+3z)))\\ &\quad+D(z-1)(8(8+\beta_z)+z(-40-6\beta_z+z(-18+\beta_z+5z)))),\\ a_4 &= -4zD(z-2)(D-1+z)(8+D^2-8z-2Dz+5z^2+\beta_z(-4+D+z)),\\ a_5 &= -4z^2(D^3+D^2(\beta_z-2z)+8(z-1)z+D(8+\beta_z(z-4)-3z^2)),\\ a_6 &= -4z(D-1)(D^3+D^2(\beta_z-2z)+8(z-1)z+D(8+\beta_z(z-4)-3z^2)),\\ a_7 &= -4z^2D(D-1+z)(8+D^2-8z-2Dz+5z^2+\beta_z(D-4+z)),\\ a_8 &= z^2(D^4+D^3(\beta_z-z)+8(z-1)z^2+D^2(8-4\beta_z+z(-8+2\beta_z+3z))\end{aligned}$$

$$+Dz(8-4\beta_z+z(-8+\beta_z+5z))).$$

In the above expression we have used the following identities for terms in the action (up to total derivatives):

$$\begin{aligned}
\frac{\nabla_i N\nabla_j N\nabla^i\nabla^j N}{N^3} &\sim \Big(\frac{\nabla_i N\nabla^i N}{N^2}\Big)^2-\frac{\nabla_i N\nabla^i N\nabla^j\nabla_j N}{2N^3},\\
\frac{\nabla_i\nabla_j N\nabla^i\nabla^j N}{N^2} &\sim \Big(\frac{\nabla_i N\nabla^i N}{N^2}\Big)^2-\frac{3\nabla_i N\nabla^i N\nabla^j\nabla_j N}{2N^3}+\Big(\frac{\nabla^i\nabla_i N}{N}\Big)^2-\frac{\nabla^i N\nabla^j N R_{ij}}{N^2},\\
\frac{\nabla^i\nabla^j N R_{ij}}{N} &\sim \frac{\nabla^i\nabla_i N R}{2N}.
\end{aligned}$$

For $D=2$ we have further simplifications because $R_{ij}=\dfrac{R}{2}\delta_{ij}$ and so:

$$\begin{aligned}
(z-2)\mathcal{L}^{(4)} &= \frac{(z-2)}{b_0}\Big[b_1\Big(\frac{\nabla_i N\nabla^i N}{N^2}\Big)^2+b_2\frac{\nabla_i N\nabla^i N}{N^2}\hat{R}+b_3\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2}\\
&\quad +b_4\frac{\nabla^i\nabla_i N}{N}\hat{R}+b_5\Big(\frac{\nabla^i\nabla_i N}{N}\Big)^2+b_6\hat{R}^2\Big],
\end{aligned}\tag{5.126}$$

where:

$$\begin{aligned}
b_0 &= -2z^4(z+1)(z-2+\beta_z)^2,\\
b_1 &= 12+36z-11z^2-2z^3+5z^4+\beta_z(-2-7z+z^3),\\
b_2 &= 4z^2(z-6+\beta_z),\\
b_3 &= -2z(36-4z-7z^2+5z^3+\beta_z(z^2-z-6)),\\
b_4 &= -4z^2(z-6+\beta_z),\\
b_5 &= -4z(z-6+\beta_z),\\
b_6 &= -z^3(z-6+\beta_z).
\end{aligned}$$

Note that in the important case where $z\to 2$ (and still $D=2$):

$$(z-2)\mathcal{L}^{(4)} = \frac{(2-z)}{64}\Big[3\Big(\frac{\nabla_i N\nabla^i N}{N^2}\Big)^2-4\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2}\Big].\tag{5.127}$$

A useful check is for $z=1$, which is the usual relativistic AdS case. The standard known result ([41] and also (4.46)) is that the 4th order term involving only spatial derivative is (up to total derivatives):

$$\begin{aligned}
\mathcal{L}^{(4)} &= \Big[\frac{1}{(D-3)(D-1)^2}\Big(R_{\alpha\beta}R^{\alpha\beta}-\frac{D+1}{4D}R^2\Big)\Big]^{(4)}\\
&= \frac{1}{(D-3)(D-1)^2}\Big[\Big(\frac{\nabla^i\nabla_i N}{N}\Big)^2+\Big(\hat{R}_{ij}-\frac{\nabla_i\nabla_j N}{N}\Big)\Big(\hat{R}^{ij}-\frac{\nabla^i\nabla^j N}{N}\Big)\\
&\quad -\frac{D+1}{4D}\Big(\hat{R}-\frac{2\nabla^i\nabla_i N}{N}\Big)^2)\Big]
\end{aligned}$$

$$
\begin{aligned}
= \; & -\frac{1}{(D-3)(D-1)^2}\Big[-\Big(\frac{\nabla_i N\nabla^i N}{N^2}\Big)^2+\frac{3\nabla_i N\nabla^i N\nabla^j\nabla_j N}{2N^3}+\frac{\nabla^i N\nabla^j N R_{ij}}{N^2} \\
& -\frac{1}{D}\frac{\nabla_i\nabla^i N}{N}\hat{R}-\frac{D-1}{D}\Big(\frac{\nabla^i\nabla_i N}{N}\Big)^2-\hat{R}_{ij}\hat{R}^{ij}+\frac{D+1}{4D}\hat{R}^2\Big]. \qquad (5.128)
\end{aligned}
$$

This agrees exactly with the general result above. Of course, for $z=1$ there will also be contributing terms at this order which come from the $4z$ and $2+2z$ order terms (these will involve time derivatives).

An easily computable case is $D=1$ (for which $\hat{R}=0$). The above expressions yield $(z-3)\mathcal{L}^{(4)}=\dfrac{(z-3)\nabla_i N\nabla^i N}{12z^3N^2}$. For $z=3$, which is when this would possibly generate a scaling anomaly, this expression vanishes.

Four-derivative counterterms involving ψ

There are many possible four-derivative counterterms involving ψ, for example:

$$
(z+D-4-\Delta_-)\mathcal{L}^{(4+\Delta_-)}=2K^{(2)}_{AB}\pi^{AB(2+\Delta_-)}+\pi^{(2)}_A\pi^{A(2+\Delta_-)}+K^{(\Delta_-)}_{AB}T^{AB(4)}. \qquad (5.129)
$$

The right hand-side has been explicitly calculated and found to be zero in the case where $z=2$ and $D=2$.

Higher-derivative counterterms

The divergent terms calculated above are all the divergent terms with $\psi=0$ for $D+z<6$. For higher values of D and z there will be further divergences, which can be calculated systematically using this method.

Chapter 6

The Anisotropic Weyl Anomaly for $D = z = 2$

In this section we use the results of the previous chapter to calculate the anisotropic Weyl anomaly for the specific case of $D = z = 2$. Before examining the specific theory from Chapter 5, we begin by classifying the possible terms that can appear in the anomaly in this case.

6.1 Classification of Anisotropic Weyl Anomalies for $D = z = 2$

Just as in the relativistic case, a theory which has the classical symmetry under anisotropic Weyl transformations can develop an anomaly in this symmetry at the quantum level. Under the transformations

$$\delta_\omega N = zN\delta\omega, \qquad \delta_\omega N_i = 2N_i\delta\omega, \qquad \delta_\omega \gamma_{ij} = 2\gamma_{ij}\delta\omega, \tag{6.1}$$

the anomaly will show up as a nonvanishing variation of the partition function $\mathcal{Z}[N, N_i, \gamma_{ij}]$, of the general form

$$\delta_\omega \log \mathcal{Z}[N, N_i, \gamma_{ij}] = -\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\, \sqrt{\gamma}\, N\, \mathcal{A}\, \delta\omega, \tag{6.2}$$

where $\mathcal{A}$ is now a function of N, N_i, and γ_{ij}.

We wish to determine what terms can arise in $\mathcal{A}$. As in the relativistic case, this question is cohomological in nature.[1] We introduce a nilpotent BRST operator Q, which acts on the metric multiplet via the infinitesimal anisotropic Weyl transformations (6.1), with $\delta\omega$

[1] The cohomological approach to the relativistic Weyl anomaly was developed in [54, 55, 56]; see [57], Chapter 22, for a general review of this approach.

replaced by a Grassmann parameter c of ghost number one. We can represent this operator as

$$Q = c\left(zN\frac{\delta}{\delta N} + 2N_i\frac{\delta}{\delta N_i} + 2\gamma_{ij}\frac{\delta}{\delta \gamma_{ij}}\right). \tag{6.3}$$

Since Q is nilpotent, the variation of the anomaly vanishes:

$$Q\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\,\mathcal{A}\,c = -Q^2\log\mathcal{Z} = 0. \tag{6.4}$$

This puts a constraint on the terms that can arise as $\mathcal{A}$.

As usual, some of these terms can be removed by including appropriate counterterms. If a term in the anomaly can be expressed as the variation some local counterterm, this (gravitational) counterterm can be subtracted from the action, thereby eliminating the associated anomaly. Therefore the physical anomaly can be considered to lie in the cohomology of Q, at ghost number one. The number of possible independent terms (*i.e.*, generalized central charges) in the anomaly will be determined by the dimension of this cohomology.

In the case of $2+1$ dimensions with $z = 2$, the anomaly must be – on dimensional grounds – a sum of terms of dimension four, lying in the cohomology of Q. The list of possible terms is rather large; however, all but two are cohomologically trivial and can therefore be eliminated using local counterterms. The only two terms that cannot be removed are:

$$\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2, \tag{6.5}$$

$$\left(\hat{R} - \frac{\nabla^i N\nabla_i N}{N^2} + \frac{\nabla^i\nabla_i N}{N}\right)^2. \tag{6.6}$$

As usual, this cohomology analysis only reveals the complete list of terms which *may* in principle occur in the anomaly. Whether or not such terms are generated in a particular theory is a dynamical question, which requires an additional calculation. Both anomaly terms should be expected to appear in the anomaly of generic $z = 2$ field theories in $2+1$ dimensions:

$$\mathcal{A} = c_K\left(K_{ij}K^{ij} - \frac{1}{2}K^2\right) + c_V\left(\hat{R} - \frac{\nabla^i N\nabla_i N}{N^2} + \frac{\nabla^i\nabla_i N}{N}\right)^2, \tag{6.7}$$

with two independent central charges, c_K and c_V. What is the form of the anomaly that appears for the setup examined in Chapter 5? We shall see that the first cohomology class (6.5) indeed arises in the holographic computation of the anisotropic Weyl anomaly ($c_K \neq 0$) but, interestingly, the second one (6.6) does not ($c_V = 0$).

6.2 $D = z = 2$ Anomaly for GR with a Massive Vector

The holographic renormalization in Lifshitz spacetime for GR with a massive vector was carried out in Chapter 5. We will rewrite the main results from the holographic renormalization for the specific case of $D = 2$ below. Although our main interest will be in $z = 2$, we start by considering general z. If we set $\psi = 0$, the terms that will give rise to divergent contributions in the on-shell action for $z < 4$ are $\mathcal{L}^{(0)}$, $\mathcal{L}^{(2)}$, $\mathcal{L}^{(2z)}$, and $\mathcal{L}^{(4)}$. As shown in Chapter 5, the holographic renormalization equations can be computed in terms of the boundary metric multiplet (N, N_i, γ_{ij}), giving (up to total derivatives)

$$\begin{aligned}
\mathcal{L}^{(0)} &= 2(z+1), && (6.8)\\
z\mathcal{L}^{(2)} &= \hat{R} + \frac{\alpha^2}{2}\frac{\nabla_i N \nabla^i N}{N^2}, && (6.9)\\
(2-z)\mathcal{L}^{(2z)} &= \hat{K}_{ij}\hat{K}^{ij} + \frac{z-3}{2}\hat{K}^2, && (6.10)\\
(2-z)\mathcal{L}^{(4)} &= \frac{z-2}{2z^4(z+1)(z-2+\beta_z)^2}\Bigg\{-4z(z-6+\beta_z)\left(\frac{\nabla^i\nabla_i N}{N}\right)^2 \\
&\quad + \left(12+36z-11z^2-2z^3+5z^4+\beta_z(z^3-7z-2)\right)\left(\frac{\nabla^i N\nabla_i N}{N^2}\right)^2 \\
&\quad - 2z\left(36-4z-7z^2+5z^3+\beta_z(z^2-z-6)\right)\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2} \\
&\quad + (z-6+\beta_z)\left[4z^2\frac{\nabla^i N\nabla_i N}{N^2}\hat{R} - 4z^2\frac{\nabla^i\nabla_i N}{N}\hat{R} - z^3\hat{R}^2\right]\Bigg\}. && (6.11)
\end{aligned}$$

When $z = 2$ is approached, the divergent terms of dimension four become logarithmic, and the residue of the $\Delta = 4$ (or $\Delta = 2z$) terms at the $z = 2$ pole give rise to $\tilde{\mathcal{L}}^{(4)}$. Specifically, we get

$$\tilde{\mathcal{L}}^{(4)} = \lim_{z\to 2}\left[(z-2)\mathcal{L}^{(4)} + (2-z)\mathcal{L}^{(2z)}\right]. \quad (6.12)$$

With this substitution, the $z = 2$ divergent terms in the on-shell action are

$$\begin{aligned}
\mathcal{L}^{(0)} &= 2(z+1) = 6, && (6.13)\\
\mathcal{L}^{(2)} &= \frac{1}{2}\hat{R} + \frac{1}{4}\frac{\nabla^i N\nabla_i N}{N^2}, && (6.14)\\
\tilde{\mathcal{L}}^{(4)} &= \hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2. && (6.15)
\end{aligned}$$

The coefficient $\tilde{\mathcal{L}}^{(4)}$ of the logarithmic divergence can be recognized as the unique kinetic term (2.14) for Lifshitz gravity with local conformal invariance in $2+1$ dimensions, invariant under the $z = 2$ anisotropic Weyl transformations (2.11). This is one of the central results of this chapter.

The expression for the counterterms has no potential term – *i.e.*, the only derivatives that appear in the counterterm are the time derivatives. This is in spite of the fact that there exists a term with spatial derivatives, written down in (2.15), which is invariant under the local $z = 2$ anisotropic Weyl transformations and which is not a total derivative. In other words, in terms of the central charges defined in (6.7), we have $c_V = 0$ and the anomaly satisfies the detailed balance condition.

It is surprising, at least at first sight, that such a potential term is not generated in the logarithmic counterterm of holographic renormalization in Lifshitz space. Indeed, as we showed in Section 6.1, this term (6.6) represents a non-trivial cohomology class appropriate to appear as an anomaly. What would be a minimal generalization of our holographic setup, which would generate such a term in the anomaly? One might suspect that a different dynamical embedding of the Lifshitz space may perhaps produce a more general set of holographic counterterms, allowing (6.6) to appear. Even in the embedding considered here, we have not turned on the most general sources in the boundary, and one can ask whether allowing nonzero ψ generates new counterterms. However, a detailed calculation (see (5.129)) reveals that turning on ψ also preserves detailed balance, and does not lead to the appearance of the second independent counterterm (6.6). In Chapter 7, we shall present a different theory where both anomaly terms do arise.

6.3 Gravity with a Massive Vector Coupled to Bulk Scalars

In order to probe further the structure of holographic counterterms in Lifshitz spacetime, it is useful to add additional matter fields in the bulk theory. The holographic renormalization procedure can be easily repeated with the inclusion of scalar fields in the bulk. We will see that for a marginal scalar at $z = 2$, there is a new logarithmically divergent counterterm, giving rise to a new, nongravitational contribution to the anisotropic Weyl anomaly. However, we will see that this new counterterm also satisfies the detailed balance condition: Even in the presence of the bulk scalars, the second gravitational counterterm (6.6) – which violates detailed balance – is not generated.

The bulk scalar action takes the standard relativistic form

$$S_{bulk,\ X} = -\frac{1}{2}\int_{\mathcal{M}} d^{d+1}x\sqrt{-G}\left(G^{\mu\nu}\partial_\mu X^a\partial_\nu X^a + \mu^2 X^a X^a\right). \tag{6.16}$$

In this section, we set $d = 3$, and again follow the procedure of [8], with appropriate modifications to include the scalar fields. The holographic renormalization equations of [8] now become

$$(z + 2 - \Delta)\mathcal{L}^{(\Delta)} = \tilde{\mathcal{Q}}^{(\Delta)} + \tilde{\mathcal{S}}^{(\Delta)}, \tag{6.17}$$

where the quadratic and source terms $\mathcal{Q}$ and $\mathcal{S}$ are modified to

$$\tilde{\mathcal{Q}}^{(\Delta)} = \mathcal{Q}^{(\Delta)} + 8\pi G_4(\tilde{\pi}^{a(\Delta/2)})^2 + 16\pi G_4 \sum_{s<\Delta/2;\ s\neq\tilde{\Delta}_-} (\tilde{\pi}^{a(s)}\tilde{\pi}^{a(\Delta-s)}) \tag{6.18}$$

and

$$\tilde{\mathcal{S}} = \mathcal{S} - 8\pi G_4(\partial_\alpha X^a\partial^\alpha X^a + \mu^2 X^a X^a). \tag{6.19}$$

In this expression, $\tilde{\pi}^a = r\partial_r X^a$ is the scalar momentum and the scalars fall of asymptotically as $r^{-\tilde{\Delta}_-}$, where

$$\mu^2 = \tilde{\Delta}_-(\tilde{\Delta}_- - 2 - z).$$

The additional source terms only contribute at orders $\Delta = 2\tilde{\Delta}_-$, $2 + 2\tilde{\Delta}_-$ and $2z + 2\tilde{\Delta}_-$:

$$\tilde{\mathcal{S}}^{(2\tilde{\Delta}_-)} = -8\pi G_4 \mu^2 X^a X^a, \tag{6.20}$$

$$\tilde{\mathcal{S}}^{(2+2\tilde{\Delta}_-)} = -\left[8\pi G_4 \partial_\alpha X^a \partial^\alpha X^a\right]^{(2+2\tilde{\Delta}_-)} = -8\pi G_4 \nabla_i X^a \nabla^i X^a, \tag{6.21}$$

$$\tilde{\mathcal{S}}^{(2z+2\tilde{\Delta}_-)} = -\left[8\pi G_4 \partial_\alpha X^a \partial^\alpha X^a\right]^{(2z+2\tilde{\Delta}_-)} = \frac{8\pi G_4}{N^2}(\partial_t X^a - N^i \nabla_i X^a)^2. \tag{6.22}$$

We now specialize to the case of a marginal scalar, that is, a scalar which has $\tilde{\Delta}_- = 0$. Note that this also means that the scalar is massless since $\mu^2 = \tilde{\Delta}_-(\tilde{\Delta}_- - 2 - z) = 0$. We are interested in calculating its contribution to the anisotropic Weyl anomaly in the case $z = 2$. The divergent pieces of the on-shell action that appear at orders $\Delta = 2 + 2\tilde{\Delta}_-$ and $\Delta = 2z + 2\tilde{\Delta}_-$ are straightforward to calculate as they only receive contributions from the source terms,

$$(z - 2\tilde{\Delta}_-)\mathcal{L}^{(2+2\tilde{\Delta}_-)} = -8\pi G_4 \nabla_i X^a \nabla^i X^a, \tag{6.23}$$

$$(2 - z - 2\tilde{\Delta}_-)\mathcal{L}^{(2z+2\tilde{\Delta}_-)} = \frac{8\pi G_4}{N^2}(\partial_t X^a - N^i \nabla_i X^a)^2. \tag{6.24}$$

By taking the functional derivative of this term in the on-shell action with respect to the metric, the contribution to the boundary stress energy tensor can be calculated. For example, for $\Delta = 2 + 2\tilde{\Delta}_-$

$$(z - 2\tilde{\Delta}_-)T_{00}^{(2+2\tilde{\Delta}_-)} = -8\pi G_4 \nabla_i X^a \nabla^i X^a, \tag{6.25}$$

$$(z - 2\tilde{\Delta}_-)T_{IJ}^{(2+2\tilde{\Delta}_-)} = -16\pi G_4 \nabla_I X^a \nabla_J X^a + 8\pi G_4 \nabla_i X^a \nabla^i X^a \delta_{IJ}, \tag{6.26}$$

$$(z - 2\tilde{\Delta}_-)T_{0I}^{(3-z+2\tilde{\Delta}_-)} = 0. \tag{6.27}$$

In addition, by taking the functional derivative with respect to the scalar, the boundary scalar momentum can be calculated, via

$$\tilde{\pi}^a = -\frac{1}{N\sqrt{\gamma}}\frac{\delta S}{\delta X^a}.$$

For example, one gets

$$(z - 2\tilde{\Delta}_-)\tilde{\pi}^{a(2+\tilde{\Delta}_-)} = -\frac{1}{N}\nabla^i(N\nabla_i X^a) = -\nabla^i\nabla_i X^a - \frac{\nabla^i N \nabla_i X^a}{N}. \tag{6.28}$$

The higher order counterterms are more involved because they receive contributions from the quadratic piece. For example,

$$(z - \Delta_- - 2\tilde{\Delta}_-)\mathcal{L}^{(2+\Delta_-+2\tilde{\Delta}_-)} = K_{AB}^{(\Delta_-)} T^{AB(2+2\tilde{\Delta}_-)}$$

$$
\begin{aligned}
&= -\frac{\alpha\psi}{2(z+1)}\left[z(3z-\Delta_-)T^{00(2+2\tilde{\Delta}_-)} + z(2z-1-\Delta_-)T_I^{I(2+2\tilde{\Delta}_-)}\right] \\
&= -\frac{\alpha\psi}{2(z+1)}\left[z(3z-\Delta_-)T^{00(2+2\tilde{\Delta}_-)}\right], \qquad (6.29)
\end{aligned}
$$

using the fact that $T_I^{I(2+2\tilde{\Delta}_-)} = 0$, as calculated above. Note that for $z = 2$ this becomes $\mathcal{L}^{(2+\Delta_-+2\tilde{\Delta}_-)} = -\psi T^{00(2+2\tilde{\Delta}_-)}$. The calculation of this term is useful even when the source for the massive vector ψ is set to zero. This is because we can determine $\pi_\psi^{(2+2\tilde{\Delta}_-)}$ by taking the functional derivative with respect to ψ:

$$
(z-\Delta_- - 2\tilde{\Delta}_-)\pi_\psi^{(2+2\tilde{\Delta}_-)} = \frac{\alpha}{2(z+1)}\left[z(3z-\Delta_-)T^{00(2+2\tilde{\Delta}_-)}\right]. \qquad (6.30)
$$

The following terms also receive contributions from the quadratic piece:

$$
\begin{aligned}
(z-2-2\tilde{\Delta}_-)\mathcal{L}^{(4+2\tilde{\Delta}_-)} &= 2K_{AB}^{(2)}T^{AB(2+2\tilde{\Delta}_-)} + \pi_A^{(2)}\pi^{A(2+2\tilde{\Delta}_-)} + 8\pi G_4\left(\tilde{\pi}^{a(2+\tilde{\Delta}_-)}\right)^2, \qquad (6.31) \\
(z-2-4\tilde{\Delta}_-)\mathcal{L}^{(4+4\tilde{\Delta}_-)} &= K_{AB}^{(2+2\tilde{\Delta}_-)}T^{AB(2+2\tilde{\Delta}_-)} + \frac{1}{2}\pi_A^{(2+2\tilde{\Delta}_-)}\pi^{A(2+2\tilde{\Delta}_-)}. \qquad (6.32)
\end{aligned}
$$

These are the terms that will contribute to the scaling anomaly when $z = 2$. After a lengthy calculation of the right hand sides for $z = 2$, the following result is obtained (up to total derivatives):

$$
\begin{aligned}
(z-2-2\tilde{\Delta}_-)\mathcal{L}^{(4+2\tilde{\Delta}_-)} &= 2\pi G_4(\nabla_i\nabla^i X^a)^2, \qquad (6.33) \\
(z-2-4\tilde{\Delta}_-)\mathcal{L}^{(4+4\tilde{\Delta}_-)} &= \frac{1}{4}T_{IJ}^{(2+2\tilde{\Delta}_-)}T^{IJ(2+2\tilde{\Delta}_-)}, \\
&= 16\pi^2 G_4^2\left(\nabla_i X^a\nabla_j X^a\nabla^i X^b\nabla^j X^b - \frac{1}{2}(\nabla_i X^a\nabla^i X^a)^2\right) \qquad (6.34)
\end{aligned}
$$

By combining all these results, the contribution of the massless scalars to the logarithmically divergent counterterm when $z = 2$ is (by (5.76)):

$$
\begin{aligned}
\tilde{\mathcal{L}}_X^{(4)} &= \lim_{z\to 2}\left((2-z)\mathcal{L}^{(2z+2\tilde{\Delta}_-)} + (z-2)\mathcal{L}^{(4+2\tilde{\Delta}_-)} + (z-2)\mathcal{L}^{(4+4\tilde{\Delta}_-)}\right) \\
&= \frac{8\pi G_4}{N^2}(\partial_t X^a - N^i\nabla_i X^a)^2 + 2\pi G_4(\nabla_i\nabla^i X^a)^2 \\
&\quad + 16\pi^2 G_4^2\left(\nabla_i X^a\nabla_j X^a\nabla^i X^b\nabla^j X^b - \frac{1}{2}(\nabla_i X^a\nabla^i X^a)^2\right). \qquad (6.35)
\end{aligned}
$$

Together with the gravitational counterterms from the previous section, the total counterterm action for $z = 2$ is given by

$$
S_{ct} = -\int_{\partial\mathcal{M}} dt\, d^2\mathbf{x}\sqrt{\gamma}N\left\{\frac{1}{16\pi G_4}\left[6 + \frac{1}{2}\hat{R} + \frac{1}{4}\frac{\nabla_i N\nabla^i N}{N^2}\right] - \frac{1}{4}\nabla_i X^a\nabla^i X^a\right.
$$

$$-\log\epsilon\left[\frac{1}{16\pi G_4}(\hat{K}_{ij}\hat{K}^{ij}-\frac{1}{2}\hat{K}^2)+\frac{1}{2N^2}(\partial_t X^a - N^i\nabla_i X^a)^2+\frac{1}{8}(\nabla^i\nabla_i X^a)^2 \right.$$
$$\left.\left. + \pi G_4\left(\nabla_i X^a\nabla_j X^a\nabla^i X^b\nabla^j X^b - \frac{1}{2}(\nabla_i X^a\nabla^i X^a)^2\right)\right]\right\}. \qquad (6.36)$$

Interestingly, this logarithmically divergent counterterm takes the form identical to the action written down in Section 2.2, describing the coupling of $z = 2$ gravity and $z = 2$ Lifshitz matter in $2 + 1$ dimensions. This action is invariant under $z = 2$ anisotropic Weyl transformations, with the scalars transforming with weight zero, and satisfies the detailed balance condition. We see that the property of detailed balance, satisfied by the logarithmic counterterms in the absence of extra matter, persists in the presence of the marginal scalar fields.

Two additional comments are worth making:

(1) The relative sign between the potential terms and the kinetic term in the logarithmic counterterm is opposite to the sign one would expect from the action of a unitary theory with $z = 2$ scaling in real time. This is not very surprising, and corresponds to the fact already appreciated in the relativistic case: The holographic counterterms do not have to reproduce the action of a unitary theory, as is clear from the appearance of the higher-derivative conformal gravity action in the holographic counterterms in AdS_5.

(2) In the classical theories with Lifshitz scaling, the coupling constants in front of the individual contributions to the potential term are not related by any symmetry to the kinetic terms. Therefore, they represent classically marginal couplings. In the structure of our counterterms, we find this freedom realized only partially: A uniform overall rescaling of all the couplings in the potential can be accomplished by a shift in r, but it appears that the interaction with the bulk relativistic system eliminates the apparent freedom of the relative rescaling between different contributions to the potential from species unrelated by any symmetry in the boundary theory. This mechanism deserves further study.

6.4 Explaining Detailed Balance

Now that we have accumulated some evidence suggesting that the appearance of the detailed balance condition in the structure of the counterterms is rather generic, it would be desirable to obtain a more systematic explanation of this fact. It would be interesting to see why this principle should be naturally satisfied in the context of holographic renormalization.

A closer look at the structure of the holographic renormalization equations reveals a possible answer: In the procedure we followed in $3 + 1$ bulk dimensions, the potential terms in the counterterm at order four are generated by quadratic terms in the stress-energy tensor and field momenta at order two. These momenta arise from the functional differentiation of the counterterm at order two. Consider the counterterm appearing above at order two:

$$S^{(2)}_{ct} = -\int_{\partial\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{\gamma}N\left\{\frac{1}{32\pi G_4}\left[\hat{R}+\frac{1}{2}\frac{\nabla^i N\nabla_i N}{N^2}\right]-\frac{1}{4}\nabla_i X^a\nabla^i X^a\right\}. \qquad (6.37)$$

This Lagrangian is exactly the one used in the detailed balance condition in [17], in the case where N does not depend upon spatial coordinates.[2] Hence, the detailed balance relation, as reviewed in Section 2.1, is simply a consequence of the relationship between two counterterms implied by the holographic renormalization in asymptotically Lifshitz spacetime.

It should be noted that in the above procedure, the presence of the massive vector complicates the equations and make the detailed-balance-like relation between the two actions less transparent. This means that the detailed balance condition is unlikely to hold in more complicated theories with Lifshitz spacetime solutions. But the logarithmic counterterm potential terms (with scaling dimension four) are nonetheless directly derivable from the counterterms with scaling dimension two.

In fact, an analogous result also holds in the relativistic case of holographic renormalization in AdS_5, where the second order counterterm is simply the Einstein-Hilbert action and the conformal anomaly is the action S_{conf} of conformal gravity in $3+1$ dimensions: It turns out that S_{conf} is obtained by squaring the functional derivative of the Einstein-Hilbert action. The reason behind this relationship is the same: S_{conf} and the Einstein-Hilbert action appear as two counterterms, linked via the holographic renormalization procedure into a condition reminiscent of detailed balance.

A closer look also reveals that the holographic justification for the detailed balance condition being satisfied by the logarithmic conterterm quickly ceases to be valid with increasing spacetime dimension. However, this property does not disappear completely: Instead, the holographic renormalization machinery implies a more complex relation between the logarithmic counterterm and the variational derivatives of the entire hierarchy of the power-law counterterms.

6.5 Analytic Continuation to the de Sitter-like Regime

In relativistic AdS/CFT correspondence, the Hamilton-Jacobi formulation of holographic renormalization – with the radial direction r as the evolution parameter – can be easily continued analytically to de Sitter space. Upon this continuation, the evolution parameter r becomes the real time η, and the analytic continuation of the counterterms gives useful information about the wavefunction Ψ of the Universe on superhorizon scales [59, 60, 61]. In particular, in the case of AdS_5 continued analytically to dS_5, the exponential of the logarithmic counterterm (known to take the form of the relativistic conformal gravity action S_{conf} in $3+1$ dimensions) is related to the wavefunction via

$$|\Psi|^2 = e^{-S_{conf}}. \tag{6.38}$$

In this chapter, we have analyzed holographic counterterms in the Lifshitz space background, and in the case of $z = 2$ and $3+1$ bulk dimensions, we also found a logarithmic counterterm

[2] Detailed balance in the nonprojectable theory has been discussed recently in [58].

in the form of a $z = 2$ multicritical conformal gravity action. It is natural to ask whether an analytic continuation exists, similar to the one studied in [59, 60, 61], so that the $z = 2$ anisotropic conformal gravity action similarly produces the square of the wavefunction of the dual system. The answer appears to be yes, and the dual system is a gravity theory with an interesting kind of spatial anisotropy.

Reintroducing the length scale L_r in the spacetime metric of the Lifshitz space at $z = 2$,

$$ds^2 = L_r^2 \left(-r^4 dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2} \right), \tag{6.39}$$

we can analytically continue our results by taking $r = i\eta$ and $L_r = -iL_\eta$ and relabeling $t = y$, which leads to the following spacetime:

$$ds^2 = L_\eta^2 \left(\eta^4 dy^2 + \eta^2 d\mathbf{x}^2 - \frac{d\eta^2}{\eta^2} \right). \tag{6.40}$$

This spacetime can be viewed as a spatially anisotropic, "multicritical" version of de Sitter space. We found the on-shell action for asymptotically Lifshitz space to be (with the cutoff at $r = 1/\epsilon_r$):

$$\begin{aligned} S &= \frac{L_r^2}{16\pi G_4} \int_{\partial\mathcal{M}_{1/\epsilon_r}} dt\, d^2\mathbf{x}\, \sqrt{\gamma}\, N(\mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} - \tilde{\mathcal{L}}^{(4)} \log \epsilon_r) \\ &= \frac{L_r^2}{16\pi G_4} \int_{\partial\mathcal{M}_{1/\epsilon_r}} dt\, d^2\mathbf{x}\, \sqrt{\gamma_{fin}} N_{fin} \left\{ \frac{\mathcal{L}^{(0)}_{fin}}{\epsilon_r^4} + \frac{\mathcal{L}^{(2)}_{fin}}{\epsilon_r^2} + \mathcal{L}^{(4)}_{fin} - \tilde{\mathcal{L}}^{(4)}_{fin} \log \epsilon_r \right\}, \end{aligned} \tag{6.41}$$

where the quantities with fins are defined to be finite as $r \to \infty$ (that is, $\mathcal{O}^{(\Delta)} = \mathcal{O}^{(\Delta)}_{fin} \epsilon_r^\Delta$). The analytic continuation implies that the cutoff changes to $\epsilon_r = -i\epsilon_\eta$, where $\epsilon_\eta < 0$. Note that all terms in the on-shell action remain real after the analytic continuation, except for the logarithm, which now has an imaginary part since $\log \epsilon_r = \log(-\epsilon_\eta) + i\pi/2$. Thus, after this analytic continuation, the square of the ground-state wavefunction for the spatially anisotropic version of de Sitter space is given solely by the coefficient of the logarithmic counterterm,

$$|\Psi|^2 = |e^{iS}|^2 = \exp\left\{ -\frac{L_\eta^2}{16G_4} \int_{\partial\mathcal{M}} d^2\mathbf{x}\, dy\, \sqrt{\gamma}\, N\, \tilde{\mathcal{L}}^{(4)} \right\}. \tag{6.42}$$

In the case of the theory studied in Section 6.2, we found that $\tilde{\mathcal{L}}^{(4)}$ is the action of $z = 2$ conformal Lifshitz gravity in detailed balance. It depends only on the y derivatives but not the $\mathbf{x}$ derivatives of the metric. Thus, the ground-state wavefunction (6.42) represents a theory with spatial anisotropy, ultralocal along all but one spatial dimension, similar to the theory discussed in [62, 63].

In the theory with bulk scalars studied in Section 6.3, $\tilde{\mathcal{L}}^{(4)}$ was found to be the action of $z = 2$ conformal Lifshitz gravity coupled to $z = 2$ scalars, still satisfying the detailed balance condition. This action has a nontrivial potential term, of fourth order in the $\mathbf{x}$ derivatives of the scalars. Notably, the sign of this potential term, which we commented on at the end of Section 6.3, is such that the analytically continued $\tilde{\mathcal{L}}^{(4)}$ appearing in (6.42) is positive definite.

6.6 Discussion

We have calculated that the $D = z = 2$ anisotropic Weyl anomaly for the action in (5.1). Furthermore, we have seen that the anomaly naturally takes the form of conformal Lifshitz gravity. This conformal gravity theory unexpectedly obeys the condition of detailed balance (because $c_V = 0$). Is it possible to find a holographic duals of more general QFTs with both central charges (c_V and c_K) independently nonzero? Before we embark on this pursuit, we should first check that QFTs whose central charges c_K and c_V are both nonzero indeed exist. Examples of strongly coupled Lifshitz field theories are very scarce to say the least, but our point can be made by considering the theory of the free $z = 2$ Lifshitz scalar Φ. When Φ is minimally coupled to background HL gravity,

$$S_\Phi = \int dt\, d^2x \sqrt{\gamma} \left\{ \frac{1}{N} \left(\partial_t \Phi - N^i \nabla_i \Phi \right)^2 - N \left(\nabla^i \nabla_i \Phi \right)^2 \right\},$$

this theory is classically invariant under (2.11) (with $\delta\Phi = 0$), but develops an anisotropic Weyl anomaly at the quantum level. This anomaly was calculated in [64], and it turns out to have $c_V = 0$. One could perhaps speculate that $c_V = 0$ might be a universal property of all consistent QFTs, hence eliminating the need for finding gravity duals with $c_V \neq 0$. A simple counterexample comes from coupling Φ to background gravity non-minimally, adding

$$-e^2 \int dt\, d^2x \sqrt{\gamma} N \left\{ \hat{R} - \frac{\nabla^i N \nabla_i N}{N^2} + \frac{\nabla^i \nabla_i N}{N} \right\}^2 \Phi^2$$

to S_Φ. Even with this non-minimal coupling, this theory stays classically invariant under the anisotropic Weyl transformations (again with $\delta\Phi = 0$), and develops a quantum anomaly. We calculated this anisotropic Weyl anomaly using the ζ-function regularization, and found $c_K = 1/(32\pi)$ and $c_V = -e^2/(8\pi)$.

Having demonstrated the existence of QFTs with $c_V \neq 0$, we can now ask how to reproduce this second central charge in a holographic gravity dual. One could look for relativistic bulk models more complicated than (7.1). Instead, in Chapter 7, we will consider a holographic setup where the bulk gravity theory is nonrelativistic. Such constructions could extend the list of nonrelativistic field theories amenable to a holographic description to a broader class, in which those nonrelativistic theories that have a relativistic bulk dual may well be only a minority.

Other various interesting open questions remain. First of all, our calculation of the Weyl anomaly[3] can be generalized to larger values of D and z. In particular, at $z = 3$ in $4 + 1$ bulk dimensions, we expect the appearance of logarithmic counterterms taking the form of the action for $z = 3$ multicritical conformal gravity in $3 + 1$ dimensions, introduced in [18]. Moreover, now that we have seen that the classical action of multicritical gravity appears from string-inspired holography, it would also be interesting to see whether the full *dynamics* of multicritical gravity can also be engineered from string theory, perhaps by taking judicious scaling limits of backgrounds without Lorentz invariance.

[3] A Weyl anomaly involving 2 time derivatives will appear whenever $z = D$. When $(z + D)$ is an even integer, there are possible anomaly terms involving $(z + D)$ spatial derivatives.

Chapter 7

Lifshitz Holography 2: HL Gravity

In Chapter 5, we used a relativistic bulk gravity coupled to matter as a holographic dual to a Lifshitz field theory. Indeed, the overwhelming share of work on Lifshitz holography (starting with [65]) follows this route. In the relativistic case, the coupling to matter is necessary, as the Lifshitz spacetime with $z \neq 1$ does not solve the Einstein equations in the vacuum. But another natural option is available: Instead of adding *ad hoc* matter to Einstein gravity so that (1.12) becomes a solution, one can modify gravity itself. In this chapter, we will follow this alternate path, and show that the Lifshitz spacetime is a vacuum solution of minimal HL gravity, with no additional matter. The preferred foliation of the Lifshitz spacetime, required for its embedding into HL gravity, is simply the foliation by leaves of constant t. Note that the value of the dynamical critical exponent of the HL gravity theory in the bulk, which we denote by z_B, is not required to be the same as the dynamical critical exponent, z for the dual Lifshitz field theory.

HL gravity may enjoy better short-distance properties than Einstein gravity (if it is dominated at high energies by its own $z_B > 1$ scaling), but here we will follow the "bottom-up" strategy common in relativistic holography, and work only in the low-energy bulk gravity approximation. This is equivalent to the large N limit in the dual field theory. In this low-energy limit, HL gravity is dominated by the most relevant terms compatible with the gauge symmetries.

7.1 The Bulk Action and Notation

We consider only the low-energy limit, of the simplest theory of HL gravity without matter (see Section 2.1 for details). Even though at low energies we are effectively driven to $z_B = 1$, this does not mean that the low-energy theory would just reproduce relativistic gravity: Even at low energies, there are generally imprints of the underlying microscopic anisotropy. The low-energy action is

$$S \quad = \quad S_K + S_{\mathcal{V}} + S_{GH}, \tag{7.1}$$

where:

$$
\begin{aligned}
S_K &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} dt\, d^D x\, dr\, \sqrt{g} N\,(K_{ab}K^{ab} - \lambda K^2),\\
S_{\mathcal{V}} &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} dt\, d^D x\, dr\, \sqrt{g} N\,(\beta(R-2\Lambda) + \frac{\alpha^2}{2}\frac{\tilde{\nabla}_a N \tilde{\nabla}^a N}{N^2}),\\
S_{GH} &= \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^D x\, \sqrt{\gamma} N\, 2\beta\mathcal{K}.
\end{aligned}
$$

The novelty compared to General Relativity is in the three couplings β, λ and α, which in GR are fixed to $\lambda = \beta = 1$ and $\alpha = 0$. This action results in the following equations of motion, when varying N, N_a and g_{ab} respectively:

$$
\begin{aligned}
-K_{ab}K^{ab} + \lambda K^2 + R + \Lambda + \frac{\alpha^2}{2}(\frac{\tilde{\nabla}_a N \tilde{\nabla}^a N}{N^2} - 2\frac{\tilde{\nabla}^a\tilde{\nabla}_a N}{N}) &= 0\\
\tilde{\nabla}_b K^{ba} - \lambda \tilde{\nabla}^a K &= 0\\
\frac{\tilde{\nabla}^c\tilde{\nabla}_c N}{N} g_{ab} + R_{ab} - \frac{\tilde{\nabla}_a\tilde{\nabla}_b N}{N} - \frac{1}{2}(R+\Lambda)g_{ab} - \frac{\alpha^2}{4}\frac{\tilde{\nabla}_c N\tilde{\nabla}^c N}{N^2} g_{ab} + \frac{\alpha^2}{2}\frac{\tilde{\nabla}_a N \tilde{\nabla}_b N}{N^2} &\\
+2(K_{ac}K^c_b - \lambda K K_{ab}) - \frac{1}{2}(K_{cd}K^{cd} - \lambda K^2)g_{ab} + \frac{g_{ac}g_{bd}}{\sqrt{g}N}\frac{\partial}{\partial t}[\sqrt{g}(K^{cd} - \lambda K g^{cd})] &\\
+\frac{1}{N}\tilde{\nabla}^c[(K_{ac} - \lambda K g_{ac})N_b + (K_{bc} - \lambda K g_{bc})N_a - (K_{ab} - \lambda K g_{ab})N_c] &= 0 \quad (7.2)
\end{aligned}
$$

Note that the third equation of motion can be simplified by substituting the first equation of motion into it:

$$
\begin{aligned}
(1 - \frac{\alpha^2}{2})\frac{\tilde{\nabla}^c\tilde{\nabla}_c N}{N} g_{ab} + R_{ab} - \frac{\tilde{\nabla}_a\tilde{\nabla}_b N}{N} + \frac{\alpha^2}{2}\frac{\tilde{\nabla}_a N\tilde{\nabla}_b N}{N^2} &\\
+2(K_{ac}K^c_b - \lambda K K_{ab}) - N(K_{cd}K^{cd} - \lambda K^2)g_{ab} + \frac{g_{ac}g_{bd}}{\sqrt{g}N}\frac{\partial}{\partial t}[\sqrt{g}(K^{cd} - \lambda K g^{cd})] &\\
+\frac{1}{N}\tilde{\nabla}^c[(K_{ac} - \lambda K g_{ac})N_b + (K_{bc} - \lambda K g_{bc})N_a - (K_{ab} - \lambda K g_{ab})N_c] &= 0 \quad (7.3)
\end{aligned}
$$

We find that Lifshitz space ($N = r^z$, $N_a = 0$, $g_{ij} = r^2\delta_{ij}$, $g_{rr} = \frac{1}{r^2}$, $g_{ri} = 0$) is a solution to the low-energy action (7.1), without any additional matter fields being necessary. Demanding that Lifshitz space is a solution of the equations of motion (7.2) in this low-energy theory determines one of the extra couplings,

$$
\alpha^2 = \frac{2\beta(z-1)}{z}, \tag{7.4}
$$

relates the value of the cosmological constant to our choice of scale,

$$
\Lambda = -\frac{(D+z-1)(D+z)}{2}, \tag{7.5}
$$

and leaves the value of λ undetermined.

In [30, 66], a bound on the possible values of α was derived from the requirement of perturbative unitarity. It is pleasing to see that this bound is respected by (7.4) for all values of z.

Let us again clarify our notation for this chapter. We take D to be the number of spatial dimensions on the boundary and so there are $D+2$ spacetime dimensions in the bulk and $d \equiv D+1$ spacetime dimensions on the boundary. For coordinate indices, i, j are used for the D spatial boundary indices (x^i), whereas a, b are used for the $D+1$ spatial bulk indices (r, x^i). The bulk fields are the metric g_{ab}, the shift vector N_a and the lapse function N. We need to separate out the radial parts of g_{ab}, that is, we introduce ADM variables in the radial direction:

$$g_{rr} = \mathcal{N}^2 + \mathcal{N}_i\mathcal{N}^i \qquad g_{ri} = \mathcal{N}_i \qquad g_{ij} = \gamma_{ij}. \tag{7.6}$$

Note that we can gauge fix the bulk foliation-preserving diffeomorphisms by setting the the bulk shift vector $\mathcal{N}_i = 0$, and the bulk lapse function $\mathcal{N} = 1/r$. This *radial gauge* is adopted throughout this chapter. In order to distinguish the lapse and shift variables in the bulk from those of the ADM decomposition on the boundary, we refer to the bulk variables CN and $\mathcal{N}_i$ as the “radial lapse” and “radial shift”.

There are now many curvature quantities which causes confusion if one is not careful. Let us now define all of them here. The Ricci tensor of the $(D+1)$-dimensional metric g_{ab} is R_{ab} and the Ricci tensor of the D-dimensional metric γ_{ij} is $\hat{R}_{ij}$. $\tilde{\nabla}_a$ represents the covariant derivative for the metric g_{ab} and ∇_i represents the covariant derivative for the metric γ_{ij}. The various extrinsic curvatures are defined as follows:

$$\begin{aligned}
K_{ab} &\equiv \frac{1}{2N}(\partial_t g_{ab} - \tilde{\nabla}_a N_b - \tilde{\nabla}_b N_a), \\
\hat{K}_{ij} &\equiv \frac{1}{2N}(\partial_t \gamma_{ij} - \nabla_i N_j - \nabla_j N_i), \\
\mathcal{K}_{ij} &\equiv \frac{1}{2\mathcal{N}}(\partial_r \gamma_{ij} - \nabla_i \mathcal{N}_j - \nabla_j \mathcal{N}_i).
\end{aligned} \tag{7.7}$$

Note that in the radial gauge, $\mathcal{K}_{ij} = \frac{r}{2}\partial_r\gamma_{ij}$. This is different notation from that used in Chapter 5.

7.2 Radial Decomposition of the Action

We need to carry out a radial decomposition of the action using the radial ADM variables introduced in (7.6). With this radial decomposition, the bulk action (7.1) becomes:

$$\begin{aligned}
S \;=\; & \frac{1}{2\kappa^2}\int dt\, d^Dx\, dr\, \sqrt{\gamma}\,\mathcal{N}\,N\Big\{\hat{K}_{ij}\hat{K}^{ij} - \lambda\hat{K}^2 - 2(\hat{K}^{ij}\mathcal{K}_{ij} - \lambda\hat{K}\mathcal{K})\phi + (\mathcal{K}_{ij}\mathcal{K}^{ij} - \lambda\mathcal{K}^2)\phi^2 \\
& + \frac{1-\lambda}{N^2\mathcal{N}^2}(\partial_t\mathcal{N} - \nabla^i\mathcal{N}N_i - \partial_r(N\phi))^2 - \frac{2\lambda}{N\mathcal{N}}(\hat{K} - \mathcal{K}\phi)(\partial_t\mathcal{N} - \nabla^i\mathcal{N}N_i - \partial_r(N\phi)) \\
& + \frac{\gamma^{ij}}{2N^2\mathcal{N}^2}(\partial_r N_i - 2\mathcal{N}\mathcal{K}_{ik}N^k + \mathcal{N}^2\nabla_i(\frac{\phi N}{\mathcal{N}}))(\partial_r N_j - 2\mathcal{N}\mathcal{K}_{jl}N^l + \mathcal{N}^2\nabla_j(\frac{\phi N}{\mathcal{N}}))
\end{aligned}$$

$$+\beta(\hat{R}+\mathcal{K}^2-\mathcal{K}_{ij}\mathcal{K}^{ij}+\frac{2\partial_r N\mathcal{K}}{N\mathcal{N}}-2\frac{\nabla^i\nabla_i N}{N}-2\Lambda)+\frac{\alpha^2}{2}\frac{\nabla_i N\nabla^i N}{N^2}+\frac{\alpha^2}{2}(\frac{\partial_r N}{N\mathcal{N}})^2$$
$$+\text{ terms proportional to }\mathcal{N}_i\Big\}, \tag{7.8}$$

where we have defined $\phi := \frac{N_r}{N\mathcal{N}}$. We will not need the terms proportional to $\mathcal{N}_i$, since we will be choosing the radial gauge $\mathcal{N}_i = 0$, $\mathcal{N} = \frac{1}{r}$ in all equations that follow.

The Hamiltonian constraint is obtained by varying the action with respect to $\mathcal{N}$:

$$\begin{aligned}
&\frac{1}{2}\mathcal{P}^i\mathcal{P}_i+\mathcal{P}_{ij}\mathcal{K}^{ij}+2\beta\mathcal{P}\mathcal{K}+\frac{\alpha^2}{2}\mathcal{P}^2+(\mathcal{K}_{ij}\mathcal{K}^{ij}-\lambda\mathcal{K}^2)\phi^2+(1-\lambda)\Pi^2-2\lambda\mathcal{K}\phi\Pi\\
&\quad=\hat{K}_{ij}\hat{K}^{ij}-\lambda\hat{K}^2+\beta(\hat{R}-2\frac{\nabla_i\nabla^i N}{N}-2\Lambda)+\frac{\alpha^2}{2}\frac{\nabla_i N\nabla^i N}{N^2}\\
&\qquad+\frac{2(1-\lambda)}{N}(\frac{\partial_t(\sqrt{\gamma}\Pi)}{\sqrt{\gamma}}-\nabla^i(N_i\Pi))+\frac{2\lambda\partial_t(\sqrt{\gamma}(\hat{K}-\mathcal{K}\phi))}{\sqrt{\gamma}N}\\
&\qquad\quad-\frac{2\lambda\nabla^i((\hat{K}-\mathcal{K}\phi)N_i)}{N}+\frac{2}{N}\nabla^i(\phi N)\mathcal{P}_i+\phi\nabla^i\mathcal{P}_i,
\end{aligned}\tag{7.9}$$

where we have defined for convenience:

$$\begin{aligned}
\mathcal{P}_{ij} &:= \beta(\mathcal{K}\gamma_{ij}-\mathcal{K}_{ij}), &\quad \mathcal{P}_i &:= \frac{(\partial_r N_i-2\mathcal{N}\mathcal{K}_{ik}N^k+\mathcal{N}^2\nabla_i(\frac{\phi N}{\mathcal{N}}))}{N\mathcal{N}},\\
\mathcal{P} &:= \frac{\partial_r N}{N\mathcal{N}}, & \Pi &:= \frac{1}{N\mathcal{N}}(-\partial_t\mathcal{N}+\nabla^i\mathcal{N}N_i+\partial_r(N\phi)).
\end{aligned}\tag{7.10}$$

Note that the Hamiltonian constraint (7.9) can be simplified to become:

$$\begin{aligned}
&\frac{1}{2}\mathcal{P}^i\mathcal{P}_i+\mathcal{P}_{ij}\mathcal{K}^{ij}+2\beta\mathcal{P}\mathcal{K}+\frac{\alpha^2}{2}\mathcal{P}^2+(\mathcal{K}_{ij}\mathcal{K}^{ij}-\lambda\mathcal{K}^2)\phi^2+(1-\lambda)\Pi^2-2\lambda\mathcal{K}\phi\Pi\\
&\quad-\frac{1}{N}\nabla^i(\phi N)\mathcal{P}_i=\hat{K}_{ij}\hat{K}^{ij}-\lambda\hat{K}^2+\beta(\hat{R}-2\Lambda)+\frac{\alpha^2}{2}\frac{\nabla_i N\nabla^i N}{N^2}+\text{tot. der.},
\end{aligned}\tag{7.11}$$

where the total derivatives are of the form $\frac{\partial_t A}{\sqrt{\gamma}N}$ or $\frac{\nabla^i A_i}{\sqrt{\gamma}N}$.

Using the action in (7.8), we can also define[1] the (radial) momenta conjugate to γ_{ij}, N_i, N and ϕ as:

$$\begin{aligned}
P^{ij} &:= \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta(\partial_r\gamma_{ij})} \qquad\qquad (7.12)\\
&= \mathcal{P}^{ij}+\beta\mathcal{P}\gamma^{ij}-(\hat{K}^{ij}-\lambda\hat{K}\gamma^{ij})\phi+(\mathcal{K}^{ij}-\lambda\mathcal{K}\gamma^{ij})\phi^2-\lambda\gamma^{ij}\phi\Pi-\frac{N^j\mathcal{P}^i}{2N}-\frac{N^i\mathcal{P}^j}{2N},
\end{aligned}$$

[1] These differ slightly from the usual definition of canonical momenta in order to simplify some of the subsequent equations.

$$\begin{aligned} P^i &:= \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta(\partial_r N_i)} = \frac{\mathcal{P}^i}{N}, && (7.13)\\ P &:= \frac{2\kappa^2}{\sqrt{\gamma}}\frac{\delta S}{\delta(\partial_r N)} = \alpha^2\mathcal{P} + 2\beta\mathcal{K} + 2(1-\lambda)\phi\Pi + 2\lambda\phi(\hat{K} - \mathcal{K}\phi), && (7.14)\\ \pi &:= \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta(\partial_r \phi)} = 2(1-\lambda)\Pi + 2\lambda(\hat{K} - \mathcal{K}\phi). && (7.15) \end{aligned}$$

7.3 Functional Derivatives and the Stress Tensor

As before, we will be applying the Hamilton-Jacobi formalism in the radial direction to perform the holographic renormalization. We form the on-shell action by taking the bulk action (7.1) and evaluating it as a function of the boundary fields:

$$S = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} dtd^D\mathbf{x}\,\sqrt{\gamma}N\mathcal{L}. \qquad (7.16)$$

As in the standard Hamilton-Jacobi theory, the momenta can be obtained by functional differentiation of the on-shell action:

$$\begin{aligned} P^{ij} &= \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta\gamma_{ij}},\\ P^i &= \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta N_i},\\ P &= \frac{2\kappa^2}{\sqrt{\gamma}}\frac{\delta S}{\delta N},\\ \pi &= \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta \phi}. \end{aligned} \qquad (7.17)$$

Equivalently, the variation of the on-shell action is:

$$\delta S = \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} dt\,d^Dx\,\sqrt{\gamma}N(\frac{P}{N}\delta N + P^i\delta N_i + P^{ij}\delta\gamma_{ij} + \pi\delta\phi). \qquad (7.18)$$

We can extract some formulae from these equations that will be useful to us later on. Contracting (7.12) with γ_{ij} and using the definitions in (7.10) gives:

$$\gamma_{ij}P^{ij} = \mathcal{K}(\beta(D-1) + (1-\lambda D)\phi^2) + \beta\mathcal{P}D - (1-\lambda D)\phi\hat{K} - \lambda D\phi\Pi - N_iP^i. \qquad (7.19)$$

Then inverting (7.14) and (7.19) for $\mathcal{P}$ and $\mathcal{K}$, we have the useful results:

$$\begin{aligned} (2\beta D - (D-1)\alpha^2)\mathcal{P} &= 2\gamma_{ij}P^{ij} + 2\mathcal{K}(\lambda-1)\phi^2 - 2(\lambda-1)\phi\hat{K}\\ &\quad +2(D+\lambda-1)\phi\Pi + 2N_iP^i - (D-1)P,\\ \beta(\alpha^2(D-1) - 2\beta D)\mathcal{K} &= \alpha^2\gamma_{ij}P^{ij} - \alpha^2(\mathcal{K}(1-\lambda D)\phi^2 - (1-\lambda D)\phi\hat{K} - \lambda D\phi\Pi - N_iP^i) \end{aligned}$$

$$-\beta DP + \beta D(2\mathcal{K}(-\lambda\phi^2) + 2(1-\lambda)\phi\Pi + 2\lambda\phi\hat{K}). \qquad (7.20)$$

We can also define the boundary stress-energy tensor $T^A{}_B = 2\kappa^2 \frac{e^A_\alpha}{\sqrt{\gamma}N}\frac{\delta S}{\delta e^B_\alpha}$ and then:

$$\begin{aligned}
T^0{}_0 &= -P = -[\alpha^2\mathcal{P} + 2\beta\mathcal{K} + 2(1-\lambda)\phi\Pi + 2\lambda\phi(\hat{K} - \mathcal{K}\phi)] && (7.21)\\
T^I{}_J &= -[e^I_t P^i e_{Ji} + e^I_i(P^i e_{Jt} + 2P^{ij}e_{Jj})] \\
&= -(P^J N_I + P^I N_J + 2P^I{}_J) \\
&= -2(\mathcal{P}^I{}_J + \beta\mathcal{P}\delta^I_J - (\hat{K}^I_J - \lambda\hat{K}\delta^I_J)\phi + (\mathcal{K}^I{}_J - \lambda\mathcal{K}\delta^I_J)\phi^2 - \lambda\delta^I_J\phi\Pi) && (7.22)\\
T^0{}_I &= -NP^i e_{Ii} = -NP_I = -\mathcal{P}_I && (7.23)
\end{aligned}$$

7.4 Analysis of Linearized Constant Modes

We have seen in the previous section that Lifshitz space is a vacuum solution in HL gravity with a negative cosmological constant. As usual, the bulk fields in HL gravity will correspond to operators in the dual quantum field theory. To elucidate these operators and their scaling dimensions it is useful to conduct an analysis of the constant linearized modes around this background.

Before doing this, recall that the bulk theory has a gauge group consisting of all foliation preserving diffeomorphisms, so some modes will be pure gauge. The foliation preserving diffeomorphisms are:

$$\begin{aligned}
\delta g_{ab} &= \partial_a\zeta^c g_{bc} + \partial_b\zeta^c g_{ac} + \zeta^c\partial_c g_{ab} + f\partial_t g_{ab}, \\
\delta N_a &= \partial_a\zeta^b N_b + \zeta^b\partial_b N_a + (\partial_t\zeta^b)g_{ab} + (\partial_t f)N_a + f\partial_t N_a, \\
\delta N &= \zeta^a\partial_a N + (\partial_t f)N + f\partial_t N. && (7.24)
\end{aligned}$$

We are just looking at constant modes (that is, the ζ are independent of $\mathbf{x}$ and t, but can depend on r). In addition, since we are only looking at the linearized modes, we can substitute in our background solution of $N = r^z$, $N_a = 0$, $g_{ij} = r^2\delta_{ij}$, $g_{rr} = \frac{1}{r^2}$, $g_{ri} = 0$ and so the gauge transformations become:

$$\begin{aligned}
\delta g_{ij} &= 2r\zeta^r\delta_{ij}, \\
\delta g_{rr} &= \frac{2}{r}\partial_r(\frac{\zeta^r}{r}), \\
\delta g_{ir} &= r^2\partial_r\zeta^k\delta_{ik}, \\
\delta N_a &= 0, \\
\delta N &= zr^{z-1}\zeta^r. && (7.25)
\end{aligned}$$

To maintain the radial gauge that we are using ($\mathcal{N} = \frac{1}{r}$, $\mathcal{N}_i = 0$) we must set $\delta g_{rr} = \delta g_{ir} = 0$. Even with this choice of radial gauge, if we take $\zeta^r = \frac{\zeta}{2}r$ for some constant ζ then we have a pure gauge scalar mode:

$$\delta g_{ij} = \zeta r^2\delta_{ij},$$

$$\begin{aligned} \delta N &= \frac{\zeta z}{2} r^z, \\ \delta g_{rr} &= \delta g_{ir} = \delta N_i = \delta N_r = 0. \end{aligned} \tag{7.26}$$

We can now analyze the linearized modes. We only consider the modes that are constant in $\mathbf{x}$ and t since this is enough to extract the scaling dimensions of the dual operators.

$$\begin{aligned} N = r^z(1 + \frac{1}{2}f), \qquad g_{ij} = r^2[(1+k)\delta_{ij} + t_{ij}], \\ g_{rr} = \frac{1}{r^2}, \qquad N_r = r^{z-1} j, \qquad N_i = r^2 v_i, \qquad g_{ri} = 0, \end{aligned} \tag{7.27}$$

where $t_{ij}\delta_{ij} = 0$. Furthermore, we are only examining the modes constant in $\mathbf{x}$ and t so k, f, j, v_i and t_{ij} are functions of r only. The powers of r in each coefficient has been chosen here for later convenience. We now substitute this into the equations of motion, keeping only terms linear in k, f, j, v_i and t_{ij}. Since the background solution preserves rotational symmetry in the D transverse spatial dimensions, we can examine the linearized scalar, vector and tensor modes decouple. The scalar modes must satisfy four linearized equations of motion, obtained from (7.2) and (7.3):

$$\begin{aligned} D(D+z+1)rk' + Dr^2k'' + \frac{\alpha^2}{2}((D+z+1)rf' + r^2 f'') &= 0, \\ (D+1-z)rf' + (1-z)r^2 f'' - Dz(2rk' + r^2k'') &= 0, \\ (D+z+1)rf' + r^2 f'' - z((D+1+z)rk' + r^2k'') &= 0, \\ (1-\lambda)r^2 j'' + (1-\lambda)(D+z+1)rj' + D(z-1)j &= 0. \end{aligned} \tag{7.28}$$

This has general solution:

$$\begin{aligned} k &= \zeta + c_3 r^{-(z+D)}, \\ f &= \zeta z + c_2 - c_3 D r^{-(z+D)}, \\ j &= c_4 r^{-\Delta_\phi^-} + c_5 r^{-\Delta_\phi^+}, \end{aligned} \tag{7.29}$$

where $\Delta_\phi^\pm = \frac{z+D}{2}\left\{1 \pm \sqrt{1 + \frac{4(z-1)D}{(\lambda-1)(z+D)^2}}\right\}$ and ζ, c_2, c_3, c_4 and c_5 are constants. As explained above, ζ is a pure gauge mode.

We can repeat the procedure for the vector and tensor modes. When substituted into the linearized equations of motion, the vector mode must satisfy $v_i' = 0$, leading to $v_i = c_{1i}$ for some constant c_{1i}. The tensor mode has equation of motion $(D+z+1)rt_{ij}' + r^2 t_{ij}'' = 0$, which is satisfied by $t_{ij} = c_{1ij} + c_{2ij} r^{-(D+z)}$ for traceless constants c_{1ij} and c_{2ij}.

c_2, c_{1i}, c_{1ij} are the asymptotic values of the fields and so, just as in previous works involving Lifshitz holography [8, 7], they act as sources for the non-relativistic stress tensor complex[2] $\mathcal{E}, \mathcal{P}_i, \Pi^i_j$. The energy density $\mathcal{E}$ and spatial stress tensor Π^i_j have dimension $z + D$, while the momentum density $\mathcal{P}_i$ has dimension $D + 1$. The novel element here is the scalar graviton mode (c_4 and c_5), which corresponds to a new dual operator of dimension $\Delta_\pm$. The $\pm$ sign here depends on whether the standard or alternate boundary condition is chosen for this mode. This additional scalar graviton is examined further in the next section.

[2]Note that there is no source for the energy flux, $\mathcal{E}_i$, a result similar to what was found in [8] when one requires the boundary to have a foliation.

7.5 The Scalar Graviton

We can obtain some interesting limits on the possible values of λ from an argument analogous to the Breitenlohner-Freedman bound.

Although we have embedded Lifshitz geometries as vacuum solutions into Lifshitz gravity without extra matter, the theory propagates an extra scalar polarization of the graviton. It is this scalar graviton whose behavior is sensitive to λ, and which plays effectively the role of "matter" that is being added in this theory to the tensor gravitons of general relativity.

The scaling dimensions associated with the asymptotic behavior of the bulk scalar graviton near anisotropic conformal infinity are

$$\Delta_{\pm} = \frac{z+D}{2}\left\{1 \pm \sqrt{1+\frac{4(z-1)D}{(\lambda-1)(z+D)^2}}\right\}. \tag{7.30}$$

In the dictionary of Lifshitz holography, these become the scaling dimensions of the dual operators in the non-relativistic CFT. If the "standard" boundary condition is chosen, the scaling dimension of the dual operator is Δ_+ whereas if the "alternate" boundary condition is chosen, the scaling dimension of the dual operator is Δ_-.

Requiring that the scaling dimensions $\Delta_\pm$ be real gives an interesting constraint on the values of λ when $z > 1$. There are two ways to satisfy this reality condition: Either $\lambda \geq 1$, or

$$\lambda \leq \frac{(z-D)^2+4D}{(z+D)^2}. \tag{7.31}$$

Recall that in the flat spacetime, perturbative unitarity of the graviton spectrum in the bulk requires either $\lambda \geq 1$ or $\lambda \leq 1/(D+1)$. Hence, (7.31) opens up a new, BF-like window of the allowed values of λ. For example, in the special case of interest when $z = D$, λ can now consistently dip into the region

$$\frac{1}{D+1} \leq \lambda \leq \frac{1}{D}, \tag{7.32}$$

previously forbidden by unitarity around the flat spacetime.

The reality condition is the only constraint if one chooses the "standard" boundary condition. However, if one chooses the "alternate" boundary condition then there is an additional restriction, corresponding to the unitarity bound [67, 24] of $\Delta \geq \frac{D-z}{2}$. In our case this corresponds to:

$$-\frac{(D+z)^2}{4} \leq \frac{(z-1)D}{\lambda-1} \leq -\frac{(D+z)^2}{4} + z^2. \tag{7.33}$$

Only in this region can either boundary condition be chosen (much like in the relativistic AdS/CFT for the values of scalar masses in the window $-m_{BF}^2 \leq m^2 \leq -m_{BF}^2 + 1$). For example, when $z = D$, there are two consistent sets of boundary conditions provided $\lambda \leq \frac{1}{D}$.

7.6 Boundary source fields and asymptotic scaling

The boundary conditions are specified by fixing the sources for the various field theory operators on the boundary. In accordance with our analysis of the falloff of the linearized modes in Section 7.4, our boundary conditions involve the following finite fixed sources as $r \to \infty$ (denoting each source with a bar):

$$N \sim \bar{N} r^z, \qquad \gamma_{ij} \sim \bar{\gamma_{ij}} r^2, \qquad N_i \sim \bar{N_i} r^2, \qquad \phi \sim \bar{\phi} r^{-\Delta_\phi}, \tag{7.34}$$

where $\Delta_\phi = \Delta_-$ if the "standard" boundary condition is chosen for the scalar graviton and $\Delta_\phi = \Delta_+$ if the "alternate" boundary condition is used.

The above scaling behavior allows us to determine the scaling behavior of other quantities near the boundary. Any boundary quantity can be written in terms of the source fields $\bar{\gamma_{ij}}, \bar{N}, \bar{N_i}$ and $\bar{\phi}$ and then the scaling behavior can be read off from the resulting exponents of r. Consider a general object $\mathcal{O}$. When written in terms of the boundary source fields, we say that the term in $\mathcal{O}$ scaling as $r^{-\Delta}$ is of "order Δ" and denote it by $\mathcal{O}^{(\Delta)}$. For example, N has order $-z$, N_i has order -2, γ_{ij} has order -2, γ^{ij} has order 2 and ϕ has order Δ_ϕ. Note that since γ_{ij} has non-trivial scaling behavior, it is useful to use tangent space indices (which we denote by $I, J = 1, ..., D$) so that raising and lowering indices does not change the scaling. For this reason, we use these tangent space indices for any operator $\mathcal{O}^{(\Delta)}$ when we need to specify how it scales. Then substituting (7.34) into (7.10) we can evaluate the leading asymptotic behavior of $\mathcal{P}$, $\mathcal{K}_{IJ}$, $\mathcal{P}^I$ and Π:

$$\mathcal{P}^{(0)} = z, \qquad {\mathcal{K}_{IJ}}^{(0)} = \delta_{IJ}, \qquad \mathcal{P}^{I\,(z-1)} = 0, \qquad \Pi^{(\Delta_\phi)} = (z - \Delta_\phi)\phi. \tag{7.35}$$

7.7 Holographic Renormalization Equations

When the action (7.1) is evaluated on shell as a function of the boundary fields we write it as:

$$S = \frac{1}{2\kappa^2} \int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\, \sqrt{\gamma}\, N\mathcal{L}. \tag{7.36}$$

A convenient way of computing the divergent part of $\mathcal{L}$ is to organize the terms with respect to how they scale with r. More precisely, we define the dilatation operator by:

$$\delta_{\mathcal{D}} = \int_{\partial\mathcal{M}} dt\, d^D x \,(zN\frac{\delta}{\delta N} + 2N_i \frac{\delta}{\delta N_i} + 2\gamma_{ij}\frac{\delta}{\delta \gamma_{ij}} - \Delta_\phi \phi \frac{\delta}{\delta\phi}). \tag{7.37}$$

This operator asymptotically represents $r\dfrac{\partial}{\partial r}$.

$\mathcal{L}$ can then be decomposed into a sum of terms as follows:

$$\mathcal{L} = \sum_{\Delta \geq 0} \mathcal{L}^{(\Delta)} + \tilde{\mathcal{L}}^{(z+D)} \log r. \tag{7.38}$$

Note that we include a logarithmic term at order $z+D$ due to the possibility of a Weyl scaling anomaly. The individual terms of the expansion (7.38) satisfy

$$\delta_{\mathcal{D}}\mathcal{L}^{(\Delta)} = -\Delta\,\mathcal{L}^{(\Delta)} \qquad \text{for} \quad \Delta \neq z+D, \tag{7.39}$$

$$\delta_{\mathcal{D}}\mathcal{L}^{(z+D)} = -(z+D)\mathcal{L}^{(z+D)} + \tilde{\mathcal{L}}^{(z+D)}, \tag{7.40}$$

$$\delta_{\mathcal{D}}\tilde{\mathcal{L}}^{(z+D)} = -(z+D)\tilde{\mathcal{L}}^{(z+D)}. \tag{7.41}$$

Applying the $\delta_{\mathcal{D}}$ to the on-shell action (7.36) and using (7.18) yields:

$$(z+D+\delta_{\mathcal{D}})\mathcal{L} = zP + 2N_iP^i + 2\gamma_{ij}P^{ij} - \Delta_\phi\phi\pi. \tag{7.42}$$

Using (7.14, 7.15, 7.19), this can be written in terms of the variables defined in (7.10):

$$\begin{aligned}(z+D+\delta_{\mathcal{D}})\mathcal{L} \;=\; & (z\alpha^2+2\beta D)\mathcal{P} + 2\beta(z+D-1)\mathcal{K} + 2((z-\Delta_\phi)(1-\lambda)-D\lambda)\phi\Pi \\ & -2(1-(z-\Delta_\phi+D)\lambda)\phi(\hat{K}-\mathcal{K}\phi).\end{aligned} \tag{7.43}$$

We will now use the Hamiltonian constraint to find the holographic renormalization equation. This will be a recursive equation allowing us to solve for the divergent pieces of the action order by order. Expanding the Hamiltonian constraint (7.11) in dilatation eigenvalues gives:

$$\begin{aligned}\sum_{s<\Delta/2}[\mathcal{P}^{I(s)}\mathcal{P}_I^{(\Delta-s)} + 2\mathcal{K}_{IJ}^{(s)}\mathcal{P}^{IJ(\Delta-s)} + 2\beta\mathcal{P}^{(s)}\mathcal{K}^{(\Delta-s)} + 2\beta\mathcal{K}^{(s)}\mathcal{P}^{(\Delta-s)} & \\ +\alpha^2\mathcal{P}^{(s)}\mathcal{P}^{(\Delta-s)} + 2(1-\lambda)\Pi^{(s)}\Pi^{(\Delta-s)} & \\ +2\phi^2(\mathcal{K}_{IJ}^{(s-\Delta_\phi)}\mathcal{K}^{IJ(\Delta-s-\Delta_\phi)} - \lambda\mathcal{K}^{(s-\Delta_\phi)}\mathcal{K}^{(\Delta-s-\Delta_\phi)}) & \\ -2\lambda\mathcal{K}^{(s-\Delta_\phi)}\phi\Pi^{(\Delta-s)} - 2\lambda\mathcal{K}^{(\Delta-s-\Delta_\phi)}\phi\Pi^{(s)}] & \\ +\frac{1}{2}\mathcal{P}^{I(\Delta/2)}\mathcal{P}_I^{(\Delta/2)} + \mathcal{K}_{IJ}^{(\Delta/2)}\mathcal{P}^{IJ(\Delta/2)} & \\ +2\beta\mathcal{P}^{(\Delta/2)}\mathcal{K}^{(\Delta/2)} + \frac{\alpha^2}{2}\mathcal{P}^{(\Delta/2)}\mathcal{P}^{(\Delta/2)} + (1-\lambda)\Pi^{(\Delta/2)}\Pi^{(\Delta/2)} & \\ +\phi^2(\mathcal{K}_{IJ}^{(\Delta/2-\Delta_\phi)}\mathcal{K}^{IJ(\Delta/2-\Delta_\phi)} - \lambda\mathcal{K}^{(\Delta/2-\Delta_\phi)}\mathcal{K}^{(\Delta/2-\Delta_\phi)}) & \\ -2\lambda\mathcal{K}^{(\Delta/2-\Delta_\phi)}\phi\Pi^{(\Delta/2)} - \frac{1}{N}\nabla^I(\phi N)\mathcal{P}_I^{(\Delta-\Delta_\phi-1)} & \;=\; \tilde{\mathcal{S}}^{(\Delta)},\end{aligned} \tag{7.44}$$

where:

$$\tilde{\mathcal{S}} \;=\; \hat{K}_{ij}\hat{K}^{ij} - \lambda\hat{K}^2 + \beta(\hat{R}-2\Lambda) + \frac{\alpha^2}{2}\frac{\nabla_iN\nabla^iN}{N^2} + \text{tot. der.}. \tag{7.45}$$

The $s=0$ terms in the sum in (7.44) are:

$$2\beta(D-1)\mathcal{K}^{(\Delta)} + 2z\beta\mathcal{K}^{(\Delta)} + 2\beta D\mathcal{P}^{(\Delta)} + z\alpha^2\mathcal{P}^{(\Delta)}. \tag{7.46}$$

The $s=\Delta_\phi$ terms in the sum in (7.44) are:

$$2[(1-\lambda)(z-\Delta_\phi) - \lambda D]\phi\Pi^{(\Delta-\Delta_\phi)} + 2\phi^2[1-\lambda(D+z-\Delta_\phi)]\mathcal{K}^{(\Delta-2\Delta_\phi)}. \tag{7.47}$$

Therefore, substituting (7.44) into (7.43) yields our final expression for $\mathcal{L}^{(\Delta)}$. Explicitly, the terms in the on-shell action are given (for $\Delta \neq 0$, Δ_ϕ, $2\Delta_\phi$, $z+D$) by:

$$(z+D-\Delta)\mathcal{L}^{(\Delta)} = \mathcal{Q}^{(\Delta)} + \mathcal{S}^{(\Delta)}, \tag{7.48}$$

where:

$$\begin{aligned}\mathcal{S} &= \hat{K}_{ij}\hat{K}^{ij} - \lambda\hat{K}^2 + \beta(\hat{R} - 2\Lambda) + \frac{\alpha^2}{2}\frac{\nabla_i N \nabla^i N}{N^2} \\ &\quad -2(1-(z-\Delta_\phi+D)\lambda)\phi\hat{K} + \text{tot. der.},\end{aligned} \tag{7.49}$$

$$\begin{aligned}-\mathcal{Q}^{(\Delta)} = \sum_{s<\Delta/2, s\neq 0,\Delta_\phi} &[\mathcal{P}^{I(s)}\mathcal{P}_I^{(\Delta-s)} + 2\mathcal{K}_{IJ}^{(s)}\mathcal{P}^{IJ(\Delta-s)} + 2\beta\mathcal{P}^{(s)}\mathcal{K}^{(\Delta-s)} \\ &+2\beta\mathcal{K}^{(s)}\mathcal{P}^{(\Delta-s)} + \alpha^2\mathcal{P}^{(s)}\mathcal{P}^{(\Delta-s)} + 2(1-\lambda)\Pi^{(s)}\Pi^{(\Delta-s)} \\ &+2\phi^2(\mathcal{K}_{IJ}^{(s-\Delta_\phi)}\mathcal{K}^{IJ(\Delta-s-\Delta_\phi)} - \lambda\mathcal{K}^{(s-\Delta_\phi)}\mathcal{K}^{(\Delta-s-\Delta_\phi)}) \\ &-2\lambda\mathcal{K}^{(s-\Delta_\phi)}\phi\Pi^{(\Delta-s)} - 2\lambda\mathcal{K}^{(\Delta-s-\Delta_\phi)}\phi\Pi^{(s)}] \\ &+\frac{1}{2}\mathcal{P}^{I(\Delta/2)}\mathcal{P}_I^{(\Delta/2)} + \mathcal{K}_{IJ}^{(\Delta/2)}\mathcal{P}^{IJ(\Delta/2)} \\ &+2\beta\mathcal{P}^{(\Delta/2)}\mathcal{K}^{(\Delta/2)} + \frac{\alpha^2}{2}\mathcal{P}^{(\Delta/2)}\mathcal{P}^{(\Delta/2)} + (1-\lambda)\Pi^{(\Delta/2)}\Pi^{(\Delta/2)} \\ &+\phi^2(\mathcal{K}_{IJ}^{(\Delta/2-\Delta_\phi)}\mathcal{K}^{IJ(\Delta/2-\Delta_\phi)} - \lambda\mathcal{K}^{(\Delta/2-\Delta_\phi)}\mathcal{K}^{(\Delta/2-\Delta_\phi)}) \\ &-2\lambda\mathcal{K}^{(\Delta/2-\Delta_\phi)}\phi\Pi^{(\Delta/2)} - \frac{1}{N}\nabla^I(\phi N)\mathcal{P}_I^{(\Delta-\Delta_\phi-1)}.\end{aligned} \tag{7.50}$$

We have the following exceptions to the above formula:

$$(z+D)\mathcal{L}^{(0)} = 2\mathcal{S}^{(0)} = -4\beta\Lambda, \tag{7.51}$$

$$(z+D-\Delta_\phi)\mathcal{L}^{(\Delta_\phi)} = 0, \tag{7.52}$$

$$(z+D-2\Delta_\phi)\mathcal{L}^{(2\Delta_\phi)} = [D(1-\lambda D) - 2D\lambda(z-\Delta_\phi) + (1-\lambda)(z-\Delta_\phi)^2]\phi^2. \tag{7.53}$$

When simplified, (7.53) becomes $\mathcal{L}^{(2\Delta_\phi)} = [(z-\Delta_\phi)(1-\lambda) - \lambda D]\phi^2$, which yields $\pi^{(\Delta_\phi)} = 2[(z-\Delta_\phi)(1-\lambda) - \lambda D]\phi$ and $\Pi^{(\Delta_\phi)} = (z-\Delta_\phi)\phi$ as expected.

Furthermore, note that when $\Delta = z+D$, $\mathcal{L}^{(z+D)}$ cannot be determined by this asymptotic analysis, but the logarithmic divergence (which contributes to the anisotropic Weyl anomaly) can be determined using [7]:

$$\tilde{\mathcal{L}}^{(z+D)} = \lim_{\Delta\to z+D}\left((z+D-\Delta)\mathcal{L}^{(\Delta)}\right). \tag{7.54}$$

From the form of the holographic renormalization equations we expect divergent terms to appear whenever $\Delta = 2n + 2mz + 2r\Delta_\phi + s(z+\Delta_\phi)$ for $n, m, r, s \in \mathbb{N}$.

7.8 Calculation of Divergences

Now that we have the general formula (7.48) needed for holographic renormalization, we proceed to calculate the divergent terms at each order. Once these divergent terms have been calculated, counterterms must be added to the action in order to subtract these divergences. With a boundary cutoff at $r = \frac{1}{\epsilon}$, the counterterms are

$$S_{ct} = -\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\left[\sum_{0\leq\Delta<z+D}\mathcal{L}^{(\Delta)} - \tilde{\mathcal{L}}^{(z+D)}\log(\epsilon)\right]. \tag{7.55}$$

We will determine $\mathcal{L}^{(\Delta)}$ and $\tilde{\mathcal{L}}^{(z+D)}$ only up to total derivatives of the form $\frac{\partial_t}{\sqrt{\gamma}N}$ or $\frac{\nabla_i}{N}$. This will not allow us to determine anomalies in $\tilde{\mathcal{L}}^{(z+D)}$ that are total derivatives (such as was found for $D = z = 1$ in (4.44)), but we shall see that this suffices for our calculation of anomalies in $D = z = 2$. Therefore, we can drop the total derivative terms in (7.49). We now proceed to calculate the divergences at each order.

At order 2 and $z + \Delta_\phi$ we have:

$$\begin{aligned}(z+D-2)\mathcal{L}^{(2)} &= \mathcal{S}^{(2)} = \beta\hat{R} + \frac{\alpha^2}{2}\frac{\nabla_i N\nabla^i N}{N^2}, &(7.56)\\ (z+D-z-\Delta_\phi)\mathcal{L}^{(z+\Delta_\phi)} &= \mathcal{S}^{(z+\Delta_\phi)} = -2(1-(z-\Delta_\phi+D)\lambda)\phi\hat{K}. &(7.57)\end{aligned}$$

For higher order divergences we now have contributions from $\mathcal{Q}^{(\Delta)}$ so we must begin to calculate the canonical momenta using (7.17). We have that $\pi = \frac{2\kappa^2}{\sqrt{\gamma}N}\frac{\delta S}{\delta\phi}$ and so:

$$(D-\Delta_\phi)\pi^{(z)} = -2(1-(z-\Delta_\phi+D)\lambda)\hat{K}. \tag{7.58}$$

Furthermore, $2(1-\lambda)\Pi^{(z)} = \pi^{(z)} - 2\lambda\hat{K}$ so:

$$2(1-\lambda)(D-\Delta_\phi)\Pi^{(z)} = -2(1-z\lambda)\hat{K}. \tag{7.59}$$

Thus at order $2z$ we have:

$$\begin{aligned}(z+D-2z)\mathcal{L}^{(2z)} &= \mathcal{S}^{(2z)} + \mathcal{Q}^{(2z)} = \hat{K}_{ij}\hat{K}^{ij} - \lambda\hat{K}^2 - (1-\lambda)\Pi^{(z)}\Pi^{(z)}\\ &= \hat{K}_{ij}\hat{K}^{ij} - \lambda\hat{K}^2 - \frac{(1-z\lambda)^2}{(1-\lambda)(D-\Delta_\phi)^2}\hat{K}^2. &(7.60)\end{aligned}$$

This generates a logarithmically divergent term when $z = D$ in which case:

$$(z+D-2z)2\kappa^2\mathcal{L}^{(2z)} = \hat{K}_{ij}\hat{K}^{ij} - \frac{1}{D}\hat{K}^2, \tag{7.61}$$

which has anisotropic Weyl invariance.

Next we move on to the calculation of the fourth order divergence, that is, divergent terms involving four spatial derivatives. We first find the necessary momenta by varying $\mathcal{L}^{(2)}$ and using (7.17):

$$\begin{aligned}
(z+D-2)P^{(2)} &= \beta\hat{R} + \frac{\alpha^2}{2}\frac{\nabla_i N\nabla^i N}{N^2} - \alpha^2\frac{\nabla^i\nabla_i N}{N}, \\
(z+D-2)P^{IJ(2)} &= \beta(-\hat{R}^{IJ} + \frac{\hat{R}\delta^{IJ}}{2} - \frac{\nabla^k\nabla_k N\delta^{IJ}}{N} + \frac{\nabla^I\nabla^J N}{N}) \\
&\quad -\frac{\alpha^2}{2}\frac{\nabla^I N\nabla^J N}{N^2} + \frac{\alpha^2}{4}\frac{\nabla_k N\nabla^k N\delta^{IJ}}{N^2}.
\end{aligned} \tag{7.62}$$

Utilizing (7.20) we then get:

$$\begin{aligned}
(z+D-4)\mathcal{L}^{(4)} &= -\mathcal{K}^{(2)}_{IJ}\mathcal{P}^{IJ(2)} - 2\beta\mathcal{P}^{(2)}\mathcal{K}^{(2)} - \frac{\alpha^2}{2}\mathcal{P}^{(2)}\mathcal{P}^{(2)} \\
&= \frac{\beta}{a_0}[a_1(\frac{\nabla_i N\nabla^i N}{N^2})^2 + a_2\frac{\nabla_i N\nabla^i N}{N^2}\hat{R} + a_3\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2} \\
&\quad +a_4\frac{\nabla^i N\nabla^j N}{N^2}\hat{R}_{ij} + a_5\frac{\nabla^i\nabla_i N}{N}\hat{R} + a_6(\frac{\nabla^i\nabla_i N}{N})^2 \\
&\quad +a_7\hat{R}_{ij}\hat{R}^{ij} + a_8\hat{R}^2],
\end{aligned} \tag{7.63}$$

where:

$$\begin{aligned}
a_0 &= 4z^2(D+z-1)(z+D-2)^2, \\
a_1 &= -4+3z+2z^2-z^3-D(z^2-2z-3), \\
a_2 &= -2z(z-1)(D+z), \\
a_3 &= 2z(z^2-3z+2+D(-4+z)), \\
a_4 &= 4z(z-2)(D+z-1), \\
a_5 &= 4z^2, \\
a_6 &= 4z(D-1), \\
a_7 &= 4z^2(D+z-1), \\
a_8 &= -z^2(D+z).
\end{aligned} \tag{7.64}$$

For $D=2$ we have:

$$\begin{aligned}
(z-2)\mathcal{L}^{(4)} &= \frac{\beta}{b_0}[b_1(\frac{\nabla_i N\nabla^i N}{N^2})^2 + b_2\frac{\nabla_i N\nabla^i N}{N^2}\hat{R} + b_3\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2} \\
&\quad +b_4\frac{\nabla^i\nabla_i N}{N}\hat{R} + b_5(\frac{\nabla^i\nabla_i N}{N})^2 + b_6\hat{R}^2],
\end{aligned} \tag{7.65}$$

where:

$$\begin{aligned}
b_0 &= 4z^4(z+1), \\
b_1 &= -z^3+7z+2, \\
b_2 &= -4z^2,
\end{aligned}$$

$$
\begin{aligned}
b_3 &= 2z(z-3)(z+2), \\
b_4 &= 4z^2, \\
b_5 &= 4z, \\
b_6 &= z^3.
\end{aligned} \tag{7.66}
$$

When $z = 2$, this becomes:

$$
(z-2)\mathcal{L}^{(4)} = \frac{\beta}{24}(\hat{R} - \frac{\nabla_i N \nabla^i N}{N^2} + \frac{\nabla^i \nabla_i N}{N})^2. \tag{7.67}
$$

Once again, this has anisotropic Weyl invariance.

7.9 Analysis and Discussion

For $z + D < 6$, the divergences in the on-shell action calculated above represent all possible divergences (when we take the source $\bar{\phi} = 0$). In particular, if $D = z = 2$, the counterterms we need to add to the action to remove these divergences for a cutoff at $r = \frac{1}{\epsilon}$ is:

$$
\begin{aligned}
S_{ct} = &-\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^2\mathbf{x}\sqrt{\gamma}N\left\{6\beta + \frac{\beta}{2}\hat{R} + \frac{\beta}{4}\frac{\nabla_i N \nabla^i N}{N^2}\right. \\
&\left. - \log\epsilon\left[(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2) + \frac{\beta}{24}(\hat{R} - \frac{\nabla_i N \nabla^i N}{N^2} + \frac{\nabla^i \nabla_i N}{N})^2\right]\right\}.
\end{aligned} \tag{7.68}
$$

Note that if we take $\bar{\phi} \neq 0$ then additional counterterms involving ϕ will likely be needed (for certain values of λ, z and D) in order to remove divergences such as (7.53) and (7.57).[3]

From the logarithmic counterterm in (7.68), we find that there is an anisotropic Weyl anomaly for $D = z = 2$ of the form:

$$
\mathcal{A} = \frac{1}{2\kappa^2}(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2) + \frac{\beta}{48\kappa^2}(\hat{R} - \frac{\nabla_i N \nabla^i N}{N^2} + \frac{\nabla^i \nabla_i N}{N})^2. \tag{7.69}
$$

It is indeed of the most general form, with the two independent central charges given in terms of two low-energy couplings in minimal HL gravity: $c_K = 1/(2\kappa^2)$ and $c_V = \beta/(48\kappa^2)$.

Now that we have seen that HL gravity provides candidate holographic duals for QFTs with anisotropic Lifshitz scaling, is it possible to apply HL gravity also to QFTs with isotropic $z = 1$ scaling? Interestingly, the limit $z \to 1$ corresponds to $\alpha \to 0$, the "unhealthy reduction" [26] of nonprojectable HL gravity, and may therefore be difficult to make sense of. This is perhaps to be expected: $z = 1$ QFTs with such gravity duals would likely exhibit isotropic dilatation symmetry without full relativistic conformal symmetry, a phenomenon whose examples are few and far between. Further study of our holographic duality in the $\alpha \to 0$ limit may shed new light on this rare class of QFTs.

[3] The case $D = z = 2$ and $\lambda = \frac{1}{3}$ has been calculated explicitly and no additional counterterms are needed here

Finally, throughout this chapter we have used the effective low-energy limit of HL gravity, dominated by the terms of the lowest dimension in the action. We have been agnostic about how the model is completed at high energies. This completion may come from additional degrees of freedom, perhaps via an embedding into string theory; or it can be via a self-completion of HL gravity, due to highly anisotropic scaling at short distances. This latter possibility would be particularly interesting, as it could open a new door away from the large N limit and small bulk curvature. Complementary results about another form of nonrelativistic holography with HL gravity have also been presented in [68, 69]. Our results, and those of [68, 69], thus provide further evidence for the picture proposed originally in [7], that the natural arena for nonrelativistic holography is nonrelativistic HL gravity. It remains to be seen whether – as suggested in [7] – the nonrelativistic field theories whose holographic duals happen to be relativistic indeed represent only a minority among all theories with gravity duals.

Chapter 8

Lifshitz Black Holes in HL Gravity

In the standard AdS/CFT correspondence, a field theory at finite temperature is dual to a static black hole in the bulk. For this reason it is interesting to search for static black hole solutions of the HL gravity action (7.1) studied in Chapter 7. Furthermore, the properties of black holes in HL gravity are also of interest and having a holographic interpretation of them would be beneficial. Before searching for Lifshitz black hole solutions in HL gravity, we begin with a simple example to illustrate the main techniques. We examine the theory introduced in Chapter 4 with action (4.1): pure Einstein gravity with cosmological constant.

8.1 Illustrative Example: General Relativity

We want to find the most general black hole solution to the action (4.1) that is static, has rotational and translational symmetry in the x^α coordinates and is asymptotically AdS. Without loss of generality, we can choose the gauge where the metric takes the form:

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -f^2r^2dt^2 + \frac{dr^2}{h^2r^2} + r^2d\mathbf{x}^2, \tag{8.1}$$

where h and f are arbitrary functions of r. Since we want the black hole to be asymptotically AdS, we require $f \to 1$ and $h \to 1$ as $r \to \infty$. We denote the number of spatial dimensions on the boundary by $D = d - 1$.

The non-zero $(D+2)$-dimensional Christoffel symbols are:

$$\begin{aligned}
\Gamma^r_{rr} &= -\frac{1}{r} - \frac{h'}{h}, \qquad \Gamma^t_{tr} = \frac{1}{r} + \frac{f'}{f}, \\
\Gamma^r_{tt} &= f^2h^2r^4(\frac{f'}{f} + \frac{1}{r}), \qquad \Gamma^j_{ir} = \frac{1}{r}\delta^j_i, \\
\Gamma^r_{ij} &= -h^2r^3\delta_{ij}.
\end{aligned} \tag{8.2}$$

The non-zero components of the $(D+2)$-dimensional Ricci tensor are:

$$\mathcal{R}_{rr} = -\frac{D+1}{r^2} - (\frac{f'}{f})' - \frac{3}{r}\frac{f'}{f} - \frac{(D+1)}{r}\frac{h'}{h} - (\frac{f'}{f})^2 - \frac{h'}{h}\frac{f'}{f},$$

$$\begin{aligned}\mathcal{R}_{tt} &= f^2h^2r^4(\frac{D+1}{r^2}+(\frac{f'}{f})'+\frac{(D+3)}{r}\frac{f'}{f}+\frac{1}{r}\frac{h'}{h}+(\frac{f'}{f})^2+\frac{h'}{h}\frac{f'}{f}),\\ \mathcal{R}_{ij} &= -h^2r^3(\frac{D+1}{r}+\frac{h'}{h}+\frac{f'}{f}).\end{aligned} \tag{8.3}$$

Einstein's equation ($\mathcal{R}_{\mu\nu}=\frac{1}{2}(\mathcal{R}-2\Lambda)G_{\mu\nu}$) gives three equations of motion:

$$\begin{aligned}0 &= -\frac{D+1}{r^2}(1-h^{-2})-(\frac{f'}{f})'-\frac{3}{r}\frac{f'}{f}-\frac{(D+1)}{r}\frac{h'}{h}-(\frac{f'}{f})^2-\frac{h'}{h}\frac{f'}{f},\\ 0 &= \frac{D+1}{r^2}(1-h^{-2})+(\frac{f'}{f})'+\frac{(D+3)}{r}\frac{f'}{f}+\frac{1}{r}\frac{h'}{h}+(\frac{f'}{f})^2+\frac{h'}{h}\frac{f'}{f},\\ 0 &= \frac{D+1}{r^2}(1-h^{-2})+\frac{h'}{rh}+\frac{f'}{rf}.\end{aligned} \tag{8.4}$$

Adding the first equation to the second gives $\frac{f'}{f}=\frac{h'}{h}$, which implies $f=h$. The third equation of motion then becomes:

$$0 = (D+1)(h^2-1)+r(h^2)'. \tag{8.5}$$

This has general solution $h^2=f^2=1-\rho$, where $\rho=\gamma r^{-a}$, $a=D+z=D+1$ and γ is an arbitrary constant of integration. It is easily checked that this solution satisfies all of the equations of motion.

In this simple case, this solution can be found analytically, but later we will be dealing with a more difficult equation without a simple analytic solution. So it will be useful here to analyze this solution without knowing its explicit form. Notice that the second and third equations in (8.4) can be combined (using $\frac{f'}{f}=\frac{h'}{h}$) to form:

$$0 = (\frac{f'}{f})'+\frac{(D+2)}{r}\frac{f'}{f}+2(\frac{f'}{f})^2. \tag{8.6}$$

If we let $\sigma=\frac{rf'}{f}$ and $\tau=\ln r$, then this becomes:

$$0 = \frac{d\sigma}{d\tau}+a\sigma+2\sigma^2, \tag{8.7}$$

where $a=D+1$. Of course, the solution can be found analytically to be $\sigma=\Sigma(r)\equiv\dfrac{a}{2}\dfrac{\rho}{1-\rho}$. But we can also analyze the solution $\Sigma(r)$ without knowing the full analytic solution. $\Sigma(r)$ satisfies the differential equation (8.7), i.e., $\frac{d\Sigma}{d\tau}+a\Sigma+2\Sigma^2=0$. The differential equation has fixed points at $\Sigma=0$ and $\Sigma=-\frac{a}{2}$. Note that $\Sigma=0$ is an attractive fixed point and $\Sigma=-\frac{a}{2}$ is a repulsive fixed point for increasing τ. For decreasing τ, these are reversed.

The solutions that have the appropriate asymptotic behavior converge to the $\Sigma=0$ fixed point as $\tau\to\infty$. If we follow this solution back by decreasing τ the solution with $\gamma>0$ must have originated at $\Sigma=\infty$ and the solution with $\gamma<0$ must have originated from the

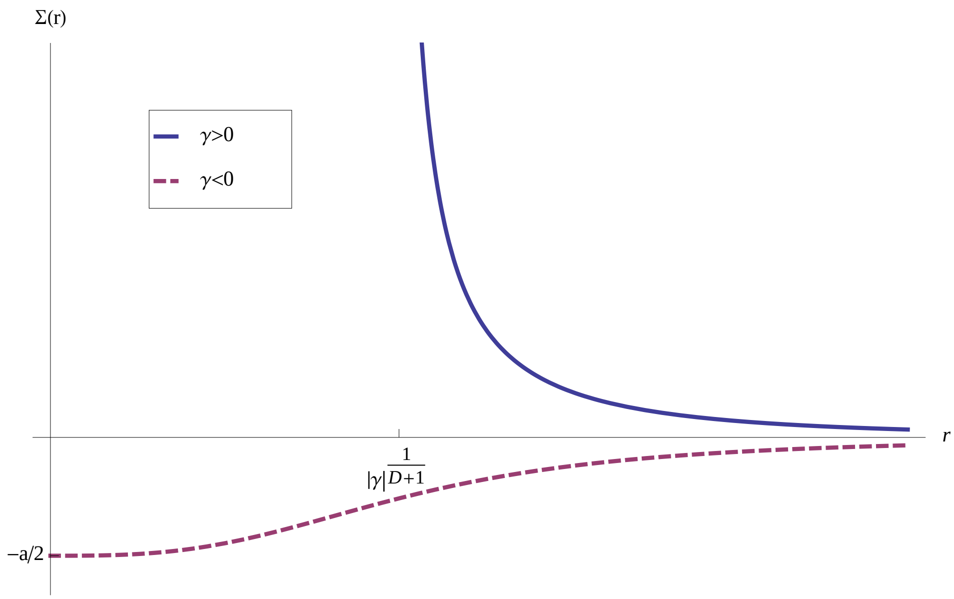

Figure 8.1: The function $\Sigma(r)$ for the AdS-Schwarschild solution in General Relativity, derived in Section 8.1. For $\gamma > 0$ the solution originates from $\Sigma = \infty$ while for $\gamma < 0$ it originates from the $\Sigma = -a/2$ fixed point.

$\Sigma = -\frac{a}{2}$ fixed point. This behavior for $\Sigma(r)$ is shown in Figure 8.1. This translates into the form of $f(r)$ and $h(r)$ that is shown in Figure 8.2. We see that the $\gamma < 0$ solution is smooth all the way back to $r = 0$ but the $\gamma > 0$ solution has a (coordinate) singularity at $r = \gamma^{\frac{1}{D+1}}$. Of course, it is well known that this coordinate singularity can be removed by a change of coordinates and the space-time can be extended. $r = \gamma^{\frac{1}{D+1}}$ is an event horizon for the black hole. The $\gamma < 0$ solution has no event horizon. In either case, there is a curvature singularity[1] at $r = 0$ for any $D > 1$. So we see that the $\gamma > 0$ solution has a singularity at $r = 0$ shielded by an event horizon at $r = \gamma^{\frac{1}{D+1}}$, whereas the $\gamma < 0$ solution has a naked singularity at $r = 0$.

Calculation of the mass

We now calculate the mass of the solutions found above using $M = \frac{-1}{16\pi G_{D+2}} \int_{\partial\mathcal{M}} d^D x \sqrt{-g}\, T_{00}^{fin}$. Here T_{00}^{fin} is the renormalized energy density after subtracting off divergent pieces. The divergent pieces can be obtained from the holographic renormalization results of Chapter 4. The static solution (8.1) is currently not written in the radial gauge ($\mathcal{N} = 1/r$, $\mathcal{N}_\alpha = 0$)

[1]The Kretschmann scalar is $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} = 2(D+2)(D+1) + D^2(D^2-1)\rho^2$, which becomes infinite at $r = 0$ ($\rho = \infty$) for any $D > 1$. In $D = 1$, there is no longer a *curvature* singularity, but it can be shown that $r = 0$ is still a singular point [70].

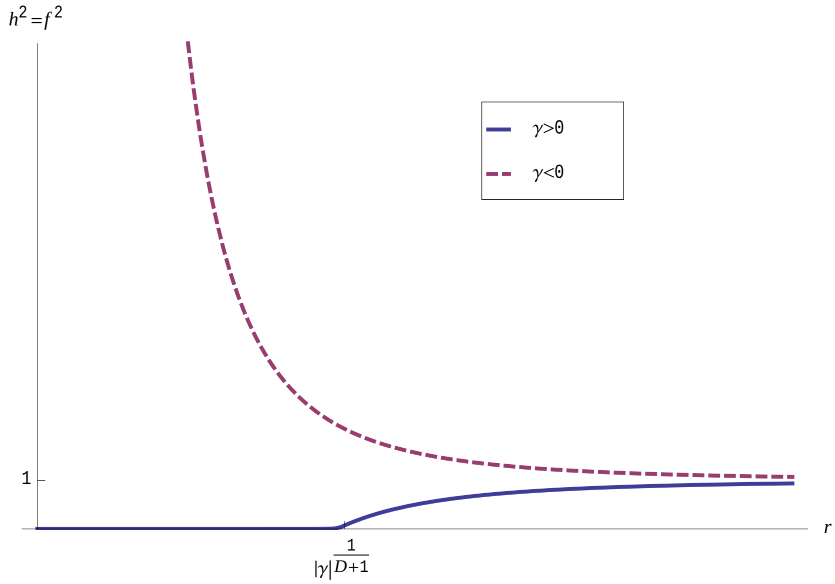

Figure 8.2: The functions $f(r)^2 = h(r)^2$ for the AdS-Schwarschild solution in General Relativity, derived in Section 8.1. The $\gamma > 0$ solution is only shown outside the event horizon at $r = |\gamma|^{1/(D+1)}$.

that was used for the holographic renormalization calculation. So in order to use the counterterms that we calculated we need to switch to this gauge. The solution we found has the asymptotic form[2] (with $\rho = \gamma r^{-a}$):

$$ds^2 = -r^2(1-\rho+...)dt^2 + \frac{dr^2}{r^2(1-\rho+...)} + r^2 d\mathbf{x}^2. \tag{8.8}$$

Changing coordinates to the radial gauge (by letting $r \to r(1+\frac{\rho}{2a}+...)$) results in:

$$ds^2 = -r^2(1-\frac{D\rho}{a}+...)dt^2 + \frac{dr^2}{r^2} + r^2(1+\frac{\rho}{a}+...)d\mathbf{x}^2. \tag{8.9}$$

Then, using $K_{\alpha\beta} = r\partial_r g_{\alpha\beta}/2$ we have:

$$\begin{aligned} K_{tt} &= -r^2(1+\frac{D(D-1)\rho}{2a}+...), \\ K_{ij} &= r^2(1-\frac{(D-1)\rho}{2a}+...)\delta_{ij}. \end{aligned} \tag{8.10}$$

[2]Of course, in this simple example, this is actually the exact solution, but we only need the asymptotic behavior.

Transforming to tangent space indices, this becomes:

$$\begin{aligned} K_{00} &= -(1+\frac{D\rho}{2}+...), \\ K_{IJ} &= (1-\frac{\rho}{2}+...)\delta_{IJ}, \\ K &= D+1+..., \end{aligned} \tag{8.11}$$

and therefore

$$T_{00} = 2(K_{00}+K) = 2D(1-\frac{\rho}{2}+...). \tag{8.12}$$

From (4.41) we know that the divergent term at order zero is $T_{00}^{(0)} = 2D$ and so the finite piece is:

$$T_{00}^{fin} = -D\rho + ..., \tag{8.13}$$

which gives a contribution to the mass of:

$$M = -\frac{1}{16\pi G_{D+2}}\int d^D\mathbf{x}\,\sqrt{-g}\,T_{00}^{fin} = \frac{D\gamma}{16\pi G_{D+2}}\int d^D\mathbf{x}\,. \tag{8.14}$$

Therefore $\gamma > 0$ corresponds to a positive mass solution while $\gamma < 0$ is a negative mass solution.

As a check of the holographic renormalization procedure, we can calculate the contribution to the on-shell action of our solution and confirm the calculation of the divergent terms. Using Einstein's equation ($\mathcal{R}_{\mu\nu} = \frac{1}{2}(\mathcal{R}-2\Lambda)G_{\mu\nu}$) and (8.11) we have that:

$$\begin{aligned} S &= \frac{1}{16\pi G_{D+2}}\int_{\mathcal{M}} dt\,d^D\mathbf{x}\,dr\,\sqrt{-G}\,(\mathcal{R}-2\Lambda) + \frac{1}{8\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\,d^D\mathbf{x}\,\sqrt{-g}\,K \\ &= \frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\,d^D\mathbf{x}\int_{r=0}^{r=1/\epsilon} dr\,r^{a-1}\,(-2(D+1)) \\ &\qquad + \frac{1}{8\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\,d^D\mathbf{x}\,(\frac{1}{\epsilon})^a(D+1+...) \\ &= \frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\,d^D\mathbf{x}\,((\frac{1}{\epsilon})^a(2D)+...). \end{aligned} \tag{8.15}$$

Indeed the only divergent term from the holographic renormalization procedure evaluated on this solution is (4.40), $\mathcal{L}^{(0)} = 2D$, and so the divergent terms match.

8.2 Static Lifshitz Solutions in HL Gravity

We want to now search for Lifshitz black hole solutions for the anisotropic theory of gravity with action (7.1). Specifically, we want to find the most general solution that is static, has planar symmetry and is asymptotically Lifshitz. Since this theory is defined on a manifold with foliation, we define a static solution to mean that the metric has a Killing vector (∂_t) that is orthogonal to the hypersurfaces of the foliation (defined by t =constant). In particular, this means that we take $N_a = 0$. We will also be searching for solutions which have $z > 1$. Without loss of generality, we can choose the gauge where the metric complex takes the form:

$$ds^2 = \frac{dr^2}{h^2 r^2} + r^2 d\mathbf{x}^2, \qquad N = f r^z, \qquad N_a = 0, \tag{8.16}$$

where h and f are arbitrary functions of r. Since we also want the black hole to be asymptotically Lifshitz, we require $f \to 1$ and $h \to 1$ as $r \to \infty$. The three equations of motions that arise from varying (7.1) can be rearranged into the form:

$$\begin{aligned} -(\frac{rh'}{h} + \frac{rf'}{f}) + a(h^{-2} - 1) &= 0, \\ \frac{rf'}{f} - \frac{rh'}{h} + c(\frac{rf'}{f})^2 &= 0, \\ (a+1)\frac{rf'}{f} + \frac{r^2 f''}{f} + \frac{rh'}{h}\frac{rf'}{f} &= 0. \end{aligned} \tag{8.17}$$

where $a = z + D$ and $c = \dfrac{(z-1)}{z(z+D-1)}$.

Now, eliminating h from the last two equations in (8.17), gives:

$$(a+1)\frac{rf'}{f} + \frac{r^2 f''}{f} + (\frac{rf'}{f})^2 + c(\frac{rf'}{f})^3 = 0, \tag{8.18}$$

and letting $\sigma = \frac{rf'}{f}$, we get

$$r\sigma' + a\sigma + 2\sigma^2 + c\sigma^3 = 0. \tag{8.19}$$

This is a first-order differential equation possessing a one parameter family of solutions. Denote the solution to this equation by $\sigma = \Sigma(r)$. Substituting $\frac{rf'}{f} = \Sigma$ into the first and second equation in (8.17) then implies that the only solution for h is:

$$h^2 = \frac{a}{a + 2\Sigma + c\Sigma^2}. \tag{8.20}$$

Also, writing f in terms of Σ gives:

$$f^2 = \exp\left(\int dr \frac{2\Sigma}{r}\right). \tag{8.21}$$

It is now easy to check that this solution solves all three equations of motion above.

An analytic form for Σ is difficult to obtain but we can expand the solution near the boundary ($r = \infty$) as a power series in $\rho = \gamma r^{-a}$, where γ is a constant of integration. The results are:

$$\begin{aligned}
\Sigma &= \frac{a}{2}(\rho + \rho^2 + (1 + ac/8)\rho^3 + (1 + ac/3)\rho^4 + \ldots), \\
f^2 &= 1 - \rho - \frac{ac}{24}\rho^3 - \frac{ac}{24}\rho^4 + \ldots, \\
h^2 &= 1 - \rho - \frac{ac}{4}\rho^2 - \frac{ac}{8}\rho^3 - \frac{ac}{12}\rho^4 + \ldots. \qquad (8.22)
\end{aligned}$$

We can also find a more explicit form for the function f. Note that the third equation of (8.17) can be rearranged to give:

$$\begin{aligned}
\frac{(a+1)}{r} + \frac{rf''}{f'} + \frac{rh'}{h} &= 0, \\
\frac{d}{dr}(r^{a+1}hf') &= 0, \\
r^{a+1}hf' &= \text{constant} = \frac{a\gamma}{2}, \qquad (8.23)
\end{aligned}$$

where the constant has been determined by the asymptotic boundary expansion given above. Since $\frac{rf'}{f} = \Sigma$, (8.23) allows us to write f as:

$$f^2 = \frac{a^2\gamma^2}{4r^{2a}h^2\Sigma^2} = \frac{a\gamma^2}{4r^{2a}\Sigma^2}(a + 2\Sigma + c\Sigma^2) = \frac{a\gamma^2}{4r^{2a}}(a\Sigma^{-2} + 2\Sigma^{-1} + c). \qquad (8.24)$$

Note that when $z \to 1$, this solution reduces to the usual AdS-Schwarzschild solution of Section 8.1: $\Sigma = \frac{a}{2}\frac{\rho}{1-\rho}$ and $f^2 = h^2 = 1 - \rho$.

8.3 Analysis of solutions

We start by investigating the behavior of the function $\Sigma(r)$. From the previous section, Σ satisfies the differential equation $r\Sigma' + a\Sigma + 2\Sigma^2 + c\Sigma^3 = 0$ and has asymptotic behavior $\Sigma \sim \frac{a\gamma}{2r^a}$ as $r \to \infty$. If we define $\tau \equiv \ln r$, then the differential equation becomes:

$$\frac{d\Sigma}{d\tau} + a\Sigma + 2\Sigma^2 + c\Sigma^3 = 0. \qquad (8.25)$$

This differential equation has three fixed points at $\Sigma = 0$, $\Sigma = \Sigma_+$ and $\Sigma = \Sigma_-$, where $\Sigma_\pm \equiv \frac{-1\pm\sqrt{1-ac}}{c}$. Note that $\Sigma_\pm < 0$. Furthermore, $\Sigma = 0$ and $\Sigma = \Sigma_-$ are attractive fixed points and $\Sigma = \Sigma_+$ is a repulsive fixed point as τ increases. For decreasing τ, the opposite occurs.

The solutions that have the appropriate asymptotic behavior ($\Sigma \sim \frac{a\gamma}{2r^a}$ as $r \to \infty$) must converge to the $\Sigma = 0$ fixed point as $\tau \to \infty$. If we follow this solution back by decreasing τ the solution with $\gamma > 0$ must have originated at $\Sigma = \infty$ and the solution with $\gamma < 0$ must have originated from the $\Sigma = \Sigma_+$ fixed point. This behavior is shown in Figure 8.3. We treat these two cases, $\gamma > 0$ and $\gamma < 0$, separately:

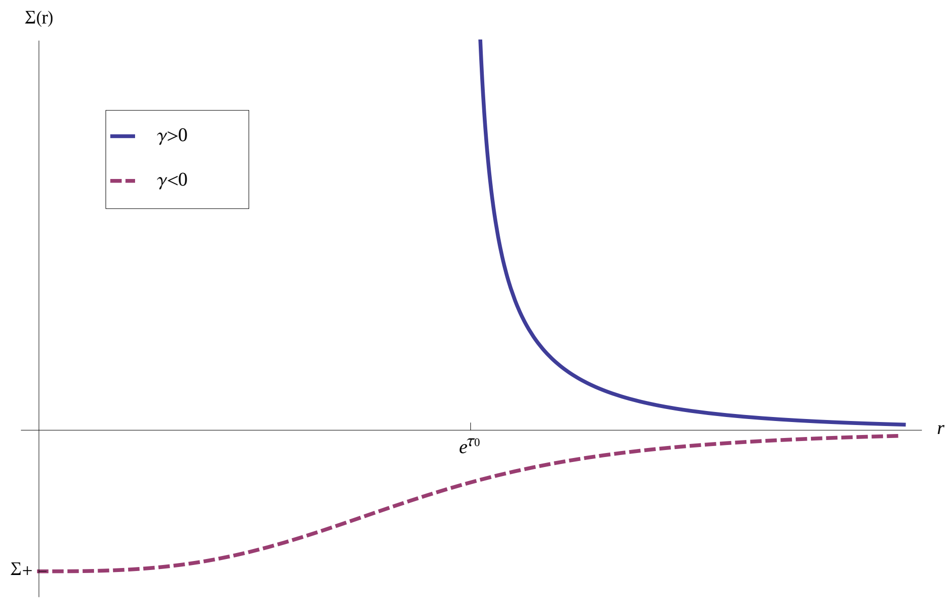

Figure 8.3: The function $\Sigma(r)$ for the static Lifshitz solution in HL gravity. For $\gamma > 0$ the solution originates from $\Sigma = \infty$ while for $\gamma < 0$ it originates from the $\Sigma = \Sigma_+$ fixed point. Compare to the AdS-Schwarzschild solution of Figure 8.1

.

$\gamma > 0$

In this case $\Sigma \to \infty$ as τ decreases and so we can understand the solution better in the $\Sigma \to \infty$ regime by approximately solving the differential equation for Σ large:

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= -(a\Sigma + 2\Sigma^2 + c\Sigma^3) \approx -c\Sigma^3, \\ \Sigma &\approx \frac{1}{\sqrt{2c(\tau - \tau_0)}}. \end{aligned} \tag{8.26}$$

where τ_0 is a constant of integration. So we can expand in a Taylor series in $u = \sqrt{\tau - \tau_0}$ to find:

$$\begin{aligned} \Sigma &= \frac{1}{\sqrt{2c}u}(1 - \frac{4u}{3\sqrt{2c}} + \frac{4 - 3ac}{6c}u^2 + ...), \\ h^2 &= 2au^2(1 - \frac{4}{3\sqrt{2c}}u + \frac{4 - 3ac}{3c}u^2 + ...), \\ f^2 &= \frac{ac\gamma^2}{4e^{2a\tau_0}}(1 + \frac{4}{\sqrt{2c}}u + \frac{8}{3c}u^2 + ...). \end{aligned} \tag{8.27}$$

This expansion is only currently valid for $u > 0$ ($\tau > \tau_0$). If we change coordinates from r to u then the spatial metric near $u = 0$ looks like:

$$\begin{aligned} ds^2 &= \frac{dr^2}{h^2r^2} + r^2 d\mathbf{x}^2 \\ &= \frac{du^2}{\frac{a}{2}(1 - \frac{4}{3\sqrt{2c}}u + \frac{4-3ac}{3c}u^2 + ...)} + e^{2\tau_0+2u^2} d\mathbf{x}^2 \\ &= \frac{du^2}{\frac{a}{2}(1 - \frac{4}{3\sqrt{2c}}u + \frac{4-3ac}{3c}u^2 + ...)} + e^{2\tau_0}(1 + 2u^2 + ...)d\mathbf{x}^2, \end{aligned} \tag{8.28}$$

and the lapse function is given by:

$$\begin{aligned} N^2 = r^2 f^2 &= e^{2z\tau_0+2zu^2}\frac{ac\gamma^2}{4e^{2a\tau_0}}(1 + \frac{4}{\sqrt{2c}}u + \frac{8}{3c}u^2 + ...) \\ &= \frac{ac\gamma^2}{4e^{2D\tau_0}}(1 + \frac{4}{\sqrt{2c}}u + (\frac{8}{3c} + 2z)u^2 + ...). \end{aligned} \tag{8.29}$$

In the u coordinate, this metric is now smooth through $u = 0$ and we can extend the spacetime. Letting $u = -\sqrt{\tilde{\tau} - \tilde{\tau}_0}$ and $\tilde{\tau} = \ln \tilde{r}$ we can now solve for the rest of the solution as $\tilde{r} \to \infty$ ($u \to -\infty$). Note that $u\Sigma$ is finite at $u = 0$ and requiring this to be continuous means that for u just less than zero, Σ is large and negative. So as $\tilde{r} \to \infty$, Σ will approach the fixed point at $\Sigma = \Sigma_-$ as shown in Figure 8.4. The Taylor series expansion in $\tilde{\rho} = \gamma_1 \tilde{r}^{2(\Sigma_-+a)}$ near this point $\Sigma = \Sigma_-$ looks like:

$$\begin{aligned} \Sigma &= \Sigma_-(1 + \tilde{\rho} + ...), \\ \tilde{h}^2 &= \frac{a}{c\Sigma_-(\Sigma_- - \Sigma_+)\tilde{\rho}}(1 + ...), \\ \tilde{f}^2 &= \frac{kc(\Sigma_- - \Sigma_+)\tilde{\rho}}{\tilde{r}^{2a}\Sigma_-}(1 + ...), \end{aligned} \tag{8.30}$$

where k and γ_1 are constants chosen to match the solution at $u = 0$. Here, $\tilde{h}$ and $\tilde{f}$ are defined exactly as in (8.16) but with a tilde on h, f and r. Note that $\tilde{h}^2 \to \infty$ and $\tilde{f}^2 \to 0$ as $\tilde{r} \to \infty$. The leading order behavior of the spatial metric as $\tilde{r} \to \infty$ is:

$$ds^2 \sim \frac{k_1}{\tilde{r}^{2(1-\Sigma_--a)}} d\tilde{r}^2 + \tilde{r}^2 d\mathbf{x}^2, \tag{8.31}$$

and the leading order behavior of the lapse function is

$$N \sim \frac{k_2}{\tilde{r}^{(-\Sigma_--z)}}, \tag{8.32}$$

for some non-zero constants k_1 and k_2.

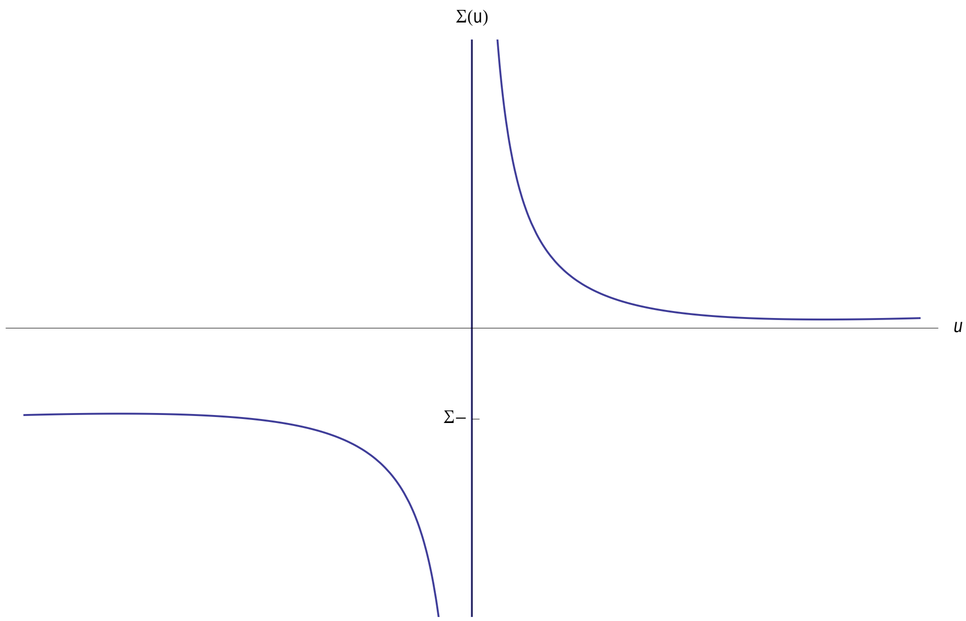

Figure 8.4: Extending Σ as a function of u. It approaches the $\Sigma = \Sigma_-$ fixed point as $u \to -\infty$ ($\tilde{r} \to \infty$).

In summary, we find that there is no horizon at $r = e^{\tau_0}$. The solution can be smoothly extended through this point (by switching to u coordinates) and f^2 is non-zero everywhere. Furthermore, from the form of (8.28), the measurement of distances in the x^i directions decrease as r decreases from infinity, until a minimum throat is reached at $r = e^{\tau_0}$ ($u = 0$). After that, the measurement of distances increase as $u \to -\infty$ ($\tilde{r} \to \infty$).

$\gamma < 0$

In this case $\Sigma \to \Sigma_+$ as τ decreases and so we can understand the solution better in the $\Sigma \to \Sigma_+$ regime by letting $\Sigma = \Sigma_+ + \epsilon$ and approximately solving the differential equation for small $\epsilon > 0$:

$$\begin{aligned} \frac{d\Sigma}{d\tau} &= -(a\Sigma + 2\Sigma^2 + c\Sigma^3), \\ \frac{d\epsilon}{d\tau} &\approx -c\Sigma_+(\Sigma_+ - \Sigma_-)\epsilon, \\ \epsilon &\approx b_0 e^{c(-\Sigma_+)(\Sigma_+ - \Sigma_-)\tau} = b_0 e^{2(\Sigma_+ + a)\tau}, \end{aligned} \tag{8.33}$$

where b_0 is a constant of integration. Then, to leading order in ϵ (that is, as $\tau \to -\infty$):

$$h^2 \approx \frac{a}{c\epsilon(\Sigma_+ - \Sigma_-)} \to \infty,$$

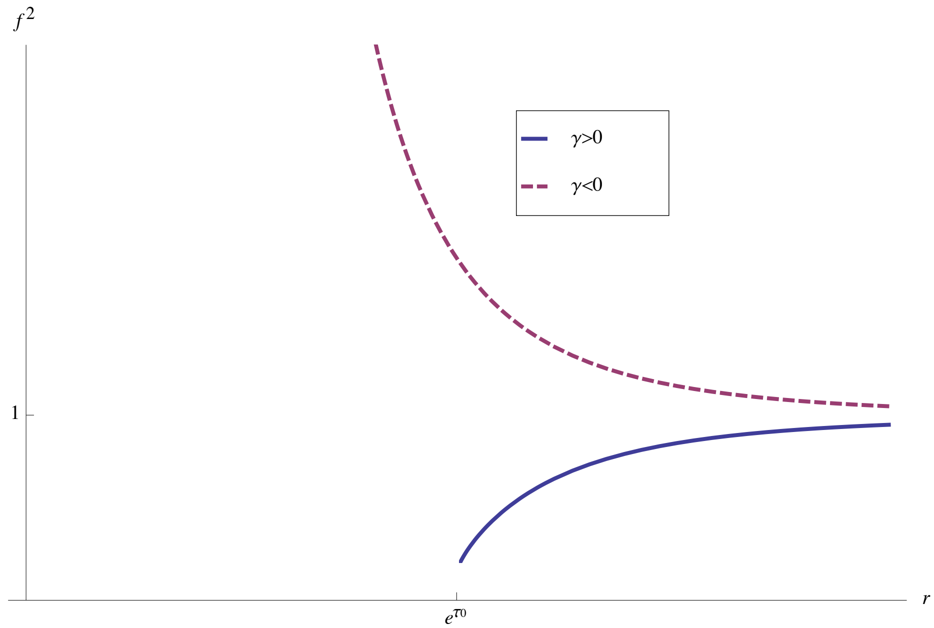

Figure 8.5: The function f^2 for the static Lifshitz solution in HL gravity. Compare to the AdS-Schwarzschild solution of Figure 8.2. Note that unlike in Figure 8.2, f^2 in the $\gamma > 0$ solution does not have a zero at $r = e^{\tau_0}$ and there is no horizon. This $\gamma > 0$ solution is only shown outside the coordinate singularity at $r = e^{\tau_0}$, where a coordinate change to first u and then $\tilde{r}$ must be implemented to move to smaller radii (see text for details).

$$f^2 \approx \frac{a\gamma^2}{4e^{2a\tau}\Sigma_+^2} c\epsilon(\Sigma_+ - \Sigma_-) \approx \frac{a\gamma^2}{4\Sigma_+^2} cb_0 e^{2\Sigma_+\tau}(\Sigma_+ - \Sigma_-) \to \infty. \tag{8.34}$$

The forms of $f(r)$ and $h(r)$, for both the $\gamma > 0$ and $\gamma < 0$ solutions, are shown in Figures 8.5 and 8.6.

Curvature singularities

For the above static solution we have that the spatial curvature is:

$$\begin{aligned}
R &= -Dh^2(D + 1 + 2\frac{rh'}{h}) \\
&= -Dh^2(D + 1 + 2(-\Sigma + a(h^{-2} - 1))) \\
&= -2aD - Dh^2(D + 1 - 2a - 2\Sigma) \\
&= -2aD - \frac{aD(D + 1 - 2a - 2\Sigma)}{c(\Sigma - \Sigma_+)(\Sigma - \Sigma_-)}.
\end{aligned} \tag{8.35}$$

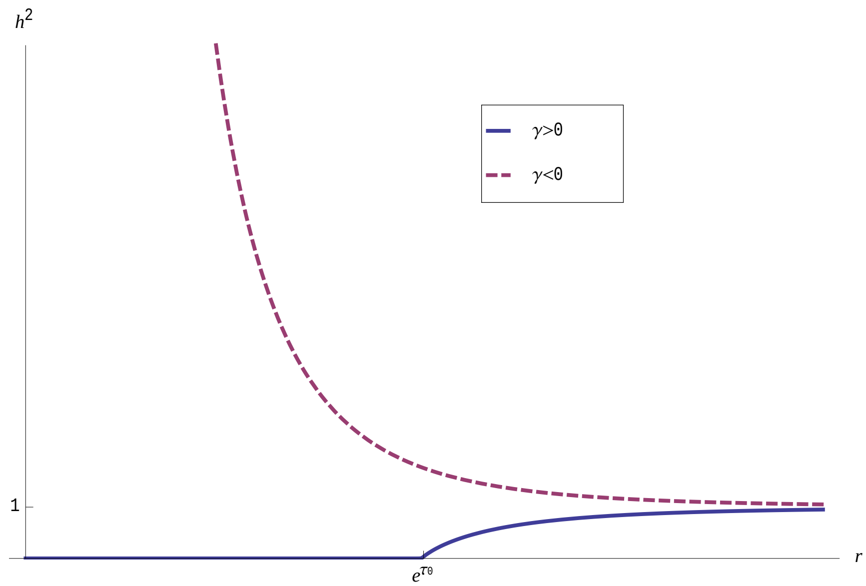

Figure 8.6: The function h^2 for the static Lifshitz solution in HL gravity. Compare to the AdS-Schwarzschild solution of Figure 8.2. The $\gamma > 0$ solution is only shown outside the coordinate singularity at $r = e^{\tau_0}$, where a coordinate change to first u and then $\tilde{r}$ must be implemented to move to smaller radii (see text for details).

Therefore $\Sigma \to \Sigma_\pm$ (or equivalently[3] $h^2 \to \infty$) always indicates a curvature singularity except possibly if $D + 1 - 2a - 2\Sigma = 0$ there. For $z > 1$, this exception occurs if and only if $D = 1$ and $\Sigma \to \Sigma_+$. Therefore, except possibly for this special case, every static solution presented above has a (naked) curvature singularity. Note also that R is finite at $\tau = \tau_0$ (where $|\Sigma| \to \infty$), as expected.

Other curvature quantities ($\partial_r R$, $\nabla_a R \nabla^a R$, $R_{ab}R^{ab}$, $R_{abcd}R^{abcd}$) have also been calculated and they share the same properties as R, being singular when $\Sigma \to \Sigma_\pm$, except for the case when $D = 1$ and $\Sigma \to \Sigma_+$.

Calculation of the mass

We now calculate the mass of the solutions found above using $M = \frac{-1}{2\kappa^2}\int_{r=\frac{1}{\epsilon}} d^D x \sqrt{\gamma} N\, T_{00}^{fin}$. Here T_{00}^{fin} is the renormalized energy density after subtracting off divergent pieces. The divergent pieces can be obtained from the holographic renormalization results of Chapter 7. The static solution (8.1) is currently not written in the radial gauge ($\mathcal{N} = 1/r$, $\mathcal{N}_\alpha = 0$) that was used for the holographic renormalization calculation. So in order to use the counterterms

[3]Recall that $h^2 = \dfrac{a}{c(\Sigma - \Sigma_+)(\Sigma - \Sigma_-)}$.

that we calculated we need to switch to this gauge. From (8.22), the solution currently has the asymptotic form near the boundary (with $\rho = \gamma r^{-a}$):

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2(1-\rho+...)} + r^2 d\mathbf{x}^2, \\ N &= r^z(1-\frac{1}{2}\rho+...). \end{aligned} \tag{8.36}$$

Changing to the radial gauge (by letting $r \to r(1+\frac{\rho}{2a}+...)$) results in:

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2} + r^2(1+\frac{\rho}{a}+...)d\mathbf{x}^2, \\ N &= r^z(1-\frac{D\rho}{2a}+...) \end{aligned} \tag{8.37}$$

Then, in this gauge:

$$\begin{aligned} R &= -D((D+1)+(z-1)\rho+...), \\ \frac{\nabla_\alpha N \nabla^\alpha N}{N^2} &= z(z+D\rho+...), \\ \mathcal{K} &= D(1-\rho/2+...), \\ \mathcal{P} &= (z+D\rho/2+...). \end{aligned} \tag{8.38}$$

We can then calculate T_{00} using (7.21):

$$\begin{aligned} T_{00} &= \alpha^2\mathcal{P} + 2\beta\mathcal{K} + 2(1-\lambda)\phi\Pi + 2\lambda\phi(\hat{K} - \mathcal{K}\phi) \\ &= \beta\frac{2(z-1)}{z}(z+D\rho/2+...) + 2\beta D(1-\rho/2+...) \\ &= \beta[2(z+D-1) - \frac{D}{z}\rho+...]. \end{aligned} \tag{8.39}$$

The results of Chapter 7 (see, for example, (7.35)) indeed give the divergent piece for this solution as $T_{00}^{(0)} = 2\beta(z+D-1)$ and so we have:

$$\begin{aligned} T_{00}^{fin} &= -\frac{\beta D\rho}{z} + ..., \\ M &= \frac{\beta D\gamma}{2z\kappa^2}\int d^D x\,. \end{aligned} \tag{8.40}$$

Therefore $\gamma > 0$ corresponds to a positive mass solution while $\gamma < 0$ is a negative mass solution. We can also calculate the contribution to the on-shell action (using the asymptotic boundary expansion) to check that the divergent terms cancel:

$$S = \frac{\beta}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^D x \int_{r=0}^{r=1/\epsilon} dr r^{a-1}\left(2(z+D)(z-1) + 0\rho + ...\right)$$

$$
\begin{aligned}
& +\frac{\beta}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^D x\, (\frac{1}{\epsilon})^a 2D(1-\rho/2+...) \\
= \quad & \frac{\beta}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^D x\, (2(\frac{1}{\epsilon})^a(z+D-1) - D\gamma + ...).
\end{aligned} \tag{8.41}
$$

Indeed from (7.51), $\mathcal{L}^{(0)} = 2\beta(z+D-1)$, so the divergent terms cancel as they should, and $S^{fin} \sim -\frac{\beta D\gamma}{2\kappa^2}\int_{\partial\mathcal{M}} dt\, d^D x$. There may be also be additional finite contributions to S^{fin} from the interior of the spacetime.

8.4 Discussion

Let us begin by summarizing the above results. We have found that, for every λ, $z > 1$ and mass M, there is a unique static, planar symmetric, asymptotically Lifshitz solution of the action (7.1). For the positive mass solutions, distances on the x^i plane decrease as r decreases until a minimum throat is reached[4]. Past this point, distances begin to increase again until a curvature singularity is reached in the IR. This solution has no horizon and thus represents a naked singularity.

The negative mass solutions for $D > 1$ are simpler and have a naked curvature singularity at $r = 0$, similar to the negative mass AdS-Schwarzschild black hole solution in Section 8.1.

Therefore all static, planar-symmetric, asymptotically Lifshitz solutions in $D > 1$ HL gravity have a naked curvature singularity, unlike the static black hole solutions found in General Relativity. It is therefore necessary to widen our search for asymptotically Lifshitz black hole type solutions in HL gravity; we must examine stationary (but non-static) black hole solutions, which have $N_r \neq 0$. These solutions have a Killing vector ∂_t which is not orthogonal to the leaves of the foliation of the manifold. In fact, in some cases it is possible for ∂_t to become tangential to the leaves of the foliation (this requires $N = 0$), creating what is known as a universal horizon [71, 72, 73, 74, 75]. Since any signal must propagate forwards in t, this can create a causal boundary of the spacetime. This occurs even though the theory is nonrelativistic and allows signals to travel faster than light. No analytic solution for a $z > 1$ Lifshitz black hole with a universal horizon has so far been found in HL gravity[5], even though an asymptotic expansion has been written down [75].

For an interesting example of a stationary (but non-static) solution, consider the Painlevé type ansatz:

$$
ds^2 = \frac{dr^2}{r^2} + r^2 d\mathbf{x}^2, \qquad N = r^z, \qquad N_r = f(r), \qquad N_i = 0. \tag{8.42}
$$

This solves the equations of motion if and only if $f = c_1 r^n$ with either:

$$
\lambda = 1 - \frac{4D(z-1)}{(z+D)^2} \qquad \text{and} \qquad n = \frac{z-D-2}{2}, \tag{8.43}
$$

[4]This is the point $r = e^{\tau_0}$ described in Section 8.3.

[5]An analytic solution for $z = 1$ has been written down [75], but HL gravity suffers from other problems in this $\alpha \to 0$ limit, as described in Section 2.1.

or

$$\lambda = \frac{1}{D+1} \qquad \text{and} \qquad n = 0. \tag{8.44}$$

This does not have a universal horizon for $r > 0$ since N is non-zero everywhere. Note, however, that a relativistic observer would experience a metric with $G_{tt} = -(N^2 - N_a N^a) = -(r^{2z} - c_1^2 r^{2(n+1)})$, implying a relativistic event horizon at $r = |c_1|^{1/(z-n-1)}$. Superluminal signals, however will escape this horizon. For $z > 1$, there is a singularity in the extrinsic curvature K at $r = 0$ and there is no universal horizon to hide this singularity.

Clearly the next step is to search for stationary, planar, asymptotically $z > 1$ Lifshitz solutions which have singularities hidden behind a universal horizon. Finding an analytic solution would allow the thermodynamic properties of these black holes to be studied.

Chapter 9

Conclusions

In this dissertation, we have seen how the holographic dictionary of the AdS/CFT correspondence can be extended to describe a more general gauge/gravity duality between gravitational theories in Lifshitz spacetime and quantum field theories with a Lifshitz fixed point. In particular, we have seen how the the procedure of holographic renormalization can be applied to theories with a Lifshitz spacetime background solution.

One useful piece of information that comes out of the holographic renormalization procedure is the form of the anisotropic Weyl anomaly. In this dissertation, we have shown that this anisotropic Weyl anomaly takes the form of conformal Hořava-Lifshitz (HL) gravity. In particular, for $z = 2$ and $D = 2$ boundary spatial dimensions, we have classified the possible anomalies and found that there are only two independent terms: A term with two time derivatives and a term with four spatial derivatives. Therefore HL gravity appears naturally in the anomaly structure of Lifshitz holography.

The first theory analyzed in Chapters 5 and 6 is one with a relativistic gravitational theory in the bulk and the Lifshitz background solution supported by a massive vector. The holographic renormalization of this theory is carried out in this dissertation and (for $z = D = 2$) results in one of the possible anomaly terms but not the other. This led us to search for a different gravitational theory in Chapter 7 which has Lifshitz spacetime as a background solution. This is the second appearance of HL gravity in this work; it was discovered here that Lifshitz spacetime is a vacuum solution of HL gravity with negative cosmological constant. This suggests that HL gravity in the bulk is the natural setting for Lifshitz holography, a view also supported by recent work [69]. Using the holographic renormalization procedure applied to the HL bulk gravity, the anisotropic Weyl anomaly was once again calculated for $z = D = 2$, and this time, both possible terms appear in the anomaly.

In order to further probe the HL gravity theory used in Lifshitz holography, Chapter 8 searches for asymptotically Lifshitz, planar, static black hole solutions of the theory. It is shown here that all such static solutions for $D > 1$ possess naked curvature singularities, that is, there is no form of horizon that hides these singularities. It is argued that one must generalize to stationary black holes in order to find black hole solutions containing a universal horizon. Finding an analytic solution for a $z > 1$ Lifshitz black hole in HL gravity would be very useful for providing a thermodynamic interpretation for these objects in HL gravity.

Throughout this dissertation, we have seen the appearance of HL gravity in Lifshitz holography. Regardless of whether HL gravity is relevant for the description of gravity in our world, it plays a key role in the holographic description of nonrelativistic quantum field theories with Lifshitz fixed points.

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