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Global, Non-scattering Solutions to Energy Critical Geometric Wave Equations

By

Mohandas Pillai

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in

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of the

University of California, Berkeley

Committee in charge:

Professor Daniel Tataru, Chair
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Abstract

Global, Non-scattering Solutions to Energy Critical Geometric Wave Equations

by

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Daniel Tataru, Chair

In this thesis, we consider two geometric, energy critical, semilinear wave equations arising from the wave maps problem (which is referred to as a nonlinear sigma model in many physics contexts) and Yang-Mills theory. The wave maps and Yang-Mills problems are families of equations, parameterized by various choices of domain and target manifolds, etc. The wave maps equation is geometric in the sense that the equation is independent of choice of coordinates on the domain and target manifolds. The Yang-Mills problem can be regarded as a geometric problem because it is invariant under gauge transformations. The specific symmetry reductions of these problems which we study are semilinear because their nonlinear terms involve only the unknown function, rather than its derivatives. Finally, both equations we consider admit a conserved energy and a scaling symmetry. The energy is invariant with respect to the scaling symmetry in the dimensions in which we study each equation, which is why the equations are said to be energy critical. We will now describe the equations considered in more detail.

The wave maps problem is an extension of the scalar linear wave equation to the case where the unknown function maps a Lorentzian manifold into a Riemannian manifold, (M, g). The wave maps equation can be derived from the least action principle applied to the following natural extension of the usual wave equation action.

$$\mathcal{S}(\Phi) = \int_{\mathbb{R}^{1+d}} \langle \partial_\alpha \Phi(t,x), \partial^\alpha \Phi(t,x) \rangle_{g(\Phi(t,x))} dt dx$$

where the α indices are contracted using the Minkowski metric, which we take to be

$$m = \text{diag}(-1, 1, 1, \dots, 1)$$

Now, we will describe the particular choice of target manifold and symmetry reductions considered. First of all, we will focus on the case $(M, g) = (\mathbb{S}^2, \mathring{g})$, where $\mathring{g}$ denotes the usual round metric on $\mathbb{S}^2$. We will regard Φ as a map into $\mathbb{R}^3$ with unit Euclidean norm. The wave maps equation then becomes

$$-\partial_\alpha \partial^\alpha \Phi(t,x) = \Box_{\mathbb{R}^{1+d}} \Phi(t,x) = \Phi(t,x) \left(\partial^\alpha \Phi(t,x) \cdot \partial_\alpha \Phi(t,x) \right), \quad \Phi(t,x) \cdot \Phi(t,x) = 1$$

where $\cdot$ is the Euclidean inner product on $\mathbb{R}^3$. We then make a symmetry reduction of the problem, and consider solutions which are 1-equivariant by using polar coordinates (r, φ) on $\mathbb{R}^2$, and writing

$$\Phi_u(t, r, \varphi) = \begin{pmatrix} \cos(\varphi)\sin(u(t, r)) \\ \sin(\varphi)\sin(u(t, r)) \\ \cos(u(t, r)) \end{pmatrix}$$

Then, the wave maps equation reduces to the following semilinear wave equation

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = 0$$

which has the following conserved energy

$$E_{WM}(u, \partial_t u) = \pi \int_0^\infty \left((\partial_t u)^2 + \frac{\sin^2(u)}{r^2} + (\partial_r u)^2 \right) r dr$$

Note that the energy is invariant if we replace u by u_λ defined by $u_\lambda(t, r) = u(\lambda t, \lambda r)$, for $\lambda > 0$. Such a scaling transformation is also a symmetry of the wave maps equation above.

Given a Lie Group G, the (free) Yang-Mills equation we will consider is an equation for a $Lie(G)$-valued one-form A defined on $\mathbb{R}^{1+d}$. We consider the Yang-Mills equation in $1+4$ dimensions, with gauge group $SO(4)$. Therefore, A (which is sometimes called the gauge field) is a Lie($SO(4)$)-valued one-form on $\mathbb{R}^{1+4}$. We write $A = A_\mu dx^\mu$, where, for each μ, A_μ is a Lie($SO(4)$)-valued function, defined on $\mathbb{R}^{1+4}$. Defining F, a Lie($SO(4)$)-valued two-form on $\mathbb{R}^{1+4}$ by

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

the Yang-Mills equation can be written as

$$-\partial_t F_{0\nu} - [A_0, F_{0\nu}] + \sum_{\mu=1}^{4} (\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}]) = 0, \qquad \text{for } \nu = 0, 1, 2, 3, 4$$

where 0 on the right-hand is the zero in Lie($SO(4)$).

The Yang-Mills equation is also invariant under gauge transformations, which are transformations of A of the form

$$A_\mu \to gA_\mu g^{-1} - \partial_\mu g g^{-1}$$

where $g : \mathbb{R}^{1+4} \to SO(4)$.

We make the equivariant ansatz (which has also been studied in prior works)

$$A_\mu^{i,j}(t, x) = \left(\delta_\mu^i x^j - \delta_\mu^j x^i\right) \left(\frac{u(t, |x|) - 1}{|x|^2}\right), \quad 0 \leqslant \mu \leqslant 4, \quad 1 \leqslant i, j \leqslant 4$$

and the Yang-Mills equation reduces to a single semilinear wave equation

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u + \frac{2u(1 - u^2)}{r^2} = 0$$

This wave equation conserves the following energy

$$E_{YM}(u, \partial_t u) = \frac{1}{2}\int_0^\infty \left((\partial_t u)^2 + (\partial_r u)^2 + \frac{(1-u^2)^2}{r^2}\right) r dr$$

which is invariant under the scaling symmetry:

$$u \to u_\lambda, \quad \text{where } u_\lambda(t,r) = u(\lambda t, \lambda r)$$

Each equation mentioned above admits time-independent, smooth solutions, with localized energy density, called solitons. For the wave maps problem considered above, the soliton is given by

$$Q_1(r) = 2\arctan(r)$$

For the Yang-Mills problem, the soliton is

$$Q_1(r) = \frac{1-r^2}{1+r^2}$$

By applying the aforementioned scaling symmetry to Q_1, one obtains a family of soliton solutions, Q_λ for $\lambda > 0$, given by

$$Q_\lambda(r) = Q(r\lambda)$$

As mentioned earlier, we consider energy critical equations, so all Q_λ have the same energy.

In this thesis, we study global in time soliton dynamics for single solitons coupled to radiation. More precisely, we construct globally defined solutions, u, which can be decomposed as follows.

$$u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + f(t,r) + v_e(t,r) \tag{0.1}$$

The function f represents radiation coupled to the soliton with time-dependent scale, and is a solution to the following linear wave equation

$$-\partial_{tt} f + \partial_{rr} f + \frac{1}{r}\partial_r f - \frac{f}{r^2} = 0, \quad \text{for wave maps}$$
$$-\partial_{tt} f + \partial_{rr} f + \frac{1}{r}\partial_r f - \frac{4}{r^2} f = 0, \quad \text{for Yang-Mills}$$

(In the main body of the thesis, f will be denoted as v_2 in the wave maps work, but as v_1 in the Yang-Mills work). The function v_e appearing in (0.1) is a correction which is small in an appropriate sense as time approaches infinity. We also provide a precise relation between the asymptotics of the time-dependent soliton length scale, $\lambda(t)$, and the coupled radiation f.

For the wave maps equation, for any choice of function $\lambda_0(t)$ in an appropriate symbol class, we construct a solution as described above, with $\lambda(t)$ asymptotically equal to $\lambda_0(t)$. Some examples of the λ_0 in our symbol class are

$$\lambda_0(t) = \frac{1}{\log^b(t)}, \quad \lambda_0(t) = \frac{2+\sin(\log(\log(t)))}{\log^b(t)}, \quad t \gg 1$$

For the Yang-Mills equation mentioned above, we construct a class of solutions as in (0.1). The main difference between the wave maps result, and the result here is that $\lambda(t)$ is asymptotically constant for all of our solutions to the Yang-Mills problem. This is true, even though our set of solutions includes ones for which the radiation f in (0.1) can be quite large, and in fact "logarithmically" close to having infinite energy. In fact, in the setup of this work, the soliton length scale asymptoting to a constant is a necessary condition for the radiation f to have finite energy. Another interesting point of this construction is that, for each choice of f in our admissible class of functions, there exists a one-parameter family of solutions as in (0.1) with $\lambda(t)$ having any asymptotic value.

Contents

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1 Background Material

In this thesis, we study two energy critical semilinear wave equations. Specifically, we study a particular wave maps equation (which is referred to as a nonlinear sigma model in many physics contexts) and a Yang-Mills equation. As will be described in more detail later on, the wave maps and Yang-Mills problems are families of equations, parameterized by various choices of domain and target manifolds, etc. The wave maps equation is geometric in the sense that the equation is independent of choice of coordinates on the domain and target manifolds. The Yang-Mills problem can be regarded as a geometric problem because it is invariant under gauge transformations. The specific symmetry reductions of these problems which we study are semilinear because their nonlinear terms involve only the unknown function, rather than its derivatives. Finally, both equations we consider admit a conserved energy and a scaling symmetry. The energy is invariant with respect to the scaling symmetry in the dimensions in which we study each equation, which is why the equations are said to be energy critical. The specific equations we consider admit smooth, time-independent solutions with localized energy density, called solitons. (Solitons appear in a variety of classical field theories in physics, as described in [20]). The soliton solutions of these equations will play a major role in the work of this thesis. We will now describe the equations considered in more detail.

1.1 Wave Maps

The (scalar) linear wave equation considers functions $\Phi : \mathbb{R}^{1+d} \to \mathbb{R}$. The wave maps problem is an extension of this wave equation to the case where the unknown function maps a Lorentzian manifold into a Riemannian manifold, (M, g). The wave maps equation can be derived from the least action principle applied to the following natural extension of the usual wave equation action. (The reference [6] contains an introduction to the wave maps problem). Choosing Minkowski space with metric $m = \text{diag}(-1, 1, 1, \dots, 1)$ as the domain, the wave maps action for maps $\Phi : \mathbb{R}^{1+d} \to (M, g)$ is

$$\mathcal{S}(\Phi) = \int_{\mathbb{R}^{1+d}} \langle \partial_\alpha \Phi(t,x), \partial^\alpha \Phi(t,x) \rangle_{g(\Phi(t,x))} dt dx$$

where the α indices are contracted using the Minkowski metric. We will now describe the particular wave maps equation, and its symmetry reduction, which is studied in this work. First of all, we will focus on the case $(M, g) = (\mathbb{S}^2, \mathring{g})$, where $\mathring{g}$ denotes the usual round metric on $\mathbb{S}^2$. We will regard Φ as a map into $\mathbb{R}^3$ with unit Euclidean norm. To fix conventions, we let

$$\Box_{\mathbb{R}^{1+d}} = \partial_t^2 - \Delta_{\mathbb{R}^d} = -\partial^\alpha \partial_\alpha$$

The linear wave equation for functions $\Phi : \mathbb{R}^{1+d} \to \mathbb{R}$ is

$$\Box_{\mathbb{R}^{1+d}} \Phi(t,x) = 0$$

On the other hand, for the choice of target manifold above, the wave maps equation can be written as

$$\Box_{\mathbb{R}^{1+d}} \Phi(t,x) \perp T_{\Phi(t,x)} \mathbb{S}^2 \subset \mathbb{R}^3$$

or equivalently, as

$$\Box_{\mathbb{R}^{1+d}}\Phi(t,x) = \Phi(t,x)\left(\partial^{\alpha}\Phi(t,x)\cdot\partial_{\alpha}\Phi(t,x)\right)$$

where $\cdot$ is the Euclidean inner product on $\mathbb{R}^3$, and we re-iterate that Φ is regarded as a unit norm map into $\mathbb{R}^3$.

Sufficiently regular solutions to the wave maps equation in this context conserve the energy

$$En(\Phi) = \frac{1}{2}\int_{\mathbb{R}^d}\left(|\partial_t\Phi(t,x)|^2 + \sum_{i=1}^{d}|\partial_i\Phi(t,x)|^2\right)dx$$

where, for $y \in \mathbb{R}^3$, $|y|$ denotes the Euclidean norm of y. Note that, if $\lambda > 0$, and Φ is a solution to the wave maps equation, then, so is Φ_{λ} defined by

$$\Phi_{\lambda}(t,r) = \Phi(\lambda t, \lambda r)$$

and

$$En(\Phi_{\lambda}) = En(\Phi), \qquad \text{for all } \lambda > 0 \text{ when } d = 2$$

For this reason, the wave maps problem considered in two spatial dimensions is called energy critical. In this thesis, we consider the energy critical case of $d = 2$. In summary, we consider the energy critical wave maps problem with target $\mathbb{S}^2$, equipped with the usual round metric.

Next, we recall that, for a finite energy map $F : \mathbb{R}^2 \to \mathbb{S}^2$, one may define the integer-valued topological degree, $N(F)$ (see, e.g. [12], and references therein) by

$$4\pi N(F) = \int_{\mathbb{R}^2} F \cdot (\partial_1 F \times \partial_2 F)\, dx \tag{1.1}$$

The main wave maps result of this thesis deals with topological degree one solutions. For sufficiently regular $F : \mathbb{R}^2 \to \mathbb{S}^2$, we have

$$En(F) \geqslant 4\pi|N(F)|$$

This can be seen in the following way (see [27], [20]). If we let R be the complexified stereographic projection coordinate representative of F:

$$R_F(x_1,x_2) = \frac{F_1(x_1,x_2) + iF_2(x_1,x_2)}{1+F_3(x_1,x_2)}, \quad F_i(x_1,x_2) = F(x_1,x_2)\cdot \mathbf{e}_i$$

then,

$$En(F) = 4\int_{\mathbb{R}^2}\frac{\left(|\frac{(\partial_1+i\partial_2)R_F}{2}|^2 + |\frac{(\partial_1-i\partial_2)R_F}{2}|^2\right)}{(1+|R_F|^2)^2}dx$$

and

$$4\pi N(F) = 4\int_{\mathbb{R}^2}\frac{\left(-|\frac{(\partial_1+i\partial_2)R_F}{2}|^2 + |\frac{(\partial_1-i\partial_2)R_F}{2}|^2\right)}{(1+|R_F|^2)^2}dx$$

If we complexify the domain of R_F, then, we see that any F such that R_F is holomorphic or anti-holomorphic, satisfies $En(F) = 4\pi|N(F)|$. These maps F are all time-independent solutions to the wave maps equation with $\mathbb{S}^2$ target, and can be regarded as soliton solutions to the wave maps equation. Since, for any sufficiently regular G, we have $En(G) \geqslant 4\pi|N(G)|$, these F have minimal energy within the class of maps of a given topological degree. The soliton which will play an important role in our work corresponds to the choice

$$R(x_1, x_2) = x_1 + ix_2 \tag{1.2}$$

We can now describe a "threshold theorem" for the energy critical wave maps problem. The works of Sterbenz and Tataru, [32] and [33] (which apply to much more general targets than $\mathbb{S}^2$) show that for data with energy strictly less than that of the lowest energy non-trivial harmonic map, one has global well-posedness and scattering. (In the case of the $\mathbb{S}^2$ target, the topological degree of any solution to the wave maps equation with such data is zero). More precisely, their result is (see also [12])

Theorem 1.1 (Sterbenz, Tataru, [32],[33])**.** *For the wave maps equation from $\mathbb{R}^{2+1}$ to a compact Riemannian manifold (M, g), the following is true.*

a. If there are no non-constant harmonic maps into (M, g), then, one has global well-posedness and scattering for large data in $\dot{H}^1 \times L^2$.

b. If E_0 is the smallest possible energy of a nontrivial harmonic map into (M, g), then, one has global well-posedness and scattering for all data in $\dot{H}^1 \times L^2$ with energy strictly less than E_0

In the case of the $\mathbb{S}^2$ target in the 1-equivariant setting (which is a symmetry reduction that will be further discussed below), Côte, Kenig, Lawrie, and Schlag [2] showed that global existence and scattering holds for smooth, topological degree zero data with energy less than twice that of the soliton described by (1.2) (which is the appropriate threshold in this setting). An analogous result, but without the equivariance assumption, and for slightly more general targets than $\mathbb{S}^2$, is implied by the work of Lawrie and Oh [19].

Finally, in this thesis, we make a symmetry reduction of the problem, and consider solutions which are 1-equivariant, by using polar coordinates (r, φ) on $\mathbb{R}^2$, and writing

$$\Phi_u(t, r, \varphi) = \begin{pmatrix} \cos(\varphi)\sin(u(t,r)) \\ \sin(\varphi)\sin(u(t,r)) \\ \cos(u(t,r)) \end{pmatrix}$$

Geometrically, this means that if R is a rotation in two spatial dimensions, $R \in SO(2)$, then $\Phi_u(t, Rx) = H(R)\Phi_u(t, x)$, where

$$H(R) = \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right] \in SO(3)$$

is the corresponding rotation in three spatial dimensions, preserving the x_3 coordinate.

We return to the topological degree defined in (1.1), and note that, for sufficiently regular u satisfying, for all t, $u(t,0)=0$ and $\lim_{r\to\infty} u(t,r)=\pi$, we have

$$N(\Phi_u(t)) = 1$$

Under the 1-equivariant symmetry reduction, the wave maps equation reduces to the following scalar, semilinear wave equation, which is the focus of the wave maps portion of this work.

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = 0 \tag{1.3}$$

We consider the above equation with the conditions $u(t,0)=0, \lim_{r\to\infty} u(t,r)=\pi$. In total, we thus consider the energy critical wave maps equation with $\mathbb{S}^2$ target for topological degree one, 1-equivariant maps.

In [31], Shatah and Tahvildar-Zadeh studied the local well-posedness of (a more general problem which includes) the Cauchy problem associated to (1.3), with data (u_0,u_1) such that

$$(x_1,x_2) \mapsto \left(\frac{x_1 u_0(r)}{r}, \frac{x_2 u_0(r)}{r}\right) \in H^1_{\text{loc}}(\mathbb{R}^2)$$
$$(x_1,x_2) \mapsto \left(\frac{x_1 u_1(r)}{r}, \frac{x_2 u_1(r)}{r}\right) \in L^2_{\text{loc}}(\mathbb{R}^2)$$

The special case of their theorem which is relevant for us is the following. Let

$$N = [0,\pi) \times \mathbb{S}^1, \quad \text{with the metric } g = d\theta^2 + \sin^2(\theta)d\phi^2, \quad (\theta,\phi) \in [0,\pi) \times \mathbb{S}^1$$

Then, consider the following Cauchy problem with

$$U^1(t,x) = \theta(t,x)\cos(\phi(t,x)), \quad U^2(t,x) = \theta(t,x)\sin(\phi(t,x))$$

$$\begin{cases} \partial_\mu \partial^\mu U^a + \sum_{b,c=1}^2 \Gamma^a_{bc}(U)\partial_\mu U^b \partial^\mu U^c = 0 \\ U(0,x) = U_0(x) \\ \partial_t U(0,x) = U_1(x) \end{cases} \tag{1.4}$$

where Γ are the Christoffel symbols of N. If the data satisfy the equivariant condition:

$$U_0^j = x_j \frac{u_0(r)}{r}, \quad U_1^j = x_j \frac{u_1(r)}{r}, \quad j=1,2$$

then, Shatah and Tahvildar-Zadeh proved

Theorem 1.2 (Shatah, Tahvildar-Zadeh, [31])**.** *There exists $T^* > 0$ such that the equivariant wave maps equation,* (1.4)*, considered with data (U_0,U_1) such that*

$$U_0 \in H^1_{loc}(\mathbb{R}^2, N), \quad U_1 \in L^2_{loc}(\mathbb{R}^2, TN)$$

has a unique solution U satisfying, for all $z_0 = (t_0,x_0) \in [0,T^*) \times \mathbb{R}^2$,

$$U \in L^\infty([0,t_0), H^1(D(t;z_0),N)) \cap L^{\frac{10}{3}}([0,t_0), \dot{B}^{\frac{1}{2},\frac{10}{3}}(D(t;z_0),N))$$
$$\partial_t U \in L^\infty(([0,t_0), L^2(D(t;z_0),TN))$$

where, for $z_0 = (t_0, x_0)$,

$$D(t; z_0) = \{z = (t, x) \in \mathbb{R}^{1+2} || x - x_0| \leqslant t_0 - t\}$$

We note that the energy for the equivariant reduction of the wave maps equation is

$$En(\Phi_u) = \pi \int_0^\infty \left((\partial_t u)^2 + \frac{\sin^2(u)}{r^2} + (\partial_r u)^2 \right) r dr$$

We also remark that the soliton described by (1.2) corresponds to a 1-equivariant map from $\mathbb{R}^2$ to $\mathbb{S}^2$, with polar angle given by

$$Q_1(r) = 2 \arctan(r)$$

1.2 Yang-Mills

Given a Lie Group G, the (free) Yang-Mills equation we will consider is an equation for a $Lie(G)$-valued one-form A defined on $\mathbb{R}^{1+d}$. We consider the Yang-Mills equation in $1+4$ dimensions, with gauge group $SO(4)$. Therefore, A (which is sometimes called the gauge field) is a Lie$(SO(4))$-valued one-form on $\mathbb{R}^{1+4}$. We write $A = A_\mu dx^\mu$, where, for each μ, A_μ is a Lie$(SO(4))$-valued function, defined on $\mathbb{R}^{1+4}$. Defining F, a Lie$(SO(4))$-valued two-form on $\mathbb{R}^{1+4}$ by

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

the Yang-Mills equation can be written as

$$-\partial_t F_{0\nu} - [A_0, F_{0\nu}] + \sum_{\mu=1}^{4} (\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}]) = 0, \quad \text{for } \nu = 0, 1, 2, 3, 4$$

where 0 on the right-hand is the zero in Lie$(SO(4))$. The Yang-Mills equation has the conserved energy

$$E_{YM} = -\frac{1}{48\pi^2} \int_{\mathbb{R}^4} \text{Tr} \left(F_{\mu\nu}(t, x) F_{\mu\nu}(t, x) \right) dx$$

The equation is invariant under the scaling symmetry

$$A_\mu(t, x) \to \lambda A_\mu(\lambda t, \lambda x)$$

The components of F transform under this symmetry as

$$F_{\mu\nu}(t, x) \to \lambda^2 F_{\mu\nu}(\lambda t, \lambda x)$$

which means that the energy E_{YM} is invariant under the scaling symmetry, because the equation is considered in 4 spatial dimensions. The Yang-Mills equation is also invariant under gauge transformations, which are transformations of A of the form

$$A_\mu \to g A_\mu g^{-1} - \partial_\mu g g^{-1}$$

where $g : \mathbb{R}^{1+4} \to SO(4)$. In particular, if A is a solution to the Yang-Mills equation, and $g : \mathbb{R}^{1+4} \to SO(4)$ satisfies

$$g(t,x) = Id, \quad \text{in a neighborhood of } t = 0$$

but $g(t,x)$ is *not* globally constant, and, for some $t_0 > 0$, $g(t_0,x) = Id$ with $\partial_\mu g(t_0,x) \neq 0$ then, $A^{(1)}_\mu(t,x) := g(t,x)A_\mu(t,x)g^{-1}(t,x) - \partial_\mu g(t,x)g^{-1}(t,x)$ is a solution to the Yang-Mills equation, with $A^{(1)}(0,x) = A(0,x)$ and $\partial_\mu A^{(1)}(0,x) = \partial_\mu A(0,x)$, but $A^{(1)}(t_0,x) \neq A(t_0,x)$. Therefore, one can not regard the Yang-Mills equation as a well-defined evolution equation for A, in the sense that there can exist distinct gauge fields solving the Yang-Mills equation with the same Cauchy data. Instead, one can regard solutions to the Yang-Mills equation as equivalence classes of gauge fields under the equivalence relation: two gauge fields are equivalent if and only if there exists a gauge transformation relating the two. The Yang-Mills equation can then be regarded as an equation which evolves Cauchy data into the space of equivalence classes of gauge fields.

When studying the Yang-Mills equation, one can then look for solutions whose equivalence class is represented by a gauge field A satisfying certain properties. For example (see [24] for more discussion), one can work in the temporal gauge, where $A_0 = 0$. The symmetry reduction we consider in this thesis is a reduction to a case satisfying, among other things, that $A_0 = 0$. One could also work in the Lorenz gauge where

$$\partial^\mu A_\mu = 0$$

or the Coulomb gauge, where

$$\sum_{k=1}^{4} \partial_k A_k = 0$$

Small energy global well posedness for the $(4+1)$ dimensional Yang-Mills problem was established by Krieger and Tataru, [18]. More precisely, in [18], the Yang-Mills equation in the Coulomb gauge is written as a Cauchy problem for the spatial components of A (with initial data $(A_{0,j}, A_{1,j}) = (A_j(0), \partial_t A_j(0)) \in \dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$) coupled to A_0, which solves an equation without any time derivatives. Then, the following theorem is proved.

Theorem 1.3 (Krieger, Tataru, [18])**.** *The Yang-Mills equation in the Coulomb gauge is globally well-posed for data $(A_{0,j}, A_{1,j})$ which is small in $\dot{H}^1(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$*

In addition, the works of Tataru and Oh, [24], [25], [21], [22], [23], established a threshold theorem and dichotomy theorem for this problem, with gauge group given by any compact, non-abelian Lie group. In order to more precisely state their result, we first note that, as stated in [24], there exists a minimal energy non-trivial solution (called a harmonic Yang-Mills connection) to

$$\sum_{k=1}^{4} \left(\partial_j F_{jk} + [A_j, F_{jk}]\right) = 0, \quad \text{in } \mathbb{R}^4, \quad 1 \leqslant j \leqslant 4$$

Let E_{GS} denote the energy of such a solution. We will also need the following observation stated in [24]: a finite energy connection A on $\mathbb{R}^4$ is topologically trivial if and only if $A \in \dot{H}^1$ in a suitable gauge. Then, the threshold theorem is

Theorem 1.4 (Oh,Tataru, [24])**.** *One has global wellposedness and scattering for the Yang-Mills equation in $\mathbb{R}^{1+4}$ for all topologically trivial initial data with energy below $2E_{GS}$.*

On the other hand, the following "dichotomy theorem" regards solutions which may not satisfy the assumptions of the threshold theorem.

Theorem 1.5 (Oh, Tataru, [24])**.** *The Yang-Mills equation in $\mathbb{R}^{1+4}$ is locally well posed in the energy space. The maximal solution satisfies either*
1. The solution is topologically trivial, globally defined, and scatters at infinity.
2. The solution "bubbles off" a soliton either at a finite blow-up time, or at infinity.

Roughly speaking, the solution A "bubbles off" a soliton at a finite time t_0, if there exists a convergent sequence of points (t_n, x_n), with $\lim_{n\to\infty} t_n = t_0$, a Lorentz transformation, L, and a harmonic Yang-Mills connection Q, such that an appropriate re-scaling and gauge transformation of A, translated by (t_n, x_n) converges to $L(Q)$. Bubbling off a soliton at infinity is defined similarly, except that the sequence (t_n, x_n) is not convergent. See [24] for the precise definitions.

With the equivariant ansatz (see also [28], [15])

$$A_\mu^{i,j}(t,x) = \left(\delta_\mu^i x^j - \delta_\mu^j x^i\right)\left(\frac{u(t,|x|)-1}{|x|^2}\right), \quad 0 \leqslant \mu \leqslant 4, \quad 1 \leqslant i,j \leqslant 4$$

the Yang-Mills equation reduces to

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u + \frac{2u(1-u^2)}{r^2} = 0 \tag{1.5}$$

The energy E_{YM} reduces to the following quantity, which is conserved by the above reduction of the Yang-Mills equation.

$$E_{YM}(u,\partial_t u) = \frac{1}{2}\int_0^\infty \left((\partial_t u)^2 + (\partial_r u)^2 + \frac{(1-u^2)^2}{r^2}\right) r dr$$

Another way to understand the equivariant ansatz above is to note that a similar ansatz for the $SU(2)$ Yang-Mills equation in $1+4$ dimensions also gives rise to (1.5). In particular, if we make the ansatz (see also [34])

$$A_0 = 0, \quad A_j = \frac{-1}{2}\left(\frac{u(t,r)-1}{r^2}\right)\sum_{i=1}^{3}\sum_{k=1}^{4}(\eta_i)_{j,k}x_k\sigma_i, \quad 1 \leqslant j \leqslant 4$$

with η_i being 't Hooft matrices

$$\eta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and σ_i being Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

then, the $1+4$ dimensional Yang-Mills equation with gauge group $SU(2)$ reduces to (1.5). One reason why the $SU(2)$ perspective is a useful one to take, is that the following soliton solution to (1.5) (which will be important for the work of this thesis):

$$Q_1(r) = \frac{1-r^2}{1+r^2} \tag{1.6}$$

corresponds to a gauge field for the $SU(2)$ Yang-Mills problem in $1+4$ dimensions, whose associated F satisfies, for $1 \leqslant i,j \leqslant 4$, $F_{ij} = \frac{1}{2}\sum_{l,m=1}^{4} \epsilon_{ijlm} F_{lm}$ where ϵ is the Levi-Civita tensor, with $\epsilon_{1234} = 1$. The topological properties of the gauge field for this soliton are described by the second Chern number

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} Tr(\overline{F} \wedge \overline{F})$$

where, $\overline{F}_{jk} = F_{jk}$ for $1 \leqslant j,k \leqslant 4$. This implies that the gauge field associated to (1.6) is a minimizer of the energy within a class of functions with a given second Chern number, because, from the antisymmetry of F_{jk}, we have $\sum_{k,j=1}^{4} Tr(\overline{F}_{jk}\overline{F}_{jk}) = \sum_{k,j=1}^{4} Tr(*\overline{F}_{jk} * \overline{F}_{jk})$, where $(*\overline{F})_{ij} = \frac{1}{2}\sum_{k,l=1}^{4} \overline{F}_{kl}\epsilon_{ijkl}$. Therefore,

$$\begin{aligned} E_{YM} &= \frac{-1}{48\pi^2}\sum_{\mu,\nu=0}^{4}\int_{\mathbb{R}^4} Tr(F_{\mu\nu}F_{\mu\nu})dx \\ &\geqslant -\frac{1}{96\pi^2}\sum_{j,k=1}^{4}\int_{\mathbb{R}^4}\left(Tr\left((\overline{F}_{jk} - *\overline{F}_{jk})(\overline{F}_{jk} - *\overline{F}_{jk})\right) + 2Tr\left(\overline{F}_{jk} * \overline{F}_{jk}\right)\right)dx \\ &\geqslant -\frac{1}{48\pi^2}\sum_{j,k=1}^{4}\int_{\mathbb{R}^4} Tr\left(\overline{F}_{jk} * \overline{F}_{jk}\right)dx \end{aligned}$$

For completeness, we note that the second Chern number of the gauge field associated to the soliton mentioned above is -1:

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} Tr\left(\overline{F} \wedge \overline{F}\right) = -\frac{2\pi^2}{8\pi^2}\int_0^\infty \frac{48p^3}{(p^2+1)^4}\,dp = -1$$

1.3 Summary of main results

We recall the soliton solutions mentioned above. For the wave maps problem, the soliton is given by

$$Q_1(r) = 2\arctan(r)$$

For the Yang-Mills problem, the soliton is

$$Q_1(r) = \frac{1-r^2}{1+r^2}$$

By applying the aforementioned scaling symmetry to Q_1, one obtains a family of soliton solutions, Q_λ for $\lambda > 0$. As mentioned earlier, we consider energy critical equations, so all Q_λ have the same energy. In appropriate regimes, it is known that if one has a globally defined solution to the equations under consideration, then, it can be decomposed as a soliton with a potentially time-dependent scaling parameter, $Q_{\frac{1}{\lambda(t)}}$, coupled to a solution to an appropriate *linear* wave equation (called radiation), plus corrections which are small in an appropriate sense, as time approaches infinity.

More precisely, such a solution u can be decomposed as follows.

$$u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + f(t,r) + v_e(t,r) \tag{1.7}$$

The function f represents radiation coupled to the soliton with time-dependent scale, and is a solution to the following linear wave equation

$$\begin{aligned} &-\partial_{tt}f + \partial_{rr}f + \frac{1}{r}\partial_r f - \frac{f}{r^2} = 0, \quad \text{for wave maps} \\ &-\partial_{tt}f + \partial_{rr}f + \frac{1}{r}\partial_r f - \frac{4}{r^2}f = 0, \quad \text{for Yang-Mills} \end{aligned} \tag{1.8}$$

The function v_e is a correction which is small in an appropriate sense as time approaches infinity. We remark that the following energy is conserved by any sufficiently regular solution to (1.8).

$$E(u,\partial_t u) = \begin{cases} \pi \int_0^\infty \left((\partial_t u)^2 + (\partial_r u)^2 + \frac{u^2}{r^2}\right) rdr, & \text{for wave maps} \\ \pi \int_0^\infty \left((\partial_t u)^2 + (\partial_r u)^2 + \frac{4u^2}{r^2}\right) rdr, & \text{for Yang-Mills} \end{cases}$$

(In Section 8, regarding the Yang-Mills equation, E appears with a slightly different normalization, but this is not important).

Even though it is known that in appropriate regimes, if one has a global solution u, then it can be decomposed as in (1.7), to the author's knowledge, there were no known actual examples of such solutions with "interesting" asymptotics of $\lambda(t)$ for the equations under consideration. (This will be made more precise for each equation later on). In addition, a relation between $\lambda(t)$ and the associated radiation was not known, to the author's knowledge. The main result of this thesis is to construct a large class of solutions of the above form, with the following properties. For the wave maps equation under consideration, we can construct solutions with a symbol class of possible choices of $\lambda(t)$ (all of which satisfy $\lambda(t) \to 0$ as t approaches infinity) by obtaining a precise relation between the radiation and the asymptotics of $\lambda(t)$.

In particular, we have the following. For $b > 0$, let Λ_b be the set of $f \in C^\infty([100,\infty))$ such that there exist $C_l, C_m, C_{m,k} > 0$ with

$$\frac{C_l}{\log^b(t)} \leqslant f(t) \leqslant \frac{C_m}{\log^b(t)}, \quad |f^{(k)}(t)| \leqslant \frac{C_{m,k}}{t^k \log^{b+1}(t)}, k \geqslant 1, \quad t \geqslant 100$$

Then, the main result of this thesis regarding wave maps is (see also Theorem 2.2, which has slightly more information)

Theorem 1.6. *Let $b > 0$. For all $\lambda_{0,0,b} \in \Lambda_b$, there exists $T_0 > 100$ and a (real-valued) finite energy solution u_b to (1.3) for $t \geqslant T_0$, of the form*

$$u_b(t,r) = Q_{\frac{1}{\lambda_b(t)}}(r) + v_2(t,r) + v_e(t,r)$$

where $\lambda_b \in C^4([T_0,\infty))$,

$$-\partial_{tt}v_2 + \partial_{rr}v_2 + \frac{1}{r}\partial_r v_2 - \frac{v_2}{r^2} = 0$$

$$E(v_e, \partial_t\left(Q_{\frac{1}{\lambda_b}} + v_e\right)) \leqslant \frac{C}{t^2\log^{2b}(t)}, \quad t \geqslant T_0$$

and

$$\lambda_b(t) = \lambda_{0,0,b}(t) + O\left(\frac{1}{\log^b(t)\sqrt{\log(\log(t))}}\right)$$

In addition to solutions corresponding to $\lambda_{0,0,b}(t) = \frac{1}{\log^b(t)}$ (for t sufficiently large) our class of solutions includes ones for which the leading part of λ has some oscillations, such as the example

$$\lambda_{0,0,b}(t) = \frac{2+\sin(\log(\log(t)))}{\log^b(t)}, \qquad t \geqslant 100$$

Moreover, the radiation, v_2 is uniquely determined by its Cauchy data at $t=0$, which is related to the leading part of $\lambda(t)$ as follows.

$$v_2(0,r) = 0, \quad \partial_t v_2(0,r) = v_{2,0}(r)$$

with

$$\widehat{v_{2,0}}(\xi) = -\frac{1}{\xi\pi}\int_0^\infty F(t)\sin(t\xi)dt$$

where

$$F(t) = \left(4\int_t^\infty \frac{\lambda_{0,0,b}''(s)ds}{1+s-t}\right), \qquad \text{for all } t \text{ sufficiently large}$$

Here, $\widehat{\cdot}$ denotes the Hankel transform of order 1:

$$\widehat{f}(\xi) = \int_0^\infty f(r)J_1(r\xi)rdr$$

For the Yang-Mills equation, we consider a large class of finite energy radiation, including functions which are "logarithmically" close to having infinite energy. We then construct solutions as in (1.7) by obtaining a precise relation between $\lambda(t)$ and the radiation. In this case, it turns out that for a given radiation, there exists a one-parameter family of corresponding choices of $\lambda(t)$, and all such $\lambda(t)$ are asymptotically constant in time, despite the "largeness" of the radiation mentioned above. In fact, in our setup, $\lambda(t)$ being asymptotically constant in time is a necessary condition for the radiation to have finite energy.

In particular, we have the following. For $b > \frac{2}{3}$, let F_b denote the set of functions f such that there exists $M > 50$, and $C_{f,k} > 0$, such that

$$f \in C^\infty([M,\infty)), \quad |f^{(k)}(t)| \leqslant \frac{C_{f,k}}{t^k\log^b(t)}, \text{ for } t \geqslant M \text{ and } k \geqslant 0$$

The class of radiation components, v_1, of our solutions can be labeled by F_b in the following way. For $f \in F_b$, we have

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi^2}\int_0^\infty \frac{(\psi \cdot f)'(t)}{t}\sin(t\xi)dt$$

(where ψ is an unimportant cutoff function, which is equal to 1 for all sufficiently large arguments), and the radiation profile v_1 is given by

$$\begin{cases} -\partial_{tt}v_1 + \partial_{rr}v_1 + \frac{1}{r}\partial_r v_1 - \frac{4v_1}{r^2} = 0 \\ v_1(0) = 0 \\ \partial_t v_1(0) = v_{1,1} \end{cases}$$

Here, $\widehat{\cdot}$ denotes the Hankel transform of order two

$$\widehat{v_{1,1}}(\xi) = \int_0^\infty v_{1,1}(r)J_2(r\xi)rdr$$

In order to describe the leading order behavior of $\lambda(t)$, we introduce the following family of functions. For $b > \frac{2}{3}$, let Λ_b denote the set of functions λ_0 for which there exists $T_{\lambda_0} > 50$ such that $\lambda_0 \in C^\infty([T_{\lambda_0}, \infty))$, and the following two conditions hold: Firstly, there exists $f \in F_b$ such that

$$\frac{\lambda_0''(t)}{\lambda_0(t)} = \frac{f'(t)}{t}, \quad t \geqslant T_{\lambda_0} \tag{1.9}$$

Secondly,

$$\lambda_0(t) > 0, \quad \frac{|\lambda_0'(t)|}{\lambda_0(t)} \leqslant \frac{C}{t\log^b(t)}, \quad t \geqslant T_{\lambda_0} \tag{1.10}$$

Note that the above conditions on λ_0 imply that $\lambda_0(t) \to \lambda_1 > 0$ as $t \to \infty$, despite the fact that some $\lambda_0 \in \Lambda_b$ (for $b \leqslant 1$) satisfy

$$\int_t^\infty \int_x^\infty \frac{|\lambda_0''(s)|}{\lambda_0(s)} dsdx = \infty$$

To see this, we write

$$f(t) = -\lim_{M\to\infty}\int_t^M \frac{s\lambda_0''(s)}{\lambda_0(s)}ds = -\lim_{M\to\infty}\int_t^M \frac{\frac{d}{ds}\left(\lambda_0'(s)s - \lambda_0(s)\right)}{\lambda_0(s)}ds, \quad t > T_{\lambda_0}$$

Integrating by parts and using the assumptions on $\frac{|\lambda_0'(s)|}{\lambda_0(s)}$, and the fact that $b > \frac{2}{3}$, we see that

$$\lim_{M\to\infty}\log(\lambda_0(M)) < \infty$$

Before we state the main result, we remark that, given any $f \in F_b$, there exists $T_{\lambda_0} > 50$, and a one-parameter family of $\lambda_0 \in \Lambda_b$ satisfying (1.9) and (1.10). This can be seen as follows. Given $f \in F_b$, we can first find ω satisfying

$$\omega'(t) + \omega(t)^2 = \frac{f'(t)}{t}, \quad |\omega(t)| \leqslant \frac{C}{t\log^b(t)}, \quad t \geqslant N$$

(where $N > 50$ is sufficiently large) with a fixed point argument. By inspection of this equation, $\omega \in C^{\infty}([N,\infty))$. Then, we can define $T_{\lambda_0} = N+1$, and let λ_0 be given by

$$\lambda_0(t) = c\exp\left(\int_{N+1}^{t} \omega(s)ds\right), \quad t \geqslant N+1, \quad \text{any } c > 0$$

Then, we have (1.9) and (1.10).

To clarify the relation between the radiation and the leading behavior of λ, we note that, for a given $f \in F_b$, and any $\lambda_0 \in \Lambda_b$ satisfying (1.9) and (1.10), the radiation is uniquely specified by its initial velocity, which is determined from

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi^2}\int_0^{\infty} \frac{(\psi \cdot f)'(t)}{t}\sin(t\xi)dt$$

where, since $\psi(t) = 1$ for all t sufficiently large,

$$\frac{(\psi \cdot f)'(t)}{t} = \frac{f'(t)}{t} = \frac{\lambda_0''(t)}{\lambda_0(t)}, \quad t \text{ sufficiently large}$$

An interesting feature of our solutions is that the radiation v_1 depends only on f (as per the formula for $v_{1,1}$ given above) which is invariant with respect to multiplying λ_0 by a constant. As we just showed, there is a one-parameter family of $\lambda_0 \in \Lambda_b$, corresponding to a given $f \in F_b$. In particular, our family of solutions includes functions of the form $Q_{\frac{1}{\lambda(t)}}(r) + v_1(t,r) + o(1)$, for a one-parameter family of possible asymptotic values of $\lambda(t)$, and the **same** v_1.

Our main result is

Theorem 1.7. *For all $b > \frac{2}{3}$ and $f \in F_b$, let λ_0 be any element of Λ_b satisfying* (1.9)*. Then, there exists $T_0 = T_0(\lambda_0)$ and a finite energy solution, u, to* (1.5)*, with the following properties.*

$$u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_1(t,r) + v_e(t,r)$$

where $\lambda(t) \in C^4([T_0,\infty))$

$$-\partial_{tt}v_1 + \partial_{rr}v_1 + \frac{1}{r}\partial_r v_1 - \frac{4}{r^2}v_1 = 0$$

$$E(v_e, \partial_t v_e) < \frac{C}{\log^{4b-2}(t)}, \quad t \geqslant T_0$$

and, for some $\epsilon_0 > 0$,

$$\lambda(t) = \lambda_0(t)\left(1 + O\left(\frac{1}{\log^{\epsilon_0}(t)}\right)\right)$$

Remark 1. The initial data for v_1 in the theorem statement is explicit in terms of $f \in F_b$, as noted above.

Remark 2. For $\frac{2}{3} < \beta < \alpha < 1$, we can let

$$f(t) = \frac{\sin(\log^{\alpha}(t))}{\log^{\beta}(t)}, \quad t \geqslant 50$$

Then, $f \in F_b$ for any $\frac{2}{3} < b < \beta$. We then carry out the procedure discussed before the main theorem, to recover a $\lambda_0 \in \Lambda_b$ satisfying (1.9) and (1.10). In this case, we have

$$\frac{\lambda_0'(t)}{\lambda_0(t)} \sim \frac{-\alpha \log^{\alpha-1}(t)\cos(\log^{\alpha}(t))}{t \log^{\beta}(t)}$$

Since $1 + \beta - \alpha < 1$, this gives rise to $\lambda_0 \in \Lambda_b$ with

$$\int_t^{\infty} \frac{|\lambda_0'(s)|}{\lambda_0(s)} ds = \infty$$

Nevertheless, as pointed out earlier in a more general context, λ_0 is asymptotically constant.

Remark 3. By choosing

$$f(t) = \frac{1}{\log^{b}(t)}, \quad t \geqslant 50, \quad b > \frac{2}{3}$$

we can show (see (11.6)) that

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi \log^{b}(\frac{1}{\xi})} + O\left(\frac{1}{\xi \log^{b+1}(\frac{1}{\xi})}\right), \quad \xi \to 0$$

which shows that we can have radiation whose initial velocity has quite a large singularity at low frequencies. In fact, the condition for the radiation to have finite energy in our setting is $\widehat{v_{1,1}}(\xi) \in L^2((0,\infty), \xi d\xi)$. The initial velocity therefore satisfies this condition only "logarithmically". In fact, given our definition of $\widehat{v_{1,1}}(\xi)$ above, $\lambda(t)$ approaching a constant as t approaches infinity is a necessary condition for $\widehat{v_{1,1}}(\xi) \in L^2((0,\infty), \xi d\xi)$ (see the discussion after (11.8).

Now that we have discussed the equations to be considered, and summarized the main results, we start by stating and proving our precise results for the wave maps equation. The Yang-Mills equation results will be stated and proven afterwards. The following is taken from a work of the author which is to appear in the Memoirs of the American Mathematical Society.

2 Introduction (Wave Maps)

We consider the wave maps equation, with domain $\mathbb{R}^{2+1}$ and target $\mathbb{S}^2$. This equation is the Euler-Lagrange equation associated to the functional

$$\mathcal{L}(\Phi) = \int_{\mathbb{R}^{2+1}} \langle \partial^{\alpha}\Phi(t,x), \partial_{\alpha}\Phi(t,x)\rangle_{g(\Phi(t,x))} dt dx$$

where g denotes the round metric on $\mathbb{S}^2$, and $\Phi : \mathbb{R}^{2+1} \to \mathbb{S}^2$. We will only work with 1-equivariant maps Φ, which we describe by first regarding Φ as a map into $\mathbb{R}^3$ with unit norm, and then writing

$$\Phi_u(t,r,\phi) = (\cos(\phi)\sin(u(t,r)), \sin(\phi)\sin(u(t,r)), \cos(u(t,r)))$$

where (r,ϕ) are polar coordinates on $\mathbb{R}^2$. Then, the wave maps equation becomes

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u = \frac{\sin(2u)}{2r^2} \tag{2.1}$$

[31] studied (a more general problem which includes) the Cauchy problem associated to (2.1), with data (u_0, u_1) such that

$$(x_1,x_2) \mapsto (\frac{x_1 u_0(r)}{r}, \frac{x_2 u_0(r)}{r}) \in H^1_{\text{loc}}(\mathbb{R}^2)$$
$$(x_1,x_2) \mapsto (\frac{x_1 u_1(r)}{r}, \frac{x_2 u_1(r)}{r}) \in L^2_{\text{loc}}(\mathbb{R}^2)$$

We will say that u is a finite energy solution to (2.1) if u is a distributional solution, with $\Phi_u \in C^0_t \dot{H}^1(\mathbb{R}^2)$ and $\partial_t \Phi_u \in C^0_t L^2(\mathbb{R}^2)$. Define the energy by

$$E_{\text{WM}}(u,v) = \pi \int_0^\infty \left(v^2 + \frac{\sin^2(u)}{r^2} + (\partial_r u)^2\right) r dr \tag{2.2}$$

Then, if u solves (2.1), $E_{\text{WM}}(u, \partial_t u)$ is formally independent of time. We recall the ground state soliton, $Q_1(r) = 2\arctan(r)$ is a solution to (2.1), with the property that the family of all $Q_\lambda(r) = Q_1(r\lambda)$ are the unique minimizers of $E_{\text{WM}}(u,0)$ among finite energy u with Φ_u having topological degree one. In order to state our main theorem, it will also be useful to consider the following non-degenerate energy which will be used to measure perturbations of Q_λ.

$$E(u,v) = \pi \int_0^\infty \left(v^2 + (\partial_r u)^2 + \frac{u^2}{r^2}\right) r dr$$

The quantity $E(u, \partial_t u)$ is formally conserved for solutions to the wave equation

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{u}{r^2} = 0 \tag{2.3}$$

We consider the problem of constructing 1-equivariant, topological degree one, global, non-scattering solutions to (2.1), which have energy strictly greater than $E_{\text{WM}}((Q_1, 0))$. The authors of [3] classified 1-equivariant topological degree one solutions to the wave maps equation with energy strictly between $E_{\text{WM}}((Q_1, 0))$ and $3E_{\text{WM}}((Q_1, 0))$. The part of their result which is relevant for this paper is the statement that any such solutions which are global in time admit a generic decomposition into the form

$$u = Q_{\frac{1}{\lambda(t)}} + \phi_L + \epsilon \tag{2.4}$$

where ϕ_L solves 2.3, $\lambda(t) = o(t), \quad t \to \infty$, and $\epsilon \to 0$ (in an appropriate sense) as $t \to \infty$. There are many possible asymptotic behaviors of λ allowed by the above result, and, according to [3], there were no known constructions of solutions of the above form, with $\lambda(t) \to 0$ or $\lambda(t) \to \infty$.

This paper constructs a family of finite energy solutions to (2.1), say $\{u_b\}_{b>0}$, where each u_b can be decomposed as in (2.4), with

$$\lambda_b(t) = \lambda_{0,0,b}(t) + O\left(\frac{1}{\log^b(t)\sqrt{\log(\log(t))}}\right)$$

(see the main theorem below for the sense in which the ϵ term for our solution, which is called v_e, vanishes as t goes to infinity). Here, $\lambda_{0,0,b} \in C^\infty([100,\infty))$ satisfies, for some $b, C_l, C_{m,k} > 0$,

$$\frac{C_l}{\log^b(t)} \leqslant \lambda_{0,0,b}(t) \leqslant \frac{C_m}{\log^b(t)}, \quad |\lambda_{0,0,b}^{(k)}(t)| \leqslant \frac{C_{m,k}}{t^k \log^{b+1}(t)}, k \geqslant 1 \quad t \geqslant 100$$

To the author's knowledge, these are the first examples of such solutions of (2.1). First, we will prove the following theorem, which corresponds to the special case of $\lambda_{0,0,b}(t) = \frac{1}{\log^b(t)}$. Solutions with more general $\lambda_{0,0,b}$ can be constructed with a slight modification of the proof of this special case, outlined in the appendix.

Theorem 2.1. *For each $b > 0$, there exists $T_0 > 0$ and a (real-valued) finite energy solution u_b to* (2.1) *for $t \geqslant T_0$, of the form*

$$u_b(t,r) = Q_{\frac{1}{\lambda_b(t)}}(r) + v_2(t,r) + v_e(t,r)$$

where $\lambda_b \in C^4([T_0,\infty))$,

$$E_{\text{WM}}(u_b, \partial_t u_b) < \infty$$

$$-\partial_{tt}v_2 + \partial_{rr}v_2 + \frac{1}{r}\partial_r v_2 - \frac{v_2}{r^2} = 0, \quad E(v_2, \partial_t v_2) < \infty, \quad v_2 \in C^\infty([T_0,\infty) \times [0,\infty))$$

$$E(v_e, \partial_t \left(Q_{\frac{1}{\lambda_b}} + v_e\right)) \leqslant \frac{C}{t^2 \log^{2b}(t)}, \quad t \geqslant T_0$$

and

$$\lambda_b(t) = \frac{1}{\log^b(t)} + e(t), \quad |e(t)| \leqslant \frac{C}{\log^b(t)\sqrt{\log(\log(t))}}$$

$$|e^{(j)}(t)| \leqslant \begin{cases} \frac{C}{t^j \log^{b+1}(t)\sqrt{\log(\log(t))}}, & j = 1,2 \\ \frac{C}{t^j \log^{b+1}(t)}, & j = 3,4 \end{cases}$$

Regarding more general $\lambda_{0,0,b}$, we have the following. For $b > 0$, let Λ_b be the set of $f \in C^\infty([100,\infty))$ such that there exist $C_l, C_m, C_{m,k} > 0$ with

$$\frac{C_l}{\log^b(t)} \leqslant f(t) \leqslant \frac{C_m}{\log^b(t)}, \quad |f^{(k)}(t)| \leqslant \frac{C_{m,k}}{t^k \log^{b+1}(t)}, k \geqslant 1, \quad t \geqslant 100$$

Theorem 2.2. *Let $b > 0$. For all $\lambda_{0,0,b} \in \Lambda_b$, there exists $T_0 > 100$ and a (real-valued) finite energy solution u_b to* (2.1) *for $t \geqslant T_0$, of the form*

$$u_b(t,r) = Q_{\frac{1}{\lambda_b(t)}}(r) + v_2(t,r) + v_e(t,r)$$

where $\lambda_b \in C^4([T_0,\infty))$,

$$E_{\mathrm{WM}}(u_b,\partial_t u_b) < \infty$$

$$-\partial_{tt}v_2 + \partial_{rr}v_2 + \frac{1}{r}\partial_r v_2 - \frac{v_2}{r^2} = 0, \quad E(v_2,\partial_t v_2) < \infty, \quad v_2 \in C^\infty([T_0,\infty)\times[0,\infty))$$

$$E(v_e, \partial_t\left(Q_{\frac{1}{\lambda_b}} + v_e\right)) \leqslant \frac{C}{t^2\log^{2b}(t)}, \quad t \geqslant T_0$$

and

$$\lambda_b(t) = \lambda_{0,0,b}(t) + e(t), \quad |e(t)| \leqslant \frac{C}{\log^b(t)\sqrt{\log(\log(t))}}$$

$$|e^{(j)}(t)| \leqslant \begin{cases} \frac{C}{t^j\log^{b+1}(t)\sqrt{\log(\log(t))}}, & j=1,2 \\ \frac{C}{t^j\log^{b+1}(t)}, & j=3,4 \end{cases}$$

Remark 1. Our proof yields more information about the regularity of v_e appearing in the main theorem. In particular, we have

$$v_e = v_{e,0} + v_6$$

where the function $v_{e,0}$ is fairly explicit (but complicated), and v_6 is constructed with a fixed point argument, but has more Sobolev regularity than what follows from u having finite energy, namely: if (r,θ) denote polar coordinates on $\mathbb{R}^2$, then

$$(t,r,\theta) \mapsto e^{i\theta}v_6(t,r) \in C^0_t([T_0,\infty), H^2(\mathbb{R}^2))$$

$$(t,r,\theta) \mapsto e^{i\theta}\partial_t v_6(t,r) \in C^0_t([T_0,\infty), H^1(\mathbb{R}^2))$$

Remark 2. Theorem 2.2 includes solutions with mildly oscillating $\lambda_{0,0,b}$, such as

$$\lambda_{0,0,b} = \frac{2+\sin(\log(\log(t)))}{\log^b(t)}, \quad t \geqslant 100$$

Remark 3. It is expected that the method used in this paper can be extended to allow for the construction of such solutions with $\lambda(t) \to \infty, \quad t \to \infty$.

Finally, this method should also be applicable to higher equivariance classes, the energy critical Yang-Mills problem in 4 dimensions with gauge group $SO(4)$, as well as the quintic, focusing semilinear wave equation in $\mathbb{R}^{1+3}$. All of these extensions are work in progress by the author.

Remark 4. The leading behavior of $\lambda(t)$ is partially motivated by the fact that, for $k=1,2$,

$$\frac{|\lambda^{(k)}(t)|}{\lambda(t)} \leqslant \frac{C}{t^k\log(t)} = o(t^{-k}), \quad t \to \infty$$

This fact is very important so that certain error terms appearing in this work which involve the "transferrence operator" defined in [14] can be treated perturbatively. In particular, the method of this work can not be directly applied to construct solutions with $\lambda(t)$ being all powers of t.

In order to understand this work in a larger context, we review previous work regarding dynamical behavior of solutions to the critical wave maps problem with $\mathbb{S}^2$ target. Firstly, the works [32] and [33] (which apply to much more general targets than $\mathbb{S}^2$) show that for data with energy strictly less than that of the lowest energy non-trivial harmonic map, one has global well-posedness and scattering. In the 1-equivariant setting, [2] showed that global existence and scattering holds for smooth, topological degree zero data with energy less than twice that of Q_1 (which is the appropriate threshold in this setting). An analogous result, but without the equivariance assumption, is implied by [19]. Then, [9] constructed, for all $k \geqslant 2$, k equivariant, topological degree zero, two-bubble solutions with energy exactly equal to 2 times the energy of $Q_k(r) := Q_1(r^k)$. In [10], all k-equivariant solutions (with $k \geqslant 2$) of topological degree zero, and energy exactly equal to twice the energy of $Q_k(r)$ were classified. A classification of 1-equivariant, topological degree 0 solutions with energy equal to twice $E(Q_1)$ was obtained in [30], and shows that the dynamical behavior of such solutions can be quite different than in the higher equivariance case. [30] also constructs a finite-time blow-up solution in this setting. The methods used in this work differ significantly from these previous works.

1-equivariant, topological degree one, finite time blow-up solutions have been constructed in [14], and the work [5] extended the range of possible blow-up rates of these solutions. The method of construction of the ansatz of this paper differs significantly from that used in [14]. However, we do eventually use some technical information, most importantly, the distorted Fourier transform, from [14] as part of the process to complete our ansatz to an exact solution. Analogs of the solutions of [14] for the $4+1$-dimensional Yang-Mills equation with gauge group $SO(4)$ and the quintic, focusing nonlinear wave equation in $\mathbb{R}^{1+3}$ were also constructed in [15] and [17]. The work of [16] studies the stability of the solutions of [14] and [5] under certain equivariant perturbations. In addition, [29] constructs finite-time blow-up, $k-$equivariant solutions with $k \geqslant 4$, and [28] constructs finite time blow-up solutions for the $(4+1)$-dimensional Yang-Mills problem with gauge group $SO(4)$, as well as in all equivariance classes for energy critical wave maps. The method of this work is quite different from the methods used in [28] and [29].

The work of [1] constructs modulated soliton solutions, where λ is bounded away from zero and infinity for all time. Some facts from [1] about the wave maps equation linearized around Q_1 will be utilized in this paper, but the ansatz construction is again quite different. Finally, infinite time blow-up and infinite time relaxation solutions to the quintic, focusing, nonlinear wave equation in $\mathbb{R}^{1+3}$ have been constructed in [4], but the method of this paper is again quite different.

3 Notation (Wave Maps)

We will occasionally use the Hankel transform of order 1, and it will be denoted as

$$\widehat{f}(\xi) = \int_0^\infty f(r) J_1(r\xi) r dr$$

The main Fourier transform we will use is the distorted Fourier transform of [14], which we denote by

$$\mathcal{F}(f)(\xi) = \int_0^\infty \phi(r,\xi) f(r) dr$$

Briefly, we will make some use of the distorted Fourier transforms of [1], which are denoted by $\mathcal{F}_H$ and $\mathcal{F}_{\tilde{H}}$. Finally, we use the same notation as [1] for the following norm

$$||f||^2_{\dot{H}^1_e} = ||\partial_r f||^2_{L^2(rdr)} + ||\frac{f}{r}||^2_{L^2(rdr)}$$

We denote by ϕ_0, the zero resonance of the elliptic part of the wave equation linearized around Q_1:

$$\phi_0(r) = \frac{d}{d\lambda}\Big|_{\lambda=1} Q_\lambda(r) = \frac{2r}{1+r^2}$$

(This notation for ϕ_0 is the same as that used in [1], but is different from that used in [14]).

4 Overview of the Proof (Wave Maps)

We remind the reader that we will prove theorem 2.1, with the extra arguments needed to establish theorem 2.2 summarized in the appendix. The argument used in this paper proceeds in two parts. First, we construct an approximate solution, u, to (2.1). Second, we construct an exact solution to (2.1) which is close to our approximate one.

Part 1: Constructing the approximate solution

One way to understand the intuition behind our approximate solution is as follows. One could start by looking for an approximate solution to (2.1) which consists of a dynamically rescaled soliton and a radiation field, along with an appropriate compatibility condition between the soliton length scale, and the radiation field. This would correspond to an approximate solution of the form $u_a = Q_{\frac{1}{\lambda(t)}} + v_2$, where v_2 is some solution to (2.3), $\lambda(t)$ is not yet chosen, and one can look for an appropriate relation between v_2 and $\lambda(t)$ so as to make the approximate solution accurate. Two difficulties immediately arise with this procedure. One is that $\partial_t Q_{\frac{1}{\lambda(t)}}(r) \notin L^2(rdr)$, which means that $E_{\text{WM}}(u_a, \partial_t u_a)$ is not finite. Another key difficulty is that the elliptic part of the wave maps equation linearized around $Q_{\frac{1}{\lambda(t)}}$ has a zero resonance, $\phi_0(\frac{r}{\lambda(t)})$. So, one would like the principal part of the error term of u_a to be orthogonal to $\phi_0(\frac{r}{\lambda(t)})$. On the other hand, the soliton error term

$$\partial_t^2 Q_{\frac{1}{\lambda(t)}}(r) = \frac{-2r\lambda''(t)}{r^2+\lambda(t)^2} + \frac{4r\lambda(t)\lambda'(t)^2}{(r^2+\lambda(t)^2)^2}$$

does not decay fast enough in r for its inner product with $\phi_0(\frac{r}{\lambda(t)})$ to even be defined in the first place.

Hence, we start by introducing an additional correction, v_1, which is independent of v_2, and whose purpose is both to make the ansatz have finite energy and to eliminate an appropriate principal part of the soliton error term, $\partial_t^2 Q_{\frac{1}{\lambda(t)}}$, for large r. More precisely the starting point for our ansatz is

$$u_{a,1} = Q_{\frac{1}{\lambda(t)}}(r) + v_1(t,r) + v_2(t,r)$$

Here, $\lambda(t)$ is not yet chosen, and v_1 solves the nondegenerate wave equation

$$-\partial_{tt}v_1 + \partial_{rr}v_1 + \frac{1}{r}\partial_r v_1 - \frac{v_1}{r^2} = -2\lambda''(t)\frac{r}{1+r^2}$$

with 0 Cauchy data at infinity. As explained in more detail later on, we do not have the entire soliton error term on the right-hand side of the v_1 equation, because doing so would be more difficult and also would hide the leading, linear part in the modulation equation for λ.

On the other hand, v_2 solves (2.3) with Cauchy data

$$v_2(0,r) = 0, \quad \partial_t v_2(0,r) = v_{2,0}(r)$$

where $v_{2,0}$ will be a prescribed function depending on a fixed parameter $b > 0$. The choices for $v_{2,0}$ and $\lambda(t)$ are closely related. Later, we will choose $\lambda(t)$ to solve a modulation equation involving $v_{2,0}$, thereby correlating the length scale of the dynamically rescaled soliton with the radiation profile.

At this stage, one could choose $\lambda(t)$ by requiring that the principal part of the error term associated to $u_{a,1}$, say $e_{a,1}$, is orthogonal to $\phi_0(\frac{r}{\lambda(t)})$. This is not exactly what is done in our argument, since we will need to add additional corrections to $u_{a,1}$ before imposing the orthogonality condition on the principal part of the error term of our final ansatz. Nevertheless, computing $\langle e_{a,1}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$ allows one to see, in a simpler context than our final equation for $\lambda(t)$, a relation between the leading order behavior of $\lambda''(t)$ and v_2. For the purposes of this discussion, the principal part of the $u_{a,1}$ error term is

$$e_{a,1}(t,r) = \partial_{tt} Q_{\frac{1}{\lambda(t)}}(r) + \frac{2\lambda''(t) r}{1+r^2} + \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r)) - 1}{r^2}\right) v_1(t,r) + \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r)) - 1}{r^2}\right) v_2(t,r)$$

Using the Hankel transform of order 1 to express v_2, the contribution to

$$\langle e_{a,1}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)}$$

from the v_2-related term above is given by

$$\begin{aligned}&\langle \left(\frac{\cos(2Q_1(R)) - 1}{R^2 \lambda(t)^2}\right) v_2(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)} \\ &= -2\int_0^\infty \sin(t\xi) \widehat{v_{2,0}}(\xi) \xi^2 K_1(\xi\lambda(t)) d\xi\end{aligned}$$

where K_1 denotes the modified Bessel function of the second kind. We thus get

$$\begin{aligned}&\langle e_{a,1}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)} \\ &= -\frac{4}{\lambda(t)} \int_t^\infty \frac{\lambda''(s)}{1+s-t} ds - 2\int_0^\infty \sin(t\xi) \widehat{v_{2,0}}(\xi) \xi^2 K_1(\xi\lambda(t)) d\xi + \frac{4\lambda''(t)\log(\lambda(t))}{\lambda(t)} \\ &+ f_1(\lambda(t), \lambda'(t), \lambda''(t))\end{aligned}$$

We will choose $v_{2,0}$ and the principal part of λ, denoted by $\lambda_{0,0}$, in order to have a leading order cancellation in the above equation. The term $f_1(\lambda(t), \lambda'(t), \lambda''(t))$ turns out to be subleading for all λ close (in a C^2 sense) to the choice of $\lambda_{0,0}$ which we make. Our choice of $v_{2,0}$ gives the following equation

$$\begin{aligned}\langle e_{a,1}(t, R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)} &= -\frac{4}{\lambda(t)} \int_t^\infty \frac{\lambda''(s)}{1+s-t} ds + \frac{4b}{\lambda(t) t^2 \log^b(t)} \\ &+ \frac{4\lambda''(t)\log(\lambda(t))}{\lambda(t)} + f_2(\lambda(t), \lambda'(t), \lambda''(t))\end{aligned}$$

Again, the term $f_2(\lambda(t),\lambda'(t),\lambda''(t))$ is subleading, for all $\lambda(t)$ close (in a C^2 sense) to $\lambda_{0,0}(t) = \frac{1}{\log^b(t)}$, which is a leading order solution to

$$-\frac{4}{\lambda(t)}\int_t^\infty \frac{\lambda''(s)}{1+s-t}ds + \frac{4b}{\lambda(t)t^2\log^b(t)} + \frac{4\lambda''(t)\log(\lambda(t))}{\lambda(t)} = 0$$

The term

$$\frac{4b}{\lambda(t)t^2\log^b(t)}$$

occurs in the above computation, as a consequence of our particular choice of Cauchy data for v_2. Despite the fact that our final equation for $\lambda(t)$ is not exactly the one given above, the leading behavior of our $\lambda(t)$ is indeed $\lambda_{0,0}(t)$, due to the same cancellation seen above.

Even if the principal part of the error term of $u_{a,1}$ is chosen to be orthogonal to $\phi_0(\frac{\cdot}{\lambda(t)})$, it does not have enough decay to be treated perturbatively via our methods. So, we need to add three more corrections to $u_{a,1}$, denoted by v_3, v_4, and v_5, in order to achieve an acceptable error term.

When the error term resulting from $Q_{\frac{1}{\lambda(t)}}$ and v_1 is computed in the renormalized spatial coordinate R, defined by $r = R\lambda(t)$, it has insufficient decay for large R (because factors of $\frac{1}{\lambda(t)}$ will turn out to correspond to logarithmic growth in t, once we choose $\lambda(t)$). On the other hand, if this error term is completely eliminated, the resulting modulation equation for λ becomes much more difficult to study. The third correction, v_3, solves an inhomogeneous version of (2.3) with 0 Cauchy data at infinity in order to correct the problem of this error term for large R while also not complicating the final modulation equation for λ. In the process of doing this, we introduce a small, positive parameter α.

The soliton, along with these three corrections can be regarded as the principal components of the ansatz. However, we introduce two more corrections, v_4 and v_5, both of which solve inhomogeneous versions of (2.3), with 0 Cauchy data at infinity, in order to improve the error terms resulting from the previous terms in the ansatz. More precisely, v_4 eliminates a large r portion of linear error terms associated to $v_k, \quad k=1,2,3$, as well as an error term arising from the combination of the right-hand sides of the v_3 and v_1 equations and $\partial_t^2 Q_{\frac{1}{\lambda(t)}}$. v_5 eliminates error terms associated to the nonlinear interaction between $v_k, \quad k=1,2,3,4$.

Our final ansatz

$$u_{ansatz} = Q_{\frac{1}{\lambda(t)}}(r) + \sum_{k=1}^5 v_k(t,r)$$

satisfies

$$E_{\text{WM}}(u_{ansatz},\partial_t u_{ansatz}) < \infty$$
$$-\partial_{tt}u_{ansatz} + \partial_{rr}u_{ansatz} + \frac{1}{r}\partial_r u_{ansatz} - \frac{\sin(2u_{ansatz})}{2r^2} = -\left(F_4+F_5+F_6\right)$$

where F_5+F_6 has sufficiently fast time decay in sufficiently many norms, and is perturbative:

$$\frac{||F_5(t,r)+F_6(t,r)||_{L^2(rdr)}}{\lambda(t)^2} \leqslant \frac{C}{t^4\log^{3b+2N-1}(t)}$$

$$\frac{||F_5+F_6||_{\dot{H}^1_e}}{\lambda(t)} \leqslant \frac{C\log^{6+b}(t)}{t^{35/8}}$$

and F_4, the principal part of the error term, does not have as fast decay as F_5+F_6, but satisfies

$$\langle F_4(t,\cdot),\phi_0(\frac{\cdot}{\lambda(t)})\rangle = 0$$

and the following estimate (in a symbol-type fashion, see theorem 5.1 for the precise statement):

$$|F_4(t,r)| \leqslant \frac{C\mathbb{1}_{\{r\leqslant \log^N(t)\}}r}{t^2\log^{3b+1-2\alpha b}(t)(r^2+\lambda(t)^2)^2}+\frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}}r}{t^2\log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2}$$

where α and N are parameters associated to v_3 and v_4, respectively. α is small relative to b and 1, while N and is large relative to b and 1.

The modulation equation for λ,

$$\langle F_4(t,\cdot),\phi_0(\frac{\cdot}{\lambda(t)})\rangle = 0$$

has a principal part which is of the form of a Volterra equation of the second kind in the variable λ'' (with kernel and coefficients weakly depending on λ). In particular, this equation is of the form

$$\begin{aligned}&-4\int_t^\infty \frac{\lambda''(s)}{1+s-t}ds+\frac{4b}{t^2\log^b(t)}+4\alpha\log(\lambda(t))\lambda''(t)\\&-4\int_t^\infty\frac{\lambda''(s)}{(\lambda(t)^{1-\alpha}+s-t)(1+s-t)^3}ds\\&=f_3(\lambda(t),\lambda'(t),\lambda''(t))\end{aligned}$$

We emphasize that the leading behavior of λ is independent of the small parameter $\alpha>0$, associated to v_3, and the crucial source term

$$\frac{4b}{t^2\log^b(t)}$$

comes from v_2, and is a consequence of the particular choice of data for v_2. The other terms on the left-hand side of the above equation come from $v_1,v_3,Q_{\frac{1}{\lambda(t)}}$. In particular, v_4 and v_5 do not contribute to the leading order part of the equation for $\lambda(t)$ because the terms contained in f_3 are subleading, for all λ close (in a C^2 sense) to $\frac{1}{\log^b(t)}$.

To the knowledge of the author, a modulation equation of the above form is quite different from that of previous works. Moreover, in the context of our ansatz, the Volterra form of the modulation equation seems to be related to the fact that $\frac{d}{d\lambda}\Big|_{\lambda=1}Q_\lambda(r)\notin L^2(rdr)$. Indeed, the integral operators acting on λ'' arising in the principal part of the modulation equation come from v_1 and v_3, which were introduced to correct soliton related error terms for large r.

As motivated by the discussion of the simpler ansatz $u_{a,1}$ above, we find an approximate solution to the modulation equation (5.63) for λ of the form

$$\lambda_{0,0}(t)=\frac{1}{\log^b(t)}$$

The equation for λ is then exactly solved around $\lambda_{0,0}$ with a fixed point argument in a weighted C^2 space, and we obtain an exact solution

$$\lambda(t) = \frac{1}{\log^b(t)} + O\left(\frac{1}{\log^b(t)\sqrt{\log(\log(t))}}\right)$$

Afterwards, we show that λ is in fact a C^4 function, and prove quantitative estimates on its derivatives. Along the way, we thus obtain estimates on higher time derivatives of the corrections v_k, whose right-hand sides depend on $\lambda(t)$ (for all $k \neq 2$).

Part 2: Construction of the exact solution

Once we construct u_{ansatz}, we substitute $u = u_{ansatz} + v_6$ into (2.1), thereby obtaining an equation for v_6. Our goal is to solve this equation for v_6 perturbatively, with 0 Cauchy data at infinity. We achieve this, by studying the distorted Fourier transform (in the sense of [14]), of v_6. In particular, we (formally) derive the equation for

$$y(t,\omega) = \mathcal{F}(\sqrt{\cdot}v_6(t,\cdot\lambda(t)))(\omega\lambda(t)^2)$$

where we recall that $\mathcal{F}$ denotes the distorted Fourier transform of [14]. The particular choice of the rescaling used in the definition of y is explained by noting that the equation for y takes the form

$$\partial_{tt}y + \omega y = -\mathcal{F}(\sqrt{\cdot}F(t,\cdot\lambda(t)))(\omega\lambda(t)^2) + F_2(y)(t,\omega) - \mathcal{F}(\sqrt{\cdot}F_3(v_6(y))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)$$

for appropriate F_2, F_3, where

$$F = F_4 + F_5 + F_6$$

is the sum of the error terms of u_{ansatz}. In deriving the y equation, a few properties about the distorted Fourier transform, most importantly, the transferrence identity from [14] are used. The equation for y is solved by showing that the map

$$T(y)(t,\omega) = -\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}} \big(F_2(y)(x,\omega) - \mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2) \\ -\mathcal{F}(\sqrt{\cdot}F_3(u(y))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\big)\, dx$$

has a fixed point in an appropriate Banach space (whose norm is roughly the sum of weighted $L^\infty_t L^2_\omega$ norms of y and $\partial_t y$, see (6.46) for the precise definition) via the contraction mapping theorem. The most delicate term to estimate is

$$-\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}} \left(-\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right) dx$$

In order to obtain sufficient estimates on this term (and its time derivative) in appropriate norms, we must utilize the previously discussed orthogonality condition on F_4. The orthogonality condition on F_4 is utilized by noting that it implies that $\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)$ has a certain degree of vanishing at small frequencies, and this allows us to integrate by parts in the x variable in the integral above. This, combined with the symbol-type nature of the pointwise estimates on F_4 turn out to provide sufficient decay of the integral above in all norms required by the fixed point argument. Since the density of the spectral measure associated to the distorted Fourier transform of [14] (which appears in the weighted norms of our iteration space) has a singularity at low frequencies, such integration by parts would be impossible without the orthogonality condition. In particular, we can not integrate by parts for the analogous integrals involving $F_5 + F_6$. The faster time decay of the L^2 and $\dot{H}^1_e$ norms of this term, relative to F_4 is what allows it to be perturbative, despite the non-orthogonality to $\phi_0(\frac{\cdot}{\lambda(t)})$.

5 Construction of the Ansatz (Wave Maps)

Fix $b > 0$, $0 < \alpha < \min\{\frac{1}{b(1040!)}, \frac{1}{1040!}\}$, and $N > (5000!)(b+1)$. We consider (2.1) for $t \geqslant T_0$, where

$$T_0 > 2e^{e^{1000(b+1)}} + T_{0,1} \tag{5.1}$$

and is otherwise arbitrary for now, and where $T_{0,1} > e^{(128000)^{\frac{4}{b}}} + e^{N^2}$ is such that

$$|\frac{d^j}{dt^j}\left(\frac{\log(\log(t))}{\log^{b+1}(t)}\right)| + |\frac{d^j}{dt^j}\left(\frac{1}{b\log^b(t)\sqrt{\log(\log(t))}}\right)| \leqslant \frac{1}{200}|\frac{d^j}{dt^j}\left(\frac{1}{\log^b(t)}\right)|$$
$$, t \geqslant T_{0,1}, \quad j = 0,1,2$$

$\lambda : [T_0, \infty) \to (0, \infty)$ denotes a $C^2([T_0, \infty))$ map satisfying, for some $C > 0$, *independent* of T_0

$$\lambda'(t) < 0, \quad |\lambda''(t)| \leqslant \frac{C}{t^2 \log^{b+1}(t)}, \quad \text{and } \frac{1}{C\log^C(t)} < \lambda(t) < \frac{1}{2}, \quad t \geqslant T_0 \tag{5.2}$$

and is otherwise arbitrary for now. (Note that the first and third requirements above are not strictly necessary for the validity of most of our procedure. However, for this work, there is no loss of generality in assuming them, since λ will later on be restricted to a class of functions of the form $\lambda = \lambda_0 + e$, where λ_0 is some explicit function to be specified later, and e belongs to a certain space of functions such that (among other things) (5.2) holds for $\lambda_0 + e$).

Also for all estimates appearing in the entire paper, we use the convention that C will always denote a positive constant *independent* of T_0, whose value may change from line to line. Although we have already summarized the ansatz construction in the overview of the proof, we now provide an outline which clarifies the logical structure of the process.

5.1 Outline

Step 1, Definitions of the corrections, v_k: For all T_0 and λ as above, and $t \geqslant T_0$, we define functions $v_k, \quad 1 \leqslant k \leqslant 5$, which were roughly described in the overview of the proof, thereby obtaining

$$u_{ansatz}(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + \sum_{k=1}^{5} v_k(t,r)$$

Step 2, Splitting of the error term of u_{ansatz}, and setup of the modulation equation for λ: We define functions F_4, F_5, and F_6, which split the error term of u_{ansatz} into $-(F_4 + F_5 + F_6)$. Then, we consider the modulation equation for $\lambda(t)$ resulting from setting

$$\langle F_4(t,\cdot), \phi_0(\frac{\cdot}{\lambda(t)})\rangle = 0, \quad t \geqslant T_0$$

(Note that F_4 depends on $\lambda(t)$). As described (to a certain extent) in the overview of the proof, this modulation equation takes the form of a Volterra equation of the second kind in the variable $\lambda''(t)$ (with kernel and coefficients weakly depending on λ), and our choice of Cauchy data for v_2 gives rise to a leading order solution

$$\lambda_0 = \frac{1}{\log^b(t)} + \int_t^\infty \int_{t_1}^\infty \frac{-b^2 \log(\log(t_2))}{t_2^2 \log^{b+2}(t_2)} dt_2 dt_1$$

to this equation. We then introduce a weighted C^2 space, X, with norm (5.30). From this point on, we further restrict $\lambda(t)$ to be of the form

$$\lambda(t) = \lambda_0(t) + e(t), \quad e \in B$$

where

$$B = \overline{B_1(0)} \subset X$$

(Note that $\lambda_0 + B$ is contained within the class of $\lambda(t)$ we initially started with (which is described by (5.2))).

Step 3, Solving the modulation equation for $\lambda(t)$**:** Writing $F_4(t,r) = F_4^{\lambda(t)}(t,r)$ to emphasize the λ dependence of F_4, we show that there exists $T_3 > 0$ such that, for all $T_0 \geqslant T_3$, the equation

$$\langle F_4^{\lambda_0(t)+e(t)}(t,\cdot), \phi_0(\frac{\cdot}{\lambda_0(t)+e(t)})\rangle = 0, \quad t \geqslant T_0$$

can be solved for $e(t) \in B$, using the contraction mapping principle. An important part of this procedure is that the kernel appearing in the Volterra equation for e (which is independent of e, modulo error terms which can be treated perturbatively) satisfies the structural condition (5.69). From here on, we work under the constraint $T_0 \geqslant T_3$.

Step 4, Estimates on higher derivatives of λ''**:** Denoting the solution to the above equation for e by $e_0(t)$, we fix $\lambda(t) = \lambda_0(t) + e_0(t)$. Then, we prove that $\lambda(t)$, which is apriori only in $C^2([T_0,\infty))$, is actually in $C^4([T_0,\infty))$, and establish quantitative estimates on λ''' and λ''''.

Step 5, Estimates on F_k**:** At this stage, we prove pointwise estimates on F_4, as well as estimates on $||F_k||_{L^2(rdr)}$, $||F_k||_{\dot{H}^1_e}$ for $k = 5,6$. This completes the ansatz construction. More precisely, our main result of this section is

Theorem 5.1 (Approximate solution to (2.1))**.** *For all $b > 0$, there exists $T_3 > 0$ such that, for all $T_0 \geqslant T_3$, there exists $v_{corr} \in C^4([T_0,\infty); C^2((0,\infty)))$, and $\lambda \in C^4([T_0,\infty))$ such that, if $u = Q_{\frac{1}{\lambda(t)}} + v_{corr}$, then*

$$E_{WM}(u, \partial_t u) < \infty$$

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = -\left(F_4 + F_5 + F_6\right)$$

where

$$\frac{1}{\lambda(t)^2}||\left(F_5+F_6\right)(t,r)||_{L^2(rdr)} \leqslant \frac{C}{t^4\log^{3b+2N-1}(t)} \tag{5.3}$$

$$\frac{||\left(F_5+F_6\right)(t)||_{\dot{H}^1_e}}{\lambda(t)} \leqslant \frac{C\log^{6+b}(t)}{t^{35/8}} \tag{5.4}$$

$$\langle F_4(t,\cdot), \phi_0(\frac{\cdot}{\lambda(t)})\rangle = 0$$

For $0 \leqslant k \leqslant 2, \quad 0 \leqslant j \leqslant 1, \quad j+k \leqslant 2$, we have

$$t^j r^k |\partial_r^k \partial_t^j F_4(t,r)| \leqslant \frac{C\mathbb{1}_{\{r \leqslant \log^N(t)\}} r}{t^2\log^{3b+1-2\alpha b}(t)(r^2+\lambda(t)^2)^2} + \frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}} r}{t^2\log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2}$$

In addition, we have

$$|\partial_t^2 F_4(t,r)| \leqslant \frac{C\mathbb{1}_{\{r \leqslant \log^N(t)\}} r}{t^4\log^{3b+1-2\alpha b}(t)(r^2+\lambda(t)^2)^2} + \frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}} r}{t^4\log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2}$$
$$+ \frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}}}{t^4\log^{5b+N-2}(t)(r^2+\lambda(t)^2)^2}$$

We also have the following estimates on v_{corr}.

$$||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)}||^2_{L^\infty}+||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)^2(1+R^2)}||_{L^\infty}\leqslant\frac{C\log(\log(x))}{x^2\log(x)} \tag{5.5}$$

$$1+||\frac{v_{corr}(x,R\lambda(x))}{R}||_{L^\infty}+||\partial_R(v_{corr}(x,R\lambda(x)))||_{L^\infty}\leqslant C \tag{5.6}$$

$$\begin{aligned}&||\frac{v_{corr}(x,R\lambda(x))\partial_R(v_{corr}(x,R\lambda(x)))}{R\lambda(x)^2}||_{L^\infty_R((0,1))}\\&+||\frac{v_{corr}(x,R\lambda(x))\partial_R(v_{corr}(x,R\lambda(x)))}{R^2\lambda(x)^2}||_{L^\infty_R((1,\infty))}+||\frac{\partial_R(v_{corr}(x,R\lambda(x)))}{(1+R^2)\lambda(x)^2}||_{L^\infty}\\&\leqslant\frac{C\log(\log(x))}{x^2\log(x)}\end{aligned} \tag{5.7}$$

Finally, λ is described by

$$\lambda(t)=\frac{1}{\log^b(t)}+e(t),\quad |e(t)|\leqslant\frac{C}{\log^b(t)\sqrt{\log(\log(t))}}$$

$$|e^{(j)}(t)|\leqslant\begin{cases}\frac{C}{t^j\log^{b+1}(t)\sqrt{\log(\log(t))}}, & j=1,2\\ \frac{C}{t^j\log^{b+1}(t)}, & j=3,4\end{cases}$$

5.2 Correcting the large r behavior of $Q_{\frac{1}{\lambda(t)}}$

The error term $\partial_t^2 Q_{\frac{1}{\lambda(t)}}$, which arises from inserting $Q_{\frac{1}{\lambda(t)}}$ into (2.1), does not decay quickly enough for its inner product with $\phi_0(\frac{\cdot}{\lambda(t)})$ to be defined. The first term in the ansatz, v_1, is designed to correct this problem. Note that we do not choose v_1 to solve an equation whose right-hand side is equal to $\partial_t^2 Q_{\frac{1}{\lambda(t)}}$, as doing so would result in a much more difficult equation to solve when we eventually choose $\lambda(t)$. Instead, v_1 is defined as the solution to the equation

$$-\partial_{tt}v_1+\partial_{rr}v_1+\frac{1}{r}\partial_r v_1-\frac{v_1}{r^2}=-2\lambda''(t)\frac{r}{1+r^2}$$

with 0 Cauchy data at infinity. I.e., by Duhamel's principle, we have

$$v_1(t,r)=\int_t^\infty v_s(t,r)ds$$

where v_s is the solution to the following Cauchy problem

$$\begin{cases}-\partial_{tt}v_s+\partial_{rr}v_s+\frac{1}{r}\partial_r v_s-\frac{v_s}{r^2}=0\\ v_s(s,r)=0\\ \partial_t v_s(s,r)=-2\lambda''(s)\frac{r}{1+r^2}\end{cases} \tag{5.8}$$

We can determine a fairly explicit formula for v_s. The procedure used to determine the formula for v_s may be slightly formal, but the final expression obtained can be seen to be the solution to the Cauchy problem (5.8). Firstly, note that if u is a solution to

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u = 0$$

then, $w = \partial_r u$ is formally a solution to

$$-\partial_{tt}w + \partial_{rr}w + \frac{1}{r}\partial_r w - \frac{w}{r^2} = 0$$

So, we will first write down the spherical means representation formula for the problem

$$\begin{cases} -\partial_{tt}u_1 + \Delta u_1 = 0 \\ u_1(s) = 0 \\ \partial_t u_1(s) = -\lambda''(s)\log(1+|x|^2) \end{cases}$$

then define u by $u(t,|x|) = u_1(t,x)$ (which is possible, since u_1 is radially symmetric). Then,

$$v_s(t,r) = \partial_r u(t,r)$$

will be seen to be the solution to (5.8). From the spherical means representation formula, for $t > s$, we have

$$\begin{aligned} u_1(t,x) &= -\frac{\lambda''(s)}{2\pi}\int_{B(0,t-s)} \frac{\log(1+|y+x|^2)}{((t-s)^2-|y|^2)^{1/2}}dy \\ &= -\frac{\lambda''(s)}{2\pi}\int_0^{t-s}\rho\int_0^{2\pi}\frac{\log(1+|x|^2+2|x|\rho\cos(\theta)+\rho^2)}{((t-s)^2-\rho^2)^{1/2}}d\theta d\rho \end{aligned}$$

Recalling that u is the radial coordinate representative of u_1, we have

$$\partial_{|x|}u = -\frac{\lambda''(s)}{2\pi}\int_0^{t-s}\rho\int_0^{2\pi}\frac{2(|x|+\rho\cos(\theta))}{(1+|x|^2+2|x|\rho\cos(\theta)+\rho^2)((t-s)^2-\rho^2)^{1/2}}d\theta d\rho \tag{5.9}$$

To do the integral over the angular coordinate, we can regard it as

$$\int_C \frac{2(|x|+\frac{\rho}{2}(z+\frac{1}{z}))}{(1+|x|^2+|x|\rho(z+\frac{1}{z})+\rho^2)((t-s)^2-\rho^2)^{1/2}}\frac{dz}{iz}$$

where C is the boundary of the unit circle in the complex plane, traversed in the counterclockwise direction.

The integrand is a meromorphic function on $\mathbb{C}$, with poles at

$$z = 0, -\left(\frac{1+|x|^2+\rho^2 \pm \sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}}{2|x|\rho}\right)$$

Note that

$$-4|x|^2\rho^2 + (1+|x|^2+\rho^2)^2 = 1 + 2(|x|^2+\rho^2) + (|x|^2-\rho^2)^2 \geqslant 1$$

So,

$$\frac{1+|x|^2+\rho^2+\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}}{2|x|\rho} > \frac{1+|x|^2+\rho^2}{2|x|\rho} > \frac{|x|^2+\rho^2}{2|x|\rho} \geqslant 1$$

On the other hand, we have

$$\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2} \leqslant (1+|x|^2+\rho^2)$$

so that

$$\frac{1+|x|^2+\rho^2-\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}}{2|x|\rho} > 0$$

and

$$\begin{aligned}&\frac{1+|x|^2+\rho^2-\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}}{2|x|\rho}\\ &= \frac{2|x|\rho}{1+|x|^2+\rho^2+\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}} \leqslant 1\end{aligned}$$

So, the only two poles of the integrand located inside the unit disk in the complex plane are $z=0$ and $z=z_1=-\left(\frac{1+|x|^2+\rho^2-\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}}{2|x|\rho}\right)$. The corresponding residues are

$$Res_0 = \frac{1}{i|x|\sqrt{(s-t)^2-\rho^2}}$$

$$Res_{z_1} = \frac{-1+|x|^2-\rho^2}{i|x|\sqrt{-4|x|^2\rho^2+(1+|x|^2+\rho^2)^2}\sqrt{(s-t)^2-\rho^2}}$$

Returning to (5.9), we get, for $t>s$,

$$(\partial_{|x|}u)(t,r) = -\frac{\lambda''(s)}{|x|}\int_0^{t-s}\frac{\rho}{\sqrt{(t-s)^2-\rho^2}}\left(1+\frac{|x|^2-1-\rho^2}{\sqrt{(1+|x|^2+\rho^2)^2-4|x|^2\rho^2}}\right)d\rho$$

By substitution, we see that $\partial_{|x|}u(t,r)$ solves (5.8) for $t>s$. We can extend the solution to $t<s$ with the same Cauchy data at $t=s$ by defining

$$v_s(t,r) = (-\partial_{|x|}u)(s-(t-s),r), \quad t<s$$

so that, for $t<s$, we have

$$v_s(t,r) = \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right)d\rho$$

We will only need to use pointwise estimates on v_1, which will be proven shortly, but to give the reader some idea of the behavior of v_s, we note that

$$v_s(t,r) \sim \frac{2\lambda''(s)\left(1-\sqrt{1-a^2}\right)}{a}, \quad 0<a=\frac{r}{(s-t)}<1 \ll s-t$$

We have

$$v_1(t,r) = \int_t^{\infty}\frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right)d\rho ds \qquad (5.10)$$

5.2.1 Pointwise estimates on $\partial_r^j v_1$

In this section, we will prove the following

Lemma 5.2.

$$v_1(t,r) = r\int_t^\infty \frac{\lambda''(s)}{1+s-t}ds + \text{Err}(t,r) \tag{5.11}$$

where

$$|\text{Err}(t,r)| \leqslant Cr\log(3+2r)\sup_{x\geqslant t}|\lambda''(x)|, \quad r>0$$

In addition,

$$|v_1(t,r)| \leqslant \frac{C}{r}\int_t^\infty |\lambda''(s)|(s-t)ds, \quad r>0 \tag{5.12}$$

Moreover, similar results are true for $\partial_r v_1$:

$$\partial_r v_1(t,r) = \int_t^\infty \frac{\lambda''(s)}{1+s-t}ds + E_{\partial_r v_1}(t,r) \tag{5.13}$$

with

$$|E_{\partial_r v_1}(t,r)| \leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)\log(3+2r), \quad r>0$$

and

$$|\partial_r v_1(t,r)| \leqslant \frac{C}{r^2}\left(\sup_{x\geqslant t}\left(|\lambda''(x)|(1+(x-t)^2)\right)+\int_t^\infty |\lambda''(s)|(s-t)ds\right), \quad r>2$$

Proof. We start with

$$\begin{aligned} v_1(t,r) &= \int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{s-t}\right)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds \\ &+ \int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds \end{aligned} \tag{5.14}$$

The first line of (5.14) can be estimated as follows

$$\begin{aligned} &|\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{s-t}\right)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds| \\ &\leqslant \frac{\sup_{x\geqslant t}\left(|\lambda''(x)|\right)}{r}\int_0^\infty \rho\int_{\rho+t}^\infty\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{s-t}\right)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)dsd\rho \\ &\leqslant \frac{\sup_{x\geqslant t}\left(|\lambda''(x)|\right)}{r}\int_0^\infty \rho\log(2)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho \\ &\leqslant C\frac{\sup_{x\geqslant t}\left(|\lambda''(x)|\right)}{r}\cdot r^2 \\ &\leqslant Cr\sup_{x\geqslant t}\left(|\lambda''(x)|\right) \end{aligned}$$

The second line of (5.14) is split into the following two terms

$$\begin{aligned}
&\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds\\
&=\int_t^{t+2(r+1)} \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds\\
&+\int_{t+2(r+1)}^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds
\end{aligned} \tag{5.15}$$

For the second line of (5.15), we have

$$\begin{aligned}
&\int_t^{t+2(r+1)} \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds\\
&=\int_t^{t+2(r+1)} \frac{\lambda''(s)}{r(s-t)}\frac{\left(1+r^2+(s-t)^2-\sqrt{(1+(r+s-t)^2)(1+(r-(s-t))^2)}\right)}{2}ds\\
&=\int_t^{t+2(r+1)} \frac{\lambda''(s)}{r(s-t)}\frac{2r^2(s-t)^2}{(1+r^2+(s-t)^2+\sqrt{(1+(r+s-t)^2)(1+(r-(s-t))^2)})}ds
\end{aligned}$$

So, we have

$$\begin{aligned}
&\left|\int_t^{t+2(r+1)} \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds\right|\\
&\leqslant Cr\sup_{x\geqslant t}\left(|\lambda''(x)|\right)\int_t^{t+2(r+1)}\frac{(s-t)}{1+r^2}ds\\
&\leqslant Cr\sup_{x\geqslant t}\left(|\lambda''(x)|\right)
\end{aligned}$$

For the third line of (5.15), we have

$$\begin{aligned}
&\int_{t+2(r+1)}^\infty \lambda''(s)\frac{2r(s-t)}{(1+r^2+(s-t)^2+\sqrt{(1+(r+s-t)^2)(1+(r-(s-t))^2)})}ds\\
&=\int_{t+2(r+1)}^\infty \lambda''(s)\frac{2r}{(s-t)}\left(\frac{1}{2}+O\left(\frac{(1+r)^2}{(s-t)^2}\right)\right)ds\\
&=\int_{t+2(r+1)}^\infty \lambda''(s)\frac{r}{(s-t)}ds+E_1(t,r)
\end{aligned}$$

where

$$\begin{aligned}
|E_1(t,r)| &\leqslant \sup_{x\geqslant t}\left(|\lambda''(x)|\right)r(1+r)^2\int_{t+2(r+1)}^\infty\frac{ds}{(s-t)^3}\\
&\leqslant Cr\sup_{x\geqslant t}\left(|\lambda''(x)|\right)
\end{aligned}$$

Next,

$$\int_{t+2(r+1)}^{\infty} \frac{\lambda''(s)r}{(s-t)}ds = \int_{t+2(r+1)}^{\infty} \frac{\lambda''(s)r}{1+s-t}ds + \int_{t+2(r+1)}^{\infty} \lambda''(s)r\left(\frac{1}{s-t} - \frac{1}{1+s-t}\right)ds$$

and

$$|\int_{t+2(r+1)}^{\infty} \lambda''(s)r\left(\frac{1}{s-t} - \frac{1}{1+s-t}\right)ds| \leqslant r\sup_{x\geqslant t}\left(|\lambda''(x)|\right)\log(1+\frac{1}{2+2r}) \leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)r$$

Finally,

$$\int_{t+2(r+1)}^{\infty} \frac{\lambda''(s)r}{1+s-t}ds = \int_{t}^{\infty} \frac{\lambda''(s)r}{1+s-t}ds - \int_{t}^{t+2(r+1)} \frac{\lambda''(s)r}{1+s-t}ds$$

with

$$|\int_{t}^{t+2(r+1)} \frac{\lambda''(s)r}{1+s-t}ds| \leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)r\log(3+2r)$$

Thus, we obtain the decomposition (5.11), with the desired estimate on Err.

Next, we have

$$|v_1(t,r)| = |\int_t^{\infty} \frac{\lambda''(s)}{r}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds| \leqslant \frac{C}{r}\int_t^{\infty}|\lambda''(s)|(s-t)ds$$

Now we treat $\partial_r v_1$:

$$\partial_r v_1(t,r) = \int_t^{\infty} \frac{-\lambda''(s)}{r^2}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds + \int_t^{\infty} \frac{\lambda''(s)}{r}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}d\rho ds \tag{5.16}$$

Even though the first line of (5.16) is equal to $\frac{-v_1}{r}$, the principal contribution to $\partial_r v_1$ near the origin actually comes from a combination of effects from appropriate parts of both the first and second lines of (5.16). So, we do not simply divide the previous v_1 estimates by $-r$ to treat the first line of (5.16). Instead, we split the first line of (5.16) as

$$\int_t^{\infty} \frac{-\lambda''(s)}{r^2}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds$$

$$= \int_t^{\infty} \frac{-\lambda''(s)}{r^2}\int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds + \int_t^{\infty} \frac{-\lambda''(s)}{r^2}\int_0^{s-t} \frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds$$

and

$$
\begin{aligned}
&|\int_t^\infty \frac{-\lambda''(s)}{r^2}\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds| \\
&\leqslant \frac{C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)}{r^2} \\
&\cdot\int_0^\infty \rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)\int_{\rho+t}^\infty\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{s-t}\right)dsd\rho \\
&\leqslant \frac{C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)}{r^2}\int_0^\infty \rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho \\
&\leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)
\end{aligned}
$$

The second line of (5.16) is split in the same way

$$
\begin{aligned}
&\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}d\rho ds \\
&=\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}d\rho ds \\
&+\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}d\rho ds
\end{aligned}
$$

and we again have

$$
\begin{aligned}
&|\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}|d\rho ds \\
&\leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)\int_0^\infty \rho\frac{\left(\rho^2+r^2+1\right)}{((\rho^2-r^2+1)^2+4r^2)^{3/2}}\int_{\rho+t}^\infty\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{s-t}\right)dsd\rho \\
&\leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)
\end{aligned}
$$

So far, we have

$$
\begin{aligned}
\partial_r v_1(t,r)=&\int_t^\infty \frac{-\lambda''(s)}{r^2}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds \\
&+\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}d\rho ds \\
&+E_{0,\partial_r v_1}
\end{aligned}
\tag{5.17}
$$

where

$$|E_{0,\partial_r v_1}(t,r)|\leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)$$

Now, we combine the first and second lines of (5.17), to get

$$\int_t^\infty \frac{-\lambda''(s)}{r^2}\int_0^{s-t}\frac{\rho}{(s-t)}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho ds$$
$$+\int_t^\infty \frac{\lambda''(s)}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}}d\rho ds$$
$$=-2\int_t^\infty \lambda''(s)(s-t)\left(\frac{(r^2-1-(s-t)^2)}{\sqrt{\beta}(1+r^2+(s-t)^2+\sqrt{\beta})}\right)ds$$

where

$$\beta=4r^2+(r^2-1-(s-t)^2)^2$$

Now, we proceed as in the estimates for v_1.

$$|-2\int_t^{t+2(r+1)}\lambda''(s)(s-t)\left(\frac{(r^2-1-(s-t)^2)}{\sqrt{\beta}(1+r^2+(s-t)^2+\sqrt{\beta})}\right)ds|$$
$$\leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)\int_t^{t+2(r+1)}\frac{|s-t|}{1+r^2}ds$$
$$\leqslant C\sup_{x\geqslant t}\left(|\lambda''(x)|\right)$$

Next, note that

$$\left(\frac{(r^2-1-(s-t)^2)}{\sqrt{\beta}(1+r^2+(s-t)^2+\sqrt{\beta})}\right)=\frac{-\left(1+\frac{1-r^2}{(s-t)^2}\right)}{(s-t)^2}\left(\frac{1}{\sqrt{1+q}}\cdot\frac{1}{1+y+\sqrt{1+q}}\right)$$

where

$$q=\frac{2(1-r^2)}{(s-t)^2}+\frac{(r^2+1)^2}{(s-t)^4}$$
$$y=\frac{1+r^2}{(s-t)^2}$$

So, for $s-t\geqslant 2(r+1)$,

$$|q|\leqslant\frac{9}{16},\quad |y|\leqslant\frac{1}{4}$$

Using this, we have

$$\left(\frac{(r^2-1-(s-t)^2)}{\sqrt{\beta}(1+r^2+(s-t)^2+\sqrt{\beta})}\right)=\frac{-1}{2(s-t)^2}\left(1+O\left(\frac{1+r^2}{(s-t)^2}\right)\right)$$

So,

$$-2\int_{t+2(r+1)}^\infty \lambda''(s)(s-t)\left(\frac{(r^2-1-(s-t)^2)}{\sqrt{\beta}(1+r^2+(s-t)^2+\sqrt{\beta})}\right)ds$$
$$=\int_{t+2(r+1)}^\infty\frac{\lambda''(s)}{(s-t)}ds+E_{1,\partial_r v_1}(t,r)$$

where

$$|E_{1,\partial_r v_1}(t,r)| \leqslant C \int_{t+2(r+1)}^{\infty} \frac{|\lambda''(s)|}{(s-t)} \frac{(1+r^2)}{(s-t)^2} ds \leqslant C \sup_{x \geqslant t} (|\lambda''(x)|)$$

Then,

$$\int_{t+2(r+1)}^{\infty} \frac{\lambda''(s)}{(s-t)} ds = \int_{t+2(r+1)}^{\infty} \frac{\lambda''(s)}{1+s-t} ds + E_{2,\partial_r v_1}(t,r)$$

with

$$|E_{2,\partial_r v_1}(t,r)| \leqslant \int_{t+2(r+1)}^{\infty} |\lambda''(s)| \left(\frac{1}{s-t} - \frac{1}{1+s-t} \right) ds \leqslant C \sup_{x \geqslant t} (|\lambda''(x)|)$$

Finally,

$$\begin{aligned} \int_{t+2(r+1)}^{\infty} \frac{\lambda''(s)}{1+s-t} ds &= \int_t^{\infty} \frac{\lambda''(s)}{1+s-t} ds - \int_t^{t+2(r+1)} \frac{\lambda''(s)}{1+s-t} ds \\ &= \int_t^{\infty} \frac{\lambda''(s)}{1+s-t} ds + E_{3,\partial_r v_1}(t,r) \end{aligned}$$

where

$$|E_{3,\partial_r v_1}(t,r)| \leqslant C \sup_{x \geqslant t} (|\lambda''(x)|) \log(3+2r)$$

and we get the desired result

$$\partial_r v_1(t,r) = \int_t^{\infty} \frac{\lambda''(s)}{1+s-t} ds + E_{\partial_r v_1}(t,r)$$

with

$$|E_{\partial_r v_1}(t,r)| \leqslant C \sup_{x \geqslant t} (|\lambda''(x)|) \log(3+2r)$$

It remains to prove the last estimate in the lemma statement: The first term of (5.16) is estimated by

$$\begin{aligned} &| \int_t^{\infty} \frac{-\lambda''(s)}{r^2} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}} \right) d\rho ds| \\ &\leqslant \frac{C}{r^2} \int_t^{\infty} |\lambda''(s)|(s-t) ds \end{aligned}$$

Turning to the second term of (5.16), we have

$$\begin{aligned} &| \int_t^{\infty} \frac{\lambda''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{4r\left(\rho^2+r^2+1\right)}{\left(\left(\rho^2-r^2+1\right)^2+4r^2\right)^{3/2}} d\rho ds| \\ &\leqslant C \int_0^{\infty} \frac{\rho(\rho^2+r^2+1)}{((\rho^2-r^2+1)^2+4r^2)^{3/2}} \int_{\rho+t}^{\infty} \frac{|\lambda''(s)|(1+(s-t)^2)}{\sqrt{(s-t)^2-\rho^2}(1+(s-t)^2)} ds d\rho \\ &\leqslant C \text{sup}_{x \geqslant t} \left(|\lambda''(x)|(1+(x-t)^2) \right) \\ &\cdot \int_0^{\infty} \frac{\rho(\rho^2+r^2+1)}{((\rho^2-r^2+1)^2+4r^2)^{3/2}} \int_{\rho+t}^{\infty} \frac{1}{\sqrt{(s-t)^2-\rho^2}} \frac{1}{(1+(s-t)^2)} ds d\rho \\ &\leqslant C \text{sup}_{x \geqslant t} \left(|\lambda''(x)|(1+(x-t)^2) \right) \int_0^{\infty} \frac{(\rho^2+r^2+1)}{((\rho^2-r^2+1)^2+4r^2)^{3/2}} \frac{d\rho}{\sqrt{1+\rho^2}} \end{aligned}$$

Let us first note that, for $\rho \leqslant \frac{r}{2}$, we have

$$r^2 - 1 - \rho^2 > Cr^2, \quad \text{for } r > 2$$

so,

$$\frac{(\rho^2 + r^2 + 1)}{((\rho^2 - r^2 + 1)^2 + 4r^2)^{3/2}} \leqslant \frac{C}{r^4}$$

and

$$\int_0^{r/2} \frac{(\rho^2 + r^2 + 1)}{((\rho^2 - r^2 + 1)^2 + 4r^2)^{3/2}} \frac{d\rho}{\sqrt{1+\rho^2}} \leqslant \frac{C}{r^4} \sinh^{-1}\left(\frac{r}{2}\right)$$

On the other hand, when $\rho \geqslant r/2$, we have

$$\begin{aligned}
&\int_{r/2}^{\infty} \frac{(\rho^2 + r^2 + 1)}{((\rho^2 - r^2 + 1)^2 + 4r^2)^{3/2}} \frac{d\rho}{\sqrt{1+\rho^2}} \\
&\leqslant C \int_{r/2}^{\infty} \frac{(\rho^2 + r^2 + 1)}{\rho((\rho^2 - r^2 + 1)^2 + 4r^2)^{3/2}} d\rho \\
&= \frac{r^2 + \frac{3r^4}{\sqrt{9r^4+40r^2+16}} - \frac{9r^2}{\sqrt{9r^4+40r^2+16}} - \frac{12}{\sqrt{9r^4+40r^2+16}}}{4\left(r^2+1\right)^2} \\
&+ \frac{2\log\left(3r^4 + \left(\sqrt{9r^4+40r^2+16}+9\right)r^2 + \sqrt{9r^4+40r^2+16}+4\right) - 4\log(r) + 1 - \log(4)}{4\left(r^2+1\right)^2} \\
&\leqslant \frac{C}{r^2}
\end{aligned}$$

So, we conclude that

$$\begin{aligned}
&\left|\int_t^{\infty} \frac{\lambda''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2 - \rho^2}} \frac{4r\left(\rho^2 + r^2 + 1\right)}{\left(\left(\rho^2 - r^2 + 1\right)^2 + 4r^2\right)^{3/2}} d\rho ds\right| \\
&\leqslant C \text{sup}_{x \geqslant t}\left(|\lambda''(x)|(1 + (x-t)^2)\right) \frac{1}{r^2}
\end{aligned}$$

Combining these estimates, we get

$$|\partial_r v_1(t,r)| \leqslant \frac{C}{r^2}\left(\text{sup}_{x \geqslant t}\left(|\lambda''(x)|(1+(x-t)^2)\right) + \int_t^{\infty} |\lambda''(s)|(s-t) ds\right), \quad r > 2$$

□

5.3 The free wave correction

The next correction is designed to produce a crucial source term in the equation that we will eventually use to choose a specific λ. In fact, it is this correction which ultimately determines the asymptotics of the λ which we will choose. Define v_2 to be the solution to

$$\begin{cases} -\partial_{tt} v_2 + \partial_{rr} v_2 + \frac{1}{r}\partial_r v_2 - \frac{v_2}{r^2} = 0 \\ v_2(0) = 0 \\ \partial_t v_2(0) = v_{2,0} \end{cases}$$

where

$$\widehat{v_{2,0}}(\xi) = \int_0^\infty v_{2,0}(r) J_1(r\xi) r dr = \begin{cases} \frac{4b}{\pi(b-1)} \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}, & b \neq 1 \\ \frac{-4}{\pi} \chi_{\leqslant \frac{1}{4}}(\xi) \log(\log(\frac{1}{\xi})), & b = 1 \end{cases}$$

where $\chi_{\leqslant \frac{1}{4}} \in C_c^\infty([0,\infty))$, $0 \leqslant \chi_{\leqslant \frac{1}{4}}(x) \leqslant 1$ for all x, and $\chi_{\leqslant \frac{1}{4}}$ satisfies

$$\chi_{\leqslant \frac{1}{4}}(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \frac{1}{8} \\ 0, & x > \frac{1}{4} \end{cases}$$

and is otherwise arbitrary. For ease of notation, let

$$c_b = \begin{cases} \frac{4b}{\pi(b-1)}, & b \neq 1 \\ \frac{-4}{\pi}, & b = 1 \end{cases}$$

Note that this particular form of c_b is due to the fact that part of $\widehat{v_{2,0}}(\xi)$ involves an antiderivative of $\frac{1}{\xi \log^b(\frac{1}{\xi})}$. We have a formula for v_2, namely:

$$v_2(t,r) = c_b \int_0^\infty \sin(t\xi) J_1(r\xi) \chi_{\leqslant \frac{1}{4}}(\xi) \cdot \begin{cases} \frac{1}{\log^{b-1}(\frac{1}{\xi})}, & b \neq 1 \\ \log(\log(\frac{1}{\xi})), & b = 1 \end{cases} d\xi \tag{5.18}$$

The significance of this particular choice of Cauchy data will be seen later on, when we identify the v_2-related contribution to the equation we use to choose λ.

We will prove pointwise estimates on v_2 later on, but, to give the reader some idea of the behavior of v_2, we note that (for instance, for $b > 1$)

$$v_2(t,r) = \frac{c_b(1-\text{sgn}(t-r))}{2\sqrt{2r}\sqrt{|t-r|}\log^{b-1}(|t-r|)} + E_2(t,r)$$

with

$$|E_2(t,r)| \leqslant C\left(\frac{1}{\sqrt{r}\sqrt{|t-r|}\log^b(|t-r|)}\right) + \frac{C}{\sqrt{r}\sqrt{t+r}\log^{b-1}(t+r)} + \frac{C}{r\log^{b-1}(r)}$$
$$r > \frac{t}{2}, \quad |t-r| > 5$$

This can be established by a procedure similar to the one which we use later on to compute the inner product of the v_2 linear error term with $\phi_0(\frac{\cdot}{\lambda(t)})$.

5.4 Further improvement of the soliton error term

If we substitute $u = Q_{\frac{1}{\lambda(t)}} + v_1 + v_2 + u_3$ into the wave maps equation

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = 0$$

we get

$$
\begin{aligned}
&- \partial_{tt} u_3 + \partial_{rr} u_3 + \frac{1}{r}\partial_r u_3 - \frac{\cos(2Q_{\frac{1}{\lambda(t)}})}{r^2} u_3 \\
&= \partial_{tt} Q_{\frac{1}{\lambda(t)}} + \frac{2\lambda''(t) r}{1+r^2} + \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right) v_1 + \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right) v_2 \\
&+ N_2(v_1+v_2) + N(u_3) + L(u_3)
\end{aligned} \tag{5.19}
$$

where

$$N(f) = \left(\frac{\sin(2f)-2f}{2r^2}\right)\cos(2Q_{\frac{1}{\lambda(t)}}) + \left(\frac{\cos(2f)-1}{2r^2}\right)\sin(2(Q_{\frac{1}{\lambda(t)}}+v_1+v_2))$$

$$L(f) = \frac{\sin(2f)}{2r^2}\cos(2Q_{\frac{1}{\lambda(t)}})(\cos(2(v_1+v_2))-1) - \frac{\sin(2f)}{2r^2}\sin(2Q_{\frac{1}{\lambda(t)}})\sin(2(v_1+v_2))$$

$$N_2(f) = \frac{\sin(2Q_{\frac{1}{\lambda(t)}})}{2r^2}(\cos(2f)-1) + \frac{\cos(2Q_{\frac{1}{\lambda(t)}})}{2r^2}(\sin(2f)-2f)$$

Note that

$$F_0(t,r) = \partial_{tt} Q_{\frac{1}{\lambda(t)}} + \frac{2\lambda''(t) r}{1+r^2}$$

appears on the right-hand side of (5.19). When the spatial coordinate is renormalized, via

$$r = R\lambda(t)$$

one term arising in the the large R expansion of F_0 has insufficient decay in time. To remedy this, we will add another correction, to be denoted v_3. On one hand, choosing v_3 to solve an equation whose right-hand side is exactly equal to F_0 would eventually lead to a much more difficult equation that we use to determine $\lambda(t)$. On the other hand, the error terms remaining after adding the correction v_3 should no longer have insufficient decay in time for large values of R. We therefore proceed as follows: recall that α has been introduced just before (5.1), and satisfies

$$0 < \alpha < \min\{\frac{1}{b(1040!)}, \frac{1}{1040!}\}$$

and let

$$F_{0,1}(t,r) = \frac{2r\lambda''(t)}{(\lambda(t)^2+r^2)}\left(\frac{-1+\lambda(t)^2}{1+r^2} + \frac{1-\lambda(t)^{2\alpha}}{1+r^2\lambda(t)^{2\alpha-2}}\right)$$

Next, we consider v_3, defined as the solution (with 0 Cauchy data at infinity) to the equation

$$-\partial_{tt} v_3 + \partial_{rr} v_3 + \frac{1}{r}\partial_r v_3 - \frac{v_3}{r^2} = F_{0,1}(t,r)$$

Following the same steps as for v_1, we get

$$
\begin{aligned}
&v_3(t,r) \\
&= -\frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{F_{0,1}(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})(r+\rho\cos(\theta))}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}} d\theta d\rho ds
\end{aligned}
$$

which gives

$$v_3(t,r) = -\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda''(s) \cdot \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + \frac{\lambda(s)^2-(r^2-\rho^2)\lambda(s)^{2\alpha}}{\lambda(s)^2\sqrt{1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}}}\right) d\rho ds \tag{5.20}$$

The main result of this section is a decomposition of v_3 which will be useful in understanding its contribution to the equation for λ:

Lemma 5.3.

$$v_3(t,r) = -r\int_{t+6r}^\infty \lambda''(s)(s-t)\left(\frac{1}{1+(s-t)^2} - \frac{1}{\lambda(t)^{2-2\alpha}+(s-t)^2}\right) ds + E_5(t,r) \tag{5.21}$$

where

$$\begin{aligned}|E_5(t,r)| \leqslant{}& C\left(\sup_{x\geqslant t}|\lambda''(x)|\right)\cdot \sup_{x\geqslant t}\left(\frac{|\lambda'(x)|\lambda(t)^\alpha}{\lambda(x)^\alpha}\right)\lambda(t)^{1-2\alpha}\\ &+ Cr\sup_{x\geqslant t}\left(|\lambda''(x)|\lambda(x)^{\alpha-1}(\lambda(x)^{\alpha-1}-\lambda(t)^{\alpha-1})\right)\lambda(t)^{2-2\alpha}\\ &+ Cr\sup_{x\geqslant t}|\lambda''(x)|\end{aligned}$$

Proof.

$$v_3(t,r) = -\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda''(s) \cdot \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + \frac{\lambda(s)^2-(r^2-\rho^2)\lambda(s)^{2\alpha}}{\lambda(s)^2\sqrt{1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}}}\right) d\rho ds$$

v_3 is then decomposed as

$$v_3(t,r) = v_{3,1}(t,r) + v_{3,2}(t,r)$$

where

$$v_{3,1}(t,r) = -\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s) \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + \frac{\lambda(s)^2-(r^2-\rho^2)\lambda(s)^{2\alpha}}{\lambda(s)^2\sqrt{1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}}}\right) d\rho ds \tag{5.22}$$

$$v_{3,2} = v_3 - v_{3,1}$$

Then, we record some pointwise estimates on $v_{3,2}$. If

$$\begin{aligned} F_3(r,\rho,\lambda(s)) &= \frac{1-(r^2-\rho^2)\lambda(s)^{2\alpha-2}}{\sqrt{1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}}} \\ &= \frac{1-(r^2-\rho^2)\lambda(s)^{2\alpha-2}}{\sqrt{4r^2\lambda(s)^{2\alpha-2}+(1-(r^2-\rho^2)\lambda(s)^{2\alpha-2})^2}} \end{aligned}$$

then,

$$\begin{aligned} v_{3,2}(t,r) = &\frac{-1}{r}\int_t^\infty \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\lambda''(s) \\ &\cdot \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(t))\right) d\rho ds \\ &-\frac{1}{r}\int_t^\infty \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\lambda''(s) \\ &\cdot \left(F_3(r,\rho,\lambda(s)) - F_3(r,\rho,\lambda(t))\right) d\rho ds \end{aligned} \tag{5.23}$$

For the first line of (5.23), we have

$$\begin{aligned} &|-\frac{1}{r}\int_t^\infty \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{s-t}\right)\lambda''(s) \\ &\qquad \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(t))\right) d\rho ds| \\ &\leqslant \frac{C}{r}\left(\sup_{x\geqslant t}|\lambda''(x)|\right) \\ &\cdot \int_0^\infty \rho \left|\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + 1 - 1 + F_3(r,\rho,\lambda(t))\right| \\ &\qquad \int_{t+\rho}^\infty \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{s-t}\right) ds d\rho \\ &\leqslant \frac{C}{r}\left(\sup_{x\geqslant t}|\lambda''(x)|\right) \Big(\int_0^\infty \rho \left(1 - \frac{1+\rho^2-r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}\right) d\rho \\ &\qquad + \int_0^\infty \rho \left(1 - F_3(r,\rho,\lambda(t))\right) d\rho \Big) \\ &\leqslant Cr\left(\sup_{x\geqslant t}|\lambda''(x)|\right) \end{aligned}$$

where we used the facts that

$$\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} \geqslant -1$$

and

$$F_3(r,\rho,\lambda(t)) \leqslant 1$$

To estimate the second line of (5.23), we first note that

$$\begin{aligned}&F_3(r,\rho,\lambda(s)) - F_3(r,\rho,\lambda(t))\\&= \int_0^1 \frac{-4r^2 z_\sigma (1+(r^2-\rho^2)z_\sigma^2)}{(1+2(\rho^2+r^2)z_\sigma^2+(\rho^2-r^2)^2 z_\sigma^4)^{3/2}} \left(\lambda(s)^{\alpha-1} - \lambda(t)^{\alpha-1}\right) d\sigma\end{aligned}$$

where

$$z_\sigma = \sigma\lambda(s)^{\alpha-1} + (1-\sigma)\lambda(t)^{\alpha-1}$$

First, we note that

$$1+2(\rho^2+r^2)z^2 + (\rho^2-r^2)^2 z^4 = 4\rho^2 z^2 + (1+(r^2-\rho^2)z^2)^2$$

So,

$$\begin{aligned}&\left|\frac{-4r^2 z_\sigma (1+(r^2-\rho^2)z_\sigma^2)}{(1+2(\rho^2+r^2)z_\sigma^2+(\rho^2-r^2)^2 z_\sigma^4)^{3/2}} \left(\lambda(s)^{\alpha-1} - \lambda(t)^{\alpha-1}\right)\right|\\&\leqslant C \frac{r^2 |z_\sigma| \left(\lambda(s)^{\alpha-1} - \lambda(t)^{\alpha-1}\right)}{(1+2(\rho^2+r^2)z_\sigma^2+(\rho^2-r^2)^2 z_\sigma^4)}\end{aligned}$$

Since λ is a decreasing function, and $0 < \alpha < \frac{1}{8}$, we have

$$\lambda(t)^{\alpha-1} \leqslant |z_\sigma| = |\sigma\lambda(s)^{\alpha-1} + (1-\sigma)\lambda(t)^{\alpha-1}| \leqslant \lambda(s)^{\alpha-1}, \quad 0 \leqslant \sigma \leqslant 1, \quad s \geqslant t$$

So,

$$\begin{aligned}&|F_3(r,\rho,\lambda(s)) - F_3(r,\rho,\lambda(t))|\\&\leqslant C \frac{r^2 \lambda(s)^{\alpha-1} |\lambda(s)^{\alpha-1} - \lambda(t)^{\alpha-1}|}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2} + (\rho^2-r^2)^2 \lambda(t)^{4\alpha-4})}, \quad s \geqslant t\end{aligned}$$

This gives

$$\begin{aligned}&\left| -\frac{1}{r}\int_t^\infty \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \lambda''(s) \left(F_3(r,\rho,\lambda(s)) - F_3(r,\rho,\lambda(t))\right) d\rho ds\right|\\&\leqslant Cr \int_0^\infty \rho \int_{\rho+t}^\infty \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \frac{|\lambda''(s)| \lambda(s)^{\alpha-1} |\lambda(s)^{\alpha-1} - \lambda(t)^{\alpha-1}| ds d\rho}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2} + (\rho^2-r^2)^2 \lambda(t)^{4\alpha-4})}\\&\leqslant Cr \sup_{x \geqslant t} \left(|\lambda''(x)| \lambda(x)^{\alpha-1} |\lambda(x)^{\alpha-1} - \lambda(t)^{\alpha-1}|\right)\\&\cdot \int_0^\infty \frac{\rho d\rho}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2} + (\rho^2-r^2)^2 \lambda(t)^{4\alpha-4})}\end{aligned}$$

Then, if $2r > \lambda(t)^{1-\alpha}$, we have

$$\begin{aligned}
&\int_0^\infty \frac{\rho d\rho}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(t)^{4\alpha-4})}\\
&\leqslant C\left(\int_0^{\lambda(t)^{1-\alpha}} \rho d\rho + \int_{\lambda(t)^{1-\alpha}}^{2r} \frac{\rho d\rho}{r^2\lambda(t)^{2\alpha-2}} + \int_{2r}^\infty \frac{\rho d\rho}{\rho^4 \lambda(t)^{4\alpha-4}}\right)\\
&\leqslant C\lambda(t)^{2-2\alpha}
\end{aligned}$$

On the other hand, if $2r \leqslant \lambda(t)^{1-\alpha}$, we have

$$\begin{aligned}
&\int_0^\infty \frac{\rho d\rho}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(t)^{4\alpha-4})}\\
&\leqslant \int_0^{\lambda(t)^{1-\alpha}} \rho d\rho + \int_{\lambda(t)^{1-\alpha}}^\infty \frac{d\rho}{\rho^3\lambda(t)^{4\alpha-4}}\\
&\leqslant C\lambda(t)^{2-2\alpha}
\end{aligned}$$

In total, we get

$$\begin{aligned}
&|-\frac{1}{r}\int_t^\infty \int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\lambda''(s)\left(F_3(r,\rho,\lambda(s)) - F_3(r,\rho,\lambda(t))\right) d\rho ds|\\
&\leqslant Cr \sup_{x\geqslant t}\left(|\lambda''(x)|\lambda(x)^{\alpha-1}|\lambda(x)^{\alpha-1}-\lambda(t)^{\alpha-1}|\right)\lambda(t)^{2-2\alpha}
\end{aligned}$$

It then suffices to study $v_{3,1}$. Firstly, we have

$$\begin{aligned}
&|-\frac{1}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds|\\
&\leqslant \frac{1}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}|\lambda''(s)| \cdot 2 d\rho ds\\
&\leqslant Cr\sup_{x\geqslant t}|\lambda''(x)|
\end{aligned}$$

Next, we have

$$\begin{aligned}
&-\frac{1}{r}\int_{t+6r}^\infty\int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds\\
&= -2r\int_{6r}^\infty \lambda''(t+w)w\left(\frac{1}{(1+w^2)\left(\frac{r^2}{1+w^2}+1+\sqrt{(\frac{r^2}{1+w^2}+1)^2-4\frac{r^2w^2}{(1+w^2)^2}}\right)}\right.\\
&\left.+\frac{-1}{(\lambda(t+w)^{2-2\alpha}+w^2)\left(\frac{r^2}{\lambda(t+w)^{2-2\alpha}+w^2}+1+\sqrt{(\frac{r^2}{\lambda(t+w)^{2-2\alpha}+w^2}+1)^2-\frac{4r^2w^2}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}}\right)}\right)dw
\end{aligned}$$

Using the fact that $w = s - t \geqslant 6r$ in the integral below, we get

$$
\begin{aligned}
&-\frac{1}{r}\int_{t+6r}^{\infty}\int_0^{s-t}\frac{\rho}{(s-t)}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds\\
&= -2r\int_{6r}^{\infty}\lambda''(t+w)w\left(\frac{1}{2(1+w^2)}-\frac{1}{2(\lambda(t+w)^{2-2\alpha}+w^2)}\right)dw + E_4
\end{aligned}
$$

where

$$
\begin{aligned}
|E_4| &\leqslant Cr\sup_{x\geqslant t}|\lambda''(x)|\int_{6r}^{\infty}\frac{r^2w}{w^2}\left(\frac{1}{1+w^2}+\frac{1}{\lambda(t+w)^{2-2\alpha}+w^2}\right)dw\\
&\leqslant Cr\sup_{x\geqslant t}|\lambda''(x)|
\end{aligned}
$$

So, it suffices to study

$$
-2r\int_{6r}^{\infty}\lambda''(t+w)w\left(\frac{1}{2(1+w^2)}-\frac{1}{2(\lambda(t+w)^{2-2\alpha}+w^2)}\right)dw
$$

We will make one more reduction, which is to replace $\lambda(t+w)$ in the above expression with $\lambda(t)$. The error in doing this replacement is

$$
-r\int_{t+6r}^{\infty}\lambda''(s)(s-t)\left(\frac{1}{(\lambda(t)^{2-2\alpha}+(s-t)^2)}-\frac{1}{(\lambda(s)^{2-2\alpha}+(s-t)^2)}\right)ds
$$

But, if

$$
F_4(x, s-t) = \frac{1}{x^2+(s-t)^2}
$$

then,

$$
\begin{aligned}
|F_4(\lambda(s)^{1-\alpha}, s-t) - F_4(\lambda(t)^{1-\alpha}, s-t)| &\leqslant C\frac{\lambda(t)^{1-\alpha}}{(s-t)^4}\cdot|\lambda(t)^{1-\alpha}-\lambda(s)^{1-\alpha}|\\
&\leqslant C\frac{\lambda(t)^{1-2\alpha}}{(s-t)^3}\cdot\sup_{x\geqslant t}\left(\frac{|\lambda'(x)|\lambda(t)^{\alpha}}{\lambda(x)^{\alpha}}\right)
\end{aligned}
$$

where we use the fact that λ is a decreasing function. So,

$$
\begin{aligned}
&|-r\int_{t+6r}^{\infty}\lambda''(s)(s-t)\left(\frac{1}{(\lambda(t)^{2-2\alpha}+(s-t)^2)}-\frac{1}{(\lambda(s)^{2-2\alpha}+(s-t)^2)}\right)ds|\\
&\leqslant Cr\left(\sup_{x\geqslant t}|\lambda''(x)|\right)\cdot\sup_{x\geqslant t}\left(\frac{|\lambda'(x)|\lambda(t)^{\alpha}}{\lambda(x)^{\alpha}}\right)\lambda(t)^{1-2\alpha}\int_{6r}^{\infty}\frac{w}{w^3}dw\\
&\leqslant C\left(\sup_{x\geqslant t}|\lambda''(x)|\right)\cdot\sup_{x\geqslant t}\left(\frac{|\lambda'(x)|\lambda(t)^{\alpha}}{\lambda(x)^{\alpha}}\right)\lambda(t)^{1-2\alpha}
\end{aligned}
$$

This finally gives (5.21). □

5.5 The linear error terms for large r

Despite the decay of

$$\frac{1-\cos(2Q_1(\frac{r}{\lambda(t)}))}{r^2}$$

for large r, we will still need to add another correction which improves the linear error terms of v_1, v_2, v_3, as well as $F_{0,2} = -F_{0,1} + \frac{2\lambda''(t)r}{1+r^2} + \partial_t^2 Q_{\frac{1}{\lambda(t)}}$, for large r. The addition of this correction will not change the leading order contribution of these error terms to the modulation equation, but, will improve the overall error term of the final ansatz for large r. Let $\chi_{\geqslant 1} \in C^{\infty}(\mathbb{R})$ satisfy

$$\chi_{\geqslant 1}(x) = \begin{cases} 1, & x \geqslant 2 \\ 0, & x < 1 \end{cases}$$

and

$$0 \leqslant \chi_{\geqslant 1}(x) \leqslant 1, \quad x \in \mathbb{R}$$

Then, we recall that N has been defined just before (5.1), let

$$v_{4,c}(t,r) = \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2}\right)(v_1+v_2+v_3) + F_{0,2}(t,r)\right)$$

and define v_4 as the solution to

$$-\partial_{tt}v_4 + \partial_{rr}v_4 + \frac{1}{r}\partial_r v_4 - \frac{v_4}{r^2} = v_{4,c}(t,r)$$

with 0 Cauchy data at infinity. In other words, we have

$$v_4(t,r) = \int_t^{\infty} v_{4,s}(t,r)ds$$

where $v_{4,s}$ solves

$$\begin{cases} -\partial_{tt}v_{4,s} + \partial_{rr}v_{4,s} + \frac{1}{r}\partial_r v_{4,s} - \frac{v_{4,s}}{r^2} = 0 \\ v_{4,s}(s,r) = 0 \\ \partial_t v_{4,s}(s,r) = v_{4,c}(s,r) \end{cases}$$

So,

$$v_{4,s}(t,r) = \partial_r w_{4,s}(t,r)$$

where

$$u_{4,s}(t,y) = w_{4,s}(t,|y|)$$

and $u_{4,s} : [T_0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ solves

$$\begin{cases} \partial_{tt}u_{4,s} - \Delta u_{4,s} = 0 \\ u_{4,s}(s,x) = 0 \\ \partial_t u_{4,s}(s,x) = -\int_{|x|}^{\infty} v_{4,c}(s,q)dq \end{cases}$$

We get, for $s \geqslant t$,

$$u_{4,s}(t,x) = \frac{1}{2\pi}\int_{B_{s-t}(x)} \frac{\left(\int_{|y|}^{\infty} v_{4,c}(s,q)dq\right)}{\sqrt{(s-t)^2-|y-x|^2}}dy = \frac{1}{2\pi}\int_{B_{s-t}(0)} \frac{\left(\int_{|z+x|}^{\infty} v_{4,c}(s,q)dq\right)}{\sqrt{(s-t)^2-|z|^2}}dz$$

We use polar coordinates in the z variable, with origin 0, and polar axis $\hat{x}$ for $x \neq 0$. Then, we obtain, apriori for $r \neq 0$,

$$\begin{aligned}&v_4(t,r)\\ &= \frac{-1}{2\pi}\int_t^{\infty}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi} \frac{v_{4,c}(s,\sqrt{r^2+2r\rho\cos(\theta)+\rho^2})\,(r+\rho\cos(\theta))}{\sqrt{r^2+2r\rho\cos(\theta)+\rho^2}}d\theta d\rho ds\end{aligned}$$

If we let

$$\begin{aligned}G(s,r,\rho) &= \int_0^{2\pi} \frac{v_{4,c}(s,\sqrt{r^2+2r\rho\cos(\theta)+\rho^2})}{\sqrt{r^2+2r\rho\cos(\theta)+\rho^2}}\,(r+\rho\cos(\theta))\,d\theta \\ &s \geqslant t, \quad r \geqslant 0, \quad s-t \geqslant \rho \geqslant 0\end{aligned} \tag{5.24}$$

Then,

$$G(s,0,\rho) = 0$$

and

$$\begin{aligned}G(s,r,\rho) &= r\int_0^1 \partial_2 G(s,r\beta,\rho)d\beta\\ &= r\int_0^1\int_0^{2\pi}\Bigg(\partial_2 v_{4,c}(s,\sqrt{\beta^2r^2+2\beta r\rho\cos(\theta)+\rho^2})\frac{(\beta r+\rho\cos(\theta))^2}{\beta^2r^2+2\beta r\rho\cos(\theta)+\rho^2}\\ &-\ v_{4,c}(s,\sqrt{\beta^2r^2+2\beta r\rho\cos(\theta)+\rho^2})\frac{(\beta r+\rho\cos(\theta))^2}{(\beta^2r^2+2\beta r\rho\cos(\theta)+\rho^2)^{3/2}}\\ &+\frac{v_{4,c}(s,\sqrt{\beta^2r^2+2\beta r\rho\cos(\theta)+\rho^2})}{\sqrt{\beta^2r^2+2\beta r\rho\cos(\theta)+\rho^2}}\Bigg)\,d\theta d\beta\end{aligned}$$

So, $v_4(t,\cdot)$ is (for instance) continuous on $[0,\infty)$ and we have, for all $r \geqslant 0$ (including $r=0$)

$$v_4(t,r) = \frac{-r}{2\pi}\int_t^{\infty}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^1 \partial_2 G(s,r\beta,\rho)d\beta d\rho ds \tag{5.25}$$

We will use this formula to prove estimates on v_4, but this will be done later on, once we further restrict the class of functions λ under consideration (which will be done once we introduce an iteration space in which to solve the eventual equation for λ).

5.6 The nonlinear error terms involving v_1, v_2, v_3, v_4

Let

$$f_{v_5} = v_1 + v_2 + v_3 + v_4$$

and

$$N_2(f_{v_5})(t,r) = \frac{\sin(2Q_{\frac{1}{\lambda(t)}}(r))}{2r^2}(\cos(2f_{v_5})-1) + \frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))}{2r^2}(\sin(2f_{v_5})-2f_{v_5})$$

Then, we consider v_5, defined as the solution with 0 Cauchy data at infinity, to the problem

$$-\partial_{tt}v_5 + \partial_{rr}v_5 + \frac{1}{r}\partial_r v_5 - \frac{v_5}{r^2} = N_2(f_{v_5})(t,r)$$

Following the same steps used to obtain (5.25), we obtain the analogous formula for v_5:

$$v_5(t,r) = \frac{-r}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^1 \partial_2 G_5(s, r\beta, \rho) d\beta d\rho ds$$

where

$$\begin{aligned} G_5(s,r,\rho) &= r\int_0^1 \partial_2 G_5(s, r\beta, \rho) d\beta \\ &= r\int_0^1 \int_0^{2\pi} \Bigg(\partial_2 N_2(f_{v_5})(s, \sqrt{\beta^2 r^2 + 2\beta r\rho\cos(\theta)+\rho^2}) \frac{(\beta r + \rho\cos(\theta))^2}{\beta^2 r^2 + 2\beta r\rho\cos(\theta)+\rho^2} \\ &- N_2(f_{v_5})(s, \sqrt{\beta^2 r^2 + 2\beta r\rho\cos(\theta)+\rho^2}) \frac{(\beta r + \rho\cos(\theta))^2}{(\beta^2 r^2 + 2\beta r\rho\cos(\theta)+\rho^2)^{3/2}} \\ &+ \frac{N_2(f_{v_5})(s, \sqrt{\beta^2 r^2 + 2\beta r\rho\cos(\theta)+\rho^2})}{\sqrt{\beta^2 r^2 + 2\beta r\rho\cos(\theta)+\rho^2}}\Bigg) d\theta d\beta \end{aligned}$$

As was the case for v_4, we will prove estimates on v_5 later on, once we further restrict the class of functions λ under consideration.

5.7 The equation resulting from u_{ansatz}

If we substitute

$$u = Q_{\frac{1}{\lambda(t)}} + v_1 + v_2 + v_3 + v_4 + v_5 + v_6$$

into the wave maps equation

$$-\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u - \frac{\sin(2u)}{2r^2} = 0$$

and use the equations solved by v_1, v_2, v_3, v_4, v_5, then, we obtain

$$\begin{aligned}
&-\partial_{tt}v_6 + \partial_{rr}v_6 + \frac{1}{r}\partial_r v_6 - \frac{\cos(2Q_{\frac{1}{\lambda(t)}})}{r^2}v_6 \\
&= F_{0,2}(t,r) + N(v_6) + L_1(v_6) + N_2(v_5) \\
&+ \frac{\sin(2(v_1+v_2+v_3+v_4))}{2r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}} + 2v_5) - \cos(2Q_{\frac{1}{\lambda(t)}})\right) \\
&+ \left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{2r^2}\right)\left(\sin(2Q_{\frac{1}{\lambda(t)}} + 2v_5) - \sin(2Q_{\frac{1}{\lambda(t)}})\right) \\
&+ \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)(v_1+v_2+v_3+v_4+v_5) \\
&- \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)(v_1+v_2+v_3) - \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}(t,r)
\end{aligned} \tag{5.26}$$

where

$$\begin{aligned}
N(f) &= \left(\frac{\sin(2f)-2f}{2r^2}\right)\cos(2Q_{\frac{1}{\lambda(t)}}) \\
&\quad + \left(\frac{\cos(2f)-1}{2r^2}\right)\sin(2(Q_{\frac{1}{\lambda(t)}} + v_1+v_2+v_3+v_4+v_5))
\end{aligned}$$

$$\begin{aligned}
L_1(f) &= \frac{\sin(2f)}{2r^2}\cos(2Q_{\frac{1}{\lambda(t)}})(\cos(2(v_1+v_2+v_3+v_4+v_5))-1) \\
&\quad - \frac{\sin(2f)}{2r^2}\sin(2Q_{\frac{1}{\lambda(t)}})\sin(2(v_1+v_2+v_3+v_4+v_5)) \\
N_2(f) &= \frac{\sin(2Q_{\frac{1}{\lambda(t)}})}{2r^2}(\cos(2f)-1) + \frac{\cos(2Q_{\frac{1}{\lambda(t)}})}{2r^2}(\sin(2f)-2f) \\
F_{0,2}(t,r) &= -F_{0,1}(t,r) + \frac{2\lambda''(t)r}{1+r^2} + \partial_t^2 Q_{\frac{1}{\lambda(t)}}
\end{aligned}$$

Note that we do not combine the terms involving $\chi_{\geqslant 1}$ with analogous terms having coefficient 1 because the terms involving $\chi_{\geqslant 1}$ will turn out to have a subleading contribution to the modulation equation for λ. When we solve the final equation, after choosing λ, we will of course make use of the fact that $1 - \chi_{\geqslant 1}(x)$ is supported on the set $x \leqslant 2$.

We can re-write the right-hand side of (5.26) as $F + F_3$ where

$$F = F_4 + F_5 + F_6$$

with

$$\begin{aligned}
F_4(t,r) &= F_{0,2}(t,r) + \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left(v_1+v_2+v_3+\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)(v_4+v_5)\right) \\
&\quad - \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)(v_1+v_2+v_3) - \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}(t,r)
\end{aligned} \tag{5.27}$$

$$F_5(t,r) = N_2(v_5) + \frac{\sin(2(v_1+v_2+v_3+v_4))}{2r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right) + \left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{2r^2}\right)\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right) \tag{5.28}$$

$$F_6(t,r) = \chi_{\geqslant 1}(\frac{4r}{t})\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)(v_4(t,r)+v_5(t,r)) \tag{5.29}$$

and

$$F_3 = N(v_6) + L_1(v_6)$$

5.8 Choosing $\lambda(t)$

λ will be chosen so that the term

$$\begin{aligned} F_4(t,r) &= F_{0,2}(t,r) \\ &+ \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)\left(v_1+v_2+v_3+\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)(v_4+v_5)\right) \\ &- \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(F_{0,2}(t,r)+(v_1(t,r)+v_2(t,r)+v_3(t,r))\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)\right)\end{aligned}$$

which appears on the right-hand side of (5.26), is orthogonal to $\phi_0\left(\frac{\cdot}{\lambda(t)}\right)$.

Define the space $(X, ||\cdot||_X)$ by

$$X = \{f \in C^2([T_0,\infty)) | ||f||_X < \infty\}$$

where

$$||f||_X = \sup_{t\geqslant T_0}\left(|f(t)|b\log^b(t)\sqrt{\log(\log(t))} + |f'(t)|t\log^{b+1}(t)\sqrt{\log(\log(t))} + |f''(t)|t^2\log^{b+1}(t)\sqrt{\log(\log(t))}\right) \tag{5.30}$$

In this section, we will first prove the following proposition

Proposition 5.1. *There exists $T_3 > 0$ such that, for all $T_0 \geqslant T_3$, there exists $\lambda \in C^2([T_0,\infty))$ which solves*

$$\langle F_4(t), \phi_0\left(\frac{\cdot}{\lambda(t)}\right)\rangle = 0, \quad t \geqslant T_0$$

Moreover,

$$\lambda(t) = \lambda_0(t) + e_0(t), \quad ||e_0||_X \leqslant 1$$

and

$$\lambda_0(t) = \frac{1}{\log^b(t)} + \int_t^\infty \int_{t_1}^\infty \frac{-b^2\log(\log(t_2))}{t_2^2\log^{b+2}(t_2)}dt_2dt_1$$

After fixing λ as in the above proposition, we will then show that λ, apriori only in $C^2([T_0,\infty))$, is actually in $C^4([T_0,\infty))$, with the estimates

$$|\lambda^{(2+k)}(t)| \leqslant \frac{C}{t^{2+k}\log^{b+1}(t)}, \quad t \geqslant T_0, \quad k=1,2$$

To prove the proposition, we will first show that the equation

$$\langle F_4(t), \phi_0\left(\frac{\cdot}{\lambda(t)}\right)\rangle = 0$$

is equivalent to

$$\begin{aligned}
&-4\int_t^\infty \frac{\lambda''(s)}{1+s-t}ds + \frac{4b}{t^2\log^b(t)} + 4\alpha\log(\lambda(t))\lambda''(t)\\
&-4\int_t^\infty \frac{\lambda''(s)}{(\lambda(t)^{1-\alpha}+s-t)(1+s-t)^3}ds\\
&= -\lambda(t)E_{0,1}(\lambda(t),\lambda'(t),\lambda''(t)) - 16\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t)) - K_{3,0}(s-t,\lambda(t))\right)ds\\
&+\frac{16}{\lambda(t)^2}\int_t^\infty K(s-t,\lambda(t))\lambda''(s)ds - \lambda(t)E_{v_2,ip}(t,\lambda(t))\\
&+\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(s)\left(K_1(s-t,\lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)}\right)ds \qquad (5.31)\\
&-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left((v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)+E_5\right)|_{r=R\lambda(t)},\phi_0\rangle\\
&+\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(v_1+v_2+v_3\right)|_{r=R\lambda(t)},\phi_0\rangle\\
&+\lambda(t)\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}(t,r)|_{r=R\lambda(t)},\phi_0\rangle\\
&:= G(t,\lambda(t))
\end{aligned}$$

Then, we will substitute $\lambda(t)=\lambda_0(t)+e_0(t)$, for $e_0 \in \overline{B_1(0)} \subset X$, and solve the resulting equation for e_0 with a fixed point argument.

We start by studying the relevant inner products of the v_1, v_2 and v_3 terms above.

5.8.1 The inner product of the (rescaled) v_1 linear error term with ϕ_0

In this section, we will prove

Lemma 5.4. *For v_1 defined by* (5.10)*, we have*

$$\begin{aligned}
&\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)v_1|_{r=R\lambda(t)},\phi_0\rangle\\
&=\frac{-16}{\lambda(t)^3}\int_t^\infty \lambda''(s)K(s-t,\lambda(t))ds + \frac{-16}{\lambda(t)^3}\int_t^\infty \lambda''(s)K_1(s-t,\lambda(t))ds
\end{aligned}$$

where

$$\int_t^\infty |K(s-t,\lambda(t))|ds \leqslant C\lambda(t)^2$$

$$|K_1(x,\lambda(t))| \leqslant \frac{C\lambda(t)^2 x}{1+x^2}$$

$$\int_t^\infty |K_1(s-t,\lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)}|ds \leqslant C\lambda(t)^2 \tag{5.32}$$

$$|K_1(s-t,\lambda(t)) - \frac{\lambda(t)^2}{4(s-t)}| \leqslant \frac{C\lambda(t)^2(1+\lambda(t)^2)}{(s-t)(1+(s-t)^2)}, \quad s-t \geqslant 1$$

Proof. We have

$$\langle \frac{(\cos(2Q_{\frac{1}{\lambda(t)}})-1)}{r^2} v_1\Big|_{r=R\lambda(t)}, \phi_0\rangle = \frac{-1}{\lambda(t)^2}\int_0^\infty v_1(t,R\lambda(t))\frac{8}{(1+R^2)^2}\frac{2R}{1+R^2}RdR$$
$$= -\frac{16}{\lambda(t)^2}\int_0^\infty v_1(t,R\lambda(t))\frac{R^2}{(1+R^2)^3}dR$$

$$-\frac{16}{\lambda(t)^2}\int_0^\infty v_1(t,R\lambda(t))\frac{R^2}{(1+R^2)^3}dR$$
$$= -\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}$$
$$\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR ds$$
$$= -\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{s-t}\frac{\rho}{s-t} \tag{5.33}$$
$$\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR ds$$
$$-\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{s-t}\right)$$
$$\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR ds$$

The last line of (5.33) is of the form

$$-\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)K(s-t,\lambda(t))ds$$

where

$$K(x,\lambda(t)) = \int_0^\infty \frac{R}{(1+R^2)^3}$$
$$\int_0^x \rho\left(\frac{1}{\sqrt{x^2-\rho^2}}-\frac{1}{x}\right)\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR \geqslant 0$$

and

$$
\begin{aligned}
&\int_t^\infty K(s-t,\lambda(t))ds \\
&= \int_t^\infty \int_0^\infty \frac{R}{(1+R^2)^3} \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{s-t} \right) \\
&\qquad \left(1 + \frac{R^2\lambda(t)^2 - 1 - \rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2 + 4R^2\lambda(t)^2}} \right) d\rho dR ds \\
&= \int_0^\infty \rho \int_0^\infty \frac{R}{(1+R^2)^3} \int_{\rho+t}^\infty \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{s-t} \right) \\
&\qquad \left(1 + \frac{R^2\lambda(t)^2 - 1 - \rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2 + 4R^2\lambda(t)^2}} \right) ds dR d\rho \\
&= \int_0^\infty \rho \int_0^\infty \frac{R}{(1+R^2)^3} \left(1 + \frac{R^2\lambda(t)^2 - 1 - \rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2 + 4R^2\lambda(t)^2}} \right) \log(2) dR d\rho \\
&= \int_0^\infty \frac{R}{(1+R^2)^3} \log(2) R^2 \lambda(t)^2 dR \\
&= \frac{\log(2)}{4} \lambda(t)^2
\end{aligned}
\tag{5.34}
$$

Hence, it remains to calculate

$$
\begin{aligned}
&-\frac{16}{\lambda(t)^3} \int_t^\infty \lambda''(s) \int_0^\infty \frac{R}{(1+R^2)^3} \int_0^{s-t} \frac{\rho}{s-t} \left(1 + \frac{R^2\lambda(t)^2 - 1 - \rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2 + 4R^2\lambda(t)^2}} \right) d\rho dR ds \\
&= -\frac{16}{\lambda(t)^3} \\
&\cdot \int_t^\infty \lambda''(s) \int_0^\infty \frac{R}{(1+R^2)^3} \\
&\qquad \left(\frac{r^2 - \sqrt{((r+s-t)^2+1)((r-s+t)^2+1)} + (s-t)^2 + 1}{2(s-t)} \right) |_{r=R\lambda(t)} dR ds \\
&= -\frac{16}{\lambda(t)^3} \int_t^\infty \lambda''(s) K_1(s-t,\lambda(t)) ds
\end{aligned}
$$

where

$$
\begin{aligned}
&K_1(w,\lambda(t))\\
&=\int_0^\infty \frac{R}{(1+R^2)^3}\left(\frac{R^2\lambda(t)^2+w^2+1-\sqrt{(1+R^2\lambda(t)^2+w^2)^2-4R^2\lambda(t)^2w^2}}{2w}\right)dR\\
&=\frac{w\lambda(t)^2\left(\lambda(t)^2+w^2+1\right)}{4(y^2+4w^2)}+\frac{w\lambda(t)^4\log\left(\lambda(t)^2\left(\sqrt{y^2+4w^2}-y\right)\right)}{2\left(y^2+4w^2\right)^{3/2}}\\
&-w\lambda(t)^4\frac{\log\left(4w^2+y^2-\lambda(t)^2y+(w^2+1)\sqrt{y^2+4w^2}\right)}{2\left(y^2+4w^2\right)^{3/2}}
\end{aligned} \tag{5.35}
$$

where

$$y=\lambda(t)^2+w^2-1$$

and, for the first line of (5.35), we used the fact that

$$((r+w)^2+1)((r-w)^2+1)=(r^2+w^2+1)^2-4r^2w^2 \tag{5.36}$$

Now, we will prove a pointwise estimate on K_1. Using (5.36), we have

$$
\begin{aligned}
0\leqslant &K_1(w,\lambda(t))\\
&=\int_0^\infty \frac{R}{2w(1+R^2)^3}\left(\frac{4R^2\lambda(t)^2w^2}{1+R^2\lambda(t)^2+w^2+\sqrt{(1+(R\lambda(t)+w)^2)(1+(R\lambda(t)-w)^2)}}\right)dR\\
&\leqslant C\int_0^\infty \frac{R^3}{(1+R^2)^3}\frac{\lambda(t)^2w}{1+w^2}dR\leqslant C\lambda(t)^2\frac{w}{1+w^2}
\end{aligned}
$$

So,

$$|K_1(x,\lambda(t))|\leqslant C\lambda(t)^2\frac{x}{1+x^2} \tag{5.37}$$

For use later on, we will also need to estimate

$$\int_t^\infty |K_1(s-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+s-t)}|ds$$

Note that $K_1(w,\lambda(t))\geqslant 0$, by its definition as the integral of a non-negative function. We start with

$$
\begin{aligned}
\int_t^{t+1}|K_1(s-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+s-t)}|ds&\leqslant\int_t^{t+1}K_1(s-t,\lambda(t))ds+\int_t^{t+1}\frac{\lambda(t)^2}{4(1+s-t)}ds\\
&\leqslant C\lambda(t)^2
\end{aligned}
$$

where we used (5.37).

Now, we consider the region $s-t \geqslant 1$. Returning to (5.35), we see that

$$K_1(w,\lambda(t)) = \int_0^\infty \frac{R}{(1+R^2)^3}\frac{R^2\lambda(t)^2}{w}dR \\ - \int_0^\infty \frac{R}{(1+R^2)^3}\frac{4R^2\lambda(t)^2 dR}{2w\left(\sqrt{\left(-R^2\lambda(t)^2+w^2+1\right)^2+4R^2\lambda(t)^2}-R^2\lambda(t)^2+w^2+1\right)} \tag{5.38}$$

The right-hand side of the first line of (5.38) is equal to

$$\int_0^\infty \frac{R}{(1+R^2)^3}\frac{R^2\lambda(t)^2}{w}dR = \frac{\lambda(t)^2}{4w}$$

So,

$$K_1(w,\lambda(t)) - \frac{\lambda(t)^2}{4w} \\ = -\int_0^\infty \frac{R}{(1+R^2)^3}\frac{4R^2\lambda(t)^2}{2w\left(\sqrt{(1+w^2-R^2\lambda(t)^2)^2+4R^2\lambda(t)^2}+1+w^2-R^2\lambda(t)^2\right)}dR$$

First, we note that

$$1+w^2-R^2\lambda(t)^2 \geqslant C(1+w^2), \quad \text{if } R\lambda(t) \leqslant \frac{w}{2}$$

So,

$$\int_0^{\frac{w}{2\lambda(t)}} \frac{R}{(1+R^2)^3}\frac{4R^2\lambda(t)^2}{2w\left(\sqrt{(1+w^2-R^2\lambda(t)^2)^2+4R^2\lambda(t)^2}+1+w^2-R^2\lambda(t)^2\right)}dR \\ \leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\frac{R^2\lambda(t)^2}{w(1+w^2)}dR \leqslant \frac{C\lambda(t)^2}{w(1+w^2)}, \quad w \geqslant 1$$

Next,

$$\frac{1}{\sqrt{(1+w^2-R^2\lambda(t)^2)^2+4R^2\lambda(t)^2}+1+w^2-R^2\lambda(t)^2} \\ = \frac{1+w^2-R^2\lambda(t)^2-\sqrt{(1+w^2-R^2\lambda(t)^2)^2+4R^2\lambda(t)^2}}{-4R^2\lambda(t)^2}$$

and

$$|\frac{1+w^2-R^2\lambda(t)^2-\sqrt{(1+w^2-R^2\lambda(t)^2)^2+4R^2\lambda(t)^2}}{-4R^2\lambda(t)^2}| \leqslant C\frac{R^2\lambda(t)^2}{R^2\lambda(t)^2} \leqslant C, \\ \text{if} \quad R\lambda(t) > \frac{w}{2}, \quad w \geqslant 1$$

So,

$$\int_{\frac{w}{2\lambda(t)}}^{\infty} \frac{R}{(1+R^2)^3} \frac{4R^2\lambda(t)^2}{2w\left(\sqrt{(1+w^2-R^2\lambda(t)^2)^2+4R^2\lambda(t)^2}+1+w^2-R^2\lambda(t)^2\right)} dR$$
$$\leqslant \int_{\frac{w}{2\lambda(t)}}^{\infty} \frac{1}{R^5} \frac{R^2\lambda(t)^2}{w} dR \leqslant \frac{C\lambda(t)^4}{w^3}, \quad w \geqslant 1$$

Combining these, we get

$$|K_1(w,\lambda(t)) - \frac{\lambda(t)^2}{4w}| \leqslant \frac{C\lambda(t)^2(1+\lambda(t)^2)}{w(1+w^2)}, \quad w \geqslant 1 \tag{5.39}$$

So,

$$\int_{t+1}^{\infty} |K_1(s-t,\lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)}| ds \leqslant C \int_{t+1}^{\infty} \frac{\lambda(t)^2}{4}\left(\frac{1}{s-t} - \frac{1}{1+s-t}\right) ds$$
$$+ C \int_{t+1}^{\infty} \frac{C\lambda(t)^2(1+\lambda(t)^2)}{(s-t)(1+(s-t)^2)} ds$$
$$\leqslant C\lambda(t)^2(1+\lambda(t)^2)$$

Recalling (5.2), we conclude

$$\int_{t}^{\infty} |K_1(s-t,\lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)}| ds \leqslant C\lambda(t)^2$$

□

5.8.2 The inner product of the (rescaled) v_2 linear error term with ϕ_0

Lemma 5.5. *For all $b > 0$, and v_2 defined by* (5.18)*, we have*

$$\int_0^{\infty} R\left(\frac{\cos(2Q_1(R))-1}{R^2\lambda(t)^2}\right)\phi_0(R) v_2(t, R\lambda(t)) dR = \frac{4b}{\lambda(t)t^2\log^b(t)} + E_{v_2,ip}(t,\lambda(t))$$

where

$$|E_{v_2,ip}(t,\lambda(t))| \leqslant \frac{C}{\lambda(t)t^2\log^{b+1}(t)} \tag{5.40}$$

Proof. We start with the case $b \neq 1$. Using our formula for v_2, we get

$$\int_0^{\infty} R\left(\frac{\cos(2Q_1(R))-1}{R^2\lambda(t)^2}\right)\phi_0(R) v_2(t, R\lambda(t)) dR$$
$$= -2c_b \int_0^{\infty} \xi^2 \sin(t\xi) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} K_1(\xi\lambda(t)) d\xi$$

where K_1 denotes the modified Bessel function of the second kind. (This follows, for instance, from equation (**6.532** 4) of the table of integrals [7]). Recalling that

$$K_1(x) = \frac{1}{x} + O(x\log(x)), \quad x \to 0$$

we can integrate by parts two times, and get

$$\begin{aligned}&-2c_b\int_0^\infty \xi^2\sin(t\xi)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}K_1(\xi\lambda(t))d\xi\\&=2c_b\int_0^\infty\frac{\sin(t\xi)}{t^2}\partial_\xi^2\left(\xi^2\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}K_1(\xi\lambda(t))\right)d\xi\end{aligned}$$

Let

$$F_b(\xi)=\partial_\xi^2\left(\xi^2\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}K_1(\xi\lambda(t))\right)$$

Note that

$$F_b(\xi)=\chi_{\leqslant\frac{1}{4}}(\xi)\partial_\xi^2\left(\xi^2\frac{K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\right)+\psi_{v_2}(\xi,\lambda(t))\tag{5.41}$$

where

$$\psi_{v_2}\in C_c^\infty([\frac{1}{8},\frac{1}{4}])$$

So, integration by parts (for instance, once) gives

$$|2c_b\int_0^\infty\frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi|\leqslant\frac{C}{t^3\lambda(t)}$$

To study

$$2c_b\int_0^\infty\chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}\partial_\xi^2\left(\xi^2\frac{K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\right)d\xi$$

we can use the asymptotics of K_1 to write

$$\partial_\xi^2\left(\xi^2\frac{K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\right)=\partial_\xi^2\left(\frac{\xi}{\lambda(t)\log^{b-1}(\frac{1}{\xi})}\right)+F_{v_2}(\xi,\lambda(t)),\quad \xi\leqslant\frac{1}{4}\tag{5.42}$$

where

$$\begin{aligned}F_{v_2}(\xi,y)=&\frac{\xi^2y^2\log\left(\frac{1}{\xi}\right)\left(-3\log\left(\frac{1}{\xi}\right)-2b+2\right)K_0(y\xi)}{\xi y\log^{b+1}(\frac{1}{\xi})}\\&+\frac{(b-1)\left(\log\left(\frac{1}{\xi}\right)+b\right)(\xi yK_1(y\xi)-1)}{\xi y\log^{b+1}(\frac{1}{\xi})}\\&+\frac{\xi^3y^3\log^2\left(\frac{1}{\xi}\right)K_1(y\xi)}{\log^{b+1}(\frac{1}{\xi})\xi y},\quad \xi\leqslant\frac{1}{4}\end{aligned}$$

Using the simple estimates, valid for all $x>0$,

$$\begin{aligned}|-1+xK_1(x)|&\leqslant Cx^2(|\log(x)|+1)\\|K_0(x)|&\leqslant C(|\log(x)|+1)\\|xK_1(x)|&\leqslant C\end{aligned}\tag{5.43}$$

we get, for any $\lambda(t) > 0$,

$$|F_{v_2}(\xi, \lambda(t))| \leqslant C \frac{\xi\lambda(t)}{\log^{b-1}(\frac{1}{\xi})} \left(|\log(\xi)| + |\log(\lambda(t))|\right), \quad \xi \leqslant \frac{1}{4}$$

Similarly, we have

$$\begin{aligned}\partial_\xi F_{v_2}(\xi, y) = & \frac{(b-1)\left(-\log^2\left(\frac{1}{\xi}\right) + b^2 + b\right)\log^{-b-2}\left(\frac{1}{\xi}\right)(\xi y K_1(y\xi) - 1)}{\xi^2 y} \\ & + y\frac{\left(\xi y \log\left(\frac{1}{\xi}\right)\left(4\log\left(\frac{1}{\xi}\right) + 3b - 3\right)K_1(y\xi)\right)}{\log^{b+1}\left(\frac{1}{\xi}\right)} \\ & - yK_0(y\xi)\frac{\left(6(b-1)\log\left(\frac{1}{\xi}\right) + 3(b-1)b + \log^2\left(\frac{1}{\xi}\right)(\xi^2y^2+3)\right)}{\log^{b+1}(\frac{1}{\xi})}\end{aligned}$$

Again, using (5.43), we get, for all $\lambda(t) > 0$,

$$|\partial_\xi F_{v_2}(\xi, \lambda(t))| \leqslant \frac{C\lambda(t)\left(|\log(\xi)| + |\log(\lambda(t))|\right)}{\log^{b-1}(\frac{1}{\xi})}\left(1 + \lambda(t)^2\xi^2\right), \quad \xi \leqslant \frac{1}{4}$$

So, we can integrate by parts (for instance) once, and recall (5.2), to get

$$|2c_b \int_0^\infty \chi_{\leqslant \frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi, \lambda(t))d\xi| \leqslant \frac{C\lambda(t)|\log(\lambda(t))|}{t^3}$$

Next, we need to consider

$$\begin{aligned}& 2c_b \int_0^\infty \chi_{\leqslant \frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}\partial_\xi^2\left(\frac{\xi}{\lambda(t)\log^{b-1}(\frac{1}{\xi})}\right)d\xi \\ & = \frac{2c_b}{\lambda(t)}\int_0^{1/2}\chi_{\leqslant \frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\end{aligned}$$

From one integration by parts we see that

$$|\frac{2c_b}{\lambda(t)}\int_0^{1/2}\left(\chi_{\leqslant \frac{1}{4}}(\xi) - 1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi| \leqslant \frac{C}{t^3\lambda(t)}$$

So, we need only consider

$$\begin{aligned}& \frac{2c_b}{\lambda(t)}\int_0^{1/2}\frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi \\ & = \frac{2c_b}{\lambda(t)}\int_0^{t/2}\frac{\sin(u)}{t^2}\left(\frac{b-1}{u\log^b(\frac{t}{u})} + \frac{b(b-1)}{u\log^{b+1}(\frac{t}{u})}\right)du\end{aligned} \tag{5.44}$$

Let us start by studying, for $a > 0$,

$$\int_0^{t/2} \frac{\sin(u)}{u \log^a(t/u)} du$$

Let

$$f(z) = \frac{e^{iz}}{z \log^a(t/z)}$$

where we use the principal branch of $\log$ and $(\cdot)^a$ Then, f is analytic on (for instance) D given by

$$D = \mathbb{C} \backslash ((-\infty, 0] \cup [2t/3, \infty))$$

For $0 < \epsilon \leqslant \frac{1}{2}$, consider the contour $C_{t,\epsilon}$ in D given by:

$$C_{t,\epsilon} = \{\frac{t}{2} e^{i\theta} | 0 \leqslant \theta \leqslant \frac{\pi}{2}\} \cup \{iy | \epsilon \leqslant y \leqslant \frac{t}{2}\} \cup \{\epsilon e^{i\theta} | 0 \leqslant \theta \leqslant \frac{\pi}{2}\} \cup [\epsilon, t/2]$$

traversed in the counter-clockwise direction. By Cauchy's residue theorem,

$$\int_{C_{t,\epsilon}} f(z) dz = 0$$

On the other hand,

$$\int_{C_{t,\epsilon}} f(z) dz = \int_0^{\frac{\pi}{2}} i \frac{e^{\frac{it}{2}\cos(\theta)} e^{-\frac{t}{2}\sin(\theta)}}{\log^a(2e^{-i\theta})} d\theta - \int_\epsilon^{t/2} \frac{e^{-y}}{y \log^a(\frac{t}{iy})} dy - \int_0^{\frac{\pi}{2}} i \frac{e^{i\epsilon e^{i\theta}}}{\log^a(\frac{te^{-i\theta}}{\epsilon})} d\theta$$
$$+ \int_\epsilon^{t/2} \frac{e^{iu}}{u \log^a(t/u)} du$$

So,

$$\int_\epsilon^{t/2} \frac{\sin(u)}{u \log^a(t/u)} du = \text{Im}\left(\int_\epsilon^{t/2} \frac{e^{-y}}{y \log^a(\frac{t}{iy})} dy\right) + \text{Im}\left(\int_0^{\frac{\pi}{2}} i \frac{e^{i\epsilon e^{i\theta}}}{\log^a(\frac{te^{-i\theta}}{\epsilon})} d\theta\right)$$
$$- \text{Im}\left(\int_0^{\frac{\pi}{2}} i \frac{e^{\frac{it}{2}\cos(\theta)} e^{-\frac{t}{2}\sin(\theta)}}{\log^a(2e^{-i\theta})} d\theta\right)$$

Letting $\epsilon \to 0$, we have

$$\int_0^{t/2} \frac{\sin(u)}{u \log^a(t/u)} du = \lim_{\epsilon \to 0} \text{Im}\left(\int_\epsilon^{t/2} \frac{e^{-y}}{y \log^a(\frac{t}{iy})} dy\right)$$
$$- \text{Im}\left(\int_0^{\frac{\pi}{2}} i \frac{e^{\frac{it}{2}\cos(\theta)} e^{-\frac{t}{2}\sin(\theta)}}{\log^a(2e^{-i\theta})} d\theta\right) \tag{5.45}$$

Note that

$$\text{Im}\left(\int_\epsilon^{t/2} \frac{e^{-y}}{y \log^a(\frac{t}{iy})} dy\right) = \int_\epsilon^{t/2} \frac{e^{-y} \sin(a \tan^{-1}(\frac{\pi}{2(\log(t) - \log(y))}))}{y((\log(t) - \log(y))^2 + \frac{\pi^2}{4})^{a/2}} dy$$

So,

$$\lim_{\epsilon \to 0} \text{Im} \left(\int_\epsilon^{t/2} \frac{e^{-y}}{y \log^a(\frac{t}{iy})} dy \right) = \int_0^{t/2} \frac{e^{-y} \sin(a \tan^{-1}(\frac{\pi}{2(\log(t) - \log(y))}))}{y((\log(t) - \log(y))^2 + \frac{\pi^2}{4})^{a/2}} dy$$

We have

$$\frac{e^{-y} \sin(a \tan^{-1}(\frac{\pi}{2(\log(t) - \log(y))}))}{y((\log(t) - \log(y))^2 + \frac{\pi^2}{4})^{a/2}} = a\frac{\pi}{2} \frac{e^{-y}}{y \log^{a+1}(t/y)} + \text{Err}, \quad 0 \leqslant y \leqslant t/2$$

where

$$|\text{Err}| \leqslant C \frac{e^{-y}}{y} \frac{1}{\log^{3+a}(t/y)}, \quad 0 \leqslant y \leqslant t/2$$

So,

$$\begin{aligned} \int_0^{t/2} \frac{e^{-y} \sin(a \tan^{-1}(\frac{\pi}{2(\log(t) - \log(y))}))}{y((\log(t) - \log(y))^2 + \frac{\pi^2}{4})^{a/2}} dy &= a\frac{\pi}{2} \int_0^{t/2} \frac{e^{-y}}{y \log^{a+1}(t/y)} dy \\ &+ \int_0^{t/2} \text{Err}\, dy \end{aligned} \tag{5.46}$$

Let us start by considering

$$\int_0^1 \frac{1}{y} \frac{e^{-y} dy}{\log^{a+1}(t/y)} = \int_0^1 \frac{dy}{y \log^{a+1}(t/y)} + \int_0^1 \frac{(e^{-y} - 1)}{y \log^{a+1}(\frac{t}{y})} dy$$

Treating each term individually, we have:

$$\int_0^1 \frac{dy}{y \log^{a+1}(t/y)} = \frac{1}{a \log^a(t)}$$

$$\left| \int_0^1 \frac{(e^{-y} - 1)}{y \log^{a+1}(\frac{t}{y})} dy \right| \leqslant \frac{C}{\log^{a+1}(t)} \int_0^1 \frac{|e^{-y} - 1|}{y} dy \leqslant \frac{C}{\log^{a+1}(t)}$$

where we used the fact that

$$y \mapsto \frac{1}{\log^{a+1}(\frac{t}{y})}, \text{ is increasing on } (0, 1]$$

So,

$$\int_0^1 \frac{1}{y} \frac{e^{-y} dy}{\log^{a+1}(t/y)} = \frac{1}{a \log^a(t)} + O\left(\frac{1}{\log^{a+1}(t)} \right)$$

Also,

$$\int_1^{\sqrt{t}} \frac{1}{y} \frac{e^{-y} dy}{(\log(t) - \log(y))^{a+1}} \leqslant 2^{a+1} \int_1^{\sqrt{t}} \frac{1}{y} \frac{e^{-y} dy}{\log^{a+1}(t)} = O(\frac{1}{\log^{a+1}(t)})$$

Finally,

$$\int_{\sqrt{t}}^{t/2} \frac{1}{y} \frac{e^{-y} dy}{(\log(t) - \log(y))^{a+1}} \leqslant \frac{1}{\sqrt{t}} \int_{\sqrt{t}}^{t/2} \frac{e^{-y}}{\log^{a+1}(2)} dy = O(\frac{e^{-\sqrt{t}}}{\sqrt{t}})$$

So,

$$\int_0^{t/2} \frac{1}{y} \frac{e^{-y} dy}{\log^{a+1}(t/y)} = \frac{1}{a \log^a(t)} + O\left(\frac{1}{\log^{a+1}(t)}\right) \tag{5.47}$$

Treating the Err term in (5.46) in the same way as was used to obtain (5.47), we have

$$\begin{aligned}\int_0^{t/2} \frac{e^{-y} \sin(a \tan^{-1}(\frac{\pi}{2(\log(t)-\log(y))}))}{y((\log(t)-\log(y))^2+\frac{\pi^2}{4})^{a/2}} dy &= a\frac{\pi}{2}\left(\frac{1}{a\log^a(t)} + O\left(\frac{1}{\log^{a+1}(t)}\right)\right)\\ &+ O\left(\frac{1}{\log^{a+2}(t)}\right)\\ &= \frac{\pi}{2\log^a(t)} + O\left(\frac{1}{\log^{a+1}(t)}\right)\end{aligned}$$

Lastly, let us estimate

$$\begin{aligned}|-\text{Im}\left(\int_0^{\frac{\pi}{2}} i \frac{e^{\frac{it}{2}\cos(\theta)} e^{-\frac{t}{2}\sin(\theta)}}{\log^a(2e^{-i\theta})} d\theta\right)| &\leqslant C \int_0^{\frac{\pi}{2}} \frac{e^{-\frac{t}{2}\sin(\theta)}}{\log^a(2)} d\theta\\ &\leqslant C \int_0^{\frac{\pi}{2}} e^{-\frac{t}{2}\frac{2}{\pi}\theta} d\theta = O\left(\frac{1}{t}\right)\end{aligned}$$

So, (5.45) can be written as

$$\int_0^{t/2} \frac{\sin(u)}{u \log^a(t/u)} du = \frac{\pi}{2\log^a(t)} + O\left(\frac{1}{\log^{a+1}(t)}\right)$$

Since (5.45) is valid for all $a > 0$, we can repeat the identical steps done in this section for the case $b = 1$, and apply the above result to (5.44) (and its analog for $b = 1$), thereby obtaining, for all $b > 0$:

$$\int_0^\infty R\left(\frac{\cos(2Q_1(R))-1}{R^2\lambda(t)^2}\right)\phi_0(R) v_2(t, R\lambda(t)) dR = \frac{4b}{\lambda(t) t^2 \log^b(t)} + E_{v_2,ip}(t,\lambda(t))$$

where, for $b \neq 1$:

$$\begin{aligned}E_{v_2,ip}(t,\lambda(t)) &= 2c_b \int_0^\infty \frac{\sin(t\xi)}{t^2} \psi_{v_2}(\xi,\lambda(t)) d\xi\\ &+ 2c_b \int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi) \frac{\sin(t\xi)}{t^2} F_{v_2}(\xi,\lambda(t)) d\xi\\ &+ 2\frac{c_b}{\lambda(t)} \int_0^{\frac{1}{2}} \left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right) \frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right) d\xi\\ &+ \frac{2c_b}{\lambda(t)}\left(\int_0^{\frac{t}{2}} \frac{\sin(u)(b-1)}{t^2 u \log^b(\frac{t}{u})} du - \frac{(b-1)\pi}{2t^2\log^b(t)} + \int_0^{\frac{t}{2}} \frac{\sin(u) b(b-1)}{t^2 u \log^{b+1}(\frac{t}{u})} du\right)\end{aligned} \tag{5.48}$$

When $b=1$, $E_{v_2,ip}$ has the same form as (5.48), except the third and fourth lines changed to

$$\frac{2c_1}{\lambda(t)}\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{\xi t^2}\left(-\frac{1}{\log^2(\frac{1}{\xi})}-\frac{1}{\log(\frac{1}{\xi})}\right)d\xi$$
$$+\frac{2c_1}{\lambda(t)}\left(\int_0^{\frac{1}{2}}\frac{\sin(t\xi)}{\xi t^2}\left(\frac{-1}{\log(\frac{1}{\xi})}-\frac{1}{\log^2(\frac{1}{\xi})}\right)d\xi+\frac{\pi}{2t^2\log(t)}\right)$$

For all $b>0$,

$$|E_{v_2,ip}(t,\lambda(t))|\leqslant\frac{C}{\lambda(t)t^2\log^{b+1}(t)}$$

□

5.8.3 Pointwise estimates on $\partial_t^i\partial_r^j v_2$

Lemma 5.6. *For any $b>0$, there exists $C>0$ such that*

For $0\leqslant j\leqslant 2,\quad 0\leqslant k\leqslant 1,\quad j+k\leqslant 2,$

$$\partial_t^j\partial_r^k v_2(t,r)=\frac{\partial_t^j\partial_r^k\left(\frac{-br}{t^2}\right)}{\log^b(t)}+E_{\partial_t^j\partial_r^k v_2}(t,r)\tag{5.49}$$

where

$$|E_{\partial_t^j\partial_r^k v_2}(t,r)|\leqslant C\partial_t^j\partial_r^k\left(\frac{r}{t^2\log^{b+1}(t)}\right)+C\partial_t^j\partial_r^k\left(\frac{r^2}{t^3\log^b(t)}\right),\quad r\leqslant\frac{t}{2}$$

For $0\leqslant j\leqslant 2,\quad 0\leqslant k\leqslant 2,\quad j+k\leqslant 2,$

$$|\partial_t^j\partial_r^k v_2(t,r)|\leqslant\frac{C}{\sqrt{r}},\quad r\geqslant\frac{t}{2}\tag{5.50}$$

For $0\leqslant j\leqslant 1,\quad 0\leqslant k\leqslant 1,\quad j+k\leqslant 2,$

$$|\partial_t^j\partial_r^k v_2(t,r)|\leqslant\frac{C\log(r)}{|t-r|^{1+j+k}},\quad r\geqslant\frac{t}{2},\quad r\neq t\tag{5.51}$$

Finally, we also have the estimates

$$|\partial_r^2 v_2(t,r)|\leqslant\begin{cases}\frac{Cr}{t^4\log^b(t)}, & r\leqslant\frac{t}{2}\\ \frac{C\log(r)}{|t-r|^3}, & \frac{t}{2}<r\neq t\end{cases}\tag{5.52}$$

Proof. In the course of proving these estimates, we will occasionally make use of the following formula, which can be found, for example, in Appendix B of [12]: for $n>-\frac{1}{2}$

$$J_n(x)=\frac{1}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\left(\frac{x}{2}\right)^n\int_0^{\pi}\cos(x\cos(\theta))\sin^{2n}(\theta)d\theta\tag{5.53}$$

Similarly, we will need, Lemma 8.1 of [12], which states

$$J_{\frac{d-2}{2}}(x) = \frac{\left(\frac{x}{2}\right)^{\frac{d-2}{2}}}{\pi^{\frac{d-1}{2}}} \mathrm{Re}\left(e^{-ix}\Phi_{\frac{d-2}{2}}(x)\right), \quad x > 0 \tag{5.54}$$

where

$$|\Phi^{(k)}_{\frac{d-2}{2}}|(x) \leqslant C_k x^{-\frac{(d-1)}{2}-k}$$

We will first prove the lemma for the case $b \neq 1$, and then remark what the identical procedures give, for the case $b = 1$.

We will start with the region $0 \leqslant r \leqslant \frac{t}{2}$, and use (5.53) for $n = 1$ in the formula for v_2:

$$\begin{aligned}
v_2(t,r) &= c_b \int_0^\infty \sin(t\xi) J_1(r\xi) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi \\
&= \frac{c_b}{\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty \sin(t\xi) r\xi \cos(r\xi\cos(\theta)) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi d\theta \\
&= \frac{c_b}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty r\xi \left(\sin(\xi(t + r\cos(\theta))) + \sin(\xi(t - r\cos(\theta)))\right) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi d\theta
\end{aligned}$$

Integrating by parts twice, we get

$$v_2(t,r) = -\frac{c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty \partial_\xi^2 \left(\frac{\xi \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right) \left(\frac{\sin(\xi t_+)}{t_+^2} + \frac{\sin(\xi t_-)}{t_-^2}\right) d\xi d\theta$$

where

$$t_{+/-} = t \pm r\cos(\theta)$$

We divide the terms analogously to how similar terms were treated in the previous subsection. In particular, we have

$$\begin{aligned}
v_2(t,r) = \frac{-c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty & \left(\chi_{\leqslant \frac{1}{4}}(\xi) \left(\frac{b-1}{\xi \log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi \log^{b+1}(\frac{1}{\xi})}\right) + \psi(\xi)\right) \\
& \left(\frac{\sin(\xi t_+)}{t_+^2} + \frac{\sin(\xi t_-)}{t_-^2}\right) d\xi d\theta
\end{aligned}$$

where $\psi \in C_c^\infty([\frac{1}{8}, \frac{1}{4}])$. So, we can integrate by parts (for example, once) to treat the ψ term:

$$\begin{aligned}
&|\frac{-c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty \psi(\xi) \left(\frac{\sin(\xi t_+)}{t_+^2} + \frac{\sin(\xi t_-)}{t_-^2}\right) d\xi d\theta| \leqslant Cr \int_0^\pi \int_0^\infty \frac{|\psi'(\xi)| d\xi d\theta}{t^3} \\
&\leqslant \frac{Cr}{t^3}, \quad r \leqslant \frac{t}{2}
\end{aligned}$$

where we use

$$\frac{1}{|t_\pm|} \leqslant \frac{C}{t}, \quad r \leqslant \frac{t}{2}$$

Then, it remains to study

$$
\begin{aligned}
&\frac{-c_b r}{2\pi}\int_0^\pi \sin^2(\theta)\int_0^\infty \left(\chi_{\leqslant\frac14}(\xi)\left(\frac{b-1}{\xi\log^b(\frac1\xi)}+\frac{b(b-1)}{\xi\log^{b+1}(\frac1\xi)}\right)\right)\left(\frac{\sin(\xi t_+)}{t_+^2}+\frac{\sin(\xi t_-)}{t_-^2}\right)d\xi d\theta\\
&=\frac{-c_b r}{2\pi}\int_0^\pi \sin^2(\theta)\int_0^{\frac12}\left(\frac{b-1}{\xi\log^b(\frac1\xi)}+\frac{b(b-1)}{\xi\log^{b+1}(\frac1\xi)}\right)\left(\frac{\sin(\xi t_+)}{t_+^2}+\frac{\sin(\xi t_-)}{t_-^2}\right)d\xi d\theta\\
&+\frac{-c_b r}{2\pi}\int_0^\pi \sin^2(\theta)\int_0^{\frac12}\left(\chi_{\leqslant\frac14}(\xi)-1\right)\left(\frac{b-1}{\xi\log^b(\frac1\xi)}+\frac{b(b-1)}{\xi\log^{b+1}(\frac1\xi)}\right)\left(\frac{\sin(\xi t_+)}{t_+^2}+\frac{\sin(\xi t_-)}{t_-^2}\right)d\xi d\theta
\end{aligned} \tag{5.55}
$$

We start with the second line of (5.55). Since all other terms will use the identical argument, we first consider the term involving $\frac{1}{\log^b(t)}$ and t_+, namely

$$
\begin{aligned}
&\frac{-c_b r}{2\pi}\int_0^\pi \sin^2(\theta)\left((b-1)\int_0^{\frac{t_+}{2}}\frac{\sin(u)du}{u\log^b(\frac{t_+}{u})t_+^2}\right)d\theta\\
&=-\frac{c_b r}{2\pi}\int_0^\pi \sin^2(\theta)(b-1)\left(\frac{\pi}{2\log^b(t_+)t_+^2}+\text{Err}(t,r,\theta)\right)d\theta
\end{aligned} \tag{5.56}
$$

where

$$
|\text{Err}(t,r,\theta)|\leqslant \frac{C}{t_+^2\log^{b+1}(t_+)}
$$

and we use our calculation of (5.45) from the previous subsection. To treat this last integral, we start with the first term:

$$
\begin{aligned}
&\frac{-c_b r}{2\pi}\cdot(b-1)\frac{\pi}{2}\int_0^\pi\frac{\sin^2(\theta)d\theta}{\log^b(t_+)t_+^2}\\
&=\frac{-c_b r(b-1)}{4}\frac{1}{t^2\log^b(t)}\int_0^\pi\frac{\sin^2(\theta)}{(1+\frac{r\cos(\theta)}{t})^2}\frac{d\theta}{\left(1+\frac{\log(1+\frac{r\cos(\theta)}{t})}{\log(t)}\right)^b}\\
&=\frac{-c_b r(b-1)}{4}\left(\frac{\pi}{2t^2\log^b(t)}+E_{v_2,1}(t,r)\right),\quad r\leqslant\frac{t}{2}
\end{aligned}
$$

with

$$
|E_{v_2,1}(t,r)|\leqslant C\frac{r}{t^3\log^b(t)},\quad r\leqslant\frac{t}{2}
$$

For the second term of (5.56), we have

$$
\begin{aligned}
&|\frac{-c_b r}{2\pi}\int_0^\pi \sin^2(\theta)(b-1)\text{Err}(t,r,\theta)d\theta|\leqslant Cr\int_0^\pi\frac{d\theta}{t^2\log^{b+1}(t)}\\
&\leqslant\frac{Cr}{t^2\log^{b+1}(t)},\quad r\leqslant\frac{t}{2}
\end{aligned}
$$

where we used

$$\frac{1}{t_+^2 \log^{b+1}(t_+)} \leqslant \frac{C}{t^2 \log^{b+1}(t)}, \quad r \leqslant \frac{t}{2}$$

Combining these, we get

$$\frac{-c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \left((b-1) \int_0^{\frac{t_+}{2}} \frac{\sin(u) du}{u \log^b(\frac{t_+}{u}) t_+^2} \right) d\theta = \frac{-br}{2t^2 \log^b(t)} + E_{v_2,2}(t,r), \quad r \leqslant \frac{t}{2}$$

$$|E_{v_2,2}(t,r)| \leqslant C \left(\frac{r^2}{t^3 \log^b(t)} + C \frac{r}{t^2 \log^{b+1}(t)} \right), \quad r \leqslant \frac{t}{2}$$

We use the identical procedure to treat all the other terms in the second line of (5.55), and get

$$\begin{aligned} &\frac{-c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^{\frac{1}{2}} \left(\frac{b-1}{\xi \log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi \log^{b+1}(\frac{1}{\xi})} \right) \left(\frac{\sin(\xi t_+)}{t_+^2} + \frac{\sin(\xi t_-)}{t_-^2} \right) d\xi d\theta \\ &= \frac{-br}{t^2 \log^b(t)} + E_{v_2,3}(t,r) \end{aligned}$$

with

$$|E_{v_2,3}(t,r)| \leqslant C \left(\frac{r}{t^2 \log^{b+1}(t)} + \frac{r^2}{t^3 \log^b(t)} \right), \quad r \leqslant \frac{t}{2}$$

Finally, for the third line of (5.55), we integrate by parts, identically to how a similar term was treated in the last subsection, and get

$$\begin{aligned} &|\frac{-c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^{\frac{1}{2}} \left(\chi_{\leqslant \frac{1}{4}}(\xi) - 1 \right) \left(\frac{b-1}{\xi \log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi \log^{b+1}(\frac{1}{\xi})} \right) \left(\frac{\sin(\xi t_+)}{t_+^2} + \frac{\sin(\xi t_-)}{t_-^2} \right) d\xi d\theta| \\ &\leqslant \frac{Cr}{t^3}, \quad r \leqslant \frac{t}{2} \end{aligned}$$

This completes the proof of (5.49) (for v_2).

We will need two more estimates on v_2. One is given by using the simple estimate

$$|J_1(x)| \leqslant \frac{C}{\sqrt{x}}$$

which yields

$$|v_2(t,r)| \leqslant \frac{C}{\sqrt{r}}$$

We will obtain another estimate in the region $r \geqslant \frac{t}{2}$, by first decomposing v_2 as follows:

$$\begin{aligned} v_2(t,r) &= c_b \int_0^\infty \sin(t\xi) J_1(r\xi) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi \\ &= c_b \int_0^\infty \chi_{\leqslant 1}(r\xi) \sin(t\xi) J_1(r\xi) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi \\ &+ c_b \int_0^\infty (1 - \chi_{\leqslant 1}(r\xi)) \sin(t\xi) J_1(r\xi) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi \end{aligned} \tag{5.57}$$

where $\chi_{\leqslant 1} \in C_c^\infty([0,\infty))$, $0 \leqslant \chi_{\leqslant 1} \leqslant 1$,

$$\chi_{\leqslant 1}(x) = \begin{cases} 1, & x \leqslant \frac{1}{2} \\ 0, & x \geqslant 1 \end{cases}$$

and $\chi_{\leqslant 1}$ is otherwise arbitrary.

We start with the second line of (5.57). Using the simple estimate

$$|J_1(x)| \leqslant Cx$$

we get

$$\begin{aligned} |\int_0^\infty \chi_{\leqslant 1}(r\xi)\sin(t\xi)J_1(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi| &\leqslant Cr\int_0^\infty \chi_{\leqslant 1}(r\xi)\xi\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi \\ &\leqslant \frac{C}{r\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2} > 4 \end{aligned}$$

We need only estimate the remaining integral in (5.57), namely,

$$\int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\sin(t\xi)J_1(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

Note that this integral vanishes if $r < 2$, by the support properties of $\chi_{\leqslant 1}$ and $\chi_{\leqslant \frac{1}{4}}$. We use (5.54), for $d = 4$. Then, for $r > 4$, we have

$$\begin{aligned} &|\int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\sin(t\xi)J_1(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi| \\ &= |\mathrm{Re}\left(\int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\sin(t\xi)\frac{r\xi}{2\pi^{3/2}}e^{-ir\xi}\Phi_1(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi\right)| \qquad (5.58) \\ &\leqslant \frac{C}{|t-r|}\int_0^\infty |\partial_\xi\left((1-\chi_{\leqslant 1}(r\xi))\, r\xi\Phi_1(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)|d\xi \end{aligned}$$

Note that

$$\begin{aligned} \int_0^\infty |\chi'_{\leqslant 1}(r\xi)|r|r\xi\Phi_1(r\xi)|\frac{|\chi_{\leqslant \frac{1}{4}}(\xi)|}{\log^{b-1}(\frac{1}{\xi})}d\xi &\leqslant Cr\int_0^{1/r}\frac{d\xi}{\log^{b-1}(\frac{1}{\xi})} \\ &\leqslant \frac{C}{\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2} \end{aligned}$$

where we used the support properties of χ and symbol property of Φ_1.

Next,

$$\begin{aligned} \int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\,|\partial_\xi(r\xi\Phi_1(r\xi))|\frac{|\chi_{\leqslant \frac{1}{4}}(\xi)|}{\log^{b-1}(\frac{1}{\xi})}d\xi &\leqslant Cr\int_0^\infty \frac{(1-\chi_{\leqslant 1}(r\xi))}{(r\xi)^{3/2}}\frac{|\chi_{\leqslant \frac{1}{4}}(\xi)|}{\log^{b-1}(\frac{1}{\xi})}d\xi \\ &\leqslant \frac{C}{\sqrt{r}}\int_{\frac{1}{2r}}^{\frac{1}{4}}\frac{d\xi}{\xi^{3/2}\log^{b-1}(\frac{1}{\xi})} \\ &\leqslant \frac{C}{\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2} \end{aligned}$$

Finally, we use the same procedure to get

$$\begin{aligned}
&\int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\,|r\xi\Phi_1(r\xi)||\partial_\xi\left(\frac{\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)|d\xi\\
&\leqslant \frac{C}{\sqrt{r}}\int_{\frac{1}{2r}}^{\frac14}\left(\frac{|\chi'_{\leqslant\frac14}(\xi)|}{\sqrt{\xi}\log^{b-1}(\frac{1}{\xi})}+\frac{1}{\xi^{3/2}\log^b(\frac{1}{\xi})}\right)d\xi\\
&\leqslant \frac{C}{\log^b(r)},\quad r\geqslant\frac{t}{2}
\end{aligned}$$

Returning to (5.58), we get, for $r\geqslant\frac{t}{2}$,

$$\frac{C}{|t-r|}\int_0^\infty|\partial_\xi\left((1-\chi_{\leqslant 1}(r\xi))\,r\xi\Phi_1(r\xi)\frac{\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)|d\xi\leqslant\frac{C}{|t-r|\log^{b-1}(r)}$$

Note that, if $\frac{t}{2}\leqslant r$, then, $|t-r|\leqslant r$, so

$$\frac{C}{r\log^{b-1}(r)}\leqslant\frac{C}{|t-r|\log^{b-1}(r)},\quad r\geqslant\frac{t}{2},\quad r\neq t$$

So, we can combine the previous two estimates to conclude (5.51), thereby concluding the proof of the v_2 estimates (in the case $b\neq 1$).

We now proceed to prove the similar statements about $\partial_r v_2$. We have

$$\partial_r v_2(t,r)=\frac{c_b}{2}\int_0^\infty\sin(t\xi)\xi\left(J_0(r\xi)-J_2(r\xi)\right)\frac{\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

and start by treating the J_0 term. Using (5.53) for $n=0$, we have

$$\begin{aligned}
&\frac{c_b}{2}\int_0^\infty\sin(t\xi)\xi J_0(r\xi)\frac{\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi\\
&=\frac{c_b}{4\pi}\int_0^\pi\int_0^\infty\left(\sin(\xi t_+)+\sin(\xi t_-)\right)\frac{\xi\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi d\theta\\
&=\frac{-c_b}{4\pi}\int_0^\pi\int_0^\infty\left(\frac{\sin(\xi t_+)}{t_+^2}+\frac{\sin(\xi t_-)}{t_-^2}\right)\partial_\xi^2\left(\frac{\xi\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)d\xi d\theta
\end{aligned}$$

We follow the identical procedure used to prove (5.49) for v_2, the only major difference being an extra $\sin^2(\theta)$ in the θ integral in the v_2 case. We get

$$\frac{c_b}{2}\int_0^\infty\sin(t\xi)\xi J_0(r\xi)\frac{\chi_{\leqslant\frac14}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi=\frac{-b}{t^2\log^b(t)}+E_{\partial_r v_2,1}(t,r)$$

where

$$|E_{\partial_r v_2}(t,r)|\leqslant\frac{Cr}{t^3\log^b(t)}+\frac{C}{t^2\log^{b+1}(t)},\quad r\leqslant\frac{t}{2}$$

Now, we treat the J_2 term, again using (5.53), this time for $n = 2$.

$$
\begin{aligned}
& -\frac{c_b}{2}\int_0^\infty \sin(t\xi)\xi J_2(r\xi)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi \\
&= \frac{-c_b r^2}{6\pi}\int_0^\infty \sin(t\xi)\xi^3\int_0^\pi \cos(r\xi\cos(\theta))\sin^4(\theta)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\theta d\xi \\
&= \frac{-c_b r^2}{12\pi}\int_0^\pi \sin^4(\theta)\int_0^\infty \left(\sin(\xi(t+r\cos(\theta))) + \sin(\xi(t-r\cos(\theta)))\right)\frac{\xi^3\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi d\theta \\
&= \frac{-c_b r^2}{12\pi}\int_0^\pi \sin^4(\theta)\int_0^\infty \left(\frac{\sin(\xi(t+r\cos(\theta)))}{(t+r\cos(\theta))^4} + \frac{\sin(\xi(t-r\cos(\theta)))}{(t-r\cos(\theta))^4}\right)\partial_\xi^4\left(\frac{\xi^3\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)d\xi d\theta
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \partial_\xi^4\left(\frac{\xi^3\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right) \\
&= \chi_{\leqslant\frac{1}{4}}(\xi)\left(\frac{11b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})} + \frac{6(b-1)}{\xi\log^{b}(\frac{1}{\xi})} + \frac{b(b-1)(b+1)(b+2)}{\xi\log^{b+3}(\frac{1}{\xi})} + \frac{6b(b-1)(b+1)}{\xi\log^{b+2}(\frac{1}{\xi})}\right) \\
&+ \psi(\xi)
\end{aligned}
$$

where

$$\psi \in C_c^\infty([\frac{1}{8},\frac{1}{4}])$$

We treat the ψ term in the same way as in the v_2 estimate, and use the same argument as in the v_2 estimate to handle the other terms. In total, we get

$$|-\frac{c_b}{2}\int_0^\infty \sin(t\xi)\xi J_2(r\xi)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi| \leqslant \frac{Cr^2}{t^4\log^b(t)}, \quad r \leqslant \frac{t}{2}$$

Combining our results, we then get (5.49) for $\partial_r v_2$.

Now, we study the region $r \geqslant \frac{t}{2}$. Again using the simple estimate

$$|J_1'(x)| \leqslant \frac{C}{\sqrt{x}}$$

we get

$$|\partial_r v_2(t,r)| \leqslant \frac{C}{\sqrt{r}}$$

Then, we decompose

$$\partial_r v_2(t,r) = I_r + II_r$$

with

$$I_r = \frac{c_b}{2}\int_0^\infty \sin(t\xi)\xi\chi_{\leqslant 1}(r\xi)\left(J_0(r\xi) - J_2(r\xi)\right)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

Using the simple estimate

$$|J_k(x)| \leqslant C, \quad k = 0, 2$$

we get

$$\begin{aligned}
|I_r| &\leqslant |\frac{c_b}{2}\int_0^\infty \sin(t\xi)\chi_{\leqslant 1}(r\xi)\xi\left(J_0(r\xi) - J_2(r\xi)\right)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi| \\
&\leqslant C\int_0^{\min\{\frac{1}{r},\frac{1}{4}\}} \frac{\xi d\xi}{\log^{b-1}(\frac{1}{\xi})} \leqslant \frac{C}{r^2\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2} \geqslant 4
\end{aligned}$$

It remains to estimate II_r. We write

$$II_r = II_{r,0} + II_{r,2}$$

with

$$II_{r,0} = \frac{c_b}{2}\int_0^\infty \sin(t\xi)\left(1 - \chi_{\leqslant 1}(r\xi)\right)\xi J_0(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

$$II_{r,2} = -\frac{c_b}{2}\int_0^\infty \sin(t\xi)\left(1 - \chi_{\leqslant 1}(r\xi)\right)\xi J_2(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

Again using (5.54) for $d = 2$, we get

$$\begin{aligned}
|II_{r,0}| &\leqslant C\int_0^\infty |\partial_\xi^2\left(\left(1 - \chi_{\leqslant 1}(r\xi)\right)\xi\Phi_0(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)|\frac{d\xi}{(t-r)^2} \\
&\leqslant \frac{C}{(t-r)^2}\int_0^\infty \frac{\mathbb{1}_{\{r\xi \geqslant \frac{1}{2}\}}\mathbb{1}_{\{\xi \leqslant \frac{1}{4}\}}}{\sqrt{r}\xi^{3/2}\log^{b-1}(\frac{1}{\xi})}d\xi \\
&\leqslant \frac{C}{(t-r)^2}\int_{\frac{1}{2r}}^{\frac{1}{4}} \frac{d\xi}{\sqrt{r}\xi^{3/2}\log^{b-1}(\frac{1}{\xi})} \leqslant \frac{C}{(t-r)^2}\frac{1}{\sqrt{r}}\int_{\frac{1}{2r}}^{\frac{1}{4}} \frac{d\xi}{\xi^{3/2}\log^{b-1}(\frac{1}{\xi})} \\
&\leqslant \frac{C}{(t-r)^2\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2}
\end{aligned} \tag{5.59}$$

For $II_{r,2}$, we use (5.54) for $d = 6$. Using the same argument as for $II_{r,0}$, we get

$$\begin{aligned}
|II_{r,2}| &\leqslant C\int_0^\infty |\partial_\xi^2\left(\left(1 - \chi_{\leqslant 1}(r\xi)\right)\xi(r\xi)^2\Phi_2(r\xi)\frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)|\frac{d\xi}{(t-r)^2} \\
&\leqslant \frac{C}{(t-r)^2\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2}
\end{aligned}$$

Combining these, we have

$$|II_r| \leqslant \frac{C}{(t-r)^2\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2}$$

Using the same reasoning as was used to prove the analogous estimate for v_2, if $r \geqslant \frac{t}{2}$, $\quad r \neq t$, we have

$$\frac{1}{r^2 \log^{b-1}(r)} \leqslant \frac{C}{(t-r)^2 \log^{b-1}(r)}$$

which gives (5.51) for $\partial_r v_2$.

Next, we study $\partial_t v_2$, in the region $0 \leqslant r \leqslant \frac{t}{2}$. Using (5.53), we have

$$\begin{aligned}
\partial_t v_2(t,r) &= c_b \int_0^\infty \cos(t\xi) \frac{\xi J_1(r\xi) \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi \\
&= \frac{c_b}{\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty \xi \cos(t\xi) r\xi \cos(r\xi\cos(\theta)) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi d\theta \\
&= \frac{c_b r}{\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty \frac{\xi^2}{2} \left(\cos(\xi(t + r\cos(\theta))) + \cos(\xi(t - r\cos(\theta)))\right) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi d\theta \\
&= \frac{c_b r}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty \left(\frac{\sin(\xi(t + r\cos(\theta)))}{(t + r\cos(\theta))^3} + \frac{\sin(\xi(t - r\cos(\theta)))}{(t - r\cos(\theta))^3}\right) \\
&\qquad \partial_\xi^3 \left(\frac{\xi^2 \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right) d\xi d\theta
\end{aligned}$$

We note that

$$\begin{aligned}
\partial_\xi^3 \left(\frac{\xi^2}{\log^{b-1}(\frac{1}{\xi})} \chi_{\leqslant \frac{1}{4}}(\xi)\right) &= \left(\frac{(b-1)b(b+1)}{\xi \log^{b+2}(\frac{1}{\xi})} + \frac{3b(b-1)}{\xi \log^{b+1}(\frac{1}{\xi})} + \frac{2(b-1)}{\xi \log^{b}(\frac{1}{\xi})}\right) \chi_{\leqslant \frac{1}{4}}(\xi) \\
&\quad + \psi(\xi)
\end{aligned}$$

where

$$\psi \in C_c^\infty([\frac{1}{8}, \frac{1}{4}])$$

The integral to estimate is therefore of exactly the same form as that treated in estimating v_2, and repeating this procedure gives (5.49) for $\partial_t v_2$. For the region $r \geqslant \frac{t}{2}$, we can again use

$$|J_1(x)| \leqslant \frac{C}{\sqrt{x}}$$

to get

$$|\partial_t v_2(t,r)| \leqslant \frac{C}{\sqrt{r}}$$

We now consider the region $r \geqslant \frac{t}{2}$. First, let us write

$$\partial_t v_2(t,r) = I_t(t,r) + II_t(t,r)$$

where

$$I_t = c_b \int_0^\infty \chi_{\leqslant 1}(r\xi) \xi \cos(t\xi) J_1(r\xi) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi$$

$$II_t = c_b \int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\,\xi\cos(t\xi)J_1(r\xi)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

Then, we use

$$|J_1(x)| \leqslant Cx$$

to get

$$|I_t| \leqslant C\int_0^{\min\{\frac{1}{r},\frac{1}{4}\}} \frac{r\xi^2}{\log^{b-1}(\frac{1}{\xi})}d\xi \leqslant \frac{C}{r^2\log^{b-1}(r)}, \quad r \geqslant 4$$

As usual, we use (5.54) with $d=4$ to get

$$II_t = \frac{c_b}{4\pi^{3/2}}\mathrm{Re}\left(\int_0^\infty (1-\chi_{\leqslant 1}(r\xi))\,\xi\left(e^{i(t-r)\xi}+e^{-i(t+r)\xi}\right)\frac{r\xi\Phi_1(r\xi)\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi\right)$$

and this gives

$$|II_t| \leqslant C\int_0^\infty \frac{1}{|t-r|^2}\left|\partial_\xi^2\left(\frac{(1-\chi_{\leqslant 1}(r\xi))\,\xi^2 r\Phi_1(r\xi)\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)\right|d\xi$$

But, comparing this expression with (5.59), we see that the only difference is

$$\Phi_0(r\xi) \text{ is replaced by } \Phi_1(r\xi)\cdot r\xi$$

Comparing the symbol-type estimates on Φ_d which follow (5.54), we can use the same procedure used to treat (5.59) to get

$$|II_t| \leqslant \frac{C}{(t-r)^2\log^{b-1}(r)}, \quad r \geqslant \frac{t}{2}$$

This gives (5.51) for $\partial_t v_2$.

Next, we obtain estimates on $\partial_r^2 v_2$, using the same procedure as above. In particular, we use

$$J_1''(x) = \frac{1}{4}(J_3(x)-3J_1(x))$$

to get

$$\partial_r^2 v_2(t,r) = \frac{c_b}{4}\int_0^\infty \sin(t\xi)\xi^2\left(-3J_1(r\xi)+J_3(r\xi)\right)\frac{\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}d\xi$$

Then, we use (5.53) for $J_k, \quad k=1,3$. For the term involving J_k, for $k=1$ or $k=3$, we integrate by parts $k+3$ times in ξ and get:

$$\begin{aligned}\partial_r^2 v_2(t,r) = &\frac{-3c_b r}{8\pi}\int_0^\pi \sin^2(\theta)\int_0^\infty \left(\frac{\sin(\xi(t+r\cos(\theta)))}{(t+r\cos(\theta))^4}+\frac{\sin((t-r\cos(\theta))\xi)}{(t-r\cos(\theta))^4}\right)\\ &\partial_\xi^4\left(\frac{\xi^3\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)d\xi d\theta\\ &-\frac{c_b r^3}{120\pi}\int_0^\pi \sin^6(\theta)\int_0^\infty\left(\frac{\sin(\xi(t+r\cos(\theta)))}{(t+r\cos(\theta))^6}+\frac{\sin(\xi(t-r\cos(\theta)))}{(t-r\cos(\theta))^6}\right)\\ &\partial_\xi^6\left(\frac{\xi^5\chi_{\leqslant\frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})}\right)d\xi d\theta\end{aligned}$$

We will not need as precise of a description of $\partial_r^2 v_2$ here, since such a description for small r will be obtained later using the equation solved by v_2. So, we use the identical procedure used for $\partial_r v_2$, and get

$$|\partial_r^2 v_2(t,r)| \leqslant \frac{Cr}{t^4 \log^b(t)}, \quad r \leqslant \frac{t}{2}$$

For the larger r estimates, we can again use

$$|J_k(x)| \leqslant \frac{C}{\sqrt{x}}, \quad k=1,3$$

to get

$$|\partial_r^2 v_2(t,r)| \leqslant C \int_0^\infty \frac{\xi^2}{\sqrt{r\xi}} \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi \leqslant \frac{C}{\sqrt{r}}$$

Lastly, for another estimate for large r, we first split $\partial_r^2 v_2$ as follows

$$\partial_r^2 v_2(t,r) = I_{rr} + II_{rr}$$

and

$$I_{rr} = \frac{c_b}{4} \int_0^\infty \sin(t\xi)\xi^2 \chi_{\leqslant 1}(r\xi)\left(-3J_1(r\xi)+J_3(r\xi)\right) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi$$

$$II_{rr} = \frac{c_b}{4} \int_0^\infty \sin(t\xi)\xi^2 \left(1-\chi_{\leqslant 1}(r\xi)\right)\left(-3J_1(r\xi)+J_3(r\xi)\right) \frac{\chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} d\xi$$

which give

$$|I_{rr}| \leqslant C \int_0^{\min\{\frac{1}{r},\frac{1}{4}\}} \frac{\xi^2 r\xi}{\log^{b-1}(\frac{1}{\xi})} d\xi \leqslant \frac{C}{r^3 \log^{b-1}(r)}, \quad r \geqslant 4$$

Next, we again use (5.54) for $J_k, \quad k=1,3$, and integrate by parts 3 times, to get

$$\begin{aligned} |II_{rr}| &\leqslant \sum_{k\in\{1,3\}} \frac{C}{|t-r|^3} \int_0^\infty |\partial_\xi^3 \left(\frac{\xi^2 \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} (r\xi)^k \Phi_k(r\xi)\left(1-\chi_{\leqslant 1}(r\xi)\right)\right)| d\xi \\ &\leqslant \frac{C}{|t-r|^3} \int_0^\infty \left(\frac{\mathbb{1}_{\{r\xi \geqslant \frac{1}{2}\}} \mathbb{1}_{\{\xi \leqslant \frac{1}{4}\}}}{\sqrt{r}\xi^{3/2} \log^{b-1}(\frac{1}{\xi})} + \sqrt{\frac{r}{\xi}} \frac{\mathbb{1}_{\{1 \geqslant r\xi \geqslant \frac{1}{2}\}} \mathbb{1}_{\{\xi \leqslant \frac{1}{4}\}}}{\log^{b-1}(\frac{1}{\xi})} \right) d\xi \\ &\leqslant \frac{C}{|t-r|^3 \log^{b-1}(r)}, \quad r > \frac{t}{2} \end{aligned}$$

Combining the above estimates, we get (5.52)

Next, we study $\partial_t^2 v_2$ in the region $r \leqslant \frac{t}{2}$. We proceed just as for v_2:

$$\partial_t^2 v_2(t,r) = \frac{-c_b}{2\pi} \int_0^\pi \sin^2(\theta) \int_0^\infty r \left(\frac{\sin(\xi t_+)}{t_+^4} + \frac{\sin(\xi t_-)}{t_-^4} \right) \partial_\xi^4 \left(\frac{\chi_{\leqslant \frac{1}{4}}(\xi) \cdot \xi^3}{\log^{b-1}(\frac{1}{\xi})} \right) d\xi d\theta$$

We then use

$$\partial_\xi^4 \left(\frac{\chi_{\leqslant \frac{1}{4}}(\xi) \cdot \xi^3}{\log^{b-1}(\frac{1}{\xi})} \right)$$
$$= \chi_{\leqslant \frac{1}{4}}(\xi) \left(\frac{6b\,(b^2-1)}{\xi \log^{b+2}\left(\frac{1}{\xi}\right)} + \frac{(b-1)b(b+1)(b+2)}{\xi \log^{b+3}\left(\frac{1}{\xi}\right)} + \frac{11(b-1)b}{\xi \log^{b+1}\left(\frac{1}{\xi}\right)} + \frac{6(b-1)}{\xi \log^{b}\left(\frac{1}{\xi}\right)} \right)$$
$$+ \psi(\xi)$$
$$, \quad \psi \in C_c^\infty([\frac{1}{8}, \frac{1}{4}])$$

The integral is of the same form as that considered during the v_2 estimates, and we get (5.49) for $\partial_t^2 v_2$. (Note that any other estimates on $\partial_t^2 v_2$ which we may need can be obtained using the other estimates of the lemma, and the equation solved by v_2).

Finally, we will need a similar formula for $\partial_{tr} v_2$: We have

$$\partial_{tr} v_2(t,r) = \frac{c_b}{2} \int_0^\infty \cos(t\xi)\xi^2 (J_0(r\xi) - J_2(r\xi)) \frac{\chi_{\leqslant \frac{1}{4}}(\xi) d\xi}{\log^{b-1}(\frac{1}{\xi})}$$

Applying (5.53), the J_0 term is

$$\frac{c_b}{4\pi} \int_0^\pi \int_0^\infty \left(\frac{\sin(\xi t_+)}{t_+^3} + \frac{\sin(\xi t_-)}{t_-^3} \right) \partial_\xi^3 \left(\frac{\xi^2 \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} \right) d\xi d\theta$$

Comparing this to the analogous integral treated while studying $\partial_t v_2$, we get

$$\frac{c_b}{4\pi} \int_0^\pi \int_0^\infty \left(\frac{\sin(\xi t_+)}{t_+^3} + \frac{\sin(\xi t_-)}{t_-^3} \right) \partial_\xi^3 \left(\frac{\xi^2 \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} \right) d\xi d\theta$$
$$= \frac{2b}{t^3 \log^b(t)} + E_{\partial_{12} v_2, 1}(t,r)$$

where

$$|E_{\partial_{12} v_2, 1}(t,r)| \leqslant C \left(\frac{r}{t^4 \log^b(t)} + \frac{1}{t^3 \log^{b+1}(t)} \right), \quad r \leqslant \frac{t}{2}$$

For the J_2 term, we use (5.53), and get

$$\frac{c_b}{2} \int_0^\infty \cos(t\xi)\xi^2(-J_2(r\xi)) \frac{\chi_{\leqslant \frac{1}{4}}(\xi) d\xi}{\log^{b-1}(\frac{1}{\xi})}$$
$$= \frac{c_b r^2}{12\pi} \int_0^\pi \sin^4(\theta) \int_0^\infty \left(\frac{\sin(\xi t_+)}{t_+^5} + \frac{\sin(\xi t_-)}{t_-^5} \right) \partial_\xi^5 \left(\frac{\xi^4 \chi_{\leqslant \frac{1}{4}}(\xi)}{\log^{b-1}(\frac{1}{\xi})} \right) d\xi d\theta$$

we then treat this integral in exactly the same way as the J_0 term, and get

$$|\frac{c_b}{2} \int_0^\infty \cos(t\xi)\xi^2(-J_2(r\xi)) \frac{\chi_{\leqslant \frac{1}{4}}(\xi) d\xi}{\log^{b-1}(\frac{1}{\xi})}| \leqslant \frac{Cr^2}{t^5 \log^b(t)}, \quad r \leqslant \frac{t}{2}$$

This gives (5.49) for $\partial_{tr}v_2$.

Finally, (5.50) is proven for $\partial_t^2 v_2$ and $\partial_{tr}v_2$, and (5.51) is proven for $\partial_{tr}v_2$ in exactly the same way as for the other derivatives.

The identical procedure shows that (5.49) is still true for $b = 1$. In the region $r \geqslant \frac{t}{2}$, (5.50) is still true for $b = 1$, and for the second estimate in the region $r \geqslant \frac{t}{2}$, we get for $0 \leqslant j \leqslant 1, \quad 0 \leqslant k \leqslant 1, \quad j + k \leqslant 2$

$$|\partial_t^j \partial_r^k v_2(t,r)| \leqslant \frac{C}{|t-r|^{1+j+k}} \log(\log(r)), \quad t \neq r \geqslant \frac{t}{2}, \quad b = 1$$

(Note the difference with the case $b \neq 1$, which had $\frac{C}{|t-r|^{1+j+k}\log^{b-1}(r)}$ on the right hand side). In any case, (5.51) is true for all $b > 0$ (for simplicity, we use $\log(r)$ instead of $\begin{cases} \log^{1-b}(r), & b \neq 1 \\ \log(\log(r)), & b = 1 \end{cases}$ in (5.51)). □

5.8.4 The inner product of the (rescaled) v_3 linear error term

Lemma 5.7. *For v_3 defined in* (5.20)*, we have*

$$\begin{aligned} &\int_0^\infty \left(\frac{\cos(2Q_1(R)) - 1}{R^2\lambda(t)^2}\right) v_3(t, R\lambda(t))\phi_0(R) R dR \\ &= \frac{16}{\lambda(t)} \int_t^\infty K_3(s-t, \lambda(t))\lambda''(s) ds \\ &+ \int_0^\infty \left(\frac{\cos(2Q_1(R)) - 1}{R^2\lambda(t)^2}\right) E_5(t, R\lambda(t))\phi_0(R) R dR \end{aligned}$$

where E_5 is as in (5.21)*,*

$$K_3(w, \lambda(t)) = \left(\frac{w}{1+w^2} - \frac{w}{\lambda(t)^{2-2\alpha} + w^2}\right) \frac{w^4}{4(w^2 + 36\lambda(t)^2)^2}$$

and

$$\int_0^\infty |K_3(w, \lambda(t)) - K_{3,0}(w, \lambda(t))| dw \leqslant C \tag{5.60}$$

where

$$K_{3,0}(w, \lambda(t)) = -\frac{1}{4(\lambda(t)^{1-\alpha} + w)(1+w)^3}$$

Proof. We start with

$$\begin{aligned} &\int_0^\infty \left(\frac{\cos(2Q_1(R)) - 1}{R^2\lambda(t)^2}\right) v_3(t, R\lambda(t))\phi_0(R) R dR \\ &= \frac{16}{\lambda(t)} \int_0^\infty \frac{R^3}{(1+R^2)^3} \int_{t+6R\lambda(t)}^\infty \lambda''(s) \left(\frac{s-t}{1+(s-t)^2} - \frac{(s-t)}{\lambda(t)^{2-2\alpha} + (s-t)^2}\right) ds dR \\ &+ \int_0^\infty \left(\frac{\cos(2Q_1(R)) - 1}{R^2\lambda(t)^2}\right) E_5(t, R\lambda(t))\phi_0(R) R dR \end{aligned} \tag{5.61}$$

where we recall the decomposition (5.21).

We study the second line of (5.61) in more detail, since the third line is as in the lemma statement.

$$\frac{16}{\lambda(t)}\int_0^\infty \frac{R^3}{(1+R^2)^3}\int_{t+6R\lambda(t)}^\infty \lambda''(s)\left(\frac{s-t}{1+(s-t)^2}-\frac{(s-t)}{\lambda(t)^{2-2\alpha}+(s-t)^2}\right)dsdR$$
$$=\frac{16}{\lambda(t)}\int_t^\infty K_3(s-t,\lambda(t))\lambda''(s)ds$$

where

$$K_3(w,\lambda(t))=\left(\frac{w}{1+w^2}-\frac{w}{\lambda(t)^{2-2\alpha}+w^2}\right)\frac{w^4}{4(w^2+36\lambda(t)^2)^2}$$

We now estimate

$$\int_0^\infty |K_3(w,\lambda(t))-K_{3,0}(w,\lambda(t))|dw$$

where

$$K_{3,0}(w,\lambda(t))=-\frac{1}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}$$

We start with

$$\int_0^{\lambda(t)^{1-\alpha}}|K_3(w,\lambda(t))|dw+\int_0^{\lambda(t)^{1-\alpha}}\frac{dw}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}$$
$$\leqslant C\int_0^{\lambda(t)^{1-\alpha}}\frac{wdw}{(1+w^2)}+C\int_0^{\lambda(t)^{1-\alpha}}\frac{wdw}{\lambda(t)^{2-2\alpha}}+C\int_0^{\lambda(t)^{1-\alpha}}\frac{dw}{\lambda(t)^{1-\alpha}}$$
$$\leqslant C$$

Next,

$$\begin{aligned}&\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}}|K_3(w,\lambda(t))+\frac{1}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}|dw\\&\leqslant\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}}|w\left(\frac{1}{1+w^2}-\frac{1}{\lambda(t)^{2-2\alpha}+w^2}\right)\left(\frac{w^4}{4(w^2+36\lambda(t)^2)^2}-\frac{1}{4}\right)|dw\\&+\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}}|\frac{w}{4}\left(\frac{1}{1+w^2}-\frac{1}{\lambda(t)^{2-2\alpha}+w^2}\right)+\frac{1}{4(\lambda(t)^{1-\alpha}+w)}|dw\\&+\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}}|\frac{-1}{4(\lambda(t)^{1-\alpha}+w)}+\frac{1}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}|dw\end{aligned}\tag{5.62}$$

For the second line of (5.62), we have

$$\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}}|w\left(\frac{1}{1+w^2}-\frac{1}{\lambda(t)^{2-2\alpha}+w^2}\right)\left(\frac{w^4}{4(w^2+36\lambda(t)^2)^2}-\frac{1}{4}\right)|dw$$
$$\leqslant C\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}}w\left(\frac{1}{\lambda(t)^{2-2\alpha}+w^2}-\frac{1}{1+w^2}\right)\frac{\lambda(t)^2}{w^2}dw$$
$$\leqslant C\lambda(t)^{2\alpha}|\log(\lambda(t)^{2-2\alpha})|\leqslant C$$

For the third line of (5.62), we have

$$\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}} |\frac{w}{4}\left(\frac{1}{1+w^2}-\frac{1}{\lambda(t)^{2-2\alpha}+w^2}\right)+\frac{1}{4(\lambda(t)^{1-\alpha}+w)}|dw$$
$$\leqslant \int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}} \frac{wdw}{4(1+w^2)}+\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}} |\frac{1}{\lambda(t)^{1-\alpha}+w}-\frac{w}{\lambda(t)^{2-2\alpha}+w^2}|dw$$
$$\leqslant C$$

Finally, the last line of (5.62) is estimated as follows:

$$\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}} |\frac{-1}{4(\lambda(t)^{1-\alpha}+w)}+\frac{1}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}|dw$$
$$\leqslant C\int_{\lambda(t)^{1-\alpha}}^{\frac{1}{2}} \frac{wdw}{(\lambda(t)^{1-\alpha}+w)}$$
$$\leqslant C$$

Then, we consider

$$\int_{\frac{1}{2}}^{\infty} |K_3(w,\lambda(t))+\frac{1}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}|dw$$
$$\leqslant \int_{\frac{1}{2}}^{\infty} |K_3(w,\lambda(t))|dw+\int_{\frac{1}{2}}^{\infty} \frac{dw}{(\lambda(t)^{1-\alpha}+w)(1+w)^3}$$
$$\leqslant C\int_{\frac{1}{2}}^{\infty} \frac{w|\lambda(t)^{2-2\alpha}-1|}{w^4}dw+C$$
$$\leqslant C$$

Combining the above, we get that

$$\int_0^{\infty} |K_3(w,\lambda(t))-K_{3,0}(w,\lambda(t))|dw \leqslant C$$

□

5.8.5 Leading behavior of λ and set-up of the modulation equation

From the previous subsections, we have

$$\begin{aligned}
&\langle F_4(t,\cdot\lambda(t)),\phi_0\rangle\\
&=\frac{-16}{\lambda(t)^3}\int_t^\infty \lambda''(s)K_1(s-t,\lambda(t))ds+\frac{4b}{\lambda(t)t^2\log^b(t)}+\frac{4\alpha\log(\lambda(t))\lambda''(t)}{\lambda(t)}\\
&+\frac{16}{\lambda(t)}\int_t^\infty \lambda''(s)K_{3,0}(s-t,\lambda(t))ds\\
&+E_{0,1}(\lambda(t),\lambda'(t),\lambda''(t))+\frac{16}{\lambda(t)}\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))\right)ds\\
&+\frac{-16}{\lambda(t)^3}\int_t^\infty \lambda''(s)K(s-t,\lambda(t))ds+E_{v_2,ip}(t,\lambda(t))\\
&+\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left((v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)+E_5\right)|_{r=R\lambda(t)},\phi_0\rangle\\
&-\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(v_1+v_2+v_3\right)|_{r=R\lambda(t)},\phi_0\rangle\\
&-\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}(t,r)|_{r=R\lambda(t)},\phi_0\rangle
\end{aligned}$$

where

$$E_{0,1}(\lambda(t),\lambda'(t),\lambda''(t))=2\frac{\lambda'(t)^2}{\lambda(t)^2}+\frac{2\lambda''(t)}{\lambda(t)}-\frac{4\alpha\lambda''(t)\log(\lambda(t))}{\lambda(t)}\left(\frac{1}{-1+\lambda(t)^{2\alpha}}+1\right)$$

So, the equation resulting from

$$\langle F_4(t),\phi_0(\frac{\cdot}{\lambda(t)})\rangle=0$$

is what we recorded at the beginning of this section, namely (5.31). For convenience, we repeat the equation here.

$$
\begin{aligned}
&-4\int_t^\infty \frac{\lambda''(s)}{1+s-t}ds+\frac{4b}{t^2\log^b(t)}+4\alpha\log(\lambda(t))\lambda''(t)\\
&-4\int_t^\infty \frac{\lambda''(s)}{(\lambda(t)^{1-\alpha}+s-t)(1+s-t)^3}ds\\
&=-\lambda(t)E_{0,1}(\lambda(t),\lambda'(t),\lambda''(t))-16\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))\right)ds\\
&+\frac{16}{\lambda(t)^2}\int_t^\infty K(s-t,\lambda(t))\lambda''(s)ds-\lambda(t)E_{v_2,ip}(t,\lambda(t))\\
&+\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(s)\left(K_1(s-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+s-t)}\right)ds \qquad (5.63)\\
&-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left((v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)+E_5\right)|_{r=R\lambda(t)},\phi_0\rangle\\
&+\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(v_1+v_2+v_3\right)|_{r=R\lambda(t)},\phi_0\rangle\\
&+\lambda(t)\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}(t,r)|_{r=R\lambda(t)},\phi_0\rangle\\
&:=G(t,\lambda(t))
\end{aligned}
$$

The key point is that there is leading order cancellation between the four terms on the first two lines of (5.63) (which means cancellation of terms of size $\frac{1}{t^2\log^b(t)}$ and terms of size $\frac{\log(\log(t))}{t^2\log^{b+1}(t)}$) when we substitute $\lambda=\lambda_0$ into these terms, where

$$\lambda_0(t)=\frac{1}{\log^b(t)}+\int_t^\infty\int_{t_1}^\infty \frac{-b^2\log(\log(t_2))}{t_2^2\log^{b+2}(t_2)}dt_2dt_1:=\lambda_{0,0}+\lambda_{0,1}$$

In order to show this cancellation, we first let $\lambda(t)=\lambda_0(t)+e(t)$, and re-write the equation for e in the following way:

$$
\begin{aligned}
&-4\int_t^\infty \frac{\lambda_{0,0}''(s)}{1+s-t}ds+\frac{4b}{t^2\log^b(t)}\\
&-4\int_t^\infty \frac{\lambda_{0,1}''(s)}{1+s-t}ds+4\alpha\log(\lambda_{0,0}(t))\lambda_{0,0}''(t)-4\int_t^\infty \frac{\lambda_{0,0}''(s)ds}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)(1+s-t)^3}\\
&-4\int_t^\infty \frac{e''(s)ds}{1+s-t}+4\alpha\log(\lambda_0(t))e''(t)-4\int_t^\infty \frac{e''(s)ds}{(\lambda_0(t)^{1-\alpha}+s-t)(1+s-t)^3}\\
&+4\alpha(\log(\lambda(t))-\log(\lambda_{0,0}(t)))\lambda_{0,0}''(t)+4\alpha\log(\lambda(t))\left(\lambda_0''(t)-\lambda_{0,0}''(t)\right)\\
&+4\alpha e''(t)\left(\log(\lambda(t))-\log(\lambda_0(t))\right) \qquad (5.64)\\
&-4\int_t^\infty \frac{e''(s)}{(1+s-t)^3}\left(\frac{1}{\lambda(t)^{1-\alpha}+s-t}-\frac{1}{\lambda_0(t)^{1-\alpha}+s-t}\right)ds\\
&-4\int_t^\infty \frac{(\lambda_0''(s)-\lambda_{0,0}''(s))ds}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)(1+s-t)^3}\\
&-4\int_t^\infty \frac{\lambda_0''(s)}{(1+s-t)^3}\left(\frac{1}{(\lambda(t)^{1-\alpha}+s-t)}-\frac{1}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\right)ds\\
&=G(t,\lambda(t))
\end{aligned}
$$

Now, we will show that the terms in each of the first two lines of (5.64) cancel to leading order. More precisely, this means that both terms in the first line of (5.64) are of size $\frac{1}{t^2\log^b(t)}$, but their sum has size bounded above by $\frac{1}{t^2\log^{b+1}(t)}$. Similarly, each term on the second line of (5.64) is of size $\frac{\log(\log(t))}{t^2\log^{b+1}(t)}$, but their sum has size bounded above by $\frac{1}{t^2\log^{b+1}(t)}$.

$$
\begin{aligned}
-4\int_t^\infty \frac{\lambda_{0,0}''(s)}{1+s-t}ds &= -4\int_t^{2t} \frac{\lambda_{0,0}''(s)}{1+s-t}ds-4\int_{2t}^\infty \frac{\lambda_{0,0}''(s)}{1+s-t}ds\\
&=-4\lambda_{0,0}''(t)\int_t^{2t}\frac{ds}{1+s-t}-4\int_t^{2t}\frac{\lambda_{0,0}''(s)-\lambda_{0,0}''(t)}{1+s-t}ds\\
&\quad-4\int_{2t}^\infty \frac{\lambda_{0,0}''(s)}{1+s-t}ds\\
&=-\frac{4b}{t^2\log^{b+1}(t)}\log(1+t)+O\left(\frac{1}{t^2\log^{b+1}(t)}\right)+\text{Err}
\end{aligned}
$$

where

$$
\begin{aligned}
|\text{Err}| &\leqslant 4\int_t^{2t}\frac{||\lambda_{0,0}'''||_{L^\infty(t,2t)}(s-t)}{1+s-t}ds+\frac{4}{1+t}\int_{2t}^\infty|\lambda_{0,0}''(s)|ds\\
&\leqslant \frac{C}{t^2\log^{b+1}(t)}
\end{aligned}
$$

So, we get

$$
-4\int_t^\infty \frac{\lambda_{0,0}''(s)}{1+s-t}ds=-\frac{4b}{t^2\log^b(t)}+E_{\lambda_{0,0}}
$$

where

$$|E_{\lambda_{0,0}}| \leqslant \frac{C}{t^2 \log^{b+1}(t)}$$

Next, we have

$$\begin{aligned}
&-4\int_t^\infty \frac{\lambda_{0,0}''(x)dx}{(\lambda_{0,0}(t)^{1-\alpha}+x-t)(1+x-t)^3}\\
&= -4\int_t^\infty \frac{b}{x^2\log^{b+1}(x)} \frac{dx}{(\lambda_{0,0}(t)^{1-\alpha}+x-t)(1+x-t)^3} + E_{v_{3,ip}}
\end{aligned}$$

where

$$|E_{v_{3,ip}}| \leqslant C\int_t^\infty \frac{dx}{x^2\log^{b+2}(x)(\lambda_{0,0}(t)^{1-\alpha}+x-t)(1+x-t)^3}$$

Then, we have

$$\begin{aligned}
&|-4b\int_t^{t+\lambda_{0,0}(t)^{1-\alpha}} \frac{dx}{x^2\log^{b+1}(x)(\lambda_{0,0}(t)^{1-\alpha}+x-t)(1+x-t)^3}|\\
&\leqslant C\int_t^{t+\lambda_{0,0}(t)^{1-\alpha}} \frac{dx}{x^2\log^{b+1}(x)}\frac{1}{\lambda_{0,0}(t)^{1-\alpha}}\\
&\leqslant \frac{C}{t^2\log^{b+1}(t)}
\end{aligned}$$

The second term to consider is

$$\begin{aligned}
&-4b\int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{1}{x^2\log^{b+1}(x)}\frac{1}{(\log^{(\alpha-1)b}(t)+x-t)}\frac{dx}{(1+x-t)^3}\\
&= \frac{-4b}{t^2\log^{b+1}(t)}\int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{dx}{(\log^{(\alpha-1)b}(t)+x-t)}\\
&-4b\int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \left(\frac{1}{x^2\log^{b+1}(x)}-\frac{1}{t^2\log^{b+1}(t)}\right)\frac{1}{(\log^{(\alpha-1)b}(t)+x-t)}dx\\
&-4b\int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{1}{x^2\log^{b+1}(x)}\frac{1}{(\log^{(\alpha-1)b}(t)+x-t)}\left(\frac{1}{(1+x-t)^3}-1\right)dx
\end{aligned} \tag{5.65}$$

The second line of (5.65) is treated as follows:

$$\begin{aligned}
&\frac{-4b}{t^2\log^{b+1}(t)}\int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{dx}{(\log^{(\alpha-1)b}(t)+x-t)}\\
&= \frac{-4b}{t^2\log^{b+1}(t)}\left(\log(2\log^{(\alpha-1)b}(t)+\frac{1}{2})-\log(2\log^{(\alpha-1)b}(t))\right)\\
&= \frac{4b^2(\alpha-1)\log(\log(t))}{t^2\log^{b+1}(t)} + E_{v_{3,ip},1b}
\end{aligned}$$

where

$$|E_{v_{3,ip},1b}| \leqslant \frac{C}{t^2 \log^{b+1}(t)}$$

The third line of (5.65) is estimated by:

$$\begin{aligned}
&|-4b \int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \left(\frac{1}{x^2 \log^{b+1}(x)} - \frac{1}{t^2 \log^{b+1}(t)}\right) \frac{1}{(\log^{(\alpha-1)b}(t) + x - t)} dx| \\
&\leqslant \frac{C}{t^3 \log^{b+1}(t)} \int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{(x-t)dx}{(\log^{(\alpha-1)b}(t) + x - t)} \\
&\leqslant \frac{C}{t^3 \log^{b+1}(t)}
\end{aligned}$$

The last line of (5.65) is estimated by

$$\begin{aligned}
&|-4b \int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{1}{x^2 \log^{b+1}(x)} \frac{1}{(\log^{(\alpha-1)b}(t) + x - t)} \left(\frac{1}{(1+x-t)^3} - 1\right) dx| \\
&\leqslant C \int_{t+\log^{(\alpha-1)b}(t)}^{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}} \frac{1}{x^2 \log^{b+1}(x)} \frac{x-t}{(\log^{(\alpha-1)b}(t) + x - t)} dx \\
&\leqslant \frac{C}{t^2 \log^{b+1}(t)}
\end{aligned}$$

The third term to consider is

$$\begin{aligned}
&|-4b \int_{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}}^{\infty} \frac{1}{x^2 \log^{b+1}(x)} \frac{1}{(\log^{(\alpha-1)b}(t) + x - t)} \frac{dx}{(1+x-t)^3}| \\
&\leqslant \frac{C}{t^2 \log^{b+1}(t)} \int_{t+\log^{(\alpha-1)b}(t)+\frac{1}{2}}^{\infty} \frac{dx}{(x-t)^4} \\
&\leqslant \frac{C}{t^2 \log^{b+1}(t)}
\end{aligned}$$

Then, we estimate $E_{v_{3,ip}}$:

$$\begin{aligned}
|E_{v_{3,ip}}| &\leqslant \frac{C}{t^2 \log^{b+2}(t)} \int_t^{\infty} \frac{dx}{(\log^{(\alpha-1)b}(t) + x - t)(1+x-t)^3} \\
&\leqslant \frac{C \log(\log(t))}{t^2 \log^{b+2}(t)}
\end{aligned}$$

In total, we have

$$-4 \int_t^{\infty} \frac{\lambda''_{0,0}(x)dx}{(\log^{(\alpha-1)b}(t) + x - t)(1+x-t)^3} = E_{v_{3,ip,f}} + \frac{4b^2(\alpha-1)\log(\log(t))}{t^2 \log^{b+1}(t)}$$

where

$$|E_{v_{3,ip,f}}| \leqslant \frac{C}{t^2 \log^{b+1}(t)}$$

By the same procedure used to study the analogous term involving $\lambda_{0,0}$, we have

$$-4\int_t^\infty \frac{\lambda_{0,1}''(s)ds}{1+s-t} = \frac{4b^2\log(\log(t))}{t^2\log^{b+1}(t)} + E_{v_{3,ip},01}$$

where

$$|E_{v_{3,ip},01}| \leqslant \frac{C\log(\log(t))}{t^2\log^{b+2}(t)}$$

Combining the above, we can show the cancellation, to leading order, of the terms on each of the first two lines of (5.64):

$$-4\int_t^\infty \frac{\lambda_{0,0}''(s)}{1+s-t}ds + \frac{4b}{t^2\log^b(t)} = E_{\lambda_{0,0}}$$

with

$$|E_{\lambda_{0,0}}| \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

and

$$\begin{aligned}&-4\int_t^\infty \frac{\lambda_{0,1}''(s)}{1+s-t}ds + 4\alpha\log(\lambda_{0,0}(t))\lambda_{0,0}''(t) - 4\int_t^\infty \frac{\lambda_{0,0}''(s)ds}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)(1+s-t)^3}\\ &= E_{v_3,ip,01} + E_{v_3,ip,f} + E_{\lambda_{0,1}}\end{aligned}$$

with

$$E_{\lambda_{0,1}}(t) = 4\alpha\log(\lambda_{0,0}(t))\left(\lambda_{0,0}''(t) - \frac{b}{t^2\log^{b+1}(t)}\right)$$

$$|E_{v_3,ip,01}| + |E_{v_3,ip,f}| + |E_{\lambda_{0,1}}| \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

We recall that we are considering (5.63) for $t \in [T_0,\infty)$, with T_0 satisfying (5.1).

The equation for e can be written as follows:

$$
\begin{aligned}
&-4\int_t^\infty \frac{e''(s)ds}{\log(\lambda_0(s))(1+s-t)} + 4\alpha e''(t) - 4\int_t^\infty \frac{e''(s)ds}{\log(\lambda_0(s))(\lambda_0(t)^{1-\alpha}+s-t)(1+s-t)^3}\\
&= \frac{1}{\log(\lambda_0(t))}\left(-E_{\lambda_{0,0}} - E_{v_3,ip,01} - E_{v_3,ip,f} - E_{\lambda_{0,1}}\right)\\
&+ \frac{1}{\log(\lambda_0(t))}\left(G(t,\lambda(t)) - 4\alpha(\log(\lambda(t)) - \log(\lambda_{0,0}(t)))\lambda_{0,0}''(t) - 4\alpha\log(\lambda(t))\left(\lambda_0''(t) - \lambda_{0,0}''(t)\right)\right)\\
&+ \frac{1}{\log(\lambda_0(t))}\left(-4\alpha e''(t)\left(\log(\lambda(t)) - \log(\lambda_0(t))\right)\right)\\
&+ \frac{1}{\log(\lambda_0(t))}\left(4\int_t^\infty \frac{e''(s)}{(1+s-t)^3}\left(\frac{1}{\lambda(t)^{1-\alpha}+s-t} - \frac{1}{\lambda_0(t)^{1-\alpha}+s-t}\right)ds\right)\\
&+ \frac{4}{\log(\lambda_0(t))}\int_t^\infty \frac{(\lambda_0''(s) - \lambda_{0,0}''(s))ds}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)(1+s-t)^3}\\
&+ \frac{4}{\log(\lambda_0(t))}\int_t^\infty \frac{\lambda_0''(s)}{(1+s-t)^3}\left(\frac{1}{(\lambda(t)^{1-\alpha}+s-t)} - \frac{1}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\right)ds\\
&+ 4\int_t^\infty e''(s)\left(\frac{1}{\log(\lambda_0(t))} - \frac{1}{\log(\lambda_0(s))}\right)\frac{1}{(1+s-t)}ds\\
&+ 4\int_t^\infty e''(s)\left(\frac{1}{\log(\lambda_0(t))} - \frac{1}{\log(\lambda_0(s))}\right)\frac{1}{(\lambda_0(t)^{1-\alpha}+s-t)(1+s-t)^3}ds\\
&:= RHS(e,t)
\end{aligned}
\tag{5.66}
$$

where $\lambda(t) = \lambda_0(t) + e(t)$. In order to study this equation, let us first consider the problem of solving an equation of the form

$$
-\int_t^\infty \frac{y(s)}{\log(\lambda_0(s))}\left(\frac{1}{1+s-t} + \frac{1}{(\lambda_0(t)^{1-\alpha}+s-t)(1+s-t)^3}\right)ds + \alpha y(t) = F(t) \quad , t \geqslant T_0
$$

where

$$
F \in C([T_0,\infty)), \quad |F(x)| \leqslant \frac{C}{x^2} \tag{5.67}
$$

If

$$
x(t) = y(-t), \quad H(t) = \frac{F(-t)}{\alpha}, \quad t \leqslant -T_0
$$

then our equation becomes

$$
\begin{aligned}
&-\int_{-\infty}^t \frac{x(s)}{\alpha\log(\lambda_0(-s))}\left(\frac{1}{1-s+t} + \frac{1}{(\lambda_0(-t)^{1-\alpha}-s+t)(1-s+t)^3}\right)ds + x(t)\\
&= H(t), \quad t \leqslant -T_0
\end{aligned}
$$

which can be written as

$$
\int_J x(s)K(t,s)ds + x(t) = H(t), \quad t \in J \tag{5.68}
$$

where

$$J = (-\infty, -T_0]$$

and, for $(t,s) \in J^2$,

$$\begin{aligned} K(t,s) &= -\frac{\mathbb{1}_{\leqslant 0}(s-t)}{\alpha \log(\lambda_0(-s))} \left(\frac{1}{1-s+t} + \frac{1}{(\lambda_0(-t)^{1-\alpha} - s + t)(1-s+t)^3} \right) \\ &= \frac{\mathbb{1}_{\leqslant 0}(s-t)}{\alpha |\log(\lambda_0(-s))|} \left(\frac{1}{1-s+t} + \frac{1}{(\lambda_0(-t)^{1-\alpha} - s + t)(1-s+t)^3} \right) \end{aligned}$$

Note that $K(t,s) = 0$ when $s > t$, so K is a (non-negative) Volterra kernel on J^2.

Moreover, K is of type L^∞_{loc} on J. To see this, it suffices to consider $C = [a,d] \subset J$ a compact subinterval, $g \in L^1(C)$, $f \in L^\infty(C)$ with $||g||_{L^1(C)} \leqslant 1, ||f||_{L^\infty(C)} \leqslant 1$ and estimate

$$\begin{aligned} &\int_C \int_C |g(t)||K(t,s)||f(s)| ds dt \\ &\leqslant \int_C \frac{|g(t)|}{2\alpha} \left(\int_a^t \frac{ds}{1-s+t} + \frac{1}{\lambda_0(-t)^{1-\alpha}} \int_{-\infty}^t \frac{ds}{(1-s+t)^3} \right) dt \\ &\leqslant \frac{\log(1+d-a) + \frac{1}{2\lambda_0(-a)^{1-\alpha}}}{2\alpha} \end{aligned}$$

where we used the facts (which follow from (5.1) and (5.2))

$$t \mapsto \frac{1}{\lambda_0(-t)^{1-\alpha}} \text{ is decreasing}, \quad \frac{1}{|\log(\lambda_0(-t))|} \leqslant \frac{1}{2}, \quad t \in J$$

Moreover, if $s \leqslant u \leqslant v \leqslant t$, and $(t,v,u,s) \in J^4$, then,

$$K(v,s)K(t,u) \leqslant K(t,s)K(v,u) \tag{5.69}$$

To verify this, let us note that, by the given conditions on s, u, v, t, we have

$$1 = \mathbb{1}_{\leqslant 0}(s-v) = \mathbb{1}_{\leqslant 0}(u-t) = \mathbb{1}_{\leqslant 0}(s-t) = \mathbb{1}_{\leqslant 0}(u-v)$$

and

$$K(t,s) = \frac{\mathbb{1}_{\leqslant 0}(s-t)}{\alpha |\log(\lambda_0(-s))|} k(t,s)$$

for

$$k(t,s) = \frac{1}{1-s+t} + \frac{1}{(\lambda_0(-t)^{1-\alpha} - s + t)(1-s+t)^3}$$

So, it suffices to show that

$$\frac{\partial^2}{\partial s \partial t} \log(k(t,s)) \leqslant 0, \quad s \leqslant t$$

Then, we note that

$$\log(k(t,s)) = \log(\frac{1}{1-s+t}) + \log\left(1 + \frac{1}{(\lambda_0(-t)^{1-\alpha} - s + t)(1-s+t)^2}\right)$$

and we have

$$
\begin{aligned}
&\partial_{st}\log\left(1+\frac{1}{(\lambda_0(-t)^{1-\alpha}-s+t)(1-s+t)^2}\right)\\
&=\left(\frac{-\left(2(1-s+t)(\lambda_0(-t)^{1-\alpha}-s+t)+(1-s+t)^2(-\lambda_0'(-t)(1-\alpha)\lambda_0(-t)^{-\alpha}+1)\right)}{((1-s+t)^2(\lambda_0(-t)^{1-\alpha}-s+t)+1)^2}\right)\\
&\cdot\left(\frac{2}{1-s+t}+\frac{1}{(\lambda_0(-t)^{1-\alpha}-s+t)}\right)\\
&+\left(\frac{1}{(1-s+t)^2(\lambda_0(-t)^{1-\alpha}-s+t)+1}\right)\left(\frac{-2}{(1-s+t)^2}-\frac{(-(1-\alpha)\lambda_0(-t)^{-\alpha}\lambda_0'(-t)+1)}{(\lambda_0(-t)^{1-\alpha}-s+t)^2}\right)\\
&\leqslant 0
\end{aligned}
$$

where we recall $\alpha\leqslant\frac{1}{4}$ and $\lambda_0'(x)\leqslant 0,\quad x\geqslant T_0$. This completes the verification of (5.69).

Now, by Theorem 8.6 (sec. 9.8, pg. 259) of [8], K has a non-negative resolvent, r, which is locally of type L^∞ on J^2. Recall that the resolvent kernel, r, satisfies (a.e) the equations

$$
r+K*r=K,\quad r+r*K=K \tag{5.70}
$$

where the $*$ operation, (as defined in [8], Definition 2.3) between two measureable functions on J^2, or between a measureable function on J^2 and one on J (when the integrands are integrable) is

$$
\begin{aligned}
(a*b)(t,s)&=\int_J a(t,u)b(u,s)du\\
(a*f)(t)&=\int_J a(t,u)f(u)du
\end{aligned}
$$

In fact, we also have

$$
\frac{K(t,s)}{K(u,s)}\leqslant 2,\quad s\leqslant u\leqslant t \tag{5.71}
$$

because

$$
\begin{aligned}
\frac{K(t,s)}{K(u,s)}&=\frac{\frac{1}{1-s+t}+\frac{1}{(\lambda_0(-t)^{1-\alpha}-s+t)(1-s+t)^3}}{\frac{1}{1-s+u}+\frac{1}{(\lambda_0(-u)^{1-\alpha}-s+u)(1-s+u)^3}}\\
&\leqslant\frac{1-s+u}{1-s+t}+\frac{(\lambda_0(-u)^{1-\alpha}-s+u)(1-s+u)^3}{(\lambda_0(-t)^{1-\alpha}-s+t)(1-s+t)^3}\\
&\leqslant 2,\quad s\leqslant u\leqslant t
\end{aligned}
$$

Hence, by Theorem 8.5(sec. 9.8, pg. 258) of [8] r is in fact a non-negative Volterra kernel of type L^∞, and we have the estimate

$$
\int_{-\infty}^{t} r(t,u)du\leqslant 2,\quad \text{a.e. } t\in J \tag{5.72}
$$

By our assumptions on F, namely (5.67), H satisfies the property that (e.g)

$$
H(\cdot)\,(\cdot)^2\in L^\infty(J) \tag{5.73}
$$

Then, we have a solution to (5.68) (a.e.) given by the formula

$$x = H - (r * H) \tag{5.74}$$

To see this, we first note that (5.68) is

$$K * x + x = H$$

Then,

$$K * (r * H) = (K * r) * H$$

by Fubini's theorem. Fubini's theorem is applicable because

$$\begin{aligned}
&\int_J \int_J |K(t,u)||r(u,s)||H(s)|dsdu = \int_{-\infty}^{t} \int_{-\infty}^{u} |K(t,u)||r(u,s)||H(s)|dsdu \\
&\leqslant ||H(\cdot)\,(\cdot)^2\,||_{L^\infty(J)} \int_{-\infty}^{t} \int_{-\infty}^{u} K(t,u) \frac{r(u,s)}{s^2} dsdu \\
&\leqslant ||H(\cdot)\,(\cdot)^2\,||_{L^\infty(J)} \int_{-\infty}^{t} \int_{-\infty}^{u} \frac{K(t,u)}{u^2} r(u,s) dsdu \\
&\leqslant 2||H(\cdot)\,(\cdot)^2\,||_{L^\infty(J)} \int_{-\infty}^{t} \frac{K(t,u)}{u^2} du \\
&\leqslant \frac{2||H(\cdot)\,(\cdot)^2\,||_{L^\infty(J)}}{\alpha|\log(\lambda_0(T_0))|} \left(1 + \frac{1}{\lambda_0(-t)^{1-\alpha}}\right) \int_{-\infty}^{t} \frac{du}{u^2} \\
&\leqslant \frac{2||H(\cdot)\,(\cdot)^2\,||_{L^\infty(J)}}{\alpha|\log(\lambda_0(T_0))|} \left(1 + \frac{1}{\lambda_0(-t)^{1-\alpha}}\right) \frac{1}{|t|}, \quad t \in J
\end{aligned}$$

by (5.72), (5.73), and inspection of the formula for K. Now, substituting

$$x = H - (r * H)$$

using Fubini's theorem as above, and using the resolvent equation

$$r + K * r = K$$

we see that (5.74) is a solution to (5.68).

But, this means that we have a solution to the equation

$$\alpha y(t) = F(t) + \int_t^\infty \frac{y(s)}{\log(\lambda_0(s))} \left(\frac{1}{1+s-t} + \frac{1}{(\lambda_0(t)^{1-\alpha} + s - t)(1+s-t)^3} \right) ds$$
$$, \quad \text{a.e. } t \geqslant T_0$$

Since we are considering this equation for F satisfying (5.67) the right-hand side of the equation above is a continuous function of t. So, y agrees with a continuous function a.e., and hence, we may extend y given a.e. by (5.74) to a continuous function of $t \in [T_0, \infty)$.

(5.74), written in terms of y reads

$$y(t) = \frac{F(t)}{\alpha} - \int_t^\infty \frac{F(s)}{\alpha} r(-t,-s) ds \tag{5.75}$$

and (5.72) implies that

$$\int_t^\infty r(-t,-z)dz \leqslant 2, \quad \text{a.e. } t \geqslant T_0 \tag{5.76}$$

Now, we are finally ready to solve (5.66). We recall the complete, normed vector space $(X, ||\cdot||_X)$ defined in the beginning of this section by

$$X = \{f \in C^2([T_0,\infty)) | ||f||_X < \infty\}$$

where

$$||f||_X = \sup_{t \geqslant T_0} \Big(|f(t)| b \log^b(t) \sqrt{\log(\log(t))} + |f'(t)| t \log^{b+1}(t) \sqrt{\log(\log(t))} \\ + |f''(t)| t^2 \log^{b+1}(t) \sqrt{\log(\log(t))} \Big)$$

We quickly remark that all previous manipulations done on v_k, including estimates and representations of inner products, are valid for all $\lambda = \lambda_0 + e, \quad e \in \overline{B_1(0)} \subset X$, since (5.2) is valid for all λ of this form. For $e \in \overline{B_1(0)} \subset X$, $RHS(e,t)$ is a continuous function of $t \in [T_0, \infty)$. We will now estimate $RHS(e,t)$ for an arbitrary $e \in \overline{B_1(0)} \subset X$. The estimate we will obtain will then allow us to define a map, T, on $\overline{B_1}(0) \subset X$, and prove some properties about it, using the discussion above. Eventually, we will show that T has a fixed point.

We start by estimating all the terms of $RHS(e,t)$, except for the one involving G, for $e \in \overline{B_1(0)} \subset X$. From our previous calculations, we have

$$|E_{\lambda_{0,0}}| + |E_{v_3,ip,01}| + |E_{v_3,ip,f}| + |E_{\lambda_{0,1}}| \leqslant \frac{C}{t^2 \log^{b+1}(t)}$$

So, the first line of $RHS(e,t)$ is bounded above in absolute value by

$$\frac{C}{\log(\log(t)) t^2 \log^{b+1}(t)}$$

Next, we note that

$$\lambda_{0,1}(t) = \int_t^\infty \int_{t_1}^\infty \frac{-b^2 \log(\log(t_2))}{t_2^2 \log^{b+2}(t_2)} dt_2 dt_1 = \frac{-b^2 \log(\log(t))}{(b+1)\log^{b+1}(t)} + O\left(\frac{1}{\log^{b+1}(t)}\right)$$

Then, using the fact that $e \in \overline{B_1(0)} \subset X$, we get that the terms in the second line of $RHS(e,t)$, except for the one depending on G are bounded above in absolute value by

$$\frac{C}{(\log(\log(t)))^{3/2} t^2 \log^{b+1}(t)}$$

Similarly, the third, fourth, fifth, sixth, seventh, and eighth lines of $RHS(e,t)$ are bounded above in absolute value (respectively) by

$$\frac{C}{(\log(\log(t)))^2 t^2 \log^{b+1}(t)} + \frac{C}{(\log(\log(t)))^2 \log^{b+1}(t) t^2} + \frac{C \log(\log(t))}{t^2 \log^{b+2}(t)} \\ + \frac{C}{(\log(\log(t)))^{3/2} t^2 \log^{b+1}(t)} + \frac{C}{t^2 \log^{b+2}(t) (\log(\log(t))^{5/2}} \\ + \frac{C}{(\log(\log(t))^{5/2} t^3 \log^{b+2}(t)}$$

Now, we proceed to estimate the terms from G. In the below expressions, we note that $\lambda(t) = \lambda_0(t) + e(t)$, and $e \in \overline{B}_1(0) \subset X$ is arbitrary. Recalling the definition of $E_{0,1}$, we have

$$|\lambda(t)E_{0,1}(\lambda(t), \lambda'(t), \lambda''(t))| \leqslant \frac{C}{t^2 \log^{b+1}(t)}$$

Then, from (5.60), we have

$$|-16\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t)) - K_{3,0}(s-t,\lambda(t))\right)ds| \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

From (5.40), we have

$$|\lambda(t)E_{v_2,ip}(t,\lambda(t))| \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

Using (5.34) and (5.32), we get

$$|\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(s)K(s-t,\lambda(t))ds| \leqslant \frac{C}{t^2\log^{b+1}(t)}$$
$$|\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)\left(K_1(x-t,\lambda(t)) - \frac{\lambda(t)^2}{4(1+x-t)}\right)dx| \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

In order to proceed, we will use some pointwise estimates on v_3, v_4 and v_5.

5.8.6 Pointwise estimates on $v_3, \partial_r^j v_3$

Here, we will prove two simple pointwise estimate on v_3 which do not directly follow from (5.21). As with all of our work from now on, every estimate is valid for any λ of the form

$$\lambda(t) = \lambda_0(t) + e(t), \quad e \in \overline{B}_1(0) \subset X$$

Lemma 5.8. *We have the following pointwise estimates on* $\partial_r^j v_3, \quad j = 0,1,2$*:*

$$|v_3(t,r)| \leqslant \frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)} \tag{5.77}$$

$$|v_3(t,r)| \leqslant \frac{C}{r}\int_t^\infty |\lambda''(s)|(s-t)ds \tag{5.78}$$

$$|\partial_r v_3(t,r)| \leqslant \frac{C}{t^2\log^b(t)} \tag{5.79}$$

Proof. We again make the decomposition

$$v_3 = v_{3,1} + v_{3,2}$$

and use the same estimates on $v_{3,2}$ proven while obtaining (5.21), to get

$$|v_{3,2}(t,r)| \leqslant \frac{Cr}{t^2\log^{b+1}(t)}$$

For $v_{3,1}$, whose definition is (5.22), we have

$$|v_{3,1}(t,r)| \leqslant \frac{Cr}{t^2\log^{b+1}(t)} + Cr\int_{6r}^{\infty} |\lambda''(t+w)| w |\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)} - \frac{1}{1+w^2}|dw$$

In order to estimate this integral, we use the fact that $\lambda'(x) \leqslant 0, \quad x \geqslant T_0$, and get

$$\begin{aligned}\int_0^1 w\frac{|\lambda''(t+w)|dw}{(\lambda(t+w)^{2-2\alpha}+w^2)} &\leqslant \frac{C}{t^2\log^{b+1}(t)}\frac{1}{\log^{(2\alpha-2)b}(t)}\int_0^1 \frac{wdw}{1+w^2\lambda(t)^{2\alpha-2}} \\ &\leqslant \frac{C\log(\log(t))}{t^2\log^{b+1}(t)}\end{aligned} \tag{5.80}$$

Also,

$$\int_0^1 \frac{w|\lambda''(t+w)|dw}{1+w^2} \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

On the other hand, if $w \geqslant 1$, then,

$$|\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)} - \frac{1}{1+w^2}| \leqslant \frac{C}{w^4}$$

which gives

$$\int_1^{\infty} |\lambda''(t+w)| w |\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)} - \frac{1}{1+w^2}|dw \leqslant \frac{C}{t^2\log^{b+1}(t)}$$

In total, we get

$$|v_{3,1}(t,r)| \leqslant \frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)}$$

which gives (5.77). For the second pointwise estimate on v_3 in the lemma, we use

$$\begin{aligned}|v_3(t,r)| &\leqslant \frac{1}{r}\int_t^{\infty}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}|\lambda''(s)| \cdot 2d\rho ds \\ &\leqslant \frac{C}{r}\int_t^{\infty} |\lambda''(s)|(s-t)ds\end{aligned}$$

We now prove the estimate on $\partial_r v_3$ in the lemma statement. We recall the definition of v_3

$$v_3(t,r) = \frac{-1}{r}\int_t^{\infty}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right)d\rho ds \tag{5.81}$$

Then, we make a decomposition analogous to $v_3 = v_{3,1} + v_{3,2}$, used previously, and treat each term separately:

$$\begin{aligned}&|\int_t^{\infty}\int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\frac{\lambda''(s)}{r}\partial_r\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}\right)d\rho ds| \\ &\leqslant C\left(\sup_{x\geqslant t}|\lambda''(x)|\right)\int_0^{\infty} \rho\left(\frac{1+\rho^2+r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}\right)d\rho \\ &\leqslant C\sup_{x\geqslant t}|\lambda''(x)|\end{aligned}$$

$$
\begin{aligned}
&|\int_t^\infty \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \frac{\lambda''(s)}{r} \partial_r \left(F_3(r,\rho,\lambda(s))\right) d\rho ds| \\
&\leqslant C \sup_{x\geqslant t} \left(|\lambda''(x)|\lambda(x)^{4\alpha-4}\right) \int_0^\infty \rho \frac{(\rho^2+r^2+\lambda(t)^{2-2\alpha})}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(t)^{4\alpha-4})^{3/2}} d\rho \\
&\leqslant C \sup_{x\geqslant t} \left(|\lambda''(x)|\lambda(x)^{4\alpha-4}\right) \lambda(t)^{4-4\alpha}
\end{aligned}
$$

where we used

$$
\frac{|\partial_r F_3|}{r} \leqslant \frac{C((\rho^2+r^2)\lambda(s)^{4\alpha-4}+\lambda(s)^{2\alpha-2})}{(1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4})^{3/2}} \tag{5.82}
$$

Next, we have the term where the r derivative acts on the $\frac{1}{r}$ factored out of the integrals in (5.81). For this term, we simply note that

$$
\begin{aligned}
&\frac{1}{r^2} \int_t^\infty \int_0^{s-t} \frac{\rho\lambda''(s)}{\sqrt{(s-t)^2-\rho^2}} \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds \\
&= \frac{-v_3(t,r)}{r}
\end{aligned}
$$

Then, using (5.77), we have

$$
\begin{aligned}
&|\frac{1}{r^2} \int_t^\infty \int_0^{s-t} \frac{\rho\lambda''(s)}{\sqrt{(s-t)^2-\rho^2}} \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds| \\
&\leqslant C \frac{\log(\log(t))}{t^2 \log^{b+1}(t)}
\end{aligned}
$$

The last term to estimate is

$$
-\int_t^\infty \frac{1}{(s-t)} \int_0^{s-t} \rho \frac{\lambda''(s)}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds
$$

If $s-t \leqslant \frac{1}{2}$, we start with

$$
\begin{aligned}
&|\int_0^{s-t} \frac{\rho}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}\right) d\rho| \\
&= |\frac{-1-r^2+(s-t)^2}{\sqrt{4(s-t)^2+(1+r^2-(s-t)^2)^2}} + 1| \\
&= \frac{4(s-t)^2}{\sqrt{4(s-t)^2+(1+r^2-(s-t)^2)^2}(1+r^2-(s-t)^2+\sqrt{(1+r^2-(s-t)^2)^2+4(s-t)^2})} \\
&\leqslant C(s-t)^2, \quad s-t \leqslant \frac{1}{2}
\end{aligned}
$$

This gives

$$|-\int_{t}^{t+\frac{1}{2}}\frac{1}{(s-t)}\int_{0}^{s-t}\rho\frac{\lambda''(s)}{r}\partial_{r}\left(\frac{-1-\rho^{2}+r^{2}}{\sqrt{(1+\rho^{2}-r^{2})^{2}+4r^{2}}}\right)d\rho ds|$$
$$\leqslant\frac{C}{t^{2}\log^{b+1}(t)}$$

On the other hand, we have

$$|-\int_{t+\frac{1}{2}}^{\infty}\frac{1}{(s-t)}\int_{0}^{s-t}\rho\frac{\lambda''(s)}{r}\partial_{r}\left(\frac{-1-\rho^{2}+r^{2}}{\sqrt{(1+\rho^{2}-r^{2})^{2}+4r^{2}}}\right)d\rho ds|$$
$$\leqslant\int_{t+\frac{1}{2}}^{\infty}\frac{1}{(s-t)}|\lambda''(s)|\cdot 2ds$$
$$\leqslant C\sup_{x\geqslant t}\left(x|\lambda''(x)|\right)\frac{\log(t)}{t}$$

Now, we have to treat the F_3 related terms. We start by recalling (5.82).
Then, we use a slightly different procedure:

$$|-\int_{t}^{\infty}\frac{1}{(s-t)}\int_{0}^{s-t}\rho\frac{\lambda''(s)}{r}\partial_{r}\left(F_{3}(r,\rho,\lambda(s))\right)d\rho ds|$$
$$\leqslant C\int_{0}^{\infty}\rho\int_{\rho+t}^{\infty}\frac{|\lambda''(s)|}{(s-t)}\frac{(\rho^{2}+r^{2}+\lambda(s)^{2-2\alpha})\lambda(s)^{4\alpha-4}}{(1+2(\rho^{2}+r^{2})\lambda(s)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(s)^{4\alpha-4})^{3/2}}dsd\rho$$
$$\leqslant C\int_{0}^{\infty}\rho\int_{\rho+t}^{\infty}\frac{|\lambda''(s)|}{(s-t)}\frac{(\rho^{2}+r^{2}+\lambda(t)^{2-2\alpha})\lambda(s)^{4\alpha-4}}{(1+2(\rho^{2}+r^{2})\lambda(t)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(t)^{4\alpha-4})^{3/2}}dsd\rho$$
$$\leqslant C\int_{0}^{\infty}\frac{\rho(\rho^{2}+r^{2}+\lambda(t)^{2-2\alpha})}{(1+2(\rho^{2}+r^{2})\lambda(t)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(t)^{4\alpha-4})^{3/2}}\frac{1}{\log^{(4\alpha-4)b}(t)t\log^{b+1}(t)}$$
$$\int_{\rho+t}^{\infty}\frac{ds}{s(s-t)}d\rho$$
$$\leqslant C\int_{0}^{t}\frac{\rho(\rho^{2}+r^{2}+\lambda(t)^{2-2\alpha})}{(1+2(\rho^{2}+r^{2})\lambda(t)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(t)^{4\alpha-4})^{3/2}}\frac{\log(1+\frac{t}{\rho})d\rho}{\log^{(4\alpha-4)b}(t)t^{2}\log^{b+1}(t)}$$
$$+C\int_{t}^{\infty}\frac{\rho(\rho^{2}+r^{2}+\lambda(t)^{2-2\alpha})}{(1+2(\rho^{2}+r^{2})\lambda(t)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(t)^{4\alpha-4})^{3/2}}\frac{d\rho}{\log^{(4\alpha-4)b}(t)t^{2}\log^{b+1}(t)}$$

We finally consider two subsets of the region $r\leqslant t$ separately:

$$\int_{0}^{\lambda(t)^{1-\alpha}}\frac{\rho(\rho^{2}+r^{2}+\lambda(t)^{2-2\alpha})}{(1+2(\rho^{2}+r^{2})\lambda(t)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(t)^{4\alpha-4})^{3/2}}\frac{(\log(t)+|\log(\rho)|)d\rho}{\log^{(4\alpha-4)b}(t)t^{2}\log^{b+1}(t)}$$
$$\leqslant C\int_{0}^{\lambda(t)^{1-\alpha}}\frac{\rho\lambda(t)^{2-2\alpha}(\log(t)+|\log(\rho)|)d\rho}{(1+2(\rho^{2}+r^{2})\lambda(t)^{2\alpha-2}+(\rho^{2}-r^{2})^{2}\lambda(t)^{4\alpha-4})^{1/2}}\frac{1}{\log^{(4\alpha-4)b}(t)t^{2}\log^{b+1}(t)}$$
$$\leqslant\frac{C}{t^{2}\log^{b}(t)}$$

Then,

$$\int_{\lambda(t)^{1-\alpha}}^{t} \frac{\rho(\rho^2+r^2+\lambda(t)^{2-2\alpha})}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(t)^{4\alpha-4})^{3/2}} \frac{(\log(t)+|\log(\rho)|)d\rho}{\log^{(4\alpha-4)b}(t)t^2\log^{b+1}(t)}$$
$$\leqslant \frac{C\log(t)}{t^2\log^{(4\alpha-4)b}(t)\log^{b+1}(t)} \int_0^\infty \frac{\rho(\rho^2+r^2+\lambda(t)^{2-2\alpha})d\rho}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(t)^{4\alpha-4})^{3/2}}$$
$$\leqslant \frac{C}{t^2\log^b(t)}$$

The final integral to estimate is then

$$\int_t^\infty \frac{\rho(\rho^2+r^2+\lambda(t)^{2-2\alpha})d\rho}{(1+2(\rho^2+r^2)\lambda(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(t)^{4\alpha-4})^{3/2}} \frac{1}{\log^{(4\alpha-4)b}(t)t^2\log^{b+1}(t)}$$
$$\leqslant \frac{C}{t^2\log^{b+1}(t)}$$

This gives (5.79).

□

5.8.7 Pointwise estimates on v_4, $\partial_r v_4$

In this section, we prove

Lemma 5.9. *For all λ of the form*

$$\lambda(t) = \lambda_0(t) + e(t), \quad e \in \overline{B}_1(0) \subset X$$

we have the pointwise estimates

$$|v_4(t,r)| \leqslant \begin{cases} \frac{Cr}{t^2\log^{3b+2N-1}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{\sqrt{r}t\log^{\frac{3N}{2}+3b-1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.83}$$

$$|\partial_r v_4(t,r)| \leqslant \begin{cases} \frac{C}{t^2\log^{3b+2N-1}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{\sqrt{r}t\log^{3b-1+\frac{5N}{2}}(t)}, & r \geqslant \frac{t}{2} \end{cases} \tag{5.84}$$

and

$$|\partial_t v_4(t,r)| \leqslant \frac{C}{\sqrt{r}t\log^{3b-1+\frac{5N}{2}}(t)}, \quad r \geqslant \frac{t}{2}$$

We also have the L^2 estimates

$$||\partial_t v_4||_{L^2(rdr)} + ||\partial_r v_4||_{L^2(rdr)} + ||\frac{v_4}{r}||_{L^2(rdr)} \leqslant \frac{C}{t\log^{2N+3b}(t)}$$

Proof. We start by considering $v_4(t,r)$, for $\frac{t}{2} > r > 0$. In order to ease notation, let $x \in \mathbb{R}^2$ be defined by $x = r\mathbf{e}_1$. Then, we recall (5.25) and get

$$v_4(t,r) = \frac{-r}{2\pi}\int_0^1\int_t^\infty\int_{B_{s-t}(0)} \frac{1}{\sqrt{(s-t)^2-|y|^2}}\left(\frac{\partial_2 v_{4,c}(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^2} - \frac{v_{4,c}(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^3} + \frac{v_{4,c}(s,|\beta x+y|)}{|\beta x+y|}\right)dA(y)dsd\beta$$

Note that, for $|x| \leqslant \frac{t}{2}$, $0 < \beta < 1$, and $|y| \leqslant s-t$, we have

$$s - |\beta x+y| \geqslant \frac{t}{2}$$

This means that, for the purposes of estimating v_4 in the region $r \leqslant \frac{t}{2}$, we can use (5.51) to estimate v_2, for all $r \geqslant \frac{t}{2}$. We then combine this with the estimates for v_1, v_3, and $F_{0,2}$, to get

$$|v_{4,c}(t,r)| \leqslant C|\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})| \begin{cases} \frac{1}{t^2r^3\log^{3b}(t)}, & r \leqslant \frac{t}{2} \\ \frac{\log(r)}{\log^{2b}(t)r^4|t-r|} + \frac{\log^{2b\alpha}(t)}{t^2r^3\log^{3b+1}(t)}, & r \geqslant \frac{t}{2} \end{cases} \tag{5.85}$$

and similarly, for the derivatives, we have

$$|\partial_r v_{4,c}(t,r)| \leqslant C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)}) \begin{cases} \frac{1}{r^4t^2\log^{3b}(t)}, & r \leqslant \frac{t}{2} \\ \frac{\log(r)}{\log^{2b}(t)r^4(t-r)^2} + \frac{1}{\log^{3b}(t)t^2r^4}, & r \geqslant \frac{t}{2} \end{cases} + \frac{C|\chi'_{\geqslant 1}(\frac{2r}{\log^N(t)})|}{\log^{5N+2b}(t)}\frac{r}{t^2\log^b(t)}$$

Then, we get

$$\begin{aligned}|v_4(t,r)| \leqslant{}& Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)} \frac{1}{\sqrt{(s-t)^2-|y|^2}}|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta \\ &+ Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)} \frac{1}{\sqrt{(s-t)^2-|y|^2}}\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta \\ &+ Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c} \frac{1}{\sqrt{(s-t)^2-|y|^2}}|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta \\ &+ Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c} \frac{1}{\sqrt{(s-t)^2-|y|^2}}\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta\end{aligned} \tag{5.86}$$

Each line of (5.86) is then further split into two terms, based on the decomposition

$$\frac{1}{\sqrt{(s-t)^2-|y|^2}} = \frac{1}{s-t} + \left(\frac{1}{\sqrt{(s-t)^2-|y|^2}} - \frac{1}{s-t}\right)$$

and estimated separately. For the first term of the first line, we have

$$
\begin{aligned}
& r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{(s-t)}|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta\\
& \leqslant Cr\int_0^1\int_t^{t+\frac{1}{2}}\int_{B_{s-t}(0)}\frac{1}{(s-t)}\frac{1}{\log^{4N}(s)s^2\log^{3b}(s)}dA(y)dsd\beta\\
& + Cr\int_0^1\int_{t+\frac{1}{2}}^\infty\frac{1}{(s-t)}\int_{B_{\frac{s}{2}}(-\beta x)}\frac{\mathbb{1}_{\{|\beta x+y|\geqslant\log^N(s)\}}}{|\beta x+y|^4s^2\log^{3b}(s)}dA(y)dsd\beta\\
& \leqslant Cr\int_0^1\int_t^{t+\frac{1}{2}}\frac{(s-t)}{s^2\log^{3b+4N}(s)}dsd\beta\\
& + Cr\int_0^1\int_{t+\frac{1}{2}}^\infty\frac{1}{(s-t)}\int_{B_{\frac{s}{2}}(0)}\frac{\mathbb{1}_{\{|z|\geqslant\log^N(s)\}}}{|z|^4s^2\log^{3b}(s)}dA(z)dsd\beta\\
& \leqslant \frac{Cr}{t^2\log^{3b+4N}(t)}+Cr\int_0^1\int_{t+\frac{1}{2}}^\infty\frac{1}{(s-t)}\int_{\log^N(s)}^{\frac{s}{2}}\rho\int_0^{2\pi}\frac{1}{\rho^4s^2\log^{3b}(s)}d\theta d\rho dsd\beta\\
& \leqslant \frac{Cr}{\log^{3b+2N-1}(t)t^2}
\end{aligned}
$$

where we used

$$
\frac{|\chi'_{\geqslant 1}(\frac{|\beta x+y|}{\log^N(s)})||\beta x+y|}{\log^{5N+3b}(s)s^2}\leqslant C\frac{\mathbb{1}_{\{|\beta x+y|\geqslant\log^N(s)\}}}{|\beta x+y|^4\log^{3b}(s)s^2}
$$

Next, we estimate

$$
\begin{aligned}
&r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)|\partial_r v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta\\
&\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\frac{1}{\log^{2N}(s)}\frac{dA(y)dsd\beta}{(\log^{2N}(t)+|\beta x+y|^2)s^2\log^{3b}(s)}\\
&\leqslant Cr\int_0^1\int_0^\infty\rho\int_0^{2\pi}\frac{1}{\log^{2N}(t)}\frac{1}{(\log^{2N}(t)+\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta))}\frac{1}{\log^{3b}(t)}\\
&\qquad\qquad\int_{\rho+t}^\infty\frac{1}{s^2}\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)dsd\theta d\rho d\beta\\
&\leqslant Cr\int_0^1\int_0^\infty\rho\int_0^{2\pi}\frac{1}{\log^{2N+3b}(t)}\frac{1}{(\log^{2N}(t)+\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta))}\frac{1}{(\rho+t)^2}d\theta d\rho d\beta\\
&\leqslant\frac{Cr}{\log^{2N+3b}(t)}\int_0^1\int_0^\infty\frac{\rho}{(\rho+t)^2}\frac{1}{\sqrt{(\log^{2N}(t)+(\beta r+\rho)^2)(\log^{2N}(t)+(\beta r-\rho)^2)}}d\rho d\beta\\
&\leqslant\frac{Cr}{\log^{2N+3b}(t)}\int_0^1\int_0^t\frac{\rho}{(\rho+t)^2}\frac{1}{\sqrt{(\log^{2N}(t)+(\beta r+\rho)^2)(\log^{2N}(t)+(\beta r-\rho)^2)}}d\rho d\beta\\
&+\frac{Cr}{\log^{2N+3b}(t)}\int_0^1\int_t^\infty\frac{\rho}{\rho^4}d\rho d\beta\\
&\leqslant\frac{Cr(\log(2+r^2)+\log(2+t^2))}{t^2\log^{2N+3b}(t)}\\
&\leqslant\frac{Cr}{t^2\log^{2N+3b-1}(t)},\quad r\leqslant\frac{t}{2}
\end{aligned}
\tag{5.87}
$$

where, we used the fact that

$$\rho\geqslant t,\quad r\leqslant\frac{t}{2}\implies|\beta r-\rho|=\rho-\beta r\geqslant\frac{\rho}{2}$$

Next, we estimate

$$
\begin{aligned}
&r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{(s-t)}\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta\\
&\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{(s-t)}\frac{\chi_{\geqslant 1}(\frac{|\beta x+y|}{\log^N(s)})}{\log^{3b}(s)s^2|\beta x+y|^4}dA(y)dsd\beta
\end{aligned}
$$

The last line in the above equation has already been estimated above, and we get

$$r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{(s-t)}\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta\leqslant\frac{Cr}{t^2\log^{3b+2N-1}(t)}$$

Next, we have

$$r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\frac{\chi_{\geqslant 1}(\frac{|\beta x+y|}{\log^N(s)})}{|\beta x+y|^4 s^2\log^{3b}(s)}dA(y)dsd\beta$$

The last line in the above equation has also been estimated above. Next, we recall that

$$r\leqslant\frac{t}{2}, |y|\leqslant s-t\implies s-|\beta x+y|\geqslant\frac{t}{2}$$

and consider

$$r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{(s-t)}|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta$$

We first treat the portion of the integral when $s-t\leqslant\frac{1}{2}$:

$$r\int_0^1\int_t^{t+\frac{1}{2}}\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{(s-t)}|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_t^{t+\frac{1}{2}}\int_{B_{s-t}(0)}\frac{1}{(s-t)}\left(\frac{1}{\log^{2b-1}(s)s^4t^2}+\frac{1}{\log^{3b}(s)s^6}\right)dA(y)dsd\beta$$
$$\leqslant\frac{Cr}{t^6\log^{2b-1}(t)}$$

We next consider the region $s-t\geqslant\frac{1}{2}$, and get

$$r\int_0^1\int_{t+\frac{1}{2}}^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{(s-t)}|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_{t+\frac{1}{2}}^\infty\frac{1}{(s-t)}\int_{(B_{\frac{s}{2}}(0))^c}\left(\frac{\log(|z|)}{\log^{2b}(s)|z|^4t^2}+\frac{1}{\log^{3b}(s)s^2|z|^4}\right)dA(z)dsd\beta$$
$$\leqslant Cr\int_{t+\frac{1}{2}}^\infty\frac{1}{(s-t)}\frac{1}{s^2t^2\log^{2b-1}(s)}ds$$
$$\leqslant\frac{Cr}{t^4\log^{2b-2}(t)}$$

Next, we have

$$
\begin{aligned}
& r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta \\
& \leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\frac{1}{s^2}\frac{\log(s)}{(t^2+|\beta x+y|^2)t^2\log^{2b}(s)}dA(y)dsd\beta \\
& \leqslant Cr\int_0^1\int_0^\infty\rho\int_0^{2\pi}\frac{1}{(\rho+t)t^3\log^{2b-1}(t)(t^2+\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta))}d\theta d\rho d\beta
\end{aligned}
$$

We can then treat the last line of the above equation in the same way as we treated (5.87). This results in

$$
\begin{aligned}
& r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)|\partial_2 v_{4,c}(s,|\beta x+y|)|dA(y)dsd\beta \\
& \leqslant \frac{Cr}{t^4\log^{2b-2}(t)}
\end{aligned}
$$

Next, we have

$$
r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{(s-t)}\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta
$$

The contribution to the integral from the region $s-t\leqslant\frac{1}{2}$ is treated in an identical manner as above, and we estimate the other contribution, using the same procedure as used above:

$$
\begin{aligned}
& r\int_0^1\int_{t+\frac{1}{2}}^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{(s-t)}\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta \\
& \leqslant Cr\int_0^1\int_{t+\frac{1}{2}}^\infty\frac{1}{(s-t)}\int_{\frac{s}{2}}^\infty\rho\int_0^{2\pi}\left(\frac{1}{t\log^{2b-1}(s)\rho^5}+\frac{\log^{2b\alpha}(s)}{s^2\rho^4\log^{3b+1}(s)}\right)d\theta d\rho dsd\beta \\
& \leqslant Cr\int_{t+\frac{1}{2}}^\infty\frac{ds}{(s-t)ts^3\log^{2b-1}(s)}+Cr\int_{t+\frac{1}{2}}^\infty\frac{\log^{2b\alpha-3b-1}(s)ds}{(s-t)s^4} \\
& \leqslant \frac{Cr}{t^4\log^{2b-2}(t)}
\end{aligned}
$$

The last term to estimate is

$$r\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\frac{|v_{4,c}(s,|\beta x+y|)|}{|\beta x+y|}dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\left(\frac{\log(s)}{\log^{2b}(s)s^5t}+\frac{\log^{2b\alpha}(s)}{s^6\log^{3b+1}(s)}\right)dA(y)dsd\beta$$
$$\leqslant Cr\int_0^\infty\rho\int_{\rho+t}^\infty\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\left(\frac{1}{(\rho+t)^4t^2\log^{2b-1}(t)}+\frac{\log^{2b\alpha-3b-1}(t)}{(\rho+t)^6}\right)dsd\rho$$
$$\leqslant\frac{Cr}{t^4\log^{2b-1}(t)}$$

Combining these estimates, we conclude that

$$|v_4(t,r)|\leqslant\frac{Cr}{t^2\log^{3b+2N-1}(t)},\quad r\leqslant\frac{t}{2}$$

Next, we estimate $\partial_r v_4$, starting with the region $r\leqslant\frac{t}{2}$. We recall the function G defined in (5.24):

$$G(s,r,\rho)=\int_0^{2\pi}\frac{v_{4,c}(s,\sqrt{r^2+2r\rho\cos(\theta)+\rho^2})}{\sqrt{r^2+2r\rho\cos(\theta)+\rho^2}}\left(r+\rho\cos(\theta)\right)d\theta$$
$$s\geqslant t,\quad r\geqslant 0,\quad s-t\geqslant\rho\geqslant 0$$

and start with

$$v_4(t,r)=\frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}G(s,r,\rho)d\rho ds$$

Note that, when we estimated v_4 in the region $r\leqslant\frac{t}{2}$, we used

$$G(s,r,\rho)=r\int_0^1\partial_2 G(s,r\beta,\rho)d\beta$$

Now, we have

$$\partial_r v_4(t,r)=\frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\partial_2 G(s,r,\rho)d\rho ds$$

Therefore, the identical procedure gives the estimate

$$|\partial_r v_4(t,r)|\leqslant\frac{C}{t^2\log^{3b+2N-1}(t)},\quad r\leqslant\frac{t}{2}$$

Now, we treat the region $r\geqslant\frac{t}{2}$. Here, we use a different combination of the v_2 estimates in the various regions, to obtain

$$|v_{4,c}(t,r)|\leqslant C|\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})|\begin{cases}\frac{1}{r^3t^2\log^{3b}(t)}, & r\leqslant\frac{t}{2}\\ \frac{\log(r)}{\log^{2b}(t)r^4|t-r|}+\frac{\log^{2b\alpha}(t)}{t^2r^3\log^{3b+1}(t)}, & \frac{t}{2}\leqslant r\leqslant t-\sqrt{t},\quad r>t+\sqrt{t}\\ \frac{1}{\log^{2b}(t)r^{9/2}}, & t-\sqrt{t}\leqslant r\leqslant t+\sqrt{t}\end{cases}\tag{5.88}$$

and

$$
\begin{aligned}
|\partial_r v_{4,c}(t,r)| &\leqslant \frac{C|\chi'_{\geqslant 1}(\frac{2r}{\log^N(t)})|}{\log^{2b+N}(t) r^4} \left(\frac{r}{t^2 \log^b(t)}\right) \\
&+ \frac{C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})}{r^4 \log^{2b}(t)} \begin{cases} \frac{1}{t^2 \log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{1}{t^2 \log^b(t)} + \frac{\log(r)}{(t-r)^2}, & \frac{t}{2} \leqslant r \leqslant t - t^{1/4}, \text{ or } r \geqslant t + t^{1/4} \\ \frac{1}{\sqrt{r}}, & t - t^{1/4} \leqslant r \leqslant t + t^{1/4} \end{cases}
\end{aligned} \tag{5.89}
$$

Next, we will use a different representation formula for v_4 to estimate v_4 and $\partial_r v_4$ in the region $r \geqslant \frac{t}{2}$. In particular, we have

$$v_4(t,r) = \int_t^\infty \int_0^\infty J_1(r\xi) \sin((t-x)\xi) \widehat{v_{4,c}}(x,\xi) d\xi dx$$

So, it suffices to estimate $\widehat{v_{4,c}}$. To do this, we consider separately the regions

$$\frac{1}{\xi} \geqslant t + \sqrt{t}, \quad t + \sqrt{t} \geqslant \frac{1}{\xi} \geqslant t - \sqrt{t}, \quad t - \sqrt{t} \geqslant \frac{1}{\xi} \geqslant \frac{t}{2}, \quad \frac{t}{2} \geqslant \frac{1}{\xi} \geqslant \log^N(t)$$

Then, for example, in the case $\frac{1}{\xi} \geqslant t + \sqrt{t}$, we have

$$\int_0^\infty J_1(r\xi) r v_{4,c}(t,r) dr = \sum_{k=1}^5 \int_{I_k} J_1(r\xi) r v_{4,c}(t,r) dr$$

where

$$I_1 = [\frac{1}{\xi}, \infty), \quad I_2 = [t+\sqrt{t}, \frac{1}{\xi}], \quad I_3 = [t - \sqrt{t}, t+\sqrt{t}], \quad I_4 = [\frac{t}{2}, t-\sqrt{t}], \quad I_5 = [0, \frac{t}{2}]$$

and we use

$$|J_1(x)| \leqslant \begin{cases} Cx, & 0 < x < 1 \\ \frac{C}{\sqrt{x}}, & x > 1 \end{cases}$$

and (5.88). The analogous decomposition is done for all cases of ξ mentioned above.

This procedure results in

$$|\widehat{v_{4,c}(t,\xi)}| \leqslant \begin{cases} \frac{C\xi \log(t)(\log(t) + |\log(\xi)|)}{t^2 \log^{2b}(t)} + \frac{C\xi^2 |\log(\xi)| \cdot |\log(1-t\xi)|}{t \log^{2b}(t)}, & \xi \leqslant \frac{1}{t+\sqrt{t}} \\ \frac{C\xi}{t^2 \log^{2b-2}(t)}, & \frac{1}{t+\sqrt{t}} \leqslant \xi \leqslant \frac{1}{t-\sqrt{t}} \\ \frac{C\xi \log^2(t)}{t^2 \log^{2b}(t)}, & \frac{1}{t-\sqrt{t}} \leqslant \xi \leqslant \frac{2}{t} \\ \frac{C\xi(|\log(\xi)| + \log(\log(t)))}{t^2 \log^{3b}(t)} + \frac{C}{\sqrt{\xi} t^{7/2} \log^{2b-2}(t)}, & \frac{2}{t} \leqslant \xi \leqslant \frac{1}{\log^N(t)} \end{cases}$$

Rather than recording pointwise estimates on $\widehat{v_{4,c}}(t,\xi)$ in the region $\frac{1}{\xi} \leqslant \log^N(t)$, we use the following argument to infer an integral estimate on $\widehat{v_{4,c}}(t,\xi)$. From (5.88) and (5.89),

$$
\begin{aligned}
&|| \left(\partial_r + \frac{1}{r}\right) v_{4,c}(t,r) ||_{L^2(rdr)} \leqslant ||\partial_r v_{4,c}(t,r)||_{L^2(rdr)} + ||\frac{v_{4,c}(t,r)}{r}||_{L^2(rdr)} \\
&\leqslant \frac{C}{t^2 \log^{3b+3N}(t)}
\end{aligned}
$$

On the other hand,

$$\int_0^\infty J_0(r\xi)\left(\partial_r v_{4,c}(t,r)+\frac{v_{4,c}(t,r)}{r}\right)rdr$$
$$=-\int_0^\infty v_{4,c}(t,r)J_0'(r\xi)\xi rdr=\xi\int_0^\infty v_{4,c}(t,r)J_1(r\xi)rdr$$
$$=\xi\widehat{v_{4,c}}(t,\xi)$$

where the vanishing of the boundary terms arising from integration by parts is justified by (5.88).

By the L^2 isometry property of the Hankel transform of order 0, this implies

$$\frac{C}{t^2\log^{3b+3N}(t)}\geqslant ||\partial_r v_{4,c}(t,r)+\frac{v_{4,c}(t,r)}{r}||_{L^2(rdr)}=||\xi\widehat{v_{4,c}}(t,\xi)||_{L^2(\xi d\xi)} \tag{5.90}$$

Then, we use

$$|J_1(x)|+|J_1'(x)|\leqslant\frac{C}{\sqrt{x}}$$

in the formulae

$$v_4(t,r)=\int_t^\infty\int_0^\infty J_1(r\xi)\sin((t-x)\xi)\widehat{v_{4,c}}(x,\xi)d\xi dx$$

and

$$\partial_r v_4(t,r)=\int_t^\infty\int_0^\infty \xi J_1'(r\xi)\sin((t-x)\xi)\widehat{v_{4,c}}(x,\xi)d\xi dx$$

(Note that the differentiation under the integral sign is justified by the pointwise estimates on $\widehat{v_{4,c}}$, as well as (5.90)).

Thus, we get

$$\begin{aligned}|v_4(t,r)|&\leqslant C\int_t^\infty\frac{1}{\sqrt{r}}\int_0^{\frac{1}{\log^N(x)}}\frac{|\widehat{v_{4,c}}(x,\xi)|}{\sqrt{\xi}}d\xi dx\\&+C\int_t^\infty\frac{1}{\sqrt{r}}\int_{\frac{1}{\log^N(x)}}^\infty\frac{|\widehat{v_{4,c}}(x,\xi)|}{\sqrt{\xi}}\cdot\frac{\xi^{3/2}}{\xi^{3/2}}d\xi dx\\&\leqslant C\int_t^\infty\frac{1}{\sqrt{r}}\int_0^{\frac{1}{\log^N(x)}}\frac{|\widehat{v_{4,c}}(x,\xi)|}{\sqrt{\xi}}d\xi dx\\&+C\int_t^\infty\frac{1}{\sqrt{r}}\left(\int_{\frac{1}{\log^N(x)}}^\infty\frac{d\xi}{\xi^4}\right)^{1/2}||\xi\widehat{v_{4,c}}(x,\xi)||_{L^2(\xi d\xi)}dx\end{aligned}$$

and

$$
\begin{aligned}
|\partial_r v_4(t,r)| &\leqslant C\int_t^\infty \frac{1}{\sqrt{r}}\int_0^{\frac{1}{\log^N(x)}} |\widehat{v_{4,c}}(x,\xi)|\sqrt{\xi}d\xi dx\\
&+C\int_t^\infty \frac{1}{\sqrt{r}}\int_{\frac{1}{\log^N(x)}}^\infty |\widehat{v_{4,c}}(x,\xi)|\sqrt{\xi}\cdot\frac{\xi}{\xi}d\xi dx\\
&\leqslant C\int_t^\infty \frac{1}{\sqrt{r}}\int_0^{\frac{1}{\log^N(x)}} |\widehat{v_{4,c}}(x,\xi)|\sqrt{\xi}d\xi dx\\
&+C\int_t^\infty \frac{1}{\sqrt{r}}\left(\int_{\frac{1}{\log^N(x)}}^\infty \frac{d\xi}{\xi^2}\right)^{1/2} ||\xi\widehat{v_{4,c}}(x,\xi)||_{L^2(\xi d\xi)}dx
\end{aligned}
$$

Using our pointwise estimates on $\widehat{v_{4,c}}$, as well as (5.90), we get

$$
|v_4(t,r)|\leqslant \frac{C}{\sqrt{r}t\log^{\frac{3N}{2}+3b-1}(t)},\quad r\geqslant\frac{t}{2}
$$

and

$$
|\partial_r v_4(t,r)|\leqslant \frac{C}{\sqrt{r}t\log^{3b-1+\frac{5N}{2}}(t)},\quad r\geqslant\frac{t}{2}
$$

In addition, we have

$$
\partial_r v_4(t,r)+\frac{v_4(t,r)}{r}=\int_t^\infty\int_0^\infty \xi J_0(r\xi)\sin((t-x)\xi)\widehat{v_{4,c}}(x,\xi)d\xi dx
$$

First using Minkowski's inequality, then the L^2 isometry property of the Hankel transform of order 0, and then that of the Hankel transform of order 1, we get

$$
\begin{aligned}
||\partial_r v_4(t,r)+\frac{v_4(t,r)}{r}||_{L^2(rdr)} &\leqslant \int_t^\infty ||\int_0^\infty \xi J_0(r\xi)\sin((t-x)\xi)\widehat{v_{4,c}}(x,\xi)d\xi||_{L^2(rdr)}dx\\
&\leqslant \int_t^\infty ||\sin((t-x)\xi)\widehat{v_{4,c}}(x,\xi)||_{L^2(\xi d\xi)}dx\\
&\leqslant \int_t^\infty ||\widehat{v_{4,c}}(x,\xi)||_{L^2(\xi d\xi)}dx\\
&\leqslant \int_t^\infty ||v_{4,c}(x)||_{L^2(rdr)}dx\leqslant\int_t^\infty \frac{Cdx}{\log^{2N+3b}(x)x^2}\\
&\leqslant \frac{C}{t\log^{2N+3b}(t)}
\end{aligned}
$$

where we used (5.88). Note that this is simply the energy estimate for the equation solved by v_4. Next, we note that the pointwise estimates established for v_4 imply that

$$
v_4(t,0)=0,\quad \lim_{r\to\infty}v_4(t,r)=0
$$

So,

$$\left|\left|\left(\partial_r + \frac{1}{r}\right) v_4\right|\right|^2_{L^2(rdr)} = \int_0^\infty \left((\partial_r v_4)^2 + \frac{\partial_r(v_4^2)}{r} + \frac{v_4^2}{r^2}\right) rdr$$
$$= ||\partial_r v_4||^2_{L^2(rdr)} + ||\frac{v_4}{r}||^2_{L^2(rdr)}$$

We now treat $\partial_t v_4$:

$$\partial_t v_4(t,r) = \int_t^\infty \int_0^\infty J_1(r\xi)\cos((t-x)\xi)\xi\widehat{v_{4,c}}(x,\xi)d\xi dx$$

where the differentiation under the integral sign is again justified by the pointwise estimates on $\widehat{v_{4,c}}$ and (5.90).

This gives

$$|\partial_t v_4(t,r)| \leqslant \frac{C}{\sqrt{r}}\int_t^\infty \int_0^\infty \sqrt{\xi}|\widehat{v_{4,c}}(x,\xi)|d\xi dx$$

So, the same exact procedure used for $\partial_r v_4$ pointwise estimates in the region $r \geqslant \frac{t}{2}$ also applies to $\partial_t v_4$.

Finally,

$$\begin{aligned}
||\partial_t v_4(t,r)||_{L^2(rdr)} &\leqslant \int_t^\infty ||\int_0^\infty \xi J_1(r\xi)\cos((t-x)\xi)\widehat{v_{4,c}}(x,\xi)d\xi||_{L^2(rdr)}dx \\
&\leqslant \int_t^\infty ||\cos((t-x)\xi)\widehat{v_{4,c}}(x,\xi)||_{L^2(\xi d\xi)}dx \\
&\leqslant \int_t^\infty ||\widehat{v_{4,c}}(x,\xi)||_{L^2(\xi d\xi)}dx \\
&\leqslant \int_t^\infty ||v_{4,c}(x)||_{L^2(rdr)}dx \leqslant \int_t^\infty \frac{Cdx}{\log^{2N+3b}(x)x^2} \\
&\leqslant \frac{C}{t\log^{2N+3b}(t)}
\end{aligned}$$

This completes the proof of the lemma. □

5.8.8 Pointwise estimates on v_5

Lemma 5.10. *For all λ of the form*

$$\lambda(t) = \lambda_0(t) + e(t), \quad e \in \overline{B_1(0)} \subset X$$

we have the pointwise estimates

$$|v_5(t,r)| \leqslant \begin{cases} \frac{Cr}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C\log^4(t)}{\sqrt{r}t^{3/2}}, & r > \frac{t}{2} \end{cases} \tag{5.91}$$

$$|\partial_r v_5(t,r)| \leqslant \begin{cases} \frac{C}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C\log^3(t)}{\sqrt{r}t^{3/2}}, & r > \frac{t}{2} \end{cases} \tag{5.92}$$

$$|\partial_t v_5(t,r)| \leqslant \frac{C\log^3(t)}{\sqrt{r}t^{3/2}}, \quad r > \frac{t}{2}$$

In addition, we have the L^2 estimates

$$||\partial_t v_5||_{L^2(rdr)} + ||\partial_r v_5||_{L^2(rdr)} + ||\frac{v_5}{r}||_{L^2(rdr)} \leqslant C\frac{\log^3(t)}{t^{7/4}} \tag{5.93}$$

Proof. We start with the region $\frac{t}{2} > r > 0$. In order to ease notation, let $x \in \mathbb{R}^2$ be defined by $x = r\mathbf{e}_1$. Then,

$$v_5(t,r) = \frac{-r}{2\pi}\int_0^1\int_t^\infty\int_{B_{s-t}(0)}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\left(\frac{\partial_2 N_2(f_{v_5})(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^2}\right.$$
$$-\frac{N_2(f_{v_5})(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^3}$$
$$\left.+\frac{N_2(f_{v_5})(s,|\beta x+y|)}{|\beta x+y|}\right)dA(y)dsd\beta$$

We then decompose

$$v_5(t,r) = v_{5,1}(t,r) + v_{5,2}(t,r)$$

where

$$v_{5,1}(t,r)$$
$$= \frac{-r}{2\pi}\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\left(\frac{\partial_2 N_2(f_{v_5})(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^2}\right.$$
$$-\frac{N_2(f_{v_5})(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^3}$$
$$\left.+\frac{N_2(f_{v_5})(s,|\beta x+y|)}{|\beta x+y|}\right)dA(y)dsd\beta$$

and

$$v_{5,2}(t,r) = v_5(t,r) - v_{5,1}(t,r)$$

Finally, we use the pointwise estimates on v_1, v_2, v_3, v_4 to record pointwise estimates on $N_2(f_{v_5})$: We start with

$$|N_2(f)(t,r)| \leqslant \frac{Cr(f(t,r))^2}{\lambda(t)(1+\frac{r^2}{\lambda(t)^2})r^2} + C\frac{|f(t,r)|^3}{r^2}$$

The first set of estimates we will require on $N_2(f_{v_5})$ concern the regions $r \leqslant \frac{t}{2}$, and $t > r > \frac{t}{2}$. In the region $r \leqslant \frac{t}{2}$, we get

$$|N_2(f_{v_5})(t,r)| \leqslant \frac{Cr}{(\lambda(t)^2+r^2)t^4\log^{3b}(t)} + \frac{Cr}{t^6\log^{3b}(t)}, \quad r \leqslant \frac{t}{2}$$

In the region $t > r > \frac{t}{2}$, we use (5.51) to estimate v_2 in the region $t > r \geqslant \frac{t}{2}$, exactly as was done while studying v_4, and we get

$$|N_2(f_{v_5})(t,r)| \leqslant \frac{C\log^3(r)}{r^2|t-r|^3} + \frac{C\lambda(t)}{r^{7/2}t^{5/2}\log^{3N+6b-2}(t)}, \quad t > r \geqslant \frac{t}{2}$$

Next, we consider $\partial_r N_2(f_{v_5})$:

$$|\partial_r(N_2(f))(t,r)| \leqslant \frac{C(f(t,r))^2}{r^2\lambda(t)\left(1+\frac{r^2}{\lambda(t)^2}\right)} + \frac{C|f(t,r)\partial_r f(t,r)|}{\lambda(t)r\left(1+\frac{r^2}{\lambda(t)^2}\right)} + \frac{C|f(t,r)|^3}{r^3} + \frac{C|f(t,r)^2\partial_r f(t,r)|}{r^2}$$

We again treat the regions $t > r > \frac{t}{2}$ and $r \leqslant \frac{t}{2}$. We get

$$|\partial_r N_2(f_{v_5})(t,r)| \leqslant \frac{C}{t^4\log^{3b}(t)(\lambda(t)^2+r^2)} + \frac{C}{t^6\log^{3b}(t)}, \quad r \leqslant \frac{t}{2}$$

$$|\partial_r N_2(f_{v_5})(t,r)| \leqslant \frac{C}{r^3t^3\log^{4N+7b-2}(t)} + \frac{C\lambda(t)\log(r)}{r^3t^{3/2}|t-r|\log^{3b-1+\frac{5N}{2}}(t)} + \frac{C\log^2(r)}{r^2t^{3/2}(t-r)^2\log^{3b-1+\frac{5N}{2}}(t)} + \frac{C\log^3(r)}{r^2(t-r)^4}, \quad t > r \geqslant \frac{t}{2}$$

where we used

$$\frac{1}{r} \leqslant \frac{1}{|t-r|}, \quad r \geqslant \frac{t}{2}$$

So,

$$|v_{5,1}(t,r)| \leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)} \frac{1}{\sqrt{(s-t)^2-|y|^2}} \left(\frac{1}{s^4\log^b(s)\left(1+\frac{|\beta x+y|^2}{\lambda(s)^2}\right)} + \frac{1}{s^6\log^{3b}(s)}\right)dA(y)dsd\beta \tag{5.94}$$

We treat several terms comprising (5.94) sperately. First, we have

$$\begin{aligned}
&r\int_0^1\int_t^{t+\frac{1}{2}}\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)} \frac{1}{(s-t)}\frac{1}{s^4\log^b(s)\left(1+\frac{|\beta x+y|^2}{\lambda(s)^2}\right)}dA(y)dsd\beta \\
&\leqslant Cr\int_0^1\int_t^{t+\frac{1}{2}}\int_0^{s-t}\frac{\rho}{(s-t)}\int_0^{2\pi}\frac{1}{s^4\log^b(s)}d\theta d\rho dsd\beta \\
&\leqslant \frac{Cr}{t^4\log^b(t)}
\end{aligned}$$

Next,

$$r\int_0^1\int_{t+\frac{1}{2}}^{\infty}\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{(s-t)}\frac{1}{s^4\log^b(s)\left(1+\frac{|\beta x+y|^2}{\lambda(s)^2}\right)}dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_{t+\frac{1}{2}}^{\infty}\frac{1}{(s-t)}\int_0^{s-t}\rho\int_0^{2\pi}\frac{1}{s^2(\rho+t)^2\log^b(s)(1+\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta))}d\theta d\rho dsd\beta$$
$$\leqslant Cr\int_0^1\int_{t+\frac{1}{2}}^{\infty}\frac{(\log(2+r^2\beta^2)+\log(2+t^2))}{(s-t)s^2\log^b(s)t^2}dsd\beta$$
$$\leqslant\frac{Cr}{t^4\log^{b-2}(t)},\quad r\leqslant\frac{t}{2}$$

where we used the same procedure that we used in (5.87).

The third term to consider is

$$r\int_0^1\int_t^{\infty}\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\left(\frac{1}{\sqrt{(s-t)^2-|y|^2}}-\frac{1}{(s-t)}\right)\left(\frac{1}{s^4\log^b(s)\left(1+\frac{|\beta x+y|^2}{\lambda(s)^2}\right)}\right)dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_0^{\infty}\frac{\rho}{(\rho+t)^2}\int_0^{2\pi}\frac{1}{(1+\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta))}\int_{\rho+t}^{\infty}\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\frac{1}{s^2\log^b(s)}dsd\theta d\rho d\beta$$
$$\leqslant Cr\int_0^1\left(\frac{\log(2+r^2\beta^2)+\log(2+t^2)}{t^2}\right)\frac{d\beta}{t^2\log^b(t)}$$
$$\leqslant\frac{Cr}{t^4\log^{b-1}(t)},\quad r\leqslant\frac{t}{2}$$

Finally, the last term to consider is

$$r\int_0^1\int_t^{\infty}\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-\beta x)}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\frac{1}{s^6\log^{3b}(s)}dA(y)dsd\beta$$
$$\leqslant Cr\int_0^1\int_t^{\infty}\frac{1}{s^6\log^{3b}(s)}\int_0^{s-t}\rho\int_0^{2\pi}\frac{1}{\sqrt{(s-t)^2-\rho^2}}d\theta d\rho dsd\beta$$
$$\leqslant Cr\int_t^{\infty}\frac{(s-t)}{s^6\log^{3b}(s)}ds$$
$$\leqslant\frac{Cr}{t^4\log^{3b}(t)}$$

Combining the above estimates, we conclude:

$$|v_{5,1}(t,r)|\leqslant C\frac{r}{t^4\log^{b-2}(t)},\quad r\leqslant\frac{t}{2}$$

Next, we treat $v_{5,2}$.

$$\begin{aligned}
&v_{5,2}(t,r)\\
&=\frac{-r}{2\pi}\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\left(\frac{\partial_2 N_2(f)(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^2}\right.\\
&\qquad-\frac{N_2(f)(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^3}\\
&\qquad\left.+\frac{N_2(f)(s,|\beta x+y|)}{|\beta x+y|}\right)dA(y)dsd\beta
\end{aligned}$$

So,

$$\begin{aligned}
&|v_{5,2}(t,r)|\\
&\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\\
&\qquad\left(|\partial_2 N_2(f)(s,|\beta x+y|)|+\frac{|N_2(f)(s,|\beta x+y|)|}{|\beta x+y|}\right)dA(y)dsd\beta\\
&\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-\beta x))^c}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\text{integrand}_{v_{5,2}}(s,|\beta x+y|)dA(y)dsd\beta
\end{aligned}$$

where

$$\begin{aligned}
&\text{integrand}_{v_{5,2}}(s,|\beta x+y|)\\
&=\frac{1}{|\beta x+y|^3 s^3\log^{4N+7b-2}(s)}+\frac{\lambda(s)\log(|\beta x+y|)}{|\beta x+y|^3|s-|\beta x+y||s^{3/2}\log^{3b-1+\frac{5N}{2}}(s)}\\
&+\frac{\log^2(|\beta x+y|)}{|\beta x+y|^2 s^{3/2}(s-|\beta x+y|)^2\log^{3b-1+\frac{5N}{2}}(s)}+\frac{\log^3(|\beta x+y|)}{|\beta x+y|^2(s-|\beta x+y|)^4}
\end{aligned}$$

Exactly as in the case of v_4, we note that, for $z\in B_{s-t}(\beta x)\cap(B_{\frac{s}{2}}(0))^c$, we have

$$|z|=|z-\beta x+\beta x|\leqslant|z-\beta x|+r<s-t+r\leqslant s-\frac{t}{2}$$

So,

$$s-|z|\geqslant\frac{t}{2}$$

We use this estimate for every term in $\text{integrand}_{v_{5,2}}$, except for the term

$$\frac{\log^3(|\beta x+y|)}{|\beta x+y|^2(s-|\beta x+y|)^4}\leqslant\frac{C\log^3(s)}{s^2t^3(s-|\beta x+y|)}$$

Then,

$$
\begin{aligned}
&|v_{5,2}(t,r)| \\
&\leqslant Cr\int_0^1\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\left(\frac{\log^2(s)}{s^{7/2}t^2\log^{3b-1+\frac{5N}{2}}(s)}+\frac{\log^3(s)}{s^3t^3}\right)d\theta d\rho ds d\beta \\
&+Cr\int_0^1\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{\log^3(s)}{s^2t^3(s-\sqrt{\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta)})}d\theta d\rho ds d\beta \\
&\leqslant \frac{Cr}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)} \\
&+Cr\int_0^1\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{\log^3(s)}{s^2t^3(s-\sqrt{\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta)})}d\theta d\rho ds d\beta
\end{aligned}
\tag{5.95}
$$

We need only continue to estimate the last line of (5.95).

$$
\begin{aligned}
&r\int_0^1\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{\log^3(s)}{s^2t^3(s-\sqrt{\beta^2r^2+\rho^2+2\beta r\rho\cos(\theta)})}d\theta d\rho ds d\beta \\
&\leqslant Cr\int_0^1\int_0^\infty\rho\int_0^{2\pi}\frac{1}{t^3}\int_{t+\rho}^\infty\frac{1}{\sqrt{(s-t)^2-\rho^2}}\frac{\log^3(s)}{s^2}\frac{1}{(s-\sqrt{\rho^2+2\rho\beta r\cos(\theta)+\beta^2r^2})}ds d\theta d\rho d\beta \\
&\leqslant Cr\int_0^1\int_0^\infty\rho\int_0^{2\pi}\frac{1}{t^3}\int_{t+\rho}^\infty\frac{1}{\sqrt{s-t+\rho}}\frac{1}{\sqrt{s-t-\rho}} \\
&\qquad\frac{\log^3(s)}{s^2}\frac{1}{(s-\sqrt{\rho^2+2\rho\beta r\cos(\theta)+\beta^2r^2})}ds d\theta d\rho d\beta \\
&\leqslant Cr\int_0^1\int_0^\infty\rho\int_0^{2\pi}\frac{1}{t^3}\frac{1}{\sqrt{\rho}}\frac{\log^3(t+\rho)}{(t+\rho)^2}\int_{t+\rho}^\infty\frac{1}{\sqrt{s-t-\rho}}\frac{ds d\theta d\rho d\beta}{(s-\sqrt{\rho^2+2\rho\beta r\cos(\theta)+\beta^2r^2})} \\
&\leqslant Cr\int_0^1\int_0^\infty\frac{\sqrt{\rho}\log^3(t+\rho)}{(t+\rho)^2}\int_0^{2\pi}\frac{1}{t^3}\frac{1}{\sqrt{t}}d\theta d\rho d\beta \\
&\leqslant C\frac{r\log^3(t)}{t^4}
\end{aligned}
\tag{5.96}
$$

So, we get

$$
|v_{5,2}(t,r)|\leqslant\frac{Cr}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)},\quad r\leqslant\frac{t}{2}
$$

In total, we have

$$
|v_5(t,r)|\leqslant\frac{Cr}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)},\quad r\leqslant\frac{t}{2}
$$

Exactly as for $\partial_r v_4$, we have

$$|\partial_r v_5(t,r)| \leqslant C \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \Big(|\partial_2 N_2(f_{v_5})(s, \sqrt{r^2+\rho^2+2r\rho\cos(\theta)})| + \frac{|N_2(f_{v_5})(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})|}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}} \Big) d\theta d\rho ds \tag{5.97}$$

As was also the case for v_4, the integrals appearing in (5.97) were already estimated above. So, we get

$$|\partial_r v_5(t,r)| \leqslant \frac{C}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)}, \quad r \leqslant \frac{t}{2}$$

We will now prove estimates on $v_5(t,r)$ and $\partial_r v_5$ in the region $r \geqslant \frac{t}{2}$. For the next steps, we will use slightly different combinations of the estimates (5.51) and (5.50) for v_2 in various subsets of the region $r \geqslant \frac{t}{2}$, in order to estimate $N_2(f_{v_5})$, resulting in

$$|N_2(f_{v_5})(t,r)| \leqslant \begin{cases} \frac{Cr}{(\lambda(t)^2+r^2)t^4\log^{3b}(t)} + \frac{Cr}{t^6\log^{3b}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C\log^3(r)}{r^2|t-r|^3} + \frac{C}{r^{7/2}t^{5/2}\log^{3N+7b-2}(t)}, & \frac{t}{2} \leqslant r \leqslant t-\sqrt{t} \text{ or } r \geqslant t+\sqrt{t} \\ \frac{C}{r^{7/2}}, & t-\sqrt{t} \leqslant r \leqslant t+\sqrt{t} \end{cases} \tag{5.98}$$

We will also need estimates on $\partial_r N_2(f_{v_5})$. To obtain these, we use (5.49) in the region $r \leqslant \frac{t}{2}$. In the region $r > \frac{t}{2}$, we use (5.50) in the region $t-\sqrt{t} \leqslant r \leqslant t+\sqrt{t}$, and (5.51) in the regions $\frac{t}{2} < r < t-\sqrt{t}$ and $r > t+\sqrt{t}$ for v_2. For $\partial_r v_2$, we use (5.50) in the region $t-t^{1/4} \leqslant r \leqslant t+t^{1/4}$ and (5.51) in the regions $\frac{t}{2} \leqslant r \leqslant t-t^{1/4}$ and $t+t^{1/4} \leqslant r$. Then, we get

$$|\partial_r N_2(f_{v_5})(t,r)| \leqslant \frac{C}{t^4\log^{3b}(t)(\lambda(t)^2+r^2)} + \frac{C}{t^6\log^{3b}(t)}, \quad r \leqslant \frac{t}{2}$$

$$|\partial_r N_2(f_{v_5})(t,r)| \leqslant \frac{C}{r^3t^3\log^{4N+7b-2}(t)} + \frac{C\log(r)}{r^3|t-r|t^{3/2}\log^{4b-1+\frac{5N}{2}}(t)} + \frac{C\log^2(r)}{r^2t^{3/2}(t-r)^2\log^{3b-1+\frac{5N}{2}}(t)} + \frac{C\log^3(r)}{r^2(t-r)^4}, \quad t-\sqrt{t} > r > \frac{t}{2} \text{ or } r > t+\sqrt{t}$$

$$|\partial_r N_2(f_{v_5})(t,r)| \leqslant \begin{cases} \frac{C\log(r)}{r^3(t-r)^2}, & t-\sqrt{t} \leqslant r \leqslant t-t^{1/4}, \text{ or } t+t^{1/4} \leqslant r \leqslant t+\sqrt{t} \\ \frac{C}{r^{7/2}}, & t-t^{1/4} \leqslant r \leqslant t+t^{1/4} \end{cases}$$

Then, we have

$$v_5(t,r) = \int_t^\infty \int_0^\infty J_1(r\xi)\sin((t-s)\xi)\widehat{N_2(f_{v_5})}(s,\xi)d\xi ds$$

and we will prove estimates on $\widehat{N_2(f_{v_5})}(t,\xi)$, for $\frac{1}{\xi} \geqslant 1$. We consider separately the cases

$$\frac{1}{\xi} \geqslant t+\sqrt{t}, \quad t-\sqrt{t} \leqslant \frac{1}{\xi} \leqslant t+\sqrt{t}, \quad \frac{t}{2} \leqslant \frac{1}{\xi} \leqslant t-\sqrt{t}, \quad 1 \leqslant \frac{1}{\xi} \leqslant \frac{t}{2}$$

and proceed in exactly the same manner as was done for $\widehat{v_{4,c}}$.

This results in

$$|\widehat{N_2(f_{v_5})}(t,\xi)| \leqslant \frac{C\xi}{t}\left(\log^3(t)+|\log(\xi)|^3\right), \quad \frac{1}{\xi} \geqslant t+\sqrt{t}$$

$$|\widehat{N_2(f_{v_5})}(t,\xi)| \leqslant \frac{C\log^3(t)}{t^2}, \quad t-\sqrt{t} \leqslant \frac{1}{\xi} \leqslant t+\sqrt{t}$$

$$|\widehat{N_2(f_{v_5})}(t,\xi)| \leqslant \frac{C\log^3(t)}{t^2}, \quad \frac{t}{2} \leqslant \frac{1}{\xi} \leqslant t-\sqrt{t}$$

$$|\widehat{N_2(f_{v_5})}(t,\xi)| \leqslant \frac{C}{\xi t^4 \log^{3b}(t)} + \frac{C}{\xi^3 t^6 \log^{3b}(t)} + \frac{C\log^3(t)}{\sqrt{\xi} t^{5/2}}, \quad 1 \leqslant \frac{1}{\xi} \leqslant \frac{t}{2}$$

For the region $\frac{1}{\xi} < 1$, we again use the following argument:

From (5.98) and the $\partial_r N_2(f_{v_5})$ estimates which follow it,

$$\begin{aligned}
&||\left(\partial_r + \frac{1}{r}\right) N_2(f_{v_5})(t,r)||_{L^2(rdr)}\\
&\leqslant ||\partial_r N_2(f_{v_5})(t,r)||_{L^2(rdr)} + ||\frac{N_2(f_{v_5})(t,r)}{r}||_{L^2(rdr)}\\
&\leqslant \frac{C\log(t)}{t^{23/8}}
\end{aligned}$$

Then, the same procedure used for $v_{4,c}$ gives

$$\frac{C\log(t)}{t^{23/8}} \geqslant ||\partial_r N_2(f_{v_5})(t,r) + \frac{N_2(f_{v_5})(t,r)}{r}||_{L^2(rdr)} = ||\xi \widehat{N_2(f_{v_5})}(t,\xi)||_{L^2(\xi d\xi)} \tag{5.99}$$

Now, we return to

$$v_5(t,r) = \int_t^\infty \int_0^\infty J_1(r\xi)\sin((t-s)\xi)\widehat{N_2(f_{v_5})}(s,\xi)d\xi ds$$

use

$$|J_1(x)| \leqslant \frac{C}{\sqrt{x}}$$

and the same procedure used for v_4:

$$\begin{aligned}
|v_5(t,r)| &\leqslant \frac{C}{\sqrt{r}}\int_t^\infty \int_0^1 \frac{1}{\sqrt{\xi}}|\widehat{N_2(f_{v_5})}(s,\xi)|d\xi ds\\
&+ \frac{C}{\sqrt{r}}\int_t^\infty \left(\int_1^\infty \frac{d\xi}{\xi^4}\right)^{1/2} ||\xi\widehat{N_2(f_{v_5})}(s,\xi)||_{L^2(\xi d\xi)} ds\\
&\leqslant \frac{C}{\sqrt{r}}\frac{\log^4(t)}{t^{3/2}}
\end{aligned}$$

Next, the pointwise estimates on $\widehat{N_2(f_{v_5})}$, as well as (5.99) justify the following differentiation under the integral sign:

$$\partial_r v_5(t,r) = \int_t^\infty \int_0^\infty \xi J_1'(r\xi) \sin((t-s)\xi) \widehat{N_2(f_{v_5})}(s,\xi) d\xi ds$$

and we also have

$$|J_1'(x)| \leqslant \frac{C}{\sqrt{x}}$$

So,

$$\begin{aligned}|\partial_r v_5(t,r)| &\leqslant \frac{C}{\sqrt{r}} \int_t^\infty \int_0^1 \sqrt{\xi} |\widehat{N_2(f_{v_5})}(s,\xi)| d\xi ds + \frac{C}{\sqrt{r}} \int_t^\infty ||\xi \widehat{N_2(f_{v_5})}(s)||_{L^2(\xi d\xi)} ds \\ &\leqslant \frac{C \log^3(t)}{\sqrt{r} t^{3/2}}\end{aligned}$$

For $\partial_t v_5$, we can similarly differentiate under the integral sign to get

$$\partial_t v_5(t,r) = \int_t^\infty \int_0^\infty J_1(r\xi) \cos((t-x)\xi) \xi \widehat{N_2(f_{v_5})}(x,\xi) d\xi dx$$

so, the same argument used for $\partial_r v_5$ also applies to $\partial_t v_5$.

Finally, the same procedure used for the energy estimates for v_4 also applies here, to give

$$\begin{aligned}||\partial_t v_5||_{L^2(rdr)} + ||\left(\partial_r + \frac{1}{r}\right) v_5||_{L^2(rdr)} &\leqslant \int_t^\infty ||N_2(f_{v_5})(x,r)||_{L^2(rdr)} dx \\ &\leqslant C \frac{\log^3(t)}{t^{7/4}}\end{aligned}$$

(We again have

$$||\left(\partial_r + \frac{1}{r}\right) v_5||^2_{L^2(rdr)} = ||\partial_r v_5||^2_{L^2(rdr)} + ||\frac{v_5}{r}||^2_{L^2(rdr)}$$

by the same argument used while studying v_4). ☐

5.8.9 Solving the modulation equation

The main result of this section is to prove Proposition 5.1, which we recall is the following statement:

There exists $T_3 > 0$ such that, for all $T_0 \geqslant T_3$, there exists $\lambda \in C^2([T_0, \infty))$ which solves (5.63). Moreover,

$$\lambda(t) = \lambda_0(t) + e_0(t), \quad ||e_0||_X \leqslant 1$$

($|| \cdot ||_X$ was defined in (5.30)).

We use our pointwise estimates on v_k, E_5 to get

$$\begin{aligned}
&|-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)v_4\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)},\phi_0\rangle|\\
&\leqslant \frac{C}{\lambda(t)}\int_0^{\frac{t}{2\lambda(t)}}\frac{R^2}{(1+R^2)^3}|v_4(t,R\lambda(t))|dR\\
&\leqslant \frac{C}{t^2\log^{3b+2N-1}(t)}
\end{aligned}$$

$$|-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)v_5\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)},\phi_0\rangle|\leqslant\frac{C}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)}$$

$$|-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)E_5|_{r=R\lambda(t)},\phi_0\rangle|\leqslant\frac{C}{t^2\log^{b+1}(t)}$$

$$\begin{aligned}
&|-\lambda(t)\langle\left(\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)(v_1+v_2+v_3)+F_{0,2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})|_{r=R\lambda(t)},\phi_0\rangle|\\
&\leqslant C\lambda(t)\int_0^\infty |v_{4,c}(t,R\lambda(t))|\phi_0(R)RdR\leqslant\frac{C}{t^2\log^{3b+2N}(t)}
\end{aligned}$$

Combining the above with our previous estimates, there exists a constant $D_1>0$ such that

$$|G(t,\lambda(t))|\leqslant\frac{D_1}{t^2\log^{b+1}(t)},\quad \lambda=\lambda_0+e,\quad e\in\overline{B}_1(0)\subset X$$

Then, combining this with the estimate of all the terms comprising RHS except for G, we get, for a constant D_2 independent of e, T_0:

$$|RHS(e,t)|\leqslant\frac{D_2}{\log(\log(t))t^2\log^{b+1}(t)},\quad e\in\overline{B}_1(0)\subset X$$

Let $T_{0,2}>e^{e^{900}}$ be such that the following three inequalities are true:

$$\frac{1}{\sqrt{\log(\log(T_{0,2}))}}\leqslant\frac{1}{100}\cdot\frac{\alpha}{3D_2}$$

$$\frac{(b+1)}{\log(T_{0,2})}+\frac{1}{2\log(T_{0,2})\log(\log(T_{0,2}))}\leqslant\frac{1}{100}$$

$$\frac{1}{2b\log(\log(T_{0,2}))}+\frac{b}{\log(T_{0,2})}+\frac{b}{2(b+1)\log(T_{0,2})\log(\log(T_{0,2}))}\leqslant\frac{1}{100}$$

Finally, there exists $T_{0,3}>700+700b$, such that

$$\frac{\log^{700+700b}(T_{0,3})}{T_{0.3}}<\frac{1}{2}$$

From now on, we will work under the additional restriction that

$$T_0 \geqslant 2e^{e^{1000(b+1)}} + T_{0,1} + T_{0,2} + T_{0,3}$$

(Recall that (5.1) was, up to now, the only constraint on T_0). From the discussion preceding (5.75), and our estimates on $RHS(e,t)$ above, for $e \in \overline{B_1}(0) \subset X$, the map

$$x \mapsto \frac{RHS(e,x)}{4\alpha} - \int_x^\infty \frac{RHS(e,z)}{4\alpha} r(-x,-z)dz, \text{ a.e. } x \in [T_0,\infty)$$

a priori defined only almost everywhere, admits a continuous extension to $[T_0,\infty)$. Then, we define T on $\overline{B_1}(0) \subset X$ by

$$T(f)(t) = \int_t^\infty \int_{q_1}^\infty S_f(q_2)dq_2dq_1$$

where we define S_f by (the above mentioned continuous extension of)

$$S_f(x) = \frac{RHS(f,x)}{4\alpha} - \int_x^\infty \frac{RHS(f,z)}{4\alpha} r(-x,-z)dz$$

By definition of T, we have

$$T(e)''(t) = S_e(t) = \frac{RHS(e,t)}{4\alpha} - \int_t^\infty \frac{RHS(e,z)}{4\alpha} r(-t,-z)dz$$

Using (5.76), we get

$$\begin{aligned}|T(e)''(t)| &\leqslant \frac{3D_2}{\alpha \log(\log(t)) t^2 \log^{b+1}(t)} \\ &\leqslant \frac{1}{100\sqrt{\log(\log(t))} t^2 \log^{b+1}(t)}, \quad e \in \overline{B}_1(0) \subset X\end{aligned}$$

Next, we have

$$\begin{aligned}T(e)'(t) &= -\int_t^\infty T(e)''(x)dx \\ |T(e)'(t)| &\leqslant \frac{1}{100} \int_t^\infty \frac{dx}{\sqrt{\log(\log(x))} x^2 \log^{b+1}(x)}\end{aligned}$$

Then, we note that

$$\begin{aligned}\int_t^\infty \frac{dx}{\sqrt{\log(\log(x))} x^2 \log^{b+1}(x)} &= \frac{1}{t \log^{b+1}(t) \sqrt{\log(\log(t))}} \\ &- (b+1) \int_t^\infty \frac{dx}{x^2 \log^{b+2}(x) \sqrt{\log(\log(x))}} \\ &- \frac{1}{2} \int_t^\infty \frac{1}{x^2} \frac{dx}{\log^{b+2}(x) (\log(\log(x)))^{3/2}} \\ &= \frac{1}{t \log^{b+1}(t) \sqrt{\log(\log(t))}} + E_{int,1}\end{aligned}$$

where

$$
\begin{aligned}
|E_{int,1}| &\leqslant \frac{(b+1)}{t\log^{b+2}(t)\sqrt{\log(\log(t))}} + \frac{1}{2t\log^{b+2}(t)(\log(\log(t)))^{3/2}} \\
&\leqslant \frac{1}{100}\frac{1}{t\log^{b+1}(t)\sqrt{\log(\log(t))}}, \quad t \geqslant T_0
\end{aligned}
$$

which gives

$$
|T(e)'(t)| \leqslant \frac{1}{100}\left(1+\frac{1}{100}\right)\left(\frac{1}{t\log^{b+1}(t)\sqrt{\log(\log(t))}}\right), \quad t \geqslant T_0
$$

Similarly,

$$
T(e)(t) = \int_t^\infty \int_{x_1}^\infty T(f)''(x_2)dx_2dx_1
$$

Then we use

$$
\int_t^\infty \int_{x_1}^\infty \frac{dx_2dx_1}{\sqrt{\log(\log(x_2))}x_2^2\log^{b+1}(x_2)} = \frac{1}{b\log^b(t)\sqrt{\log(\log(t))}} + E_{int,2}
$$

where

$$
\begin{aligned}
|E_{int,2}| &\leqslant \frac{1}{2b^2(\log(\log(t)))^{3/2}\log^b(t)} + \frac{1}{\sqrt{\log(\log(t))}\log^{b+1}(t)} \\
&\quad + \frac{1}{2(b+1)(\log(\log(t)))^{3/2}\log^{b+1}(t)} \\
&\leqslant \frac{1}{100}\frac{1}{b\log^b(t)\sqrt{\log(\log(t))}}, \quad t \geqslant T_0
\end{aligned}
$$

to get

$$
|T(e)(t)| \leqslant \frac{1}{100}\left(1+\frac{1}{100}\right)\frac{1}{b\log^b(t)\sqrt{\log(\log(t))}}, \quad t \geqslant T_0
$$

Then, we conclude that $T(e) \in \overline{B_1(0)}$ for $e \in \overline{B_1(0)}$.

Now, we will show that T is a contraction on the space $\overline{B_1(0)} \subset X$. We start by estimating, for $e_1, e_2 \in \overline{B_1(0)} \subset X$, the expression

$$
RHS(e_1(t),t) - RHS(e_2(t),t)
$$

In particular, we will prove the following proposition.

Proposition 5.2. *There exists* $C_{lip} > 0$ independent *of* T_0*, such that for* $e_1, e_2 \in \overline{B_1(0)} \subset X$*,*

$$
|RHS(e_1,t) - RHS(e_2,t)| \leqslant \frac{C_{lip}||e_1-e_2||_X}{t^2\log^{b+1}(t)(\log(\log(t)))^{3/2}}
$$

To prove the proposition, it will be useful to define the functions

$$\lambda_i(t) = \lambda_0(t) + e_i(t), \quad i = 1, 2$$

We start with the terms of RHS which don't involve G:

$$\begin{aligned} \left|\frac{-4\alpha\lambda_0''(t)\left(\log(\lambda_1(t)) - \log(\lambda_2(t))\right)}{\log(\lambda_0(t))}\right| &\leqslant \frac{C}{t^2\log^{b+1}(t)\log(\log(t))}\frac{|\lambda_1(t)-\lambda_2(t)|}{\lambda_{0,0}(t)} \\ &\leqslant \frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)(\log(\log(t)))^{3/2}} \end{aligned}$$

$$\begin{aligned} &\left|\frac{-4\alpha}{\log(\lambda_0(t))}\left(e_1''(t)\left(\log(\lambda_1(t)) - \log(\lambda_0(t))\right) - e_2''(t)\left(\log(\lambda_2(t)) - \log(\lambda_0(t))\right)\right)\right| \\ &\leqslant \frac{C}{\log(\log(t))}\left(|e_1''(t)-e_2''(t)|\log(\frac{\lambda_0(t)+e_1(t)}{\lambda_0(t)}) + |e_2''(t)|\left(\log(\lambda_1(t)) - \log(\lambda_2(t))\right)\right) \\ &\leqslant C\frac{||e_1-e_2||_X}{t^2\log^{b+1}(t)(\log(\log(t)))^2} \end{aligned}$$

$$\begin{aligned} &\left|\frac{4}{\log(\lambda_0(t))}\left(\int_t^\infty \frac{(e_1''(s)-e_2''(s))}{(1+s-t)^3}\left(\frac{1}{\lambda_1(t)^{1-\alpha}+s-t} - \frac{1}{\lambda_0(t)^{1-\alpha}+s-t}\right)ds\right.\right. \\ &\left.\left.+\int_t^\infty \frac{e_2''(s)}{(1+s-t)^3}\left(\frac{1}{\lambda_1(t)^{1-\alpha}+s-t} - \frac{1}{\lambda_2(t)^{1-\alpha}+s-t}\right)ds\right)\right| \\ &\leqslant C\frac{||e_1-e_2||_X}{(\log(\log(t)))^2t^2\log^{b+1}(t)} \end{aligned}$$

$$\begin{aligned} &\frac{1}{\log(\log(t))}\int_t^\infty \frac{1}{s^2\log^{b+1}(s)(1+s-t)^3}\left|\frac{1}{\lambda_1(t)^{1-\alpha}+s-t} - \frac{1}{\lambda_2(t)^{1-\alpha}+s-t}\right|ds \\ &\leqslant \frac{C||e_1-e_2||_X}{(\log(\log(t)))^{3/2}t^2\log^{b+1}(t)} \end{aligned}$$

$$\int_t^\infty |e_1''(s)-e_2''(s)|\left|\frac{1}{\log(\lambda_0(t))} - \frac{1}{\log(\lambda_0(s))}\right|\frac{ds}{1+s-t} \leqslant \frac{C||e_1-e_2||_X}{t^2\log^{b+2}(t)(\log(\log(t)))^{5/2}}$$

$$\begin{aligned} &\int_t^\infty |e_1''(s)-e_2''(s)|\left|\frac{1}{\log(\lambda_0(t))} - \frac{1}{\log(\lambda_0(s))}\right|\frac{ds}{(\lambda_0(t)^{1-\alpha}+s-t)(1+s-t)^3} \\ &\leqslant \frac{C||e_1-e_2||_X}{t^3\log^{b+2}(t)(\log(\log(t)))^{5/2}} \end{aligned}$$

We now proceed to study each term appearing in the expression

$$G(t,\lambda_1(t)) - G(t,\lambda_2(t))$$

By writing

$$\begin{aligned}&\lambda_1(t)E_{0,1}(\lambda_1(t),\lambda_1'(t),\lambda_1''(t))-\lambda_2(t)E_{0,1}(\lambda_2(t),\lambda_2'(t),\lambda_2''(t))\\&=\int_0^1 DF_{0,0,1}(\lambda_\sigma)\cdot(\lambda_1(t)-\lambda_2(t),\lambda_1'(t)-\lambda_2'(t),\lambda_1''(t)-\lambda_2''(t))d\sigma\end{aligned}$$

where

$$F_{0,0,1}(x,y,z)=xE_{0,1}(x,y,z)$$
$$\lambda_\sigma=\sigma(\lambda_1(t),\lambda_1'(t),\lambda_1''(t))+(1-\sigma)(\lambda_2(t),\lambda_2'(t),\lambda_2''(t))$$

we get

$$\begin{aligned}&\lambda_1(t)E_{0,1}(\lambda_1(t),\lambda_1'(t),\lambda_1''(t))-\lambda_2(t)E_{0,1}(\lambda_2(t),\lambda_2'(t),\lambda_2''(t))\\&\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\end{aligned}$$

Next, we consider

$$\begin{aligned}&K_3(w,\lambda_1(t))-K_{3,0}(w,\lambda_1(t))-(K_3(w,\lambda_2(t))-K_{3,0}(w,\lambda_2(t)))\\&=\frac{w^5}{4(1+w^2)}\left(\frac{1}{(w^2+36\lambda_1(t)^2)^2}-\frac{1}{(w^2+36\lambda_2(t)^2)^2}\right)\\&-\frac{1}{4}\left(\frac{w^5}{(\lambda_1(t)^{2-2\alpha}+w^2)(w^2+36\lambda_1(t)^2)^2}-\frac{w^5}{(\lambda_2(t)^{2-2\alpha}+w^2)(w^2+36\lambda_1(t)^2)^2}\right.\\&+\frac{w^5}{(\lambda_2(t)^{2-2\alpha}+w^2)(w^2+36\lambda_1(t)^2)^2}-\frac{w^5}{(\lambda_2(t)^{2-2\alpha}+w^2)(w^2+36\lambda_2(t)^2)^2}\\&\left.-\frac{1}{(\lambda_1(t)^{1-\alpha}+w)(1+w)^3}+\frac{1}{(\lambda_2(t)^{1-\alpha}+w)(1+w)^3}\right)\end{aligned}$$

So,

$$\begin{aligned}&\int_t^\infty|K_3(s-t,\lambda_1(t))-K_{3,0}(s-t,\lambda_1(t))-(K_3(s-t,\lambda_2(t))-K_{3,0}(s-t,\lambda_2(t)))|ds\\&\leqslant\frac{C||e_1-e_2||_X}{\log^b(t)\sqrt{\log(\log(t))}}+C\frac{||e_1-e_2||_X}{\sqrt{\log(\log(t))}}+\frac{C||e_1-e_2||_X}{\log^{b\alpha}(t)\sqrt{\log(\log(t))}}+\frac{C||e_1-e_2||_X}{\sqrt{\log(\log(t))}}\\&\leqslant\frac{C||e_1-e_2||_X}{\sqrt{\log(\log(t))}}\end{aligned}$$

From this, we get

$$\begin{aligned}
&|\int_t^\infty \lambda_1''(s)\left(K_3(s-t,\lambda_1(t)) - K_{3,0}(s-t,\lambda_1(t))\right)ds \\
&- \int_t^\infty \lambda_2''(s)\left(K_3(s-t,\lambda_2(t)) - K_{3,0}(s-t,\lambda_2(t))\right)ds| \\
&\leqslant C\int_t^\infty |e_1''(s)-e_2''(s)||K_3(s-t,\lambda_1(t)) - K_{3,0}(s-t,\lambda_1(t))|ds \\
&+ C\int_t^\infty |\lambda_2''(s)||K_3(s-t,\lambda_1(t)) - K_{3,0}(s-t,\lambda_1(t)) \\
&\qquad\qquad - \left(K_3(s-t,\lambda_2(t)) - K_{3,0}(s-t,\lambda_2(t))\right)|ds \\
&\leqslant C\frac{||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\int_t^\infty |K_3(s-t,\lambda_1(t)) - K_{3,0}(s-t,\lambda_1(t))|ds \\
&+ \frac{C}{t^2\log^{b+1}(t)}\int_t^\infty |K_3(s-t,\lambda_1(t)) - K_{3,0}(s-t,\lambda_1(t)) \\
&\qquad\qquad - \left(K_3(s-t,\lambda_2(t)) - K_{3,0}(s-t,\lambda_2(t))\right)|ds \\
&\leqslant C\frac{||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}} + \frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}} \\
&\leqslant \frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}
\end{aligned}$$

(where we recall that $\int_t^\infty |K_3(s-t,\lambda_1(t)) - K_{3,0}(s-t,\lambda_1(t))|ds$ was previously estimated).

Next, let us consider the following term which appears in the expression $G(t,\lambda_0(t)+e_1(t)) - G(t,\lambda_0(t)+e_2(t))$:

$$\begin{aligned}
&\frac{16}{(\lambda_0(t)+e_1(t))^2}\int_t^\infty (\lambda_0''(x)+e_1''(x))\left(K_1(x-t,\lambda_0(t)+e_1(t)) - \frac{(\lambda_0(t)+e_1(t))^2}{4(1+x-t)}\right)dx \\
&- \frac{16}{(\lambda_0(t)+e_2(t))^2}\int_t^\infty (\lambda_0''(x)+e_2''(x))\left(K_1(x-t,\lambda_0(t)+e_2(t)) - \frac{(\lambda_0(t)+e_2(t))^2}{4(1+x-t)}\right)dx
\end{aligned}$$

which we re-write as

$$\begin{aligned}
&\left(\frac{16}{(\lambda_0(t)+e_1(t))^2} - \frac{16}{(\lambda_0(t)+e_2(t))^2}\right) \\
&\cdot\int_t^\infty (\lambda_0''(x)+e_1''(x))\left(K_1(x-t,\lambda_0(t)+e_1(t)) - \frac{(\lambda_0(t)+e_1(t))^2}{4(1+x-t)}\right)dx \\
&+ \frac{16}{(\lambda_0(t)+e_2(t))^2}\int_t^\infty (e_1''(x)-e_2''(x))\left(K_1(x-t,\lambda_0(t)+e_1(t)) - \frac{(\lambda_0(t)+e_1(t))^2}{4(1+x-t)}\right)dx \\
&+ \frac{16}{(\lambda_0(t)+e_2(t))^2}\int_t^\infty (\lambda_0''(x)+e_2''(x))\left(L_1(x-t,\lambda_0(t)+e_1(t)) - L_1(x-t,\lambda_0(t)+e_2(t))\right)dx
\end{aligned} \tag{5.100}$$

where

$$L_1(w,z) = K_1(w,z) - \frac{z^2}{4(1+w)}$$

From (5.37) and (5.39) there exists $C > 0$, independent of z such that

$$|K_1(w,z) - \frac{z^2}{4(1+w)}| \leqslant \frac{Cz^2}{1+w^2}, \quad |z| \leqslant \frac{1}{2}$$

(Recall that, by the choice of T_0, $|\lambda_0(t) + f_i(t)| \leqslant \frac{1}{2}, \quad i = 1,2$). Finally, let us note that

$$|\partial_z \left(K_1(w,z) - \frac{z^2}{4(1+w)}\right)| \leqslant \frac{Cz}{1+w^2}, \quad w \geqslant 0$$

Using the facts that

$$|\frac{16}{(\lambda_0(t)+e_1(t))^2} - \frac{16}{(\lambda_0(t)+e_2(t))^2}| \leqslant C\frac{\log^{3b}(t)||e_1-e_2||_X}{\sqrt{\log(\log(t))}\log^b(t)}$$

and

$$|L_1(w,z_1) - L_1(w,z_2)| \leqslant ||\partial_2 L_1(w, sz_1 + (1-s)z_2)||_{L^\infty_s([0,1])}|z_1 - z_2|$$

we get that the absolute value of (5.100) is bounded above by

$$\frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}$$

Another term arising in
$G(t,\lambda_0(t)+e_1(t)) - G(t,\lambda_0(t)+e_2(t))$ is

$$\frac{16}{(\lambda_0(t)+e_1(t))^2}\int_t^\infty (\lambda_0''(x)+e_1''(x))K(x-t,\lambda_0(t)+e_1(t))dx$$
$$-\frac{16}{(\lambda_0(t)+e_2(t))^2}\int_t^\infty (\lambda_0''(x)+e_2''(x))K(x-t,\lambda_0(t)+e_2(t))dx$$

This can be split analogously to the previous term:

$$\begin{aligned}&\left(\frac{16}{(\lambda_0(t)+e_1(t))^2} - \frac{16}{(\lambda_0(t)+e_2(t))^2}\right)\int_t^\infty (\lambda_0''(x)+e_1''(x))K(x-t,\lambda_0(t)+e_1(t))dx\\&+\frac{16}{(\lambda_0(t)+e_2(t))^2}\int_t^\infty (e_1''(x)-e_2''(x))K(x-t,\lambda_0(t)+e_1(t))dx\\&+\frac{16}{(\lambda_0(t)+e_2(t))^2}\int_t^\infty (\lambda_0''(x)+e_2''(x))\left(K(x-t,\lambda_0+e_1(t)) - K(x-t,\lambda_0(t)+e_2(t))\right)dx\end{aligned} \tag{5.101}$$

Note that

$$K(x-t,z_1) - K(x-t,z_2) = (z_1-z_2)\int_0^1 \partial_2 K(x-t, z_2 + q(z_1-z_2))dq$$

So,

$$\begin{aligned}&\int_t^\infty |K(x-t,z_1)-K(x-t,z_2)|dx\\&\leqslant |z_1-z_2|\int_0^1\int_0^\infty |\partial_2 K(w,z_2+(z_1-z_2)q)|dwdq\end{aligned} \tag{5.102}$$

But, using the formula for K from the previous sections, we see that $\partial_z K(w,z)$ has a sign:

$$\begin{aligned}&\partial_z K(w,z)\\&=\int_0^\infty\int_0^w \left(\frac{4\rho R^3 z\left(\rho^2+R^2z^2+1\right)}{\left(R^2+1\right)^3\left(\left(\rho^2-R^2z^2+1\right)^2+4R^2z^2\right)^{3/2}}\right)\left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)d\rho dR\\&\geqslant 0\end{aligned}$$

So, to control the integral in (5.102), we can note the following: If

$$z_q = z_2+(z_1-z_2)q$$

then

$$\begin{aligned}&\int_0^1\int_0^\infty |\partial_2 K(w,z_q)|dwdq\\&=\int_0^1\int_0^\infty\int_0^\infty\int_0^w \left(\frac{4\rho R^3 z_q\left(\rho^2+R^2z_q^2+1\right)}{\left(R^2+1\right)^3\left(\left(\rho^2-R^2z_q^2+1\right)^2+4R^2z_q^2\right)^{3/2}}\right)\\&\qquad\left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)d\rho dR dw dq\\&=\int_0^1\int_0^\infty\int_0^\infty\int_\rho^\infty \left(\frac{4\rho R^3 z_q\left(\rho^2+R^2z_q^2+1\right)}{\left(R^2+1\right)^3\left(\left(\rho^2-R^2z_q^2+1\right)^2+4R^2z_q^2\right)^{3/2}}\right)\\&\qquad\left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)dw d\rho dR dq\\&=\int_0^1\int_0^\infty\int_0^\infty \frac{4\rho R^3 z_q\log(2)\left(\rho^2+R^2z_q^2+1\right)}{\left(R^2+1\right)^3\left(\left(\rho^2-R^2z_q^2+1\right)^2+4R^2z_q^2\right)^{3/2}}d\rho dR dq\\&=\int_0^1\int_0^\infty \frac{R^3 z_q\log(4)}{\left(R^2+1\right)^3}dRdq\\&=\int_0^1 \frac{1}{2}z_q\log(2)dq\\&=\frac{1}{4}\log(2)(z_1+z_2)\end{aligned} \tag{5.103}$$

Using (5.34), we see that the absolute value of (5.101) is bounded above by

$$\frac{C||e_1 - e_2||_X}{t^2 \log^{b+1}(t)\sqrt{\log(\log(t))}}$$

Next, we will consider the terms involving $E_{v_2,ip}$, starting with the case $b \neq 1$. We recall

$$\begin{aligned}\lambda(t)E_{v_2,ip}(t,\lambda(t)) &= 2c_b\lambda(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi\\ &+ 2c_b\lambda(t)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda(t))d\xi\\ &+ 2c_b\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{(b-1)}{\xi\log^b(\frac{1}{\xi})}+\frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\\ &+ 2c_b\left(\int_0^{\frac{t}{2}}\frac{\sin(u)(b-1)du}{t^2u\log^b(\frac{t}{u})}-\frac{(b-1)\pi}{2t^2\log^b(t)}+\int_0^{\frac{t}{2}}\frac{\sin(u)b(b-1)du}{t^2u\log^{b+1}(\frac{t}{u})}\right)\end{aligned} \tag{5.104}$$

where F_{v_2} and ψ_{v_2} were defined in (5.42) and (5.41), respectively. Notice that the last two lines of (5.104) are *independent* of $\lambda(t)$. So,

$$\begin{aligned}&\lambda_1(t)E_{v_2,ip}(t,\lambda_1(t)) - \lambda_2(t)E_{v_2,ip}(t,\lambda_2(t))\\ &= 2c_b\lambda_1(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_1(t))d\xi - 2c_b\lambda_2(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_2(t))d\xi\\ &+ 2c_b\lambda_1(t)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda_1(t))d\xi - 2c_b\lambda_2(t)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda_2(t))d\xi\end{aligned} \tag{5.105}$$

First, we consider the second line of (5.105):

$$\begin{aligned}&|2c_b\lambda_1(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_1(t))d\xi - 2c_b\lambda_2(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_2(t))d\xi|\\ &\leqslant |2c_b(e_1(t)-e_2(t))\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_1(t))d\xi|\\ &+ |2c_b\lambda_2(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\left(\psi_{v_2}(\xi,\lambda_1(t))-\psi_{v_2}(\xi,\lambda_2(t))\right)d\xi|\end{aligned} \tag{5.106}$$

The second line of (5.106) is estimated as follows:

$$\begin{aligned}&|2c_b(e_1(t)-e_2(t))\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_1(t))d\xi|\\ &\leqslant C|e_1(t)-e_2(t)||\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda_1(t))d\xi|\\ &\leqslant C\frac{|e_1(t)-e_2(t)|}{t^3\lambda_1(t)}\end{aligned}$$

where we use the same procedure as used in the previous subsections to estimate the integral. For the third line of (5.106), we first note that

$$\psi_{v_2}(\xi,\lambda(t)) = 2\chi'_{\leqslant\frac{1}{4}}(\xi)\partial_\xi\left(\frac{\xi^2 K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\right) + \frac{\chi''_{\leqslant\frac{1}{4}}(\xi)\xi^2 K_1(\xi\lambda(t))}{\log^{b+1}(\frac{1}{\xi})}$$

$$\begin{aligned}\partial_\xi\psi_{v_2}(\xi,\lambda(t)) &= 3\chi''_{\leqslant\frac{1}{4}}(\xi)\partial_\xi\left(\frac{\xi^2 K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\right) + 2\chi'_{\leqslant\frac{1}{4}}(\xi)\partial_\xi^2\left(\frac{\xi^2 K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\right)\\ &+ \chi'''_{\leqslant\frac{1}{4}}(\xi)\frac{\xi^2 K_1(\xi\lambda(t))}{\log^{b-1}(\frac{1}{\xi})}\end{aligned}$$

Then, we use

$$\begin{aligned}K_1'(y) &= -\left(\frac{yK_0(y)+K_1(y)}{y}\right)\\ K_1''(y) &= K_1(y) + \frac{K_2(y)}{y}\\ K_1'''(y) &= \frac{-(3+y^2)}{y^2}K_2(y)\end{aligned}$$

and the estimates, for $0 < x < \frac{1}{2}$ and $\frac{1}{8} \leqslant \xi \leqslant \frac{1}{4}$,

$$|K_1(\xi x)| \leqslant \frac{C}{\xi x}, \quad |K_0(\xi x)| \leqslant C|\log(\xi x)|, \quad |K_2(\xi x)| \leqslant \frac{C}{\xi^2 x^2} \tag{5.107}$$

Since $\text{supp}\left(\chi^{(j)}_{\leqslant\frac{1}{4}}\right) \subset [\frac{1}{8},\frac{1}{4}]$, we have

$$|\partial_{12}\psi_{v_2}(\xi,y)| \leqslant \frac{C\mathbb{1}_{[\frac{1}{8},\frac{1}{4}]}(\xi)}{\xi y^2\log^{b-1}(\frac{1}{\xi})}, \quad 0 < y < \frac{1}{2} \tag{5.108}$$

Note that

$$\begin{aligned}&\int_0^\infty \frac{\sin(t\xi)}{t^2}\left(\psi_{v_2}(\xi,\lambda_1(t)) - \psi_{v_2}(\xi,\lambda_2(t))\right)d\xi\\ &= \int_0^\infty \frac{\cos(t\xi)}{t^3}\partial_\xi\left(\psi_{v_2}(\xi,\lambda_1(t)) - \psi_{v_2}(\xi,\lambda_2(t))\right)d\xi\end{aligned}$$

and

$$|\partial_\xi\left(\psi_{v_2}(\xi,\lambda_1(t)) - \psi_{v_2}(\xi,\lambda_2(t))\right)| \leqslant \sup_{x\in[\frac{\lambda_0(t)}{2},\frac{1}{2}]}|\partial_{12}\psi_{v_2}(\xi,x)||\lambda_1(t)-\lambda_2(t)|$$

where we used the fact that, for $e_i \in \overline{B_1(0)} \subset X, \quad i = 1,2$, we have

$$\frac{1}{2}\lambda_0(t) < \lambda_i(t) < 2\lambda_0(t) < \frac{1}{2}, \quad i = 1,2$$

Now, we use (5.108) to get

$$
\begin{aligned}
&|\int_0^\infty \frac{\cos(t\xi)}{t^3}\partial_\xi \left(\psi_{v_2}(\xi,\lambda_1(t)) - \psi_{v_2}(\xi,\lambda_2(t))\right) d\xi| \\
&\leqslant \frac{C}{t^3}\int_0^\infty \sup_{x\in[\frac{\lambda_0(t)}{2},\frac{1}{2}]} |\partial_{12}\psi_{v_2}(\xi,x)||\lambda_1(t)-\lambda_2(t)|d\xi \\
&\leqslant C\frac{|e_1(t)-e_2(t)|}{t^3}\int_0^\infty \frac{\mathbb{1}_{[\frac{1}{8},\frac{1}{4}]}(\xi)d\xi}{\lambda_0(t)^2\xi\log^{b-1}(\frac{1}{\xi})} \\
&\leqslant C\frac{|e_1(t)-e_2(t)|\log^{2b}(t)}{t^3}
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&|2c_b(\lambda_0(t)+e_2(t))\int_0^\infty \frac{\cos(t\xi)}{t^3}\left(\partial_\xi\psi_{v_2}(\xi,\lambda_1(t)) - \partial_\xi\psi_{v_2}(\xi,\lambda_2(t))\right) d\xi| \\
&\leqslant \frac{C||e_1-e_2||_X}{t^3\sqrt{\log(\log(t))}}
\end{aligned}
$$

We now need to consider the third line of (5.105). Here, we use a similar procedure as above:

$$
\begin{aligned}
&|2c_b\lambda_1(t)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda_1(t))d\xi - 2c_b\lambda_2(t)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda_2(t))d\xi| \\
&\leqslant C|\left(e_1(t)-e_2(t)\right)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda_1(t))d\xi| \\
&+ C\lambda_2(t)|\int_0^\infty \frac{\chi_{\leqslant\frac{1}{4}}(\xi)\sin(t\xi)}{t^2}\left(F_{v_2}(\xi,\lambda_1(t)) - F_{v_2}(\xi,\lambda_2(t))\right) d\xi|
\end{aligned} \tag{5.109}
$$

For the second line of (5.109), we use the same procedure to estimate the integral as that used in previous subsections, to get

$$
\begin{aligned}
&|\left(e_1(t)-e_2(t)\right)\int_0^\infty \chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}F_{v_2}(\xi,\lambda_1(t))d\xi| \\
&\leqslant C|e_1(t)-e_2(t)|\frac{\lambda_1(t)|\log(\lambda_1(t))|}{t^3}
\end{aligned}
$$

For the third line of (5.109), we first integrate by parts once to get

$$
\begin{aligned}
&\int_0^\infty \frac{\chi_{\leqslant\frac{1}{4}}(\xi)\sin(t\xi)}{t^2}\left(F_{v_2}(\xi,\lambda_1(t)) - F_{v_2}(\xi,\lambda_2(t))\right) d\xi \\
&= \int_0^\infty \frac{\cos(t\xi)}{t^3}\Big(\chi'_{\leqslant\frac{1}{4}}(\xi)\left(F_{v_2}(\xi,\lambda_1(t)) - F_{v_2}(\xi,\lambda_2(t))\right) \\
&\qquad + \chi_{\leqslant\frac{1}{4}}(\xi)\partial_\xi\left(F_{v_2}(\xi,\lambda_1(t)) - F_{v_2}(\xi,\lambda_2(t))\right)\Big) d\xi
\end{aligned}
$$

Now, we recall that

$$F_{v_2}(\xi,y)=\partial_\xi^2\left(\frac{\xi^2}{\log^{b-1}(\frac{1}{\xi})}\left(K_1(\xi y)-\frac{1}{\xi y}\right)\right)$$

Using the same estimates on K_j as used to obtain (5.108), we get

$$|\partial_2 F_{v_2}(\xi,y)|\leqslant\frac{C\xi(|\log(\xi)|+|\log(y)|)}{\log^{b-1}(\frac{1}{\xi})},\quad 0<\xi<\frac{1}{4},\quad 0<y<\frac{1}{2}$$

and

$$|\partial_{12}F_{v_2}(\xi,y)|\leqslant\frac{C(|\log(\xi)|+|\log(y)|)}{\log^{b-1}(\frac{1}{\xi})},\quad 0<y<\frac{1}{2},\quad 0<\xi<\frac{1}{4}$$

We then get

$$|F_{v_2}(\xi,\lambda_2(t))-F_{v_2}(\xi,\lambda_1(t))|\leqslant\left(\sup_{x\in[\frac{\lambda_0(t)}{2},\frac{1}{2}]}|\partial_2 F_{v_2}(\xi,x)|\right)|\lambda_1(t)-\lambda_2(t)|$$

and

$$|\partial_1 F_{v_2}(\xi,\lambda_2(t))-\partial_1 F_{v_2}(\xi,\lambda_1(t))|\leqslant\left(\sup_{x\in[\frac{\lambda_0(t)}{2},\frac{1}{2}]}|\partial_{12}F_{v_2}(\xi,x)|\right)|\lambda_1(t)-\lambda_2(t)|$$

Using the above estimates, we obtain,

$$\begin{aligned}&|2c_b\lambda_2(t)\int_0^\infty\chi_{\leqslant\frac{1}{4}}(\xi)\frac{\sin(t\xi)}{t^2}\left(F_{v_2}(\xi,\lambda_1(t))-F_{v_2}(\xi,\lambda_2(t))\right)d\xi|\\&\leqslant C\frac{|\lambda_1(t)-\lambda_2(t)|\lambda_0(t)|\log(\lambda_0(t))|}{t^3}\leqslant C\frac{||e_1-e_2||_X\sqrt{\log(\log(t))}}{t^3\log^{2b}(t)}\end{aligned}$$

The same procedure for the case $b=1$ yields the same estimates, since ψ_{v_2} and F_{v_2} have the same form for all b.

Combining the above estimates then gives, for all $b>0$:

$$|-\lambda_1(t)E_{v_2,ip}(t,\lambda_1(t))+\lambda_2(t)E_{v_2,ip}(t,\lambda_2(t))|\leqslant\frac{C||e_1-e_2||_X}{t^3\sqrt{\log(\log(t))}}$$

If

$$\begin{aligned}G_2(t,\lambda(t))&=-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left((v_4+v_5)\left(1-\chi_{\geqslant1}(\frac{4r}{t})\right)\right)|_{r=R\lambda(t)},\phi_0\rangle\\&-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)E_5|_{r=R\lambda(t)},\phi_0\rangle\\&+\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant1}(\frac{2r}{\log^N(t)})\left(v_1+v_2+v_3\right)|_{r=R\lambda(t)},\phi_0\rangle\end{aligned}$$

Then, we need to estimate

$$G_2(t,\lambda_1(t))-G_2(t,\lambda_2(t))$$

From now on, the fact that $v_k, \quad k\neq 2$ depends on $\lambda(t)$ is important, so we will denote these functions by v_k^{λ} to emphasize the dependence of these functions on λ.

We first note that

$$\begin{aligned}&\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)v_4^{\lambda}\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)},\phi_0\rangle\\&=\frac{1}{\lambda(t)}\int_0^{\infty}\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2}\right)v_4^{\lambda}(t,r)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)\phi_0(\frac{r}{\lambda(t)})rdr\end{aligned}$$

so,

$$\begin{aligned}&\lambda_1(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda_1(t)}})-1}{r^2}\right)v_4^{\lambda_1}\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda_1(t)},\phi_0\rangle\\&-\left(\lambda_2(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda_2(t)}})-1}{r^2}\right)v_4^{\lambda_2}\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda_2(t)},\phi_0\rangle\right)\\&=-16\int_0^{\infty}\left(\frac{\lambda_1(t)^2}{(r^2+\lambda_1(t)^2)^3}-\frac{\lambda_2(t)^2}{(r^2+\lambda_2(t)^2)^3}\right)v_4^{\lambda_1}(t,r)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)r^2dr\\&+\int_0^{\infty}\left(\frac{\cos(2Q_1(\frac{r}{\lambda_2(t)})-1}{r^2\lambda_2(t)}\right)\phi_0(\frac{r}{\lambda_2(t)})\left(v_4^{\lambda_1}-v_4^{\lambda_2}\right)(t,r)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)rdr\end{aligned}$$

and we have the analogous formulae for all the other terms in G_2. Using the pointwise estimates for $v_i, E_5, \quad 1\leqslant i\leqslant 5$, we get

$$\begin{aligned}&|-16\int_0^{\infty}\left(\frac{\lambda_1(t)^2}{(r^2+\lambda_1(t)^2)^3}-\frac{\lambda_2(t)^2}{(r^2+\lambda_2(t)^2)^3}\right)\\&\qquad\left(\left(v_4^{\lambda_1}+v_5^{\lambda_1}\right)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)+E_5^{\lambda_1}-\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(v_1^{\lambda_1}+v_2+v_3^{\lambda_1}\right)\right)r^2dr|\\&\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{3b+2N-1}(t)\sqrt{\log(\log(t))}}+\frac{C||e_1-e_2||_X}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)\sqrt{\log(\log(t))}}\\&+\frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}+\frac{C||e_1-e_2||_X}{t^2\log^{3b+2N}(t)\sqrt{\log(\log(t))}}\\&\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\end{aligned}$$

In order to estimate the rest of the terms arising in $G_2(t,\lambda_1)-G_2(t,\lambda_2)$, we must first prove estimates on $v_{3,1,2}$ defined by

$$v_{3,1,2}:=v_3^{\lambda_1}-v_3^{\lambda_2}$$

Lemma 5.11.

$$|v_{3,1,2}(t,r)|\leqslant\begin{cases}\frac{Cr||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^b(t)}, & r\leqslant\frac{t}{2}\\ \frac{C||e_1-e_2||_X}{r\log^b(t)\sqrt{\log(\log(t))}}, & r>\frac{t}{2}\end{cases}\tag{5.110}$$

Proof. Note that $v_{3,1,2}$ solves the following equation with 0 Cauchy data at infinity:

$$-\partial_{tt}v_{3,1,2} + \partial_{rr}v_{3,1,2} + \frac{1}{r}\partial_r v_{3,1,2} - \frac{v_{3,1,2}}{r^2} = F_{0,1}^{\lambda_1}(t,r) - F_{0,1}^{\lambda_2}(t,r)$$

With the definitions from the v_3 subsection:

$$v_{3,1}^{\lambda}(t,r) = \frac{-1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds$$

and

$$v_{3,2}^{\lambda} = v_3^{\lambda} - v_{3,1}^{\lambda}$$

we then consider

$$\begin{aligned} &v_{3,2,a}^{\lambda}(t,r) \\ &:= \frac{-1}{r}\int_t^\infty \int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + 1\right) d\rho ds \end{aligned}$$

$$v_{3,2,b}^{\lambda}(t,r) := \frac{-1}{r}\int_t^\infty \int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\lambda''(s)\left(-1 + F_3(r,\rho,\lambda(s))\right) d\rho ds$$

and get

$$\begin{aligned} &|v_{3,2,a}^{\lambda_1}(t,r) - v_{3,2,a}^{\lambda_2}(t,r)| \\ &\leqslant \frac{C}{r}\int_t^\infty \int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)|\lambda_1''(s) - \lambda_2''(s)| \\ &\qquad \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + 1\right) d\rho ds \\ &\leqslant \frac{C}{r}\int_0^\infty \rho\left(1 + \frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}\right)\int_{\rho+t}^\infty \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \\ &\qquad \frac{||e_1-e_2||_X ds d\rho}{s^2\log^{b+1}(s)\sqrt{\log(\log(s))}} \end{aligned}$$

which gives

$$|v_{3,2,a}^{\lambda_1}(t,r) - v_{3,2,a}^{\lambda_2}(t,r)| \leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}$$

For $v_{3,2,b}$, we have

$$\begin{aligned} &v_{3,2,b}^{\lambda_1}(t,r) - v_{3,2,b}^{\lambda_2}(t,r) \\ &= \frac{-1}{r}\int_t^\infty \int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)(\lambda_1''(s) - \lambda_2''(s))\left(-1 + F_3(r,\rho,\lambda_1(s)\right) d\rho ds \\ &+ \frac{-1}{r}\int_t^\infty \int_0^{s-t} \rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right)\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s)) - F_3(r,\rho,\lambda_2(s))\right) d\rho ds \end{aligned} \tag{5.111}$$

The first line on the right-hand side of (5.111) is treated identically to the analogous term arising in the pointwise estimates for v_3, and it is bounded above in absolute value by

$$\frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}$$

For the second line on the right-hand side of (5.111), we start with

$$|F_3(r,\rho,z_1)-F_3(r,\rho,z_2)|\leqslant \max_{\sigma\in[0,1]}|\partial_3F_3(r,\rho,\sigma z_1+(1-\sigma)z_2)||z_1-z_2|$$

we then note that

$$\partial_3F_3(r,\rho,z)=\frac{-4(-1+\alpha)r^2z^{-3+2\alpha}(1+(r^2-\rho^2)z^{2\alpha-2})}{((1+(r^2-\rho^2)z^{2\alpha-2})^2+4\rho^2z^{2\alpha-2})^{3/2}}$$

so, for $z=\sigma\lambda_1(s)+(1-\sigma)\lambda_2(s)$,

$$\begin{aligned}|\partial_3F_3(r,\rho,z)|&\leqslant\frac{Cr^2\lambda_{0,0}(s)^{2\alpha-3}}{(1+(-\rho^2+r^2)\lambda_{0,0}(s)^{2\alpha-2})^2+4\rho^2\lambda_{0,0}(s)^{2\alpha-2}}\\&\leqslant\frac{Cr^2\lambda_{0,0}(s)^{2\alpha-3}}{1+2(\rho^2+r^2)\lambda_{0,0}(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda_{0,0}(t)^{4\alpha-4}}\end{aligned}$$

Then, we get

$$\begin{aligned}&|\frac{-1}{r}\int_t^\infty\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s))-F_3(r,\rho,\lambda_2(s))\right)d\rho ds|\\&\leqslant\frac{C}{r}\int_0^\infty\frac{\rho}{t^2\log^{b+1}(t)}\frac{r^2||e_1-e_2||_X}{\log^{b(2\alpha-3)}(t)\log^b(t)\sqrt{\log(\log(t))}}\\&\quad\cdot\frac{1}{(1+2(\rho^2+r^2)\lambda_{0,0}(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda_{0,0}(t)^{4\alpha-4})}d\rho\end{aligned}$$

The ρ integral appearing on the last line of the above estimate has been treated in the v_3 pointwise estimates, and, in total, we get

$$|v_{3,2,b}^{\lambda_1}(t,r)-v_{3,2,b}^{\lambda_2}(t,r)|\leqslant\frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}$$

We similarly divide $v_{3,1}^{\lambda_1}-v_{3,1}^{\lambda_2}$ into two parts. First, we consider

$$\begin{aligned}&|\frac{-1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{(s-t)}\left(\lambda_1''(s)-\lambda_2''(s)\right)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+F_3(r,\rho,\lambda_1(s))\right)d\rho ds|\\&\leqslant\frac{C}{r}\int_t^{t+6r}\int_0^{s-t}\frac{\rho}{(s-t)}|\lambda_1''(s)-\lambda_2''(s)|d\rho ds\\&+|\int_{t+6r}^\infty\frac{1}{r}\int_0^{s-t}\frac{\rho}{(s-t)}\left(\lambda_1''(s)-\lambda_2''(s)\right)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda_1(s))\right)d\rho ds|\\&\leqslant\frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}+II\end{aligned}$$

where, using a procedure similar to the analogous term treated in the v_3 pointwise estimates, we have

$$II = |-2r\int_{6r}^{\infty} (\lambda_1''(t+w)-\lambda_2''(t+w))\left(\frac{w}{2(1+w^2)}-\frac{w}{2(\lambda_1(t+w)^{2-2\alpha}+w^2)}\right)dw| + E_{II}$$

with

$$|E_{II}| \leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}$$

We already estimated the integral appearing in the above expression in (5.80) (except with the replacement of λ'' with $\lambda_1''-\lambda_2''$). Therefore, the same procedure used there shows

$$\begin{aligned}|II| &\leqslant Cr\log(\log(t))\sup_{x\geqslant t}|\lambda_1''(x)-\lambda_2''(x)| + \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\\ &\leqslant \frac{Cr\log(\log(t))||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}} + \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\end{aligned}$$

Next, we consider the second part of $v_{3,1}^{\lambda_1}-v_{3,1}^{\lambda_2}$:

$$\begin{aligned}&|\frac{-1}{r}\int_t^{\infty}\int_0^{s-t}\frac{\rho}{(s-t)}\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s))-F_3(r,\rho,\lambda_2(s))\right)d\rho ds|\\ &\leqslant \frac{C}{r}\int_t^{\infty}\int_0^{s-t}\frac{\rho}{(s-t)}|\lambda_2''(s)|\frac{r^2(\lambda_{0,0}(s)^{2\alpha-3})|\lambda_1(s)-\lambda_2(s)|d\rho ds}{(1+2(\rho^2+r^2)\lambda_{0,0}(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda_{0,0}(t)^{4\alpha-4})}\\ &\leqslant Cr\int_t^{t+\lambda_{0,0}(t)^{1-\alpha}}\int_0^{s-t}\frac{\rho}{(s-t)}|\lambda_2''(s)||e_1(s)-e_2(s)|\lambda_{0,0}(s)^{2\alpha-3}d\rho ds\\ &+ Cr\int_{t+\lambda_{0,0}(t)^{1-\alpha}}^{\infty}\frac{1}{(s-t)}|\lambda_2''(s)|\frac{||e_1-e_2||_X\lambda_{0,0}(s)^{2\alpha-3}ds}{\log^b(s)\sqrt{\log(\log(s))}}\\ &\qquad\int_0^{\infty}\frac{\rho d\rho}{(1+2(\rho^2+r^2)\lambda_{0,0}(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda_{0,0}(t)^{4\alpha-4})}\\ &\leqslant Cr\int_t^{t+\lambda_{0,0}(t)^{1-\alpha}}(s-t)|\lambda_2''(s)||e_1(s)-e_2(s)|\lambda_{0,0}(s)^{2\alpha-3}ds\\ &+\frac{Cr||e_1-e_2||_X}{\log^b(t)\sqrt{\log(\log(t))}}\int_{t+\lambda_{0,0}(t)^{1-\alpha}}^{\infty}\frac{\lambda_{0,0}(s)^{2\alpha-3}\lambda_{0,0}(t)^{2-2\alpha}ds}{(s-t)s^2\log^{b+1}(s)}\\ &\leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}} + \frac{Cr||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^b(t)}\\ &\leqslant \frac{Cr||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^b(t)}\end{aligned}$$

Combining the above estimates, we get

$$|(v_3^{\lambda_1}-v_3^{\lambda_2})(t,r)| \leqslant \frac{Cr||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^b(t)}$$

Next, we prove an estimate on $v_{3,1,2}$ which is more useful for large r.

$$v_3^{\lambda}(t,r) = \frac{-1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds$$

and get

$$\begin{aligned}&\left(v_3^{\lambda_1}-v_3^{\lambda_2}\right)(t,r)\\&=\frac{-1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho\left(\lambda_1''(s)-\lambda_2''(s)\right)}{\sqrt{(s-t)^2-\rho^2}}\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda_1(s))\right)d\rho ds\\&-\frac{1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s))-F_3(r,\rho,\lambda_2(s))\right)d\rho ds\end{aligned} \tag{5.112}$$

For the first line of the right-hand side of (5.112), we have

$$\begin{aligned}&\left|\frac{-1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho\left(\lambda_1''(s)-\lambda_2''(s)\right)}{\sqrt{(s-t)^2-\rho^2}}\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda_1(s))\right)d\rho ds\right|\\&\leqslant \frac{C}{r}\int_t^\infty (s-t)\frac{||e_1-e_2||_X}{s^2\log^{b+1}(s)\sqrt{\log(\log(s))}}ds\\&\leqslant \frac{C||e_1-e_2||_X}{r\log^b(t)\sqrt{\log(\log(t))}}\end{aligned}$$

For the second line on the right-hand side of (5.112), we use the same method used previously to estimate

$$F_3(r,\rho,\lambda_1)-F_3(r,\rho,\lambda_2)$$

and we get

$$\begin{aligned}&\left|-\frac{1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s))-F_3(r,\rho,\lambda_2(s))\right)d\rho ds\right|\\&\leqslant \frac{C}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}|\lambda_2''(s)|\frac{\lambda_{0,0}(s)^{-3+2\alpha}}{\log^b(s)\sqrt{\log(\log(s))}}\\&\qquad\frac{r^2||e_1-e_2||_X}{(1+\lambda_{0,0}(s)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda_{0,0}(s)^{2\alpha-2}(\rho^2+r^2))}d\rho ds\\&\leqslant \frac{C}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\frac{|\lambda_2''(s)|\lambda_{0,0}(s)^{-3+2\alpha}}{\log^b(s)\sqrt{\log(\log(s))}}\frac{r^2||e_1-e_2||_X}{r^2\lambda_{0,0}(s)^{2\alpha-2}}d\rho ds\\&\leqslant \frac{C||e_1-e_2||_X}{r\log^b(t)\sqrt{\log(\log(t))}}\end{aligned}$$

whence, we conclude the final estimate

$$|v_3^{\lambda_1}-v_3^{\lambda_2}|(t,r)\leqslant \frac{C||e_1-e_2||_X}{r\log^b(t)\sqrt{\log(\log(t))}},\quad r>\frac{t}{2}$$

□

Next, we recall the function E_5 defined in the v_3 subsection:

$$v_3^{\lambda}(t,r) = -r\int_{t+6r}^{\infty} \lambda''(s)(s-t)\left(\frac{1}{1+(s-t)^2} - \frac{1}{\lambda(t)^{2-2\alpha}+(s-t)^2}\right) ds + E_5^{\lambda}(t,r)$$

We will need to estimate $E_5^{\lambda_1} - E_5^{\lambda_2}$. For this purpose, some of the estimates already proven for $v_3^{\lambda_1} - v_3^{\lambda_2}$ will suffice, but we will need to use a slightly more complicated argument for some terms. In particular, we have

$$E_5^{\lambda}(t,r) = v_{3,1,a}^{\lambda}(t,r) + v_{3,1,b,i,1}^{\lambda}(t,r) + v_{3,1,b,ii}^{\lambda}(t,r) + v_{3,2}^{\lambda}(t,r)$$

with

$$v_{3,1,b,i,1}^{\lambda}(t,r) = -2r\int_{6r}^{\infty} \lambda''(t+w)w\left(\frac{-1}{2(\lambda(t+w)^{2-2\alpha}+w^2)} + \frac{1}{2(\lambda(t)^{2-2\alpha}+w^2)}\right) dw$$

$$\begin{aligned}v_{3,1,b,ii}^{\lambda}(t,r) = &\frac{-1}{r}\int_{t+6r}^{\infty}\int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s)\left(\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right)\right.\\ &\left. - \left(\frac{2r^2}{(1+\rho^2)^2} - \frac{2r^2\lambda(s)^{2-2\alpha}}{(\lambda(s)^{2-2\alpha}+\rho^2)^2}\right)\right) d\rho ds\end{aligned}$$

and

$$v_{3,1,a}^{\lambda}(t,r) = \frac{-1}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds$$

(Note that $v_{3,2}$ was already defined when proving estimates on $v_3^{\lambda_1} - v_3^{\lambda_2}$).

$$\begin{aligned}&v_{3,1,a}^{\lambda_1} - v_{3,1,a}^{\lambda_2}\\ &= -\frac{1}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}\left(\lambda_1''(s)-\lambda_2''(s)\right)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda_1(s))\right) d\rho ds\\ &-\frac{1}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s)) - F_3(r,\rho,\lambda_2(s))\right) d\rho ds\end{aligned}$$

This gives

$$\begin{aligned}|v_{3,1,a}^{\lambda_1} - v_{3,1,a}^{\lambda_2}| &\leqslant \frac{C}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}|e_1''(s)-e_2''(s)| d\rho ds\\ &+ \frac{C}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)} \frac{|\lambda_2''(s)|(r^2\lambda_{0,0}(t)^{2\alpha-2})\frac{\lambda_{0,0}(s)^{2\alpha-3}}{\lambda_{0,0}(t)^{2\alpha-2}}|e_1(s)-e_2(s)| d\rho ds}{(1+2(\rho^2+r^2)\lambda_{0,0}(t)^{2\alpha-2}+(\rho^2-r^2)^2\lambda_{0,0}(t)^{4\alpha-4})}\\ &\leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\\ &+ \frac{C}{r}\int_t^{t+6r}\int_0^{s-t} \frac{\rho}{(s-t)}|\lambda_2''(s)|\frac{\lambda_{0,0}(s)^{2\alpha-3}}{\lambda_{0,0}(t)^{2\alpha-2}}|e_1(s)-e_2(s)| d\rho ds\\ &\leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\end{aligned}$$

For the next term, we have

$$\begin{aligned}
&v^{\lambda_1}_{3,1,b,i,1} - v^{\lambda_2}_{3,1,b,i,1}\\
&= -2r \int_{6r}^{\infty} \left(\lambda_1''(t+w) - \lambda_2''(t+w)\right) \left(\frac{-w}{2(\lambda_1(t+w)^{2-2\alpha}+w^2)} + \frac{w}{2(\lambda_1(t)^{2-2\alpha}+w^2)}\right) dw\\
&- 2r \int_{6r}^{\infty} \lambda_2''(t+w) \left(\frac{-w}{2(\lambda_1(t+w)^{2-2\alpha}+w^2)} + \frac{w}{2(\lambda_1(t)^{2-2\alpha}+w^2)}\right) dw\\
&+ 2r \int_{6r}^{\infty} \lambda_2''(t+w) \left(\frac{-w}{2(\lambda_2(t+w)^{2-2\alpha}+w^2)} + \frac{w}{2(\lambda_2(t)^{2-2\alpha}+w^2)}\right) dw
\end{aligned} \tag{5.113}$$

For the first line on the right-hand side of (5.113), we have

$$\begin{aligned}
&|-2r \int_{6r}^{\infty} \left(\lambda_1''(t+w) - \lambda_2''(t+w)\right) \left(\frac{-w}{2(\lambda_1(t+w)^{2-2\alpha}+w^2)} + \frac{w}{2(\lambda_1(t)^{2-2\alpha}+w^2)}\right) dw|\\
&\leqslant Cr \int_{6r}^{\infty} \frac{|e_1''(t+w) - e_2''(t+w)| w \left(w \lambda_{0,0}(t)^{1-2\alpha} |\lambda_{0,0}'(t)|\right)}{w^2 \lambda_{0,0}(t+w)^{2-2\alpha}(1+w^2\lambda_{0,0}(t)^{2\alpha-2})} dw\\
&\leqslant \frac{Cr||e_1-e_2||_X}{t^3 \log^{b+2}(t)\sqrt{\log(\log(t))}} \int_0^{\infty} \frac{dw}{(1+w^2\lambda_{0,0}(t)^{2\alpha-2})}\\
&\leqslant \frac{Cr||e_1-e_2||_X}{t^3 \log^{b+2}(t)\sqrt{\log(\log(t))}}
\end{aligned}$$

The second and third lines of (5.113) can be treated together:

$$\begin{aligned}
&- 2r \int_{6r}^{\infty} \lambda_2''(t+w) \left(\frac{-w}{2(\lambda_1(t+w)^{2-2\alpha}+w^2)} + \frac{w}{2(\lambda_1(t)^{2-2\alpha}+w^2)}\right) dw\\
&+ 2r \int_{6r}^{\infty} \lambda_2''(t+w) \left(\frac{-w}{2(\lambda_2(t+w)^{2-2\alpha}+w^2)} + \frac{w}{2(\lambda_2(t)^{2-2\alpha}+w^2)}\right) dw\\
&= -r \int_{6r}^{\infty} \lambda_2''(t+w) w \left(F(t+w) - F(t)\right) dw
\end{aligned}$$

where

$$F(x) = \frac{\lambda_1(x)^{2-2\alpha} - \lambda_2(x)^{2-2\alpha}}{(\lambda_1(x)^{2-2\alpha}+w^2)(\lambda_2(x)^{2-2\alpha}+w^2)}$$

We first note that

$$|F'(\sigma s + (1-\sigma)t)| \leqslant \frac{C\lambda_{0,0}(t)^{1-2\alpha}||e_1-e_2||_X}{t \log^b(t) \log^{-b}(s) \log^{b+1}(t)\sqrt{\log(\log(t))}(w^2+\lambda_{0,0}(s)^{2-2\alpha})^2}$$
$$, \quad 0 \leqslant \sigma \leqslant 1$$

Then, we get

$$
\begin{aligned}
&|-r\int_{6r}^{\infty}\lambda_2''(t+w)w\left(F(t+w)-F(t)\right)dw|\\
&\leqslant \frac{Cr||e_1-e_2||_X\log^{-b(1-2\alpha)}(t)}{t\log^{b+1}(t)\sqrt{\log(\log(t))}}\int_{6r}^{\infty}\frac{|\lambda_2''(t+w)|dw}{(w^2+\lambda_{0,0}(s)^{2-2\alpha})\log^{b}(t)\log^{-b}(t+w)}\\
&\leqslant \frac{Cr\log^{-b(1-2\alpha)}(t)||e_1-e_2||_X}{t\log^{b+1}(t)\sqrt{\log(\log(t))}\log^{b}(t)}\int_{t+6r}^{\infty}\frac{ds}{s^2\log(s)\lambda_{0,0}(s)^{2-2\alpha}(\lambda_{0,0}(s)^{2\alpha-2}(s-t)^2+1)}\\
&\leqslant \frac{Cr||e_1-e_2||_X}{t^3\log^{b+2}(t)\sqrt{\log(\log(t))}}
\end{aligned}
$$

It then remains to consider

$$
\begin{aligned}
&v_{3,1,b,ii}^{\lambda_1}-v_{3,1,b,ii}^{\lambda_2}\\
&=\frac{-1}{r}\int_{t+6r}^{\infty}\int_0^{s-t}\frac{\rho}{(s-t)}\left(\lambda_1''(s)-\lambda_2''(s)\right)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda_1(s))\right.\\
&\left.-\left(\frac{2r^2}{(1+\rho^2)^2}-\frac{2r^2\lambda_1(s)^{2-2\alpha}}{(\lambda_1(s)^{2-2\alpha}+\rho^2)^2}\right)\right)d\rho ds\\
&-\frac{1}{r}\int_{t+6r}^{\infty}\int_0^{s-t}\frac{\rho}{(s-t)}\lambda_2''(s)\left(F_3(r,\rho,\lambda_1(s))+\frac{2r^2\lambda_1(s)^{2-2\alpha}}{(\lambda_1(s)^{2-2\alpha}+\rho^2)^2}\right.\\
&\qquad\left.-\left(F_3(r,\rho,\lambda_2(s))+\frac{2r^2\lambda_2(s)^{2-2\alpha}}{(\lambda_2(s)^{2-2\alpha}+\rho^2)^2}\right)\right)d\rho ds
\end{aligned}
\tag{5.114}
$$

The first two lines on the right-hand side of (5.114) are estimated in the same way as the analogous term appearing in the v_3 pointwise estimates, and we get

$$
\begin{aligned}
&|\frac{-1}{r}\int_{t+6r}^{\infty}\int_0^{s-t}\frac{\rho}{(s-t)}\left(\lambda_1''(s)-\lambda_2''(s)\right)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda_1(s))\right.\\
&\left.-\left(\frac{2r^2}{(1+\rho^2)^2}-\frac{2r^2\lambda_1(s)^{2-2\alpha}}{(\lambda_1(s)^{2-2\alpha}+\rho^2)^2}\right)\right)d\rho ds|\\
&\leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}
\end{aligned}
$$

To treat the third line on the right-hand side of (5.114), we start with defining $G_{3,1}$ by

$$G_{3,1}(w,r,\lambda(s)):=\int_0^w\rho\left(F_3(r,\rho,\lambda(s))+\frac{2r^2\lambda(s)^{2-2\alpha}}{(\lambda(s)^{2-2\alpha}+\rho^2)^2}\right)d\rho$$

Then, we get

$$|G_{3,1}(w,r,\lambda_1(s))-G_{3,1}(w,r,\lambda_2(s))|\leqslant\frac{|e_1(s)-e_2(s)|\lambda_{0,0}(s)^{-3+2\alpha}r^4}{(1+\lambda_{0,0}(s)^{2\alpha-2}w^2)^2}$$

which gives

$$
\begin{aligned}
&|-\frac{1}{r}\int_{t+6r}^{\infty}\int_{0}^{s-t}\frac{\rho}{(s-t)}\lambda_2''(s)\left(\,F_3(r,\rho,\lambda_1(s))+\frac{2r^2\lambda_1(s)^{2-2\alpha}}{(\lambda_1(s)^{2-2\alpha}+\rho^2)^2}\right.\\
&\qquad\qquad\qquad\qquad\left.-\left(F_3(r,\rho,\lambda_2(s))+\frac{2r^2\lambda_2(s)^{2-2\alpha}}{(\lambda_2(s)^{2-2\alpha}+\rho^2)^2}\right)\right)d\rho ds|\\
&\leqslant \frac{C}{r}\int_{t+6r}^{\infty}\frac{|\lambda_{0,0}''(s)|}{(s-t)}\frac{|e_1(s)-e_2(s)|\lambda_{0,0}(s)^{1-2\alpha}r^4}{(\lambda_{0,0}(s)^{2-2\alpha})((s-t)^2)}ds\\
&\leqslant \frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}
\end{aligned}
$$

Combining the above, we get

$$
|\left(E_5^{\lambda_1}-E_5^{\lambda_2}\right)(t,r)|\leqslant\frac{Cr||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}
$$

Lemma 5.12. *We have the following estimates*

$$
|v_4^{\lambda_1}-v_4^{\lambda_2}|(t,r)\leqslant\begin{cases}\frac{C||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^{N+3b-1}(t)}, & r\leqslant\frac{t}{2}\\ \frac{C||e_1-e_2||_X}{t\log^{3b+2N}(t)\sqrt{\log(\log(t))}}, & r\geqslant\frac{t}{2}\end{cases}
$$

Proof. We proceed to estimate $v_{4,1,2}:=v_4^{\lambda_1}-v_4^{\lambda_2}$, by noting that $v_{4,1,2}$ solves the following equation with 0 Cauchy data at infinity:

$$
-\partial_{tt}v_{4,1,2}+\partial_{rr}v_{4,1,2}+\frac{1}{r}\partial_r v_{4,1,2}-\frac{v_{4,1,2}}{r^2}=v_{4,c}^{\lambda_1}-v_{4,c}^{\lambda_2}
$$

So, we start by estimating $v_{4,c}^{\lambda_1}-v_{4,c}^{\lambda_2}$:

$$
\begin{aligned}
|v_{4,c}^{\lambda_1}-v_{4,c}^{\lambda_2}|\leqslant{}& C\frac{\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})||e_1-e_2||_X}{\log^{2b}(t)\sqrt{\log(\log(t))}}\frac{|v_1^{\lambda_1}+v_2+v_3^{\lambda_1}|}{(\lambda_{0,0}(t)^2+r^2)^2}\\
&+C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\frac{||e_1-e_2||_X}{\log^b(t)\sqrt{\log(\log(t))}}\frac{1}{t^2\log^{2b+1-2\alpha b}(t)r^3}\\
&+C\frac{\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\lambda_{0,0}(t)^2}{(r^2+\lambda_{0,0}(t)^2)^2}\left(|v_1^{\lambda_1}-v_1^{\lambda_2}|+|v_3^{\lambda_1}-v_3^{\lambda_2}|\right)
\end{aligned}
$$

where we used the explicit formula for $F_{0,2}$.

We note that the right-hand side of the equation for v_1^{λ} depends linearly on λ''. Therefore $v_1^{\lambda_1}-v_1^{\lambda_2}=v_1^{\lambda_1-\lambda_2}$, so we can use our estimates for v_1^{λ} which were previously recorded.

We thus have, in addition to (5.110), the following estimates, valid for any λ of the form

$$
\lambda(t)=\lambda_0(t)+f(t),\quad f\in\overline{B}_1(0)\subset X
$$

$$
|v_1^{\lambda}+v_2+v_3^{\lambda}|\leqslant\begin{cases}\frac{Cr}{t^2\log^b(t)}, & r\leqslant\frac{t}{2}\\ \frac{C\log(r)}{|t-r|}, & t>r\geqslant\frac{t}{2}\end{cases}
$$

$$|v_1^{\lambda_1}-v_1^{\lambda_2}| \leqslant \begin{cases} \frac{Cr||e_1-e_2||_X}{t^2\log^b(t)\sqrt{\log(\log(t))}}, & r\leqslant \frac{t}{2}\\ \frac{C||e_1-e_2||_X}{r\sqrt{\log(\log(t))}\log^b(t)}, & r>\frac{t}{2}\end{cases}$$

This gives

$$|v_{4,c}^{\lambda_1}-v_{4,c}^{\lambda_2}| \leqslant C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})||e_1-e_2||_X \cdot \begin{cases} \frac{1}{r^3t^2\log^{3b}(t)\sqrt{\log(\log(t))}}, & r\leqslant\frac{t}{2}\\ \frac{\log(r)}{r^4\log^{2b}(t)\sqrt{\log(\log(t))}|t-r|}+\frac{1}{t^2\log^{3b+1-2\alpha b}(t)\sqrt{\log(\log(t))}r^3}, & t>r>\frac{t}{2}\end{cases} \tag{5.115}$$

Now, we start with

$$\begin{aligned}&\left(v_4^{\lambda_1}-v_4^{\lambda_2}\right)(t,r)\\ &=\frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\\ &\qquad\int_0^{2\pi}\frac{(r+\rho\cos(\theta))(v_{4,c}^{\lambda_1}-v_{4,c}^{\lambda_2})(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}}d\theta d\rho ds\end{aligned}$$

which gives

$$\begin{aligned}&|v_4^{\lambda_1}-v_4^{\lambda_2}|\\ &\leqslant C\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}|v_{4,c}^{\lambda_1}-v_{4,c}^{\lambda_2}|(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})d\theta d\rho ds\end{aligned}$$

Note that (5.115) has the same right-hand side as (5.85), except for an extra factor of $\frac{||e_1-e_2||_X}{\sqrt{\log(\log(t))}}$. So, we can inspect the intermediate steps of the procedure used when obtaining pointwise estimates on v_4, thereby getting

$$|v_4^{\lambda_1}-v_4^{\lambda_2}|(t,r)\leqslant\frac{C||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^{N+3b-1}(t)}, \quad r\leqslant\frac{t}{2}$$

In the region $r\geqslant\frac{t}{2}$, we use the following, slightly different estimate for $v_1^\lambda+v_2+v_3^\lambda$, again valid for all $\lambda(t)=\lambda_0(t)+f(t), \quad f\in\overline{B}_1(0)\subset X$:

$$|v_1^\lambda+v_2+v_3^\lambda|\leqslant\begin{cases}\frac{Cr}{t^2\log^b(t)}, & r\leqslant\frac{t}{2}\\ \frac{C}{\sqrt{r}}, & t-\sqrt{t}\leqslant r\leqslant t+\sqrt{t}\\ \frac{C\log(r)}{|t-r|}, & \frac{t}{2}\leqslant r\leqslant t-\sqrt{t}, \text{ or } r\geqslant t+\sqrt{t}\end{cases}$$

We then get

$$|v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}| \leqslant C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})||e_1 - e_2||_X \cdot \begin{cases} \frac{1}{r^3 t^2 \log^{3b}(t)\sqrt{\log(\log(t))}}, & r \leqslant \frac{t}{2} \\ \frac{1}{\log^{2b}(t)\sqrt{\log(\log(t))} r^{9/2}}, & t-\sqrt{t} \leqslant r \leqslant t+\sqrt{t} \\ \frac{\log(r)}{r^4 \log^{2b}(t)\sqrt{\log(\log(t))}|t-r|} + \frac{1}{t^2 \log^{3b+1-2\alpha b}(t)\sqrt{\log(\log(t))} r^3} \\ , \quad t-\sqrt{t} > r > \frac{t}{2} \text{ or } r > t+\sqrt{t} \end{cases} \tag{5.116}$$

Note that (5.116) again has the same right-hand side as (5.88), except for the extra factor of $\frac{||e_1-e_2||_X}{\sqrt{\log(\log(t))}}$, so, we can infer the following estimates from our study of v_4:

$$||\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}(x,\xi)||_{L^2(\xi d\xi)} \leqslant \frac{C||e_1-e_2||_X}{x^2 \log^{3b+2N}(x)\sqrt{\log(\log(x))}}$$

Then, we simply note:

$$v_{4,1,2}(t,r) = \int_t^\infty \int_0^\infty J_1(r\xi)\sin((t-s)\xi)\left(\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}(s,\xi)\right) d\xi ds$$

So,

$$\begin{aligned} |v_{4,1,2}(t,r)| &\leqslant C\int_t^\infty \int_0^{\frac{1}{r}} r\xi |\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}|(s,\xi) d\xi ds \\ &+ C\int_t^\infty \int_{\frac{1}{r}}^\infty \frac{|\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}|(s,\xi)}{\sqrt{r\xi}} d\xi ds \\ &\leqslant Cr\int_t^\infty \left(\int_0^{\frac{1}{r}} \xi d\xi\right)^{1/2} \left(\int_0^{\frac{1}{r}} \xi |\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}(s,\xi)|^2 d\xi\right)^{1/2} ds \\ &+ \frac{C}{\sqrt{r}}\int_t^\infty \left(\int_{\frac{1}{r}}^\infty \frac{d\xi}{\xi^2}\right)^{1/2} \left(\int_{\frac{1}{r}}^\infty \xi |\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}(s,\xi)|^2 d\xi\right)^{1/2} ds \\ &\leqslant C\int_t^\infty ||\widehat{v_{4,c}^{\lambda_1} - v_{4,c}^{\lambda_2}}(s,\xi)||_{L^2(\xi d\xi)} ds \\ &\leqslant \frac{C||e_1-e_2||_X}{t\sqrt{\log(\log(t))}\log^{3b+2N}(t)} \end{aligned}$$

which finishes the proof of the lemma. □

Lemma 5.13. *We have the following pointwise estimate*

$$|v_5^{\lambda_1} - v_5^{\lambda_2}|(t,r) \leqslant C\frac{||e_1-e_2||_X \log^2(t)}{t^3\sqrt{\log(\log(t))}\log^b(t)}, \quad r \leqslant \frac{t}{2}$$

Proof. As was the case for v_4, we start with noting that

$$v_{5,1,2} := v_5^{\lambda_1} - v_5^{\lambda_2}$$

solves the equation (with 0 Cauchy data at infinity)

$$-\partial_{tt} v_{5,1,2} + \partial_{rr} v_{5,1,2} + \frac{1}{r}\partial_r v_{5,1,2} - \frac{v_{5,1,2}}{r^2} = N_2(f_{v_5}^{\lambda_1}) - N_2(f_{v_5}^{\lambda_2})$$

where

$$f_{v_5}^{\lambda} = v_1^{\lambda} + v_2 + v_3^{\lambda} + v_4^{\lambda}$$

Collecting the estimates from previous subsections, one estimate on $f(v_5^{\lambda})$, valid for all $\lambda = \lambda_0 + f, \quad f \in \overline{B}_1(0) \subset X$, is

$$|f_{v_5}^{\lambda}| \leqslant \begin{cases} \frac{Cr}{t^2 \log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C\log(r)}{|t-r|}, & t > r > \frac{t}{2} \end{cases}$$

Now, we start to estimate the right-hand side of the $v_{5,1,2}$ equation.

$$\begin{aligned} N_2(f_{v_5}^{\lambda_1}) - N_2(f_{v_5}^{\lambda_2}) &= \left(\frac{\sin(2Q_{\frac{1}{\lambda_1(t)}}(r))}{2r^2} - \frac{\sin(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\right)\left(\cos(2f_{v_5}^{\lambda_1}) - 1\right) \\ &+ \frac{\sin(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\left(\cos(2f_{v_5}^{\lambda_1}) - \cos(2f_{v_5}^{\lambda_2})\right) \\ &+ \left(\frac{\cos(2Q_{\frac{1}{\lambda_1(t)}}(r)) - \cos(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\right)\left(\sin(2f_{v_5}^{\lambda_1}) - 2f_{v_5}^{\lambda_1}\right) \\ &+ \frac{\cos(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\left(\sin(2f_{v_5}^{\lambda_1}) - 2f_{v_5}^{\lambda_1} - \left(\sin(2f_{v_5}^{\lambda_2}) - 2f_{v_5}^{\lambda_2}\right)\right) \end{aligned} \tag{5.117}$$

For the first line of (5.117), we have

$$\begin{aligned} &|\left(\frac{\sin(2Q_{\frac{1}{\lambda_1(t)}}(r))}{2r^2} - \frac{\sin(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\right)\left(\cos(2f_{v_5}^{\lambda_1}) - 1\right)| \\ &\leqslant \frac{C||e_1 - e_2||_X}{r\log^b(t)\sqrt{\log(\log(t))}} \frac{\left(f_{v_5}^{\lambda_1}\right)^2}{(\lambda_{0,0}(t)^2 + r^2)} \\ &\leqslant \frac{C||e_1 - e_2||_X}{r\log^b(t)\sqrt{\log(\log(t))}} \begin{cases} \frac{r^2}{t^4\log^{2b}(t)(\lambda_{0,0}(t)^2 + r^2)}, & r \leqslant \frac{t}{2} \\ \frac{\log^2(r)}{r^2(t-r)^2}, & t > r \geqslant \frac{t}{2} \end{cases} \end{aligned}$$

For the second line of (5.117), we have

$$\begin{aligned} &|\frac{\sin(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\left(\cos(2f_{v_5}^{\lambda_1}) - \cos(2f_{v_5}^{\lambda_2})\right)| \\ &\leqslant \frac{C\lambda_2(t)}{(r^2 + \lambda_2(t)^2)r}\left(|f_{v_5}^{\lambda_1}| + |f_{v_5}^{\lambda_2}|\right)\left(|v_1^{\lambda_1} - v_1^{\lambda_2}| + |v_3^{\lambda_1} - v_3^{\lambda_2}| + |v_4^{\lambda_1} - v_4^{\lambda_2}|\right) \\ &\leqslant \begin{cases} \frac{Cr\lambda_{0,0}(t)||e_1 - e_2||_X}{(r^2 + \lambda_{0,0}(t)^2)t^4\log^{2b}(t)\sqrt{\log(\log(t))}} + \frac{C||e_1 - e_2||_X\lambda_{0,0}(t)}{(r^2 + \lambda_{0,0}(t)^2)t^4\sqrt{\log(\log(t))}\log^{N+4b-1}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C\log(t)||e_1 - e_2||_X}{r^3|t-r|t\log^{2b}(t)\sqrt{\log(\log(t))}}, & t > r > \frac{t}{2} \end{cases} \end{aligned}$$

We consider the third line of (5.117), and get

$$|\left(\frac{\cos(2Q_{\frac{1}{\lambda_1(t)}}(r))-\cos(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\right)\left(\sin(2f_{v_5}^{\lambda_1})-2f_{v_5}^{\lambda_1}\right)|$$
$$\leqslant \frac{C||e_1-e_2||_X}{\log^{2b}(t)\sqrt{\log(\log(t))}(r^2+\lambda_{0,0}(t)^2)^2}\begin{cases}\frac{r^3}{t^6\log^{3b}(t)}, & r\leqslant\frac{t}{2}\\ \frac{\log^3(r)}{|t-r|^3}, & t>r>\frac{t}{2}\end{cases}$$

For the fourth line of (5.117), we have

$$|\frac{\cos(2Q_{\frac{1}{\lambda_2(t)}}(r))}{2r^2}\left(\sin(2f_{v_5}^{\lambda_1})-2f_{v_5}^{\lambda_1}-\left(\sin(2f_{v_5}^{\lambda_2})-2f_{v_5}^{\lambda_2}\right)\right)|$$
$$\leqslant \frac{C}{r^2}\left(|f_{v_5}^{\lambda_1}|^2+|f_{v_5}^{\lambda_2}|^2\right)|f_{v_5}^{\lambda_1}-f_{v_5}^{\lambda_2}|$$
$$\leqslant\begin{cases}\frac{Cr||e_1-e_2||_X}{t^6\sqrt{\log(\log(t))}\log^{3b}(t)}+\frac{C||e_1-e_2||_X}{t^6\sqrt{\log(\log(t))}\log^{N+5b-1}(t)}, & r\leqslant\frac{t}{2}\\ \frac{C||e_1-e_2||_X\log^2(t)}{r^3(t-r)^2\log^b(t)\sqrt{\log(\log(t))}}, & t>r>\frac{t}{2}\end{cases}$$

Combining all of the above estimates, we obtain

$$|N_2(f_{v_5}^{\lambda_1})(t,r)-N_2(f_{v_5}^{\lambda_2})(t,r)|$$
$$\leqslant\begin{cases}\frac{Cr\lambda_{0,0}(t)||e_1-e_2||_X}{(r^2+\lambda_{0,0}(t)^2)t^4\log^{2b}(t)\sqrt{\log(\log(t))}}+\frac{C||e_1-e_2||_X\lambda_{0,0}(t)}{(r^2+\lambda_{0,0}(t)^2)t^4\sqrt{\log(\log(t))}\log^{N+4b-1}(t)}, & r\leqslant\frac{t}{2}\\ \frac{C\log(t)||e_1-e_2||_X}{r^3\log^b(t)\sqrt{\log(\log(t))}}\left(\frac{\log(t)}{(t-r)^2}+\frac{1}{t|t-r|\log^b(t)}+\frac{\log^2(t)}{\log^b(t)r|t-r|^3}\right), & t>r>\frac{t}{2}\end{cases}$$

We now proceed to estimate $v_{5,1,2}$ in the region $r\leqslant\frac{t}{2}$. We start with

$$v_{5,1,2}(t,r)$$
$$=\frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}$$
$$\left(\int_0^{2\pi}\left(N_2(f_{v_5}^{\lambda_1})-N_2(f_{v_5}^{\lambda_2})\right)(s,\sqrt{r^2+\rho^2+2\rho r\cos(\theta)})\frac{(r+\rho\cos(\theta))}{\sqrt{r^2+2r\rho\cos(\theta)+\rho^2}}\right)d\theta d\rho ds$$

We then estimate as follows

$$\begin{aligned}|v_{5,1,2}(t,r)|&\leqslant C\int_t^\infty\int_{B_{s-t}(0)}\frac{|N_2(f_{v_5}^{\lambda_1})-N_2(f_{v_5}^{\lambda_2})|(s,|x+y|)}{\sqrt{(s-t)^2-|y|^2}}dA(y)ds\\ &\leqslant C\int_t^\infty\int_{B_{s-t}(0)\cap B_{\frac{s}{2}}(-x)}\frac{|N_2(f_{v_5}^{\lambda_1})-N_2(f_{v_5}^{\lambda_2})|(s,|x+y|)}{\sqrt{(s-t)^2-|y|^2}}dA(y)ds\\ &+C\int_t^\infty\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-x))^c}\frac{|N_2(f_{v_5}^{\lambda_1})-N_2(f_{v_5}^{\lambda_2})|(s,|x+y|)}{\sqrt{(s-t)^2-|y|^2}}dA(y)ds\end{aligned}$$

Now, we carry out the identical steps which were done to obtain the v_5^λ estimates, and get

$$|v_{5,1,2}(t,r)|\leqslant C\frac{||e_1-e_2||_X\log^2(t)}{t^3\sqrt{\log(\log(t))}\log^b(t)},\quad r\leqslant\frac{t}{2}$$

which completes the proof of the lemma. □

We can now use all of the above estimates to get

$$
\begin{aligned}
&|\int_0^\infty \left(\frac{\cos(2Q_1(\frac{r}{\lambda_2(t)}))-1}{r^2\lambda_2(t)}\right)\phi_0(\frac{r}{\lambda_2(t)})r\left(v_4^{\lambda_1}-v_4^{\lambda_2}\right)(t,r)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)dr|\\
&\leqslant C\lambda_0(t)^2\int_0^{\frac{t}{2}}\frac{r^2}{(r^2+\lambda_0(t)^2)^3}\frac{||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^{N+3b-1}(t)}dr\\
&\leqslant \frac{C||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^{N+2b-1}(t)}
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^\infty \left(\frac{\cos(2Q_1(\frac{r}{\lambda_2(t)}))-1}{r^2\lambda_2(t)}\right)\phi_0(\frac{r}{\lambda_2(t)})r\left(v_5^{\lambda_1}-v_5^{\lambda_2}\right)(t,r)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)dr|\\
&\leqslant C\lambda_0(t)^2\int_0^{\frac{t}{2}}\frac{r^2}{(r^2+\lambda_0(t)^2)^3}\frac{||e_1-e_2||_X\log^2(t)}{t^3\sqrt{\log(\log(t))}\log^b(t)}dr\leqslant\frac{C||e_1-e_2||_X\log^2(t)}{t^3\sqrt{\log(\log(t))}}
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^\infty \left(\frac{\cos(2Q_1(\frac{r}{\lambda_2(t)}))-1}{r^2\lambda_2(t)}\right)\phi_0(\frac{r}{\lambda_2(t)})r\left(E_5^{\lambda_1}-E_5^{\lambda_2}\right)(t,r)dr|\\
&\leqslant C\int_0^\infty\frac{\lambda_2(t)^2r^2}{(\lambda_2(t)^2+r^2)^3}\left(\frac{r||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\right)dr\\
&\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}\int_0^\infty\frac{u^3du}{(1+u^2)^3}\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{b+1}(t)\sqrt{\log(\log(t))}}
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^\infty \left(\frac{\cos(2Q_1(\frac{r}{\lambda_2(t)}))-1}{r^2\lambda_2(t)}\right)\phi_0(\frac{r}{\lambda_2(t)})r\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(-v_1^{\lambda_1}+v_1^{\lambda_2}-v_3^{\lambda_1}+v_3^{\lambda_2}\right)dr|\\
&\leqslant C\int_{\frac{\log^N(t)}{2}}^{\frac{t}{2}}\frac{\lambda_2(t)^2r^2}{r^6}\left(\frac{||e_1-e_2||_Xr}{t^2\log^b(t)\sqrt{\log(\log(t))}}\right)dr\\
&+C\int_{\frac{t}{2}}^\infty\frac{\lambda_2(t)^2r^2}{r^6}\frac{||e_1-e_2||_X}{r\sqrt{\log(\log(t))}\log^b(t)}dr\\
&\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{3b+2N}(t)\sqrt{\log(\log(t))}}
\end{aligned}
$$

$$
\begin{aligned}
&|\int_0^\infty\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})(F_{0,2}^{\lambda_1}-F_{0,2}^{\lambda_2})(t,r)\frac{\phi_0(\frac{r}{\lambda_1(t)})}{\lambda_1(t)}rdr|\\
&+|\int_0^\infty\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}^{\lambda_2}(t,r)\left(\frac{\phi_0(\frac{r}{\lambda_1(t)})}{\lambda_1(t)}-\frac{\phi_0(\frac{r}{\lambda_2(t)})}{\lambda_2(t)}\right)rdr|\\
&\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{3b+1-2\alpha b}(t)\sqrt{\log(\log(t))}\log^{2N}(t)}
\end{aligned}
$$

Combining the above estimates, we firstly have

$$|G(t,\lambda_1(t)) - G(t,\lambda_2(t))| \leqslant \frac{C||e_1-e_2||_X}{t^2\sqrt{\log(\log(t))}\log^{b+1}(t)}$$

Then, combining this with the earlier estimates of the terms in $RHS(e_1,t)-RHS(e_2,t)$ which don't involve G, we finally get: there exists $C_{lip} > 0$ *independent* of T_0, such that for $e_1, e_2 \in \overline{B_1(0)} \subset X$

$$|RHS(e_1,t) - RHS(e_2,t)| \leqslant \frac{C_{lip}||e_1-e_2||_X}{t^2\log^{b+1}(t)(\log(\log(t)))^{3/2}}$$

(Recall that G appears in the expression of RHS in the term $\frac{G(t,\lambda(t))}{\log(\lambda_0(t))}$). This concludes the proof of Proposition 5.2.

Using the L^1 estimate on the resolvent kernel, r, we get

$$\begin{aligned}&|(T(e_1)-T(e_2))''(t)|\\ &\leqslant \frac{|RHS(e_1,t)-RHS(e_2,t)|}{\alpha} + \frac{2}{\alpha}\sup_{z\geqslant t}\left(|RHS(e_1,z)-RHS(e_2,z)|\right)\\ &\leqslant \frac{3C_{lip}}{\alpha}\frac{||e_1-e_2||_X}{t^2\log^{b+1}(t)(\log(\log(t)))^{3/2}}\end{aligned}$$

Then, with the same procedure used when estimating $|T(e)'(t)|$ earlier, we get

$$|(T(e_1)-T(e_2))'(t)| \leqslant \frac{3C_{lip}}{\alpha}\left(1+\frac{1}{100}\right)\frac{||e_1-e_2||_X}{t\log^{b+1}(t)(\log(\log(t)))^{3/2}}$$

$$|(T(e_1)-T(e_2))(t)| \leqslant \frac{3C_{lip}}{\alpha}\left(1+\frac{1}{100}\right)\frac{||e_1-e_2||_X}{b\log^{b}(t)(\log(\log(t)))^{3/2}}$$

whence,

$$||T(e_1)-T(e_2)||_X \leqslant \frac{10C_{lip}}{\alpha\log(\log(T_0))}||e_1-e_2||_X, \quad e_1,e_2 \in \overline{B_1(0)}$$

Since C_{lip} is independent of T_0, and the above estimates are valid for all T_0 satisfying

$$T_0 > 2e^{e^{1000(b+1)}} + T_{0,1} + T_{0,2} + T_{0,3}$$

if we further restrict T_0 to satisfy

$$T_0 > 2e^{e^{1000(b+1)}} + T_{0,1} + T_{0,2} + T_{0,3} + e^{e^{\frac{1500(C_{lip}+1000(1+\frac{1}{b}))}{\alpha}}}$$

then, T is a strict contraction on the complete metric space $\overline{B_1(0)} \subset X$. By Banach's fixed point theorem, T has a fixed point, say e_0, in $\overline{B_1(0)} \subset X$. Then, if

$$\lambda(t) = \lambda_0(t) + e_0(t)$$

we have

$$\lambda \in C^2([T_0,\infty)), \quad ||e_0||_X \leqslant 1$$

and λ solves (5.63).

5.8.10 Estimating $\lambda'''(t)$

The main proposition of this section is:

Proposition 5.3. *$\lambda \in C^3([T_0, \infty))$, with*

$$|\lambda'''(t)| \leqslant \frac{C}{t^3 \log^{b+1}(t)}, \quad t \geqslant T_0$$

In addition, we have

$$|\partial_t v_1(t,r)| \leqslant \begin{cases} \frac{Cr}{t^3 \log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{rt \log^{b+1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.118}$$

$$|\partial_t v_3(t,r)| \leqslant \begin{cases} \frac{Cr \log(\log(t))}{t^3 \log^{b+1}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{rt \log^{b+1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.119}$$

$$|\partial_{tr} v_3(t,r)| \leqslant \frac{C}{t^3 \log^b(t)} \tag{5.120}$$

$$|\partial_{tr} v_1(t,r)| \leqslant \begin{cases} \frac{C}{t^3 \log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{r^2 t \log^{b+1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.121}$$

$$|\partial_t v_4(t,r)| \leqslant \begin{cases} \frac{Cr}{t^3 \log^{3b+2N-2}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{t^2 \log^{3b+2N-1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.122}$$

Proof. Since $\lambda(t) = \lambda_0(t) + e_0(t)$, it suffices to show that $e_0 \in C^3([T_0, \infty))$, and estimate $e_0'''(t)$. Recall that λ solves (5.63), which can be re-written as

$$\lambda''(t) = \frac{RHS_2(t)}{g_2(t)} \tag{5.123}$$

where

$$g_2(t) = \frac{-2}{\lambda(t)} + \frac{4\alpha \log(\lambda(t))}{\lambda(t)} \left(\frac{1}{-1 + \lambda(t)^{2\alpha}}\right) - \frac{2(\lambda(t)^{-2\alpha} - 1)}{\lambda(t)} g_5(t)$$

$$g_5(t) = \int_0^\infty \frac{\chi_{\geqslant 1}(\frac{2R\lambda(t)}{\log^N(t)})\phi_0(R) R^2 dR}{(R^2+1)(\lambda(t)^{-2\alpha} + R^2)}$$

and

$$\begin{aligned} RHS_2(t) &= -\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\left(K_1(s-t,\lambda(t))+K(s-t,\lambda(t))\right)ds + 2\frac{(\lambda'(t))^2}{\lambda(t)^2}\\ &+ \frac{4b}{\lambda(t)t^2\log^b(t)} + E_{v_2,ip}(t,\lambda(t))\\ &+ \langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left(v_3+(v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)\right)|_{r=R\lambda(t)},\phi_0\rangle\\ &- \langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}\left(\frac{2r}{\log^N(t)}\right)(v_1+v_2+v_3)|_{r=R\lambda(t)},\phi_0\rangle\\ &- 4\int_0^\infty \chi_{\geqslant 1}(\frac{2R\lambda(t)}{\log^N(t)})\frac{(\lambda'(t))^2R^2\phi_0(R)dR}{\lambda(t)^2(R^2+1)^2} \end{aligned}$$

The point of this re-writing is that the right-hand side of (5.123) is in $C^1([T_0,\infty))$, since $\lambda \in C^2([T_0,\infty))$. So, $\lambda \in C^3([T_0,\infty))$, and we have

$$\lambda'''(t) = \partial_t\left(\frac{RHS_2(t)}{g_2(t)}\right)$$

We will first prove a preliminary estimate on $\lambda'''(t)$, which will then allow us to obtain a more useful formula for e_0'''. Using the formula for g_2, and estimates on e_0 from the fact that $e_0 \in \overline{B_1(0)}$, we get

$$|\frac{1}{g_2(t)}| \leqslant \frac{C}{\log^b(t)\log(\log(t))}, \quad |\left(\frac{1}{g_2}\right)'(t)| \leqslant \frac{C}{t\log^{b+1}(t)\log(\log(t))}$$

and

$$|\lambda'''(t)| \leqslant \frac{C|RHS_2'(t)|}{\log^b(t)\log(\log(t))} + \frac{C|RHS_2(t)|}{t\log^{b+1}(t)\log(\log(t))}$$

For the preliminary estimate, it will suffice to estimate each term in RHS_2 separately, despite the fact that there is some cancellation between some terms in RHS_2. When we prove the final estimate on $\lambda'''(t)$, we will take this cancellation into account.

Using the same procedures and estimates used to estimate $G(t,\lambda_0(t)+f(t))$ for arbitrary $f \in \overline{B_1(0)} \subset X$, we get

$$|RHS_2(t)| \leqslant \frac{C}{t^2}$$

Now, we estimate $|RHS_2'(t)|$. Using estimates on $\lambda^{(j)}(t), \quad j=0,1,2$, we get

$$\begin{aligned}&|RHS_2'(t)|\\&\leqslant |\partial_t\left(\frac{-16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\left(K_1(s-t,\lambda(t))+K(s-t,\lambda(t))\right)ds\right)|\\&+\frac{C}{t^3\log^2(t)}+\frac{C}{t^3}+|\partial_t E_{v_2,ip}(t,\lambda(t))|\\&+|\partial_t\left(\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left(v_3+(v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)\right)|_{r=R\lambda(t)},\phi_0\rangle\right)|\\&+|\partial_t\left(\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\left(v_1+v_2+v_3\right)|_{r=R\lambda(t)},\phi_0\rangle\right)|\end{aligned} \tag{5.124}$$

For the $E_{v_2,ip}(t)$ term, we recall the definition in (5.104), and start with

$$\begin{aligned}&\partial_t\left(\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{(b-1)}{\xi\log^b(\frac{1}{\xi})}+\frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\right)\\&=\partial_t\left(\int_0^{\frac{t}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\frac{u}{t})-1\right)\frac{\sin(u)}{t^2}\left(\frac{b-1}{u\log^b(\frac{t}{u})}+\frac{b(b-1)}{u\log^{b+1}(\frac{t}{u})}\right)du\right)\end{aligned}$$

After taking the derivative, and integrating by parts once in the remaining integrals, we get

$$\begin{aligned}&\partial_t\left(\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{(b-1)}{\xi\log^b(\frac{1}{\xi})}+\frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\right)\\&=\frac{-\sin(\frac{t}{2})}{t^3}\left(\frac{b-1}{\log^b(2)}+\frac{b(b-1)}{\log^{b+1}(2)}\right)+\text{Err}\end{aligned}$$

where

$$|\text{Err}|\leqslant\frac{C}{t^4}$$

On the other hand,

$$\begin{aligned}&\partial_t\left(\int_0^{\frac{t}{2}}\frac{\sin(u)b(b-1)du}{t^2u\log^{b+1}(\frac{t}{u})}\right)\\&=\frac{\sin(\frac{t}{2})b(b-1)}{t^3\log^{b+1}(2)}+\int_0^{\frac{t}{2}}\frac{\sin(u)b(b-1)}{u}\left(\frac{-2}{t^3\log^{b+1}(\frac{t}{u})}-\frac{(b+1)}{\log^{b+2}(\frac{t}{u})t^3}\right)du\end{aligned}$$

So,

$$\begin{aligned}E_{t,v_2,ip,1} &:= \partial_t \left(2c_b \int_0^{\frac{1}{2}} \left(\chi_{\leqslant \frac{1}{4}}(\xi)-1\right) \frac{\sin(t\xi)}{t^2} \left(\frac{b-1}{\xi \log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi \log^{b+1}(\frac{1}{\xi})}\right) d\xi\right.\\
&\left. + 2c_b \left(\int_0^{\frac{t}{2}} \frac{\sin(u)(b-1)du}{t^2 u \log^b(\frac{t}{u})} - \frac{(b-1)\pi}{2t^2 \log^b(t)} + \int_0^{\frac{t}{2}} \frac{\sin(u)b(b-1)du}{t^2 u \log^{b+1}(\frac{t}{u})}\right)\right)\\
&= 2c_b\text{Err} + 2c_b \left(\int_0^{\frac{t}{2}} \sin(u)(b-1) \left(\frac{-2}{t^3 u \log^b(\frac{t}{u})} - \frac{b}{\log^{b+1}(\frac{t}{u}) t^3 u}\right) du\right.\\
&\left. + \int_0^{\frac{t}{2}} \frac{1}{u} \sin(u) b(b-1) \left(\frac{-2}{t^3 \log^{b+1}(t)} - \frac{(b+1)}{t^3 \log^{b+2}(\frac{t}{u})}\right) du\right)\\
&+ 2c_b \left(\frac{(b-1)\pi}{t^3 \log^b(t)} + \frac{b(b-1)\pi}{2t^3 \log^{b+1}(t)}\right)\end{aligned}$$

The asymptotics of the integrals in lines 3 and 4 of the above expression were computed previously, in the section which constructed v_2. The following is the most important asymptotic, which shows that there is some cancellation between the integral in line 3 and the terms on line 5.

$$-2\int_0^{\frac{t}{2}} \frac{\sin(u)(b-1)du}{t^3 u \log^b(\frac{t}{u})} = \frac{-2(b-1)}{t^3}\frac{\pi}{2}\left(\frac{1}{\log^b(t)} + O\left(\frac{1}{\log^{b+1}(t)}\right)\right)$$

In total, we get

$$|E_{t,v_2,ip,1}(t)| \leqslant \frac{C}{t^3 \log^{b+1}(t)}$$

Next, we estimate

$$\partial_t \left(2c_b \lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2} \psi_{v_2}(\xi, \lambda(t)) d\xi\right)$$

Here, we use estimates on the derivatives of ψ_{v_2}, proven earlier, when showing that the map T was a contraction, as well as (5.107), which shows (among other things) that

$$|\partial_\xi^2 \psi_{v_2}(\xi, \lambda(t))| \leqslant C\frac{p(\xi)}{\lambda(t)}, \quad p \in C_c^\infty([\frac{1}{8}, \frac{1}{4}])$$

This gives us

$$|\partial_t \left(2c_b \lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2} \psi_{v_2}(\xi, \lambda(t)) d\xi\right)| \leqslant \frac{C}{t^4}$$

Similarly, we use estimates on derivatives of F_{v_2} which were proven while showing that T is a contraction, as well as (5.107), and asymptotics of K_1, to get

$$|\partial_t \left(2c_b \lambda(t) \int_0^\infty \chi_{\leqslant \frac{1}{4}}(\xi) \frac{\sin(t\xi)}{t^2} F_{v_2}(\xi, \lambda(t)) d\xi\right)| \leqslant \frac{C \log(\log(t))}{t^4 \log^{2b}(t)}$$

In total, we have

$$|\partial_t \left(\lambda(t) E_{v_2,ip}(t, \lambda(t))\right)| \leqslant \frac{C}{t^3 \log^{b+1}(t)}$$

By the same procedure, this estimate is also true for the case $b=1$.

Some of the terms in (5.124) which remain to be estimated will be estimated in two different ways. One way of estimating these terms will give rise to preliminary estimates on e''', which will then allow us to obtain stronger estimates. For this preliminary estimate, we will start with the first line of (5.124):

$$\partial_t \left(\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\left(K(s-t,\lambda(t))+K_1(s-t,\lambda(t))\right)ds\right) = I+II+III+IV$$

where

$$I = \frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\partial_2 K(s-t,\lambda(t))\lambda'(t)ds$$

$$II = \frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\partial_2 K_1(s-t,\lambda(t))\lambda'(t)ds$$

$$III = -\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(s)\left(\partial_1 K(s-t,\lambda(t))+\partial_1 K_1(s-t,\lambda(t))\right)ds$$

$$IV = \frac{-48\lambda'(t)}{\lambda(t)^4}\int_t^\infty \lambda''(s)\left(K(s-t,\lambda(t))+K_1(s-t,\lambda(t))\right)ds$$
$$|IV| \leqslant \frac{C}{t^3\log(t)}$$

In order to estimate I, let us start with

$$\partial_2 K(x,\lambda(t))$$
$$= \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^x \rho\left(\frac{1}{\sqrt{x^2-\rho^2}}-\frac{1}{x}\right)\left(\frac{4\lambda(t)R^2(1+\rho^2+\lambda(t)^2R^2)}{(4\lambda(t)^2R^2+(1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}}\right)d\rho dR$$

Then, the same procedure used to obtain (5.103) gives

$$|\int_t^\infty \lambda''(s)\partial_2 K(s-t,\lambda(t))\lambda'(t)ds| \leqslant \frac{C}{t^3\log^{2b+2}(t)}\int_t^\infty |\partial_2 K(s-t,\lambda(t))|ds$$
$$\leqslant \frac{C}{t^3\log^{3b+2}(t)}$$

So,

$$|I| \leqslant \frac{C}{t^3\log^2(t)}$$

Next, we recall the definition of K_1:

$$K_1(x,\lambda(t)) = \int_0^\infty \frac{r}{\lambda(t)^2(1+\frac{r^2}{\lambda(t)^2})^3}\int_0^x \frac{\rho}{x}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr$$

So,

$$
\begin{aligned}
&|\partial_2 K_1(x,\lambda(t))\lambda'(t)| \\
&\leqslant C\lambda(t)^3|\lambda'(t)| \int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3} \int_0^x \frac{\rho}{x}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) d\rho dr
\end{aligned}
$$

and we get

$$
\begin{aligned}
&|II| \leqslant |\frac{16}{\lambda(t)^3}\int_t^\infty \lambda''(x)\partial_2 K_1(x-t,\lambda(t))\lambda'(t)dx| \\
&\leqslant C|\lambda'(t)|\int_t^\infty |\lambda''(x)| \\
&\qquad \int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3}\begin{cases}\frac{1}{(x-t)}\int_0^{x-t} 2\rho d\rho, & x-t\leqslant 1\\ \frac{1}{(x-t)}\int_0^\infty \rho d\rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right), & x-t\geqslant 1\end{cases} drdx \\
&\leqslant C|\lambda'(t)|\int_t^\infty |\lambda''(x)|\int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3}\begin{cases}(x-t), & x-t\leqslant 1\\ \frac{r^2}{(x-t)}, & x-t\geqslant 1\end{cases} drdx \\
&\leqslant C|\lambda'(t)|\left(\int_t^{t+1}\frac{1}{x^2\log^{b+1}(x)}\frac{(x-t)dx}{\lambda(t)^4}+\int_{t+1}^\infty \frac{dx}{x^2\log^{b+1}(x)(x-t)}\int_0^\infty \frac{r^3 dr}{(r^2+\lambda(t)^2)^3}\right) \\
&\leqslant \frac{C}{t^3\log^{1-2b}(t)}
\end{aligned}
$$

We will now treat the term III. Let $L = K + K_1$. Then, we have

$$\partial_1 L(x,\lambda(t)) = L_a(x,\lambda(t)) + L_b(x,\lambda(t))$$

with

$$
\begin{aligned}
L_a(x,\lambda(t)) &= \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^1 \frac{u}{\sqrt{1-u^2}}\left(1+\frac{R^2\lambda(t)^2-1-u^2x^2}{\sqrt{(R^2\lambda(t)^2-1-u^2x^2)^2+4R^2\lambda(t)^2}}\right) dudR \\
&= \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^x \frac{\rho}{x\sqrt{x^2-\rho^2}}\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right) d\rho dR
\end{aligned}
$$

$$
\begin{aligned}
L_b(x,\lambda(t)) &= \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^1 \frac{ux}{\sqrt{1-u^2}}\left(\frac{-8R^2\lambda(t)^2u^2x}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+u^2x^2)^2)^{3/2}}\right) dudR \\
&= \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^x \frac{\rho}{\sqrt{x^2-\rho^2}}\left(\frac{-8R^2\lambda(t)^2\rho^2}{x(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}\right) d\rho dR
\end{aligned}
$$

Now,

$$\begin{aligned}
&\int_t^\infty |L_a(s-t,\lambda(t))|ds \\
&\leqslant \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^\infty \rho\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)\int_{\rho+t}^\infty \frac{dsd\rho dR}{(s-t)\sqrt{(s-t)^2-\rho^2}} \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^\infty \left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_{2(R\lambda(t)+1)}^\infty \left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR \\
&+C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{2(R\lambda(t)+1)} \left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR
\end{aligned} \tag{5.125}$$

When $\rho > 2(R\lambda(t)+1)$, we have

$$\begin{aligned}
&|1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}| \\
&=|1+\frac{(-1-\frac{1}{\rho^2}+\frac{R^2\lambda(t)^2}{\rho^2})}{\sqrt{1-2(\frac{R^2\lambda(t)^2}{\rho^2}-\frac{1}{\rho^2})+\frac{(R^2\lambda(t)^2+1)^2}{\rho^4}}}| \leqslant C\frac{(R\lambda(t)+1)^2}{\rho^2}
\end{aligned}$$

If $\rho < 2(R\lambda(t)+1)$, then, we use

$$|1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}| \leqslant 2$$

to continue the estimate in (5.125)

$$\begin{aligned}
&\int_t^\infty |L_a(s-t,\lambda(t))|ds \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_{2(R\lambda(t)+1)}^\infty \left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR \\
&+C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{2(R\lambda(t)+1)} \left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_{2(R\lambda(t)+1)}^\infty \frac{(R\lambda(t)+1)^2}{\rho^2}d\rho dR + C\int_0^\infty \frac{R}{(1+R^2)^3}(R\lambda(t)+1)dR \\
&\leqslant C
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&|\int_t^\infty L_b(s-t,\lambda(t))ds| \\
&\leqslant \int_0^\infty \frac{R}{(1+R^2)^3}\int_0^\infty \rho\left(\frac{8R^2\lambda(t)^2\rho^2}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}\right)\int_{\rho+t}^\infty \frac{1}{\sqrt{(s-t)^2-\rho^2}}\frac{dsd\rho dR}{(s-t)} \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^\infty \frac{R^2\lambda(t)^2\rho^2}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}d\rho dR \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_{2(R\lambda(t)+1)}^\infty \frac{R^2\lambda(t)^2\rho^2}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}d\rho dR \\
&+C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{2(R\lambda(t)+1)} \frac{R^2\lambda(t)^2\rho^2}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}d\rho dR
\end{aligned}
\tag{5.126}
$$

When $\rho > 2(R\lambda(t)+1)$, we have

$$
\frac{1}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}} \leqslant \frac{C}{\rho^6}
$$

When $\rho < 2(R\lambda(t)+1)$, we use

$$
\frac{1}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}} \leqslant \frac{C}{R^3\lambda(t)^3}
$$

and then continue the estimate from (5.126):

$$
\begin{aligned}
&|\int_t^\infty L_b(s-t,\lambda(t))ds| \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}\int_{2(R\lambda(t)+1)}^\infty \frac{R^2\lambda(t)^2\rho^2}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}d\rho dR \\
&+C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{2(R\lambda(t)+1)} \frac{R^2\lambda(t)^2\rho^2}{(4R^2\lambda(t)^2+(1-R^2\lambda(t)^2+\rho^2)^2)^{3/2}}d\rho dR \\
&\leqslant C\int_0^\infty \frac{R}{(1+R^2)^3}R^2\lambda(t)^2\int_{2(R\lambda(t)+1)}^\infty \frac{\rho^2}{\rho^6}d\rho dR \\
&+C\int_0^\infty \frac{R}{(1+R^2)^3}\int_0^{2(R\lambda(t)+1)} \frac{R^2\lambda(t)^2\rho^2}{R^3\lambda(t)^3}d\rho dR \\
&\leqslant \frac{C}{\lambda(t)}
\end{aligned}
$$

Then, we combine the above estimates to get

$$
\int_t^\infty |\partial_1 L(s-t,\lambda(t))|ds \leqslant \frac{C}{\lambda(t)}
$$

So,

$$|III| \leqslant C\frac{\log^{3b-1}(t)}{t^2}$$

We finally conclude that

$$|I|+|II|+|III|+|IV| \leqslant \frac{C\log^{3b-1}(t)}{t^2}$$

Next, we estimate the last two lines of (5.124). First, we note that

$$\begin{aligned}&\partial_t\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)f|_{r=R\lambda(t)},\phi_0\rangle\\&=\int_0^\infty\partial_t\left(\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)})-1}{r^2}\right)\frac{\phi_0(\frac{r}{\lambda(t)})}{\lambda(t)^2}\right)f(t,r)rdr\\&+\int_0^\infty\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2}\right)\frac{\phi_0(\frac{r}{\lambda(t)})}{\lambda(t)^2}\partial_t f(t,r)rdr\end{aligned}$$

which gives

$$\begin{aligned}|\partial_t\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)f|_{r=R\lambda(t)},\phi_0\rangle|\leqslant C\int_0^\infty\frac{r|\lambda'(t)|}{(r^2+\lambda(t)^2)^3}|f(t,r)|rdr\\+C\int_0^\infty\frac{r\lambda(t)}{(r^2+\lambda(t)^2)^3}|\partial_t f(t,r)|rdr\end{aligned} \tag{5.127}$$

We now proceed to estimate $\partial_t v_k, \quad k=1,3,4,5$. We first prove a preliminary estimate on these quantities in order to obtain the preliminary estimate on λ'''. Once this is done, we can improve our estimates for $\partial_t v_k$.

Lemma 5.14 (Preliminary Estimates on $\partial_t v_k, \quad k=1,3,4,5$)**.** *We have the following* preliminary *estimates on* $\partial_t v_k, \quad k=1,3,4,5$.

$$\begin{aligned}|\partial_t v_1(t,r)|&\leqslant\frac{Cr}{t^2\log^{b+1}(t)}\\|\partial_t v_3(t,r)|&\leqslant\frac{Cr}{t^2\log^{1+b\alpha}(t)}\end{aligned} \tag{5.128}$$

$$|\partial_t v_4(t,r)|\leqslant\frac{C}{t^2\log^{3b+2N-1}(t)},\quad r\leqslant\frac{t}{2} \tag{5.129}$$

$$|\partial_t v_5(t,r)|\leqslant\frac{C}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)}+\frac{Cr\log(t)}{t^4\log^b(t)},\quad r\leqslant\frac{t}{2} \tag{5.130}$$

Proof. For $\partial_t v_3$, we have

$$\begin{aligned}
&\partial_t v_3(t,r)\\
&= \frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \lambda''(s) \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s))\right) d\rho ds\\
&- \frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \lambda''(s) \left(\frac{8\rho^2 r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}\right.\\
&+ \left.\frac{-8\lambda(s)^{4\alpha-4}\rho^2 r^2}{(4\lambda(s)^{2\alpha-2}r^2+(1+\lambda(s)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\right) d\rho ds
\end{aligned} \tag{5.131}$$

For the first line on the right-hand side of (5.131), we first note that

$$|1-F_3(r,\rho,\lambda(s))| = 1-F_3(r,\rho,\lambda(s)) = F_3(0,\rho,\lambda(s)) - F_3(r,\rho,\lambda(s))$$

So,

$$|1-F_3(r,\rho,\lambda(s))| \leqslant Cr \int_0^1 \frac{r_\sigma \lambda(s)^{2\alpha-2}(1+\lambda(s)^{2\alpha-2}(\rho^2+r_\sigma^2))}{(1+2(\rho^2+r_\sigma^2)\lambda(s)^{2\alpha-2}+(\rho^2-r_\sigma^2)^2\lambda(s)^{4\alpha-4})^{3/2}} d\sigma$$

where

$$r_\sigma = (1-\sigma)r$$

Then, because $\lambda'(x) \leqslant 0, \quad x \geqslant T_0$, we have $\lambda(s)^{2\alpha-2} \geqslant \lambda(t)^{2\alpha-2}, \quad s \geqslant t$. So,

$$\begin{aligned}
&\left|\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho\lambda''(s)}{(s-t)\sqrt{(s-t)^2-\rho^2}} \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + 1 + F_3(r,\rho,\lambda(s)) - 1\right) d\rho ds\right|\\
&\leqslant Cr \sup_{x\geqslant t} |\lambda''(x)| \int_0^\infty \int_0^1 \frac{d\sigma d\rho}{(1+2(\rho^2+r_\sigma^2)+(\rho^2-r_\sigma^2)^2)^{1/2}}\\
&+ Cr \sup_{x\geqslant t} \left(|\lambda''(x)|\lambda(x)^{4\alpha-4}\right)\\
&\cdot \int_0^\infty \int_0^1 \frac{(\lambda(t)^{2-2\alpha}+\rho^2+r_\sigma^2)}{(1+2(\rho^2+r_\sigma^2)\lambda(t)^{2\alpha-2}+(\rho^2-r_\sigma^2)^2\lambda(t)^{4\alpha-4})^{3/2}} d\sigma d\rho
\end{aligned}$$

and where we obtained the second, third and fourth lines by switching the s and ρ integration order, and using

$$\int_{\rho+t}^\infty \frac{ds}{(s-t)\sqrt{(s-t)^2-\rho^2}} \leqslant \frac{C}{\rho}$$

So, it suffices to estimate

$$B_3(y) = \int_0^\infty \frac{d\rho}{(1+2(\rho^2+y^2)+(\rho^2-y^2)^2)^{1/2}} \leqslant C\int_0^{2y} \frac{d\rho}{y} + \int_{2y}^\infty \frac{d\rho}{(1+\rho^2)} \leqslant C \tag{5.132}$$

This gives

$$|\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho\lambda''(s)}{(s-t)\sqrt{(s-t)^2-\rho^2}}\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+1+F_3(r,\rho,\lambda(s))-1\right)d\rho ds|$$
$$\leqslant C\frac{r}{t^2\log^{b+1}(t)}+C\frac{r}{t^2\log^{1+b\alpha}(t)}$$

We then consider the second line on the right-hand side of (5.131), and use the same procedure as above:

$$\begin{aligned}&|\frac{-1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)}\frac{\lambda''(s)}{\sqrt{(s-t)^2-\rho^2}}\Big(\frac{8\rho^2r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}\\&\qquad-\frac{8\lambda(s)^{4\alpha-4}\rho^2r^2}{(4\lambda(s)^{2\alpha-2}r^2+(1+\lambda(s)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\Big)d\rho ds|\\
&\leqslant Cr\int_0^\infty\left(\frac{\rho^2\left(\sup_{x\geqslant t}|\lambda''(x)|\right)}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}+\frac{\rho^2\sup_{x\geqslant t}\left(|\lambda''(x)|\lambda(x)^{4\alpha-4}\right)}{(1+2\lambda(t)^{2\alpha-2}(r^2+\rho^2)+\lambda(t)^{4\alpha-4}(\rho^2-r^2)^2)^{3/2}}\right)d\rho\\
&\leqslant Cr\int_0^\infty \frac{1}{t^2\log^{b+1}(t)(1+2(\rho^2+r^2)+(\rho^2-r^2)^2)^{1/2}}d\rho\\
&+Cr\int_0^\infty \frac{\lambda(t)^{3-3\alpha}dq}{t^2\log^{b+1}(t)\log^{-4b+4\alpha b}(t)(1+2(q^2+r^2\lambda(t)^{2\alpha-2})+(q^2-r^2\lambda(t)^{2\alpha-2})^2)^{1/2}}\\
&\leqslant Cr\frac{B_3(r)}{t^2\log^{b+1}(t)}+Cr\frac{B_3(r\lambda(t)^{\alpha-1})}{t^2\log^{b+1}(t)\log^{-b+\alpha b}(t)}\end{aligned}$$

This gives

$$\begin{aligned}&|\frac{-1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)}\frac{\lambda''(s)}{\sqrt{(s-t)^2-\rho^2}}\Big(\frac{8\rho^2r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}\\&\qquad-\frac{8\lambda(s)^{4\alpha-4}\rho^2r^2}{(4\lambda(s)^{2\alpha-2}r^2+(1+\lambda(s)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\Big)d\rho ds|\\
&\leqslant C\frac{r}{t^2\log^{b+1}(t)}+C\frac{r}{t^2\log^{1+b\alpha}(t)}\end{aligned}$$

In total, we get (5.128).

Next, we have to estimate $\partial_t v_4$ in the region $r \leqslant \frac{t}{2}$. Again, we write the formula for v_4 as

$$\begin{aligned}&v_4(t,r)\\&=\frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{v_{4,c}(s,\sqrt{r^2+2r\rho\cos(\theta)+\rho^2})}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}}(r+\rho\cos(\theta))d\theta d\rho ds\end{aligned}$$

Like done previously for v_4, we introduce $x = r\mathbf{e}_1 \in \mathbb{R}^2$ to ease notation, and get

$$\partial_t v_4(t,r) = -\int_t^\infty \frac{v_{4,s}(t,r)}{(s-t)}ds - \frac{1}{2\pi}\int_t^\infty\int_{B_{s-t}(0)}\frac{\text{integrand}_{v_{4,1}}}{\sqrt{(s-t)^2-|y|^2}}dA(y)ds \tag{5.133}$$

where we recall

$$v_{4,s}(t,r) = \frac{-1}{2\pi}\int_{B_{s-t}(0)} \frac{1}{\sqrt{(s-t)^2-|y|^2}} \frac{v_{4,c}(s,|x+y|)\left((x+y)\cdot\widehat{x}\right)}{|x+y|}dA(y)$$

$$\begin{aligned}\text{integrand}_{v_{4,1}} &= \frac{-\partial_2 v_{4,c}(s,|x+y|)}{|x+y|^2(s-t)}\left((x+y)\cdot y\right)\left(\hat{x}\cdot(x+y)\right)\\ &+ \frac{v_{4,c}(s,|x+y|)}{(s-t)|x+y|}\left(-y\cdot\hat{x}+\frac{\left(\hat{x}\cdot(x+y)\right)\left(y\cdot(y+x)\right)}{|x+y|^2}\right)\end{aligned}$$

So,

$$|\text{integrand}_{v_{4,1}}| \leqslant C|\partial_2 v_{4,c}(s,|x+y|)| + \frac{C|v_{4,c}(s,|x+y|)|}{|x+y|}$$

This is the same estimate obtained when estimating the integrand of v_4, except for a factor of $\frac{1}{r}$. So, the second term of (5.133) is estimated as follows:

$$|-\frac{1}{2\pi}\int_t^\infty\int_{B_{s-t}(0)}\frac{\text{integrand}_{v_{4,1}}}{\sqrt{(s-t)^2-|y|^2}}dA(y)ds| \leqslant \frac{C}{t^2\log^{3b+2N-1}(t)}, \quad r\leqslant\frac{t}{2}$$

For the first term in (5.133), we have

$$\begin{aligned}&|\int_t^\infty \frac{v_{4,s}(t,r)}{(s-t)}ds|\\ &\leqslant C\int_t^\infty \frac{1}{(s-t)}\int_{B_{s-t}(0)}\frac{|v_{4,c}(s,|x+y|)|}{\sqrt{(s-t)^2-|y|^2}}\left(\mathbb{1}_{B_{\frac{s}{2}}(-x)}(y)+\mathbb{1}_{(B_{\frac{s}{2}}(-x))^c}(y)\right)dA(y)ds\\ &\leqslant C\int_t^\infty\int_0^{s-t}\rho\int_0^{2\pi}\frac{1}{(s-t)\sqrt{(s-t)^2-\rho^2}}\frac{1}{s^2\log^{N+3b}(s)}\frac{d\theta d\rho ds}{(\log^{2N}(s)+r^2+\rho^2+2r\rho\cos(\theta))}\\ &+C\int_t^\infty\int_0^{s-t}\rho\int_0^{2\pi}\frac{1}{(s-t)\sqrt{(s-t)^2-\rho^2}}\frac{\log(s)}{\log^{2b}(s)s^2 t}\frac{1}{(s^2+\rho^2+r^2+2rp\cos(\theta))}d\theta d\rho ds\end{aligned} \tag{5.134}$$

We first estimate the third line of (5.134):

$$
\begin{aligned}
&|\int_t^\infty \int_0^{s-t} \rho \int_0^{2\pi} \frac{1}{(s-t)\sqrt{(s-t)^2-\rho^2}} \frac{1}{s^2 \log^{N+3b}(s)} \frac{d\theta d\rho ds}{(\log^{2N}(s)+r^2+\rho^2+2r\rho\cos(\theta))}| \\
&\leqslant C \int_0^\infty \rho \int_0^{2\pi} \frac{1}{(t+\rho)^2 \log^{3b+N}(t)(\log^{2N}(t)+\rho^2+r^2+2r\rho\cos(\theta))} \\
&\qquad\qquad \int_{\rho+t}^\infty \frac{ds d\theta d\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \\
&\leqslant C \int_0^\infty \frac{1}{(t+\rho)^2 \log^{3b+N}(t)} \int_0^{2\pi} \frac{d\theta d\rho}{(\log^{2N}(t)+\rho^2+r^2+2r\rho\cos(\theta))} \\
&\leqslant C \int_0^\infty \frac{1}{(t+\rho)^2 \log^{3b+N}(t)} \frac{d\rho}{\sqrt{(\log^{2N}(t)+(r+\rho)^2)(\log^{2N}(t)+(r-\rho)^2)}} \\
&\leqslant \frac{C}{\log^{3b+N}(t)} \int_0^\infty \frac{1}{(t+\rho)^2} \frac{1}{\sqrt{\log^{2N}(t)+(r+\rho)^2}} \frac{d\rho}{\log^N(t)} \\
&\leqslant \frac{C}{t^2 \log^{3b+2N-1}(t)}
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&|\int_t^\infty \int_0^{s-t} \rho \int_0^{2\pi} \frac{1}{(s-t)\sqrt{(s-t)^2-\rho^2}} \frac{\log(s) d\theta d\rho ds}{\log^{2b}(s)(s^2+\rho^2+r^2+2r\rho\cos(\theta))s^2 t}| \\
&\leqslant C \int_0^\infty \rho \int_0^{2\pi} \frac{1}{\log^{2b}(t)} \frac{\log(t)}{(\rho^2+r^2+2r\rho\cos(\theta)+t^2)} \frac{1}{(\rho+t)} \frac{1}{t^2} \int_{\rho+t}^\infty \frac{ds d\theta d\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \\
&\leqslant \frac{C}{t^2 \log^{2b}(t)} \int_0^\infty \int_0^{2\pi} \frac{\log(t) d\theta d\rho}{(\rho^2+r^2+2r\rho\cos(\theta)+t^2)(\rho+t)} \\
&\leqslant \frac{C \log(t)}{t^3 \log^{2b}(t)} \int_0^\infty \frac{d\rho}{(\rho+t)\sqrt{t^2+\rho^2}} \\
&\leqslant \frac{C \log(t)}{t^4 \log^{2b}(t)}
\end{aligned}
$$

Combining these, we get (5.129).

Now, we are ready to estimate $\partial_t v_5$. Again, we have

$$
\begin{aligned}
\partial_t v_5 = &\frac{1}{2\pi} \int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)} \frac{1}{\sqrt{(s-t)^2-\rho^2}} \\
&\qquad \int_0^{2\pi} \frac{N_2(f_{v_5})(s, \sqrt{r^2+\rho^2+2r\rho\cos(\theta)})(r+\rho\cos(\theta))}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}} d\theta d\rho ds \\
&- \frac{1}{2\pi} \int_t^\infty \int_{B_{s-t}(0)} \frac{\text{integrand}_{v_{5,1}}}{\sqrt{(s-t)^2-|y|^2}} dA(y) ds
\end{aligned}
$$

with

$$|\text{integrand}_{v_{5,1}}| \leqslant C|\partial_2 N_2(f_{v_5})(s,|x+y|)| + C\frac{|N_2(f_{v_5})(s,|x+y|)}{|x+y|}$$

Again, this is the same estimate for the integrand which appeared in the v_5 pointwise estimates, aside from an extra factor of $\frac{1}{r}$. So, we get

$$|-\frac{1}{2\pi}\int_t^\infty \int_{B_{s-t}(0)} \frac{\text{integrand}_{v_{5,1}}}{\sqrt{(s-t)^2-|y|^2}} dA(y)ds| \leqslant \frac{C}{t^{7/2}\log^{3b-3+\frac{5N}{2}}(t)}, \quad r \leqslant \frac{t}{2}$$

Then, we need to estimate

$$\begin{aligned}
&\int_t^\infty \int_{B_{s-t}(0)} \frac{1}{\sqrt{(s-t)^2-|y|^2}} \frac{|N_2(f_{v_5})|(s,|x+y|)}{(s-t)} \left(\mathbb{1}_{B_{\frac{s}{2}}(-x)}(y) + \mathbb{1}_{(B_{\frac{s}{2}}(-x))^c}(y)\right) dA(y)ds \\
&\leqslant C\int_t^\infty \int_0^{s-t} \rho \int_0^{2\pi} \frac{1}{\sqrt{(s-t)^2-\rho^2}} \frac{1}{(s-t)} \Big(\frac{r+\rho}{(\lambda(s)^2+r^2+\rho^2+2r\rho\cos(\theta))s^4\log^{3b}(s)} \\
&\qquad + \frac{s}{s^6\log^{3b}(s)}\Big) d\theta d\rho ds \\
&+ C\int_t^\infty \int_0^{s-t} \rho \int_0^{2\pi} \frac{1}{\sqrt{(s-t)^2-\rho^2}} \frac{1}{(s-t)} \left(\frac{\log^3(s)}{s^2t^3} + \frac{1}{s^6\log^{3N+7b-2}(s)}\right) d\theta d\rho ds \\
&\leqslant C\int_0^\infty \rho \int_0^{2\pi} \frac{1}{\rho} \frac{(r+\rho)}{(1+r^2+\rho^2+2r\rho\cos(\theta))} \frac{1}{\log^b(t)(\rho+t)^4} d\theta d\rho + C\int_t^\infty \frac{ds}{s^5\log^{3b}(s)} \\
&+ C\int_t^\infty \left(\frac{\log^3(s)}{s^2t^3} + \frac{1}{s^6\log^{3N+7b-2}(s)}\right) ds \\
&\leqslant C\frac{(r+1)\log(t)}{t^4\log^b(t)} + C\frac{\log^3(t)}{t^4} \\
&\leqslant C\frac{r\log(t)}{t^4\log^b(t)} + C\frac{\log^3(t)}{t^4}, \quad r \leqslant \frac{t}{2}
\end{aligned}$$

In total, we have (5.130).

Finally, we estimate $\partial_t v_1$. Using the same procedure used to estimate $\partial_t v_3$, we get

$$\begin{aligned}
&\partial_t v_1(t,r) \\
&= \int_t^\infty \frac{-\lambda''(s)}{r} \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) d\rho ds \\
&+ \int_t^\infty \frac{\lambda''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{8\rho^2 r^2}{(s-t)(4r^2+(1+\rho^2-r^2)^2)^{3/2}} d\rho ds
\end{aligned}$$

So, recalling (5.132), we have

$$|\partial_t v_1(t,r)| \leqslant \frac{Cr}{t^2\log^{b+1}(t)} \int_0^1 B_3(r(1-\sigma))d\sigma + \frac{CrB_3(r)}{t^2\log^{b+1}(t)} \leqslant \frac{Cr}{t^2\log^{b+1}(t)}$$

□

Now, we return to (5.127), and get

$$|\partial_t \langle \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right) v_3|_{r=R\lambda(t)}, \phi_0\rangle| \leqslant \frac{C\log(\log(t))}{t^3\log^2(t)} + \frac{C}{t^2\log^{-b+1+b\alpha}(t)}$$

$$|\partial_t \langle \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right) v_4 \left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)}, \phi_0\rangle|$$
$$\leqslant \frac{C}{t^3\log^{2b+2N}(t)} + \frac{C}{t^2\log^{b+2N-1}(t)}$$

$$|\partial_t \langle \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right) v_5 \left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)}, \phi_0\rangle|$$
$$\leqslant \frac{C}{t^{9/2}\log^{2b-2+\frac{5N}{2}}(t)} + \frac{C}{t^{7/2}\log^{b-3+\frac{5N}{2}}(t)}$$

$$|\partial_t \langle \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right) \chi_{\geqslant 1}(\frac{2r}{\log^N(t)}) (v_1+v_2+v_3)|_{r=R\lambda(t)}, \phi_0\rangle|$$
$$\leqslant \frac{C}{t^3\log^{2b+2N+1}(t)} + \frac{C}{t^2\log^{1+b\alpha+b+2N}(t)}$$

Combining these estimates, we finally conclude

$$|RHS_2'(t)| \leqslant \frac{C}{t^2\log^{1-3b}(t)}$$

which implies

$$|\lambda'''(t)| \leqslant \frac{C}{t^2\log(\log(t))\log^{1-2b}(t)}$$

Since $\lambda(t) = \lambda_0(t) + e_0(t)$, we get

$$|e_0'''(t)| \leqslant \frac{C}{t^2\log(\log(t))\log^{1-2b}(t)} \tag{5.135}$$

Now that we have this preliminary estimate, we return to (5.123), and write

$$\lambda(t) = \lambda_{0,0}(t) + e(t)$$

Note that the function e defined in the above expression is different from the function e which appeared in the construction of λ as a solution to the modulation equation. It will suffice for our purposes to have an estimate of the form $|e'''(t)| \leqslant C|\lambda_{0,0}'''(t)|$, and this is why we will not need to use the slightly more complicated decomposition

$$\lambda(t) = \lambda_0(t) + e_0(t) = \lambda_{0,0}(t) + \lambda_{0,1}(t) + e_0(t)$$

Then, we differentiate (5.123), and write the equation in the following way, with any differentiations under the integral sign justified by the preliminary estimate on e'''. (Note that $e'''(t) = e_0'''(t) + \lambda_{0,1}'''(t)$).

$$\begin{aligned}
&-4\int_t^\infty \frac{e'''(s)ds}{\log(\lambda_{0,0}(s))(1+s-t)} - 4\int_t^\infty \frac{e'''(s)ds}{(1+s-t)^3\log(\lambda_{0,0}(s))(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\\
&+4\alpha e'''(t)\\
&= -4\alpha\lambda_{0,0}'''(t) + \frac{-4(1-\alpha)\lambda(t)^{-\alpha}\lambda'(t)}{\log(\lambda_{0,0}(t))}\int_t^\infty \frac{e''(s)ds}{(\lambda(t)^{1-\alpha}+s-t)^2(1+s-t)^3}\\
&+\frac{1}{\log(\lambda_{0,0}(t))}\left(\frac{-4\alpha\lambda'(t)\lambda''(t)}{\lambda(t)} + \partial_t G(t,\lambda(t)) + 4\partial_t\left(\int_0^\infty \frac{\lambda_{0,0}''(t+w)dw}{(\lambda(t)^{1-\alpha}+w)(1+w)^3}\right)\right)\\
&+\frac{1}{\log(\lambda_{0,0}(t))}\left(-E_{\lambda_{0,0}}'(t) + 4\alpha(\log(\lambda_{0,0}(t)) - \log(\lambda(t)))\lambda'''(t)\right) \qquad (5.136)\\
&-\frac{4}{\log(\lambda_{0,0}(t))}\int_t^\infty \frac{e'''(s)}{(1+s-t)^3}\left(\frac{1}{\lambda_{0,0}(t)^{1-\alpha}+s-t} - \frac{1}{\lambda(t)^{1-\alpha}+s-t}\right)ds\\
&-4\int_t^\infty \frac{e'''(s)}{1+s-t}\left(\frac{1}{\log(\lambda_{0,0}(s))} - \frac{1}{\log(\lambda_{0,0}(t))}\right)ds\\
&-4\int_t^\infty \frac{e'''(s)}{(1+s-t)^3(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\left(\frac{1}{\log(\lambda_{0,0}(s))} - \frac{1}{\log(\lambda_{0,0}(t))}\right)ds\\
&:= RHS_3(t)
\end{aligned}$$

We now proceed to estimate RHS_3, starting with the terms which do not involve G.

$$|-4\alpha\lambda_{0,0}'''(t)| \leqslant \frac{C}{t^3\log^{b+1}(t)}$$

$$\begin{aligned}
&\left|\frac{-4(1-\alpha)\lambda(t)^{-\alpha}\lambda'(t)}{\log(\lambda_{0,0}(t))}\int_t^\infty \frac{e''(s)ds}{(\lambda(t)^{1-\alpha}+s-t)^2(1+s-t)^3}\right|\\
&\leqslant \frac{C}{t^3\log^{b+2}(t)(\log(\log(t)))^{3/2}}
\end{aligned}$$

$$\left|\frac{-4\alpha\lambda'(t)\lambda''(t)}{\log(\lambda_{0,0}(t))\lambda(t)}\right| \leqslant \frac{C}{t^3\log^{b+2}(t)\log(\log(t))}$$

$$\begin{aligned}
&\left|\frac{4}{\log(\lambda_{0,0}(t))}\partial_t\left(\int_0^\infty \frac{\lambda_{0,0}''(t+w)dw}{(\lambda(t)^{1-\alpha}+w)(1+w)^3}\right)\right|\\
&\leqslant \frac{C}{\log(\log(t))}\left(\int_0^\infty \frac{|\lambda_{0,0}'''(t+w)|dw}{(\lambda(t)^{1-\alpha}+w)(1+w)^3} + \int_0^\infty \frac{|\lambda_{0,0}''(t+w)|\lambda(t)^{-\alpha}|\lambda'(t)|dw}{(\lambda(t)^{1-\alpha}+w)^2(1+w)^3}\right)\\
&\leqslant \frac{C}{t^3\log^{b+1}(t)}
\end{aligned}$$

By the same procedure used to estimate $E_{\lambda_{0,0}}$, we have

$$\frac{|E_{\lambda_{0,0}}'(t)|}{|\log(\lambda_{0,0}(t))|} \leqslant \frac{C}{t^3\log^{b+1}(t)\log(\log(t))}$$

In addition,

$$\begin{aligned}
&|\frac{4\alpha(\log(\lambda_{0,0}(t))-\log(\lambda(t)))\lambda'''(t)}{\log(\lambda_{0,0}(t))}|\\
&\leqslant \frac{C}{\log(\log(t))}|\log(1+\frac{e(t)}{\lambda_{0,0}(t)})|\left(\frac{1}{t^3\log^{b+1}(t)}+|e'''(t)|\right)\\
&\leqslant \frac{C}{(\log(\log(t)))^{3/2}}\left(\frac{1}{t^3\log^{b+1}(t)}+|e'''(t)|\right)
\end{aligned}$$

$$\begin{aligned}
&|\frac{-4}{\log(\lambda_{0,0}(t))}\int_t^\infty \frac{e'''(s)}{(1+s-t)^3}\left(\frac{1}{\lambda_{0,0}(t)^{1-\alpha}+s-t}-\frac{1}{\lambda(t)^{1-\alpha}+s-t}\right)ds|\\
&\leqslant \frac{C}{\log(\log(t))}\int_t^\infty \frac{|e'''(s)|}{(1+s-t)^3}\frac{|\lambda(t)^{1-\alpha}-\lambda_{0,0}(t)^{1-\alpha}|}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)^2}ds\\
&\leqslant \frac{C\sup_{x\geqslant t}|e'''(x)|}{\log(\log(t))}\int_t^\infty \frac{1}{\sqrt{\log(\log(t))}}\frac{ds}{(1+s-t)^3(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\\
&\leqslant \frac{C\sup_{x\geqslant t}|e'''(x)|}{\sqrt{\log(\log(t))}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|-4\int_t^\infty \frac{e'''(s)}{(1+s-t)}\left(\frac{1}{\log(\lambda_{0,0}(s))}-\frac{1}{\log(\lambda_{0,0}(t))}\right)ds|\\
&\leqslant \frac{C}{t(\log(\log(t)))^2\log(t)}\int_t^\infty |e'''(s)|ds \leqslant \frac{C\sup_{x\geqslant t}\left(|e'''(x)|x^{3/2}\right)}{t^{3/2}(\log(\log(t)))^2\log(t)}
\end{aligned}$$

Finally,

$$\begin{aligned}
&|-4\int_t^\infty \frac{e'''(s)}{(1+s-t)^3(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\left(\frac{1}{\log(\lambda_{0,0}(s))}-\frac{1}{\log(\lambda_{0,0}(t))}\right)ds|\\
&\leqslant \frac{C\sup_{x\geqslant t}|e'''(x)|}{t\log(t)(\log(\log(t)))^2}
\end{aligned}$$

Now, we start to estimate the terms arising from $\partial_t G(t,\lambda(t))$:

$$|\partial_t\left(\lambda(t)E_{0,1}(\lambda(t),\lambda'(t),\lambda''(t))\right)|\leqslant \frac{C}{t^3\log^{b+1}(t)}+C|e'''(t)|$$

$$\begin{aligned}
&\partial_t\left(-16\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))\right)ds\right)\\
&=-16\int_t^\infty \lambda'''(s)\left(K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))\right)ds\\
&-16\int_t^\infty \lambda''(s)\left(\partial_2(K_3-K_{3,0})(s-t,\lambda(t))\right)\lambda'(t)ds
\end{aligned} \tag{5.137}$$

For the first line of the right-hand side of (5.137), we have

$$\begin{aligned}
&|-16\int_t^\infty \lambda'''(s)\left(K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))\right)ds| \\
&\leqslant C\sup_{x\geqslant t}|\lambda'''(x)|\int_t^\infty |K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))|ds \\
&\leqslant \frac{C}{t^3\log^{b+1}(t)}+C\sup_{x\geqslant t}|e'''(x)|
\end{aligned}$$

where we recall that $\int_t^\infty |K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))|ds$ was estimated in the v_3 inner product subsection. For the second line of the right-hand side of (5.137), we start with

$$\begin{aligned}
&K_3(w,\lambda(t))-K_{3,0}(w,\lambda(t)) \\
&=\left(\frac{w}{1+w^2}-\frac{w}{\lambda(t)^{2-2\alpha}+w^2}\right)\frac{w^4}{4(w^2+36\lambda(t)^2)^2}+\frac{1}{4(\lambda(t)^{1-\alpha}+w)(1+w)^3}
\end{aligned}$$

Then, we get

$$\begin{aligned}
|\partial_2(K_3-K_{3,0})(w,\lambda(t))| \leqslant\, & \frac{C\lambda(t)^\alpha}{(1+w)^3(\lambda(t)+\lambda(t)^\alpha w)^2} \\
&+\frac{C\lambda(t)^{1+2\alpha}w^5}{(36\lambda(t)^2+w^2)^2(\lambda(t)^2+\lambda(t)^{2\alpha}w^2)^2} \\
&+\frac{C\lambda(t)w^5}{(36\lambda(t)^2+w^2)^3}|\frac{1}{1+w^2}-\frac{1}{(\lambda(t)^{2-2\alpha}+w^2)}|
\end{aligned}$$

Then, we note that

$$\int_0^\infty \frac{\lambda(t)^{-\alpha}dw}{(1+w)^3(\lambda(t)^{1-\alpha}+w)^2}\leqslant C\log(\log(t))\log^b(t)$$

$$\int_0^\infty \frac{\lambda(t)^{1-2\alpha}w^5dw}{(36\lambda(t)^2+w^2)^2(\lambda(t)^{2-2\alpha}+w^2)^2}\leqslant \frac{C}{\lambda(t)}$$

$$\begin{aligned}
&\int_0^\infty \frac{\lambda(t)w^5}{(36\lambda(t)^2+w^2)^3}|\frac{1}{1+w^2}-\frac{1}{(\lambda(t)^{2-2\alpha}+w^2)}|dw \\
&\leqslant \frac{C\log(\log(t))}{\lambda(t)}
\end{aligned}$$

This gives

$$\begin{aligned}
&|\partial_t\left(-16\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t))-K_{3,0}(s-t,\lambda(t))\right)ds\right)| \\
&\leqslant C\sup_{x\geqslant t}|e'''(x)|+\frac{C}{t^3\log^{b+1}(t)}
\end{aligned}$$

As mentioned before, some terms arising in $\partial_t G(t, \lambda_0(t) + e(t))$ will be treated differently, now that we have the preliminary estimate on e'''. We start with the term

$$\begin{aligned}
&\partial_t \left(\frac{16}{\lambda(t)^2} \int_t^\infty \lambda''(s) \left(K_1(s-t, \lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)} \right) ds \right) \\
&= \frac{-32\lambda'(t)}{\lambda(t)^3} \int_t^\infty \lambda''(s) \left(K_1(s-t, \lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)} \right) ds \\
&+ \frac{16}{\lambda(t)^2} \int_t^\infty \lambda'''(s) \left(K_1(s-t, \lambda(t)) - \frac{\lambda(t)^2}{4(1+s-t)} \right) ds \\
&+ \frac{16}{\lambda(t)^2} \int_t^\infty \lambda''(s) \left(\partial_2 K_1(s-t, \lambda(t))\lambda'(t) - \frac{\lambda(t)\lambda'(t)}{2(1+s-t)} \right) ds
\end{aligned}$$

For $s - t \leqslant 1$, we estimate $\partial_2 K_1$ as follows:

$$\begin{aligned}
&|\partial_2 K_1(s-t, \lambda(t))| \\
&\leqslant \int_0^\infty \frac{R}{(1+R^2)^3} \int_0^{s-t} \frac{\rho}{(s-t)} \frac{4\lambda(t)R^2\left(1+\rho^2+R^2\lambda(t)^2\right)}{(4\lambda(t)^2R^2 + (1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}} d\rho dR
\end{aligned}$$

We then note that

$$4R^2\lambda(t)^2 + (1+\rho^2-R^2\lambda(t)^2)^2 = (1+(\rho+R\lambda(t))^2)(1+(\rho-R\lambda(t))^2)$$

So,

$$\left|\frac{4\lambda(t)R^2\left(1+\rho^2+R^2\lambda(t)^2\right)}{(4\lambda(t)^2R^2 + (1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}}\right| \leqslant CR^2\lambda(t)$$

Then, we obtain

$$|\partial_2 K_1(s-t, \lambda(t))| \leqslant C\lambda(t)(s-t), \quad s-t \leqslant 1$$

From here, we get

$$\int_t^{t+1} |\lambda''(x)| |\partial_2 K_1(x-t, \lambda(t)) - \frac{\lambda(t)}{2(1+x-t)}| |\lambda'(t)| dx \leqslant \frac{C}{t^3 \log^{3b+2}(t)}$$

For $s - t \geqslant 1$, we use

$$\begin{aligned}
&\partial_2 K_1(s-t, \lambda(t)) \\
&= \int_0^\infty \frac{R}{(1+R^2)^3} \int_0^\infty \frac{\rho}{(s-t)} \frac{4\lambda(t)R^2(1+\rho^2+\lambda(t)^2R^2)}{(4\lambda(t)^2R^2 + (1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}} d\rho dR \\
&- \int_0^\infty \frac{R}{(1+R^2)^3} \int_{s-t}^\infty \frac{\rho}{(s-t)} \frac{4\lambda(t)R^2(1+\rho^2+R^2\lambda(t)^2)}{(4R^2\lambda(t)^2 + (1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}} d\rho dR \\
&= \frac{\lambda(t)}{2(s-t)} - \int_0^\infty \frac{R}{(1+R^2)^3} \int_{s-t}^\infty \frac{\rho}{(s-t)} \frac{4\lambda(t)R^2(1+\rho^2+R^2\lambda(t)^2)}{(4R^2\lambda(t)^2 + (1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}} d\rho dR
\end{aligned}$$

Then,

$$\int_{t+1}^{\infty}|\lambda''(s)||\partial_2 K_1(s-t,\lambda(t))-\frac{\lambda(t)}{2(1+s-t)}|ds|\lambda'(t)|$$
$$\leqslant \frac{C}{t^3\log^{2b+2}(t)}\int_{t+1}^{\infty}|\partial_2 K_1(s-t,\lambda(t))-\frac{\lambda(t)}{2(1+s-t)}|ds$$
$$\leqslant \frac{C}{t^3\log^{2b+2}(t)}\int_1^{\infty}\left(\frac{\lambda(t)}{2w}-\frac{\lambda(t)}{2(1+w)}\right)dw$$
$$+\frac{C}{t^3\log^{2b+2}(t)}\int_0^{\infty}\frac{R}{(1+R^2)^3}\int_1^{\infty}\rho\log(\rho)\frac{4\lambda(t)R^2(1+\rho^2+R^2\lambda(t)^2)}{(4R^2\lambda(t)^2+(1+\rho^2-R^2\lambda(t)^2)^2)^{3/2}}d\rho dR$$
$$\leqslant \frac{C}{t^3\log^{3b+2}(t)}+\frac{C}{t^3\log^{3b+2}(t)}\int_0^{\infty}\frac{R^3}{(1+R^2)^3}\log(2+R\lambda(t))dR$$
$$\leqslant \frac{C}{t^3\log^{3b+2}(t)}$$

We also have

$$|\frac{-32\lambda'(t)}{\lambda(t)^3}\int_t^{\infty}\lambda''(s)\left(K_1(s-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+s-t)}\right)ds|$$
$$\leqslant \frac{C}{t^3\log^{2-b}(t)}\int_t^{\infty}|K_1(s-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+s-t)}|ds$$
$$\leqslant \frac{C}{t^3\log^{b+2}(t)}$$

Finally, we have

$$|\frac{16}{\lambda(t)^2}\int_t^{\infty}\lambda'''(x)\left(K_1(x-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+x-t)}\right)dx|$$
$$\leqslant \frac{C}{\lambda(t)^2}\sup_{x\geqslant t}(|\lambda'''(x)|)\int_t^{\infty}|K_1(x-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+x-t)}|dx$$
$$\leqslant C\sup_{x\geqslant t}|\lambda'''(x)|$$

Combining these, we get

$$|\partial_t\left(\frac{16}{\lambda(t)^2}\int_t^{\infty}\lambda''(s)\left(K_1(s-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+s-t)}\right)ds\right)|$$
$$\leqslant \frac{C}{t^3\log^{b+1}(t)}+C\sup_{x\geqslant t}|e'''(x)|$$

Next, we consider

$$\partial_t\left(\frac{16}{\lambda(t)^2}\int_t^{\infty}\lambda''(x)K(x-t,\lambda(t))dx\right)$$
$$=\frac{-32\lambda'(t)}{\lambda(t)^3}\int_t^{\infty}\lambda''(x)K(x-t,\lambda(t))dx+\frac{16}{\lambda(t)^2}\int_t^{\infty}\lambda'''(x)K(x-t,\lambda(t))dx$$
$$+\frac{16}{\lambda(t)^2}\int_t^{\infty}\lambda''(x)\partial_2 K(x-t,\lambda(t))\lambda'(t)dx$$

Note that

$$\begin{aligned}|\frac{-32\lambda'(t)}{\lambda(t)^3}\int_t^\infty \lambda''(x)K(x-t,\lambda(t))dx| &\leqslant \frac{C\log^{3b}(t)}{t^3\log^{2b+2}(t)}\int_t^\infty |K(x-t,\lambda(t))|dx\\ &\leqslant \frac{C}{t^3\log^{b+2}(t)}\end{aligned}$$

Next,

$$\begin{aligned}|\frac{16}{\lambda(t)^2}\int_t^\infty \lambda'''(x)K(x-t,\lambda(t))dx| &\leqslant C\sup_{x\geqslant t}|\lambda'''(x)|\\ &\leqslant \frac{C}{t^3\log^{b+1}(t)}+C\sup_{x\geqslant t}|e'''(x)|\end{aligned}$$

Finally, the integral

$$\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(s)\partial_2 K(s-t,\lambda(t))\lambda'(t)ds$$

was treated during the preliminary estimates (it is equal to $\lambda(t)\cdot I$ in the notation of that section). So, in total, we get

$$|\partial_t\left(\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)K(x-t,\lambda(t))dx\right)|\leqslant \frac{C}{t^3\log^{b+1}(t)}+C\sup_{x\geqslant t}|e'''(x)|$$

Now, we return to (5.77) to prove an estimate on $\partial_t v_3$ which will be sufficient to estimate $\partial_t v_4$. We start with

$$\begin{aligned}&\partial_t v_3(t,r)\\ &=\frac{-1}{r}\int_0^\infty\int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}}\lambda'''(w+t)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho dw\\ &-\frac{1}{r}\int_0^\infty\int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}}\lambda''(t+w)\partial_3F_3(r,\rho,\lambda(t+w))\lambda'(t+w)d\rho dw\end{aligned} \tag{5.138}$$

The first line of (5.138) is treated in the same manner as v_3 was, while obtaining (5.77). In particular, the analog of $v_{3,2}$ which appeared in an intermediate step in obtaining (5.77) is estimated by

$$\begin{aligned}&Cr\sup_{x\geqslant t}|\lambda'''(x)|+Cr\sup_{x\geqslant t}\left(|\lambda'''(x)|\lambda(x)^{\alpha-1}\left(\lambda(x)^{\alpha-1}-\lambda(t)^{\alpha-1}\right)\right)\lambda(t)^{2-2\alpha}\\ &\leqslant \frac{Cr}{t^3\log^{b+1}(t)}+\frac{Cr}{t\log^{b-b\alpha}(t)}\sup_{x\geqslant t}\left(\frac{|e'''(x)|x}{\lambda(x)^{1-\alpha}}\right)\end{aligned}$$

while the analog of $v_{3,1}$ is estimated by

$$Cr\sup_{x\geqslant t}\left(\frac{|\lambda'''(x)|}{\lambda(x)^{2-2\alpha}}\right)\log(\log(t))\lambda(t)^{2-2\alpha}$$

To treat the second line of (5.138), we start with

$$|\partial_3 F_3(r,\rho,\lambda(s))| \leqslant \frac{Cr^2\lambda(s)^{2\alpha-3}}{(1+\lambda(s)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda(s)^{2\alpha-2}(\rho^2+r^2))}$$

Then, we get

$$\begin{aligned}&|\frac{-1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)}\lambda''(s)\partial_3 F_3(r,\rho,\lambda(s))\lambda'(s)d\rho ds|\\
&\leqslant \frac{C}{r}\int_t^{t+1} \frac{1}{(s-t)}\frac{1}{s^3\log^{2b+2}(s)}\int_0^{s-t}\rho\frac{r^2\lambda(s)^{2\alpha-3}}{(1+2\lambda(s)^{2\alpha-2}\rho^2)}d\rho ds\\
&+\frac{C}{r}\int_{t+1}^\infty \frac{1}{s^3\log^{2b+2}(s)(s-t)}\int_0^{s-t}\rho|\partial_3 F_3(r,\rho,\lambda(s))|d\rho ds\\
&\leqslant \frac{Cr}{t^3\log^{b+1}(t)}\end{aligned}$$

and

$$\begin{aligned}&|\frac{-1}{r}\int_t^\infty\int_0^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\lambda''(s)\partial_3 F_3(r,\rho,\lambda(s))\lambda'(s)d\rho ds|\\
&\leqslant Cr\int_0^\infty \frac{\rho d\rho}{(1+\lambda(t)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda(t)^{2\alpha-2}(\rho^2+r^2))}\frac{\log^{(3-2\alpha)b}(t)}{t^3\log^{2b+2}(t)}\\
&\leqslant \frac{Cr}{t^3\log^{b+2}(t)}\end{aligned}$$

Combining these, we get

$$|\partial_t v_3(t,r)| \leqslant \frac{Cr\log(\log(t))}{t^3\log^{b+1}(t)}+\frac{Cr}{t}\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{2-2\alpha}}\right)\log(\log(t))\lambda(t)^{2-2\alpha} \tag{5.139}$$

Next, we estimate $\partial_t E_5$. First, we recall the definition of E_5:

$$\begin{aligned}&E_5(t,r)\\
&=-\frac{1}{r}\int_t^{t+6r}\int_0^{s-t}\frac{\rho}{(s-t)}\lambda''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds\\
&-\frac{1}{r}\int_{6r}^\infty\left(\int_0^w \frac{\rho}{w}\lambda''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho\right.\\
&\left.-\frac{\lambda''(t+w)}{w}r^2w^2\left(\frac{1}{1+w^2}-\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)}\right)\right)dw\\
&-r\int_{t+6r}^\infty \lambda''(s)(s-t)\left(\frac{1}{(\lambda(t)^{2-2\alpha}+(s-t)^2)}-\frac{1}{\lambda(s)^{2-2\alpha}+(s-t)^2}\right)ds\\
&+v_{3,2}(t,r)\end{aligned} \tag{5.140}$$

We then estimate the time derivative of each of the lines of (5.140). For the first line, we have

$$\begin{aligned}
&|\partial_t\left(-\frac{1}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}\lambda''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho dw\right)|\\
&\leqslant \frac{C}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}|\lambda'''(t+w)|\cdot|\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))|d\rho dw\\
&+\frac{C}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}|\lambda''(t+w)\partial_3 F_3(r,\rho,\lambda(t+w))\lambda'(t+w)|d\rho dw\\
&\leqslant \frac{C}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}|\lambda'''(t+w)|d\rho dw+\frac{Cr}{t^3\log^{b+1}(t)}
\end{aligned}$$

where we note that the third line of the inequality above has been estimated already. Next, we have

$$\begin{aligned}
&\partial_t\left(-\frac{1}{r}\int_{6r}^{\infty}\left(\int_0^w \frac{\rho}{w}\lambda''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho\right.\right.\\
&\left.\left.-\frac{\lambda''(t+w)}{w}r^2w^2\left(\frac{1}{1+w^2}-\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)}\right)\right)dw\right)\\
&=\frac{1}{r}\int_{6r}^{\infty}\lambda''(t+w)r^2w\left(\frac{(2-2\alpha)}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}\lambda(t+w)^{1-2\alpha}\lambda'(t+w)\right)dw\\
&-\frac{1}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda'''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho dw\\
&+\frac{1}{r}\int_{6r}^{\infty}\lambda'''(t+w)r^2w\left(\frac{1}{1+w^2}-\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)}\right)dw\\
&-\frac{1}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda''(t+w)\partial_3 F_3(r,\rho,\lambda(t+w))\lambda'(t+w)d\rho dw
\end{aligned} \tag{5.141}$$

We get

$$\begin{aligned}
&|\frac{1}{r}\int_{6r}^{\infty}\lambda''(t+w)r^2w\left(\frac{(2-2\alpha)}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}\lambda(t+w)^{1-2\alpha}\lambda'(t+w)\right)dw|\\
&\leqslant Cr\int_{6r}^{\infty}\frac{|\lambda''(t+w)|w}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}\lambda(t+w)^{1-2\alpha}|\lambda'(t+w)|dw\\
&\leqslant \frac{Cr}{t^3\log^{b+2}(t)}
\end{aligned}$$

By the identical procedure used to estimate E_4 in the v_3 subsection, we have

$$\begin{aligned}
&|-\frac{1}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda'''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho dw\\
&+\frac{1}{r}\int_{6r}^{\infty}\lambda'''(t+w)r^2w\left(\frac{1}{1+w^2}-\frac{1}{(\lambda(t+w)^{2-2\alpha}+w^2)}\right)dw|\\
&\leqslant \frac{Cr}{t^3\log^{b+1}(t)}+Cr\sup_{x\geqslant t}|e'''(x)|
\end{aligned}$$

We then note that the last line of (5.141) was already estimated, and is bounded above in absolute value by

$$\frac{Cr}{t^3 \log^{b+1}(t)}$$

Then, we estimate

$$\begin{aligned}
&|\partial_t \left(r \int_{t+6r}^{\infty} \lambda''(s)(s-t) \left(\frac{1}{(\lambda(t)^{2-2\alpha} + (s-t)^2)} - \frac{1}{\lambda(s)^{2-2\alpha} + (s-t)^2)} \right) ds \right) | \\
&\leqslant Cr \int_{6r}^{\infty} |\lambda'''(t+w)| \frac{|\lambda(t+w)^{2-2\alpha} - \lambda(t)^{2-2\alpha}|}{(\lambda(t+w)^{2-2\alpha} + w^2)^2} w dw \\
&+ Cr | \int_{6r}^{\infty} \lambda''(t+w) w \partial_t \left(\frac{1}{(\lambda(t)^{2-2\alpha} + w^2)} - \frac{1}{\lambda(t+w)^{2-2\alpha} + w^2} \right) dw |
\end{aligned}$$

The second line of the above expression is estimated by

$$\begin{aligned}
&r \int_{6r}^{\infty} |\lambda'''(t+w)| \frac{|\lambda(t+w)^{2-2\alpha} - \lambda(t)^{2-2\alpha}|}{(\lambda(t+w)^{2-2\alpha} + w^2)^2} w dw \\
&\leqslant Cr \int_{6r}^{\infty} |\lambda'''(t+w)| \frac{\lambda(t)^{1-2\alpha} |\lambda'(t)| w^2}{(\lambda(t+w)^{2-2\alpha} + w^2)^2} dw \\
&\leqslant \frac{Cr}{t \log^{3b+1-3b\alpha}(t)} \left(\frac{1}{t^3 \log^{1-b+2\alpha b}(t)} + \sup_{x \geqslant t} \left(\frac{|e'''(x)|}{\lambda(x)^{2-2\alpha}} \right) \right)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&|r \int_{6r}^{\infty} \lambda''(t+w) w \partial_t \left(\frac{1}{(\lambda(t)^{2-2\alpha} + w^2)} - \frac{1}{\lambda(t+w)^{2-2\alpha} + w^2} \right) dw | \\
&\leqslant \frac{Cr}{t^3 \log^{b+2}(t)}
\end{aligned}$$

Finally, we recall

$$\begin{aligned}
v_{3,2}(t,r) = \frac{-1}{r} \int_t^{\infty} \int_0^{s-t} \rho & \left(\frac{1}{\sqrt{(s-t)^2 - \rho^2}} - \frac{1}{(s-t)} \right) \lambda''(s) \\
& \left(\frac{-1 - \rho^2 + r^2}{\sqrt{(-1-\rho^2+r^2)^2 + 4r^2}} + F_3(r, \rho, \lambda(s)) \right) d\rho ds
\end{aligned}$$

After taking the time derivative, we estimate the following term with the same argument used for $v_{3,2}$.

$$\begin{aligned}
&|\frac{-1}{r} \int_0^{\infty} \int_0^{w} \rho \left(\frac{1}{\sqrt{w^2 - \rho^2}} - \frac{1}{w} \right) \lambda'''(t+w) \\
&\qquad \left(\frac{-1 - \rho^2 + r^2}{\sqrt{(-1-\rho^2+r^2)^2 + 4r^2}} + F_3(r, \rho, \lambda(t+w)) \right) d\rho dw | \\
&\leqslant \frac{Cr}{t^3 \log^{b+1}(t)} + \frac{Cr \sup_{x \geqslant t} \left(\frac{|e'''(x)| x}{\lambda(x)^{3-2\alpha}} \right)}{t \log^{3b-2b\alpha}(t)}
\end{aligned}$$

Finally,

$$|\frac{-1}{r}\int_0^\infty \int_0^w \rho\left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)\lambda''(t+w)\partial_3 F_3(r,\rho,\lambda(t+w))\lambda'(t+w)d\rho dw|$$
$$\leqslant \frac{Cr}{t^3\log^{b+2}(t)}$$

where we note that the integral has already been estimated while studying $\partial_t v_3$.

In total, we get

$$|\partial_t E_5(t,r)| \leqslant \frac{Cr}{t^3\log^{b+1}(t)} + \frac{Cr\sup_{x\geqslant t}\left(\frac{|e'''(x)|x}{\lambda(x)^{3-2\alpha}}\right)}{t\log^{(3-2\alpha)b}(t)}$$

Now, we will obtain an estimate on $\partial_t v_4$ which is better than the preliminary one, (5.129). In particular, now that we have the preliminary estimate on λ''', we can use the same procedure used to estimate $\partial_t v_3$, to see that $\partial_t v_1$ solves, with 0 Cauchy data at infinity, the same equation as v_1 does, except with λ'' replaced by λ'''. This observation, combined with the estimates on v_1 gives

$$|\partial_t v_1(t,r)| \leqslant \frac{Cr\left(\log(t)+\log(3+2r)\right)}{t}\sup_{x\geqslant t}\left(|\lambda'''(x)|x\right) \tag{5.142}$$

(Note that we can not directly use the $\partial_t v_1$ analog of (5.12), because the preliminary estimate on $|e'''|$ is not good enough to justify the steps which would produce such an estimate).

Now, we use the estimate (5.139), combined with the above estimate on $\partial_t v_1$, and the previous estimate on $\partial_t v_2$, to get

$$\begin{aligned}|\partial_t v_{4,c}(t,r)| &\leqslant \frac{C|\chi_{\geqslant 1}'(\frac{2r}{\log^N(t)})|}{t\log(t)}\left(\frac{\lambda(t)^2}{(\lambda(t)^2+r^2)^2}|v_1+v_2+v_3|+|F_{0,2}(t,r)|\right)\\ &+\frac{C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\lambda(t)|\lambda'(t)|}{(r^2+\lambda(t)^2)^2}|v_1+v_2+v_3|\\ &+\frac{C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\lambda(t)^2|\partial_t(v_1+v_2+v_3)|}{(\lambda(t)^2+r^2)^2}\\ &+C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})|\partial_t F_{0,2}(t,r)|\end{aligned}$$

Using $|\chi_{\geqslant 1}'(\frac{2r}{\log^N(t)})|\cdot\frac{r}{t\log^{N+1}(t)} \leqslant C\frac{\mathbb{1}_{\{r\geqslant\frac{\log^N(t)}{2}\}}}{t\log(t)}$, we get

$$\begin{aligned}|\partial_t v_{4,c}(t,r)| &\leqslant \frac{C\mathbb{1}_{\{r\geqslant\frac{\log^N(t)}{2}\}}}{\log^{2b}(t)r^4}\begin{cases}\frac{r}{t^3\log^b(t)}, & r\leqslant\frac{t}{2}\\ \frac{\log(r)}{|t-r|}\left(\frac{1}{|t-r|}+\frac{1}{t\log(t)}\right), & t>r>\frac{t}{2}\end{cases}\\ &+\frac{C\mathbb{1}_{\{r\geqslant\frac{\log^N(t)}{2}\}}}{r^3t^3}\left(\frac{\log(t)+\log(r)}{\log^{3b+1}(t)}\right)\\ &+C\mathbb{1}_{\{r\geqslant\frac{\log^N(t)}{2}\}}\frac{\lambda(t)^{2-2\alpha}(\log(t)+\log(r))}{r^3t\log^{2b}(t)}\sup_{x\geqslant t}\left(\frac{|e'''(x)|x}{\lambda(x)^{2-2\alpha}}\right)\end{aligned}$$

Then, we note that

$$\partial_t v_4(t,r)$$
$$= \frac{-1}{2\pi}\int_0^\infty \int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}}\int_0^{2\pi} \frac{\partial_1 v_{4,c}(t+w,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}}(r+\rho\cos(\theta))d\theta d\rho dw$$

and, carry out the identical procedure done to obtain the estimate on $v_4^{\lambda_1}-v_4^{\lambda_2}$. This results in

$$|\partial_t v_4(t,r)| \leqslant \frac{C}{t^3\log^{3b+N-2}(t)} + \frac{C\lambda(t)^{2-2\alpha}}{t\log^{2b+N-3}(t)}\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{2-2\alpha}}\right), \quad r\leqslant \frac{t}{2}$$

Like previously, in order to estimate $\partial_t v_4$ in the region $r\geqslant \frac{t}{2}$, we first record a slightly different estimate on $\partial_t v_{4,c}$. To obtain this, we use (5.50) in the region $t-\sqrt{t}\leqslant r\leqslant t+\sqrt{t}$, and (5.51) for the other parts of the region $r\geqslant \frac{t}{2}$ to estimate v_2. For $\partial_t v_2$, we use (5.50) in the region $t-t^{1/4}\leqslant r\leqslant t+t^{1/4}$, and (5.51) in the other parts of the region $r\geqslant \frac{t}{2}$.

This leads to

$$\begin{aligned}|\partial_t v_{4,c}(t,r)| &\leqslant \frac{C\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}}{\log^{2b}(t)r^4}\\ &\cdot\begin{cases}\frac{r}{t^3\log^b(t)}, & r\leqslant \frac{t}{2}\\ \frac{\log(r)}{|t-r|}\left(\frac{1}{|t-r|}+\frac{1}{t\log(t)}\right), & \frac{t}{2}\leqslant r\leqslant t-\sqrt{t}, \text{ or } r\geqslant t+\sqrt{t}\\ \frac{\log(r)}{(t-r)^2}+\frac{1}{\sqrt{r}t\log(t)}, & t-\sqrt{t}\leqslant r\leqslant t-t^{1/4}, \text{ or } t+t^{1/4}\leqslant r\leqslant t+\sqrt{t}\\ \frac{1}{\sqrt{r}}, & t-t^{1/4}\leqslant r\leqslant t+t^{1/4}\end{cases}\\ &+\frac{C\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}}{r^3t^3}\left(\frac{\log(t)+\log(r)}{\log^{3b+1}(t)}\right)\\ &+C\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}\frac{\lambda(t)^{2-2\alpha}(\log(t)+\log(r))}{r^3t\log^{2b}(t)}\sup_{x\geqslant t}\left(\frac{|e'''(x)|x}{\lambda(x)^{2-2\alpha}}\right)\end{aligned}$$

First, we obtain

$$||\partial_t v_{4,c}||_{L^2(rdr)} \leqslant \frac{C}{t^3\log^{3b+2N}(t)} + \frac{C\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{2-2\alpha}}\right)}{t\log^{4b-2b\alpha+2N-1}(t)} \tag{5.143}$$

Then, we use the same procedure used to estimate $v_4^{\lambda_1}-v_4^{\lambda_2}$ in the region $r\geqslant \frac{t}{2}$, and get

$$|\partial_t v_4(t,r)| \leqslant \frac{C}{t^2\log^{3b+2N}(t)} + \frac{C\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{2-2\alpha}}\right)}{\log^{4b-2\alpha b+2N-2}(t)}, \quad r\geqslant \frac{t}{2} \tag{5.144}$$

Now, we recall (5.127), and prove new estimates on the terms involving $v_4, E_5, v_1+v_2+v_3$, and $F_{0,2}$

$$\begin{aligned}&|\partial_t\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)v_4\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)},\phi_0\rangle|\\ &\leqslant \frac{C}{t^3\log^{b+N-2}(t)} + \frac{C\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{2-2\alpha}}\right)}{t\log^{N-3+2b-2b\alpha}(t)}\end{aligned}$$

$$|\partial_t\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)E_5|_{r=R\lambda(t)},\phi_0\rangle| \leqslant \frac{C}{t^3\log(t)}+\frac{C\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{3-2\alpha}}\right)}{\log^{(2-2\alpha)b}(t)t}$$

$$|\partial_t\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\,(v_1+v_2+v_3)\,|_{r=R\lambda(t)},\phi_0\rangle|$$

$$\leqslant \frac{C}{t^3\log^{2b+2N}(t)}+\frac{C\lambda(t)^{2-2\alpha}\sup_{x\geqslant t}\left(\frac{|e'''(x)|x}{\lambda(x)^{2-2\alpha}}\right)}{t\log^{2N+b-1}(t)}$$

$$|\partial_t\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}|_{r=R\lambda(t)},\phi_0\rangle| \leqslant \frac{C}{t^3\log^{2b+1-2b\alpha+2N}(t)}+\frac{C|e'''(t)|}{\log^{b-2b\alpha+2N}(t)}$$

From our previous estimates, we also have

$$\begin{aligned}&|\lambda'(t)||\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left((v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)+E_5\right)|_{r=R\lambda(t)},\phi_0\rangle|\\&+|\lambda'(t)||\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\,(v_1+v_2+v_3)\,|_{r=R\lambda(t)},\phi_0\rangle|\\&+|\lambda'(t)||\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}|_{r=R\lambda(t)},\phi_0\rangle|\\&\leqslant \frac{C}{t^3\log^{b+2}(t)}\end{aligned}$$

Combining these, we get

$$\begin{aligned}&|\partial_t\left(\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\left((v_4+v_5)\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)+E_5\right)|_{r=R\lambda(t)},\phi_0\rangle\right)|\\&+|\partial_t\left(\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}})-1}{r^2}\right)\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\,(v_1+v_2+v_3)\,|_{r=R\lambda(t)},\phi_0\rangle\right)|\\&+|\partial_t\left(\lambda(t)\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}|_{r=R\lambda(t)},\phi_0\rangle\right)|\\&\leqslant \frac{C}{t^3\log^{b+1}(t)}+\frac{C\sup_{x\geqslant t}\left(\frac{x|e'''(x)|}{\lambda(x)^{3-2\alpha}}\right)}{t\log^{(3-2\alpha)b}(t)}\end{aligned}$$

where we use (5.130) to estimate $\partial_t v_5$. Finally, for all remaining terms in RHS_3, we use the same estimate that was used in obtaining the preliminary estimate on e''', and conclude

$$|RHS_3(t)| \leqslant \frac{C}{t^3\log^{b+1}(t)}+\frac{C\sup_{x\geqslant t}\left(x^{3/2}|e'''(x)|\right)}{t^{3/2}\sqrt{\log(\log(t))}}$$

Note that the left-hand side of (5.136) is the same as that of (5.66), except with e'' replaced with e''', and λ_0 replaced with $\lambda_{0,0}$. This will not cause any major differences in the study of (5.136) relative to (5.66), because the key estimates (5.69) and (5.71) are invariant under multiplication of $K(t,s)$ by any non-negative function of s, and (5.69) is still true with the replacement of $\frac{1}{(\lambda_0(t)^{1-\alpha}+s-t)}$ by $\frac{1}{(\lambda_{0,0}(t)^{1-\alpha}+s-t)}$. In particular, if we define

$$K_2(t,s) := \frac{\mathbb{1}_{\leqslant 0}(s-t)}{\alpha|\log(\lambda_{0,0}(-s))|}\left(\frac{1}{1-s+t}+\frac{1}{(1-s+t)^3(\lambda_{0,0}(-t)^{1-\alpha}-s+t)}\right)$$

then, by the above discussion, the resolvent kernel associated to K_2 in the same way that r was associated to K, exists, and satisfies the same estimate as r: (5.72). Let us denote this resolvent kernel by r_2.

So far, we have that e''' is a solution to (5.136), which can be re-cast into the form

$$K_2 * x + x = H_2$$

similarly to (5.66). Next, we carry out the following computation, noting that the step which requires Fubini's theorem is justified by the preliminary estimate on e''', (5.135).

$$\begin{aligned} r_2 * (K_2 * x) + r_2 * x &= r_2 * H_2 \\ (r_2 * K_2) * x + r_2 * x &= r_2 * H_2 \\ H_2 - x = K_2 * x &= r_2 * H_2 \end{aligned}$$

where we used the equation

$$r_2 + r_2 * K_2 = K_2$$

(Recall that r_2 solves the r_2 and K_2 analogs of (5.70)). Translating x and H_2 back to e''' and RHS_3, we have

$$e'''(t) = \frac{RHS_3(t)}{4\alpha} - \int_t^\infty \frac{RHS_3(z)}{4\alpha} r_2(-t,-z)dz, \quad \text{a.e. } t \geqslant T_0 \tag{5.145}$$

e''', RHS_3 are continuous functions, and the above equation can be rearranged to yield

$$\int_t^\infty \frac{RHS_3(z)}{4\alpha} r_2(-t,-z)dz = \frac{RHS_3(t)}{4\alpha} - e'''(t), \quad \text{a.e. } t \geqslant T_0$$

So,

$$t \mapsto \int_t^\infty \frac{RHS_3(z)}{4\alpha} r_2(-t,-z)dz$$

agrees with a continuous function almost everywhere, and can thus be extended to a continuous function. Moreover, using (5.72), for r_2 instead of r, we have

$$|\int_t^\infty \frac{RHS_3(z)}{4\alpha} r_2(-t,-z)dz| \leqslant ||\frac{RHS_3}{2\alpha}||_{L^\infty([t,\infty))}, \quad \text{a.e. } t \geqslant T_0$$

So, the same estimate holds for all $t \geqslant T_0$, where we now identify

$$t \mapsto \int_t^\infty \frac{RHS_3(z)}{4\alpha} r_2(-t,-z)dz$$

with its continuous extension described above. Such an identification will be performed without further mention. Returning to (5.145), we get

$$\begin{aligned}|e'''(t)| &\leqslant C \sup_{x \geqslant t} |RHS_3(x)| \\ &\leqslant \frac{C}{t^3 \log^{b+1}(t)} + \frac{C \sup_{x \geqslant t} \left(x^{3/2}|e'''(x)|\right)}{t^{3/2}\sqrt{\log(\log(t))}}\end{aligned}$$

Thus,

$$t^{3/2}|e'''(t)| \leqslant \frac{C}{t^{3/2} \log^{b+1}(t)} + \frac{C \sup_{x \geqslant t} \left(x^{3/2}|e'''(x)|\right)}{\sqrt{\log(\log(t))}}$$

But, as mentioned earlier, e''' is a continuous function on $[T_0, \infty)$,
so $t \mapsto t^{3/2}|e'''(t)|$ is also continuous on $[T_0, \infty)$, and

$$|e'''(t)|t^{3/2} \to 0, \quad t \to \infty$$

by the preliminary estimate on e'''. So, for all $t \geqslant T_0$, there exists $y(t) \geqslant t$ such that

$$\sup_{x \geqslant t} \left(x^{3/2}|e'''(x)|\right) = y(t)^{3/2}|e'''(y(t))|$$

But, then,

$$\begin{aligned}&\sup_{x \geqslant t} \left(x^{3/2}|e'''(x)|\right) = y(t)^{3/2}|e'''(y(t))| \\ &\leqslant \frac{C}{y(t)^{3/2} \log^{b+1}(y(t))} + \frac{C \sup_{x \geqslant y(t)} \left(x^{3/2}|e'''(x)|\right)}{\sqrt{\log(\log(y(t))}} \\ &\leqslant \frac{C}{t^{3/2} \log^{b+1}(t)} + \frac{C \sup_{x \geqslant t} \left(x^{3/2}|e'''(x)|\right)}{\sqrt{\log(\log(t))}}\end{aligned}$$

So, there exists some absolute constants $C_{p_2}, C_{p_1} > 0$ such that, for all

$$t \geqslant C_{p_1} + T_0$$

we have

$$\sup_{x \geqslant t} \left(x^{3/2}|e'''(x)|\right) \leqslant \frac{C_{p_2}}{t^{3/2} \log^{b+1}(t)}$$

But, $e''' \in C([T_0, \infty))$, so, there exists some absolute constant $C_{p_3} > 0$ such that

$$|e'''(t)| \leqslant \frac{C_{p_3}}{t^3 \log^{b+1}(t)}, \quad t \geqslant T_0$$

Recalling $\lambda(t) = \lambda_{0,0}(t) + e(t)$, we have

$$|\lambda'''(t)| \leqslant \frac{C}{t^3 \log^{b+1}(t)}, \quad t \geqslant T_0 \tag{5.146}$$

To finish, we only need to establish the estimates on $\partial_t \partial_r^j v_k$ in the proposition statement. We recall that $\partial_t v_1$ solves the same equation as v_1, also with 0 Cauchy data at infinity, except with λ'' on the right-hand side replaced with λ'''. Now that we have established (5.146), we can justify the steps leading to the $\partial_t v_1$ analog of (5.12), which gives

$$|\partial_t v_1(t,r)| \leqslant \frac{C}{rt\log^{b+1}(t)}, \quad r > \frac{t}{2}$$

Combined with (5.142), this gives (5.118).

A similar large r estimate can be also proven for $\partial_t v_3$: Starting with (5.138), we get

$$\begin{aligned}
&|\partial_t v_3(t,r)|\\
&\leqslant \frac{C}{r}\int_t^\infty |\lambda'''(s)|(s-t)ds\\
&+\frac{C}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\frac{|\lambda''(s)||\lambda'(s)|r^2\lambda(s)^{2\alpha-3}}{(1+\lambda(s)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda(s)^{2\alpha-2}(\rho^2+r^2))}d\rho ds\\
&\leqslant \frac{C}{rt\log^{b+1}(t)}, \quad r>\frac{t}{2}
\end{aligned}$$

Combining this with (5.139) gives (5.119).

Next, we estimate $\partial_{tr} v_k, \quad k=1,3$. We start with $\partial_{tr} v_3$:

$$\begin{aligned}
&\partial_t v_3(t,r)\\
&=\frac{-1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda'''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds\\
&-\frac{1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda''(s)\partial_3 F_3(r,\rho,\lambda(s))\lambda'(s)d\rho ds
\end{aligned} \tag{5.147}$$

Since the first line of (5.147) is the same expression as v_3, except with λ''' replacing λ'', we use the same procedure for this line, as was used to estimate $\partial_r v_3$. For the second line of (5.147), we start by noting that

$$\begin{aligned}
&|\partial_r\left(\frac{\partial_3 F_3(r,\rho,\lambda(s))}{r}\right)|\\
&\leqslant \frac{C\lambda(s)^{2\alpha+1}(\lambda(s)^6+(\rho^2+r^2)\lambda(s)^{4+2\alpha}+(\rho^2+r^2)^2\lambda(s)^{2+4\alpha})}{\lambda(s)^{10}(1+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}+2(\rho^2+r^2)\lambda(s)^{2\alpha-2})^{5/2}}\\
&+\frac{C\lambda(s)^{8\alpha+1}(\rho^2-r^2)^2(\rho^2+r^2)}{\lambda(s)^{10}(1+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}+2(\rho^2+r^2)\lambda(s)^{2\alpha-2})^{5/2}}
\end{aligned}$$

We then estimate the partial r derivative of the second line of (5.147) using the same procedure used before to estimate $\partial_r v_3$, and get

$$|\partial_r\left(-\frac{1}{r}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\lambda''(s)\partial_3 F_3(r,\rho,\lambda(s))\lambda'(s)d\rho ds\right)| \leqslant \frac{C}{t^3\log^{1+b}(t)}$$

(Note that we have the factor $|\lambda'(s)| \leqslant \frac{C}{s\log^{b+1}(s)}$ in the integrand of the second line of (5.147), which explains the gain relative to the analogous term arising in the $\partial_r v_3$ estimates). In total, we get (5.120).

As observed before, $\partial_t v_1$ has the same exact representation formulae as v_1, except with an extra derivative on λ''. Therefore, we can use the identical procedure used to estimate $\partial_r v_1$, to get (5.121).

This gives

$$|\partial_{tr} v_{4,c}(t,r)| \leqslant \frac{C|\chi''_{\geqslant 1}(\frac{2r}{\log^N(t)})|}{r^3t^3\log^{1+N+3b}(t)} + \frac{C|\chi'_{\geqslant 1}(\frac{2r}{\log^N(t)})|}{r^3t^3\log^{3b+N}(t)} + C\chi_{\geqslant 1}(\frac{2r}{\log^N(t)}) \begin{cases} \frac{1}{r^4t^3\log^{3b}(t)}, & r \leqslant \frac{t}{2} \\ \frac{\log(r)}{\log^{2b}(t)r^4}\left(\frac{1}{\log(t)r|t-r|t} + \frac{1}{(t-r)^2t} + \frac{1}{(t-r)^3}\right) \\ +\frac{1}{t^3r^4\log^{3b}(t)}, & t > r > \frac{t}{2} \end{cases} \tag{5.148}$$

Using the final estimates on λ''', we also get

$$\frac{|\partial_t v_{4,c}(t,r)|}{r} \leqslant \frac{C\mathbb{1}_{\{r \geqslant \frac{\log^N(t)}{2}\}}}{r^4t^3\log^{3b+1-2\alpha b}(t)} + \frac{C\mathbb{1}_{\{r \geqslant \frac{\log^N(t)}{2}\}}}{\log^{2b}(t)r^5} \begin{cases} \frac{r}{t^3\log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{\log(r)}{|t-r|}\left(\frac{1}{t\log(t)} + \frac{1}{|t-r|}\right) + \frac{1}{rt\log^{b+1}(t)}, & t > r \geqslant \frac{t}{2} \end{cases}$$

Now, we can prove (5.122) in the region $r \leqslant \frac{t}{2}$, by writing

$$\begin{aligned} &\partial_t v_4(t,r) \\ &= \frac{-r}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\ &\qquad \int_0^1\int_0^{2\pi} \partial_r\left(\frac{\partial_1 v_{4,c}(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}}(r+\rho\cos(\theta))\right)(s,\rho,\beta r)d\theta d\beta d\rho ds \end{aligned}$$

and using the same procedure used for v_4. In total, we get

$$|\partial_t v_4(t,r)| \leqslant \frac{Cr}{t^3\log^{3b+2N-2}(t)}, \quad r \leqslant \frac{t}{2} \tag{5.149}$$

Combining this with (5.144) (and the final estimate on λ''') gives (5.122).

□

5.8.11 Estimating λ''''

Proposition 5.4. $\lambda \in C^4([T_0,\infty))$ *and we have the following estimates:*

$$|\lambda''''(t)| \leqslant \frac{C}{t^4\log^{b+1}(t)}, \quad t \geqslant T_0$$

$$|\partial_t^2 v_1(t,r)| \leqslant \begin{cases} \frac{Cr}{t^4 \log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{rt^2 \log^{b+1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.150}$$

$$|\partial_t^2 v_3(t,r)| \leqslant \frac{Cr \log(\log(t))}{t^4 \log^{b+1}(t)}$$

$$|\partial_t^2 v_4(t,r)| \leqslant \begin{cases} \frac{C}{t^4 \log^{3b-2+N}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{t^{35/12} \log^{2b-1}(t)}, & r > \frac{t}{2} \end{cases} \tag{5.151}$$

$$|\partial_t^2 v_5(t,r)| \leqslant \frac{C}{t^4 \log^{3N+b-2}(t)}, \quad r \leqslant \frac{t}{2} \tag{5.152}$$

Proof. Recalling that $\lambda(t) = \lambda_{0,0}(t) + e(t)$, we will show that $e \in C^4([T_0,\infty))$, and estimate e''''. Returning to (5.123), we have

$$\lambda'''(t) = \frac{RHS_2'(t)}{g_2(t)} - \frac{g_2'(t)}{g_2(t)^2} RHS_2(t)$$

Because $\lambda''' \in C^0([T_0,\infty))$, an inspection of the definition of RHS_2 shows that $RHS_2 \in C^2([T_0,\infty))$; so, $\lambda''' \in C^1([T_0,\infty))$, with

$$\lambda''''(t) = \partial_t \left(\frac{RHS_2'(t)}{g_2(t)} - \frac{g_2'(t)}{g_2(t)^2} RHS_2(t) \right)$$

The previously obtained estimates on RHS_2, RHS_2' then show that

$$|\lambda''''(t)| \leqslant \frac{C|RHS_2''(t)|}{\log(\log(t)) \log^b(t)} + \frac{C}{t^3 \log^{2-2b}(t) \log(\log(t))} \tag{5.153}$$

Again, we will first obtain a preliminary estimate on λ'''', which will be improved afterwards. We start by recalling the definition of RHS_2:

$$\begin{aligned} RHS_2(t) &= -\frac{16}{\lambda(t)^3} \int_t^\infty \lambda''(s) \left(K_1(s-t,\lambda(t)) + K(s-t,\lambda(t)) \right) ds + 2\frac{(\lambda'(t))^2}{\lambda(t)^2} \\ &+ \frac{4b}{\lambda(t) t^2 \log^b(t)} + E_{v_2,ip}(t,\lambda(t)) \\ &+ \langle \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}) - 1}{r^2} \right) \left(v_3 + (v_4+v_5) \left(1 - \chi_{\geqslant 1}(\frac{4r}{t}) \right) \right) |_{r=R\lambda(t)}, \phi_0 \rangle \\ &- \langle \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}) - 1}{r^2} \right) \chi_{\geqslant 1} \left(\frac{2r}{\log^N(t)} \right) (v_1+v_2+v_3) |_{r=R\lambda(t)}, \phi_0 \rangle \\ &- 4 \int_0^\infty \chi_{\geqslant 1}(\frac{2R\lambda(t)}{\log^N(t)}) \frac{(\lambda'(t))^2 R^2 \phi_0(R) dR}{\lambda(t)^2 (R^2+1)^2} \end{aligned}$$

We then note

$$|\partial_t^2\left(\frac{2\lambda'(t)^2}{\lambda(t)^2}\right)| \leqslant \frac{C}{t^4\log^2(t)}$$

$$|\partial_t^2\left(\frac{4b}{\lambda(t)t^2\log^b(t)}\right)| \leqslant \frac{C}{t^4}$$

$$|\partial_t^2\left(\frac{-4\lambda'(t)^2}{\lambda(t)^2}\int_0^\infty \chi_{\geqslant 1}(\frac{2R\lambda(t)}{\log^N(t)})\frac{R^2\phi_0(R)dR}{(R^2+1)^2}\right)| \leqslant \frac{C}{t^4\log^{2+2b+2N}(t)}$$

To continue estimating RHS_2'', we start with

$$\begin{aligned}
&\partial_t^2\left(\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)K_1(x-t,\lambda(t))dx\right)\\
&=\left(\frac{96}{\lambda(t)^4}\lambda'(t)^2-\frac{32\lambda''(t)}{\lambda(t)^3}\right)\int_t^\infty \lambda''(x)K_1(x-t,\lambda(t))dx\\
&-\frac{64\lambda'(t)}{\lambda(t)^3}\int_t^\infty \lambda'''(x)K_1(x-t,\lambda(t))dx\\
&-\frac{64\lambda'(t)}{\lambda(t)^3}\int_t^\infty \lambda''(x)\partial_2K_1(x-t,\lambda(t))\lambda'(t)dx\\
&-\frac{16}{\lambda(t)^2}\int_t^\infty \lambda'''(x)\partial_1K_1(x-t,\lambda(t))dx\\
&+\frac{32}{\lambda(t)^2}\int_t^\infty \lambda'''(x)\partial_2K_1(x-t,\lambda(t))\lambda'(t)dx\\
&+\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)\left(\partial_2^2K_1(x-t,\lambda(t))\lambda'(t)^2+\partial_2K_1(x-t,\lambda(t))\lambda''(t)\right)dx
\end{aligned} \tag{5.154}$$

The only term in (5.154) which we can not immediately estimate from previous estimates is the one involving $\partial_2^2 K_1$. The analogous term involving K is also present in $RHS_3''(t)$. We start with

$$\begin{aligned}
&|\partial_{22}K(x,\lambda(t))(\lambda'(t))^2+\partial_2K(x,\lambda(t))\lambda''(t)|\\
&\leqslant \int_0^\infty \frac{Cr\lambda(t)^2\left(\lambda'(t)^2+\lambda(t)\lambda''(t)\right)}{(r^2+\lambda(t)^2)^3}\int_0^x \rho\left(\frac{1}{\sqrt{x^2-\rho^2}}-\frac{1}{x}\right)\\
&\qquad\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr
\end{aligned}$$

which gives

$$
\begin{aligned}
&|\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)\left(\partial_2^2 K(x-t,\lambda(t))\lambda'(t)^2+\partial_2 K(x-t,\lambda(t))\lambda''(t)\right)dx| \\
&\leqslant \frac{C}{t^4\log^{3b+2}(t)}\int_t^\infty\int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3}\int_0^{x-t}\rho\left(\frac{1}{\sqrt{(x-t)^2-\rho^2}}-\frac{1}{x-t}\right) \\
&\qquad\qquad \left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr dx \\
&\leqslant \frac{C}{t^4\log^{3b+2}(t)}\int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3}\int_0^\infty \rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) \\
&\qquad\qquad \int_{\rho+t}^\infty\left(\frac{1}{\sqrt{(x-t)^2-\rho^2}}-\frac{1}{x-t}\right)dx d\rho dr \\
&\leqslant \frac{C}{t^4\log^{3b+2}(t)}\int_0^\infty \frac{r^3}{(r^2+\lambda(t)^2)^3}dr \\
&\leqslant \frac{C}{t^4\log^{2+b}(t)}
\end{aligned}
$$

Then, we treat the K_1 term:

$$
K_1(x,\lambda(t))=\int_0^\infty \frac{r}{\lambda(t)^2(1+\frac{r^2}{\lambda(t)^2})^3}\int_0^x \frac{\rho}{x}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr
$$

Proceeding as for K, we get

$$
\begin{aligned}
&|\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)\left(\partial_2^2 K_1(x-t,\lambda(t))\lambda'(t)^2+\partial_2 K_1(x-t,\lambda(t))\lambda''(t)\right)dx| \\
&\leqslant C\int_t^\infty |\lambda''(x)|\frac{1}{t^2\log^{2b+1}(t)}\int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3}\frac{1}{(x-t)} \\
&\qquad\qquad \int_0^{x-t}\rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr dx \\
&\leqslant C\int_t^\infty |\lambda''(x)|\frac{1}{t^2\log^{2b+1}(t)}\int_0^\infty \frac{r}{(r^2+\lambda(t)^2)^3}\frac{1}{(x-t)} \\
&\qquad\qquad \cdot\begin{cases}\int_0^{x-t}2\rho d\rho, & x-t\leqslant 1\\ \int_0^\infty \rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho, & x-t\geqslant 1\end{cases} dr dx \\
&\leqslant C\int_t^{t+1}\frac{(x-t)}{x^2\log^{b+1}(x)t^2\log^{2b+1}(t)}\frac{dx}{\lambda(t)^4} \\
&+C\int_{t+1}^\infty \frac{dx}{x^2\log^{b+1}(x)(x-t)t^2\log(t)} \\
&\leqslant \frac{C}{t^4\log^{1-b}(t)}
\end{aligned}
$$

Combining this with estimates for the other terms in (5.154) (which are deduced from the procedure used in obtaining preliminary estimates on e'''), we get

$$\begin{aligned}&|\partial_t^2\left(\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)K_1(x-t,\lambda(t))dx+\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)K(x-t,\lambda(t))dx\right)|\\&\leqslant \frac{C}{t^3\log^{1-2b}(t)}\end{aligned}$$

Next, we consider terms arising in $\partial_t^2\left(\lambda(t)E_{v_2,ip}(t,\lambda(t))\right)$.

$$\begin{aligned}&\partial_t^2\left(2c_b\lambda(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi\right)\\&=2c_b\lambda''(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi+4c_b\lambda'(t)\int_0^\infty\frac{-2\sin(t\xi)}{t^3}\psi_{v_2}(\xi,\lambda(t))d\xi\\&+4c_b\lambda'(t)\int_0^\infty\frac{\xi\cos(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi+4c_b\lambda'(t)\int_0^\infty\frac{\sin(t\xi)}{t^2}\partial_2\psi_{v_2}\lambda'(t)d\xi\\&+4c_b\lambda(t)\int_0^\infty\frac{-2\xi\cos(t\xi)}{t^3}\psi_{v_2}(\xi,\lambda(t))d\xi+2c_b\lambda(t)\int_0^\infty\frac{6\sin(t\xi)}{t^4}\psi_{v_2}(\xi,\lambda(t))d\xi\\&+4c_b\lambda(t)\int_0^\infty\frac{-2\sin(t\xi)}{t^3}\partial_2\psi_{v_2}\lambda'(t)d\xi\\&-2c_b\lambda(t)\int_0^\infty\frac{\xi^2\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi+4c_b\lambda(t)\int_0^\infty\frac{\xi\cos(t\xi)}{t^2}\partial_2\psi_{v_2}\lambda'(t)d\xi\\&+2c_b\lambda(t)\int_0^\infty\frac{\sin(t\xi)}{t^2}\left(\partial_{22}\psi_{v_2}\lambda'(t)^2+\partial_2\psi_{v_2}\lambda''(t)\right)d\xi\end{aligned}$$

Only three integrals can not be immediately estimated based on our previous estimates. These three integrals are estimated using the same procedure used before, by estimating $K_1(x)$ and various of its derivatives, using the fact that K_1 appears in $\psi_{v_2}(\xi\lambda(t))$ with its argument $x=\xi\lambda(t)$ satisfying $0<x<\frac{1}{4}$. We have

$$-2c_b\lambda(t)\int_0^\infty\frac{\xi^2\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi=\frac{2c_b\lambda(t)}{t^5}\int_0^\infty\cos(t\xi)\partial_\xi^3\left(\xi^2\psi_{v_2}(\xi,\lambda(t))\right)d\xi$$

$$4c_b\lambda(t)\int_0^\infty\frac{\xi\cos(t\xi)}{t^2}\partial_2\psi_{v_2}\lambda'(t)d\xi=-4c_b\lambda(t)\int_0^\infty\frac{\cos(t\xi)}{t^4}\partial_\xi^2\left(\xi\partial_2\psi_{v_2}\right)\lambda'(t)d\xi$$

and

$$2c_b\lambda(t)\int_0^\infty\frac{\sin(t\xi)}{t^2}\left(\partial_{22}\psi_{v_2}\lambda'(t)^2\right)d\xi=2c_b\lambda(t)\int_0^\infty\frac{\cos(t\xi)}{t^3}\partial_{122}\psi_{v_2}(\xi,\lambda(t))\lambda'(t)^2d\xi$$

Using these, and the procedure described above, we get

$$\begin{aligned}&|-2c_b\lambda(t)\int_0^\infty\frac{\xi^2\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi|+|4c_b\lambda(t)\int_0^\infty\frac{\xi\cos(t\xi)}{t^2}\partial_2\psi_{v_2}\lambda'(t)d\xi|\\&+|2c_b\lambda(t)\int_0^\infty\frac{\sin(t\xi)}{t^2}\left(\partial_{22}\psi_{v_2}\lambda'(t)^2\right)d\xi|\\&\leqslant\frac{C}{t^5}\end{aligned}$$

Every other term arising in

$$\partial_t^2 \left(2c_b\lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi\right)$$

can be estimated by using estimates for $\lambda, \lambda', \lambda''$, and the terms estimated when considering

$$\partial_t \left(2c_b\lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi\right)$$

In total, we then get

$$|\partial_t^2 \left(2c_b\lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2}\psi_{v_2}(\xi,\lambda(t))d\xi\right)| \leqslant \frac{C}{t^5}$$

For the term

$$\partial_t^2 \left(2c_b\lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2}\chi_{\leqslant\frac{1}{4}}(\xi)F_{v_2}(\xi,\lambda(t))d\xi\right)$$

we use the identical procedure as above, noting that the only difference is that $\psi_{v_2}(\xi,\lambda(t))$ is replaced here by $\chi_{\leqslant\frac{1}{4}}(\xi)F_{v_2}(\xi,\lambda(t))$. Then, we get

$$\begin{aligned}
&|-2c_b\lambda(t)\int_0^\infty \frac{\xi^2\sin(t\xi)}{t^2}\chi_{\leqslant\frac{1}{4}}(\xi)F_{v_2}(\xi,\lambda(t))d\xi|\\
&+|4c_b\lambda(t)\int_0^\infty \frac{\xi\cos(t\xi)}{t^2}\chi_{\leqslant\frac{1}{4}}(\xi)\partial_2\left(F_{v_2}(\xi,\lambda(t))\right)\lambda'(t)d\xi|\\
&+|2c_b\lambda(t)\int_0^\infty \frac{\sin(t\xi)}{t^2}\left(\chi_{\leqslant\frac{1}{4}}(\xi)\partial_{22}\left(F_{v_2}(\xi,\lambda(t))\right)\lambda'(t)^2\right)d\xi|\\
&\leqslant \frac{C\log(\log(t))}{t^5\log^{2b}(t)}
\end{aligned}$$

In total, we have

$$|\partial_t^2 \left(2c_b\lambda(t) \int_0^\infty \frac{\sin(t\xi)}{t^2}\chi_{\leqslant\frac{1}{4}}(\xi)F_{v_2}(\xi,\lambda(t))d\xi\right)| \leqslant \frac{C\log(\log(t))}{t^5\log^{2b}(t)}$$

The next term to consider is

$$\begin{aligned}
&\partial_t^2\left(\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})}+\frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\right)\\
&=\frac{4}{t^4}\left(\frac{b-1}{\log^b(2)}+\frac{b(b-1)}{\log^{b+1}(2)}\right)\sin(\frac{t}{2})+\text{Err}\\
&-\left(\frac{2(b-1)}{\log^b(2)}+\frac{2b(b-1)}{\log^{b+1}(2)}\right)\frac{\cos(\frac{t}{2})}{4t^3}+\frac{\sin(\frac{t}{2})}{t^4}\left(\partial_\xi\left(\frac{\xi(b-1)}{\log^b(\frac{1}{\xi})}+\frac{\xi b(b-1)}{\log^{b+1}(\frac{1}{\xi})}\right)|_{\xi=\frac{1}{2}}\right)\\
&+E_2
\end{aligned}$$

where we integrate by parts after differentiating the integral, and

$$|\text{Err}| \leqslant \frac{C}{t^5}, \quad |E_2| \leqslant \frac{C}{t^5}$$

On the other hand, we have

$$\partial_t^2\left(\int_0^{\frac{t}{2}} \frac{\sin(u)b(b-1)du}{t^2 u \log^{b+1}(\frac{t}{u})}\right)$$
$$= \frac{\cos(\frac{t}{2})b(b-1)}{2t^3\log^{b+1}(2)} - \frac{3\sin(\frac{t}{2})b(b-1)}{t^4\log^{b+1}(2)}$$
$$+ \frac{\sin(\frac{t}{2})b(b-1)}{t}\left(\frac{-2}{t^3\log^{b+1}(2)} - \frac{(b+1)}{t^3\log^{b+2}(2)}\right)$$
$$+ \int_0^{\frac{t}{2}} \frac{\sin(u)b(b-1)}{u}\left(\frac{6}{t^4\log^{b+1}(\frac{t}{u})} + \frac{5(b+1)}{t^4\log^{b+2}(\frac{t}{u})} + \frac{(b+2)(b+1)}{t^4\log^{b+3}(\frac{t}{u})}\right)du$$

So,

$$\partial_t^2\left(\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\right)$$
$$+ \partial_t^2\left(\int_0^{\frac{t}{2}}\frac{\sin(u)(b-1)}{t^2 u}\left(\frac{b}{\log^{b+1}(\frac{t}{u})} + \frac{1}{\log^b(\frac{t}{u})}\right)du\right)$$
$$= O\left(\frac{1}{t^5}\right) + \int_0^{\frac{t}{2}}\frac{\sin(u)(b-1)}{u}\left(\frac{6}{t^4\log^b(\frac{t}{u})} + \frac{5b}{t^4\log^{b+1}(\frac{t}{u})} + \frac{b(b+1)}{t^4\log^{b+2}(\frac{t}{u})}\right)du$$
$$+ \int_0^{\frac{t}{2}}\frac{\sin(u)b(b-1)}{u}\left(\frac{6}{t^4\log^{b+1}(\frac{t}{u})} + \frac{5(b+1)}{t^4\log^{b+2}(\frac{t}{u})} + \frac{(b+1)(b+2)}{t^4\log^{b+3}(\frac{t}{u})}\right)du$$
$$= O\left(\frac{1}{t^5}\right) + \frac{3(b-1)\pi}{t^4\log^b(t)} + O\left(\frac{1}{t^4\log^{b+1}(t)}\right)$$

where we used the computations from the section which constructed v_2, to obtain the $\frac{1}{t^4\log^b(t)}$ term above. We conclude the following estimate on one of the terms arising in $\partial_t^2\left(\lambda(t)E_{v_2,ip}(t,\lambda(t))\right)$:

$$|\partial_t^2\left(\int_0^{\frac{1}{2}}\left(\chi_{\leqslant\frac{1}{4}}(\xi)-1\right)\frac{\sin(t\xi)}{t^2}\left(\frac{b-1}{\xi\log^b(\frac{1}{\xi})} + \frac{b(b-1)}{\xi\log^{b+1}(\frac{1}{\xi})}\right)d\xi\right)$$
$$+ \partial_t^2\left(\int_0^{\frac{t}{2}}\frac{\sin(u)(b-1)}{t^2 u}\left(\frac{b}{\log^{b+1}(\frac{t}{u})} + \frac{1}{\log^b(\frac{t}{u})}\right)du - \left(\frac{(b-1)\pi}{2t^2\log^b(t)}\right)\right)|$$
$$\leqslant \frac{C}{t^4\log^{b+1}(t)}$$

In total, we finally get

$$|\partial_t^2\left(\lambda(t)E_{v_2,ip}(t,\lambda(t))\right)| \leqslant \frac{C}{t^4\log^{b+1}(t)}$$

By the same procedure, this estimate is also true for the case $b=1$.

Now, we record preliminary estimates on $\partial_t^2 v_k, \quad k=1,3,4,5$

Lemma 5.15 (Preliminary estimates on $\partial_t^2 v_k$). *We have the following* preliminary *estimates on* $\partial_t^2 v_k$*:*

$$|\partial_t^2 v_1(t,r)| \leqslant \frac{Cr}{(1+r)} \frac{1}{t^3 \log^{b+1}(t)} \tag{5.155}$$

$$|\partial_t^2 v_3(t,r)| \leqslant \frac{Cr}{t^4 \log^{b+1}(t)} + \frac{C}{t^3 \log^{1+2\alpha b-b}(t)} \tag{5.156}$$

$$|\partial_t^2 v_4(t,r)| \leqslant \begin{cases} \frac{C}{t^3 \log^{b+2N-2}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{t^2 \log^{3N+b}(t)}, & r \geqslant \frac{t}{2} \end{cases} \tag{5.157}$$

$$|\partial_t^2 v_5(t,r)| \leqslant \frac{C}{t^4 \log^{3N+b-2}(t)}, \quad r \leqslant \frac{t}{2} \tag{5.158}$$

Proof. First, we estimate $\partial_t^2 v_3$. We start with the formula for $\partial_t v_3$ used in the process of proving the final estimate on λ''':

$$\begin{aligned} &\partial_t v_3(t,r) \\ &= \frac{-1}{r} \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \lambda'''(s) \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s)) \right) d\rho ds \\ &- \frac{1}{r} \int_0^\infty \int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}} \lambda''(t+w) \partial_3 F_3(r,\rho,\lambda(t+w)) \lambda'(t+w) d\rho dw \end{aligned} \tag{5.159}$$

Denote the first line on the right-hand side of (5.159) by $v_{3,1,t}$. Then, we have

$$\begin{aligned} &\partial_t v_{3,1,t}(t,r) \\ &= \frac{1}{r} \int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \lambda'''(s) \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s)) \right) d\rho ds \\ &- \frac{1}{r} \int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \lambda'''(s) \left(\frac{8\rho^2 r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}} \right. \\ &+ \left. \frac{-8\lambda(s)^{4\alpha-4}\rho^2 r^2}{(4\lambda(s)^{2\alpha-2}r^2+(1+\lambda(s)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}} \right) d\rho ds \end{aligned} \tag{5.160}$$

We will need to use a more complicated procedure for (5.160) than what was used to treat the analogous term which arose in the course of obtaining (5.128). For the first line on the right-hand side of (5.160), we start with

$$
\begin{aligned}
&|\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))| \\
&\leqslant \begin{cases} 2, & \rho \leqslant 4r \\ |\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+1|+|-1+F_3(r,\rho,\lambda(s))|, & \rho > 4r \end{cases} \\
&\leqslant \begin{cases} 2, & \rho \leqslant 4r \\ \frac{r^2}{\rho^2}, & \rho > 4r \end{cases}
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&|\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}}\lambda'''(s)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds| \\
&\leqslant \frac{C}{r}\int_0^\infty \frac{\rho}{t^3\log^{b+1}(t)}\left(\begin{cases} 2, & \rho \leqslant 4r \\ \frac{r^2}{\rho^2}, & \rho > 4r \end{cases}\right)\int_{\rho+t}^\infty \frac{ds d\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \\
&\leqslant \frac{C}{r}\int_0^{4r}\frac{d\rho}{t^3\log^{b+1}(t)}+\frac{C}{r}\int_{4r}^\infty \frac{r^2 d\rho}{\rho^2 t^3 \log^{b+1}(t)} \\
&\leqslant \frac{C}{t^3\log^{b+1}(t)}
\end{aligned}
$$

We then consider the second line on the right-hand side of (5.160). First, note that, because $\lambda'(x) \leqslant 0, \quad x \geqslant T_0$, we have $\lambda(s)^{2\alpha-2} \geqslant \lambda(t)^{2\alpha-2}, \quad s \geqslant t$. Then, we start with the case $r \leqslant \frac{\lambda(t)^{1-\alpha}}{4} \leqslant \frac{1}{4}$. We have $1+\rho^2 \geqslant 16r^2$, and $1+\lambda(t)^{2\alpha-2}\rho^2 \geqslant 16\lambda(t)^{2\alpha-2}r^2$. Whence, we have

$$\frac{1}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}} \leqslant \frac{C}{(1+\rho^2)^3}$$

$$\frac{1}{(4\lambda(t)^{2\alpha-2}r^2+(1+\lambda(t)^{2\alpha-2}\rho^2-\lambda(t)^{2\alpha-2}r^2)^2)^{3/2}} \leqslant \frac{C}{(1+\rho^2\lambda(t)^{2\alpha-2})^3}$$

So,

$$
\begin{aligned}
&|-\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}}\lambda'''(s)\left(\frac{8\rho^2 r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}\right. \\
&\left.+\ \frac{-8\lambda(s)^{4\alpha-4}\rho^2 r^2}{(4\lambda(s)^{2\alpha-2}r^2+(1+\lambda(s)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\right)d\rho ds| \\
&\leqslant \frac{C}{r}\int_0^\infty \frac{1}{t^3\log^{b+1}(t)}\left(\frac{\rho^2 r^2}{(1+\rho^2)^3}+\frac{\rho^2 r^2}{(1+\lambda(t)^{2\alpha-2}\rho^2)^3\log^{4\alpha b-4b}(t)}\right)d\rho \\
&\leqslant \frac{Cr}{t^3\log^{1+b\alpha}(t)}, \quad r \leqslant \frac{\lambda(t)^{1-\alpha}}{4}
\end{aligned}
$$

If $r \geqslant \frac{\lambda(t)^{1-\alpha}}{4}$, then,

$$\begin{aligned}
&|-\frac{1}{r}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}}\lambda'''(s)\left(\frac{8\rho^2 r^2}{(4r^2+(1+\rho^2-r^2)^2)^{3/2}}\right.\\
&+\left.\frac{-8\lambda(s)^{4\alpha-4}\rho^2 r^2}{(4\lambda(s)^{2\alpha-2}r^2+(1+\lambda(s)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\right)d\rho ds|\\
&\leqslant \frac{C}{r}\left(\frac{r^2}{t^3\log^{b+1}(t)}\int_0^{4r}\frac{\rho^2 d\rho}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}}\right.\\
&\left.+\frac{r^2}{t^3\log^{b+1}(t)\log^{4\alpha b-4b}(t)}\int_0^{4r}\frac{\rho^2 d\rho}{(4\lambda(t)^{2\alpha-2}r^2+(1+\lambda(t)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\right)\\
&+\frac{C}{rt^3\log^{b+1}(t)}\int_{4r}^\infty\left(\frac{\rho^2 r^2}{\rho^6}+\frac{\rho^2 r^2}{\log^{4\alpha b-4b}(t)\lambda(t)^{6\alpha-6}\rho^6}\right)d\rho\\
&\leqslant \frac{C}{r}\left(\frac{r^3}{t^3\log^{b+1}(t)}\int_0^\infty\frac{\rho d\rho}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}}\right.\\
&\left.+\frac{r^3}{t^3\log^{b+1}(t)\log^{4\alpha b-4b}(t)}\int_0^\infty\frac{\rho d\rho}{(4\lambda(t)^{2\alpha-2}r^2+(1+\lambda(t)^{2\alpha-2}(\rho^2-r^2))^2)^{3/2}}\right)\\
&+\frac{C}{rt^3\log^{b+1}(t)}\left(\frac{1}{r}+\frac{1}{r\log^{2b-2\alpha b}(t)}\right)\\
&\leqslant \frac{C}{t^3\log^{-b+2\alpha b+1}(t)}
\end{aligned}$$

In total, we get

$$|\partial_t v_{3,1,t}(t,r)| \leqslant \frac{C}{t^3\log^{1+2\alpha b-b}(t)}$$

For the t derivative of the second line of (5.159), we start with

$$\begin{aligned}
|\partial_3^2 F_3(r,\rho,\lambda(s))| &\leqslant \frac{C|\partial_3 F_3(r,\rho,\lambda(s))|}{\lambda(s)}\\
&+\frac{C\lambda(s)^{4\alpha-6}r^2}{\lambda(s)^{2\alpha-2}\left(1+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}\right)}
\end{aligned}$$

Then, we use our estimate for $\partial_3 F_3$ proven while estimating $\partial_t v_3$ previously, and note that the estimate above for $\partial_3^2 F_3$ gives rise to an estimate of the t derivative of the integrand of the second line of (5.159) which is of the same form as the estimate on the integrand used to obtain the final estimate on $\partial_t v_3$. We can therefore read off an estimate from our previous computations, and get that the partial t derivative of the second line of (5.159) is bounded above in absolute value by

$$\frac{Cr}{t^4\log^{b+1}(t)}$$

This gives (5.156).

For $\partial_t^2 v_1$, we have

$$\begin{aligned}
&\partial_{tt} v_1(t,r) \\
&= \int_t^\infty \frac{-\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}(s-t)} \left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) d\rho ds \\
&+ \int_t^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{8\rho^2 r^2}{(s-t)(4r^2+(1+\rho^2-r^2)^2)^{3/2}} d\rho ds
\end{aligned} \tag{5.161}$$

For the first line of (5.161),we get

$$\begin{aligned}
&\int_t^\infty \frac{-\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) d\rho ds \\
&= -\int_0^\infty \rho \left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) \int_{\rho+t}^\infty \frac{\lambda'''(s)}{r(s-t)\sqrt{(s-t)^2-\rho^2}} ds d\rho
\end{aligned}$$

Next, if

$$f_{v_1,t}(x,y) = 1+\frac{x-y}{\sqrt{(x-y)^2+4x}}, \quad x \geqslant 0, y \geqslant 1$$

then,

$$|f_{v_1,t}(x,y)| \leqslant \frac{Cx}{y^2}, \quad x \leqslant \frac{y}{16}$$

Also, we have

$$|f_{v_1,t}(x,y)| \leqslant 2$$

Note that if $r \leqslant \frac{1}{4}$, then, $r^2 \leqslant \frac{1+\rho^2}{16}$. So, we get

$$\begin{aligned}
&|\int_t^\infty \frac{-\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right) d\rho ds| \\
&\leqslant C \frac{\sup_{x\geqslant t} |\lambda'''(x)|}{r} \begin{cases} \int_0^\infty |f_{v_1,t}(r^2, 1+\rho^2)| d\rho, & r < \frac{1}{4} \\ \int_0^{4r} 2 d\rho + C \int_{4r}^\infty \frac{r^2 d\rho}{\rho^4}, & r > \frac{1}{4} \end{cases} \\
&\leqslant C \sup_{x \geqslant t} |\lambda'''(x)| \begin{cases} r, & r < \frac{1}{4} \\ 1, & r > \frac{1}{4} \end{cases} \\
&\leqslant C \frac{r}{1+r} \sup_{x\geqslant t} |\lambda'''(x)|
\end{aligned}$$

For the second line of (5.161), we use a similar procedure as above, again treating first the case

$r \leqslant \frac{1}{4}$, and get

$$
\begin{aligned}
&|\int_t^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{8\rho^2 r^2}{(s-t)(4r^2+(1-r^2+\rho^2)^2)^{3/2}} d\rho ds| \\
&\leqslant \frac{C \sup_{x\geqslant t} |\lambda'''(x)|}{r} \int_0^\infty \frac{\rho^3 r^2}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}} \int_{\rho+t}^\infty \frac{1}{\sqrt{(s-t)^2-\rho^2}} \frac{ds d\rho}{(s-t)} \\
&\leqslant C \frac{\sup_{x\geqslant t} |\lambda'''(x)|}{r} r^2 \int_0^\infty \frac{\rho^2 d\rho}{(1+\rho^2)^3} \\
&\leqslant C r \sup_{x\geqslant t} |\lambda'''(x)|, \quad r \leqslant \frac{1}{4}
\end{aligned}
$$

where we used the fact that

$$r \leqslant \frac{1}{4} \implies 1+\rho^2-r^2 \geqslant C(1+\rho^2)$$

Next, we consider the second line of (5.161) for $r > \frac{1}{4}$. Here, we get

$$
\begin{aligned}
&|\int_t^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{8\rho^2 r^2}{(s-t)(4r^2+(1-r^2+\rho^2)^2)^{3/2}} d\rho ds| \\
&\leqslant C \frac{\sup_{x\geqslant t} |\lambda'''(x)|}{r} \left(\int_0^{2r} \frac{\rho^2 r^2 d\rho}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}} + \int_{2r}^\infty \frac{\rho^2 r^2 d\rho}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}}\right) \\
&\leqslant \frac{C \sup_{x\geqslant t} |\lambda'''(x)|}{r} \left(2r \int_0^{2r} \frac{r^2 \rho d\rho}{(4r^2+(1-r^2+\rho^2)^2)^{3/2}} + r^2 \int_{2r}^\infty \frac{\rho^2 d\rho}{\rho^6}\right) \\
&\leqslant C \sup_{x\geqslant t} |\lambda'''(x)|, \quad r > \frac{1}{4}
\end{aligned}
$$

In total, we get (5.155).

Now, we will obtain an estimate on $\partial_t^2 v_4$. We write

$$v_4 = v_4^0(t,r) + v_4^1(t,r)$$

where v_4^0 solves the same equation as v_4, with 0 Cauchy data at infinity, except with right-hand side equal to $v_{4,c}^0$, which is given by

$$v_{4,c}^0(t,r) = \chi_{\geqslant 1}\left(\frac{2r}{\log^N(t)}\right) \left(\left(\frac{\cos(2Q_1(\frac{r}{\lambda(t)}))-1}{r^2}\right)(v_1+v_2+v_3)\right)$$

and similarly for v_4^1, where

$$v_{4,c}^1(t,r) = \chi_{\geqslant 1}\left(\frac{2r}{\log^N(t)}\right) F_{0,2}(t,r)$$

We start with

$$
\begin{aligned}
&\partial_t^2 v_4^0(t,r) \\
&= \frac{-1}{2\pi} \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{\partial_1^2 v_{4,c}^0(s, \sqrt{r^2+\rho^2+2r\rho\cos(\theta)})\,(r+\rho\cos(\theta))}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}} d\theta d\rho ds
\end{aligned}
$$

Then, we note that

$$|\partial_t^2 v_{4,c}^0(t,r)| \leqslant C\frac{\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}|v_1+v_2+v_3|}{t^2\log^{2b+1}(t)r^4} + \frac{C\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}|\partial_t(v_1+v_2+v_3)|}{t\log^{2b+1}(t)r^4} + \frac{C\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}|\partial_t^2(v_1+v_2+v_3)|}{r^4\log^{2b}(t)}$$

For the purposes of obtaining a preliminary estimate on $\partial_t^2 v_4^0$ in the region $r \leqslant \frac{t}{2}$, we combine all of our previous estimates in the following way:

$$|\partial_t^2 v_{4,c}^0(t,r)| \leqslant \mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}\begin{cases}\frac{C}{r^4\log^{2b}(t)t^3}\left(\frac{1}{\log^{1+2b\alpha-b}(t)}+\frac{1}{\log^b(t)}\right), & r\leqslant \frac{t}{2}\\ \frac{C\log(r)}{r^4\log^{2b}(t)}\left(\frac{1}{|t-r|^3}+\frac{1}{t\log(t)(t-r)^2}+\frac{1}{t^2\log(t)|t-r|}\right) & \\ +\frac{C}{r^4t^3\log^{1+b+2b\alpha}(t)}, & t>r>\frac{t}{2}\end{cases}$$

Directly inserting this estimate into our previous formula for $\partial_t^2 v_4^0(t,r)$, and estimating as in previous sections, we get

$$|\partial_t^2 v_4^0(t,r)| \leqslant \frac{C}{t^3\log^{2N+b-2}(t)}, \quad r\leqslant \frac{t}{2}$$

We will use the same procedure used for $\partial_t v_4$ to estimate $\partial_t^2 v_4^0$ in the region $r \geqslant \frac{t}{2}$. As in previous cases, we will use a different combination of $\partial_t^j v_2$ estimates to get a different estimate for $\partial_t^2 v_{4,c}^0$:

$$|\partial_t^2 v_{4,c}^0(t,r)| \leqslant \begin{cases}\frac{C\mathbb{1}_{\{r\geqslant \frac{\log^N(t)}{2}\}}}{r^4\log^b(t)t^3}, & r\leqslant \frac{t}{2}\\ \frac{C}{\log^{2b}(t)r^{9/2}}, & t-t^{1/6}\leqslant r\leqslant t+t^{1/6}\\ \frac{C\log(r)}{r^4\log^{2b}(t)}\left(\frac{1}{|t-r|^3}+\frac{1}{t\log(t)(t-r)^2}+\frac{1}{t^2\log(t)|t-r|}\right) & \\ +\frac{C}{r^4t^3\log^{1+b+2b\alpha}(t)}, & t-t^{1/6}>r>\frac{t}{2}, \text{ or } r>t+t^{1/6}\end{cases}$$

We then get

$$|\partial_t^2 v_4^0(t,r)| \leqslant \frac{C}{t^2\log^{3N+b}(t)}, \quad r\geqslant \frac{t}{2}$$

which gives

$$|\partial_t^2 v_4^0(t,r)| \leqslant \begin{cases}\frac{C}{t^3\log^{2N+b-2}(t)}, & r\leqslant \frac{t}{2}\\ \frac{C}{t^2\log^{3N+b}(t)}, & r\geqslant\frac{t}{2}\end{cases}$$

Now, we estimate $\partial_t^2 v_4^1$. This time, we follow the procedure used to obtain the preliminary estimate on $\partial_t v_4$:

$$\begin{aligned}&\partial_t^2 v_4^1(t,r)\\ &= \frac{1}{2\pi}\int_t^\infty \frac{1}{(s-t)}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\\ &\qquad\int_0^{2\pi}\frac{\partial_1 v_{4,c}^1(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})(r+\rho\cos(\theta))}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}}d\theta d\rho ds \qquad (5.162)\\ &- \frac{1}{2\pi}\int_t^\infty\int_{B_{s-t}(0)}\frac{\text{integrand}_{v_{4,2}^1}}{\sqrt{(s-t)^2-|y|^2}}dA(y)ds\end{aligned}$$

where

$$\text{integrand}_{v^1_{4,2}} = \frac{-\partial_{12} v^1_{4,c}(s,|x+y|)}{|x+y|^2(s-t)}((x+y)\cdot y)(\widehat{x}\cdot(x+y)) + \frac{\partial_1 v^1_{4,c}(s,|x+y|)}{(s-t)|x+y|}\left(-y\cdot\widehat{x} + \frac{(\widehat{x}\cdot(x+y))(y\cdot(y+x))}{|x+y|^2}\right)$$

which gives

$$|\text{integrand}_{v^1_{4,2}}| \leqslant C|\partial_{12} v^1_{4,c}| + C\frac{|\partial_1 v^1_{4,c}|}{|x+y|}$$

We start with the term on the first line of (5.162):

$$\begin{aligned}
&|\frac{1}{2\pi}\int_t^\infty \frac{1}{(s-t)}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\
&\qquad \int_0^{2\pi} \frac{\partial_1 v^1_{4,c}(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})(r+\rho\cos(\theta))}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}} d\theta d\rho ds| \\
&\leqslant C\int_t^\infty \frac{1}{(s-t)}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\
&\qquad \int_0^{2\pi} \frac{d\theta d\rho ds}{s^3\log^{3b+1+N-2b\alpha}(s)(\log^{2N}(s)+\rho^2+r^2+2r\rho\cos(\theta))} \\
&\leqslant C\int_0^\infty \rho\int_0^{2\pi}\frac{1}{(\rho+t)^3\log^{3b+1+N-2b\alpha}(t)(\log^{2N}(t)+r^2+\rho^2+2r\rho\cos(\theta))} \\
&\qquad \int_{\rho+t}^\infty \frac{ds d\theta d\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \\
&\leqslant \frac{C}{t^3\log^{3b+2N-2b\alpha}(t)}
\end{aligned}$$

For the second line of (5.162), we use the same procedure used to estimate v_4 previously, and we get

$$|-\frac{1}{2\pi}\int_t^\infty\int_{B_{s-t}(0)}\frac{\text{integrand}_{v^1_{4,2}}}{\sqrt{(s-t)^2-|y|^2}} dA(y)ds| \leqslant \frac{C}{t^3\log^{3b-1+2N-2\alpha b}(t)}$$

In total, we get

$$|\partial_t^2 v_4^1(t,r)| \leqslant \frac{C}{t^3\log^{3b+2N-1-2\alpha b}(t)}, \quad r\geqslant 0$$

Combining with the estimate on $\partial_t^2 v_4^0$ above, we get (5.157).

Next, we estimate $\partial_t^2 v_5$. We start with

$$\begin{aligned}|\partial_t^2 N_2(f)(t,r)| \leqslant & \left(\frac{C\lambda(t)\lambda'(t)^2}{r(r^2+\lambda(t)^2)^2}+\frac{C|\lambda''(t)|}{r(r^2+\lambda(t)^2)}\right)|f(t,r)|^2\\ &+\frac{C|\lambda'(t)|}{r(r^2+\lambda(t)^2)}|f(t,r)\partial_t f(t,r)|\\ &+\frac{C\lambda(t)}{r(r^2+\lambda(t)^2)}\left((\partial_t f(t,r))^2+|f(t,r)\partial_{tt}f(t,r)|\right)\\ &+\frac{C\left((\lambda'(t))^2+\lambda(t)|\lambda''(t)|\right)}{(r^2+\lambda(t)^2)^2}|f(t,r)^3|+\frac{C\lambda(t)|\lambda'(t)|}{(r^2+\lambda(t)^2)^2}|f(t,r)^2\partial_t f(t,r)|\\ &+\frac{C}{r^2}\left(|f(t,r)|(\partial_t f(t,r))^2+(f(t,r))^2|\partial_{tt}f(t,r)|\right)\end{aligned} \tag{5.163}$$

where $f = v_1 + v_2 + v_3 + v_4$.

For the purposes of estimating $\partial_t^2 v_5$ in the region $r \leqslant \frac{t}{2}$, we return to (5.163), and use (5.51) for all quantities involving v_2 in the region $r \geqslant \frac{t}{2}$, with the following exceptions. For the term involving $|f| \cdot |\partial_t f|^2$ on the last line of (5.163), we take the following average of (5.51) and (5.50) to estimate $(\partial_t v_2)^2$ in the region $r \geqslant \frac{t}{2}$:

$$(\partial_t v_2(t,r))^2 \leqslant C\left(\frac{\log(r)}{|t-r|^2}\right)^{3/2}\cdot\frac{1}{r^{1/4}}$$

Similarly, for the term involving $f^2|\partial_{tt}f|$, we estimate $v_2^2|\partial_{tt}v_2|$ by

$$(v_2(t,r))^2|\partial_{tt}v_2| \leqslant \frac{C\log(r)^2}{|t-r|^4\sqrt{r}}$$

(This is so that we will not have to have an extra argument for a term analogous to the last line of (5.95)). This procedure gives

$$|\partial_t^2 N_2(f)(t,r)| \leqslant C\begin{cases}\frac{1}{t^5\log^b(t)(r^2+\lambda(t)^2)}, & r \leqslant \frac{t}{2}\\ \frac{\log(r)}{r^3t^2|t-r|\log^{3N+2b}(t)}+\frac{\log^2(r)}{r^2(t-r)^2t^2\log^{3N+b}(t)}+\frac{\log^3(r)}{t^2r^2\log(t)|t-r|^3} & \\ +\frac{\log^{5/2}(r)}{r^{9/4}(t-r)^4}, & \frac{t}{2} \leqslant r < t\end{cases}$$

Using these estimates in the formula

$$\begin{aligned}&\partial_t^2 v_5(t,r)\\ &=\frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{\partial_1^2 N_2(f)(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)})}{\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}}(r+\rho\cos(\theta))d\theta d\rho ds\end{aligned}$$

gives (5.158). □

Then, we use all of our estimates above to get

$$|\partial_t^2\left(\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)v_3|_{r=R\lambda(t)},\phi_0\rangle\right)| \leqslant \frac{C}{t^3\log^{1+2\alpha b-3b}(t)}$$

$$|\partial_t^2\left(\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)v_4\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)},\phi_0\rangle\right)|\leqslant \frac{C}{t^3\log^{2N-2-b}(t)}$$

$$|\partial_t^2\left(\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)v_5\left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)},\phi_0\rangle\right)|\leqslant \frac{C}{t^4\log^{3N-b-2}(t)}$$

$$|\partial_t^2\left(\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)\left(\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})(v_1+v_2+v_3)\right)|_{r=R\lambda(t)},\phi_0\rangle\right)| \leqslant \frac{C}{t^3\log^{1+3N+2\alpha b}(t)}$$

Finally, assembling together all of our previous estimates, we conclude

$$|RHS_2''(t)|\leqslant \frac{C}{t^3\log^{1-3b}(t)}$$

Returning to (5.153), we get

$$|\lambda''''(t)|\leqslant \frac{C}{t^3\log^{1-2b}(t)\log(\log(t))}$$

Now that we have this preliminary estimate, we return to (5.63), substitute

$$\lambda(t)=\lambda_{0,0}(t)+e(t)$$

and differentiate twice in t, to get

$$\begin{aligned}
&-4\int_t^\infty \frac{e''''(s)ds}{\log(\lambda_{0,0}(s))(1+s-t)}+4\alpha e''''(t)\\
&-4\int_t^\infty \frac{e''''(s)ds}{\log(\lambda_{0,0}(s))(\lambda_{0,0}(t)^{1-\alpha}+s-t)(1+s-t)^3}\\
&=\frac{-E''_{\lambda_{0,0}}(t)}{\log(\lambda_{0,0}(t))}-\frac{4\alpha}{\log(\lambda_{0,0}(t))}\left(\frac{\lambda''(t)}{\lambda(t)}-\frac{\lambda'(t)^2}{\lambda(t)^2}\right)\lambda''(t)-\frac{8\alpha\lambda'(t)\lambda'''(t)}{\lambda(t)\log(\lambda_{0,0}(t))}\\
&-4\alpha\left(\frac{\log(\lambda(t))}{\log(\lambda_{0,0}(t))}-1\right)e''''(t)-\frac{4\alpha\log(\lambda(t))}{\log(\lambda_{0,0}(t))}\lambda''''_{0,0}(t)\\
&+\frac{4}{\log(\lambda_{0,0}(t))}\partial_t^2\left(\int_0^\infty \frac{\lambda''_{0,0}(t+w)}{(\lambda(t)^{1-\alpha}+w)(1+w^3)}dw\right)\\
&+4\int_0^\infty \frac{e''''(t+w)}{\log(\lambda_{0,0}(t))}\left(\frac{1}{(\lambda(t)^{1-\alpha}+w)}-\frac{1}{(\lambda_{0,0}(t)^{1-\alpha}+w)}\right)\frac{dw}{(1+w)^3}\\
&-\frac{8(1-\alpha)\lambda(t)^{-\alpha}\lambda'(t)}{\log(\lambda_{0,0}(t))}\int_0^\infty \frac{e'''(t+w)}{(\lambda(t)^{1-\alpha}+w)^2(1+w)^3}dw\\
&-\frac{4(1-\alpha)(-\alpha\lambda(t)^{-\alpha-1}(\lambda'(t))^2+\lambda(t)^{-\alpha}\lambda''(t))}{\log(\lambda_{0,0}(t))}\int_0^\infty \frac{e''(t+w)dw}{(1+w)^3(\lambda(t)^{1-\alpha}+w)^2}\\
&+\frac{8}{\log(\lambda_{0,0}(t))}\int_0^\infty \frac{e''(t+w)}{(1+w)^3}\frac{(1-\alpha)^2\lambda(t)^{-2\alpha}\lambda'(t)^2dw}{(\lambda(t)^{1-\alpha}+w)^3}+\frac{\partial_t^2G(t,\lambda_{0,0}(t)+e(t))}{\log(\lambda_{0,0}(t))}\\
&+4\int_t^\infty e''''(s)\left(\frac{1}{\log(\lambda_{0,0}(t))}-\frac{1}{\log(\lambda_{0,0}(s))}\right)\frac{ds}{(1+s-t)}\\
&+4\int_t^\infty \frac{e''''(s)}{(1+s-t)^3(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\left(\frac{1}{\log(\lambda_{0,0}(t))}-\frac{1}{\log(\lambda_{0,0}(s))}\right)ds\\
&:=RHS_4(t)
\end{aligned}\tag{5.164}$$

Like before, we start by estimating all the terms on the right-hand side of (5.164) which do not involve G:

$$|\frac{-E''_{\lambda_{0,0}}(t)}{\log(\lambda_{0,0}(t))}|\leqslant\frac{C}{t^4\log^{b+1}(t)\log(\log(t))}$$

$$|\frac{-4\alpha}{\log(\lambda_{0,0}(t))}\left(\frac{\lambda''(t)}{\lambda(t)}-\frac{\lambda'(t)^2}{\lambda(t)^2}\right)\lambda''(t)|\leqslant\frac{C}{t^4\log^{b+2}(t)\log(\log(t))}$$

$$|\frac{-8\alpha\lambda'(t)\lambda'''(t)}{\lambda(t)\log(\lambda_{0,0}(t))}|\leqslant\frac{C}{t^4\log^{b+2}(t)\log(\log(t))}$$

$$|-4\alpha\left(\frac{\log(\lambda_{0,0}(t)+e(t))}{\log(\lambda_{0,0}(t))}-1\right)e''''(t)|\leqslant\frac{C|e''''(t)|}{(\log(\log(t)))^{3/2}}$$

$$|\frac{-4\alpha\log(\lambda(t))}{\log(\lambda_{0,0}(t))}\lambda''''_{0,0}(t)|\leqslant\frac{C}{t^4\log^{b+1}(t)}$$

Then, we note that the following estimate:

$$|\partial_t^2\left(\frac{\lambda_{0,0}''(t+w)}{w+\lambda(t)^{1-\alpha}}\right)| \leqslant \frac{C}{t^2(t+w)^2\log^{b+1}(t+w)(w+\lambda(t)^{1-\alpha})}$$

implies

$$|\frac{4}{\log(\lambda_{0,0}(t))}\int_0^\infty \partial_t^2\left(\frac{\lambda_{0,0}''(t+w)}{w+\lambda(t)^{1-\alpha}}\right)\frac{dw}{(1+w)^3}| \leqslant \frac{C}{t^4\log^{b+1}(t)}$$

Next, we have

$$\begin{aligned}&|4\int_0^\infty \frac{e''''(t+w)}{\log(\lambda_{0,0}(t))}\left(\frac{\lambda_{0,0}(t)^{1-\alpha}-\lambda(t)^{1-\alpha}}{(\lambda(t)^{1-\alpha}+w)(\lambda_{0,0}(t)^{1-\alpha}+w)}\right)\frac{dw}{(1+w)^3}|\\&\leqslant \frac{C\sup_{x\geqslant t}|e''''(x)|}{\sqrt{\log(\log(t))}}\end{aligned}$$

$$|\frac{-8(1-\alpha)\lambda(t)^{-\alpha}\lambda'(t)}{\log(\lambda_{0,0}(t))}\int_0^\infty \frac{e'''(t+w)dw}{(\lambda(t)^{1-\alpha}+w)^2(1+w)^3}| \leqslant \frac{C}{t^4\log^{b+2}(t)}$$

$$\begin{aligned}&|\frac{-4(1-\alpha)(-\alpha\lambda(t)^{-\alpha-1}(\lambda'(t))^2+\lambda(t)^{-\alpha}\lambda''(t))}{\log(\lambda_{0,0}(t))}\int_0^\infty \frac{e''(t+w)dw}{(1+w)^3(\lambda(t)^{1-\alpha}+w)^2}|\\&\leqslant \frac{C}{t^4\log^{b+2}(t)}\end{aligned}$$

$$|\frac{8}{\log(\lambda_{0,0}(t))}\int_0^\infty \frac{e''(t+w)}{(1+w)^3}\frac{(1-\alpha)^2\lambda(t)^{-2\alpha}\lambda'(t)^2dw}{(\lambda(t)^{1-\alpha}+w)^3}| \leqslant \frac{C}{t^4\log^{b+3}(t)}$$

$$|4\int_t^\infty e''''(s)\left(\frac{1}{\log(\lambda_{0,0}(t))}-\frac{1}{\log(\lambda_{0,0}(s))}\right)\frac{ds}{1+s-t}| \leqslant \frac{C\sup_{x\geqslant t}(x^2|e''''(x)|)}{t^2\log(t)(\log(\log(t)))^2}$$

and

$$\begin{aligned}&|4\int_t^\infty \frac{e''''(s)}{(1+s-t)^3(\lambda_{0,0}(t)^{1-\alpha}+s-t)}\left(\frac{1}{\log(\lambda_{0,0}(t))}-\frac{1}{\log(\lambda_{0,0}(s))}\right)ds|\\&\leqslant \frac{C\sup_{x\geqslant t}(|e''''(x)|x^2)}{t^3\log(t)\log(\log(t))}\end{aligned}$$

Combining these, we get that all of the terms on the right-hand side of (5.164) which do not involve G are bounded above in absolute value by

$$\frac{C}{t^4\log^{b+1}(t)}+\frac{C\sup_{x\geqslant t}(x^2|e''''(x)|)}{\sqrt{\log(\log(t))}t^2}$$

We now estimate terms in $\partial_t^2 G(t,\lambda(t))$, starting with the term involving K_1. We again use the preliminary estimate on e'''' to justify any differentiations under the integral sign, to get

$$\begin{aligned}
&\partial_t^2\left(\frac{16}{\lambda(t)^2}\int_t^\infty \lambda''(x)\left(K_1(x-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+x-t)}\right)dx\right)\\
&=\left(\frac{96\lambda'(t)^2}{\lambda(t)^4}-\frac{32\lambda''(t)}{\lambda(t)^3}\right)\int_0^\infty \lambda''(t+w)\left(K_1(w,\lambda(t))-\frac{\lambda(t)^2}{4(1+w)}\right)dw\\
&-\frac{64\lambda'(t)}{\lambda(t)^3}\int_0^\infty \lambda'''(w+t)\left(K_1(w,\lambda(t))-\frac{\lambda(t)^2}{4(1+w)}\right)dw\\
&-\frac{64\lambda'(t)}{\lambda(t)^3}\int_0^\infty \lambda''(w+t)\left(\partial_2 K_1(w,\lambda(t))-\frac{\lambda(t)}{2(1+w)}\right)\lambda'(t)dw\\
&+\frac{16}{\lambda(t)^2}\int_0^\infty \lambda''''(w+t)\left(K_1(w,\lambda(t))-\frac{\lambda(t)^2}{4(1+w)}\right)dw\\
&+\frac{32}{\lambda(t)^2}\int_0^\infty \lambda'''(w+t)\left(\partial_2 K_1(w,\lambda(t))-\frac{\lambda(t)}{2(1+w)}\right)\lambda'(t)dw\\
&+\frac{16}{\lambda(t)^2}\int_0^\infty \lambda''(w+t)\Bigg(\left(\partial_2^2 K_1(w,\lambda(t))\lambda'(t)-\frac{\lambda'(t)}{2(1+w)}\right)\lambda'(t)\\
&\qquad+\left(\partial_2 K_1(w,\lambda(t))-\frac{\lambda(t)}{2(1+w)}\right)\lambda''(t)\Bigg)dw
\end{aligned} \tag{5.165}$$

The only term not involving λ'''' which has not already been estimated is the first term on the last line of the above expression. For $w \geqslant 1$, we start with

$$K_1(w,\lambda(t))=\int_0^\infty \frac{r}{\lambda(t)^2(1+\frac{r^2}{\lambda(t)^2})^3}\int_0^w \frac{\rho}{w}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr$$

whence

$$\begin{aligned}
&\partial_2^2 K_1(w,\lambda(t))\\
&=\int_0^\infty \frac{6\lambda(t)^2 r\left(\lambda(t)^4-5\lambda(t)^2r^2+2r^4\right)}{(\lambda(t)^2+r^2)^5}\left(\frac{r^2}{w}-\int_w^\infty \frac{\rho}{w}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho\right)dr\\
&=\frac{1}{2w}-\int_0^\infty \frac{6\lambda(t)^2r(\lambda(t)^4-5\lambda(t)^2r^2+2r^4)}{(\lambda(t)^2+r^2)^5}\int_w^\infty \frac{\rho}{w}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)d\rho dr
\end{aligned}$$

Then,

$$\begin{aligned}
&|\frac{16}{\lambda(t)^2}\int_1^\infty \lambda''(w+t)\left(\partial_{22}K_1(w,\lambda(t))\lambda'(t)-\frac{\lambda'(t)}{2(1+w)}\right)\lambda'(t)dw|\\
&\leqslant \frac{C|\lambda'(t)|}{t^3\log^2(t)}\int_1^\infty\left(\frac{1}{2w}-\frac{1}{2(1+w)}\right)dw\\
&+\frac{C|\lambda'(t)|}{t^3\log^2(t)}\int_0^\infty \frac{r\lambda(t)^2}{(\lambda(t)^2+r^2)^3}\int_1^\infty\int_1^\rho \frac{\rho}{w}\left(1+\frac{r^2-1-\rho^2}{\sqrt{(r^2-1-\rho^2)^2+4r^2}}\right)dwd\rho dr\\
&\leqslant \frac{C}{t^4\log^{b+3}(t)}+\frac{C\lambda(t)^2}{t^4\log^{b+3}(t)}\int_0^\infty\frac{r^3\log(2+r^2)dr}{(\lambda(t)^2+r^2)^3}\\
&\leqslant \frac{C}{t^4\log^{b+3}(t)}
\end{aligned}$$

For $w\leqslant 1$, we first change variables, and start with

$$K_1(w,\lambda(t))=\int_0^\infty\frac{R}{(1+R^2)^3}\int_0^w\frac{\rho}{w}\left(1+\frac{R^2\lambda(t)^2-1-\rho^2}{\sqrt{(R^2\lambda(t)^2-1-\rho^2)^2+4R^2\lambda(t)^2}}\right)d\rho dR$$

Then,

$$|\partial_2^2K_1(w,\lambda(t))|\leqslant C\int_0^\infty\frac{R}{(1+R^2)^3}\int_0^w\frac{\rho}{w}R^2d\rho dR$$

which gives

$$\begin{aligned}
&|\frac{16}{\lambda(t)^2}\int_0^1\lambda''(w+t)\left(\partial_{22}K_1(w,\lambda(t))\lambda'(t)-\frac{\lambda'(t)}{2(1+w)}\right)\lambda'(t)dw|\\
&\leqslant\frac{C}{t^3\log^2(t)}\int_0^1\left(w+\frac{1}{1+w}\right)\frac{dw}{t\log^{b+1}(t)}\leqslant\frac{C}{t^4\log^{b+3}(t)}
\end{aligned}$$

We now consider the expression (5.165), except with all instances of

$$K_1(x-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+x-t)}$$

replaced with

$$K(x-t,\lambda(t))$$

(This expression also appears in $\partial_t^2(G(t,\lambda(t)))$). All of the integrals involving K which do not involve λ'''' have already been estimated, and we get

$$\begin{aligned}
&|\partial_t^2\left(\frac{16}{\lambda(t)^2}\int_t^\infty\lambda''(x)\left(K_1(x-t,\lambda(t))-\frac{\lambda(t)^2}{4(1+x-t)}\right)dx\right)|\\
&+|\partial_t^2\left(\frac{16}{\lambda(t)^2}\int_t^\infty\lambda''(x)K(x-t,\lambda(t))dx\right)|\\
&\leqslant\frac{C}{t^4\log^{b+1}(t)}+C\sup_{x\geqslant t}|e''''(x)|
\end{aligned}$$

From the explicit expressions, we also get

$$|\partial_t^2\left(-\lambda(t)E_{0,1}(\lambda(t),\lambda'(t),\lambda''(t))\right)| \leqslant \frac{C}{t^4\log^{b+1}(t)} + C|e''''(t)|$$

Next, we start with

$$\begin{aligned}|\partial_t^2\left(K_3(w,\lambda(t))\right)| \leqslant &\frac{C}{t^2\log(t)}|\frac{w}{1+w^2} - \frac{w}{\lambda(t)^{2-2\alpha}+w^2}|\\ &+ \frac{Cw}{t^2\log^{(2-2\alpha)b+1}(t)(\lambda(t)^{2-2\alpha}+w^2)^2}\end{aligned}$$

and

$$|\partial_t^2 K_{3,0}(w,\lambda(t))| \leqslant \frac{C}{t^2\log^{b+1-b\alpha}(t)}\frac{1}{(\lambda(t)^{1-\alpha}+w)^2(1+w)^3}$$

and use our previous estimates on $K_3 - K_{3,0}$, and $\partial_t\left((K_3-K_{3,0})(w,\lambda(t))\right)$ to get

$$\begin{aligned}&|\partial_t^2\left(-16\int_t^\infty \lambda''(s)\left(K_3(s-t,\lambda(t)) - K_{3,0}(s-t,\lambda(t))\right)ds\right)|\\ &\leqslant C\sup_{x\geqslant t}|e''''(x)| + \frac{C}{t^4\log^{b+1}(t)}\end{aligned}$$

Next, we will need an estimate on $\partial_t^2 v_3$ which is different from that recorded while obtaining the preliminary estimate on λ''''. We start with

$$\begin{aligned}&\partial_t^2 v_3(t,r)\\ &= \frac{-1}{r}\int_0^\infty\int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}}\lambda''''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(t+w))\right)d\rho dw\\ &-\frac{2}{r}\int_0^\infty\int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}}\lambda'''(t+w)\partial_t(F_3(r,\rho,\lambda(t+w)))d\rho dw\\ &-\frac{1}{r}\int_0^\infty\int_0^w \frac{\rho}{\sqrt{w^2-\rho^2}}\lambda''(t+w)\partial_t^2(F_3(r,\rho,\lambda(t+w)))d\rho dw\end{aligned} \tag{5.166}$$

The second and third lines of (5.166) were already estimated in the course of obtaining the preliminary estimate on λ'''', and do not need to be estimated any differently here. The first line of (5.166), with the replacement of λ'''' with λ''' has already been estimated while obtaining the final estimate on λ'''. Therefore, we can read off that

$$|\partial_t^2 v_3(t,r)| \leqslant Cr\sup_{x\geqslant t}\left(\frac{x|e''''(x)|}{\lambda(x)^{2-2\alpha}}\right)\frac{\log(\log(t))\lambda(t)^{2-2\alpha}}{t} + \frac{Cr\log(\log(t))}{t^4\log^{b+1}(t)} \tag{5.167}$$

Next, we estimate $\partial_t^2 E_5(t,r)$ by proceeding line-by-line on the expression (5.140). Denoting, by

$E_{5,t,t,i}$, the second partial t derivative of the ith line of (5.140), we get

$$\begin{aligned}E_{5,t,t,1}(t,r) = &\frac{-1}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}\lambda''''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho dw\\&-\frac{2}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}\lambda'''(t+w)\partial_t(F_3(r,\rho,\lambda(t+w)))d\rho dw\\&-\frac{1}{r}\int_0^{6r}\int_0^w \frac{\rho}{w}\lambda''(t+w)\partial_t^2(F_3(r,\rho,\lambda(t+w)))d\rho dw\end{aligned} \tag{5.168}$$

Note that, with the replacement of λ'''' with λ''', the first line of (5.168) was already estimated in the course of obtaining the final estimate on λ'''. The same is true for the second line of (5.168), except with the replacement of λ''' with λ''. On the other hand, for the third line of (5.168), we use our previous estimate of

$$\partial_t^2(F_3(r,\rho,\lambda(t+w)))$$

to get

$$|E_{5,t,t,1}(t,r)| \leqslant \frac{Cr}{t^4\log^{b+1}(t)}+Cr\sup_{x\geqslant t}\left(|e''''(x)|\right)$$

Next, we have

$$\begin{aligned}&E_{5,t,t,2}(t,r)\\&=\frac{-1}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda''''(t+w)\left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}}+F_3(r,\rho,\lambda(t+w))\right)d\rho dw\\&-\frac{2}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda'''(t+w)\partial_t(F_3(r,\rho,\lambda(t+w)))d\rho dw\\&-\frac{1}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda''(t+w)\partial_t^2(F_3(r,\rho,\lambda(t+w)))d\rho dw\\&+\frac{1}{r}\int_{6r}^{\infty}\lambda''''(t+w)r^2w\left(\frac{1}{1+w^2}-\frac{1}{\lambda(t+w)^{2-2\alpha}+w^2}\right)dw\\&+\frac{2}{r}\int_{6r}^{\infty}\frac{\lambda'''(t+w)r^2w}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}(2-2\alpha)\lambda(t+w)^{1-2\alpha}\lambda'(t+w)dw\\&+\frac{1}{r}\int_{6r}^{\infty}\lambda''(t+w)r^2w(2-2\alpha)\\&\qquad\frac{((1-2\alpha)\lambda(t+w)^{-2\alpha}(\lambda'(t+w))^2+\lambda(t+w)^{1-2\alpha}\lambda''(t+w))}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}dw\\&-\frac{2}{r}\int_{6r}^{\infty}\frac{\lambda''(t+w)r^2w\left((2-2\alpha)\lambda(t+w)^{1-2\alpha}\lambda'(t+w)\right)^2}{(\lambda(t+w)^{2-2\alpha}+w^2)^3}dw\end{aligned} \tag{5.169}$$

Only two lines in (5.169) can not be estimated simply by comparing to an analogous term arising in the $\partial_t E_5$ estimates. They are estimated as follows:

$$\begin{aligned}
&|\frac{-1}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w}\lambda''(t+w)\partial_t^2(F_3(r,\rho,\lambda(t+w)))d\rho dw| \\
&\leqslant \frac{C}{r}\int_{6r}^{\infty}\int_0^w \frac{\rho}{w(t+w)^4\log^{2b+2}(t+w)} \\
&\qquad \left(\frac{Cr^2\lambda(t+w)^{2\alpha-3}}{(1+\lambda(t+w)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda(t+w)^{2\alpha-2}(\rho^2+r^2))}\right)d\rho dw \\
&\leqslant \frac{C}{r}\int_0^{\lambda(t)^{1-\alpha}} \frac{1}{wt^4\log^{2b+2}(t)}\int_0^w r^2\lambda(t+w)^{2\alpha-3}\rho d\rho dw \\
&+\frac{C}{r}\int_{\lambda(t)^{1-\alpha}}^{\infty}\frac{1}{w}\int_0^w \frac{\rho}{(t+w)^4\log^{2b+2}(t+w)} \\
&\qquad \frac{r^2\lambda(t+w)^{2\alpha-3}d\rho dw}{(1+\lambda(t+w)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda(t+w)^{2\alpha-2}(\rho^2+r^2))} \\
&\leqslant \frac{Cr}{t^4\log^{b+1}(t)}
\end{aligned}$$

$$\begin{aligned}
&|\frac{-1}{r}\int_{6r}^{\infty}\lambda''(t+w)r^2w(2-2\alpha) \\
&\qquad \frac{((1-2\alpha)\lambda(t+w)^{-2\alpha}(\lambda'(t+w))^2+\lambda(t+w)^{1-2\alpha}\lambda''(t+w))\,dw}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}| \\
&\leqslant \frac{Cr}{t^4\log^{b+2}(t)}
\end{aligned}$$

where we use the identical procedure used to estimate an analogous term arising while obtaining the final estimate on λ'''. Then, we get

$$|E_{5,t,t,2}(t,r)| \leqslant \frac{Cr}{t^4\log^{b+1}(t)} + Cr\sup_{x\geqslant t}|e''''(x)|$$

Next, we have

$$\begin{aligned}
E_{5,t,t,3}(t,r) &= -r\int_{6r}^{\infty}\lambda''''(t+w)w\left(\frac{1}{\lambda(t)^{2-2\alpha}+w^2}-\frac{1}{\lambda(t+w)^{2-2\alpha}+w^2)}\right)dw \\
&-2r\int_{6r}^{\infty}\lambda'''(t+w)w \\
&\qquad \left(\frac{-(2-2\alpha)\lambda(t)^{1-2\alpha}\lambda'(t)}{(\lambda(t)^{2-2\alpha}+w^2)^2}+\frac{(2-2\alpha)\lambda(t+w)^{1-2\alpha}\lambda'(t+w)}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}\right)dw \\
&-r\int_{6r}^{\infty}\lambda''(t+w)w \\
&\qquad \partial_t\left(\frac{-(2-2\alpha)\lambda(t)^{1-2\alpha}\lambda'(t)}{(\lambda(t)^{2-2\alpha}+w^2)^2}+\frac{(2-2\alpha)\lambda(t+w)^{1-2\alpha}\lambda'(t+w)}{(\lambda(t+w)^{2-2\alpha}+w^2)^2}\right)dw
\end{aligned} \tag{5.170}$$

The first and second lines of (5.170) can be estimated based on estimates of analogous terms arising in the estimation of $\partial_t E_5$ done while obtaining the final estimate on λ'''. On the other hand, by appropriately using

$$|f(t+w)-f(t)| \leqslant \sup_{\sigma\in[0,1]} |f'(t+\sigma w)| \cdot w$$

we estimate the third line on the right-hand side of (5.170), and in total, we get

$$|E_{5,t,t,3}(t,r)| \leqslant \frac{Cr}{t^4 \log^{b+2}(t)} + \frac{Cr}{t\log^{3b+1-3b\alpha}(t)} \sup_{x\geqslant t}\left(\frac{|e''''(x)|}{\lambda(x)^{2-2\alpha}}\right)$$

Finally, we have

$$\begin{aligned}
&E_{5,t,t,4}(t,r)\\
&= \frac{-1}{r}\int_0^\infty \int_0^w \rho \left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)\lambda''''(t+w)\\
&\qquad \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(t+w))\right) d\rho dw\\
&- \frac{2}{r}\int_0^\infty \int_0^w \rho \left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)\lambda'''(t+w)\partial_t \left(F_3(r,\rho,\lambda(t+w))\right) d\rho dw\\
&- \frac{1}{r}\int_0^\infty \int_0^w \rho \left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)\lambda''(t+w)\partial_t^2 \left(F_3(r,\rho,\lambda(t+w))\right) d\rho dw
\end{aligned} \tag{5.171}$$

The only term in (5.171) which can not be estimated simply by reading off estimates of analogous terms arising in $\partial_t E_5$ estimates from before is the one involving $\partial_t^2 \left(F_3(r,\rho,\lambda(t+w))\right)$. For this term, we have

$$\begin{aligned}
&|-\frac{1}{r}\int_0^\infty \int_0^w \rho \left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)\lambda''(t+w)\partial_t^2 \left(F_3(r,\rho,\lambda(t+w))\right) d\rho dw|\\
&\leqslant Cr\int_0^\infty \rho \int_\rho^\infty \left(\frac{1}{\sqrt{w^2-\rho^2}}-\frac{1}{w}\right)\\
&\qquad \frac{\log^{3b-2\alpha b}(t)}{t^4\log^{2b+2}(t)(1+\lambda(t)^{4\alpha-4}(\rho^2-r^2)^2+2\lambda(t)^{2\alpha-2}(\rho^2+r^2))} dw d\rho\\
&\qquad \leqslant \frac{Cr}{t^4\log^{b+2}(t)}
\end{aligned}$$

Combining these, we get

$$|\partial_t^2 E_5(t,r)| \leqslant \frac{Cr}{t^4\log^{b+1}(t)} + \frac{Cr}{t\log^{(3-2\alpha)b}(t)}\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{3-2\alpha}}\right)$$

Next, we need to record new estimates on $\partial_t^2 v_4$. First, we note that, by the same procedure used to obtain estimates on $\partial_t v_1$, we have

$$|\partial_t^2 v_1(t,r)| \leqslant Cr\left(\frac{\log(t)+\log(2+r^2)}{t}\right)\sup_{x\geqslant t}\left(|\lambda''''(x)|x\right) \tag{5.172}$$

Note that this time, the preliminary estimate on λ'''' is indeed strong enough to justify the steps leading up to a $\partial_t^2 v_1$ analog of (5.12), but this will be unnecessary for our purposes. We then use (5.167), and the same estimates for all other $\partial_t^j v_k$ used to obtain the preliminary estimate on $\partial_t^2 v_{4,c}$, and get

$$\begin{aligned}&|\partial_t^2 v_{4,c}(t,r)|\\&\leqslant \frac{C\mathbb{1}_{\geqslant 1}(\frac{2r}{\log^N(t)})}{r^4\log^{2b}(t)}\\&\cdot\begin{cases}\frac{r(\log(t)+\log(2+r^2))}{t}\left(\frac{1}{t^3\log^{b+1}(t)}+\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right)\lambda(t)^{2-2\alpha}\right), & r\leqslant \frac{t}{2}\\ \frac{r(\log(t)+\log(2+r^2))}{t}\left(\frac{1}{t^3\log^{b+1}(t)}+\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right)\lambda(t)^{2-2\alpha}\right)\\ +\frac{\log(r)}{|t-r|^3}+\frac{\log(r)}{t\log(t)(t-r)^2}+\frac{\log(r)}{t^2|t-r|\log(t)}, & t>r>\frac{t}{2}\end{cases}\end{aligned} \tag{5.173}$$

which leads, via the same procedure used to obtain the preliminary estimate on $\partial_t^2 v_4$, to

$$|\partial_t^2 v_4(t,r)|\leqslant \frac{C}{t^4\log^{3b+N-2}(t)}+\frac{C\lambda(t)^{2-2\alpha}}{t\log^{2b-3+N}(t)}\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right),\quad r\leqslant\frac{t}{2}$$

As usual, for the following estimate, we modify (5.173) by using $\frac{C}{r^{9/2}\log^{2b}(t)}$ in the region $t-t^{1/6}\leqslant r\leqslant t+t^{1/6}$. (Note that the preliminary estimate on λ'''' implies the following).

$$\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right)\lambda(t)^{2-2\alpha}\frac{1}{r^4\log^{2b-1}(t)}\leqslant\frac{C}{r^{9/2}\log^{2b}(t)},\quad t-t^{1/6}\leqslant r\leqslant t+t^{1/6}$$

This gives

$$||\partial_t^2 v_{4,c}||_{L^2(rdr)}\leqslant\frac{C}{t^{47/12}\log^{2b-1}(t)}+\frac{C\lambda(t)^{2-2\alpha}}{\log^{2N+2b-1}(t)t}\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right)$$

In total, this gives

$$|\partial_t^2 v_4(t,r)|\leqslant\begin{cases}\frac{C}{t^4\log^{3b+N-2}(t)}+\frac{C\lambda(t)^{2-2\alpha}}{t\log^{2b-3+N}(t)}\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right), & r\leqslant\frac{t}{2}\\ \frac{C}{t^{35/12}\log^{2b-1}(t)}+\frac{C\lambda(t)^{2-2\alpha}}{\log^{2N+2b-2}(t)}\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right), & r\geqslant\frac{t}{2}\end{cases}$$

Using the same estimates that were used for $\partial_t^j v_5$ in the course of proving the preliminary estimate on λ'''', we get

$$\begin{aligned}&|\partial_t^2\left(-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)E_5|_{r=R\lambda(t)},\phi_0\rangle\right)|\\&\leqslant\frac{C}{t^4\log^{b+1}(t)}+\frac{C\sup_{x\geqslant t}\left(\frac{|e''''(x)|x}{\lambda(x)^{3-2\alpha}}\right)}{t\log^{(3-2\alpha)b}(t)}\end{aligned}$$

$$|\partial_t^2 \left(-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right) v_4 \left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)}, \phi_0\rangle\right)|$$
$$\leqslant \frac{C}{t^4 \log^{2b+N-2}(t)} + \frac{C\lambda(t)^{2-2\alpha} \sup_{x\geqslant t} \left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right)}{t \log^{b-3+N}(t)}$$

$$|\partial_t^2 \left(-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right) v_5 \left(1-\chi_{\geqslant 1}(\frac{4r}{t})\right)|_{r=R\lambda(t)}, \phi_0\rangle\right)|$$
$$\leqslant \frac{C}{t^4 \log^{3N-2}(t)}$$

$$|\partial_t^2 \left(-\lambda(t)\langle\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right) \chi_{\geqslant 1}(\frac{2r}{\log^N(t)})\,(v_1+v_2+v_3)\,|_{r=R\lambda(t)}, \phi_0\rangle\right)|$$
$$\leqslant \frac{C}{t^4 \log^{3b+2N}(t)} + \frac{C\lambda(t)^{2-2\alpha} \sup_{x\geqslant t} \left(\frac{|e''''(x)|x}{\lambda(x)^{2-2\alpha}}\right)}{t \log^{2N+2b-1}(t)}$$

$$|\partial_t^2 \left(\lambda(t)\langle\chi_{\geqslant 1}(\frac{2r}{\log^N(t)})F_{0,2}|_{r=R\lambda(t)}, \phi_0\rangle\right)| \leqslant \frac{C}{t^4 \log^{3b+1+2N-2b\alpha}(t)} + \frac{C|e''''(t)|}{\log^{2b+2N-2b\alpha}(t)}$$

Combining these, we get

$$|\partial_t^2 G(t, \lambda_{0,0}(t)+e(t))| \leqslant \frac{C}{t^4 \log^{b+1}(t)} + C \sup_{x\geqslant t} \left(\frac{|e''''(x)|x}{\lambda(x)^{3-2\alpha}}\right) \frac{1}{t \log^{(3-2\alpha)b}(t)}$$

and

$$|RHS_4(t)| \leqslant \frac{C}{t^4 \log^{b+1}(t)} + \frac{C \sup_{x\geqslant t}\left(\frac{x^2|e''''(x)|}{\lambda(x)^{3-2\alpha}}\right)}{\sqrt{\log(\log(t))} t^2 \log^{(3-2\alpha)b}(t)}$$

We now return to (5.164), and note that it is of the form

$$\begin{aligned}
&-4\int_t^\infty \frac{e''''(s)ds}{\log(\lambda_{0,0}(s))(1+s-t)} + 4\alpha e''''(t) \\
&-4\int_t^\infty \frac{e''''(s)ds}{\log(\lambda_{0,0}(s))(\lambda_{0,0}(t)^{1-\alpha}+s-t)(1+s-t)^3} \\
&= RHS_4(t)
\end{aligned}$$

with

$$|RHS_4(t)| \leqslant \frac{C}{t^4 \log^{b+1}(t)} + C\frac{\sup_{x \geqslant t}\left(\frac{x^2|e''''(x)|}{\lambda(x)^{3-2\alpha}}\right)}{\sqrt{\log(\log(t))}t^2 \log^{(3-2\alpha)b}(t)}$$

We are now in the same situation as for e''', and repeating the procedure used there, we get

$$|e''''(t)| \leqslant \frac{C}{t^4 \log^{b+1}(t)}$$

Using the explicit formula for $\lambda_{0,0}$, we finally conclude

$$|\lambda''''(t)| \leqslant \frac{C}{t^4 \log^{b+1}(t)}, \quad t \geqslant T_0$$

To finish the proof of the proposition, we recall (5.12), and the fact that $\partial_t^2 v_1$ has the same formula as v_1, except for λ'''' replacing λ''. Using our estimate on λ'''', we get

$$|\partial_t^2 v_1(t,r)| \leqslant \frac{C}{rt^2 \log^{b+1}(t)}$$

Combining this with (5.172), we get (5.150). □

5.9 Estimates on $\partial_r^k \partial_t^j F_4$

Later on, when we start to construct the exact solution to (2.1), we will utilize the orthogonality

$$\langle F_4, \phi_0(\frac{\cdot}{\lambda(t)})\rangle = 0, \quad t \geqslant T_0$$

by integrating by parts in various oscillatory integrals involving F_4. Therefore, we will need estimates on certain derivatives of F_4 in order to control the integrands which will arise in this process:

Proposition 5.5. *For* $0 \leqslant k \leqslant 2, \quad 0 \leqslant j \leqslant 1, \quad j+k \leqslant 2$*, we have*

$$t^j r^k |\partial_r^k \partial_t^j F_4(t,r)| \leqslant \frac{C\mathbb{1}_{\{r \leqslant \log^N(t)\}} r}{t^2 \log^{3b+1-2\alpha b}(t)(r^2+\lambda(t)^2)^2} + \frac{C\mathbb{1}_{\{r \leqslant \frac{t}{2}\}} r}{t^2 \log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2} \tag{5.174}$$

In addition, we have

$$\begin{aligned}|\partial_t^2 F_4(t,r)| \leqslant &\frac{C\mathbb{1}_{\{r \leqslant \log^N(t)\}} r}{t^4 \log^{3b+1-2\alpha b}(t)(r^2+\lambda(t)^2)^2} + \frac{C\mathbb{1}_{\{r \leqslant \frac{t}{2}\}} r}{t^4 \log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2}\\ &+ \frac{C\mathbb{1}_{\{r \leqslant \frac{t}{2}\}}}{t^4 \log^{5b+N-2}(t)(r^2+\lambda(t)^2)^2}\end{aligned} \tag{5.175}$$

Proof. We start with a lemma which captures a delicate leading order cancellation near the origin between v_1 and v_2:

Lemma 5.16 (Near origin cancellation between v_1 and v_2)**.** *For* $0 \leqslant k, j \leqslant 2, \quad k+j \leqslant 2$, *we have*

$$t^j r^k |\partial_r^k \partial_t^j (v_1+v_2)| \leqslant \frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)}, \quad r \leqslant \log^N(t)$$

Proof. We use (5.11) and (5.49), to get

$$v_1(t,r)+v_2(t,r) = r\left(\int_t^\infty \frac{\lambda''(x)}{1+x-t}dx - \frac{b}{t^2\log^b(t)}\right) + \text{Err}(t,r) + E_{v_2}(t,r)$$
$$, \quad 0 \leqslant r \leqslant \frac{t}{2}$$

Then, using the modulation equation, and the previous estimates on its terms, we get

$$|v_1(t,r)+v_2(t,r)| \leqslant \frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)} + \frac{Cr\log(3+2r)}{t^2\log^{b+1}(t)} + \frac{Cr}{t^2\log^{b+1}(t)} + \frac{Cr^2}{t^3\log^b(t)} \quad , \quad r \leqslant \frac{t}{2} \tag{5.176}$$

Note that, for (e.g) $r \leqslant \log^N(t)$, this estimate is slightly better than what would result from estimating v_1 and v_2 separately.

We use (5.13) and (5.49) to obtain the analogous estimate after taking an r derivative:

$$\partial_r v_1(t,r) + \partial_r v_2(t,r) = \int_t^\infty \frac{\lambda''(s)ds}{1+s-t} - \frac{b}{t^2\log^b(t)} + E_{\partial_r v_1}(t,r) + E_{\partial_r v_2}(t,r)$$

Again, the modulation equation, combined with the estimates for $E_{\partial_r v_1}$ and $E_{\partial_r v_2}$ from the previous subsections, give

$$|\partial_r v_1(t,r) + \partial_r v_2(t,r)| \leqslant \frac{C\log(\log(t))}{t^2\log^{b+1}(t)} + \frac{C\log(3+2r)}{t^2\log^{b+1}(t)} + \frac{C}{t^2\log^{b+1}(t)} + \frac{Cr}{t^3\log^b(t)} \quad , \quad r \leqslant \frac{t}{2} \tag{5.177}$$

We first note that, for $j=1,2$, $\partial_t^j v_1$ solves the same equation with 0 Cauchy data at infinity as v_1 does, except with $\lambda^{(2+j)}(t)$ on the right-hand side. Then, we use the fact that λ solves

$$\begin{aligned} &-4\int_t^\infty \frac{\lambda''(x)dx}{1+x-t} + \frac{4b}{t^2\log^b(t)} + 4\alpha\log(\lambda(t))\lambda''(t) \\ &-4\int_t^\infty \frac{\lambda''(s)ds}{(\lambda(t)^{1-\alpha}+s-t)(1+s-t)^3} \\ &= G(t,\lambda(t)) \end{aligned}$$

and differentiate j times for $j=1,2$, to get

$$|-4\int_t^\infty \frac{\lambda'''(s)}{1+s-t}ds - \frac{8b}{t^3\log^b(t)}| \leqslant \frac{C\log(\log(t))}{t^3\log^{b+1}(t)}$$

and

$$|\int_t^\infty \frac{\lambda''''(s)ds}{1+s-t} - \frac{6b}{t^4\log^b(t)}| \leqslant \frac{C\log(\log(t))}{t^4\log^{b+1}(t)}$$

Then, using (5.11),(5.49), we get

$$|\partial_t(v_1+v_2)(t,r)| \leqslant C\frac{r\log(\log(t))}{t^3\log^{b+1}(t)}, \quad r\leqslant \log^N(t) \tag{5.178}$$

and

$$|\partial_t^2(v_1+v_2)(t,r)| \leqslant C\frac{r\log(\log(t))}{t^4\log^{b+1}(t)}, \quad r\leqslant \log^N(t) \tag{5.179}$$

Next, we use the equations solved by v_1, v_2, to get

$$r^2\partial_{rr}(v_1+v_2) = r^2\left(\frac{-2r\lambda''(t)}{1+r^2}\right) + r^2\partial_{tt}(v_1+v_2) - r\partial_r(v_1+v_2) + (v_1+v_2)$$

which gives

$$r^2|\partial_{rr}(v_1+v_2)| \leqslant C\frac{r\log(\log(t))}{t^2\log^{b+1}(t)}, \quad r\leqslant \log^N(t) \tag{5.180}$$

Finally, we study $\partial_{tr}(v_1+v_2)$. Because $\partial_t v_1$ has the same representation formula as v_1, except with λ'' replaced by λ''', we can use the same procedure used for $\partial_r v_1$, to get

$$\partial_{tr}v_1(t,r) = \int_t^\infty \frac{\lambda'''(s)ds}{1+s-t} + E_{\partial_{tr}v_1}(t,r)$$

with

$$|E_{\partial_{tr}v_1}(t,r)| \leqslant \frac{C\log(3+2r)}{t^3\log^{b+1}(t)}$$

Then, we use (5.49) to conclude

$$|\partial_{tr}(v_1+v_2)(t,r)| \leqslant \frac{C\log(\log(t))}{t^3\log^{b+1}(t)}, \quad r\leqslant \log^N(t) \tag{5.181}$$

□

Next, we note that our previous estimate for $\partial_r v_3$, (5.79), sufficed for all of our purposes up to now. For estimating derivatives of F_4, however, we will require a different estimate which will turn out to be better in the region $r\leqslant \log^N(t)$, and which will require a more complicated argument than before. This refinement will also lead to slightly better estimates on $\partial_{tr}v_3$ and $r^2\partial_{rr}v_3$ near the origin.

Lemma 5.17 (Near origin refinements of various v_3 related estimates)**.**

$$|\partial_r v_3(t,r)| \leqslant \frac{C}{t^2\log^{b+1}(t)}\left(\log(\log(t)) + \log(6+6r)\right) \tag{5.182}$$

$$|\partial_{tr}v_3(t,r)| \leqslant \frac{C\log(\log(t))}{t^3\log^{b+1}(t)}, \quad r\leqslant \log^N(t)$$

$$r^2|\partial_{rr}v_3| \leqslant \frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)}, \quad r\leqslant \log^N(t) \tag{5.183}$$

Proof. We first use the same decomposition on $\partial_r v_3$ used before, and only need to use a different argument for one term, namely:

$$-\int_t^\infty \frac{1}{(s-t)} \int_0^{s-t} \rho \frac{\lambda''(s)}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s)) \right) d\rho ds$$

We treat different regions of the variable $s-t$. First, from our previous $\partial_r v_3$ estimate, we have

$$|\int_0^{s-t} \rho \frac{\lambda''(s)}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} \right) d\rho| \leqslant C(s-t)^2 |\lambda''(s)|, \quad s-t \leqslant \frac{1}{2}$$

which gives

$$\begin{aligned}
&|-\int_t^{t+\frac{1}{2}} \frac{1}{(s-t)} \int_0^{s-t} \rho \frac{\lambda''(s)}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} \right) d\rho ds| \\
&\leqslant \frac{C}{t^2 \log^{b+1}(t)}
\end{aligned}$$

Next, we have

$$\begin{aligned}
&|-\int_{t+\frac{1}{2}}^{t+6} \frac{1}{(s-t)} \int_0^{s-t} \rho \frac{\lambda''(s)}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} \right) d\rho ds| \\
&\leqslant \int_{t+\frac{1}{2}}^{t+6} \frac{1}{(s-t)} \int_0^{\infty} \rho \frac{|\lambda''(s)|}{r} |\partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} \right) | d\rho ds \\
&\leqslant C \int_{t+\frac{1}{2}}^{t+6} \frac{|\lambda''(s)|}{(s-t)} ds \\
&\leqslant C \frac{1}{t^2 \log^{b+1}(t)}
\end{aligned}$$

For convenience, we recall the definition of F_3:

$$F_3(r,\rho,\lambda(s)) = \frac{1-(r^2-\rho^2)\lambda(s)^{2\alpha-2}}{\sqrt{4r^2\lambda(s)^{2\alpha-2} + (1-(r^2-\rho^2)\lambda(s)^{2\alpha-2})^2}}$$

Similarly, we have

$$|\partial_r F_3(r,\rho,\lambda(s))| \leqslant \frac{Cr\lambda(s)^{2+2\alpha}((\rho^2+r^2)\lambda(s)^{2\alpha-2}+1)}{\lambda(s)^4(1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4})^{3/2}}$$

For the region $s-t \leqslant \lambda(t)^{1-\alpha}$, we continue this estimate, and get

$$|\partial_r F_3(r,\rho,\lambda(s))| \leqslant Cr\lambda(s)^{2\alpha-2}$$

So,

$$
\begin{aligned}
&|\int_t^{t+\lambda(t)^{1-\alpha}} \frac{1}{(s-t)} \int_0^{s-t} \frac{\rho\lambda''(s)}{r} \partial_r F_3(r,\rho,\lambda(s)) d\rho ds| \\
&\leqslant C \int_t^{t+\lambda(t)^{1-\alpha}} \frac{1}{(s-t)} \int_0^{s-t} \frac{\rho}{r} |\lambda''(s)| r \lambda(s)^{2\alpha-2} d\rho ds \\
&\leqslant \frac{C}{t^2 \log^{b+1}(t)}
\end{aligned}
$$

$$
\begin{aligned}
&|\int_{t+\lambda(t)^{1-\alpha}}^{t+6} \frac{1}{(s-t)} \int_0^{s-t} \frac{\rho\lambda''(s)}{r} \partial_r F_3(r,\rho,\lambda(s)) d\rho ds| \\
&\leqslant C \int_{t+\lambda(t)^{1-\alpha}}^{t+6} \frac{1}{(s-t)} \int_0^{\infty} \frac{\rho|\lambda''(s)|}{r} |\partial_r F_3(r,\rho,\lambda(s))| d\rho ds \\
&\leqslant \frac{C}{r} \int_{t+\lambda(t)^{1-\alpha}}^{t+6} \frac{|\lambda''(s)|}{(s-t)} r ds \leqslant \frac{C \log(\log(t))}{t^2 \log^{b+1}(t)}
\end{aligned}
$$

Then, we treat the region $6 \leqslant s-t \leqslant 6+6r$ in a similar fashion:

$$
\begin{aligned}
&|-\int_{t+6}^{t+6+6r} \frac{1}{(s-t)} \int_0^{s-t} \rho \frac{\lambda''(s)}{r} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s)) \right) d\rho ds| \\
&\leqslant \int_{t+6}^{t+6+6r} \frac{1}{(s-t)} \int_0^{\infty} \rho \frac{|\lambda''(s)|}{r} \left(|\partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}} \right)| + |\partial_r F_3(r,\rho,\lambda(s))| \right) d\rho ds \\
&\leqslant C \int_{t+6}^{t+6+6r} \frac{|\lambda''(s)|}{(s-t)} ds \\
&\leqslant \frac{C \log(6+6r)}{t^2 \log^{b+1}(t)}
\end{aligned}
$$

Now, we study the region $s-t \geqslant 6+6r$:
We first note that

$$
\int_0^{\infty} \partial_r \left(\frac{-1-\rho^2+r^2}{\sqrt{(1+\rho^2-r^2)^2+4r^2}} + F_3(r,\rho,\lambda(s)) \right) \rho d\rho = 0
$$

which follows from the fact that the integrand of the expression above is equal to

$$
\begin{aligned}
\partial_\rho \Bigg(r \Bigg(&\frac{-1+\rho^2-r^2}{\sqrt{\rho^4-2\rho^2(-1+r^2)+(1+r^2)^2}} \\
&+ \frac{1+(r^2-\rho^2)\lambda(s)^{2\alpha-2}}{\sqrt{1+2(\rho^2+r^2)\lambda(s)^{2\alpha-2}+(\rho^2-r^2)^2\lambda(s)^{4\alpha-4}}} \Bigg) \Bigg)
\end{aligned}
$$

Then, we use

$$r\left(\frac{-1+(s-t)^2-r^2}{\sqrt{(s-t)^4-2(s-t)^2(-1+r^2)+(1+r^2)^2}}\right)=r\left(1+E_{\partial_r v_3,1}\right)$$

with

$$|E_{\partial_r v_3,1}|\leqslant C\frac{(1+r)^2}{(s-t)^2},\quad s-t\geqslant 6+6r$$

and

$$r\left(\frac{1+(r^2-(s-t)^2)\lambda(s)^{2\alpha-2}}{\sqrt{1+2((s-t)^2+r^2)\lambda(s)^{2\alpha-2}+((s-t)^2-r^2)^2\lambda(s)^{4\alpha-4}}}\right)=r\left(-1+E_{\partial_r v_3,2}\right)$$

with

$$|E_{\partial_r v_3,2}|\leqslant C\frac{(1+r)^2}{(s-t)^2},\quad s-t\geqslant 6+6r$$

which gives

$$\begin{aligned}
&|-\int_{t+6+6r}^{\infty}\frac{1}{(s-t)}\int_0^{s-t}\rho\frac{\lambda''(s)}{r}\partial_r\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds|\\
&=|\int_{t+6+6r}^{\infty}\frac{1}{(s-t)}\int_{s-t}^{\infty}\rho\frac{\lambda''(s)}{r}\partial_r\left(\frac{-1-\rho^2+r^2}{\sqrt{(-1-\rho^2+r^2)^2+4r^2}}+F_3(r,\rho,\lambda(s))\right)d\rho ds|\\
&=|\int_{t+6+6r}^{\infty}\frac{1}{(s-t)}\frac{\lambda''(s)}{r}r\left(E_{\partial_r v_3,1}+E_{\partial_r v_3,2}\right)ds|\\
&\leqslant C\int_{t+6+6r}^{\infty}\frac{|\lambda''(s)|}{(s-t)}\frac{(1+r)^2}{(s-t)^2}ds\\
&\leqslant\frac{C}{t^2\log^{b+1}(t)}
\end{aligned}$$

In total, we get

$$|\partial_r v_3(t,r)|\leqslant\frac{C}{t^2\log^{b+1}(t)}\left(\log(\log(t))+\log(6+6r)\right)$$

Note that this estimate is better than (5.79), in the region $r\leqslant\log^N(t)$. For $\partial_{tr}v_3$, we recall that the partial r derivative of the second line of (5.147) was bounded above in absolute value by $\frac{C}{t^3\log^{b+1}(t)}$, and an estimate of the r derivative of the first line of (5.147) was inferred from an estimate on $\partial_r v_3$. Using the above near origin refinement of the $\partial_r v_3$ estimate, instead of the previous one, gives

$$|\partial_{tr}v_3(t,r)|\leqslant\frac{C\log(\log(t))}{t^3\log^{b+1}(t)},\quad r\leqslant\log^N(t)$$

Now, we can prove a different estimate for $r^2\partial_r^2 v_3$ than what follows from previous work.

$$r^2\partial_{rr}v_3=r^2F_{0,1}(t,r)+r^2\partial_{tt}v_3-r\partial_r v_3+v_3$$

Using (5.182), as well as our previous estimates on v_3, we get

$$r^2|\partial_{rr}v_3| \leqslant \frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)}, \quad r \leqslant \log^N(t)$$

□

Finally, the estimate for $\partial_t v_5$ used when estimating λ''' was based on estimates for $N_2(f)$ and $\partial_r N_2(f)$ which only used at most two derivatives of λ. For future use, we will prove a stronger estimate on $\partial_t v_5$, which uses the final estimate on λ'''. (Note that this is the reason why no estimates on $\partial_t v_5$ were presented in the proposition statement in the λ''' section). As part of the process, we will also obtain an estimate on $\partial_{tr}v_4$.

Lemma 5.18 (Improved $\partial_t v_5$, $\partial_{tr}v_4$ estimates)**.**

$$|\partial_t v_5(t,r)| \leqslant \frac{Cr}{t^{9/2}\log^{\frac{3N}{2}+3b-2}(t)}, \quad r \leqslant \frac{t}{2} \tag{5.184}$$

$$|\partial_{tr}v_4(t,r)| \leqslant \begin{cases} \frac{C}{t^3\log^{3b+2N-2}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{t^{35/12}\log^{2b-1}(t)}, & r \geqslant \frac{t}{2} \end{cases} \tag{5.185}$$

Proof. For this estimate, we start with

$$\partial_t v_5(t,r) = \frac{-r}{2\pi}\int_0^1\int_t^\infty\int_{B_{s-t}(0)} \frac{1}{\sqrt{(s-t)^2-|y|^2}} \left(\frac{\partial_{12}N_2(f_{v_5})(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^2} - \frac{\partial_1 N_2(f_{v_5})(s,|\beta x+y|)\left((\beta x+y)\cdot\hat{x}\right)^2}{|\beta x+y|^3} + \frac{\partial_1 N_2(f_{v_5})(s,|\beta x+y|)}{|\beta x+y|}\right) dA(y)dsd\beta$$

This gives

$$\begin{aligned}&|\partial_t v_5(t,r)| \\ &\leqslant Cr\int_0^1\int_t^\infty\int_{B_{s-t}(0)} \frac{1}{\sqrt{(s-t)^2-|y|^2}} \\ &\qquad \left(|\partial_{12}N_2(f_{v_5})|(s,|\beta x+y|) + \frac{|\partial_1 N_2(f_{v_5})|(s,|\beta x+y|)}{|\beta x+y|}\right) dA(y)dsd\beta\end{aligned} \tag{5.186}$$

In order to proceed, we will have to estimate $\partial_{tr}v_4$, which was not done previously. We start by noting that, the procedure used to obtain (5.149) was to write

$$\partial_t v_4(t,r) = r\int_0^1 \partial_{tr}v_4(t,r\beta)d\beta$$

and then, to estimate $\partial_{tr} v_4(t, r\beta)$ uniformly for $0 \leqslant \beta \leqslant 1$. So, in the region $r \leqslant \frac{t}{2}$, we have

$$|\partial_{tr} v_4(t,r)| \leqslant \frac{C}{t^3 \log^{3b+2N-2}(t)}, \quad r \leqslant \frac{t}{2}$$

For the region $r \geqslant \frac{t}{2}$, we start by introducing the vector field

$$V = t\partial_t + r\partial_r$$

and recalling that

$$-\partial_{tt} v_4 + \partial_{rr} v_4 + \frac{1}{r}\partial_r v_4 - \frac{v_4}{r^2} = v_{4,c}$$

So, we have

$$\left(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\left(V\left(\partial_t v_4\right)\right) = V\left(\partial_t v_{4,c}\right) + 2\partial_t v_{4,c}$$

We use (5.148), (5.173), and

$$|\partial_t^2 v_{4,c}(t,r)| + |\partial_{tr} v_{4,c}(t,r)| \leqslant \frac{C}{t^{9/2}\log^{2b}(t)}, \quad t - t^{1/6} \leqslant r \leqslant t + t^{1/6}$$

to get

$$|V(\partial_t v_{4,c})|(t,r) \leqslant C \begin{cases} \frac{\mathbb{1}_{\{r \geqslant \frac{\log^N(t)}{2}\}}}{r^3 t^3 \log^{3b}(t)}, \quad r \leqslant \frac{t}{2} \\ \frac{\log(r)}{r^3 \log^{2b}(t)|t-r|^3} + \frac{\log(r)}{r^3 t \log^{2b}(t)(t-r)^2} + \frac{\log(r)}{\log^{2b+1}(t) r^4 t |t-r|} \\ + \frac{\log(r)}{r^3 \log^{3b+1}(t) t^3}, \quad \frac{t}{2} \leqslant r \leqslant t - t^{1/6}, \text{ or } r \geqslant t + t^{1/6} \\ \frac{1}{t^{7/2}\log^{2b}(t)}, \quad t - t^{1/6} \leqslant r \leqslant t + t^{1/6} \end{cases}$$

and we get

$$||V(\partial_t v_{4,c})||_{L^2(rdr)} \leqslant \frac{C}{t^{35/12}\log^{2b-1}(t)}$$

Next, we recall (5.143), and get

$$||\partial_t v_{4,c}||_{L^2(rdr)} \leqslant \frac{C}{t^3 \log^{2N+3b}(t)}$$

Then, we apply the same procedure used before to estimate (e.g.) $\partial_t v_4$ in the region $r \geqslant \frac{t}{2}$, to the equation for $V(\partial_t v_4)$, and get

$$|V(\partial_t v_4)|(t,r) \leqslant \frac{C}{t^{23/12}\log^{2b-1}(t)}$$

But, then,

$$\begin{aligned} \partial_{tr} v_4(t,r) &= \frac{(t\partial_t^2 v_4 + r\partial_{tr} v_4)}{r} - \frac{t}{r}\partial_t^2 v_4 \\ &= \frac{V(\partial_t v_4)}{r} - \frac{t}{r}\partial_t^2 v_4 \end{aligned}$$

and we have already estimated $\partial_t^2 v_4$, during our study of λ''''. So, we get

$$|\partial_{tr} v_4(t,r)| \leqslant \frac{C}{t^{35/12} \log^{2b-1}(t)}, \quad r \geqslant \frac{t}{2}$$

Now, we can estimate $\partial_{tr} N_2(f)$, and get

$$|\partial_{tr} N_2(f)|(t,r) \leqslant \begin{cases} \frac{C}{t^5 \log^{3b}(t)(r^2+\lambda(t)^2)}, & r \leqslant \frac{t}{2} \\ \frac{C\log^3(r)}{r^2|t-r|^5} + \frac{C\log^3(r)}{r^{5/2}\sqrt{t}(t-r)^4} + \frac{C\log^2(r)}{r^2 t^{3/2}|t-r|^3 \log^{3b-1+\frac{5N}{2}}(t)}, & t > r > \frac{t}{2} \end{cases}$$

(Note that we only need to estimate $\partial_{tr} N_2(f_{v_5})$ in the region $r < t$, which is why we used $\frac{1}{|t-r|} \geqslant \frac{1}{t}$ to simply the above estimate. In previous estimates, we did not proceed analogously because we eventually used a single estimate for the entire region $\frac{t}{2} \leqslant r \leqslant t - t^{\gamma}$, or $r \geqslant t + t^{\gamma}$ for some $\gamma > 0$).

Using the estimates on $\partial_t^j v_k, \quad j = 0,1, \quad k = 1,2,3,4$, we get

$$|\partial_t N_2(f)(t,r)| \leqslant \begin{cases} \frac{Cr}{t^5 \log^{3b}(t)(r^2+\lambda(t)^2)}, & r \leqslant \frac{t}{2} \\ \frac{C}{r^3 t^{7/2} \log^{\frac{3N}{2}+5b}(t)} + \frac{C\log^3(r)}{r^2(t-r)^4}, & t > r > \frac{t}{2} \end{cases}$$

Now, we return to (5.186), and use the same procedure used to estimate v_5 in the region $r \leqslant \frac{t}{2}$ earlier. (In particular, we handle the contribution to $\partial_t v_5$ coming from the term $\frac{\log^3(r)}{r^2|t-r|^5}$, which arises in the estimate for $\partial_{tr} N_2(f_{v_5})$, using the same procedure as in (5.96)). This results in

$$|\partial_t v_5(t,r)| \leqslant \frac{Cr}{t^{9/2} \log^{\frac{3N}{2}+3b-2}(t)}, \quad r \leqslant \frac{t}{2}$$

□

Now, we proceed to estimate F_4, and various of its derivatives. We recall

$$F_4(t,r) = \left(1 - \chi_{\geqslant 1}\left(\frac{2r}{\log^N(t)}\right)\right)\left(F_{0,2}(t,r) + \left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r)) - 1}{r^2}\right)(v_1+v_2+v_3)\right) + \left(1 - \chi_{\geqslant 1}\left(\frac{4r}{t}\right)\right)\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r)) - 1}{r^2}\right)(v_4+v_5)$$

Then, using (5.77), (5.83), (5.91), and (5.176), we get

$$|F_4(t,r)| \leqslant C\frac{\left(1-\chi_{\geqslant 1}\left(\frac{2r}{\log^N(t)}\right)\right)}{(r^2+\lambda(t)^2)^2}\left(\frac{r}{t^2 \log^{3b+1-2b\alpha}(t)}\right) + C\frac{\left(1-\chi_{\geqslant 1}\left(\frac{4r}{t}\right)\right)}{(r^2+\lambda(t)^2)^2}\frac{r}{t^2\log^{5b+2N-1}(t)} \tag{5.187}$$

Next, we will need to record estimates on $\partial_t F_4$, and $r\partial_r F_4$. By combining (5.122), and (5.184), we get

$$|\partial_t v_4 + \partial_t v_5|(t,r) \leqslant \frac{Cr}{t^3 \log^{3b+2N-2}(t)}, \quad r \leqslant \frac{t}{2}$$

This, combined with the explicit formula for $F_{0,2}$, (5.178), and (5.119), gives

$$|\partial_t F_4(t,r)| \leqslant \frac{C\mathbb{1}_{\{r\leqslant \log^N(t)\}} r}{t^3 \log^{3b+1-2b\alpha}(t)(r^2+\lambda(t)^2)^2} + \frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}}}{(r^2+\lambda(t)^2)^2} \frac{r}{t^3 \log^{5b+2N-2}(t)}$$

where we estimate derivatives on the cutoff functions by (e.g.)

$$|\chi'_{\geqslant 1}(\frac{2r}{\log^N(t)})| \frac{Nr}{t\log^{N+1}(t)} \leqslant \frac{C\mathbb{1}_{\{r\leqslant \log^N(t)\}}}{t\log(t)}$$

Similarly, for the partial r derivative of F_4, we use the explicit formula for $F_{0,2}$, (5.182), (5.177), (5.84), and (5.92), to get

$$r|\partial_r F_4(t,r)| \leqslant \frac{Cr\mathbb{1}_{\{r\leqslant \log^N(t)\}}}{t^2 \log^{3b+1-2b\alpha}(t)(r^2+\lambda(t)^2)^2} + C\frac{\mathbb{1}_{\{r\leqslant \frac{t}{2}\}}}{(r^2+\lambda(t)^2)^2} \frac{r}{t^2 \log^{5b+2N-1}(t)} \tag{5.188}$$

Treating the terms involving $\chi_{\geqslant 1}$ and $F_{0,2}$ as before, and using (5.179), (5.151), and (5.152), we get

$$\begin{aligned}|\partial_t^2 F_4(t,r)| &\leqslant \frac{C\mathbb{1}_{\{r\leqslant \log^N(t)\}} r}{t^4 (r^2+\lambda(t)^2)^2 \log^{3b+1-2b\alpha}(t)} + \frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}} r}{t^4 \log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2}\\ &+ \frac{C\mathbb{1}_{\{r\leqslant \frac{t}{2}\}}}{t^4 \log^{5b+N-2}(t)(r^2+\lambda(t)^2)^2}\end{aligned}$$

Next, we note that, exactly as was the case with $\partial_{tr} v_4$, we can infer the following estimate on $\partial_{tr} v_5$ by inspecting (5.184), and the procedure used to obtain it:

$$|\partial_{tr} v_5(t,r)| \leqslant \frac{C}{t^{9/2} \log^{\frac{3N}{2}+3b-2}(t)}, \quad r \leqslant \frac{t}{2} \tag{5.189}$$

We use the same procedure, (5.181), (5.189), and (5.185), to get

$$r|\partial_{rt} F_4(t,r)| \leqslant C\frac{\mathbb{1}_{\{r\leqslant \log^N(t)\}} r}{t^3 \log^{3b+1-2b\alpha}(t)(r^2+\lambda(t)^2)^2} + C\frac{\mathbb{1}_{\{r\leqslant \frac{t}{2}\}} r}{t^3 \log^{5b+2N-2}(t)(r^2+\lambda(t)^2)^2}$$

Next, we use the same procedure used for v_1+v_2, v_3 to estimate $r^2\partial_r^2 v_4, r^2\partial_r^2 v_5$:

$$r^2 \partial_r^2 v_4 = r^2 v_{4,c}(t,r) + r^2 \partial_{tt} v_4 - r\partial_r v_4 + v_4$$

So,

$$\begin{aligned}&|r^2\partial_r^2 v_4(t,r)|\\ &\leqslant \frac{Cr}{t^2 \log^{3b+2N}(t)} + \frac{Cr^2}{t^4 \log^{3b-2+N}(t)} + \frac{Cr}{t^2 \log^{3b+2N-1}(t)} + \frac{Cr}{t^2 \log^{3b+2N-1}(t)}\\ &\leqslant \frac{Cr}{t^2 \log^{3b+2N-1}(t)}, \quad r \leqslant \frac{t}{2}\end{aligned} \tag{5.190}$$

and

$$\begin{aligned}
&|r^2\partial_r^2 v_5(t,r)|\\
&\leqslant \frac{Cr}{t^4\log^{3b}(t)}+\frac{Cr^2}{t^4\log^{3N+b-2}(t)}+\frac{Cr}{t^{7/2}\log^{\frac{5N}{2}+3b-3}(t)}+\frac{Cr}{t^{7/2}\log^{\frac{5N}{2}+3b-3}(t)}\\
&\leqslant \frac{Cr}{t^3\log^{3N+b-2}(t)}, \quad r\leqslant\frac{t}{2}
\end{aligned} \tag{5.191}$$

By using (5.180), (5.183), (5.190), and (5.191), we get

$$r^2|\partial_{rr}F_4(t,r)|\leqslant C\frac{\mathbb{1}_{\{r\leqslant \log^N(t)\}}r}{t^2\log^{3b+1-2b\alpha}(t)(r^2+\lambda(t)^2)^2}+\frac{C\mathbb{1}_{\{r\leqslant\frac{t}{2}\}}r}{t^2\log^{5b+2N-1}(t)(r^2+\lambda(t)^2)^2}$$

□

5.10 Estimates on F_5

Recall that F_5 was one of the v_6-independent error terms on the right-hand side of (5.26) which was not included in the modulation equation, and is therefore, not necessarily orthogonal to $\phi_0(\frac{\cdot}{\lambda(t)})$. Hence, we must prove that it decays sufficiently quickly in sufficiently many norms. This is the result of the following lemma.

Lemma 5.19. *We have the following estimates on F_5:*

$$\frac{1}{\lambda(x)^2}||F_5(x,r)||_{L^2(rdr)}\leqslant\frac{C\log^{6+2b}(x)}{x^{17/4}}$$

$$\frac{||F_5(x,\cdot\lambda(x))||_{\dot{H}^1_e}}{\lambda(x)}\leqslant\frac{C\log^{6+b}(x)}{x^{35/8}}$$

Proof. We recall (5.28):

$$\begin{aligned}
F_5(t,r) &= N_2(v_5)(t,r)\\
&+\frac{\sin(2(v_1+v_2+v_3+v_4))}{2r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)\\
&+\left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{2r^2}\right)\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right)
\end{aligned}$$

Then, we start with

$$|N_2(v_5)|(t,r)\leqslant\begin{cases}\frac{Cr}{(r^2+\lambda(t)^2)t^7\log^{7b-6+5N}(t)}, & r\leqslant\frac{t}{2}\\ \frac{C\log^8(t)}{r^{7/2}\log^b(t)t^{7/2}}, & r\geqslant\frac{t}{2}\end{cases}$$

Next, we have

$$\begin{aligned}
&\left|\frac{\sin(2(v_1+v_2+v_3+v_4))}{2r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}})\left(\cos(2v_5)-1\right)-\sin(2Q_{\frac{1}{\lambda(t)}})\sin(2v_5)\right)\right|\\
&\leqslant C\frac{|v_1+v_2+v_3+v_4|}{r^2}\left(v_5(t,r)^2+\frac{r\lambda(t)}{r^2+\lambda(t)^2}|v_5(t,r)|\right)
\end{aligned}$$

$$|\left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{2r^2}\right)\left(\sin(2Q_{\frac{1}{\lambda(t)}})\left(\cos(2v_5)-1\right)+\cos(2Q_{\frac{1}{\lambda(t)}})\sin(2v_5)\right)|$$
$$\leqslant \frac{C(v_1+v_2+v_3+v_4)^2}{r^2}\left(\frac{r\lambda(t)v_5^2}{r^2+\lambda(t)^2}+|v_5|\right)$$

Using our previous pointwise estimates on v_1, v_2, v_3, v_4, v_5, we get

$$|F_5(t,r)| \leqslant \begin{cases} \frac{Cr}{(r^2+\lambda(t)^2)t^{11/2}\log^{5b-3+\frac{5N}{2}}(t)}, \quad r \leqslant \frac{t}{2} \\ \frac{C\log(r)\log^4(t)}{r^3t^2\log^b(t)|t-r|} + \frac{C\log^4(t)\log^2(r)}{(t-r)^2r^{5/2}t^{3/2}} \\ +\frac{C\log^4(t)}{r^{7/2}t^3\log^{\frac{3N}{2}+4b-1}(t)}, \quad \frac{t}{2} \leqslant r \leqslant t-\sqrt{t}, \text{ or } r > t+\sqrt{t} \\ \frac{\log^4(t)}{r^{7/2}t^{3/2}}, \quad t-\sqrt{t} \leqslant r \leqslant t+\sqrt{t} \end{cases}$$

This gives

$$\frac{1}{\lambda(x)^2}||F_5(x,r)||_{L^2(rdr)} \leqslant \frac{C\log^{6+2b}(x)}{x^{17/4}}$$

We then proceed to estimate $||F_5(t,\cdot\lambda(t))||_{\dot{H}^1_e}$

We start with

$$\begin{aligned} |\partial_r N_2(v_5)(t,r)| \leqslant \frac{C\lambda(t)}{r^2(r^2+\lambda(t)^2)}v_5(t,r)^2 + \frac{C\lambda(t)|v_5(t,r)\partial_r v_5(t,r)|}{r(r^2+\lambda(t)^2)} \\ + \frac{C|\partial_r v_5(t,r)|v_5(t,r)^2}{r^2} + \frac{C|v_5(t,r)|^3}{r^3} \end{aligned} \tag{5.192}$$

For the two terms on the right-hand side of the above equation which involve $\partial_r v_5$, we estimate as follows:

$$||\frac{\lambda(t)|v_5(t,r)\partial_r v_5(t,r)|}{r(r^2+\lambda(t)^2)}|_{r=R\lambda(t)}||_{L^2(RdR)}$$
$$\leqslant C\lambda(t)||\frac{v_5(t,r)}{r(r^2+\lambda(t)^2)}||_{L^\infty}\cdot||(\partial_2 v_5)(t,\cdot\lambda(t))||_{L^2(RdR)} \leqslant \frac{C}{t^{21/4}\log^{b-6+\frac{5N}{2}}(t)}$$

$$||\frac{|\partial_r v_5(t,r)|v_5(t,r)^2}{r^2}|_{r=R\lambda(t)}||_{L^2(RdR)} \leqslant ||(\partial_2 v_5)(t,\cdot\lambda(t))||_{L^2(RdR)}\cdot||\frac{v_5^2}{r^2}||_{L^\infty}$$
$$\leqslant \frac{C\log^{11+b}(t)}{t^{31/4}}$$

For the other terms in (5.192), we use the v_5 pointwise estimates. In total, we get

$$||(\partial_2 N_2(v_5))(t,\cdot\lambda(t))||_{L^2(RdR)} \leqslant \frac{C}{t^{21/4}\log^{b-6+\frac{5N}{2}}(t)}$$

Next, we have

$$\begin{aligned}&\partial_r\left(\frac{\sin(2(v_1+v_2+v_3+v_4))}{2r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)\right)\\&=\frac{\cos(2(v_1+v_2+v_3+v_4))\partial_r(v_1+v_2+v_3+v_4)}{r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)\\&-\frac{\sin(2(v_1+v_2+v_3+v_4))}{r^3}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)\\&+\frac{\sin(2(v_1+v_2+v_3+v_4))}{r^2}\left(-\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)\partial_r(Q_{\frac{1}{\lambda(t)}}+v_5)+\sin(2Q_{\frac{1}{\lambda(t)}})\partial_r Q_{\frac{1}{\lambda(t)}}\right)\end{aligned}$$

For $\partial_r v_k, \quad k=1,2,4$, we use the pointwise estimates from the previous subsections. On the other hand, for $\partial_r v_3$, we use the energy estimate procedure, previously used for $\partial_r v_5$, to get

$$\begin{aligned}||\partial_r v_3(t)||_{L^2(rdr)}&\leqslant C\int_t^\infty ||F_{0,1}(s)||_{L^2(rdr)}ds\leqslant C\int_t^\infty \frac{\sqrt{\log(\log(s))}}{s^2\log^{b+1}(s)}ds\\&\leqslant \frac{C\sqrt{\log(\log(t))}}{t\log^{b+1}(t)}\end{aligned}$$

Then, we have

$$\begin{aligned}&||\frac{\cos(2(v_1+v_2+v_3+v_4))\partial_r(v_1+v_2+v_3+v_4)}{r^2}|_{r=R\lambda(t)}\\&\cdot\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\&\leqslant ||\frac{\partial_r(v_1+v_2+v_4)}{r^2}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\&+||(\partial_2 v_3)(t,R\lambda(t))||_{L^2(RdR)}||\frac{\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)}{r^2}||_{L^\infty}\\&\leqslant\frac{C\sqrt{\log(\log(t))}}{t^{9/2}\log^{2b-2+\frac{5N}{2}}(t)}\end{aligned}$$

Using the pointwise estimates on v_5, we get

$$\begin{aligned}&||-\frac{\sin(2(v_1+v_2+v_3+v_4))}{r^3}\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\&\leqslant\frac{C}{t^{11/2}\log^{3b-3+\frac{5N}{2}}(t)}\end{aligned}$$

We use

$$\begin{aligned}&|\frac{\sin(2(v_1+v_2+v_3+v_4))}{r^2}\left(-\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)\partial_r(Q_{\frac{1}{\lambda(t)}}+v_5)+\sin(2Q_{\frac{1}{\lambda(t)}})\partial_r Q_{\frac{1}{\lambda(t)}}\right)|\\&\leqslant\frac{C|v_1+v_2+v_3+v_4|}{r^2}\left(\frac{r\lambda(t)^2v_5^2}{(r^2+\lambda(t)^2)^2}+\frac{\lambda(t)|v_5(t,r)|}{r^2+\lambda(t)^2}+\frac{|\partial_r v_5|r\lambda(t)}{r^2+\lambda(t)^2}+|v_5\partial_r v_5|\right)\end{aligned}\tag{5.193}$$

The first term in (5.193) which involves $\partial_r v_5$ is estimated as follows:

$$\begin{aligned}&||\frac{|v_1+v_2+v_3+v_4|}{r^2}\left(\frac{|\partial_r v_5| r\lambda(t)}{r^2+\lambda(t)^2}\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\&\leqslant C||\frac{|v_1+v_2+v_3+v_4|}{r^2}\left(\frac{|\partial_r v_5| r\lambda(t)}{r^2+\lambda(t)^2}\right)|_{r=R\lambda(t)}||_{L^2((0,\frac{t}{2\lambda(t)}),RdR)}\\&+C||(\partial_2 v_5)(t,R\lambda(t))||_{L^2(RdR)}\cdot||\frac{(v_1+v_2+v_3+v_4)}{r(r^2+\lambda(t)^2)}\lambda(t)||_{L^\infty(r\geqslant\frac{t}{2})}\end{aligned}\tag{5.194}$$

where the second line of (5.194) is estimated using the pointwise estimates on $\partial_r v_5$ in the region $r \leqslant \frac{t}{2}$. In total, we get

$$||\frac{|v_1+v_2+v_3+v_4|}{r^2}\left(\frac{|\partial_r v_5| r\lambda(t)}{r^2+\lambda(t)^2}\right)|_{r=R\lambda(t)}||_{L^2(RdR)} \leqslant \frac{C\log^4(t)}{t^{21/4}}$$

The second term of (5.193) which involves $\partial_r v_5$ is estimated as follows:

$$||\frac{v_1+v_2+v_3+v_4}{r}||_{L^\infty}\cdot||\frac{v_5}{r}||_{L^\infty}\cdot||(\partial_2 v_5)(t,\cdot\lambda(t))||_{L^2(RdR)} \leqslant \frac{C\log^8(t)}{t^{\frac{25}{4}}\lambda(t)}$$

The other terms of (5.193) are estimated using the v_k pointwise estimates for $1 \leqslant k \leqslant 5$. In total, we get

$$\begin{aligned}&||\frac{\sin(2(v_1+v_2+v_3+v_4))}{r^2}|_{r=R\lambda(t)}\\&\cdot\left(-\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)\partial_r(Q_{\frac{1}{\lambda(t)}}+v_5)+\sin(2Q_{\frac{1}{\lambda(t)}})\partial_r Q_{\frac{1}{\lambda(t)}}\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\&\leqslant\frac{C\log^4(t)}{t^{21/4}}\end{aligned}$$

Finally, we have

$$\begin{aligned}&\partial_r\left(\left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{2r^2}\right)\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right)\right)\\&=\frac{-\sin(2(v_1+v_2+v_3+v_4))}{r^2}\partial_r(v_1+v_2+v_3+v_4)\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right)\\&-\frac{(\cos(2(v_1+v_2+v_3+v_4))-1)}{r^3}\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right)\\&+\left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{r^2}\right)\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)(\partial_r Q_{\frac{1}{\lambda(t)}}+\partial_r v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\partial_r Q_{\frac{1}{\lambda(t)}}\right)\end{aligned}\tag{5.195}$$

The second line of (5.195) is estimated by

$$\begin{aligned}
&||\frac{-\sin(2(v_1+v_2+v_3+v_4))}{r^2}\partial_r(v_1+v_2+v_3+v_4)|_{r=R\lambda(t)}\\
&\quad\cdot\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\
&\leqslant C||\frac{v_1+v_2+v_3+v_4}{r}||_{L^\infty}\left(\,||\frac{v_5\partial_r(v_1+v_2+v_4)}{r}|_{r=R\lambda(t)}||_{L^2(RdR)}\right.\\
&\qquad\qquad\left.+||\frac{v_5}{r}||_{L^\infty}||(\partial_2 v_3)(t,\cdot\lambda(t))||_{L^2(RdR)}\right)\\
&\leqslant\frac{C\log^{6+b}(t)}{t^{35/8}}
\end{aligned}$$

The third line of (5.195) is estimated by

$$\begin{aligned}
&||-\frac{(\cos(2(v_1+v_2+v_3+v_4))-1)}{r^3}\left(\sin(2Q_{\frac{1}{\lambda(t)}}+2v_5)-\sin(2Q_{\frac{1}{\lambda(t)}})\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\
&\leqslant C||\frac{v_1+v_2+v_3+v_4}{r}||^2_{L^\infty}||\frac{v_5}{r}|_{r=R\lambda(t)}||_{L^2(RdR)}\\
&\leqslant\frac{C\log^{6+b}(t)}{t^5}
\end{aligned}$$

Finally, the fourth line of (5.195) is estimated by

$$\begin{aligned}
&||\left(\frac{\cos(2(v_1+v_2+v_3+v_4))-1}{r^2}\right)|_{r=R\lambda(t)}\\
&\quad\cdot\left(\cos(2Q_{\frac{1}{\lambda(t)}}+2v_5)(\partial_r Q_{\frac{1}{\lambda(t)}}+\partial_r v_5)-\cos(2Q_{\frac{1}{\lambda(t)}})\partial_r Q_{\frac{1}{\lambda(t)}}\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\\
&\leqslant||\frac{v_1+v_2+v_3+v_4}{r}||^2_{L^\infty}\left(\,||\left(\lambda(t)\frac{v_5^2}{\lambda(t)^2+r^2}+\frac{r\lambda(t)^2 v_5}{(r^2+\lambda(t)^2)^2}\right)|_{r=R\lambda(t)}||_{L^2(RdR)}\right.\\
&\qquad\qquad\left.+||(\partial_2 v_5)(t,\cdot\lambda(t))||_{L^2(RdR)}\right)\\
&\leqslant\frac{C\log^{5+b}(t)}{t^{19/4}}
\end{aligned}$$

Finally, we use our pointwise estimates on all v_k to get

$$\left(\frac{1}{\lambda(t)^2}\int_0^\infty\frac{(F_5(t,R\lambda(t)))^2}{R^2}RdR\right)^{1/2}\leqslant\frac{C\log^{6+b}(t)}{t^{21/4}}$$

Combining all of these estimates, we get

$$\frac{1}{\lambda(t)}||F_5(t,\cdot\lambda(t))||_{\dot{H}^1_e}\leqslant\frac{C\log^{6+b}(t)}{t^{35/8}}$$

□

5.11 Estimates on F_6

We prove the analogous estimates on F_6:

Lemma 5.20.

$$\frac{1}{\lambda(t)^2}||F_6(t,r)||_{L^2(rdr)} \leqslant \frac{C}{t^4 \log^{3b+2N-1}(t)} \tag{5.196}$$

$$\frac{1}{\lambda(t)}||F_6(t,r)||_{\dot{H}^1_e} \leqslant \frac{C}{t^{9/2} \log^{4b-1+\frac{5N}{2}}(t)}$$

Proof. We start with

$$\begin{aligned}||F_6(t,r)||^2_{L^2(rdr)} &\leqslant C\int_{\frac{t}{4}}^{\frac{t}{2}} \frac{\lambda(t)^4 r(v_4(t,r)^2+v_5(t,r)^2)}{r^8}dr + C\int_{\frac{t}{2}}^{\infty} \frac{\lambda(t)^4(v_4^2+v_5^2)rdr}{r^8}\\ &\leqslant \frac{C}{t^8\log^{10b+4N-2}(t)}\end{aligned}$$

where we used (5.83), (5.91). This concludes the proof of (5.196).

Next, we have

$$\begin{aligned}\partial_r F_6(t,r) &= \chi'_{\geqslant 1}(\frac{4r}{t})\left(\frac{4}{t}\right)\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)(v_4+v_5)\\ &+ \chi_{\geqslant 1}(\frac{4r}{t})\,(v_4+v_5)\,\partial_r\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)\\ &+ \chi_{\geqslant 1}(\frac{4r}{t})\left(\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))-1}{r^2}\right)\partial_r(v_4+v_5)\end{aligned}$$

We estimate the L^2 norm as follows.

$$\begin{aligned}||\partial_r F_6(t,r)||_{L^2(rdr)} &\leqslant \frac{C}{t^{9/2}\log^{5b-1+\frac{5N}{2}}(t)} + ||\partial_r v_5||_{L^2(rdr)}\cdot||\frac{\chi_{\geqslant 1}(\frac{4r}{t})\lambda(t)^2}{(r^2+\lambda(t)^2)^2}||_{L^\infty_r}\\ &\leqslant \frac{C}{t^{9/2}\log^{5b-1+\frac{5N}{2}}(t)}\end{aligned}$$

where we used (5.83), (5.84), (5.91), and (5.93). The last term to estimate is

$$||\frac{F_6(t,r)}{r}||_{L^2(rdr)} \leqslant \frac{C}{t^5\log^{5b+2N-1}(t)}$$

where we used (5.83) and (5.91). Combining these, we get

$$\frac{1}{\lambda(t)}||F_6(t,r)||_{\dot{H}^1_e} \leqslant \frac{C}{t^{9/2}\log^{4b-1+\frac{5N}{2}}(t)}$$

□

5.12 Estimates on v_{corr}-dependent quantities

Finally, we define $v_{corr} := v_1 + v_2 + v_3 + v_4 + v_5$ and record some estimates on v_{corr}-dependent quantities which will appear as coefficients of various error terms involving the final correction, which is to be constructed in the next section.

Lemma 5.21. *We have the following estimates*

$$||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)}||_{L^\infty}^2 + ||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)^2(1+R^2)}||_{L^\infty} \leqslant \frac{C\log(\log(x))}{x^2\log(x)}$$

$$1 + ||\frac{v_{corr}(x,R\lambda(x))}{R}||_{L^\infty} + ||\partial_R(v_{corr}(x,R\lambda(x)))||_{L^\infty} \leqslant C$$

$$\begin{aligned}&||\frac{v_{corr}(x,R\lambda(x))\partial_R(v_{corr}(x,R\lambda(x)))}{R\lambda(x)^2}||_{L^\infty_R((0,1))}\\&+||\frac{v_{corr}(x,R\lambda(x))\partial_R(v_{corr}(x,R\lambda(x)))}{R^2\lambda(x)^2}||_{L^\infty_R((1,\infty))} + ||\frac{\partial_R(v_{corr}(x,R\lambda(x)))}{(1+R^2)\lambda(x)^2}||_{L^\infty}\\&\leqslant \frac{C\log(\log(x))}{x^2\log(x)}\end{aligned}$$

Proof. From (5.176), (5.77), (5.78), (5.83), and (5.91), we get

$$|v_{corr}(t,r)| \leqslant \begin{cases}\frac{Cr\log(\log(t))}{t^2\log^{b+1}(t)}, & r \leqslant \log^N(t)\\ \frac{Cr}{t^2\log^b(t)}, & \log^N(t) \leqslant r \leqslant \frac{t}{2}\\ \frac{C}{\sqrt{r}}, & \frac{t}{2} < r\end{cases}$$

This gives

$$||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)}||_{L^\infty}^2 + ||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)^2(1+R^2)}||_{L^\infty} \leqslant \frac{C\log(\log(x))}{x^2\log(x)}$$

Then, we use (5.177),(5.182), (5.79), (5.84), and (5.92) to get

$$|\partial_R(v_{corr}(x,R\lambda(x)))| \leqslant C\lambda(x)\begin{cases}\frac{\log(\log(x))}{x^2\log^{b+1}(x)}, & R\lambda(x) \leqslant \log^N(x)\\ \frac{1}{x^2\log^b(x)}, & \log^N(x) \leqslant R\lambda(x) \leqslant \frac{x}{2}\\ \frac{1}{\sqrt{x}}, & R\lambda(x) > \frac{x}{2}\end{cases}$$

This implies

$$1 + ||\frac{v_{corr}(x,R\lambda(x))}{R}||_{L^\infty} + ||\partial_R(v_{corr}(x,R\lambda(x)))||_{L^\infty} \leqslant C$$

Next, we have

$$|\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)}| \cdot |\frac{\partial_R(v_{corr}(x,R\lambda(x)))}{\lambda(x)}| \leqslant C\frac{(\log(\log(x)))^2}{x^4\log^{2b+2}(x)}, \quad R \leqslant 1$$

$$|\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)}|\cdot|\frac{\partial_R(v_{corr}(x,R\lambda(x)))}{R\lambda(x)}|\leqslant\begin{cases}\frac{C(\log(\log(x)))^2}{Rx^4\log^{2b+2}(x)}, & 1\leqslant R\leqslant\frac{\log^N(x)}{\lambda(x)}\\ \frac{C}{Rx^4\log^{2b}(x)}, & \log^N(x)\leqslant R\lambda(x)\leqslant\frac{x}{2}\\ \frac{C}{R^{5/2}\lambda(x)^{3/2}\sqrt{x}}, & R\lambda(x)>\frac{x}{2}\end{cases}$$

and

$$\frac{|\partial_R(v_{corr}(x,R\lambda(x)))|}{(1+R^2)\lambda(x)^2}\leqslant\begin{cases}\frac{C\log(\log(x))}{x^2\log(x)(1+R^2)}, & R\lambda(x)\leqslant\log^N(x)\\ \frac{1}{x^2(1+R^2)}, & \log^N(x)\leqslant R\lambda(x)\leqslant\frac{x}{2}\\ \frac{\log^b(x)}{\sqrt{x}}\cdot\frac{1}{1+R^2}, & R\lambda(x)>\frac{x}{2}\end{cases}$$

which imply

$$\begin{aligned}&||\frac{v_{corr}(x,R\lambda(x))\partial_R(v_{corr}(x,R\lambda(x)))}{R\lambda(x)^2}||_{L^\infty_R((0,1))}\\ &+||\frac{v_{corr}(x,R\lambda(x))\partial_R(v_{corr}(x,R\lambda(x)))}{R^2\lambda(x)^2}||_{L^\infty_R((1,\infty))}+||\frac{\partial_R(v_{corr}(x,R\lambda(x)))}{(1+R^2)\lambda(x)^2}||_{L^\infty}\\ &\leqslant\frac{C\log(\log(x))}{x^2\log(x)}\end{aligned}$$

□

6 Solving the Final Equation (Wave Maps)

The full equation to solve is (5.26) with 0 Cauchy data at infinity. For ease of notation, let us set $u=v_6$, and re-write (5.26) as

$$-\partial_{tt}u+\partial_{rr}u+\frac{1}{r}\partial_r u-\frac{\cos(2Q_{\frac{1}{\lambda(t)}}(r))}{r^2}u=F(t,r)+F_3(t,r)$$

where

$$F(t,r)=F_4(t,r)+F_5(t,r)+F_6(t,r)$$

and we recall that F_4, F_5, and F_6 are defined in (5.27), (5.28), and (5.29), and are estimated in theorem 5.1. and

$$F_3=N(u)+L_1(u) \tag{6.1}$$

with

$$N(f)=\left(\frac{\sin(2f)-2f}{2r^2}\right)\cos(2Q_{\frac{1}{\lambda(t)}})+\left(\frac{\cos(2f)-1}{2r^2}\right)\sin(2(Q_{\frac{1}{\lambda(t)}}+v_{corr}))$$

$$L_1(f)=\frac{\sin(2f)}{2r^2}\cos(2Q_{\frac{1}{\lambda(t)}})(\cos(2v_{corr})-1)-\frac{\sin(2f)}{2r^2}\sin(2Q_{\frac{1}{\lambda(t)}})\sin(2v_{corr})$$

$$v_{corr}=v_1+v_2+v_3+v_4+v_5$$

Note that we will utilize the crucial fact that

$$\langle F_4(t,\cdot),\phi_0(\frac{\cdot}{\lambda(t)})\rangle=0$$

6.1 The equation for $\mathcal{F}(u)$

We will make appropriate changes of variables in order to (formally) derive the equation for the distorted Fourier transform, discussed in section 4 of [14], of u. (Note, however, that we will not renormalize the time variable, unlike in [14].) We will denote the distorted Fourier transform of a function f by $\mathcal{F}(f)$. Let

$$u(t,r) = v(t, \frac{r}{\lambda(t)})$$

Then, if we evaluate the equation for u at the point $(t, R\lambda(t))$, we obtain

$$\begin{aligned}
&-\partial_{11}v(t,R) + 2\frac{\lambda'(t)}{\lambda(t)}R\partial_{12}v(t,R) + \left(\frac{\lambda''(t)}{\lambda(t)} - 2\frac{\lambda'(t)^2}{\lambda(t)^2}\right)R\partial_2 v(t,R)\\
&-\frac{\lambda'(t)^2}{\lambda(t)^2}R^2\partial_{22}v(t,R) + \frac{1}{\lambda(t)^2}\left(\partial_{22}v(t,R) + \frac{1}{R}\partial_2 v(t,R) - \frac{\cos(2Q_1(R))}{R^2}v(t,R)\right)\\
&= F(t,R\lambda(t)) + F_3(t,R\lambda(t))
\end{aligned}$$

Now, let

$$v(t,R) = \frac{w(t,R)}{\sqrt{R}}$$

to get

$$\begin{aligned}
&-\partial_{11}w(t,R) - \frac{\lambda'(t)}{\lambda(t)}\partial_1 w(t,R) + 2\frac{\lambda'(t)}{\lambda(t)}\partial_1(R\partial_2 w)(t,R)\\
&+\left(\frac{-\lambda''(t)}{2\lambda(t)} + \frac{1}{4}\frac{\lambda'(t)^2}{\lambda(t)^2}\right)w(t,R) + \left(\frac{\lambda''(t)}{\lambda(t)} - \frac{\lambda'(t)^2}{\lambda(t)^2}\right)R\partial_2 w(t,R)\\
&-\frac{\lambda'(t)^2}{\lambda(t)^2}R^2\partial_{22}w(t,R) + \frac{1}{\lambda(t)^2}\left(\partial_{22}w(t,R) - \left(\frac{3}{4R^2} - \frac{8}{(1+R^2)^2}\right)w(t,R)\right)\\
&= \sqrt{R}F(t,R\lambda(t)) + \sqrt{R}F_3(t,R\lambda(t))
\end{aligned} \tag{6.2}$$

Next, from (5.1) of [14], we have

$$\mathcal{F}(R\partial_R w) = -2\xi\partial_\xi \mathcal{F}(w) + K(\mathcal{F}(w))$$

where we will use various estimates on K, proven in [14], later on. Making the final change of variable

$$y(t,\xi) = \mathcal{F}(w)(t,\xi\lambda(t)^2)$$

and evaluating the distorted Fourier transform of (6.2) at the point $(t,\omega\lambda(t)^2)$, we get

$$\begin{aligned}
&-\partial_{tt}y(t,\omega) - \omega y(t,\omega) - \frac{\lambda'(t)}{\lambda(t)}\partial_t y(t,\omega) + \frac{2\lambda'(t)}{\lambda(t)}K\left(\partial_1 y(t,\frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2)\\
&+\left(\frac{-\lambda''(t)}{2\lambda(t)} + \frac{\lambda'(t)^2}{4\lambda(t)^2}\right)y(t,\omega) + \frac{\lambda''(t)}{\lambda(t)}K\left(y(t,\frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2)\\
&+2\frac{\lambda'(t)^2}{\lambda(t)^2}\left([\xi\partial_\xi, K](y(t,\frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2) - \frac{\lambda'(t)^2}{\lambda(t)^2}K\left(K(y(t,\frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2)\\
&= \mathcal{F}(\sqrt{\cdot}F(t,\cdot\lambda(t)))(\omega\lambda(t)^2) + \mathcal{F}(\sqrt{\cdot}F_3(u(y))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)
\end{aligned} \tag{6.3}$$

where we write $F_3(u(y))$ to emphasize the dependence of F_3 on y, which is related to u via

$$y(t,\xi) = \mathcal{F}(\sqrt{\cdot}u(t,\cdot\lambda(t)))(\xi\lambda(t)^2)$$

6.2 Estimates on F_2

Let

$$\begin{aligned}F_2(y)(t,\omega) = &-\frac{\lambda'(t)}{\lambda(t)}\partial_t y(t,\omega) + \frac{2\lambda'(t)}{\lambda(t)}K\left(\partial_1 y(t,\frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2)\\ &+\left(\frac{-\lambda''(t)}{2\lambda(t)}+\frac{\lambda'(t)^2}{4\lambda(t)^2}\right)y(t,\omega)+\frac{\lambda''(t)}{\lambda(t)}K\left(y(t,\frac{\cdot}{\lambda(t)^2})\right)(\omega\lambda(t)^2)\\ &+2\frac{\lambda'(t)^2}{\lambda(t)^2}\left([\xi\partial_\xi,K](y(t,\frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2)\\ &-\frac{\lambda'(t)^2}{\lambda(t)^2}K\left(K(y(t,\frac{\cdot}{\lambda(t)^2}))\right)(\omega\lambda(t)^2)\end{aligned}$$

Note that (6.3) becomes

$$\partial_{tt}y+\omega y = -\mathcal{F}(\sqrt{\cdot}F(t,\cdot\lambda(t)))(\omega\lambda(t)^2)+F_2(y)(t,\omega)-\mathcal{F}(\sqrt{\cdot}F_3(u(y))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)$$

Our goal is to prove the following proposition:

Proposition 6.1. *There exists $C_1>0$ such that, for all y satisfying*

$$y(t,\omega)\sqrt{\rho(\omega\lambda(t)^2)}\langle\omega\lambda(t)^2\rangle \in C^0_t([T_0,\infty),L^2(d\omega))$$

and

$$\partial_t y(t,\omega)\langle\sqrt{\omega}\lambda(t)\rangle\sqrt{\rho(\omega\lambda(t)^2)} \in C^0_t([T_0,\infty),L^2(d\omega))$$

we have the following inequalities, for $x \geqslant T_0$:

$$\begin{aligned}||F_2(y)(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{C_1}{x\log(x)}\Big(&||\partial_t y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\ &+\frac{1}{x}||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\Big)\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}&||\sqrt{\omega}\lambda(x)F_2(y)(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\ &\leqslant \frac{C_1}{x\log(x)}\left(||\sqrt{\omega}\lambda(x)\partial_t y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}+||\partial_t y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)\\ &+\frac{C_1}{x^2\log(x)}\left(||\sqrt{\omega}\lambda(x)y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}+||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)\end{aligned} \tag{6.5}$$

Proof. By the symbol type bounds on a from Proposition 4.7 of [14], we have

$$|\frac{\xi\rho'(\xi)}{\rho(\xi)}| = |\frac{\xi(a\overline{a}'+\overline{a}a')}{|a|^2}| \leqslant C$$

Then, by (5.3) of [14], there exists a constant C such that, for $f \in C^\infty_c((0,\infty))$ and $\alpha=0,\frac{1}{2}$, we have

$$||\langle\xi\rangle^\alpha(Kf)||_{L^2(\rho d\xi)} \leqslant C\left(||\langle\xi\rangle^\alpha f||_{L^2(\rho d\xi)}+||\langle\xi\rangle^\alpha(K_0(f))||_{L^2(\rho d\xi)}\right)$$

In addition, we use Proposition 5.2 of [14] to get, for $\alpha = 0, \frac{1}{2}$,

$$||\langle\xi\rangle^{\alpha+1/2} K_0 f||_{L^2(\rho d\xi)} \leqslant C||\langle\xi\rangle^{\alpha} f||_{L^2(\rho d\xi)}$$

$$||\langle\xi\rangle^{\alpha} [\xi\partial_\xi, K_0] f||_{L^2(\rho d\xi)} \leqslant C||\langle\xi\rangle^{\alpha} f||_{L^2(\rho d\xi)}$$

So, for $\alpha = 0, \frac{1}{2}$

$$||\langle\xi\rangle^{\alpha} K f||_{L^2(\rho d\xi)} \leqslant C||\langle\xi\rangle^{\alpha} f||_{L^2(\rho d\xi)}$$

Also, since

$$[\xi\partial_\xi, K] f = [\xi\partial_\xi, K_0] f - \xi\partial_\xi \left(\frac{\xi\rho'(\xi)}{\rho(\xi)}\right) f$$

the symbol type bounds on a imply that for $\alpha = 0, \frac{1}{2}$,

$$||\langle\xi\rangle^{\alpha} [\xi\partial_\xi, K] f||_{L^2(\rho d\xi)} \leqslant C||\langle\xi\rangle^{\alpha} f||_{L^2(\rho d\xi)}$$

For later convenience, we will record estimates on some terms appearing in (6.3), treating homogeneous components of a norm with weight $\langle\omega\lambda(x)^2\rangle^{1/2}$ separately:

$$\begin{aligned}
&||K(\partial_1 y(x, \frac{\cdot}{\lambda(x)^2}))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \\
&= \left(\int_0^\infty \rho(\xi) \left(K(\partial_1 y(x, \frac{\cdot}{\lambda(x)^2}))(\xi)\right)^2 \frac{d\xi}{\lambda(x)^2}\right)^{1/2} \\
&\leqslant \frac{C}{\lambda(x)} \left(\int_0^\infty \rho(\xi) \left(\partial_1 y(x, \frac{\xi}{\lambda(x)^2})\right)^2 d\xi\right)^{1/2} \\
&\leqslant C||\partial_1 y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}
\end{aligned}$$

$$\begin{aligned}
&||\sqrt{\omega}\lambda(x) K(\partial_1 y(x, \frac{\cdot}{\lambda(x)^2}))(\omega\lambda(x)^2)||^2_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \\
&\leqslant C \int_0^\infty \left(K(\partial_1 y(x, \frac{\cdot}{\lambda(x)^2}))(\omega\lambda(x)^2)\right)^2 \left(1 + \lambda(x)^4\omega^2\right)^{1/2} \rho(\omega\lambda(x)^2) d\omega \\
&\leqslant C \int_0^\infty \left(K(\partial_1 y(x, \frac{\cdot}{\lambda(x)^2}))(\xi)\right)^2 \sqrt{1+\xi^2} \frac{\rho(\xi) d\xi}{\lambda(x)^2} \\
&\leqslant \frac{C}{\lambda(x)^2} \int_0^\infty \left(\partial_1 y(x, \frac{\xi}{\lambda(x)^2})\right)^2 \sqrt{1+\xi^2} \rho(\xi) d\xi \\
&\leqslant C \int_0^\infty \left(\partial_1 y(x,\omega)\right)^2 \left(1 + \omega\lambda(x)^2\right) \rho(\omega\lambda(x)^2) d\omega
\end{aligned}$$

So,

$$\begin{aligned}
&||\sqrt{\omega}\lambda(x) K(\partial_1 y(x, \frac{\cdot}{\lambda(x)^2}))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \\
&\leqslant C \left(||\partial_1 y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} + ||\sqrt{\omega}\lambda(x)\partial_1 y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)
\end{aligned}$$

Similarly,

$$||\left([\xi\partial_\xi, K](y(x,\frac{\cdot}{\lambda(x)^2}))\right)(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant C||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}$$

$$||K\left(K(y(x,\frac{\cdot}{\lambda(x)^2}))\right)(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant C||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}$$

$$||K\left(y(x,\frac{\cdot}{\lambda(x)^2})\right)(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant C||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}$$

$$\begin{aligned}&||\sqrt{\omega}\lambda(x)\left([\xi\partial_\xi, K](y(x,\frac{\cdot}{\lambda(x)^2}))\right)(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\&\leqslant C\left(||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} + ||\sqrt{\omega}\lambda(x)y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)\end{aligned}$$

$$\begin{aligned}&||\sqrt{\omega}\lambda(x)\left(K\left(y(x,\frac{\cdot}{\lambda(x)^2})\right)(\omega\lambda(x)^2)\right)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\&\leqslant C\left(||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} + ||\sqrt{\omega}\lambda(x)y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)\end{aligned}$$

$$\begin{aligned}&||\sqrt{\omega}\lambda(x)\left(K\left(K(y(x,\frac{\cdot}{\lambda(x)^2}))\right)(\omega\lambda(x)^2)\right)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\&\leqslant C\left(||y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} + ||\sqrt{\omega}\lambda(x)y(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)\end{aligned}$$

We thus conclude the inequalities (6.4) and (6.5). □

6.3 F_3 Estimates

Our goal in this subsection is to prove the following proposition.

Proposition 6.2. *For y satisfying*

$$y(t,\omega)\sqrt{\rho(\omega\lambda(t)^2)}\langle\omega\lambda(t)^2\rangle \in C^0_t([T_0,\infty), L^2(d\omega))$$

let $F_3(u(y))$ be given by the expression (6.1)*, where*

$$u(t,r) = \sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}\left(y(t,\frac{\cdot}{\lambda(t)^2})\right)(\frac{r}{\lambda(t)}), \quad r > 0$$

Then, there exists an absolute constant $C > 0$ independent *of y, such that*

$$\begin{aligned}&||\mathcal{F}(\sqrt{\cdot}F_3(u(y))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C||y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)}||^2_{L^\infty} + ||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)^2(1+R^2)}||_{L^\infty}\right)\\&+ C||\langle\omega\lambda(t)^2\rangle y(t)||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&+\frac{C}{\lambda(t)}||\langle\sqrt{\omega}\lambda(t)\rangle y(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\end{aligned} \tag{6.6}$$

and

$$||\sqrt{\omega}\lambda(t)\mathcal{F}(\sqrt{\cdot}F_3(u(y))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant C||\langle\sqrt{\omega}\lambda(t)\rangle y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\cdot\Bigg(||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)}||^2_{L^\infty}+||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)^2(1+R^2)}||_{L^\infty}$$
$$+||\frac{\partial_R(v_{corr}(t,R\lambda(t)))}{(1+R^2)\lambda(t)^2}||_{L^\infty}+||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R\lambda(t)^2}||_{L^\infty_R((0,1))}$$
$$+||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R^2\lambda(t)^2}||_{L^\infty_R((1,\infty))}\Bigg) \tag{6.7}$$
$$+C||\langle\omega\lambda(t)^2\rangle y(t)||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$+\frac{C}{\lambda(t)}||\langle\omega\lambda(t)^2\rangle y(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty}\right)$$

where

$$v_{corr}=v_1+v_2+v_3+v_4+v_5$$

Proof. To prove these estimates, it will be convenient to use the distorted Fourier transforms of [1], associated to the operators L^*L and LL^*, where

$$L=\partial_r-\frac{\cos(Q_1)}{r},\quad L^*=-\partial_r-\frac{(\cos(Q_1)+1)}{r}$$

Let us use $\tilde{\phi}_\xi(r)$ to denote the eigenfunctions denoted by $\phi_\xi(r)$ in [1], and $\phi(r,\xi)$ to denote the eigenfunctions of [14]. By the definitions of these eigenfunctions, there exists f such that

$$\sqrt{r}\tilde{\phi}_{\sqrt{\xi}}(r)=f(\sqrt{\xi})\phi(r,\xi) \tag{6.8}$$

To determine the expression for f, let $g\in\mathcal{S}$, with $g(0)=0$. Using the definitions of the Fourier transforms from [1] and [14], and (6.8), we have

$$\mathcal{F}(g)(\xi)=\int_0^\infty\phi(r,\xi)g(r)dr=\int_0^\infty\frac{\sqrt{r}\tilde{\phi}_{\sqrt{\xi}}(r)}{f(\sqrt{\xi})}g(r)dr=\frac{1}{f(\sqrt{\xi})}\mathcal{F}_H(\frac{g(\cdot)}{\sqrt{\cdot}})(\sqrt{\xi})$$

where $\mathcal{F}_H$ is the operator defined in section 3.1 of [1] Using the inversion formula from [14] we get

$$g(r)=\int_0^\infty\phi(r,\xi)\mathcal{F}(g)(\xi)\rho(\xi)d\xi=\int_0^\infty\frac{\sqrt{r}}{f(\sqrt{\xi})^2}\mathcal{F}_H(\frac{g(\cdot)}{\sqrt{\cdot}})(\sqrt{\xi})\rho(\xi)\tilde{\phi}_{\sqrt{\xi}}(r)d\xi$$

So, for $g\in\mathcal{S}$, with $g(0)=0$, we have, for all $r\neq 0$,

$$\frac{g(r)}{\sqrt{r}}=\int_0^\infty\frac{2}{f(u)^2}\mathcal{F}_H(\frac{g(\cdot)}{\sqrt{\cdot}})(u)\rho(u^2)\tilde{\phi}_u(r)udu$$

Comparing this to the inversion formula from [1], we get

$$f(u)=\sqrt{2u\rho(u^2)}$$

In order to estimate the F_3 terms, we will also use the following lemma

Lemma 6.1. *There exists $C > 0$ such that, for all $\overline{y}$ with $\overline{y}(\xi)\langle\xi\rangle \in L^2((0,\infty), \rho(\xi)d\xi)$, if $\overline{v}$ is given by*

$$\overline{v}(r) = \frac{1}{\sqrt{r}} \int_0^\infty \overline{y}(\xi)\phi(r,\xi)\rho(\xi)d\xi, \quad r > 0$$

then, $\overline{v} \in C^0(0,\infty)$, and $\frac{\overline{v}(r)}{r\langle\log(r)\rangle}$ admits a continuous extension to $[0,\infty)$ with $\frac{\overline{v}(r)}{r\langle\log(r)\rangle}(0) = \lim_{r\to\infty}\overline{v}(r) = 0$.

$$\overline{v}, L(\overline{v}), L^*L(\overline{v}) \in L^2((0,\infty), rdr)$$

with

$$||\overline{v}(r)||_{L^2(rdr)} = ||\mathcal{F}^{-1}(\overline{y})(r)||_{L^2(dr)} = ||\overline{y}||_{L^2(\rho(\xi)d\xi)} \tag{6.9}$$

$$||\sqrt{\xi}\overline{y}(\xi)||^2_{L^2(\rho(\xi)d\xi)} = ||L\overline{v}||^2_{L^2(rdr)} \tag{6.10}$$

and

$$||\xi\overline{y}(\xi)||^2_{L^2(\rho(\xi)d\xi)} = ||L^*L\overline{v}||^2_{L^2(rdr)} \tag{6.11}$$

Moreover, $\overline{v} \in \dot{H}^1_e$, with

$$||\overline{v}||_{\dot{H}^1_e} \leqslant C\left(||\overline{v}||_{L^2(rdr)} + ||L(\overline{v})||_{L^2(rdr)}\right) \tag{6.12}$$

$$|\overline{v}(r)| \leqslant C||\overline{v}||_{\dot{H}^1_e}, \quad r \geqslant 0 \tag{6.13}$$

$$|\frac{\overline{v}(r)}{r\langle\log(r)\rangle}| \leqslant C\left(||\overline{v}||_{L^2(rdr)} + ||L(\overline{v})||_{L^2(rdr)} + ||L^*L\overline{v}||_{L^2(rdr)}\right), \quad r > 0 \tag{6.14}$$

$$\int_0^\infty \left(\frac{(L(\overline{v})(r))^2}{r^2(1+r^2)}\right) rdr \leqslant C||L^*L\overline{v}||^2_{L^2(rdr)} \tag{6.15}$$

Proof. If $f \in C^1([0,\infty)) \cap L^2((0,\infty), rdr)$, $Lf \in L^2((0,\infty), rdr)$, and $f(0) = \lim_{r\to\infty} f(r) = 0$, then, since

$$|\frac{\cos(Q_1(r)) - 1}{r}| \leqslant 1, \quad r > 0$$

we have

$$\partial_r f - \frac{f}{r} = Lf + \left(\frac{\cos(Q_1(r)) - 1}{r}\right) f \in L^2((0,\infty), rdr)$$

Then, if $M > 1$,

$$\begin{aligned}\int_{\frac{1}{M}}^M (\partial_r f - \frac{f}{r})^2 rdr &= \int_{\frac{1}{M}}^M \left((\partial_r f)^2 - \frac{\partial_r(f^2)}{r} + \frac{f^2}{r^2}\right) rdr \\ &= \int_0^\infty \mathbb{1}_{[\frac{1}{M},M]}(r)\left((\partial_r f)^2 + \frac{f^2}{r^2}\right) rdr - (f(M))^2 + (f(\frac{1}{M}))^2\end{aligned}$$

Letting $M \to \infty$, and using the monotone convergence theorem, we have

$$||f||^2_{\dot{H}^1_e} = \int_0^\infty \left((\partial_r f)^2 + \frac{f^2}{r^2}\right) rdr \leqslant C\left(||Lf||^2_{L^2(rdr)} + ||f||^2_{L^2(rdr)}\right) \tag{6.16}$$

Next, for $f \in C^1([0,\infty)) \cap \dot{H}^1_e$ satisfying $f(0) = 0$, we have

$$f^2(r) = \int_0^r (2f(s)f'(s))\, ds = 2\int_0^r \frac{f(s)}{\sqrt{s}}(f'(s)\sqrt{s})ds \leqslant 2||\frac{f}{r}||_{L^2(rdr)}||f'||_{L^2(rdr)}$$

So,

$$||f||_\infty \leqslant C||\frac{f}{r}||^{1/2}_{L^2(rdr)}||f'||^{1/2}_{L^2(rdr)} \leqslant C||f||_{\dot{H}^1_e} \tag{6.17}$$

Next, for any $g \in C^1((0,\infty)) \cap C^0([0,\infty))$ such that $g(0) = \lim_{r\to\infty} g(r) = 0$, $L^* g \in L^2((0,\infty), rdr)$, and for any $M \geqslant 1$,

$$\begin{aligned}\int_{\frac{1}{M}}^M (L^*g)^2 rdr &= \int_{\frac{1}{M}}^M (g'(r))^2 rdr + \int_{\frac{1}{M}}^M (\cos(Q_1(r))+1)\frac{d}{dr}\left(g(r)^2\right)dr\\ &+ \int_{\frac{1}{M}}^M \left(\frac{\cos(Q_1(r))+1}{r}\right)^2 g(r)^2 rdr\\ &= (\cos(Q_1(M))+1)g(M)^2 - \left(\cos(Q_1(\frac{1}{M}))+1\right)g(\frac{1}{M})^2\\ &+ \int_0^\infty \mathbb{1}_{[\frac{1}{M},M]}(r)\left((g'(r))^2 + \frac{4g(r)^2}{r^2(1+r^2)}\right)rdr\end{aligned}$$

By the monotone convergence theorem,

$$||L^*g||^2_{L^2(rdr)} = \int_0^\infty \left((g'(r))^2 + \frac{4g(r)^2}{r^2(1+r^2)}\right)rdr \tag{6.18}$$

If $\overline{y}$ is as in the lemma statement, for $M \geqslant 4$, define

$$\overline{v}_M(r) := \frac{1}{\sqrt{r}}\int_0^\infty \overline{y}(\xi)\phi(r,\xi)\chi_{\leqslant 1}(\frac{\xi}{M})\rho(\xi)d\xi, \quad r > 0$$

We will now record some estimates on $\partial_r^k \phi(r,\xi), \quad k \leqslant 1$, which will allow us to prove a certain regularity of $\overline{v}$ and $\overline{v_M}$. From [14], we have

$$\frac{1}{\sqrt{r}}\phi(r,\xi) = \frac{1}{2}\phi_0(r) + \frac{1}{r}\sum_{j=1}^\infty (r^2\xi)^j \phi_j(r^2), \quad r^2\xi \leqslant 4$$

and

$$\frac{1}{\sqrt{r}}\phi(r,\xi) = \frac{2\text{Re}\,(a(\xi)\psi^+(r,\xi))}{\sqrt{r}}, \quad r^2\xi > 4$$

Therefore,

$$|\frac{1}{r^{3/2}\langle\log(r)\rangle}\phi(r,\xi)| \leqslant \begin{cases}\frac{C}{\langle\log(r)\rangle}\left(\frac{\phi_0(r)}{r} + \frac{\log(1+r^2)}{r^2}\right), & r^2\xi \leqslant 4\\ \frac{C|a(\xi)|}{\xi^{1/4}r^{1/2}\cdot r\langle\log(r)\rangle}, & r^2\xi > 4\end{cases} \tag{6.19}$$

We use

$$\frac{\langle\log(\xi)\rangle^2}{\langle\log(r)\rangle^2} \leqslant C, \quad r^2\xi \leqslant 4, \quad \xi \geqslant 1$$

and the fact that

$$r \mapsto \frac{1}{\sqrt{r}\langle\log(r)\rangle} \text{ is decreasing on } (0,\infty)$$

which gives

$$\frac{1}{\sqrt{r}\langle\log(r)\rangle} \leqslant \frac{C\xi^{1/4}}{\langle\log(\xi)\rangle}, \quad r^2\xi > 4$$

in (6.19) to get, for all $r \geqslant 0$

$$|\frac{\phi(r,\xi)}{r^{3/2}\langle\log(r)\rangle}| \leqslant C\frac{|a(\xi)|\sqrt{\xi}}{\langle\log(\xi)\rangle} + C\begin{cases}\frac{1}{\langle\log(\xi)\rangle}, & \xi \geqslant 1\\ 1, & \xi \leqslant 1\end{cases}$$

Then, if

$$g(\xi) = \left(\frac{|a(\xi)|^2\xi}{\langle\log(\xi)\rangle^2} + \begin{cases}\frac{1}{\langle\log(\xi)\rangle^2}, & \xi \geqslant 1\\ 1, & \xi \leqslant 1\end{cases}\right)\frac{\rho(\xi)}{\langle\xi\rangle^2}$$

we have, for all $r \geqslant 0$

$$\begin{aligned}|\overline{y}(\xi)\frac{\phi(r,\xi)}{r^{3/2}\langle\log(r)\rangle}\rho(\xi)| &\leqslant |\overline{y}(\xi)|\langle\xi\rangle\sqrt{\rho(\xi)}\cdot\left(|\frac{\phi(r,\xi)}{r^{3/2}\langle\log(r)\rangle}|\frac{\sqrt{\rho(\xi)}}{\langle\xi\rangle}\right)\\ &\leqslant C|\overline{y}(\xi)|\langle\xi\rangle\sqrt{\rho(\xi)}\cdot\sqrt{g(\xi)}\end{aligned}$$

But,

$$|\overline{y}(\xi)|\langle\xi\rangle\sqrt{\rho(\xi)}\cdot\sqrt{g(\xi)} \in L^1((0,\infty),d\xi)$$

by Cauchy-Schwartz, due to the assumptions on $\overline{y}$ and the fact that

$$\int_0^\infty g(\xi)d\xi \leqslant C$$

Therefore, by the dominated convergence theorem, $\overline{v}$, defined in the lemma statement satisfies $\frac{\overline{v}(r)}{r\langle\log(r)\rangle}$ is continuous on $(0,\infty)$, and we have

$$|\frac{\overline{v}(r)}{r\langle\log(r)\rangle}| \leqslant \int_0^\infty |\overline{y}(\xi)\frac{\phi(r,\xi)}{r^{3/2}\langle\log(r)\rangle}\rho(\xi)|d\xi \leqslant C||\overline{y}(\xi)\langle\xi\rangle||_{L^2(\rho(\xi)d\xi)}, \quad r > 0$$

and

$$\lim_{r\to 0}\frac{\overline{v}(r)}{r\langle\log(r)\rangle} = \int_0^\infty \overline{y}(\xi)\lim_{r\to 0}\left(\frac{\phi(r,\xi)}{r^{3/2}\langle\log(r)\rangle}\right)\rho(\xi)d\xi = 0$$

Similarly,

$$|\frac{\phi(r,\xi)}{\sqrt{r}}\frac{\sqrt{\rho(\xi)}}{\langle\xi\rangle}| \leqslant C\sqrt{g_2(\xi)}$$

with

$$g_2(\xi) = \left(|a(\xi)|^2 + \begin{cases}1, & \xi \leqslant 1\\ \frac{1}{\xi}, & \xi > 1\end{cases}\right)\frac{\rho(\xi)}{\langle\xi\rangle^2} \in L^1((0,\infty),d\xi)$$

So, again by the dominated convergence theorem,

$$\lim_{r\to\infty} \overline{v}(r) = \int_0^\infty \overline{y}(\xi) \lim_{r\to\infty} \left(\frac{\phi(r,\xi)}{\sqrt{r}} \right) \rho(\xi) d\xi = 0$$

The same argument shows that

$$\lim_{r\to\infty} \overline{v_M}(r) = 0$$

From (6.19), we also have

$$\left|\frac{\phi(r,\xi)}{r^{3/2}}\right| \leqslant \begin{cases} C, & r^2\xi \leqslant 4 \\ C|a(\xi)|\sqrt{\xi}, & r^2\xi > 4 \end{cases}$$

whence,

$$\left|\frac{\phi(r,\xi)}{r^{3/2}}\right| \leqslant C + C|a(\xi)|\sqrt{\xi}, \quad r \geqslant 0$$

Then, the dominated convergence theorem shows that $\overline{v_M}$ satisfies that $\frac{\overline{v_M}(r)}{r}$ admits a continuous extension to a function defined on $[0, \infty)$. Similarly,

$$\left|\partial_r \left(\frac{\phi(r,\xi)}{\sqrt{r}} \right)\right| \leqslant \begin{cases} C\left(|\phi_0'(r)| + \xi \log(1+r^2)\right), & r^2\xi \leqslant 4 \\ \frac{C|a(\xi)|\xi^{1/4}}{r^{1/2}}, & r^2\xi > 4 \end{cases}$$

which shows

$$\left|\partial_r \left(\frac{\phi(r,\xi)}{\sqrt{r}} \right)\right| \leqslant C + C|a(\xi)|\sqrt{\xi}, \quad r \geqslant 0$$

Therefore, $\overline{v_M} \in C^1([0,\infty))$. Moreover,

$$L(\overline{v_M})(0) = \int_0^\infty \overline{y}(\xi) \lim_{r\to 0} L \left(\frac{\phi(r,\xi)}{\sqrt{r}} \right) \chi_{\leqslant 1} \left(\frac{\xi}{M} \right) \rho(\xi) d\xi = 0$$

where we used $L(\phi_0) = 0$. Again, by the dominated convergence theorem,

$$\lim_{r\to\infty} \partial_r \overline{v_M}(r) = \int_0^\infty \overline{y}(\xi) \lim_{r\to\infty} \partial_r \left(\frac{\phi(r,\xi)}{\sqrt{r}} \right) \chi_{\leqslant 1} \left(\frac{\xi}{M} \right) \rho(\xi) d\xi = 0$$

Finally, we have

$$\left|\partial_r^2 \left(\frac{\phi(r,\xi)}{\sqrt{r}} \right)\right| \leqslant C \left(1 + \sqrt{\xi} + \xi|a(\xi)| \right), \quad r \geqslant 0$$

and the same dominated convergence theorem based procedure used above shows that $\overline{v_M} \in C^2((0,\infty))$.

Next, we have

$$||\overline{v}(r)||_{L^2(rdr)} = ||\mathcal{F}^{-1}(\overline{y})(r)||_{L^2(dr)} = ||\overline{y}||_{L^2(\rho(\xi)d\xi)}$$

$$\begin{aligned} ||\sqrt{\xi}\overline{y}(\xi)||^2_{L^2(\rho(\xi)d\xi)} &= \int_0^\infty \xi |\mathcal{F}(\sqrt{\cdot}\overline{v}(\cdot))(\xi)|^2 \rho(\xi) d\xi = \int_0^\infty \xi \frac{|\mathcal{F}_H(\overline{v})(\sqrt{\xi})|^2}{2\sqrt{\xi}} d\xi \\ &= \int_0^\infty \eta^2 |\mathcal{F}_H(\overline{v})(\eta)|^2 d\eta = ||\eta \mathcal{F}_H(\overline{v})(\eta)||^2_{L^2(d\eta)} \\ &= ||L\overline{v}||^2_{L^2(rdr)} \end{aligned}$$

where we used the L^2 isometry property of $\mathcal{F}_H$ from [1]. Finally,

$$\begin{aligned}||\xi\overline{y}(\xi)||^2_{L^2(\rho(\xi)d\xi)} &= \int_0^\infty \xi^2 \frac{|\mathcal{F}_H(\overline{v})(\sqrt{\xi})|^2}{2\sqrt{\xi}} d\xi = \int_0^\infty \eta^4 |\mathcal{F}_H(\overline{v})(\eta)|^2 d\eta \\ &= \int_0^\infty |\mathcal{F}_H(L^* L\overline{v})(\eta)|^2 d\eta = ||L^* L\overline{v}||^2_{L^2(rdr)}\end{aligned}$$

This shows that $\overline{v}, \overline{v_M}, L(\overline{v}), L(\overline{v_M}), L^* L(\overline{v}), L^* L(\overline{v_M}) \in L^2((0,\infty), rdr)$. Combining these facts with our estimates on $\frac{\phi(r,\xi)}{\sqrt{r}} \frac{\sqrt{\rho(\xi)}}{\langle \xi \rangle}$ and

$$|\overline{v}(r) - \overline{v_M}(r)| \leqslant \int_0^\infty |\overline{y}(t,\xi) \frac{\phi(r,\xi)}{\sqrt{r}}| \cdot |\chi_{\leqslant 1}(\frac{\xi}{M}) - 1| \rho(\xi) d\xi$$

the Dominated convergence theorem gives

$$\overline{v_M} \to \overline{v}, \text{ pointwise, and in } L^2((0,\infty), rdr), \quad M \to \infty$$

$$L(\overline{v_M}) \to L(\overline{v}), \text{in } L^2((0,\infty), rdr), \quad M \to \infty$$
$$L^* L(\overline{v_M}) \to L^* L(\overline{v}), \text{in } L^2((0,\infty), rdr), \quad M \to \infty$$

We conclude the proof of the lemma by noting that (6.16) and (6.17) hold for $\overline{v_M}$, and (6.18) holds for $g = L(\overline{v_M})$. So, by approximation, we have (6.12), (6.13), and (6.15). □

Now, we can estimate the $F_3(u(y))$ terms, for y such that $y(t,\omega)\sqrt{\rho(\omega\lambda(t)^2)}\langle\omega\lambda(t)^2\rangle \in C^0_t([T_0,\infty), L^2(d\omega))$. This is sufficient for our purposes, since all y in the space Z (which is the space in which we will construct a solution to (6.3), and is defined later on) satisfy this condition. Recall that

$$F_3(u)(t,r) = N(u)(t,r) + L_1(u)(t,r)$$

where u and y are related by

$$y(t,\xi) = \mathcal{F}(\sqrt{\cdot} u(t, \cdot\lambda(t)))(\xi\lambda(t)^2) \tag{6.20}$$

and

$$L_1(u)(t,r) = \frac{\sin(2u(t,r))}{2r^2} \left(\cos(2Q_{\frac{1}{\lambda(t)}}(r)) \left(\cos(2v_{corr}) - 1\right) - \sin(2Q_{\frac{1}{\lambda(t)}}) \sin(2v_{corr})\right) \tag{6.21}$$

$$\begin{aligned}N(u)(t,r) &= \left(\frac{\sin(2u(t,r)) - 2u(t,r)}{2r^2}\right) \cos(2Q_{\frac{1}{\lambda(t)}}(r)) \\ &+ \left(\frac{\cos(2u(t,r)) - 1}{2r^2}\right) \sin(2Q_{\frac{1}{\lambda(t)}}(r) + 2v_{corr})\end{aligned}$$

We start with the L^2 estimate on the $L_1(u)$ term. Using the same procedure as in (6.9), we get

$$||\mathcal{F}(\sqrt{\cdot} L_1(u)(t, \cdot\lambda(t)))(\omega\lambda(t)^2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} = \frac{1}{\lambda(t)^2} \int_0^\infty R(L_1(u)(t, R\lambda(t)))^2 dR$$

Then,

$$\begin{aligned}&\frac{1}{\lambda(t)^2}\int_0^\infty R(L_1(u)(t,R\lambda(t)))^2 dR\\&\leqslant \frac{C}{\lambda(t)^2}\int_0^\infty R\frac{(u(t,R\lambda(t)))^2}{R^4\lambda(t)^4}\left(v_{corr}(t,R\lambda(t))\right)^4 dR\\&+\frac{C}{\lambda(t)^2}\int_0^\infty R\frac{(u(t,R\lambda(t)))^2}{R^4\lambda(t)^4}\frac{R^2}{(1+R^2)^2}\left(v_{corr}(t,R\lambda(t))\right)^2 dR\end{aligned}$$

Using the functions v and w (introduced when deriving the equation for y) defined by

$$u(t,r)=v(t,\frac{r}{\lambda(t)})$$

$$v(t,R)=\frac{w(t,R)}{\sqrt{R}}$$

we have

$$\begin{aligned}||u(t,\cdot\lambda(t))||^2_{L^2(RdR)}&=||v(t,\cdot)||^2_{L^2(RdR)}=||w(t)||^2_{L^2(dR)}=||\mathcal{F}(w)(t)||^2_{L^2(\rho(\xi)d\xi)}\\&=\lambda(t)^2||y(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\end{aligned}$$

So, we end up with

$$\begin{aligned}&||\mathcal{F}(\sqrt{\cdot}L_1(u)(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C||y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\left(||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)}||^2_{L^\infty_R}+||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)^2(1+R^2)}||_{L^\infty_R}\right)\end{aligned}$$

Next, we apply (6.10) to our current setting to get

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\mathcal{F}(\sqrt{\cdot}L_1(u)(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&=\frac{1}{\lambda(t)^2}||L(L_1(u)(t,\cdot\lambda(t)))||^2_{L^2(RdR)}\end{aligned}$$

Using (6.21), we get

$$\begin{aligned}&|\partial_R(L_1(u)(t,R\lambda(t)))|\\&\leqslant C\left(|Lv(t,R)|+\frac{|v(t,R)|}{R}\right)\left(\frac{(v_{corr}(t,R\lambda(t)))^2}{R^2\lambda(t)^2}+\frac{|v_{corr}(t,R\lambda(t))|}{R\lambda(t)^2(1+R^2)}\right)\\&+C\frac{|v(t,R)|}{R^2\lambda(t)^2}\left(\frac{R|\partial_R(v_{corr}(t,R\lambda(t)))|}{1+R^2}+|\partial_R(v_{corr}(t,R\lambda(t)))v_{corr}(t,R\lambda(t))|\right)\end{aligned}$$

On the other hand,

$$|\frac{L_1(u)(t,R\lambda(t))}{R}|\leqslant\frac{C|v(t,R)|}{R^3\lambda(t)^2}\left(v_{corr}(t,R\lambda(t))^2+\frac{|v_{corr}(t,R\lambda(t))|R}{(1+R^2)}\right)$$

So, we get

$$
\begin{aligned}
&||L(L_1(u)(t,\cdot\lambda(t)))||_{L^2(RdR)}\\
&\leqslant C\left(||Lv(t)||_{L^2(RdR)}+||v(t)||_{\dot{H}^1_e}\right)\left(||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)}||^2_{L^\infty_R}+||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)^2(1+R^2)}||_{L^\infty_R}\right)\\
&+C||v(t)||_{\dot{H}^1_e}\cdot||\frac{\partial_R(v_{corr}(t,R\lambda(t)))}{(1+R^2)\lambda(t)^2}||_{L^\infty_R}\\
&+C||v(t)||_{\dot{H}^1_e}\cdot||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R\lambda(t)^2}||_{L^\infty_R((0,1))}\\
&+C||v(t)||_{L^2((1,\infty),RdR)}\cdot||\frac{\partial_R(v_{corr}(t,R\lambda(t)))\cdot v_{corr}(t,R\lambda(t))}{R^2\lambda(t)^2}||_{L^\infty_R((1,\infty))}
\end{aligned}
$$

Now, we use (6.9) and (6.10) to translate the right-hand side in terms of y:

$$||Lv(t)||_{L^2(RdR)}=\lambda(t)||\sqrt{\omega}\lambda(t)y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

$$||v(t)||_{L^2(RdR)}=\lambda(t)||y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

Then, we get

$$
\begin{aligned}
&||\sqrt{\omega}\lambda(t)\mathcal{F}(\sqrt{\cdot}L_1(u)(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C\left(||\sqrt{\omega}\lambda(t)y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||y(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\\
&\cdot\left(||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)}||^2_{L^\infty_R}+||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)^2(1+R^2)}||_{L^\infty_R}+||\frac{\partial_R(v_{corr}(t,R\lambda(t)))}{(1+R^2)\lambda(t)^2}||_{L^\infty_R}\right.\\
&+||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R\lambda(t)^2}||_{L^\infty_R((0,1))}\\
&\left.+||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R^2\lambda(t)^2}||_{L^\infty_R((1,\infty))}\right)
\end{aligned}
$$

Now, we treat the N terms. Recall that

$$
\begin{aligned}
N(u)(t,R\lambda(t))=&\left(\frac{\sin(2v(t,R))-2v(t,R)}{2R^2\lambda(t)^2}\right)\cos(2Q_1(R))\\
&+\left(\frac{\cos(2v(t,R))-1}{2R^2\lambda(t)^2}\right)\sin(2Q_1(R)+2v_{corr}(t,R\lambda(t)))
\end{aligned}
$$

We then use (6.14) and the previous estimates to conclude that, if $R<1$, then,

$$
\begin{aligned}
|N(u)(t,R\lambda(t))|\leqslant &\, C\left(||v(t)||^2_{\dot{H}^1_e}+||L^*Lv(t)||^2_{L^2(RdR)}\right)\left(\log^2(\frac{1}{R})+1\right)\cdot\frac{|v(t,R)|}{\lambda(t)^2}\\
&+C||v(t)||_{\dot{H}^1_e}\frac{|v(t,R)|}{R\lambda(t)^2}\left(1+\frac{|v_{corr}(t,R\lambda(t))|}{R}\right)
\end{aligned}
$$

On the other hand, if $R>1$, then,

$$|N(u)(t,R\lambda(t))|\leqslant C\frac{|v(t,R)|^3}{\lambda(t)^2}+C\frac{(v(t,R))^2}{\lambda(t)^2}\left(1+\frac{|v_{corr}(t,R\lambda(t))|}{R}\right)$$

Then, we get

$$\begin{aligned}
&\frac{1}{\lambda(t)^2}||N(u)(t,\cdot\lambda(t))||^2_{L^2(RdR)}\\
&\leqslant \frac{C}{\lambda(t)^6}\left(||v(t)||^6_{L^2(RdR)}+||Lv(t)||^6_{L^2(RdR)}+||L^*Lv(t)||^6_{L^2(RdR)}\right)\\
&+\frac{C}{\lambda(t)^6}\left(||v(t)||^4_{L^2(RdR)}+||Lv(t)||^4_{L^2(RdR)}\right)\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||^2_{L^\infty_R}\right)
\end{aligned} \tag{6.22}$$

Finally, we apply (6.11) in our current setting:

$$\int_0^\infty (L^*Lv(t,R))^2 RdR = \lambda(t)^2||\omega\lambda(t)^2 y(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

Using the same procedure as used for L_1 above, we then translate the rest of the right hand side, and the left hand side of (6.22) in terms of y, and $\mathcal{F}(\sqrt{\cdot}N(u)(t,\cdot\lambda(t)))(\omega\lambda(t)^2)$, respectively, to get

$$\begin{aligned}
&||\mathcal{F}(\sqrt{\cdot}N(u)(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C\left(||y(t)||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\sqrt{\omega}\lambda(t)y(t)||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\omega\lambda(t)^2y(t)||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\\
&+\frac{C}{\lambda(t)}\left(||y(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\sqrt{\omega}\lambda(t)y(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)
\end{aligned}$$

We will now estimate $||\sqrt{\omega}\lambda(t)\mathcal{F}(\sqrt{\cdot}N(u)(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$, starting with

$$\begin{aligned}
&|\partial_R(N(u)(t,R\lambda(t)))|\\
&\leqslant \frac{C|v(t,R)|^3}{R^3\lambda(t)^2}+C\frac{|\partial_R v(t,R)|(v(t,R))^2}{R^2\lambda(t)^2}\\
&+C\frac{|Q_1'(R)+\partial_R(v_{corr}(t,R\lambda(t)))|}{R^2\lambda(t)^2}(v(t,R))^2\\
&+C\frac{|Q_1(R)|+|v_{corr}(t,R\lambda(t))|}{R^2\lambda(t)^2}|v(t,R)|\left(\frac{|v(t,R)|}{R}+|\partial_R v(t,R)|\right)
\end{aligned}$$

Then, we use our previous estimates to get, for $R \leqslant 1$:

$$\begin{aligned}
&|\partial_R(N(u)(t,R\lambda(t)))|\\
&\leqslant \frac{C}{\lambda(t)^2}\left(||v(t)||_{\dot{H}^1_e}^3 + ||L^* Lv(t)||_{L^2(RdR)}^3\right)\left(1+\log^2(\frac{1}{R})\right)^{3/2}\\
&+ C\frac{\langle \log(R)\rangle^2}{\lambda(t)^2}\left(||v(t)||_{L^2(RdR)}^2 + ||Lv(t)||_{L^2(RdR)}^2 + ||L^* Lv(t)||_{L^2(RdR)}^2\right)\\
&\cdot\left(|Lv(t,R)| + \frac{|v(t,R)|}{R}\right)\\
&+ C\frac{(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty})}{\lambda(t)^2}\left(||v(t)||_{L^2(RdR)}^2 + ||Lv(t)||_{L^2(RdR)}^2 + ||L^* Lv(t)||_{L^2(RdR)}^2\right)\\
&\cdot\left(\log^2(\frac{1}{R})+1\right)\\
&+ C\frac{\left(||v(t)||_{L^2(RdR)} + ||Lv(t)||_{L^2(RdR)} + ||L^* Lv(t)||_{L^2(RdR)}\right)\langle\log(R)\rangle|\partial_R v|}{\lambda(t)^2}\\
&\cdot\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\\
&+ \frac{C}{\lambda(t)^2}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\left(||v(t)||_{\dot{H}^1_e}^2 + ||L^* Lv(t)||_{L^2(RdR)}^2\right)\\
&\cdot\left(\log^2(\frac{1}{R})+1\right)
\end{aligned}$$

For $R \geqslant 1$, we have

$$\begin{aligned}
&|\partial_R(N(u)(t,R\lambda(t)))|\\
&\leqslant \frac{C}{\lambda(t)^2}||v(t)||_{\dot{H}^1_e}^2|v(t,R)| + \frac{C||v(t)||_{\dot{H}^1_e}^2}{\lambda(t)^2}\left(|Lv(t,R)| + \frac{|v(t,R)|}{R}\right)\\
&+ C\frac{\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty} + ||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)}{\lambda(t)^2}||v(t)||_{\dot{H}^1_e}|v(t,R)|\\
&+ \frac{C}{\lambda(t)^2}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\left(|Lv(t,R)| + \frac{|v(t,R)|}{R}\right)||v(t)||_{\dot{H}^1_e}
\end{aligned}$$

Then, we get

$$\begin{aligned}
&||\partial_R(N(u)(t,R\lambda(t)))||_{L^2(RdR)}\\
&\leqslant \frac{C}{\lambda(t)^2}\left(||v(t)||_{L^2(RdR)}^3 + ||Lv(t)||_{L^2(RdR)}^3 + ||L^* Lv(t)||_{L^2(RdR)}^3\right)\\
&+ \frac{C}{\lambda(t)^2}\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty} + ||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\\
&\cdot\left(||v(t)||_{L^2(RdR)}^2 + ||Lv(t)||_{L^2(RdR)}^2 + ||L^* Lv(t)||_{L^2(RdR)}^2\right)
\end{aligned}$$

Finally, we consider, for $R \leqslant 1$:

$$
\begin{aligned}
|\frac{N(u)(t,R\lambda(t))}{R}| \leqslant & \frac{C}{\lambda(t)^2}\left(||v(t)||_{\dot{H}^1_e}^3 + ||L^*Lv(t)||^3_{L^2(RdR)}\right)\left(1+\log^2(\frac{1}{R})\right)^{3/2}\\
&+\frac{C}{\lambda(t)^2}\left(||v(t)||_{\dot{H}^1_e}^2 + ||L^*Lv(t)||^2_{L^2(RdR)}\right)\left(1+\log^2(\frac{1}{R})\right)\\
&\cdot\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)
\end{aligned}
$$

and for $R > 1$, we have

$$
\begin{aligned}
|\frac{N(u)(t,R\lambda(t))}{R}| \leqslant & C\frac{||v(t)||^2_{\dot{H}^1_e}}{\lambda(t)^2}|v(t,R)|\\
&+C\frac{||v(t)||_{\dot{H}^1_e}|v(t,R)|}{\lambda(t)^2}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&||L(N(u)(t,\cdot\lambda(t)))||_{L^2(RdR)}\\
&\leqslant \frac{C}{\lambda(t)^2}\left(||v(t)||^3_{L^2(RdR)} + ||Lv(t)||^3_{L^2(RdR)} + ||L^*Lv(t)||^3_{L^2(RdR)}\right)\\
&+\frac{C}{\lambda(t)^2}\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty} + ||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\\
&\cdot\left(||v(t)||^2_{L^2(RdR)} + ||Lv(t)||^2_{L^2(RdR)} + ||L^*Lv(t)||^2_{L^2(RdR)}\right)
\end{aligned}
$$

We use the same procedure as in the previous estimates, to translate the left and right hand sides of the above estimate, and combine everything, to get (6.6) and (6.7). □

Before we proceed, we will need one more estimate. First, because λ is decreasing, if $x \geqslant t$, then, $\lambda(x) \leqslant \lambda(t)$. Next, we use Proposition 4.7 b of [14] to conclude that there exists a constant $C_1 > 0$ such that

$$
\begin{aligned}
&\frac{1}{C_1\xi\log^2(\xi)} \leqslant \rho(\xi) \leqslant \frac{C_1}{\xi\log^2(\xi)}, \quad 0 < \xi < \frac{1}{2e^2}\\
&\frac{\xi}{C_1} \leqslant \rho(\xi) \leqslant C_1\xi, \quad \xi > \frac{1}{2e^2}
\end{aligned}
$$

Then, if $x \geqslant t$, if $\omega\lambda(t)^2 \leqslant \frac{1}{2e^2}$, then, $\omega\lambda(x)^2 \leqslant \frac{1}{2e^2}$, and

$$
\begin{aligned}
\frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)} &\leqslant C\frac{\omega\lambda(x)^2\log^2(\omega\lambda(x)^2)}{\omega\lambda(t)^2\log^2(\omega\lambda(t)^2)}\\
&\leqslant C
\end{aligned}
$$

where we used the fact that $x \mapsto x\log^2(x)$ is increasing on $(0, \frac{1}{2e^2})$.

If $\omega\lambda(t)^2 \geqslant \frac{1}{2e^2}$, but $\omega\lambda(x)^2 \leqslant \frac{1}{2e^2}$, then,

$$\begin{aligned}\frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)} &\leqslant C\omega\lambda(t)^2\omega\lambda(x)^2\log^2(\omega\lambda(x)^2)\\ &\leqslant C\frac{\lambda(t)^2}{\lambda(x)^2}\left(\omega\lambda(x)^2\right)^2\log^2(\omega\lambda(x)^2)\\ &\leqslant \frac{C\lambda(t)^2}{\lambda(x)^2}\end{aligned}$$

Finally, if $\omega\lambda(t)^2 \geqslant \frac{1}{2e^2}$, and $\omega\lambda(x)^2 \geqslant \frac{1}{2e^2}$, then,

$$\frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)} \leqslant C\frac{\lambda(t)^2}{\lambda(x)^2}$$

In all cases, we have: there exists $C_\rho > 0$ such that, if $x \geqslant t$, then,

$$\frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)} \leqslant C_\rho\frac{\lambda(t)^2}{\lambda(x)^2} \tag{6.23}$$

Before setting up the final iteration, we will need to estimate various oscillatory integrals involving F_4:

6.4 Estimates on F_4-related oscillatory integrals

Lemma 6.2. *We have the following estimates*

$$\begin{aligned}&||\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C(\log(\log(t))^2}{t^2\log^{b+1-2\alpha b}(t)}\end{aligned}$$

$$\begin{aligned}&||\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}\end{aligned}$$

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}\end{aligned}$$

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\partial_t\left(\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}\end{aligned}$$

$$||\omega\lambda(t)^2\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant \frac{C}{t^2\log^{1+b-2b\alpha}(t)}$$

Proof. We start with

$$\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx$$
$$= -\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\omega}$$
$$-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\Big(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)\cdot 2\lambda(x)\lambda'(x)\omega + \mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)\Big)dx$$

Then,

$$-\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\omega} = \frac{-1}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}}\phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr - \frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^\infty \phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr$$

In the region $r\sqrt{\omega\lambda(t)^2}\leqslant 2$, we use proposition 4.4 of [14] to make the decomposition

$$\phi(r,\omega\lambda(t)^2) = \widetilde{\phi_0}(r) + \frac{1}{\sqrt{r}}\sum_{j=1}^\infty (r^2\omega\lambda(t)^2)^j\phi_j(r^2)$$

where we denote, by $\widetilde{\phi_0}$, what was denoted by ϕ_0 in [14]. In our notation,

$$\widetilde{\phi_0}(r) = \frac{\sqrt{r}}{2}\phi_0(r)$$

The first term to consider is then

$$-\frac{1}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}}\widetilde{\phi_0}(r)\sqrt{r}F_4(t,r\lambda(t))dr = \frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr$$

We will then consider several cases of ω.

Case 1: $1\leqslant \frac{2}{\sqrt{\omega}\lambda(t)}\leqslant \frac{\log^N(t)}{\lambda(t)}$. In this region, we use the orthogonality of $F_4(t,\cdot\lambda(t))$ to ϕ_0, to get

$$\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr = \frac{1}{2\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^\infty \phi_0(r)F_4(t,r\lambda(t))rdr$$

which gives

$$\begin{aligned}
&|\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr|\\
&\leqslant \frac{C}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\frac{\log^N(t)}{\lambda(t)}} \frac{\phi_0(r)}{\lambda(t)^4 r^4}\left(\frac{r\lambda(t)}{t^2\log^{3b+1-2\alpha b}(t)}\right) rdr\\
&+\frac{C}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\frac{t}{2\lambda(t)}} \frac{\phi_0(r)}{r^4\lambda(t)^4\log^{5b+2N-2}(t)}\frac{r\lambda(t)}{t^2} rdr\\
&\leqslant \frac{C}{t^2\log^{2b+1-2\alpha b}(t)}, \quad 1\leqslant \frac{2}{\sqrt{\omega}\lambda(t)}\leqslant \frac{\log^N(t)}{\lambda(t)}
\end{aligned}$$

Case 2: $\frac{2}{\sqrt{\omega}\lambda(t)}\leqslant 1$. Here, we have

$$\begin{aligned}
&|\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr|\\
&\leqslant \frac{C}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} \phi_0(r)\left(\frac{r\lambda(t)}{\lambda(t)^4(r^2+1)^2t^2\log^{3b+1-2\alpha b}(t)}\right) rdr\\
&+\frac{C}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} \frac{\phi_0(r)r\lambda(t)rdr}{\lambda(t)^4(r^2+1)^2\log^{5b+2N-2}(t)t^2}\\
&\leqslant \frac{C}{\omega\left(\sqrt{\omega}\lambda(t)\right)^4 t^2\log^{1-2\alpha b}(t)}, \quad \frac{2}{\sqrt{\omega}\lambda(t)}\leqslant 1
\end{aligned}$$

Case 3: $\frac{\log^N(t)}{\lambda(t)}\leqslant \frac{2}{\sqrt{\omega}\lambda(t)}\leqslant \frac{t}{2\lambda(t)}$. We use the orthogonality condition here, and recall that the $v_1+v_2+v_3$ and $F_{0,2}$ terms are supported in the region $r\leqslant \log^N(t)$. Then, in the integral below, only the v_4+v_5 term contributes:

$$\begin{aligned}
&|\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr| = |\frac{1}{2\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\phi_0(r)F_4(t,r\lambda(t))rdr|\\
&\leqslant \frac{C}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\frac{t}{2\lambda(t)}} \frac{rdr}{t^2\log^{2b+2N-2}(t)(r^2+1)^2}\\
&\leqslant \frac{C}{t^2\log^{4b+2N-2}(t)}, \quad \frac{\log^N(t)}{\lambda(t)}\leqslant \frac{2}{\sqrt{\omega}\lambda(t)}\leqslant \frac{t}{2\lambda(t)}
\end{aligned}$$

Case 4: $\frac{2}{\sqrt{\omega}\lambda(t)}\geqslant \frac{t}{2\lambda(t)}$. We use the orthogonality condition, and note that $F_4(t,r)=0, \quad r\geqslant \frac{t}{2}$. This gives

$$|\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr| = |\frac{1}{2\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\phi_0(r)F_4(t,r\lambda(t))rdr| = 0$$

Combining these estimates, we get

$$|\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr| \leqslant \begin{cases} \frac{C}{\omega^3\lambda(t)^4t^2\log^{1-2\alpha b}(t)}, & \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant 1 \\ \frac{C}{t^2\log^{2b+1-2\alpha b}(t)}, & 1 \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{\log^N(t)}{\lambda(t)} \\ \frac{C}{t^2\log^{4b+2N-2}(t)}, & \frac{\log^N(t)}{\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{t}{2\lambda(t)} \\ 0, & \frac{t}{2\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \end{cases}$$

Using proposition 4.7 b of [14], we have (for example)

$$\rho(x) \leqslant \begin{cases} \frac{C}{x\log^2(x)}, & x \leqslant \frac{1}{4} \\ Cx, & x \geqslant \frac{1}{4} \end{cases}$$

This leads to

$$||\frac{-1}{2\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r\phi_0(r)F_4(t,r\lambda(t))dr||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C}{t^2\log^{b+1-2\alpha b}(t)} \tag{6.24}$$

The next integral to consider is

$$-\frac{1}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^j\lambda(t)^{2j}\phi_j(r^2)F_4(t,r\lambda(t))dr$$

For this integral, there is of course no orthogonality to exploit. In all cases of ω, we will use the estimate (from proposition 4.4 of [14])

$$|\phi_j(r^2)| \leqslant \frac{3C_1^j}{(j-1)!}\log(1+r^2)$$

We again treat various cases of ω.

Case 1: $1 \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{\log^N(t)}{\lambda(t)}$. Then, we have

$$\begin{aligned} &|-\frac{1}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^j\lambda(t)^{2j}\phi_j(r^2)F_4(t,r\lambda(t))dr| \\ &\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^j\lambda(t)^{2j}\frac{C_1^j}{(j-1)!}\Big(\frac{\log(\log(t))r}{\lambda(t)^3(r^2+1)^2t^2\log^{3b+1-2b\alpha}(t)} \\ &\qquad +\frac{\log(t)r\lambda(t)}{t^2\lambda(t)^4(r^2+1)^2\log^{5b+2N-2}(t)}\Big)dr \\ &\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\left(\int_0^1 \frac{r^{2j+1}\omega^j\lambda(t)^{2j}C_1^j}{(j-1)!}\frac{\log(\log(t))}{t^2\log^{1-2b\alpha}(t)}dr + \int_1^{\frac{2}{\sqrt{\omega}\lambda(t)}}\frac{\omega^j\lambda(t)^{2j}C_1^j}{(j-1)!}\frac{\log(\log(t))}{t^2\log^{1-2b\alpha}(t)}r^{2j-3}dr\right) \\ &\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+2b-2b\alpha}(t)} \end{aligned}$$

where we used the support properties of the various terms in F_4, and the largeness of N.

Case 2: $\frac{2}{\sqrt{\omega}\lambda(t)} \leqslant 1$. Here, we have

$$
\begin{aligned}
&|-\frac{1}{\omega}\sum_{j=1}^{\infty}\int_{0}^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^{j}\lambda(t)^{2j}\phi_{j}(r^{2})F_{4}(t,r\lambda(t))dr| \\
&\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\int_{0}^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^{j}\lambda(t)^{2j}\frac{C_{1}^{j}}{(j-1)!}\log(1+r^{2})\left(\frac{r\lambda(t)}{\lambda(t)^{4}(r^{2}+1)^{2}}\frac{1}{t^{2}\log^{3b+1-2b\alpha}(t)}\right. \\
&\qquad\qquad\left.+\frac{r\lambda(t)}{\lambda(t)^{4}(r^{2}+1)^{2}}\frac{1}{t^{2}\log^{5b+2N-2}(t)}\right)dr \\
&\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\left(\int_{0}^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j+3}dr\right)\frac{\omega^{j}\lambda(t)^{2j}C_{1}^{j}}{(j-1)!t^{2}\log^{1-2b\alpha}(t)} \\
&\leqslant \frac{C}{\omega^{3}\lambda(t)^{4}t^{2}\log^{1-2b\alpha}(t)}, \quad \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant 1
\end{aligned}
$$

Case 3: $\frac{\log^{N}(t)}{\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{t}{2\lambda(t)}$. Again, we use the support properties of the various terms in F_4, to get

$$
\begin{aligned}
&|-\frac{1}{\omega}\sum_{j=1}^{\infty}\int_{0}^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^{j}\lambda(t)^{2j}\phi_{j}(r^{2})F_{4}(t,r\lambda(t))dr| \\
&\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\left(\int_{0}^{\frac{\log^{N}(t)}{\lambda(t)}}\frac{r^{2j+1}dr}{(r^{2}+1)^{2}}\right)\frac{\omega^{j}\lambda(t)^{2j}C_{1}^{j}}{(j-1)!}\frac{\log(\log(t))}{t^{2}\log^{1-2b\alpha}(t)} \\
&+\frac{C}{\omega}\sum_{j=1}^{\infty}\left(\int_{0}^{\frac{2}{\sqrt{\omega}\lambda(t)}}\frac{r^{2j+1}dr}{(r^{2}+1)^{2}}\right)\frac{\omega^{j}\lambda(t)^{2j}C_{1}^{j}}{(j-1)!t^{2}\log^{2b+2N-3}(t)} \\
&\leqslant \frac{C(\log(\log(t)))^{2}}{t^{2}\log^{1+2b-2b\alpha}(t)}, \quad \frac{\log^{N}(t)}{\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{t}{2\lambda(t)}
\end{aligned}
$$

Case 4: $\frac{2}{\sqrt{\omega}\lambda(t)} \geqslant \frac{t}{2\lambda(t)}$.

$$
\begin{aligned}
&|-\frac{1}{\omega}\sum_{j=1}^{\infty}\int_{0}^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^{j}\lambda(t)^{2j}\phi_{j}(r^{2})F_{4}(t,r\lambda(t))dr| \\
&\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\left(\int_{0}^{\frac{\log^{N}(t)}{\lambda(t)}}\frac{r^{2j+1}dr}{(r^{2}+1)^{2}}\right)\frac{\omega^{j}\lambda(t)^{2j}C_{1}^{j}}{(j-1)!}\frac{\log(\log(t))}{t^{2}\log^{1-2b\alpha}(t)} \\
&+\frac{C}{\omega}\sum_{j=1}^{\infty}\left(\int_{0}^{\frac{t}{2\lambda(t)}}\frac{r^{2j+1}dr}{(r^{2}+1)^{2}}\right)\frac{\omega^{j}\lambda(t)^{2j}C_{1}^{j}}{(j-1)!t^{2}\log^{2b+2N-3}(t)} \\
&\leqslant \frac{C(\log(\log(t)))^{2}}{t^{2}\log^{1+2b-2b\alpha}(t)}, \quad \frac{2}{\sqrt{\omega}\lambda(t)} \geqslant \frac{t}{2\lambda(t)}
\end{aligned}
$$

Then, we get

$$||-\frac{1}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}} r^{2j}\omega^j\lambda(t)^{2j}\phi_j(r^2)F_4(t,r\lambda(t))dr||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2b\alpha}(t)} \tag{6.25}$$

The third integral to estimate is

$$\begin{aligned}&-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr\\&=-\frac{2}{\omega}\text{Re}\left(\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}a(\omega\lambda(t)^2)\psi^+(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr\right)\end{aligned}$$

where we used propositions 4.6 and 4.7 of [14]. The estimates from these propositions also imply

$$\begin{aligned}&|-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr|\\&\leqslant\frac{C|a(\omega\lambda(t)^2)|}{\omega^{5/4}\lambda(t)^{1/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\sqrt{r}|F_4(t,r\lambda(t))|dr\end{aligned}$$

Case 1: $1\leqslant\frac{2}{\sqrt{\omega}\lambda(t)}\leqslant\frac{\log^N(t)}{\lambda(t)}$.

$$\begin{aligned}&|-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr|\\&\leqslant\frac{C|a(\omega\lambda(t)^2)|}{\omega^{5/4}\lambda(t)^{1/2}}\left(\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\frac{\log^N(t)}{\lambda(t)}}\frac{\sqrt{r}dr}{\lambda(t)^3r^3t^2\log^{3b+1-2b\alpha}(t)}+\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\frac{r^{3/2}dr}{\lambda(t)^3r^4\log^{5b+2N-2}(t)t^2}\right)\\&\leqslant C\frac{|a(\omega\lambda(t)^2)|}{\omega^{5/4}\lambda(t)^{1/2}}\cdot\frac{\omega^{3/4}\lambda(t)^{3/2}}{t^2\log^{3b+1-2\alpha b}(t)\lambda(t)^3}\\&\leqslant\frac{C|a(\omega\lambda(t)^2)|}{\omega}\cdot\frac{\sqrt{\omega}\lambda(t)}{t^2\log^{1-2\alpha b}(t)},\quad 1\leqslant\frac{2}{\sqrt{\omega}\lambda(t)}\leqslant\frac{\log^N(t)}{\lambda(t)}\end{aligned}$$

Case 2: $\frac{2}{\sqrt{\omega}\lambda(t)}\leqslant 1$.

$$\begin{aligned}&|-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr|\\&\leqslant\frac{C|a(\omega\lambda(t)^2)|}{\omega^{5/4}\lambda(t)^{1/2}}\left(\frac{1}{t^2\log^{1-2\alpha b}(t)}\int_0^{\frac{\log^N(t)}{\lambda(t)}}\frac{r^{3/2}dr}{(r^2+1)^2}+\frac{1}{t^2\log^{2b+2N-2}(t)}\int_0^{\infty}\frac{r^{3/2}dr}{(r^2+1)^2}\right)\\&\leqslant\frac{C|a(\omega\lambda(t)^2)|}{\omega^{5/4}\lambda(t)^{1/2}t^2\log^{1-2\alpha b}(t)},\quad\frac{2}{\sqrt{\omega}\lambda(t)}\leqslant 1\end{aligned}$$

Case 3: $\frac{\log^N(t)}{\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{t}{2\lambda(t)}$. In this region, the $F_{0,2}$ and $v_1+v_2+v_3$ terms do not contribute to the integral because of the support properties of $1-\chi_{\geqslant 1}(\frac{r}{\log^N(t)})$. We get

$$\begin{aligned}
&|-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty} \phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr| \\
&\leqslant \frac{C|a(\omega\lambda(t)^2)|}{\omega^{5/4}\lambda(t)^{1/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty} \frac{\sqrt{r}dr}{\lambda(t)^3 r^3 \log^{5b+2N-2}(t)t^2} \\
&\leqslant \frac{C|a(\omega\lambda(t)^2)|}{\omega}\cdot\frac{\sqrt{\omega}\lambda(t)}{t^2\log^{2b+2N-2}(t)}
\end{aligned}$$

Case 4: $\frac{2}{\sqrt{\omega}\lambda(t)} \geqslant \frac{t}{2\lambda(t)}$. In this case, the integral is zero, because of the support properties of F_4.

Combining the above estimates, we get

$$\begin{aligned}
&|-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty} \phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr| \\
&\leqslant C|a(\omega\lambda(t)^2)| \begin{cases} \frac{1}{\omega^{5/4}\lambda(t)^{1/2}t^2\log^{1-2\alpha b}(t)}, & \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant 1 \\ \frac{1}{\sqrt{\omega}t^2\log^{1+b-2\alpha b}(t)}, & 1 \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{\log^N(t)}{\lambda(t)} \\ \frac{1}{\sqrt{\omega}t^2\log^{3b+2N-2}(t)}, & \frac{\log^N(t)}{\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \leqslant \frac{t}{2\lambda(t)} \\ 0, & \frac{t}{2\lambda(t)} \leqslant \frac{2}{\sqrt{\omega}\lambda(t)} \end{cases}
\end{aligned}$$

which gives

$$||-\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty} \phi(r,\omega\lambda(t)^2)\sqrt{r}F_4(t,r\lambda(t))dr||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C\sqrt{\log(\log(t))}}{t^2\log^{1+b-2\alpha b}(t)} \tag{6.26}$$

(Note that, by proposition 4.7 b of [14], $\rho(\xi)=\frac{1}{\pi|a(\xi)|^2}$). Now, we consider

$$-\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\omega}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)\cdot 2\omega\lambda(x)\lambda'(x)dx$$

We start by noting that, in the region $r\leqslant\frac{2}{\sqrt{\xi}}$,

$$\partial_2\phi(r,\xi)=\partial_\xi\left(\widetilde{\phi_0}(r)+\frac{1}{\sqrt{r}}\sum_{j=1}^{\infty}(r^{2j}\xi^j\phi_j(r^2))\right)$$

So,

$$\begin{aligned}
&\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2) \\
&=\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\sqrt{r}F_4(x,r\lambda(x))\cdot\frac{1}{\sqrt{r}}\sum_{j=1}^{\infty}(r^{2j}j(\omega\lambda(x)^2)^{j-1}\phi_j(r^2))dr \\
&+\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))\partial_2\phi(r,\omega\lambda(x)^2)dr
\end{aligned} \tag{6.27}$$

We start with the first term:

$$
\begin{aligned}
&|-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} \sqrt{r}F_4(x,r\lambda(x))\cdot\frac{1}{\sqrt{r}}\sum_{j=1}^\infty (r^{2j}j(\omega\lambda(x)^2)^{j-1}\phi_j(r^2))dr\right)\\
&\qquad\cdot 2\omega\lambda(x)\lambda'(x)dx|\\
&\leqslant C\sum_{j=1}^\infty j\frac{\omega^j}{\omega}\int_t^\infty \lambda(x)^{2j-1}|\lambda'(x)|\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} |F_4(x,r\lambda(x))|r^{2j}\phi_j(r^2)drdx\\
&\leqslant C\sum_{j=1}^\infty \frac{j\omega^j C_1^j}{(j-1)!}\int_t^\infty \frac{\lambda(x)^{2j-1}|\lambda'(x)|}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} |F_4(x,r\lambda(x))|r^{2j}\log(1+r^2)drdx
\end{aligned}
$$

The r integral in the last line of the above expression was estimated before, and we get

$$
\begin{aligned}
&|-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} \sqrt{r}F_4(x,r\lambda(x))\cdot\frac{1}{\sqrt{r}}\sum_{j=1}^\infty (r^{2j}j(\omega\lambda(x)^2)^{j-1}\phi_j(r^2))dr\right)\\
&\qquad\cdot 2\omega\lambda(x)\lambda'(x)dx|\\
&\leqslant \int_t^\infty \frac{1}{x\log(x)}\begin{cases}\frac{C}{\omega^3\lambda(x)^4x^2\log^{1-2b\alpha}(x)}, & \frac{2}{\sqrt{\omega}\lambda(x)}\leqslant 1\\ \frac{C(\log(\log(x)))^2}{x^2\log^{1+2b-2b\alpha}(x)}, & 1\leqslant \frac{2}{\sqrt{\omega}\lambda(x)}\end{cases} dx
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&||-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} \sqrt{r}F_4(x,r\lambda(x))\cdot\frac{1}{\sqrt{r}}\sum_{j=1}^\infty (r^{2j}j(\omega\lambda(x)^2)^{j-1}\phi_j(r^2))dr\right)\\
&\qquad\cdot 2\omega\lambda(x)\lambda'(x)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C\int_t^\infty ||\frac{1}{\omega}\cdot\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} \sqrt{r}F_4(x,r\lambda(x))\cdot\frac{1}{\sqrt{r}}\sum_{j=1}^\infty (r^{2j}j(\omega\lambda(x)^2)^{j-1}\phi_j(r^2))dr\right)\\
&\qquad\cdot 2\omega\lambda(x)\lambda'(x)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}dx\\
&\leqslant C\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)||\frac{1}{\omega}\cdot\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} \sqrt{r}F_4(x,r\lambda(x))\cdot\frac{1}{\sqrt{r}}\sum_{j=1}^\infty (r^{2j}j(\omega\lambda(x)^2)^{j-1}\phi_j(r^2))dr\right)\\
&\qquad\cdot 2\omega\lambda(x)\lambda'(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}dx\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{2+b-2b\alpha}(t)}
\end{aligned}
\tag{6.28}
$$

where we used (6.23). The next integral to treat is

$$
\begin{aligned}
&|-2\omega\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\lambda(x)\lambda'(x)\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty \sqrt{r}F_4(x,r\lambda(x))\partial_2\phi(r,\omega\lambda(x)^2)drdx|\\
&\leqslant C\int_t^\infty \frac{1}{x\log^{2b+1}(x)}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty \sqrt{r}|F_4(x,r\lambda(x))||\partial_2\phi(r,\omega\lambda(x)^2)|drdx
\end{aligned}
$$

We again use

$$\phi(r,\xi) = 2\text{Re}(a(\xi)\psi^{+}(r,\xi)), \quad r \geqslant \frac{2}{\sqrt{\xi}}$$

and the following symbol type estimate on a from proposition 4.7 of [14]:

$$|a^{(k)}(\xi)| \leqslant \frac{C_k |a(\xi)|}{\xi^k}$$

as well as, from proposition 4.6 of [14]

$$\psi^{+}(r,\xi) = \frac{e^{ir\sqrt{\xi}}}{\xi^{1/4}}\sigma(r\sqrt{\xi},r), \quad r\sqrt{\xi} \geqslant 2$$

and the asymptotic series representation for σ, to get

$$|\partial_2 \phi(r,\xi)| \leqslant \frac{C|a(\xi)|}{\xi^{3/4}} r, \quad r\sqrt{\xi} \geqslant 2$$

This gives

$$\begin{aligned}
&|-2\omega \int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega} \lambda(x)\lambda'(x) \int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty \sqrt{r} F_4(x, r\lambda(x)) \partial_2 \phi(r, \omega\lambda(x)^2) dr dx| \\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}} \int_t^\infty \frac{1}{x\log^{2b+1}(x)} \int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty |F_4(x,r\lambda(x))| r^{3/2} dr dx
\end{aligned}$$

Note that estimates on this integral can not quite be inferred from estimates on a previously treated integral. So, we start considering cases of ω:

Case 1: $1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{\log^N(x)}{\lambda(x)}$. Here, we get

$$\begin{aligned}
&\frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}} \int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty |F_4(x,r\lambda(x))| r^{3/2} dr \\
&\leqslant C \frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}} \Bigg(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\frac{\log^N(x)}{\lambda(x)}} \frac{r^{5/2}\lambda(x)}{\lambda(x)^4 (r^2+1)^2 x^2 \log^{3b+1-2b\alpha}(x)} dr \\
&\qquad + \int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\frac{x}{2\lambda(x)}} \frac{r^{3/2} dr}{\lambda(x)^3 r^3 x^2 \log^{5b+2N-2}(x)} \Bigg) \\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{x^2 \log^{1-2b\alpha}(x)\sqrt{\omega}\lambda(x)}, \quad 1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{\log^N(x)}{\lambda(x)}
\end{aligned}$$

Case 2: $\frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1$.

$$\begin{aligned}
&\frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}} \int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty |F_4(x,r\lambda(x))| r^{3/2} dr \\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}} \left(\int_0^{\frac{\log^N(x)}{\lambda(x)}} \frac{r^{5/2} dr}{(r^2+1)^2 x^2 \log^{1-2b\alpha}(x)} + \int_0^\infty \frac{r^{5/2} dr}{x^2 (r^2+1)^2 \log^{2b+2N-2}(x)} \right) \\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2} x^2 \log^{1-2b\alpha}(x)}, \quad \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1
\end{aligned}$$

Case 3: $\frac{x}{2\lambda(x)} \geqslant \frac{2}{\sqrt{\omega}\lambda(x)} \geqslant \frac{\log^N(x)}{\lambda(x)}$. In this region, only the v_4+v_5 term in F_4 contributes:

$$\begin{aligned}&\frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}|F_4(x,r\lambda(x))|r^{3/2}dr\\&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\frac{x}{2\lambda(x)}}\frac{r^{3/2}dr}{\lambda(x)^3r^3x^2\log^{5b+2N-2}(x)}\\&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\sqrt{\omega}\lambda(x)^{3/2}x^2\log^{\frac{5b}{2}+2N-2}(x)},\quad \frac{x}{2\lambda(x)}\geqslant\frac{2}{\sqrt{\omega}\lambda(x)}\geqslant\frac{\log^N(x)}{\lambda(x)}\end{aligned}$$

Case 4: $\frac{2}{\sqrt{\omega}\lambda(x)} \geqslant \frac{x}{2\lambda(x)}$. In this region, the integral to estimate is zero, because of the support properties of F_4.

Combining these estimates, we get

$$\begin{aligned}&\frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}|F_4(x,r\lambda(x))|r^{3/2}dr\\&\leqslant C|a(\omega\lambda(x)^2)|\cdot\begin{cases}\frac{1}{x^2\log^{1-2b\alpha}(x)\sqrt{\omega}\lambda(x)}, & 1\leqslant\frac{2}{\sqrt{\omega}\lambda(x)}\leqslant\frac{\log^N(x)}{\lambda(x)}\\ \frac{1}{\omega^{3/4}\lambda(x)^{3/2}x^2\log^{1-2b\alpha}(x)}, & \frac{2}{\sqrt{\omega}\lambda(x)}\leqslant 1\\ \frac{1}{\sqrt{\omega}\lambda(x)^{3/2}x^2\log^{\frac{5b}{2}+2N-2}(x)}, & \frac{x}{2\lambda(x)}\geqslant\frac{2}{\sqrt{\omega}\lambda(x)}\geqslant\frac{\log^N(x)}{\lambda(x)}\\ 0, & \frac{2}{\sqrt{\omega}\lambda(x)}\geqslant\frac{x}{2\lambda(x)}\end{cases}\end{aligned}\tag{6.29}$$

Using the same procedure as in (6.28), we get

$$\begin{aligned}&||-2\omega\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\omega}\lambda(x)\lambda'(x)\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))\partial_2\phi(r,\omega\lambda(x)^2)drdx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant\frac{C\sqrt{\log(\log(t))}}{t^2\log^{2+b-2b\alpha}(t)}\end{aligned}\tag{6.30}$$

To treat the next integral:

$$-\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\omega}\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)dx$$

we first note that

$$\int_0^{\infty}\phi_0(r)F_4(x,r\lambda(x))rdr=0,\quad x\geqslant T_0$$

implies

$$\int_0^{\infty}\phi_0(r)\partial_x(F_4(x,r\lambda(x)))rdr=0,\quad x\geqslant T_0$$

So, $\partial_x(F_4(x,r\lambda(x)))$ is still orthogonal to $\phi_0(r)$. Also, by noting the symbol-type estimates, (5.174), and inspecting the procedure used to estimate $\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)}{\omega}$, we get

$$||\frac{\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)}{\omega}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\leqslant\frac{C\lambda(t)}{\lambda(x)}\left(\frac{(\log(\log(x)))^2}{x^3\log^{b+1-2\alpha b}(x)}\right)$$

where we again use (6.23). Using Minkowski's inequality, we get

$$\begin{aligned}&||-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t))^2}{t^2\log^{b+1-2\alpha b}(t)}\end{aligned} \tag{6.31}$$

Combining this with our other estimates in this section, we finally get

$$\begin{aligned}&||\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t))^2}{t^2\log^{b+1-2\alpha b}(t)}\end{aligned}$$

Next, we estimate

$$\begin{aligned}&\partial_t\left(\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\right)\\&=\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\\&=\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\Big(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)\cdot 2\lambda(x)\lambda'(x)\omega\\&\qquad+\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)\Big)dx\\&=-\left(\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))'(\omega\lambda(t)^2)\cdot 2\lambda(t)\lambda'(t)+\frac{\mathcal{F}(\sqrt{\cdot}\partial_t(F_4(t,\cdot\lambda(t))))(\omega\lambda(t)^2)}{\omega}\right)\\&-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\Big(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''(\omega\lambda(x)^2)\cdot 4\lambda(x)^2\lambda'(x)^2\omega^2\\&\qquad+2\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)\cdot 2\lambda(x)\lambda'(x)\omega\\&\qquad+\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)\cdot 2\omega((\lambda'(x))^2+\lambda(x)\lambda''(x))\\&\qquad+\mathcal{F}(\sqrt{\cdot}\partial_x^2(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)\Big)dx\end{aligned}$$

Note that, while obtaining (6.28) and (6.30), we showed that

$$||\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)}{\omega}\cdot 2\omega\lambda(x)\lambda'(x)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\leqslant C\frac{(\log(\log(x)))^2}{x^3\log^{2+b-2b\alpha}(x)}$$

Similarly, from the procedure used to obtain (6.31), we infer

$$||\frac{\mathcal{F}(\sqrt{\cdot}\partial_t(F_4(t,\cdot\lambda(t))))(\omega\lambda(t)^2)}{\omega}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}$$

Next, we consider the term involving $\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''$. For this, we start by studying, with $f(x,r)=\sqrt{r}F_4(x,r\lambda(x))$,

$$\mathcal{F}(f(x))''(\xi)=\int_0^{\frac{2}{\sqrt{\xi}}} f(x,r)\partial_2^2\phi(r,\xi)dr+\int_{\frac{2}{\sqrt{\xi}}}^\infty f(x,r)\partial_2^2\phi(r,\xi)dr$$

In the region $r\sqrt{\xi} \leqslant 2$, we use

$$\partial_2^2 \phi(r,\xi) = \frac{1}{\sqrt{r}} \sum_{j=2}^{\infty} j(j-1)\xi^{j-2} r^{2j} \phi_j(r^2), \quad r\sqrt{\xi} \leqslant 2$$

Then,

$$\begin{aligned}
&|-4\int_t^\infty \cos((t-x)\sqrt{\omega})\lambda(x)^2\lambda'(x)^2\omega \\
&\quad \cdot \sum_{j=2}^{\infty} j(j-1)(\omega\lambda(x)^2)^{j-2} \int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} F_4(x, r\lambda(x)) r^{2j}\phi_j(r^2) dr dx| \\
&\leqslant C\sum_{j=2}^{\infty} j(j-1)\omega^{j-1} \int_t^\infty \left(\frac{\lambda'(x)}{\lambda(x)}\right)^2 \lambda(x)^{2j} \int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} |F_4(x, r\lambda(x))| r^{2j} |\phi_j(r^2)| dr dx
\end{aligned}$$

we then apply the same procedure as in (6.28), and get

$$\begin{aligned}
&||-4\int_t^\infty \cos((t-x)\sqrt{\omega})\lambda(x)^2\lambda'(x)^2\omega \\
&\quad \cdot \sum_{j=2}^{\infty} j(j-1)(\omega\lambda(x)^2)^{j-2} \int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} F_4(x, r\lambda(x)) r^{2j}\phi_j(r^2) dr dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3 \log^{3+b-2b\alpha}(t)}
\end{aligned}$$

Next, we have

$$\partial_2^2 \phi(r,\xi) = (\partial_2^2\phi(r,\xi))_0 + (\partial_2^2\phi(r,\xi))_1$$

with

$$\begin{aligned}
(\partial_2^2\phi(r,\xi))_0 &= 2\text{Re}\left(a''(\xi)\psi^+(r,\xi) + 2a'(\xi)\partial_2\psi^+(r,\xi)\right) \\
&\quad + 2\text{Re}\left(a(\xi)\left(\partial_\xi^2\left(\frac{e^{ir\sqrt{\xi}}}{\xi^{1/4}}\sigma(r\sqrt{\xi}, r)\right) + \frac{r^2}{4\xi^{5/4}} e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi}, r)\right)\right)
\end{aligned}$$

and

$$(\partial_2^2\phi(r,\xi))_1 = -2\text{Re}\left(a(\xi)\frac{r^2}{4\xi^{5/4}} e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi}, r)\right)$$

$$|(\partial_2^2\phi(r,\xi))_0| \leqslant \frac{Cr|a(\xi)|}{\xi^{7/4}}$$

Again, with

$$f(x,r) = \sqrt{r}F_4(x, r\lambda(x))$$

we have

$$\int_{\frac{2}{\sqrt{\xi}}}^{\infty} f(x,r)(\partial_2^2\phi(r,\xi))_1 dr = -2\text{Re}\left(\frac{a(\xi)}{4\xi^{5/4}}\int_{\frac{2}{\sqrt{\xi}}}^{\infty} f(x,r)r^2 e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi},r)dr\right)$$

$$= \frac{1}{2}\text{Re}\left(\frac{a(\xi)}{\xi^{5/4}}\frac{4f(x,\frac{2}{\sqrt{\xi}})}{i\xi^{3/2}}\sigma(2,\frac{2}{\sqrt{\xi}})e^{2i}\right)$$

$$+\frac{1}{2}\text{Re}\left(\frac{a(\xi)}{\xi^{5/4}}\int_{\frac{2}{\sqrt{\xi}}}^{\infty}\frac{e^{ir\sqrt{\xi}}}{i\sqrt{\xi}}\left(\partial_r f(x,r)r^2\sigma(r\sqrt{\xi},r)+2f(x,r)r\sigma(r\sqrt{\xi},r)\right.\right.$$
$$\left.\left.+r^2 f(x,r)\left(\sqrt{\xi}\partial_1\sigma(r\sqrt{\xi},r)+\partial_2\sigma(r\sqrt{\xi},r)\right)\right)dr\right)$$

Again, using Proposition 4.6 of [14] to estimate the σ terms, we get

$$|\int_{\frac{2}{\sqrt{\xi}}}^{\infty} f(x,r)(\partial_2^2\phi(r,\xi))_1 dr|$$
$$\leqslant \frac{C|a(\xi)|}{\xi^{11/4}}|f(x,\frac{2}{\sqrt{\xi}})| + C\frac{|a(\xi)|}{\xi^{7/4}}\int_{\frac{2}{\sqrt{\xi}}}^{\infty}\left(r|\partial_r f(x,r)|+|f(x,r)|\right)rdr$$

Then,

$$|-\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\omega}\left(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))\left((\partial_2^2\phi(r,\xi))_0+(\partial_2^2\phi(r,\xi))_1\right)|_{\xi=\omega\lambda(x)^2}dr\right)$$
$$\cdot 4\lambda(x)^2\lambda'(x)^2\omega^2 dx|$$

$$\leqslant C\int_t^{\infty}\left(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\left(|F_4(x,r\lambda(x))|r^{3/2}+r^{3/2}\cdot r\lambda(x)|\partial_2F_4(x,r\lambda(x))|\right)dr\right)$$
$$\cdot\frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\frac{dx}{x^2\log^{2b+2}(x)}$$
$$+C\int_t^{\infty}\frac{|a(\omega\lambda(x)^2)|}{\omega^2\lambda(x)^6}\frac{|F_4(x,\frac{2}{\sqrt{\omega}})|}{x^2\log^{4b+2}(x)}dx$$

For the last term, we have

$$\int_t^{\infty}||\frac{|a(\omega\lambda(x)^2)|}{\omega^2\lambda(x)^6}\frac{|F_4(x,\frac{2}{\sqrt{\omega}})|}{x^2\log^{4b+2}(x)}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}dx$$
$$\leqslant C\int_t^{\infty}\frac{\lambda(t)}{\lambda(x)}\frac{1}{x^2\log^{2b+2}(x)\lambda(x)^4}\left(\int_0^{\infty}|F_4(x,y)|^2y^8\frac{dy}{y^3}\right)^{1/2}dx$$
$$\leqslant\frac{C\sqrt{\log(\log(t))}}{t^3\log^{b+3-2\alpha b}(t)}$$

So,

$$||\int_t^{\infty}\frac{|a(\omega\lambda(x)^2)|}{\omega^2\lambda(x)^6}\frac{|F_4(x,\frac{2}{\sqrt{\omega}})|}{x^2\log^{4b+2}(x)}dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\leqslant\frac{C\sqrt{\log(\log(t))}}{t^3\log^{b+3-2\alpha b}(t)}$$

On the other hand, by the same procedure used to obtain (6.29) and (6.30), we get

$$||\int_t^\infty \left(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty \left(|F_4(x,r\lambda(x))|r^{3/2}+r^{3/2}\cdot r\lambda(x)|\partial_2 F_4(x,r\lambda(x))|\right)dr\right) \cdot \frac{|a(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\frac{dx}{x^2\log^{2b+2}(x)}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C\sqrt{\log(\log(t))}}{t^3\log^{3+b-2b\alpha}(t)}$$

Next, by noting the symbol-type character of the estimates (5.174), and inspecting (6.28) and (6.30), we deduce the following estimate

$$||-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\cdot 2\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)\cdot 2\lambda(x)\lambda'(x)\omega dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{2+b-2b\alpha}(t)}$$

The following term was already estimated via (6.28), (6.30) (except with different λ-dependent coefficients). Taking into account the estimates on λ', λ'', we get

$$||-\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)2\omega\left(\lambda'(x)^2+\lambda(x)\lambda''(x)\right)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{2+b-2b\alpha}(t)}$$

Finally, we start with estimating $\frac{\mathcal{F}(\sqrt{\cdot}\partial_x^2(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)}{\omega}$. Firstly, we note that $\partial_x^2(F_4(x,r\lambda(x)))$ is still orthogonal to $\phi_0(r)$. Then, we repeat the same procedure used to estimate $\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)}{\omega}$. By again noting the symbol-type behavior of (5.174), the only contributions to $\frac{\mathcal{F}(\sqrt{\cdot}\partial_x^2(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)}{\omega}$ which need to be checked are those due to the last term in (5.175), which is not directly comparable to terms arising in (5.174). To be clear, we still use the orthogonality of the full function $\partial_x^2(F_4(x,r\lambda(x)))$ to $\phi_0(r)$; but, after using the orthogonality as needed, we can then deduce estimates on all integrals which do not involve the last term in (5.175) just by comparison with an analogous term in (5.174), as described above. We start with

Case 1: $\frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1$.

$$|\frac{-1}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\widetilde{\phi_0}(r)\sqrt{r}\frac{\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr| \leqslant \frac{C}{\omega}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\frac{\phi_0(r)rdr}{x^4\log^{b+N-2}(x)(r^2+1)^2} \leqslant \frac{C}{\omega^{5/2}\lambda(x)^3x^4\log^{b+N-2}(x)}, \quad \frac{2}{\sqrt{\omega}\lambda(x)}\leqslant 1$$

Case 2: $1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{x}{2\lambda(x)}$. Here, we first use the orthogonality of $\partial_x^2(F_4(x, r\lambda(x)))$ to $\phi_0(r)$. Using the procedure described above, we only need to estimate

$$|\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty} \widetilde{\phi_0}(r)\sqrt{r}\frac{\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr| \leqslant \frac{C}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\frac{x}{2\lambda(x)}} \frac{dr}{x^4\log^{b+N-2}(x)r^4}$$
$$\leqslant \frac{C\sqrt{\omega}}{x^4\log^{4b+N-2}(x)}, \quad 1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{x}{2\lambda(x)}$$

Case 3: $\frac{x}{2\lambda(x)} \leqslant \frac{2}{\sqrt{\omega}\lambda(x)}$. In this case, after again using the orthogonality of $\partial_x^2(F_4(x, r\lambda(x)))$ to $\phi_0(r)$, the only integral to be checked is zero, due to the support properties of F_4. Next, we have

Case 1: $1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{x}{2\lambda(x)}$.

$$\frac{1}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} r^{2j}\omega^j\lambda(x)^{2j}|\phi_j(r^2)|\frac{\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr$$
$$\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\frac{\omega^j\lambda(x)^{2j}C_1^j}{(j-1)!}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\frac{r^{2j}\log(1+r^2)dr}{x^4\log^{b+N-2}(x)(r^2+1)^2}$$
$$\leqslant \frac{C}{x^4\log^{3b+N-3}(x)}, \quad 1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{x}{2\lambda(x)}$$

where we treat the integral in exactly the same way we did in obtaining (6.24)

Case 2: $\frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1$.

$$\frac{1}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} r^{2j}\omega^j\lambda(x)^{2j}|\phi_j(r^2)|\frac{\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr$$
$$\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\frac{\omega^j\lambda(x)^{2j}C_1^j}{(j-1)!x^4\log^{b+N-2}(x)}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} r^{2j+2}dr$$
$$\leqslant \frac{C}{\omega^{5/2}\lambda(x)^3x^4\log^{b+N-2}(x)}, \quad \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1$$

Case 3: $\frac{2}{\sqrt{\omega}\lambda(x)} \geqslant \frac{x}{2\lambda(x)}$.

$$\frac{1}{\omega}\sum_{j=1}^{\infty}\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}} r^{2j}\omega^j\lambda(x)^{2j}|\phi_j(r^2)|\frac{\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr$$
$$\leqslant \frac{C}{\omega}\sum_{j=1}^{\infty}\frac{\omega^j\lambda(x)^{2j}C_1^j}{(j-1)!}\int_0^{\frac{x}{2\lambda(x)}}\frac{r^{2j}\log(1+r^2)dr}{x^4\log^{b+N-2}(x)(r^2+1)^2}$$
$$\leqslant \frac{C}{x^4\log^{3b+N-3}(x)}, \quad \frac{2}{\sqrt{\omega}\lambda(x)} \geqslant \frac{x}{2\lambda(x)}$$

Finally, we consider the following integral in multiple cases:

Case 1: $\frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1$.

$$\begin{aligned}&\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty} \frac{2|\text{Re}\left(a(\omega\lambda(x)^2)\psi^+(r,\omega\lambda(x)^2)\right)|\sqrt{r}\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr\\
&\leqslant \frac{C}{\omega}\frac{|a(\omega\lambda(x)^2)|}{\omega^{1/4}\lambda(x)^{1/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\frac{x}{2\lambda(x)}} \frac{\sqrt{r}dr}{x^4\log^{b+N-2}(x)(r^2+1)^2}\\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\omega^{5/4}\sqrt{\lambda(x)}x^4\log^{b+N-2}(x)}, \quad \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant 1\end{aligned}$$

Case 2: $1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{x}{2\lambda(x)}$

$$\begin{aligned}&\frac{1}{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty} 2|\text{Re}\left(a(\omega\lambda(x)^2)\psi^+(r,\omega\lambda(x)^2)\right)|\sqrt{r}\frac{\mathbb{1}_{\{r\lambda(x)\leqslant\frac{x}{2}\}}}{x^4\log^{b+N-2}(x)(r^2+1)^2}dr\\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{\omega^{5/4}\lambda(x)^{1/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\frac{x}{2\lambda(x)}} \frac{dr}{r^{7/2}x^4\log^{b+N-2}(x)}\\
&\leqslant \frac{C|a(\omega\lambda(x)^2)|}{x^4\log^{3b+N-2}(x)}, \quad 1 \leqslant \frac{2}{\sqrt{\omega}\lambda(x)} \leqslant \frac{x}{2\lambda(x)}\end{aligned}$$

Case 3: $\frac{2}{\sqrt{\omega}\lambda(x)} > \frac{x}{2\lambda(x)}$. In this case, the integral to estimate is zero. Combining all of the above estimates, and using the procedure described above, we get

$$\begin{aligned}&||-\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\omega}\mathcal{F}(\sqrt{\cdot}\partial_x^2(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}\end{aligned}$$

In total, we finally get

$$||\int_t^{\infty}\cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)} \quad (6.32)$$

The next integral to estimate is

$$\begin{aligned}&\lambda(t)\int_t^{\infty}\sin((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\\
&= \lambda(t)\Bigg(-\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\sqrt{\omega}}\\
&\qquad -\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\sqrt{\omega}}\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)dx\Bigg)\\
&= \frac{-\lambda(t)\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\sqrt{\omega}}\\
&- \lambda(t)\int_t^{\infty}\frac{\sin((t-x)\sqrt{\omega})}{\omega}\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)dx\end{aligned}$$

In order to estimate $\frac{-\lambda(t)\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\sqrt{\omega}}$ it suffices to multiply the pointwise in ω estimates that we already obtained on $\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\omega}$, by $\sqrt{\omega}\lambda(t)$, and then take the $L^2(\rho(\omega\lambda(t)^2)d\omega)$ norm. In fact, we only need to consider the region $\frac{2}{\sqrt{\omega}\lambda(t)} \leqslant 1$, since, in the other regions, the factor $\sqrt{\omega}\lambda(t)$ that we multiply by, is less than 2. Doing this procedure for each of the terms appearing in (6.24), (6.25), and (6.26), and combining the resulting estimates, we get

$$\begin{aligned}&|\frac{-\lambda(t)\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\sqrt{\omega}}|\\&\leqslant \frac{C}{\omega^{5/2}\lambda(t)^3t^2\log^{1-2\alpha b}(t)}+\frac{C\lambda(t)^{1/2}|a(\omega\lambda(t)^2)|}{\omega^{3/4}t^2\log^{1-2\alpha b}(t)}, \quad \frac{2}{\sqrt{\omega}\lambda(t)}\leqslant 1\end{aligned}$$

and

$$\left(\int_{\frac{4}{\lambda(t)^2}}^{\infty}\rho(\omega\lambda(t)^2)\frac{d\omega}{\omega^5\lambda(t)^6t^4\log^{2-4\alpha b}(t)}\right)^{1/2}\leqslant \frac{C}{t^2\log^{1+b-2\alpha b}(t)}$$

$$\left(\int_{\frac{4}{\lambda(t)^2}}^{\infty}\frac{|a(\omega\lambda(t)^2)|^2}{\omega^{3/2}}\rho(\omega\lambda(t)^2)d\omega\right)^{1/2}\cdot\frac{C}{t^2\log^{1+\frac{b}{2}-2\alpha b}(t)}\leqslant \frac{C}{t^2\log^{1+b-2\alpha b}(t)}$$

Then, by the remarks preceding these estimates, we get

$$||\frac{\lambda(t)\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)}{\sqrt{\omega}}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}$$

Then, we use the identical procedure as was used to obtain (6.32), and get

$$\begin{aligned}&||\lambda(t)\int_t^{\infty}\frac{\sin((t-x)\sqrt{\omega})}{\omega}\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{2b+1-2\alpha b}(t)}\end{aligned}$$

Combining these, we get

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\int_t^{\infty}\frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}\end{aligned}$$

Next, we consider

$$\begin{aligned}&\sqrt{\omega}\lambda(t)\partial_t\left(\int_t^{\infty}\frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\right)\\&=\frac{-\lambda(t)\partial_t\left(\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)\right)}{\sqrt{\omega}}\\&-\int_t^{\infty}\lambda(t)\cdot\frac{\cos((t-x)\sqrt{\omega})}{\sqrt{\omega}}\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)dx\end{aligned}\tag{6.33}$$

For $\frac{-\lambda(t)\left(\mathcal{F}(\sqrt{\cdot}\partial_t(F_4(t,\cdot\lambda(t))))(\omega\lambda(t)^2)\right)}{\sqrt{\omega}}$, we again need only multiply the pointwise estimates for $\frac{\mathcal{F}(\sqrt{\cdot}\partial_t(F_4(t,\cdot\lambda(t))))(\omega\lambda(t)^2)}{\omega}$ (which were previously inferred from pointwise estimates for $\frac{\mathcal{F}(\sqrt{\cdot}(F_4(t,\cdot\lambda(t))))(\omega\lambda(t)^2)}{\omega}$ and noting the symbol-type nature of the estimate (5.174)) by $\sqrt{\omega}\lambda(t)$ in the region $\sqrt{\omega}\lambda(t) \geqslant 2$, and then take the $L^2(\rho(\omega\lambda(t)^2)d\omega)$ norm. This results in

$$|| - \sqrt{\omega}\lambda(t)\left(\frac{\mathcal{F}(\sqrt{\cdot}\partial_t(F_4(t,\cdot\lambda(t))))(\omega\lambda(t)^2)}{\omega}\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}$$

Note, however, that doing this same procedure for the term

$$-\sqrt{\omega}\lambda(t)\left(\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))'(\omega\lambda(t)^2)\cdot 2\omega\lambda(t)\lambda'(t)}{\omega}\right)$$

would result in an estimate which is not square integrable against the measure $\rho(\omega\lambda(t)^2)d\omega$. Instead, we have to integrate by parts in an appropriate r integral, to gain extra decay in ω. In particular, we first make the decomposition as in (6.27). Then, we can multiply pointwise estimates on the first term in the decomposition (6.27) by $\sqrt{\omega}\lambda(t)$, and proceed as with our previous estimates. This results in a contribution to the overall $L^2(\rho(\omega\lambda(t)^2)d\omega)$ norm of

$$-\sqrt{\omega}\lambda(t)\left(\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))'(\omega\lambda(t)^2)\cdot 2\omega\lambda(t)\lambda'(t)}{\omega}\right)$$

bounded above by:

$$\begin{aligned}&|| - 2\lambda(t)^2\lambda'(t)\sqrt{\omega}\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(t)}}\sqrt{r}F_4(t,r\lambda(t))\partial_2\phi(r,\omega\lambda(t)^2)dr\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+2-2\alpha b}(t)}\end{aligned}$$

For the second term in the decomposition as in (6.27),namely,

$$-2\sqrt{\omega}\lambda(t)^2\lambda'(t)\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\sqrt{r}F_4(t,r\lambda(t))\partial_2\phi(r,\omega\lambda(t)^2)dr$$

we start with

$$\partial_2\phi(r,\xi) = (\partial_2\phi(r,\xi))_0 + (\partial_2\phi(r,\xi))_1$$

where

$$(\partial_2\phi(r,\xi))_1 = \text{Re}\left(\frac{a(\xi)ir}{\xi^{3/4}}e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi},r)\right)$$

and

$$|(\partial_2\phi(r,\xi))_0| \leqslant \frac{C|a(\xi)|}{\xi^{5/4}}, \quad r\sqrt{\xi} \geqslant 2$$

Then,

$$\begin{aligned}
&-2\lambda(t)^2\lambda'(t)\sqrt{\omega}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\sqrt{r}F_4(t,r\lambda(t))(\partial_2\phi(r,\omega\lambda(t)^2))_1 dr\\
&=-2\lambda(t)^2\lambda'(t)\sqrt{\omega}\text{Re}\left(\frac{a(\omega\lambda(t)^2)i}{\omega^{3/4}\lambda(t)^{3/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty} r^{3/2}F_4(t,r\lambda(t))e^{ir\sqrt{\omega}\lambda(t)}\sigma(r\sqrt{\omega}\lambda(t),r)dr\right)\\
&=\frac{-2\lambda(t)^{1/2}\lambda'(t)}{\omega^{1/4}}\text{Re}\left(\frac{-a(\omega\lambda(t)^2)2^{3/2}F_4(t,\frac{2}{\sqrt{\omega}})\sigma(2,\frac{2}{\sqrt{\omega}\lambda(t)})e^{2i}}{\omega^{5/4}\lambda(t)^{5/2}}\right)\\
&+\frac{2\lambda(t)^{1/2}\lambda'(t)}{\omega^{1/4}}\text{Re}\left(a(\omega\lambda(t)^2)\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\frac{e^{ir\sqrt{\omega}\lambda(t)}}{\sqrt{\omega}\lambda(t)}\partial_r\left(r^{3/2}F_4(t,r\lambda(t))\sigma(r\sqrt{\omega}\lambda(t),r)\right)dr\right)
\end{aligned}$$

So,

$$\begin{aligned}
&|-2\sqrt{\omega}\lambda(t)^2\lambda'(t)\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\sqrt{r}F_4(t,r\lambda(t))\partial_2\phi(r,\omega\lambda(t)^2)dr|\\
&\leqslant\frac{C|\lambda'(t)||a(\omega\lambda(t)^2)|}{\omega^{3/2}\lambda(t)^2}|F_4(t,\frac{2}{\sqrt{\omega}})|\\
&+\frac{C|\lambda'(t)||a(\omega\lambda(t)^2)|}{\lambda(t)^{1/2}\omega^{3/4}}\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty} r^{1/2}\left(|F_4(t,r\lambda(t))|+r\lambda(t)|\partial_2F_4(t,r\lambda(t))|\right)dr
\end{aligned}$$

where we used the estimates on σ following from proposition 4.6 of [14].

This gives

$$\begin{aligned}
&||-2\sqrt{\omega}\lambda(t)^2\lambda'(t)\int_{\frac{2}{\sqrt{\omega}\lambda(t)}}^{\infty}\sqrt{r}F_4(t,r\lambda(t))\partial_2\phi(r,\omega\lambda(t)^2)dr||_{L^2((\frac{4}{\lambda(t)^2},\infty),\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant\frac{C|\lambda'(t)|}{\lambda(t)^2}\left(\int_0^{\lambda(t)}|F_4(t,y)|^2y^3dy\right)^{1/2}\\
&+\frac{C|\lambda'(t)|}{\lambda(t)^{1/2}}\left(\int_{\frac{4}{\lambda(t)^2}}^{\infty}\frac{1}{\omega^{3/2}}\left(\int_0^{\infty}\frac{r^{3/2}}{t^2\log^{1-2\alpha b}(t)(r^2+1)^2}dr\right)^2d\omega\right)^{1/2}\\
&\leqslant\frac{C}{t^3\log^{b+2-2\alpha b}(t)}
\end{aligned}$$

Combining these estimates, and appropriately using (6.30), for the region $\sqrt{\omega}\lambda(t)\leqslant 2$, we get

$$\begin{aligned}
&||-\sqrt{\omega}\lambda(t)\left(\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))'(\omega\lambda(t)^2)\cdot 2\omega\lambda(t)\lambda'(t)}{\omega}\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant\frac{C(\log(\log(t)))^2}{t^3\log^{b+2-2b\alpha}(t)}
\end{aligned}\tag{6.34}$$

Now, we start to treat the term inside the x integral in (6.33). Here, we expand

$$\begin{aligned}&\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)\\&=\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''(\omega\lambda(x)^2)\cdot(2\omega\lambda(x)\lambda'(x))^2\\&+2\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)\cdot 2\omega\lambda(x)\lambda'(x)\\&+\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)\cdot 2\omega((\lambda'(x))^2+\lambda(x)\lambda''(x))\\&+\mathcal{F}(\sqrt{\cdot}\partial_x^2(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)\end{aligned}$$

We start by considering

$$\frac{\sqrt{\omega}\lambda(t)}{\omega}\cdot\left(2\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)\cdot 2\omega\lambda(x)\lambda'(x)\right)$$

Recall that the last term we estimated was

$$-\sqrt{\omega}\lambda(t)\cdot 2\lambda(t)\lambda'(t)\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))'(\omega\lambda(t)^2)$$

and we used only (5.187), as well as (5.188), because we needed to integrate by parts in the r variable in one of the terms. We then repeat the same procedure, with the only difference being

$$\partial_1 F_4(x,r\lambda(x))+r\lambda'(x)\partial_2F_4(x,r\lambda(x))\text{ replacing }F_4(t,r\lambda(t))$$

and

$$\begin{aligned}&\lambda(t)\partial_{12}F_4(x,r\lambda(x))+\lambda'(x)\partial_2F_4(x,r\lambda(x))+r\lambda(x)\lambda'(x)\partial_2^2F_4(x,r\lambda(x))\\&\text{replacing }\lambda(x)\partial_2F_4(x,r\lambda(x))\end{aligned}$$

By noting the symbol-type nature of the estimate (5.174), we get

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\cdot 4\lambda(x)\lambda'(x)\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2||\sqrt{\omega}\lambda(x)\cdot 4\lambda(x)\lambda'(x)\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{(\log(\log(x)))^2}{x^4\log^{b+2-2b\alpha}(x)}\end{aligned}$$

and this gives

$$\begin{aligned}&||-\lambda(t)\int_t^\infty\frac{\cos((t-x)\sqrt{\omega})}{\sqrt{\omega}}2\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))'(\omega\lambda(x)^2)\cdot 2\omega\lambda(x)\lambda'(x)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant\frac{C(\log(\log(t)))^2}{t^3\log^{b+2-2b\alpha}(t)}\end{aligned}$$

Next, we consider

$$\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2)\cdot 2\omega((\lambda'(x))^2+\lambda(x)\lambda''(x))\cdot\sqrt{\omega}\lambda(t)}{\omega}$$

We treat this term identically to how (6.34) was treated, noting that the only difference between the two terms is a coefficient which depends on absolute constants and λ. We therefore get

$$\begin{aligned}
&|| -\lambda(t)\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\sqrt{\omega}} \Big(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))'(\omega\lambda(x)^2) \\
&\qquad\qquad \cdot 2\omega((\lambda'(x))^2+\lambda(x)\lambda''(x))\Big)\, dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \\
&\leqslant \int_t^\infty \frac{C}{x}\left(\frac{\lambda(t)}{\lambda(x)}\right)^2 \frac{(\log(\log(x)))^2}{x^3\log^{b+2-2b\alpha}(x)}dx \\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+2-2b\alpha}(t)}
\end{aligned}$$

Next, we study

$$\frac{\sqrt{\omega}\lambda(t)}{\omega}\cdot\mathcal{F}(\partial_x^2\left(\sqrt{\cdot}F_4(x,\cdot\lambda(x))\right))(\omega\lambda(x)^2)$$

by multiplying our previous pointwise in ω estimates on $\frac{\mathcal{F}(\partial_x^2\left(\sqrt{\cdot}F_4(x,\cdot\lambda(x))\right))(\omega\lambda(x)^2)}{\omega}$ by $\left(\frac{\lambda(t)}{\lambda(x)}\right)\sqrt{\omega}\lambda(x)$. We need only check the contributions to

$$\left(\frac{\lambda(t)}{\lambda(x)}\right)\cdot||\frac{\sqrt{\omega}\lambda(x)}{\omega}\cdot\mathcal{F}(\partial_x^2\left(\sqrt{\cdot}F_4(x,\cdot\lambda(x))\right))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

coming from the region $\sqrt{\omega}\lambda(x)\geqslant 2$, just as for previous terms:

The integrals to check are:

$$\begin{aligned}
&\left(\frac{\lambda(t)}{\lambda(x)}\right)\left(\int_{\frac{4}{\lambda(x)^2}}^\infty \frac{\rho(\omega\lambda(t)^2)d\omega}{\omega^4\lambda(x)^4x^8\log^{2b+2N-4}(x)}\right)^{1/2} \\
&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\left(\int_{\frac{4}{\lambda(x)^2}}^\infty \frac{\rho(\omega\lambda(x)^2)d\omega}{\omega^4\lambda(x)^4x^8\log^{2b+2N-4}(x)}\right)^{1/2} \\
&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{2b+N-2}(x)}
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\left(\int_{\frac{4}{\lambda(x)^2}}^\infty \frac{\omega\lambda(x)^2d\omega}{\omega^5}\right)^{1/2}\frac{1}{\lambda(x)^3x^4\log^{1-2\alpha b}(x)} \\
&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{b+1-2\alpha b}(x)}
\end{aligned}$$

and

$$\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\left(\int_{\frac{4}{\lambda(x)^2}}^\infty \frac{d\omega}{\omega^{3/2}x^8\log^{b+2-4\alpha b}(x)}\right)^{1/2}\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{b+1-2\alpha b}(x)}$$

In total, we get

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\frac{\mathcal{F}(\partial_x^2(\sqrt{\cdot}F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)}{\omega}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{(\log(\log(x)))^2}{x^4\log^{1+b-2\alpha b}(x)}\end{aligned}$$

and this gives

$$\begin{aligned}&||\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}\partial_x^2(F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)\frac{\sqrt{\omega}\lambda(t)}{\omega}dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}\end{aligned}$$

The last term we need to consider here is

$$\frac{\sqrt{\omega}\lambda(t)}{\omega}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''(\omega\lambda(x)^2)\cdot(2\omega\lambda(x)\lambda'(x))^2$$

We already estimated

$$\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''(\omega\lambda(x)^2)\cdot(2\omega\lambda(x)\lambda'(x))^2}{\omega}$$

So, we need only prove new estimates in the region $\sqrt{\omega}\lambda(x)\geqslant 2$ (by writing $\sqrt{\omega}\lambda(t)=\left(\frac{\lambda(t)}{\lambda(x)}\right)\cdot\sqrt{\omega}\lambda(x)$).

So, we again write

$$\begin{aligned}&\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''(\omega\lambda(x)^2)\\&=\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr\\&+\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty\sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr\end{aligned}\tag{6.35}$$

and for the second line of (6.35), we simply multiply our previous pointwise in ω estimate on $\left(\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr\right)\cdot\frac{(2\omega\lambda(x)\lambda'(x))^2}{\omega}$ by $\frac{\lambda(t)}{\lambda(x)}\sqrt{\omega}\lambda(x)$ in the region $\sqrt{\omega}\lambda(x)\geqslant 2$, and estimate the $L^2(\rho(\omega\lambda(t)^2)d\omega)$ norm, as before. We have

$$\begin{aligned}&\sqrt{\omega}\lambda(x)\cdot\frac{\omega^2\lambda(x)^2\lambda'(x)^2}{\omega}\cdot|\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr|\\&\leqslant\frac{C}{x^4\log^{3-2b\alpha}(x)\omega^{5/2}\lambda(x)^3},\quad \sqrt{\omega}\lambda(x)\geqslant 2\end{aligned}$$

and this leads to

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)\left(\frac{\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr}{\omega}\cdot\omega^2\lambda(x)^2(\lambda'(x))^2\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\left(\frac{\lambda(t)^2}{\lambda(x)^2}\right)\frac{\log(\log(x)))^2}{x^4\log^{3+b-2b\alpha}(x)}\end{aligned}$$

For the third line of (6.35), we can not simply multiply our pointwise in ω estimates on

$$\left(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty} \sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr\right)\cdot\frac{(2\omega\lambda(x)\lambda'(x))^2}{\omega}$$

by $\sqrt{\omega}\lambda(t)$, since doing so would result in an estimate that is not square integrable against the measure $\rho(\omega\lambda(t)^2)d\omega$. So, we have to integrate by parts in appropriate r integrals, just as in a previous situation. To be precise, we consider

$$\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty} \sqrt{r}F_4(x,r\lambda(x))\partial_2^2\phi(r,\omega\lambda(x)^2)dr \tag{6.36}$$

and write, for $r\sqrt{\xi}\geqslant 2$:

$$\begin{aligned}\partial_2^2\phi(r,\xi) &= 2\mathrm{Re}\Bigg(\Bigg(\frac{a''(\xi)}{\xi^{1/4}}\sigma(r\sqrt{\xi},r)+2a'(\xi)\left(\frac{-\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}+\frac{r\partial_1\sigma(r\sqrt{\xi},r)}{2\xi^{3/4}}\right)\\ &\quad+\ a(\xi)\left(\frac{5\sigma(r\sqrt{\xi},r)}{16\xi^{9/4}}-\frac{r\partial_1\sigma(r\sqrt{\xi},r)}{2\xi^{7/4}}+\frac{ir^2\partial_1\sigma(r\sqrt{\xi},r)}{2\xi^{5/4}}+\frac{r^2\partial_1^2\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}\right)\Bigg)e^{ir\sqrt{\xi}}\Bigg)\\ &\quad+2\mathrm{Re}\left(\left(2\frac{a'(\xi)ir\sigma(r\sqrt{\xi},r)}{2\xi^{3/4}}-\frac{ira(\xi)\sigma(r\sqrt{\xi},r)}{2\xi^{7/4}}\right)e^{ir\sqrt{\xi}}\right)\\ &\quad-2\mathrm{Re}\left(\frac{a(\xi)r^2}{4\xi^{5/4}}e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi},r)\right)\end{aligned} \tag{6.37}$$

For each term on the third line of (6.37), inserted into (6.36), we will integrate by parts once in r. The insertion of the first term on the third line of (6.37) gives

$$\begin{aligned}&2\mathrm{Re}\left(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))\cdot\frac{a'(\omega\lambda(x)^2)ir}{\omega^{3/4}\lambda(x)^{3/2}}\sigma(r\sqrt{\omega}\lambda(x),r)e^{ir\sqrt{\omega}\lambda(x)}dr\right)\\ &=2\mathrm{Re}\Bigg(\frac{ia'(\omega\lambda(x)^2)}{\omega^{3/4}\lambda(x)^{3/2}}\Bigg(\frac{-2^{3/2}F_4(x,\frac{2}{\sqrt{\omega}})}{\omega^{3/4}\lambda(x)^{3/2}i\sqrt{\omega}\lambda(x)}\sigma(2,\frac{2}{\sqrt{\omega}\lambda(x)})e^{2i}\\ &\quad-\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\frac{e^{ir\sqrt{\omega}\lambda(x)}}{i\sqrt{\omega}\lambda(x)}\partial_r\left(r^{3/2}F_4(x,r\lambda(x))\sigma(r\sqrt{\omega}\lambda(x),r)\right)dr\Bigg)\Bigg)\end{aligned} \tag{6.38}$$

Then we estimate each term separately:

$$\begin{aligned}&\left|2\mathrm{Re}\left(\frac{ia'(\omega\lambda(x)^2)}{\omega^{3/4}\lambda(x)^{3/2}}\left(\frac{-2^{3/2}F_4(x,\frac{2}{\sqrt{\omega}})}{\omega^{3/4}\lambda(x)^{3/2}i\sqrt{\omega}\lambda(x)}\sigma(2,\frac{2}{\sqrt{\omega}\lambda(x)})e^{2i}\right)\right)\right|\\ &\leqslant C\frac{|a'(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\frac{|F_4(x,\frac{2}{\sqrt{\omega}})|}{\omega^{3/4}\lambda(x)^{3/2}\sqrt{\omega}\lambda(x)}\end{aligned}$$

The contribution to the $L^2((\frac{4}{\lambda(x)^2},\infty),\rho(\omega\lambda(t)^2)d\omega)$ norm of the integrand of (6.33) due to the above term is thus estimated by

$$\left(\int_{\frac{4}{\lambda(x)^2}}^{\infty}\left(\frac{|a'(\omega\lambda(x)^2)|}{\omega^{3/4}\lambda(x)^{3/2}}\frac{|F_4(x,\frac{2}{\sqrt{\omega}})|}{\omega^{3/4}\lambda(x)^{3/2}\sqrt{\omega}\lambda(x)}\right)^2\omega^4\cdot\left(\frac{\sqrt{\omega}\lambda(t)}{\omega}\right)^2\frac{\rho(\omega\lambda(t)^2)d\omega}{\left(x^2\log^{4b+2}(x)\right)^2}\right)^{1/2}$$
$$\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\left(\int_0^{\lambda(x)}y^6|F_4(x,y)|^2\frac{dy}{y^3}\right)^{1/2}\frac{1}{\lambda(x)^5x^2\log^{4b+2}(x)}$$
$$\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{b+3-2b\alpha}(x)}$$

For the next term of (6.38), we use Proposition 4.6 of [14] again, to get

$$|2\text{Re}\left(\frac{ia'(\omega\lambda(x)^2)}{\omega^{3/4}\lambda(x)^{3/2}}\left(-\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\frac{e^{ir\sqrt{\omega}\lambda(x)}}{i\sqrt{\omega}\lambda(x)}\partial_r\left(r^{3/2}F_4(x,r\lambda(x))\sigma(r\sqrt{\omega}\lambda(x),r)\right)dr\right)\right)|$$
$$\leqslant\frac{C|a(\omega\lambda(x)^2)|}{\omega^{7/4}\lambda(x)^{7/2}}\frac{1}{\sqrt{\omega}\lambda(x)}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}\left(|F_4(x,r\lambda(x))|+r\lambda(x)|(\partial_2F_4)(x,r\lambda(x))|\right)dr$$
$$\leqslant\frac{C|a(\omega\lambda(x)^2)|}{\omega^{9/4}\lambda(x)^{9/2}x^2\log^{1-2b\alpha}(x)}$$

Then, the contribution of this term to the $L^2((\frac{4}{\lambda(x)^2},\infty),\rho(\omega\lambda(t)^2)d\omega)$ norm of the integrand of (6.33) is estimated by:

$$\left(\int_{\frac{4}{\lambda(x)^2}}^{\infty}\frac{\omega^4}{\omega^{9/2}}\left(\frac{\sqrt{\omega}\lambda(x)}{\omega}\right)^2d\omega\right)^{1/2}\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^2\log^{1-2b\alpha}(x)}\cdot\frac{1}{\lambda(x)^{9/2}}\cdot\frac{1}{x^2\log^{4b+2}(x)}$$
$$\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{b+3-2b\alpha}(x)}$$

By comparing the first and second terms on the third line of (6.37), and recalling the symbol-type estimates on a from Proposition 4.7 of [14], we can estimate the second term on the third line of 6.37 by repeating the same exact procedure used to estimate the first term of the third line of (6.37).

Next, we treat the fourth line of (6.37). Here, we integrate by parts twice in the r integral resulting from substitution of the fourth line of (6.37) into (6.36). With $\xi=\omega\lambda(t)^2$, we have

$$\int_{\frac{2}{\sqrt{\xi}}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))\cdot 2\text{Re}\left(\frac{-a(\xi)r^2e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}\right)dr$$
$$=\text{Re}\left(\frac{-a(\xi)}{2\xi^{5/4}}\left(\frac{-2^{5/2}F_4(x,\frac{2}{\sqrt{\xi}}\lambda(x))\sigma(2,\frac{2}{\sqrt{\xi}})e^{2i}}{i\xi^{7/4}}\right.\right.$$
$$\left.\left.+\frac{1}{i\sqrt{\xi}}\cdot\left(\partial_r\left(r^{5/2}F_4(x,r\lambda(x))\sigma(r\sqrt{\xi},r)\right)\frac{e^{ir\sqrt{\xi}}}{i\sqrt{\xi}}\right)|_{r=\frac{2}{\sqrt{\xi}}}\right)\right)$$
$$+\text{Re}\left(\frac{-a(\xi)}{2\xi^{5/4}}\cdot\frac{-1}{\xi}\int_{\frac{2}{\sqrt{\xi}}}^{\infty}e^{ir\sqrt{\xi}}\partial_r^2\left(r^{5/2}F_4(x,r\lambda(x))\sigma(r\sqrt{\xi},r)\right)dr\right)$$

Next, we again note the symbol-type character of the estimate (5.174), to get

$$\begin{aligned}&|\int_{\frac{2}{\sqrt{\xi}}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))2\text{Re}\left(\frac{-a(\xi)r^2e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}\right)dr|\cdot 4\omega^2\lambda(x)^2\lambda'(x)^2\frac{\sqrt{\omega}\lambda(x)}{\omega}\\&\leqslant \frac{C\omega^2\lambda(x)}{\sqrt{\omega}x^2\log^{4b+2}(x)}\left(\frac{|a(\omega\lambda(x)^2)|}{\omega^3\lambda(x)^6}F_{4,est}(x,\frac{2\lambda(x)}{\sqrt{\xi}})\right.\\&\left.+\frac{|a(\omega\lambda(x)^2)|}{\omega^{9/4}\lambda(x)^{9/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}r^{1/2}F_{4,est}(x,r\lambda(x))dr\right)\end{aligned} \tag{6.39}$$

where $F_{4,est}$ is the expression which appears on the right-hand side of (5.174). Then, the same procedure used to treat (6.38) also applies to treat (6.39), and we get

$$\begin{aligned}&||\int_{\frac{2}{\sqrt{\xi}}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))2\text{Re}\left(\frac{-a(\xi)r^2e^{ir\sqrt{\xi}}\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}\right)dr\\&\quad\cdot 4\omega^2\lambda(x)^2\lambda'(x)^2\frac{\sqrt{\omega}\lambda(x)}{\omega}\left(\frac{\lambda(t)}{\lambda(x)}\right)||_{L^2((\frac{4}{\lambda(x)^2},\infty),\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{b+3-2b\alpha}(x)}\end{aligned}$$

Next, we study the first two lines of (6.37), which are given below:

$$\begin{aligned}&(\partial_2^2\phi(r,\xi))_2\\&=2\text{Re}\left(\left(\frac{a''(\xi)}{\xi^{1/4}}\sigma(r\sqrt{\xi},r)+2a'(\xi)\left(\frac{-\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}+\frac{r\partial_1\sigma(r\sqrt{\xi},r)}{2\xi^{3/4}}\right)\right.\right.\\&\left.\left.+\ a(\xi)\left(\frac{5\sigma(r\sqrt{\xi},r)}{16\xi^{9/4}}-\frac{r\partial_1\sigma(r\sqrt{\xi},r)}{2\xi^{7/4}}+\frac{ir^2\partial_1\sigma(r\sqrt{\xi},r)}{2\xi^{5/4}}+\frac{r^2\partial_1^2\sigma(r\sqrt{\xi},r)}{4\xi^{5/4}}\right)\right)e^{ir\sqrt{\xi}}\right)\end{aligned}$$

We note that

$$|(\partial_2^2\phi(r,\xi))_2|\leqslant\frac{C|a(\xi)|}{\xi^{9/4}},\quad r\geqslant\frac{2}{\sqrt{\xi}}$$

So,

$$\begin{aligned}&|\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))(\partial_2^2\phi(r,\omega\lambda(x)^2))_2dr|\\&\leqslant\frac{C|a(\omega\lambda(x)^2)|}{\omega^{9/4}\lambda(x)^{9/2}}\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}|F_4(x,r\lambda(x))|dr\end{aligned}$$

Finally,

$$\begin{aligned}&\left(\int_{\frac{4}{\lambda(x)^2}}^{\infty}\rho(\omega\lambda(t)^2)\left(\left(\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty}\sqrt{r}F_4(x,r\lambda(x))(\partial_2^2\phi(r,\omega\lambda(x)^2))_2dr\right)\right.\right.\\&\qquad\left.\left.\cdot 4\omega^2\lambda(x)^2\lambda'(x)^2\frac{\sqrt{\omega}\lambda(t)}{\omega}\right)^2d\omega\right)^{1/2}\\&\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{b+3-2b\alpha}(x)}\end{aligned}$$

Combining all of our estimates, we get

$$||\sqrt{\omega}\lambda(t)\left(\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))''(\omega\lambda(x)^2)}{\omega}\omega^2\lambda(x)^2\lambda'(x)^2\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant C\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{\log(\log(x))^2}{x^4\log^{3+b-2b\alpha}(x)}$$

and

$$\begin{aligned}&||-\int_t^\infty \cos((t-x)\sqrt{\omega})\frac{\sqrt{\omega}\lambda(t)}{\omega}\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}\end{aligned}\tag{6.40}$$

which imply

$$||\sqrt{\omega}\lambda(t)\partial_t\left(\int_t^\infty\frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant\frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}$$

The last quantity to estimate is

$$\begin{aligned}&\omega\lambda(t)^2\left(\int_t^\infty\frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx\right)\\&=-\lambda(t)^2\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)\\&-\sqrt{\omega}\lambda(t)^2\int_t^\infty\frac{\sin((t-x)\sqrt{\omega})}{\omega}\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)dx\end{aligned}\tag{6.41}$$

We can estimate the third line of (6.41), simply by using the estimates which gave (6.40). For the second line of (6.41), we have, with $\xi=\omega\lambda(t)^2$,

$$\begin{aligned}&\lambda(t)^2\left(\int_0^\infty\rho(\omega\lambda(t)^2)\left(\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\omega\lambda(t)^2)\right)^2d\omega\right)^{1/2}\\&=\lambda(t)^2\left(\int_0^\infty\rho(\xi)\left(\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))(\xi)\right)^2\frac{d\xi}{\lambda(t)^2}\right)^{1/2}\\&=\lambda(t)\left(\int_0^\infty r\left(F_4(t,r\lambda(t))\right)^2dr\right)^{1/2}\\&\leqslant C\lambda(t)\left(\int_0^{\frac{\log^N(t)}{\lambda(t)}}\frac{r^3\lambda(t)^2dr}{\lambda(t)^8(r^2+1)^4t^4\log^{6b+2-4b\alpha}(t)}+\int_0^{\frac{t}{2\lambda(t)}}\frac{r^3\lambda(t)^2dr}{\lambda(t)^8(r^2+1)^4t^4\log^{10b+4N-4}(t)}\right)^{1/2}\\&\leqslant\frac{C}{t^2\log^{b+1-2b\alpha}(t)}+\frac{C}{t^2\log^{3b+2N-2}(t)}\\&\leqslant\frac{C}{t^2\log^{b+1-2b\alpha}(t)}\end{aligned}$$

So,

$$||\omega\lambda(t)^2 \int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}} \mathcal{F}(\sqrt{\cdot} F_4(x, \cdot\lambda(x)))(\omega\lambda(x)^2) dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant \frac{C}{t^2 \log^{1+b-2b\alpha}(t)}$$

which finishes the proof of the lemma □

To proceed, we quickly translate our estimates (5.3) and (5.4), by noting

$$||\mathcal{F}(\sqrt{\cdot} (F_5 + F_6)(x, \cdot\lambda(x)))(\omega\lambda(x)^2)||^2_{L^2(\rho(\omega\lambda(x)^2)d\omega)}$$
$$= \int_0^\infty \frac{r}{\lambda(x)^4} (F_5 + F_6)^2 (x, r) dr$$

$$||\sqrt{\omega}\lambda(x)\mathcal{F}(\sqrt{\cdot} (F_5 + F_6)(x, \cdot\lambda(x)))(\omega\lambda(x)^2)||^2_{L^2(\rho(\omega\lambda(x)^2)d\omega)}$$
$$= \frac{1}{\lambda(x)^2} \int_0^\infty (L((F_5 + F_6)(x, \cdot\lambda(x))))^2(R) R dR$$

and

$$|L(f)(r)| \leqslant C \left(|f'(r)| + \frac{|f(r)|}{r} \right)$$

So, we have

$$||\mathcal{F}(\sqrt{\cdot} (F_5 + F_6)(x, \cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{C}{x^4 \log^{3b+2N-1}(x)}$$

and

$$||\sqrt{\omega}\lambda(x)\mathcal{F}(\sqrt{\cdot} (F_5 + F_6)(x, \cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant C \frac{\log^{6+b}(x)}{x^{35/8}}$$

Now, we recall $F(t,r) = F_4(t,r) + F_5(t,r) + F_6(t,r)$, and estimate the following quantities:

$$\begin{aligned}
&||\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&= ||\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\Big(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\\
&\qquad\qquad + \mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\Big) dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant ||\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&+ ||\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2\alpha b}(t)}\\
&+ C\int_t^\infty (x-t)\left(||\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)dx\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2\alpha b}(t)} + C\int_t^\infty (x-t)\left(\frac{\lambda(t)}{\lambda(x)}\right)\frac{dx}{x^4\log^{3b+2N-1}(x)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2\alpha b}(t)}
\end{aligned} \tag{6.42}$$

where we used the fact that,

$$\begin{aligned}
&||\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&= \left(\int_0^\infty \left(\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\right)^2 \frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)}\rho(\omega\lambda(x)^2)d\omega\right)^{1/2}\\
&\leqslant C\frac{\lambda(t)}{\lambda(x)}||\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}
\end{aligned}$$

where we used (6.23).

Similarly,

$$\begin{aligned}
&||\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant ||\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&+||\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}\\
&+C\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)\left(||\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)dx\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}+\frac{C}{t^3\log^{3b+2N-1}(t)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}
\end{aligned}$$

$$\begin{aligned}
&||\sqrt{\omega}\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant ||\sqrt{\omega}\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&+||\sqrt{\omega}\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}\\
&+C\lambda(t)\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)\left(||\mathcal{F}(\sqrt{\cdot}\left(F_5+F_6\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\right)dx\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}+\frac{C}{t^3\log^{4b+2N-1}(t)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}
\end{aligned}$$

$$
\begin{aligned}
&||\int_{t}^{\infty}\cos((t-x)\sqrt{\omega})\sqrt{\omega}\lambda(x)\left(\frac{\lambda(t)}{\lambda(x)}\right)\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})dx||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\
&\leqslant ||\int_{t}^{\infty}\cos((t-x)\sqrt{\omega})\sqrt{\omega}\lambda(x)\left(\frac{\lambda(t)}{\lambda(x)}\right)\mathcal{F}(\sqrt{\cdot}F_{4}(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})dx||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\
&+||\int_{t}^{\infty}\cos((t-x)\sqrt{\omega})\sqrt{\omega}\lambda(x)\left(\frac{\lambda(t)}{\lambda(x)}\right)\mathcal{F}(\sqrt{\cdot}\left(F_{5}+F_{6}\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})dx||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^{2}}{t^{3}\log^{1+b-2\alpha b}(t)}\\
&+C\int_{t}^{\infty}\left(\frac{\lambda(t)}{\lambda(x)}\right)^{2}\left(||\sqrt{\omega}\lambda(x)\mathcal{F}(\sqrt{\cdot}\left(F_{5}+F_{6}\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})||_{L^{2}(\rho(\omega\lambda(x)^{2})d\omega)}\right)dx\\
&\leqslant \frac{C(\log(\log(t)))^{2}}{t^{3}\log^{1+b-2\alpha b}(t)}+C\int_{t}^{\infty}\left(\frac{\lambda(t)}{\lambda(x)}\right)^{2}\frac{\log^{b+6}(x)}{x^{35/8}}dx\\
&\leqslant \frac{C(\log(\log(t)))^{2}}{t^{3}\log^{1+b-2\alpha b}(t)}
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
&||\int_{t}^{\infty}\sin((t-x)\sqrt{\omega})\sqrt{\omega}\lambda(t)^{2}\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})dx||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\
&\leqslant ||\int_{t}^{\infty}\sin((t-x)\sqrt{\omega})\sqrt{\omega}\lambda(t)^{2}\mathcal{F}(\sqrt{\cdot}F_{4}(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})dx||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\
&+||\int_{t}^{\infty}\sin((t-x)\sqrt{\omega})\sqrt{\omega}\lambda(t)^{2}\mathcal{F}(\sqrt{\cdot}\left(F_{5}+F_{6}\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})dx||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\
&\leqslant \frac{C}{t^{2}\log^{1+b-2\alpha b}(t)}\\
&+C\int_{t}^{\infty}\lambda(t)\left(\frac{\lambda(t)}{\lambda(x)}\right)^{2}\left(||\lambda(x)\sqrt{\omega}\mathcal{F}(\sqrt{\cdot}\left(F_{5}+F_{6}\right)(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})||_{L^{2}(\rho(\omega\lambda(x)^{2})d\omega)}\right)dx\\
&\leqslant \frac{C}{t^{2}\log^{1+b-2\alpha b}(t)}+C\int_{t}^{\infty}\frac{\lambda(t)^{3}}{\lambda(x)^{2}}\frac{\log^{b+6}(x)}{x^{35/8}}dx\\
&\leqslant \frac{C}{t^{2}\log^{1+b-2\alpha b}(t)}
\end{aligned}
$$

6.5 Setup of the final iteration

Let ϵ be given by

$$\epsilon=2b+\frac{1}{2}(1-2\alpha b)$$

Note that $2b+\frac{1}{2}>\epsilon>2b$. Also, note that (5.5), (5.6), and (5.7) show that

$$||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)}||_{L^{\infty}}^{2}+||\frac{v_{corr}(x,R\lambda(x))}{R\lambda(x)^{2}(1+R^{2})}||_{L^{\infty}}\leqslant\frac{C}{x^{2}\log^{\epsilon-2b}(x)} \tag{6.43}$$

$$1+||\frac{v_{corr}(x,R\lambda(x))}{R}||_{L^{\infty}}+||\partial_{R}(v_{corr}(x,R\lambda(x)))||_{L^{\infty}}\leqslant C \tag{6.44}$$

$$\begin{aligned}&||\frac{v_{corr}(x,R\lambda(x))\partial_{R}(v_{corr}(x,R\lambda(x)))}{R\lambda(x)^{2}}||_{L^{\infty}_{R}((0,1))}\\&+||\frac{v_{corr}(x,R\lambda(x))\partial_{R}(v_{corr}(x,R\lambda(x)))}{R^{2}\lambda(x)^{2}}||_{L^{\infty}_{R}((1,\infty))}+||\frac{\partial_{R}(v_{corr}(x,R\lambda(x)))}{(1+R^{2})\lambda(x)^{2}}||_{L^{\infty}}\\&\leqslant\frac{C}{x^{2}\log^{\epsilon-2b}(x)}\end{aligned} \tag{6.45}$$

Let $(Z,||\cdot||_{Z})$ be the normed vector space defined as follows. Z is the set of (equivalence classes) of measureable functions $y:[T_{0},\infty)\times(0,\infty)\to\mathbb{R}$ such that

$$y(t,\omega)t^{2}\log^{\frac{\epsilon}{2}}(t)\sqrt{\rho(\omega\lambda(t)^{2})}\langle\omega\lambda(t)^{2}\rangle\in C^{0}_{t}([T_{0},\infty),L^{2}(d\omega))$$

$$\partial_{t}y(t,\omega)t^{3}\log^{\frac{\epsilon}{2}}(t)\langle\sqrt{\omega}\lambda(t)\rangle\sqrt{\rho(\omega\lambda(t)^{2})}\in C^{0}_{t}([T_{0},\infty),L^{2}(d\omega))$$

and $||y||_{Z}<\infty$ where

$$\begin{aligned}||y||_{Z}=\sup_{t\geqslant T_{0}}\Big(&t^{2}\log^{\frac{\epsilon}{2}}(t)\,(\,||y(t)||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}+||\lambda(t)\sqrt{\omega}y(t)||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\\&\qquad+||\omega\lambda(t)^{2}y(t)||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)})\\&+t^{3}\log^{\frac{\epsilon}{2}}(t)\left(||\partial_{t}y(t)||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}+||\lambda(t)\sqrt{\omega}\partial_{t}y(t)||_{L^{2}(\rho(\omega\lambda(t)^{2})d\omega)}\right)\Big)\end{aligned} \tag{6.46}$$

Define T on Z by

$$\begin{aligned}T(y)(t,\omega)=-\int_{t}^{\infty}\frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}&\left(F_{2}(y)(x,\omega)-\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})\right.\\&\left.-\mathcal{F}(\sqrt{\cdot}F_{3}(u(y))(x,\cdot\lambda(x)))(\omega\lambda(x)^{2})\right)dx\end{aligned}$$

Note that a fixed point of T is a solution to (6.3) with 0 Cauchy data at infinity. We will prove the following proposition which implies that T indeed has a fixed point in $\overline{B_{1}(0)}\subset Z$.

Proposition 6.3. *There exists $T_{4}>0$ such that, for all $T_{0}>T_{4}$, T is a strict contraction on $\overline{B}_{1}(0)\subset Z$*

We use (6.42), Propositions 6.1 and 6.2, and the equations (6.43), (6.44), and (6.45), to get the following (note that in the following estimates, $C>0$ denotes a constant (which might involve C_{ρ}) whose value may change from line to line, but which is *independent* of T_{0}). Also, for ease of notation, we will denote $F_3(u(y))$ by F_3 until otherwise mentioned.

$$\begin{aligned}&||T(y)(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2\alpha b}(t)}+C\int_t^\infty \frac{\lambda(t)}{\lambda(x)}\frac{1}{x^3\log^{\frac{\epsilon}{2}+1}(x)}dx\\
&+C\int_t^\infty x\left(\frac{\lambda(t)}{\lambda(x)}\right)\left(\frac{1}{x^2\log^{\frac{\epsilon}{2}}(x)}\left(\frac{1}{x^2\log^{\epsilon-2b}(x)}\right)+\frac{1}{\lambda(x)x^4\log^{\epsilon}(x)}\right)dx\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2\alpha b}(t)}+\frac{C}{t^2\log^{\frac{\epsilon}{2}+1}(t)}+\frac{C}{t^2\log^{3\frac{\epsilon}{2}-2b}(t)}+\frac{C}{t^2\log^{\epsilon-b}(t)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{b+1-2\alpha b}(t)}+C\frac{\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)+\log^{-1}(t)}{t^2\log^{\frac{\epsilon}{2}}(t)}\end{aligned}$$

Next,

$$\begin{aligned}\partial_t T(y)(t,\omega)=-\int_t^\infty \cos((t-x)\sqrt{\omega})\big(&F_2(y)(x,\omega)-\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\\
&-\mathcal{F}(\sqrt{\cdot}F_3(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\big)\,dx\end{aligned}$$

and the same procedure as above gives

$$\begin{aligned}&||\partial_t T(y)(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}+C\int_t^\infty \frac{\lambda(t)}{\lambda(x)}\frac{1}{x^4\log^{\frac{\epsilon}{2}+1}(x)}dx\\
&+C\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)\left(\frac{1}{x^2\log^{\frac{\epsilon}{2}}(x)}\left(\frac{1}{x^2\log^{\epsilon-2b}(x)}\right)+\frac{1}{\lambda(x)x^4\log^{\epsilon}(x)}\right)dx\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{b+1-2\alpha b}(t)}+C\frac{\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)+\log^{-1}(t)}{t^3\log^{\frac{\epsilon}{2}}(t)}\end{aligned}$$

Similarly,

$$\begin{aligned}\sqrt{\omega}\lambda(t)T(y)(t,\omega)=-\lambda(t)\int_t^\infty \sin((t-x)\sqrt{\omega})\,\big(&F_2(y)(x,\omega)\\
&-\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\\
&-\mathcal{F}(\sqrt{\cdot}F_3(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\big)\,dx\end{aligned}$$

and the identical argument as for the previous two terms gives

$$\begin{aligned}&||\sqrt{\omega}\lambda(t)T(y)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}+C\lambda(t)\frac{\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)+\log^{-1}(t)}{t^3\log^{\frac{\epsilon}{2}}(t)}\\
&\leqslant \frac{C(\log(\log(t)))^2}{t^2\log^{1+b-2\alpha b}(t)}+C\frac{\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)+\log^{-1}(t)}{t^3\log^{\frac{\epsilon}{2}+b}(t)}\end{aligned}$$

The next term is

$$\begin{aligned}&\lambda(t)\sqrt{\omega}\partial_t T(y)(t,\omega)\\&=-\int_t^\infty \cos((t-x)\sqrt{\omega})\left(\frac{\lambda(t)}{\lambda(x)}\right)\sqrt{\omega}\lambda(x)\,\big(\,F_2(y)(x,\omega)-\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\\&\qquad -\mathcal{F}(\sqrt{\cdot}F_3(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\big)\,dx\end{aligned}$$

and we get

$$\begin{aligned}&||\lambda(t)\sqrt{\omega}\partial_t T(y)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}+C\int_t^\infty\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{C}{x^4\log^{\frac{\epsilon}{2}+1}(x)}dx\\&+C\int_t^\infty\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\frac{1}{x^4\log^{\frac{3\epsilon}{2}-2b}(x)}dx+C\int_t^\infty\left(\frac{\lambda(t)}{\lambda(x)}\right)^2\log^b(x)\frac{1}{x^4\log^{\epsilon}(x)}dx\\&\leqslant \frac{C(\log(\log(t)))^2}{t^3\log^{1+b-2\alpha b}(t)}+C\frac{\log^{b-\frac{\epsilon}{2}}(t)+\log^{-1}(t)+\log^{-\epsilon+2b}(t)}{t^3\log^{\frac{\epsilon}{2}}(t)}\end{aligned}$$

$$\begin{aligned}&\omega\lambda(t)^2T(y)(t,\omega)\\&=-\int_t^\infty \sin((t-x)\sqrt{\omega})\frac{\lambda(t)^2}{\lambda(x)}\lambda(x)\sqrt{\omega}\,\big(\,F_2(y)(x,\omega)-\mathcal{F}(\sqrt{\cdot}F(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\\&\qquad -\mathcal{F}(\sqrt{\cdot}F_3(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\big)\,dx\end{aligned}$$

and the same procedure as above gives

$$\begin{aligned}&||\omega\lambda(t)^2T(y)(t,\omega)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant\frac{C}{t^2\log^{1+b-2\alpha b}(t)}+C\int_t^\infty\left(\frac{\lambda(t)^2}{\lambda(x)}\right)\left(\frac{\lambda(t)}{\lambda(x)}\right)\frac{C}{x^3\log^{\frac{\epsilon}{2}}(x)}\frac{1}{x\log(x)}dx\\&+C\int_t^\infty\left(\frac{\lambda(t)^2}{\lambda(x)}\right)\left(\frac{\lambda(t)}{\lambda(x)}\right)\frac{C}{x^2\log^{\frac{\epsilon}{2}}(x)}\frac{1}{x^2\log^{\epsilon-2b}(x)}dx\\&+C\int_t^\infty\left(\frac{\lambda(t)^2}{\lambda(x)}\right)\left(\frac{\lambda(t)}{\lambda(x)}\right)\frac{\log^b(x)}{x^4\log^{\epsilon}(x)}dx\\&\leqslant\frac{C}{t^2\log^{1+b-2\alpha b}(t)}+C\frac{\log^{-b-1}(t)+\log^{-b-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}}(t)}{\log^{\frac{\epsilon}{2}}(t)t^3}\end{aligned}$$

Moreover, by, for example the Dominated convergence theorem,

$$T(y)(t,\omega)t^2\log^{\frac{\epsilon}{2}}(t)\sqrt{\rho(\omega\lambda(t)^2)}\langle\omega\lambda(t)^2\rangle\in C^0_t([T_0,\infty),L^2(d\omega))$$

and

$$\partial_t T(y)(t,\omega)t^3\log^{\frac{\epsilon}{2}}(t)\langle\sqrt{\omega}\lambda(t)\rangle\sqrt{\rho(\omega\lambda(t)^2)}\in C^0_t([T_0,\infty),L^2(d\omega))$$

So, if T_0 is large enough, then, $T(y)\in\overline{B}_{\frac{1}{2}}(0)\subset Z$ if $y\in\overline{B}_1(0)\subset Z$. We will now show that T is a strict contraction on $\overline{B}_1(0)\subset Z$. Let $y_1,y_2\in Z$ satisfy

$$||y_1||_Z,||y_2||_Z\leqslant 1$$

$$
\begin{aligned}
&T(y_1) - T(y_2) \\
&= -\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}} \\
&\qquad \cdot \left(F_2(y_1) - F_2(y_2) - \left(\mathcal{F}(\sqrt{\cdot}(F_3(u(y_1)) - F_3(u(y_2)))(x, \cdot\lambda(x))))(\omega\lambda(x)^2)\right)\right) dx
\end{aligned}
\tag{6.47}
$$

First, note that F_2 is linear in y, so

$$F_2(y_1) - F_2(y_2) = F_2(y_1 - y_2)$$

Next, we treat F_3 starting with the L_1 terms. We will denote by u_i the function associated to y_i via (6.20). We will also use $\overline{v}_i, w_i$ to denote the functions associated to u_i in the same way $\overline{v}$ and w were used in the above discussion

Recall that

$$
\begin{aligned}
&(L_1(u_1) - L_1(u_2))(t,r) \\
&= \left(\frac{\sin(2u_1(t,r)) - \sin(2u_2(t,r))}{2r^2}\right) \\
&\cdot \left(\cos(2Q_1(\frac{r}{\lambda(t)}))\left(\cos(2v_{corr}(t,r)) - 1\right) - \sin(2Q_1(\frac{r}{\lambda(t)}))\sin(2v_{corr}(t,r))\right)
\end{aligned}
$$

Since

$$|\sin(2u_1) - \sin(2u_2)| \leqslant 2|u_1 - u_2|$$

we get (after estimating in terms of u_i and then translating to y_i in exactly the same manner as done above)

$$
\begin{aligned}
&||\mathcal{F}(\sqrt{\cdot}(L_1(u_1) - L_1(u_2))(t, \cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \\
&\leqslant C||y_1(t) - y_2(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \left(||\frac{v_{corr}(t, R\lambda(t))}{R\lambda(t)}||_{L^\infty}^2 + ||\frac{v_{corr}(t, R\lambda(t))}{R(1+R^2)\lambda(t)^2}||_{L^\infty}\right)
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
&|\partial_R\left((L_1(u_1) - L_1(u_2))(t, R\lambda(t))\right)| \\
&\leqslant \frac{C\left(|\overline{v}_1 - \overline{v}_2| \cdot |\partial_R \overline{v}_1| + |\partial_R(\overline{v}_1 - \overline{v}_2)|\right)}{R^2\lambda(t)^2} \left(|v_{corr}(t, R\lambda(t))|^2 + \frac{R}{(1+R^2)}|v_{corr}(t, R\lambda(t))|\right) \\
&+ \frac{C|\overline{v}_1 - \overline{v}_2|}{R^3\lambda(t)^2}\left(|v_{corr}(t, R\lambda(t))|^2 + \frac{R|v_{corr}(t, R\lambda(t))|}{(1+R^2)}\right) \\
&+ \frac{C|\overline{v}_1 - \overline{v}_2|}{R^2\lambda(t)^2}\left(|v_{corr}(t, R\lambda(t))\partial_R(v_{corr}(t, R\lambda(t)))| + \frac{R|\partial_R(v_{corr}(t, R\lambda(t)))|}{(1+R^2)}\right)
\end{aligned}
$$

Then, we get

$$\begin{aligned}&||\partial_R((L_1(u_1)-L_1(u_2))(t,R\lambda(t)))||_{L^2(RdR)}\\&\leqslant C\left(||\overline{v}_1-\overline{v}_2||_{L^2(RdR)}+||L(\overline{v}_1-\overline{v}_2)||_{L^2(RdR)}\right)\\&\cdot\left(\left(1+||\partial_R(\overline{v}_1(t,R\lambda(t)))||_{L^2(RdR)}\right)\left(||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)}||^2_{L^\infty}+||\frac{v_{corr}(t,R\lambda(t))}{R\lambda(t)^2(1+R^2)}||_{L^\infty}\right)\right.\\&+||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R\lambda(t)^2}||_{L^\infty(R\leqslant 1)}\\&\left.+\,||\frac{v_{corr}(t,R\lambda(t))\partial_R(v_{corr}(t,R\lambda(t)))}{R^2\lambda(t)^2}||_{L^\infty(R\geqslant 1)}+||\frac{\partial_R(v_{corr}(t,R\lambda(t)))}{\lambda(t)^2(1+R^2)}||_{L^\infty}\right)\end{aligned}$$

Similarly,

$$\begin{aligned}&|\frac{L_1(u_1)(t,R\lambda(t))-L_1(u_2)(t,R\lambda(t))}{R}|\\&\leqslant\frac{C|\overline{v}_1-\overline{v}_2|(t,R)}{R^3\lambda(t)^2}\left(|v_{corr}(t,R\lambda(t))|^2+\frac{R|v_{corr}(t,R\lambda(t))|}{(R^2+1)}\right)\end{aligned}$$

We combine these to estimate $||L\left((L_1(u_1)-L_1(u_2))(t,R\lambda(t))\right)||_{L^2(RdR)}$, and then, as in the previous estimates, translate the right-hand side in terms of y_i, and use the estimates on our ansatz to get

$$\begin{aligned}&||\lambda(t)\sqrt{\omega}\mathcal{F}(\sqrt{\cdot}\left((L_1(u_1)-L_1(u_2))(t,\cdot\lambda(t))\right))\left(\omega\lambda(t)^2\right)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\frac{||y_1-y_2||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\sqrt{\omega}\lambda(t)(y_1-y_2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}}{t^2\log^{\epsilon-2b}(t)}\\&\leqslant C\frac{||y_1-y_2||_Z}{t^4\log^{\frac{3\epsilon}{2}-2b}(t)}\end{aligned}$$

Now, we treat the nonlinear terms. First, note that, if

$$n_1(x)=\sin(2x)-2x$$

then, by the mean value theorem,

$$\begin{aligned}|n_1(x_1)-n_1(x_2)|&\leqslant|x_1-x_2|\text{max}_{\theta\in[0,1]}|n_1'(\theta x_1+(1-\theta)x_2)|\\&\leqslant|x_1-x_2|\text{max}_{\theta\in[0,1]}|4\sin^2(\theta x_1+(1-\theta)x_2)|\\&\leqslant C|x_1-x_2|\left(|x_1|^2+|x_2|^2\right)\end{aligned}$$

Similarly,

$$|\left(\cos(2x_1)-1\right)-\left(\cos(2x_2)-1\right)|\leqslant C\left(|x_1|+|x_2|\right)|x_1-x_2|$$

So,

$$\begin{aligned}|(N(u_1)-N(u_2))(t,R\lambda(t))|\leqslant\,&C\frac{|\overline{v}_1-\overline{v}_2|\left(\overline{v}_1^2+\overline{v}_2^2\right)}{R^2\lambda(t)^2}\\&+C\frac{|\overline{v}_1-\overline{v}_2|\left(|\overline{v}_1|+|\overline{v}_2|\right)}{R^2\lambda(t)^2}\left(|Q_1(R)|+|v_{corr}(t,R\lambda(t))|\right)\end{aligned}$$

If $R<1$, we use the estimates (6.14) to get

$$\frac{|\overline{v}_1(t,R)|}{\sqrt{R}\lambda(t)} \leqslant C\frac{||\overline{v}_1(t)||_{\dot{H}^1_e}}{\lambda(t)} + C\frac{||L^* L\overline{v}_1(t)||_{L^2(RdR)}}{\lambda(t)}$$

and if $R>1$, then, we use (6.17) to get

$$\frac{|\overline{v}_1(t,R)|}{\sqrt{R}\lambda(t)} \leqslant \frac{C}{\lambda(t)}||\overline{v}_1(t)||_{\dot{H}^1_e}$$

In total, we obtain

$$\begin{aligned}&||N(u_1)-N(u_2)(t,\cdot\lambda(t))||_{L^2(RdR)}\\ &\leqslant C||\overline{v}_1-\overline{v}_2||_{\dot{H}^1_e}\left(||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\\ &+C||\overline{v}_1-\overline{v}_2||_{\dot{H}^1_e}\left(\frac{||\langle\sqrt{\omega}\lambda(t)\rangle y_1||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\sqrt{\omega}\lambda(t)\rangle y_2||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}}{\lambda(t)}\right)\\ &\cdot\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\end{aligned}$$

which gives

$$||\mathcal{F}(\sqrt{\cdot}(N(u_1)-N(u_2))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C}{\lambda(t)}\frac{||y_1-y_2||_Z}{t^4\log^\epsilon(t)}$$

where we used the estimates on the ansatz, and the fact that

$$||y_i||_Z \leqslant 1, \quad i=1,2$$

Next, we will estimate $\partial_R\left((N(u_1)-N(u_2))(t,R\lambda(t))\right)$, treating the following expression one line at a time.

$$\begin{aligned}&\partial_R\left((N(u_1)-N(u_2))(t,R\lambda(t))\right)\\ &=\frac{\cos(2Q_1(R))}{2R^2\lambda(t)^2}\left(2\left(\cos(2\overline{v}_1(t,R))-1\right)\partial_R\overline{v}_1 - 2\left(\cos(2\overline{v}_2)-1\right)\partial_R\overline{v}_2\right)\\ &+\partial_R\left(\frac{\cos(2Q_1(R))}{2R^2\lambda(t)^2}\right)\left(n_1(\overline{v}_1)-n_2(\overline{v}_2)\right)\\ &-2\left(\frac{\sin(2Q_1(R)+2v_{corr}(t,R\lambda(t)))}{2R^2\lambda(t)^2}\right)\left(\sin(2\overline{v}_1)\partial_R\overline{v}_1-\sin(2\overline{v}_2)\partial_R\overline{v}_2\right)\\ &+\left(\cos(2\overline{v}_1)-\cos(2\overline{v}_2)\right)\partial_R\left(\frac{\sin(2Q_1(R)+2v_{corr}(t,R\lambda(t)))}{2R^2\lambda(t)^2}\right)\end{aligned} \tag{6.48}$$

For $R\leqslant 1$, we estimate the first line of the right-hand side of (6.48) by

$$\begin{aligned}&|\frac{\cos(2Q_1(R))}{2R^2\lambda(t)^2}\left(2\left(\cos(2\overline{v}_1(t,R))-1\right)\partial_R\overline{v}_1 - 2\left(\cos(2\overline{v}_2)-1\right)\partial_R\overline{v}_2\right)|\\ &\leqslant \frac{C}{\lambda(t)^2}||\frac{\overline{v}_1}{\sqrt{R}}||^2_{L^\infty}|\frac{L(\overline{v}_1-\overline{v}_2)}{R}| + \frac{C}{\lambda(t)^2}\frac{\overline{v}_1(t,R)^2}{R^2}\frac{|\overline{v}_1-\overline{v}_2|}{R}\\ &+\frac{C}{\lambda(t)^2}||\frac{\overline{v}_1-\overline{v}_2}{\sqrt{R}}||_{L^\infty}||\frac{|\overline{v}_1|+|\overline{v}_2|}{\sqrt{R}}||_{L^\infty}\frac{|L\overline{v}_2|}{R} + \frac{C}{\lambda(t)^2}|\frac{\overline{v}_1-\overline{v}_2}{R}|\left(\frac{|\overline{v}_1|+|\overline{v}_2|}{R}\right)\frac{|\overline{v}_2|}{R}\end{aligned}$$

where we used the fact that

$$\begin{aligned}&2(\cos(2\overline{v}_1(t,R))-1)\partial_R\overline{v}_1(t,R)-2(\cos(2\overline{v}_2(t,R))-1)\partial_R\overline{v}_2(t,R)\\&=2\left(\cos(2\overline{v}_1(t,R))-1\right)\left(\partial_R(\overline{v}_1-\overline{v}_2)\right)+2\partial_R\overline{v}_2\left(\cos(2\overline{v}_1)-\cos(2\overline{v}_2)\right)\end{aligned}$$

to get

$$\begin{aligned}&|2(\cos(2\overline{v}_1(t,R))-1)\partial_R\overline{v}_1(t,R)-2(\cos(2\overline{v}_2(t,R))-1)\partial_R\overline{v}_2(t,R)|\\&\leqslant C\overline{v}_1(t,R)^2|\partial_R(\overline{v}_1-\overline{v}_2)|+C|\overline{v}_1-\overline{v}_2||\partial_R\overline{v}_2|\left(|\overline{v}_1|+|\overline{v}_2|\right)\end{aligned}$$

The second line of the right-hand side of (6.48) is estimated by

$$|\partial_R\left(\frac{\cos(2Q_1(R))}{2R^2\lambda(t)^2}\right)(n_1(\overline{v}_1)-n_2(\overline{v}_2))\,|\leqslant\frac{C}{\lambda(t)^2}\left(\frac{\overline{v}_1^2}{R^2}+\frac{\overline{v}_2^2}{R^2}\right)\frac{|\overline{v}_1-\overline{v}_2|}{R}$$

The third line of the right-hand side of (6.48) is estimated by

$$\begin{aligned}&|-2\left(\frac{\sin(2Q_1(R)+2v_{corr}(t,R\lambda(t)))}{2R^2\lambda(t)^2}\right)(\sin(2\overline{v}_1)\partial_R\overline{v}_1-\sin(2\overline{v}_2)\partial_R\overline{v}_2)\,|\\&\leqslant\frac{C}{\lambda(t)^2}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\Big(\,||\overline{v}_1||_{\dot{H}^1_e}\frac{|L(\overline{v}_1-\overline{v}_2)|}{R}+||\overline{v}_1-\overline{v}_2||_{\dot{H}^1_e}|\frac{L\overline{v}_2}{R}|\\&\qquad+\left(\frac{|\overline{v}_1|+|\overline{v}_2|}{R}\right)\frac{|\overline{v}_1-\overline{v}_2|}{R}\Big)\end{aligned}$$

where we used

$$\sin(2\overline{v}_1)\partial_R\overline{v}_1-\sin(2\overline{v}_2)\partial_R\overline{v}_2=\sin(2\overline{v}_1)\left(\partial_R(\overline{v}_1-\overline{v}_2)\right)+\left(\sin(2\overline{v}_1)-\sin(2\overline{v}_2)\right)\partial_R\overline{v}_2$$

The final line of (6.48) is estimated by

$$\begin{aligned}&\left(\cos(2\overline{v}_1)-\cos(2\overline{v}_2)\right)\partial_R\left(\frac{\sin(2Q_1(R)+2v_{corr}(t,R\lambda(t)))}{2R^2\lambda(t)^2}\right)\\&\leqslant C\left(\frac{|\overline{v}_1|+|\overline{v}_2|}{R}\right)\frac{|\overline{v}_1-\overline{v}_2|}{R\lambda(t)^2}\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty}+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\end{aligned}$$

Now, we will estimate the contribution of each of these terms to

$$||\partial_R\left((N(u_1)-N(u_2))(t,R\lambda(t))\right)||_{L^2((0,1),RdR)}$$

We use (6.14) to control $||\frac{\overline{v}_1}{\sqrt{R}}||_{L^\infty}$, and get

$$\begin{aligned}&\int_0^1\frac{C}{\lambda(t)^4}||\frac{\overline{v}_1}{\sqrt{R}}||_{L^\infty}^4\left(\frac{L(\overline{v}_1-\overline{v}_2)}{R}\right)^2RdR\\&\leqslant C\lambda(t)^2||\langle\omega\lambda(t)^2\rangle y_1||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\end{aligned}$$

Next, we get

$$\int_0^1 \frac{C}{\lambda(t)^4}\frac{(\overline{v}_1(t,R))^4}{R^4}\frac{|\overline{v}_1-\overline{v}_2|^2}{R}dR$$
$$\leqslant \frac{C}{\lambda(t)^4}\int_0^1 \left(\left(||\overline{v}_1||^4_{\dot{H}^1_e}+||L^*L\overline{v}_1||^4_{L^2(RdR)}\right)\left(\log^2(\frac{1}{R})+1\right)^2\right)$$
$$\cdot\left(||\overline{v}_1-\overline{v}_2||^2_{\dot{H}^1_e}+||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)R\left(\log^2(\frac{1}{R})+1\right)dR$$
$$\leqslant C\lambda(t)^2||\langle\omega\lambda(t)^2\rangle y_1||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

The next term to consider is

$$\int_0^1 \frac{C}{\lambda(t)^4}||\frac{\overline{v}_1-\overline{v}_2}{\sqrt{R}}||^2_{L^\infty}||\frac{|\overline{v}_1|+|\overline{v}_2|}{\sqrt{R}}||^2_{L^\infty}\frac{(L\overline{v}_2)^2}{R^2}RdR$$
$$\leqslant \frac{C}{\lambda(t)^4}\left(||\overline{v}_1-\overline{v}_2||^2_{\dot{H}^1_e}+||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)$$
$$\cdot\left(||\overline{v}_1||^2_{\dot{H}^1_e}+||L^*L\overline{v}_1||^2_{L^2(RdR)}+||\overline{v}_2||^2_{\dot{H}^1_e}+||L^*L\overline{v}_2||^2_{L^2(RdR)}\right)\int_0^1\frac{(L\overline{v}_2(t,R))^2}{R^2}RdR$$
$$\leqslant C\lambda(t)^2||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$
$$\cdot||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$

Next, we have

$$\frac{C}{\lambda(t)^4}\int_0^1\left(|\frac{\overline{v}_1-\overline{v}_2}{R}|\left(\frac{|\overline{v}_1|+|\overline{v}_2|}{R}\right)\frac{|\overline{v}_2|}{R}\right)^2RdR$$
$$\leqslant \frac{C}{\lambda(t)^4}\left(||\overline{v}_1-\overline{v}_2||^2_{\dot{H}^1_e}+||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)$$
$$\cdot\left(||\overline{v}_1||^4_{\dot{H}^1_e}+||L^*L\overline{v}_1||^4_{L^2(RdR)}+||\overline{v}_2||^4_{\dot{H}^1_e}+||L^*L\overline{v}_2||^4_{L^2(RdR)}\right)\int_0^1\left(\log^2(\frac{1}{R})+1\right)^3RdR$$
$$\leqslant C\lambda(t)^2||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(||\langle\omega\lambda(t)^2\rangle y_1||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\langle\omega\lambda(t)^2\rangle y_2||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

For the second line of the right-hand side of (6.48),

$$\frac{C}{\lambda(t)^4}\int_0^1\left(\frac{\overline{v}_1^4}{R^4}+\frac{\overline{v}_2^4}{R^4}\right)\frac{|\overline{v}_1-\overline{v}_2|^2}{R^2}RdR$$
$$\leqslant \frac{C}{\lambda(t)^4}\left(||\overline{v}_1||^4_{\dot{H}^1_e}+||L^*L\overline{v}_1||^4_{L^2(RdR)}+||\overline{v}_2||^4_{\dot{H}^1_e}+||L^*L\overline{v}_2||^4_{L^2(RdR)}\right)$$
$$\cdot\left(||\overline{v}_1-\overline{v}_2||^2_{\dot{H}^1_e}+||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)\int_0^1\left(\log^2(\frac{1}{R})+1\right)^3RdR$$
$$\leqslant C\lambda(t)^2||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(||\langle\omega\lambda(t)^2\rangle y_1||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+||\langle\omega\lambda(t)^2\rangle y_2||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

Next, we treat the third line of the right-hand side of (6.48),

$$\int_0^1 \frac{1}{\lambda(t)^4} \left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}^2\right)$$
$$\cdot \left(||\overline{v}_1||_{\dot{H}_e^1}^2 \frac{(L(\overline{v}_1-\overline{v}_2))^2}{R^2} + ||\overline{v}_1-\overline{v}_2||_{\dot{H}_e^1}^2 \frac{(L\overline{v}_2)^2}{R^2} + \frac{(\overline{v}_1)^2+(\overline{v}_2)^2}{R^2}\frac{|\overline{v}_1-\overline{v}_2|^2}{R^2}\right) RdR$$
$$\leqslant C\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}^2\right) ||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot \left(||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

The final line of (6.48) is treated as follows.

$$\int_0^1 \left(\frac{|\overline{v}_1|+|\overline{v}_2|}{R}\frac{|\overline{v}_1-\overline{v}_2|}{R\lambda(t)^2}\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty}+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)\right)^2 RdR$$
$$\leqslant \frac{C}{\lambda(t)^4}\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty}^2+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}^2\right)$$
$$\cdot\left(||\overline{v}_1||_{\dot{H}_e^1}^2+||L^*L\overline{v}_1||^2_{L^2(RdR)}+||\overline{v}_2||_{\dot{H}_e^1}^2+||L^*L\overline{v}_2||^2_{L^2(RdR)}\right)$$
$$\cdot\left(||\overline{v}_1-\overline{v}_2||_{\dot{H}_e^1}^2+||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)\int_0^1\left(\log^2(\frac{1}{R})+1\right)^2 RdR$$
$$\leqslant C\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty}^2+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}^2\right)$$
$$\cdot ||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot \left(||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

We combine the above estimates to get

$$\int_0^1 \left(\partial_R\left((N(u_1)-N(u_2))(t,R\lambda(t))\right)\right)^2 RdR$$
$$\leqslant C\lambda(t)^2\left(||\langle\omega\lambda(t)^2\rangle y_1||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_2||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$
$$\cdot ||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$+C\left(1+||\partial_R(v_{corr}(t,R\lambda(t)))||_{L^\infty}^2+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}^2\right)$$
$$\cdot ||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot \left(||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

When $R \geqslant 1$, we can estimate (6.48) by

$$\begin{aligned}
&|\partial_R \left((N(u_1) - N(u_2))(t, R\lambda(t))\right)| \\
&\leqslant \frac{C}{\lambda(t)^2}||\overline{v}_1||^2_{\dot{H}^1_e}|L(\overline{v}_1 - \overline{v}_2)| + \frac{C}{\lambda(t)^2}||\overline{v}_1||^2_{\dot{H}^1_e}|\overline{v}_1 - \overline{v}_2| \\
&+ \frac{C}{\lambda(t)^2}||\overline{v}_1 - \overline{v}_2||_{\dot{H}^1_e}\left(||\overline{v}_1||_{\dot{H}^1_e} + ||\overline{v}_2||_{\dot{H}^1_e}\right)(|L\overline{v}_2| + |\overline{v}_2|) \\
&+ \frac{C}{\lambda(t)^2}\left(||\overline{v}_1||^2_{\dot{H}^1_e} + ||\overline{v}_2||^2_{\dot{H}^1_e}\right)|\overline{v}_1 - \overline{v}_2| \\
&+ \frac{C}{\lambda(t)^2}\left(||\overline{v}_1||_{\dot{H}^1_e} + ||\overline{v}_2||_{\dot{H}^1_e}\right)|\overline{v}_1 - \overline{v}_2| \\
&\cdot \left(1 + ||\partial_R(v_{corr}(t, R\lambda(t)))||_{L^\infty} + ||\frac{v_{corr}(t, R\lambda(t))}{R}||_{L^\infty}\right) \\
&+ \frac{C}{\lambda(t)^2}\left(1 + ||\frac{v_{corr}(t, R\lambda(t))}{R}||_{L^\infty}\right) \\
&\cdot \left(||\overline{v}_1||_{\dot{H}^1_e}|L(\overline{v}_1 - \overline{v}_2)| + ||\overline{v}_1 - \overline{v}_2||_{\dot{H}^1_e}|L\overline{v}_2| + \left(||\overline{v}_1||_{\dot{H}^1_e} + ||\overline{v}_2||_{\dot{H}^1_e}\right)|\overline{v}_1 - \overline{v}_2|\right)
\end{aligned}$$

Taking the L^2 norm over $R \geqslant 1$, and combining with our previous estimates, we get

$$\begin{aligned}
&||\partial_R \left((N(u_1) - N(u_2))(t, R\lambda(t))\right)||_{L^2(RdR)} \\
&\leqslant C\lambda(t)||\langle \omega\lambda(t)^2\rangle(y_1 - y_2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \\
&\cdot \left(||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right) \\
&+ C\left(1 + ||\frac{v_{corr}(t, R\lambda(t))}{R}||_{L^\infty} + ||\partial_R(v_{corr}(t, R\lambda(t)))||_{L^\infty}\right) \\
&\cdot \left(||\langle\omega\lambda(t)^2\rangle y_2||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_1||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right) \\
&\cdot ||\langle\omega\lambda(t)^2\rangle(y_1 - y_2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}
\end{aligned}$$

It only remains to consider the following.

$$\begin{aligned}
&|\frac{N(u_1)(t, R\lambda(t)) - N(u_2)(t, R\lambda(t))}{R}| \\
&\leqslant C\frac{|\overline{v}_1 - \overline{v}_2|\left((\overline{v}_1)^2 + (\overline{v}_2)^2\right)}{R^3\lambda(t)^2} + \frac{C\left(|\overline{v}_1| + |\overline{v}_2|\right)|\overline{v}_1 - \overline{v}_2|}{R^3\lambda(t)^2}\left(|Q_1(R)| + |v_{corr}(t, R\lambda(t))|\right)
\end{aligned} \tag{6.49}$$

So,

$$\int_0^1 \frac{(N(u_1)(t,R\lambda(t)) - N(u_2)(t,R\lambda(t)))^2}{R^2} RdR$$
$$\leqslant \frac{C}{\lambda(t)^4}\left(||\overline{v}_1-\overline{v}_2||^2_{\dot{H}^1_e} + ||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)$$
$$\cdot\left(||\overline{v}_1||^4_{\dot{H}^1_e} + ||L^*L\overline{v}_1||^4_{L^2(RdR)} + ||\overline{v}_2||^4_{\dot{H}^1_e} + ||L^*L\overline{v}_2||^4_{L^2(RdR)}\right)\int_0^1\left(\log^2(\frac{1}{R})+1\right)^3 RdR$$
$$+\frac{C}{\lambda(t)^4}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||^2_{L^\infty}\right)$$
$$\cdot\left(||\overline{v}_1||^2_{\dot{H}^1_e} + ||L^*L\overline{v}_1||^2_{L^2(RdR)} + ||\overline{v}_2||^2_{\dot{H}^1_e} + ||L^*L\overline{v}_2||^2_{L^2(RdR)}\right)$$
$$\cdot\left(||\overline{v}_1-\overline{v}_2||^2_{\dot{H}^1_e} + ||L^*L(\overline{v}_1-\overline{v}_2)||^2_{L^2(RdR)}\right)\int_0^1\left(\log^2(\frac{1}{R})+1\right)^2 RdR$$
$$\leqslant C\lambda(t)^2||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(||\langle\omega\lambda(t)^2\rangle y_1||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_2||^4_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$
$$+C\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||^2_{L^\infty}\right)$$
$$\cdot||\langle\omega\lambda(t)^2\rangle(y_1-y_2)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\cdot\left(||\langle\omega\lambda(t)^2\rangle y_1||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + ||\langle\omega\lambda(t)^2\rangle y_2||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)$$

We return to (6.49), and study

$$\left(\int_1^\infty \frac{((N(u_1)-N(u_2))(t,R\lambda(t)))^2}{R^2} RdR\right)^{1/2}$$
$$\leqslant \frac{C}{\lambda(t)^2}||\overline{v}_1-\overline{v}_2||_{L^2(RdR)}\left(||\overline{v}_1||^2_{\dot{H}^1_e} + ||\overline{v}_2||^2_{\dot{H}^1_e}\right)$$
$$+\frac{C}{\lambda(t)^2}\left(||\overline{v}_1||_{\dot{H}^1_e} + ||\overline{v}_2||_{\dot{H}^1_e}\right)||\overline{v}_1-\overline{v}_2||_{L^2(RdR)}\left(1+||\frac{v_{corr}(t,R\lambda(t))}{R}||_{L^\infty}\right)$$

We now combine the above estimates, translating between norms of $\overline{v_i}$ and norms of y_i as previously, and use the fact that $||y_i||_Z \leqslant 1$, as well as the estimates of the ansatz to get

$$||\lambda(t)\sqrt{\omega}\mathcal{F}(\sqrt{\cdot}(N(u_1)-N(u_2))(t,\cdot\lambda(t)))(\omega\lambda(t)^2)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant C\frac{||y_1-y_2||_Z}{t^4\log^{\epsilon-b}(t)}$$

We now return to (6.47) to get

$$\begin{aligned}
&||(T(y_1)-T(y_2))(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C\int_t^\infty x\left(\frac{\lambda(t)}{\lambda(x)}\right)\left(\frac{||y_1-y_2||_Z}{x^4\log^{1+\frac{\epsilon}{2}}(x)}\right)dx+C\int_t^\infty x\left(\frac{\lambda(t)}{\lambda(x)}\right)\frac{||y_1-y_2||_Z}{x^4\log^{3\frac{\epsilon}{2}-2b}(x)}dx\\
&+C\int_t^\infty x\left(\frac{\lambda(t)}{\lambda(x)}\right)\frac{||y_1-y_2||_Z}{x^4\log^{\epsilon-b}(x)}dx\\
&\leqslant C||y_1-y_2||_Z\frac{\log^{-1}(t)+\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)}{t^2\log^{\frac{\epsilon}{2}}(t)}
\end{aligned}$$

$$\begin{aligned}
&\partial_t(T(y_1)-T(y_2))\\
&=-\int_t^\infty\cos((t-x)\sqrt{\omega})F_2(y_1-y_2)dx\\
&+\int_t^\infty\cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}(F_3(y_1)-F_3(y_2))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx
\end{aligned}$$

So, the same argument as in the previous estimate gives

$$\begin{aligned}
&||\partial_t(T(y_1)-T(y_2))||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C||y_1-y_2||_Z\frac{\log^{-1}(t)+\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)}{t^3\log^{\frac{\epsilon}{2}}(t)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sqrt{\omega}\lambda(t)(T(y_1)-T(y_2))\\
&=-\int_t^\infty\lambda(t)\sin((t-x)\sqrt{\omega})F_2(y_1-y_2)dx\\
&+\lambda(t)\int_t^\infty\sin((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}(F_3(y_1)-F_3(y_2))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx
\end{aligned}$$

and the identical argument gives us

$$\begin{aligned}
&||\sqrt{\omega}\lambda(t)(T(y_1)-T(y_2))(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C||y_1-y_2||_Z\frac{\log^{-1}(t)+\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)}{t^3\log^{\frac{\epsilon}{2}+b}(t)}
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\sqrt{\omega}\lambda(t)\partial_t(T(y_1)-T(y_2))\\
&=-\int_t^\infty\left(\frac{\lambda(t)}{\lambda(x)}\right)\cos((t-x)\sqrt{\omega})\lambda(x)\sqrt{\omega}F_2(y_1-y_2)dx\\
&+\int_t^\infty\left(\frac{\lambda(t)}{\lambda(x)}\right)\cos((t-x)\sqrt{\omega})\lambda(x)\sqrt{\omega}\mathcal{F}(\sqrt{\cdot}(F_3(y_1)-F_3(y_2))(x,\cdot\lambda(x)))(\omega\lambda(x)^2)dx
\end{aligned}$$

So,

$$\begin{aligned}
&||\sqrt{\omega}\lambda(t)\partial_t(T(y_1)-T(y_2))(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)^2 \frac{||y_1-y_2||_Z}{x^4\log^{1+\frac{\epsilon}{2}}(x)}dx + C\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)^2 \frac{||y_1-y_2||_Z}{x^4\log^{\frac{3\epsilon}{2}-2b}(x)}dx\\
&+C\int_t^\infty \left(\frac{\lambda(t)}{\lambda(x)}\right)^2 \frac{||y_1-y_2||_Z}{x^4\log^{\epsilon-b}(x)}dx\\
&\leqslant C||y_1-y_2||_Z\frac{\log^{-1}(t)+\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)}{t^3\log^{\frac{\epsilon}{2}}(t)}
\end{aligned}$$

The identical procedure shows that

$$\begin{aligned}
&||\omega\lambda(t)^2(T(y_1)-T(y_2))(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant C||y_1-y_2||_Z\frac{\log^{-1}(t)+\log^{-\epsilon+2b}(t)+\log^{-\frac{\epsilon}{2}+b}(t)}{t^3\log^{\frac{\epsilon}{2}+b}(t)}
\end{aligned}$$

Thus, T is a strict contraction on $\overline{B}_1(0)\subset Z$, for T_0 large enough; so, T has a fixed point, say y_0, in $\overline{B}_1(0)\subset Z$ □

7 The Energy of the Solution, and its Decomposition as in Theorem 1.1 (Wave Maps)

Let us define

$$v_6(t,r):=\begin{cases}\sqrt{\frac{\lambda(t)}{r}}\left(\mathcal{F}^{-1}(y_0(t,\frac{\cdot}{\lambda(t)^2}))\right)(\frac{r}{\lambda(t)}), & r>0\\ 0, & r=0\end{cases}$$

Note that $v_6(t,\cdot)\in C^0([0,\infty))$, by the same argument as in Lemma (6.1). Inspecting the derivation of (6.3), we see that we have a solution to (2.1):

$$u(t,r)=Q_{\frac{1}{\lambda(t)}}(r)+\sum_{k=1}^{6}v_k(t,r) \tag{7.1}$$

Here, we study the energy, (2.2), of our solution, and prove that it has a decomposition as in the main theorem statement. First, we note that $\partial_t Q_{\frac{1}{\lambda(t)}}\notin L^2((0,\infty),rdr)$, because $\phi_0\notin L^2((0,\infty),rdr)$, so we have to first capture a delicate cancellation between the large r behavior of $\partial_t Q_{\frac{1}{\lambda(t)}}$ and $\partial_t v_1$ before we can even show that the solution has finite energy. We consider the region $r \geqslant t$, and use the representation formula for v_1, (5.10), which implies

$$\begin{aligned}
\partial_t v_1(t,r) &= \int_t^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds \\
&= \int_t^{t+\frac{r}{6}} \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds \\
&+ \int_{t+\frac{r}{6}}^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{(s-t)} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds \\
&+ \int_{t+\frac{r}{6}}^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \\
&\qquad \cdot \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds
\end{aligned} \tag{7.2}$$

For the second line of (7.2), we have

$$\begin{aligned}
&\int_t^{t+\frac{r}{6}} \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds \\
&= \int_t^{t+\frac{r}{6}} \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \left(1+1+O\left(\frac{r^2}{(r^2-1-\rho^2)^2}\right)\right) d\rho ds \\
&= 2\int_t^{t+\frac{r}{6}} \frac{\lambda'''(s)}{r}(s-t)ds + E_{\partial_t v_1}(t,r)
\end{aligned} \tag{7.3}$$

where

$$|E_{\partial_t v_1}(t,r)| \leqslant \frac{C}{r^3} \int_t^{t+\frac{r}{6}} |\lambda'''(s)|(s-t)ds \leqslant \frac{C}{r^3 t \log^{b+1}(t)}, \quad r \geqslant t$$

For the third line of (7.2), we have

$$\begin{aligned}
&|\int_{t+\frac{r}{6}}^\infty \frac{\lambda'''(s)}{r} \int_0^{s-t} \frac{\rho}{(s-t)} \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds| \\
&\leqslant C \int_{t+\frac{r}{6}}^\infty \frac{|\lambda'''(s)|}{r(s-t)} \int_0^\infty \rho \left(1 + \frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right) d\rho ds \\
&\leqslant \frac{C}{r^2 \log^{b+1}(r)}, \quad r \geqslant t
\end{aligned}$$

Finally, the fourth line of (7.2) is treated as follows:

$$\begin{aligned}&|\int_{t+\frac{r}{6}}^{\infty}\frac{\lambda'''(s)}{r}\int_{0}^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\left(1+\frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right)d\rho ds|\\
&\leqslant \frac{C}{r^3\log^{b+1}(r)}\cdot\frac{1}{r}\int_{t+\frac{r}{6}}^{\infty}\int_{0}^{s-t}\rho\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)\\
&\qquad\qquad\left(1+\frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right)d\rho ds\\
&\leqslant \frac{C}{r^4\log^{b+1}(r)}\int_{0}^{\infty}\rho\left(1+\frac{r^2-1-\rho^2}{\sqrt{(1+r^2+\rho^2)^2-4r^2\rho^2}}\right)\int_{\rho+t}^{\infty}\left(\frac{1}{\sqrt{(s-t)^2-\rho^2}}-\frac{1}{(s-t)}\right)dsd\rho\\
&\leqslant \frac{C}{r^2\log^{b+1}(r)},\quad r\geqslant t\end{aligned}$$

Finally, we further treat the first term on the last line of (7.3)

$$\begin{aligned}2\int_{t}^{t+\frac{r}{6}}\frac{\lambda'''(s)}{r}(s-t)ds&=\frac{2}{r}\left(\frac{r}{6}\lambda''(t+\frac{r}{6})-\lambda'(t+\frac{r}{6})+\lambda'(t)\right)\\
&=\frac{2\lambda'(t)}{r}+E_{\partial_t v_1,1}(t,r)\end{aligned}$$

where

$$|E_{\partial_t v_1,1}(t,r)|\leqslant\frac{C}{r^2\log^{b+1}(r)},\quad r\geqslant t$$

Then, we note that

$$\partial_t Q_{\frac{1}{\lambda(t)}}(r)=\frac{-2r\lambda'(t)}{r^2+\lambda(t)^2}$$

So,

$$|\partial_t\left(Q_{\frac{1}{\lambda(t)}}(r)+v_1(t,r)\right)|\leqslant\frac{C}{r^2\log^{b+1}(r)},\quad r\geqslant t$$

Using the estimates on $\partial_t v_1$, we then get

$$||\partial_t\left(Q_{\frac{1}{\lambda(t)}}(r)+v_1(t,r)\right)||_{L^2(rdr)}^2\leqslant\frac{C}{t^2\log^{2b}(t)}$$

Then, we recall

$$E(u,v)=\pi\left(||v||_{L^2(rdr)}^2+||u||_{\dot{H}_e^1}^2\right)$$

and note that energy estimates for the equations solved by v_k for $k=3,4,5$ give

$$E(v_3(t),\partial_t v_3(t))\leqslant C\left(\int_t^{\infty}||F_{0,1}(s)||_{L^2(rdr)}ds\right)^2\leqslant\frac{C\log(\log(t))}{t^2\log^{2b+2}(t)}$$

$$E(v_4(t),\partial_t v_4(t)) \leqslant C\left(\int_t^\infty ||v_{4,c}(s)||_{L^2(rdr)}ds\right)^2 \leqslant \frac{C}{t^2\log^{4N+4b}(t)}$$

$$E(v_5(t),\partial_t v_5(t)) \leqslant C\left(\int_t^\infty ||N_2(f_{v_5})(s)||_{L^2(rdr)}ds\right)^2 \leqslant \frac{C\log^6(t)}{t^{7/2}}$$

Next, we consider v_2 for $b \neq 1$. By Plancherel, we have

$$E(v_2(t),\partial_t v_2(t)) = E(v_2(0),\partial_t v_2(0)) = \frac{16b^2}{\pi(b-1)^2}\int_0^\infty \frac{(\chi_{\leqslant \frac{1}{4}}(\xi))^2}{\log^{2b-2}(\frac{1}{\xi})}d\xi$$

$$\frac{16b^2}{\pi(b-1)^2}\int_0^{\frac{1}{8}} \frac{d\xi}{\log^{2b-2}(\frac{1}{\xi})} \leqslant E(v_2,\partial_t v_2) \leqslant \frac{16b^2}{\pi(b-1)^2}\int_0^{\frac{1}{4}} \frac{d\xi}{\log^{2b-2}(\frac{1}{\xi})}$$

where we used properties of $\chi_{\leqslant \frac{1}{4}}$. This gives, for $b \neq 1$,

$$\frac{16b^2}{\pi(b-1)^2}\Gamma(3-2b,\log(8)) \leqslant E(v_2,\partial_t v_2) \leqslant \frac{16b^2}{\pi(b-1)^2}\Gamma(3-2b,\log(4))$$

By inspection of $v_{2,0}$ for $b=1$, we see that $E(v_2,\partial_t v_2) < \infty$ for $b=1$.

Using our estimates on v_1 and $\partial_r v_1$, we get

$$||v_1||^2_{\dot{H}^1_e} \leqslant \frac{C}{t^2\log^{2b}(t)}$$

Finally, we treat the v_6 term in (7.1). Theorem 5.1 of [14] (the transferrence identity) shows that

$$\begin{aligned}
||\partial_t v_6||_{L^2(rdr)} &\leqslant C||\frac{\lambda'(t)}{\sqrt{r\lambda(t)}}\mathcal{F}^{-1}(y_0(t,\frac{\cdot}{\lambda(t)^2}))(\frac{r}{\lambda(t)})||_{L^2(rdr)} \\
&+ C||\sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}(\partial_1 y_0(t,\frac{\cdot}{\lambda(t)^2}))(\frac{r}{\lambda(t)})||_{L^2(rdr)} \\
&+ C||\frac{\lambda'(t)}{\lambda(t)}\sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}(\mathcal{K}(y_0(t,\frac{\cdot}{\lambda(t)^2})))(\frac{r}{\lambda(t)})||_{L^2(rdr)}
\end{aligned} \tag{7.4}$$

The first line of (7.4) is handled by re-scaling, and applying the L^2 isometry property of $\mathcal{F}(\cdot)$:

$$\begin{aligned}
&\int_0^\infty \frac{(\lambda'(t))^2}{r\lambda(t)}\left(\mathcal{F}^{-1}(y_0(t,\frac{\cdot}{\lambda(t)^2}))(\frac{r}{\lambda(t)})\right)^2 rdr \\
&= (\lambda'(t))^2\lambda(t)^2\int_0^\infty |y_0(t,\omega)|^2\rho(\omega\lambda(t)^2)d\omega \\
&\leqslant C\frac{\lambda'(t)^2\lambda(t)^2}{t^4\log^\epsilon(t)}
\end{aligned}$$

The second and third lines of (7.4) are treated by again re-scaling and using the L^2 isometry property of $\mathcal{F}(\cdot)$, as well as the same estimates on $\mathcal{K}$ that we used while estimating F_2 to get

$$\begin{aligned}
&\int_0^\infty \frac{\lambda(t)}{r}|\mathcal{F}^{-1}(\partial_1 y_0(t,\frac{\cdot}{\lambda(t)^2}))|^2(\frac{r}{\lambda(t)})rdr\\
&+\frac{\lambda'(t)^2}{\lambda(t)}\int_0^\infty \frac{1}{r}|\mathcal{F}^{-1}(\mathcal{K}(y_0(t,\frac{\cdot}{\lambda(t)})))|^2(\frac{r}{\lambda(t)})rdr\\
&\leqslant C\lambda(t)^4||\partial_t y_0(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}+C\lambda'(t)^2\lambda(t)^2||y_0(t)||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
&\leqslant \frac{C}{t^6\log^{4b+\epsilon}(t)}
\end{aligned}$$

This gives

$$||\partial_t v_6||^2_{L^2(rdr)}\leqslant \frac{C}{t^6\log^{4b+\epsilon}(t)}$$

From the definition of v_6, we have (using the same argument used when estimating F_2, F_3)

$$||v_6(t)||_{\dot{H}^1_e}=||v_6(t,\cdot\lambda(t))||_{\dot{H}^1_e}\leqslant C\left(||v_6(t,\cdot\lambda(t))||_{L^2(RdR)}+||L(v_6(t,\cdot\lambda(t)))||_{L^2(RdR)}\right)$$

and

$$\begin{gathered}
||v_6(t,\cdot\lambda(t))||_{L^2(RdR)}=\lambda(t)||y_0(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\
||L(v_6(t,\cdot\lambda(t)))||_{L^2(RdR)}=\lambda(t)||\sqrt{\omega}\lambda(t)y_0(t)||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}
\end{gathered}$$

which gives

$$||v_6(t)||_{\dot{H}^1_e}\leqslant \frac{C}{t^2\log^{b+\frac{\epsilon}{2}}(t)}$$

Finally, we have

$$E_{\text{WM}}(u,\partial_t u)=\pi\left(||\partial_t u||^2_{L^2(rdr)}+||\frac{\sin(u)}{r}||^2_{L^2(rdr)}+||\partial_r u||^2_{L^2(rdr)}\right)$$

Then, we use

$$\int_0^\infty \frac{\sin^2(u-Q_{\frac{1}{\lambda(t)}}+Q_{\frac{1}{\lambda(t)}})}{r^2}rdr\leqslant C\left(\int_0^\infty\frac{\sin^2(Q_{\frac{1}{\lambda(t)}})}{r^2}rdr+\int_0^\infty\frac{|u-Q_{\frac{1}{\lambda(t)}}|^2}{r^2}rdr\right)$$

to get

$$\begin{aligned}
E_{\text{WM}}(u,\partial_t u)\leqslant C\Bigg(&||\partial_t\left(Q_{\frac{1}{\lambda(t)}}+v_1\right)||^2_{L^2(rdr)}+\sum_{k=2}^6||\partial_t v_k||^2_{L^2(rdr)}\\
&+||\partial_r Q_{\frac{1}{\lambda(t)}}||^2_{L^2(rdr)}+||\frac{\sin(Q_{\frac{1}{\lambda(t)}})}{r}||^2_{L^2(rdr)}+\sum_{k=1}^6||v_k||^2_{\dot{H}^1_e}\Bigg)
\end{aligned}$$

By combining our above estimates, and recalling

$$||\partial_r Q_{\frac{1}{\lambda(t)}}||^2_{L^2(rdr)}+||\frac{\sin(Q_{\frac{1}{\lambda(t)}})}{r}||^2_{L^2(rdr)}=4$$

we get

$$E_{\mathrm{WM}}(u, \partial_t u) < \infty$$

and

$$||\partial_t(u - v_2)||^2_{L^2(rdr)} + ||u - Q_{\frac{1}{\lambda(t)}} - v_2||^2_{\dot{H}^1_e} \leqslant \frac{C}{t^2 \log^{2b}(t)}$$

Finally, we note that the remark after the main theorem concerning the regularity of v_6 follows from the definition of the space Z, the continuity of dilation on L^2, and lemma 9.1 of [14]. This completes the proof of the main theorem.

A Proof of Theorem 1.2 (Wave Maps)

In this appendix, we will summarize the extra arguments needed to prove Theorem 2.2. To prove Theorem 2.2, we use a slightly different starting point than for the proof of Theorem 2.1. In particular, fix $b > 0$, $\lambda_{0,0,b} \in \Lambda$, and let $T_0 > C_1$, where $C_1 > 100$ is some sufficiently large constant depending on $\lambda_{0,0,b}$. Then, we define the X norm as in the proof of Theorem 2.1, and write $\lambda = \lambda_{0,0,b} + e_1$, where $e_1 \in \overline{B}_1(0) \subset X$. For all such λ, we then define v_1 exactly as in the main body of the paper. The main difference is a modification of v_2: Let $\psi \in C^\infty([0, \infty))$ satisfy

$$\psi(t) = \begin{cases} 1, & t \geqslant 200 \\ 0, & t \leqslant 100 \end{cases} \qquad 0 \leqslant \psi(t) \leqslant 1$$

and define

$$F(t) = \left(4 \int_t^\infty \frac{\lambda''_{0,0,b}(s) ds}{1 + s - t}\right) \psi(t)$$

Then, v_2 is exactly as in the main body of the paper, except with $v_{2,0}$, the initial velocity of v_2, given below

$$\widehat{v_{2,0}}(\xi) = -\frac{1}{\xi \pi} \int_0^\infty F(t) \sin(t\xi) dt$$

The point of this definition is that, by the sine transform inversion, $\widehat{v_{2,0}}$ solves

$$-2 \int_0^\infty \sin(t\xi) \xi \widehat{v_{2,0}}(\xi) d\xi = F(t), \quad t \geqslant 0$$

v_3 is defined exactly as in the main body of the paper. Several estimates on v_k in the main body of the paper used the fact that $\lambda'(x) < 0$, which we no longer have in the setting of this appendix. Unless specified later on, analogs of all of these estimates are still true in the setting of this appendix, and can be proven by instead using that, for some $C_2, C_3 > 0$,

$$\frac{C_2}{\log^b(t)} \leqslant \lambda(t) \leqslant \frac{C_3}{\log^b(t)} \tag{A.1}$$

and $t \to \frac{1}{\log^b(t)}$ is decreasing. The definitions of v_4, v_5, and the equation resulting from u_{ansatz} are the same as previously. The inner product of the v_1 linear error term with the re-scaled ϕ_0 is unchanged. On the other hand, for v_2, we have

$$\begin{aligned}
&\int_0^\infty \left(\frac{\cos(2Q_1(R))-1}{R^2\lambda(t)^2}\right) v_2(t,R\lambda(t))\phi_0(R)RdR \\
&= -2\int_0^\infty \frac{\sin(t\xi)}{\lambda(t)}\xi\widehat{v_{2,0}}(\xi)d\xi - 2\int_0^\infty \sin(t\xi)\xi^2\left(K_1(\xi\lambda(t)) - \frac{1}{\xi\lambda(t)}\right)\widehat{v_{2,0}}(\xi)d\xi \\
&= \frac{4}{\lambda(t)}\int_t^\infty \frac{\lambda''_{0,0,b}(s)ds}{1+s-t} + E_{v_2,ip}(t,\lambda(t)), \quad t \geqslant T_0
\end{aligned}$$

where

$$E_{v_2,ip}(t,\lambda(t)) = -2\int_0^\infty \sin(t\xi)\xi^2\left(K_1(\xi\lambda(t)) - \frac{1}{\xi\lambda(t)}\right)\widehat{v_{2,0}}(\xi)d\xi$$

and we used the fact that $\psi(t) = 1, \quad t \geqslant 100$. In order to prove the pointwise estimates on v_2, we require some estimates on $\partial_\xi\left(\xi\widehat{v_{2,0}}\right)$. For these, we use

$$\xi\widehat{v_{2,0}}(\xi) = \frac{-1}{\pi}\int_0^\infty F(\frac{\sigma}{\xi})\sin(\sigma)\frac{d\sigma}{\xi}$$

and can then differentiate under the integral sign, using the symbol type estimates on $\lambda_{0,0,b}$. The most crucial point to mention regarding the pointwise behavior of v_2 is the analogue of (5.49), since it is this which allows us to prove the crucial near origin estimates on F_4. In the setting of this appendix, we prove the analogue of (5.49) by noting, for $r \leqslant \frac{t}{2}$,

$$\begin{aligned}
v_2(t,r) &= \int_0^\infty J_1(r\xi)\sin(t\xi)\widehat{v_{2,0}}(\xi)d\xi \\
&= \frac{r}{\pi}\int_0^\pi \frac{\sin^2(\theta)}{2}\int_0^\infty \xi\left(\sin(\xi(t+r\cos(\theta))) + \sin(\xi(t-r\cos(\theta)))\right)\widehat{v_{2,0}}(\xi)d\xi d\theta \qquad \text{(A.2)} \\
&= \frac{-r}{4\pi}\int_0^\pi \sin^2(\theta)\left(F(t+r\cos(\theta)) + F(t-r\cos(\theta))\right)d\theta
\end{aligned}$$

and then using

$$F(t\pm r\cos(\theta)) = F(t) + \text{Err}(t,r), \quad |\text{Err}(t,r)| \leqslant \frac{Cr}{t^3\log^b(t)}, \quad r \leqslant \frac{t}{2}, \quad t \geqslant T_0$$

The arguments for derivatives of v_2 are done similarly. The linear error term associated to v_3 is studied similarly to the main body of the paper. The modulation equation for λ then has the same form as previously, except with the modified definition of $E_{v_2,ip}$ given above, and the replacement

$$\frac{4b}{\lambda(t)t^2\log^b(t)} \to \frac{4}{\lambda(t)}\int_t^\infty \frac{\lambda''_{0,0,b}(s)ds}{1+s-t}$$

Recalling that $\lambda = \lambda_{0,0,b} + e_1$, we then substitute $e_1(t) = \lambda_{0,1}(t) + e(t)$, where

$$\lambda_{0,1}(t) = \int_t^\infty \int_{t_1}^\infty \frac{\lambda''_{0,0,b}(t_2)\log(\lambda_{0,0,b}(t_2)))}{\log(t_2)}dt_2dt_1$$

and proceed exactly as previously. The crucial kernel estimate (5.69) was previously proven using an argument which used $\lambda_{0,0}'(x)+\lambda_{0,1}'(x) \leqslant 0$, which is no longer true in this context. On the other hand, (5.69) is still true in the context of this appendix, and is proven with the same calculation of $\partial_{st}k(s,t)$ as before. Instead of using $\lambda_{0,0}'(t)+\lambda_{0,1}'(t) \leqslant 0$, we simply use

$$-\lambda_0'(-t)(1-\alpha)\lambda_0(-t)^{-\alpha}+1 \geqslant \frac{7}{8}, \quad t \leqslant -T_0$$

which follows from the lower bound on T_0 imposed at the beginning of the entire argument of this appendix, and where $\lambda_0(t) = \lambda_{0,0,b}(t) + \lambda_{0,1}(t)$. Similarly, using the fact that λ_0 is comparable to $\frac{1}{\log^b(t)}$, which is a decreasing function, we carry out the same procedure as previously, to obtain the important resolvent kernel estimate (5.72). In this context, the analog of (5.72) has some absolute constant C, rather than precisely 2, appearing on the right-hand side. Recalling the comment just above (A.1), the rest of the estimates on v_k required to construct $\lambda(t)$ and estimate $\lambda^{(k)}(t), \quad k \leqslant 4$ follow from an argument similar to that used previously.

Recall that the modulation equation in the setting of this appendix is of the same form as the one in the main body of the paper. In addition, the definition of v_1 is the same in both settings. Finally, (A.2) is true. Combined, these imply that we can prove the crucial pointwise estimates on F_4, and its derivatives with the same procedure as before. Also, all other subsections involved in the construction of the ansatz section can be established similarly. Finally, we can then complete the argument exactly as before, with $E(v_e, \partial_t \left(Q_{\frac{1}{\lambda(t)}} + v_e\right))$ having the same decay in t as previously.

We proceed to describe the Yang-Mills component of this thesis. The main differences in the procedure are extra steps needed to address technical problems caused by the much worse radiation considered here. On the other hand, the modulation equation for $\lambda(t)$ in terms of the radiation is simpler in this problem. As mentioned earlier, in our setup, a necessary condition for the radiation to have finite energy is that $\lambda(t)$ approaches a constant.

8 Introduction (Yang-Mills)

For the reader's convenience, some information appearing in the Background Material section is repeated here. We consider the Yang-Mills equation in $4+1$ dimensions, with gauge group $SO(4)$. This equation can be described by a gauge field, A, which is a Lie$(SO(4))$-valued one-form on $\mathbb{R}^{4+1}$. We write $A = A_\mu dx^\mu$, where, for each μ, A_μ is a Lie$(SO(4))$-valued function, defined on $\mathbb{R}^{4+1}$. Defining F, a Lie$(SO(4))$-valued two-form on $\mathbb{R}^{4+1}$ by

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

the Yang-Mills equation can be written as

$$-\partial_t F_{0\nu} - [A_0, F_{0\nu}] + \sum_{\mu=1}^{4} \left(\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}]\right) = 0, \quad \text{for } \nu = 0,1,2,3,4$$

where 0 on the right-hand is the zero in Lie$(SO(4))$. The Yang-Mills equation has the conserved energy

$$E_{YM} = -\frac{1}{48\pi^2} \int_{\mathbb{R}^4} \text{Tr}\left(F_{\mu\nu}(t,x) F_{\mu\nu}(t,x)\right) dx$$

The equation is invariant under the scaling symmetry

$$A_\mu(t,x) \to \lambda A_\mu(\lambda t, \lambda x)$$

The components of F transform under this symmetry as

$$F_{\mu\nu}(t,x) \to \lambda^2 F_{\mu\nu}(\lambda t, \lambda x)$$

which means that the energy E_{YM} is invariant under the scaling symmetry, because the equation is considered in 4 spatial dimensions. The Yang-Mills equation is also invariant under gauge transformations, which are transformations of A of the form

$$A_\mu \to g A_\mu g^{-1} - \partial_\mu g g^{-1}$$

where $g : \mathbb{R}^{1+4} \to SO(4)$.

Small energy global well posedness for the $(4+1)$ dimensional Yang-Mills problem was established by Krieger and Tataru, [18]. In addition, the works of Tataru and Oh, [24], [25], [21], [22], [23], established a threshold theorem and dichotomy theorem for this problem, with any compact, non-abelian gauge group.

With the equivariant ansatz (see also [28], [15])

$$A_\mu^{i,j}(t,x) = \left(\delta_\mu^i x^j - \delta_\mu^j x^i\right) \left(\frac{u(t,|x|)-1}{|x|^2}\right), \quad 0 \leqslant \mu \leqslant 4, \quad 1 \leqslant i,j \leqslant 4$$

the Yang-Mills equation reduces to

$$-\partial_{tt} u + \partial_{rr} u + \frac{1}{r}\partial_r u + \frac{2u(1-u^2)}{r^2} = 0 \tag{8.1}$$

The energy E_{YM} reduces to the following quantity, which is conserved by (8.1).

$$E_{YM}(u, \partial_t u) = \frac{1}{2} \int_0^\infty \left((\partial_t u)^2 + (\partial_r u)^2 + \frac{(1-u^2)^2}{r^2} \right) r dr$$

The equation (8.1) admits a soliton solution, namely $u(t,r) = Q_1(r) = \frac{1-r^2}{1+r^2}$. In addition, for any $\lambda > 0$, $Q_\lambda(r) = Q_1(r\lambda)$ is a solution. We will study perturbations of $Q_{\frac{1}{\lambda(t)}}$, and it will turn out that the "main" component of such perturbations will involve solutions to the following linear wave equation

$$-\partial_{tt} u + \partial_{rr} u + \frac{1}{r}\partial_r u - \frac{4}{r^2} u = 0 \tag{8.2}$$

The formally conserved energy for this equation is

$$E(u, \partial_t u) = \frac{1}{2}\int_0^\infty \left((\partial_t u)^2 + (\partial_r u)^2 + \frac{4u^2}{r^2}\right) rdr$$

Our goal in this work is to construct global, non-scattering solutions to (8.1). For the 1-equivariant, critical wave maps equation with $\mathbb{S}^2$ target, such solutions with topological degree 0 or 1, and energy in an appropriate range were classified in [2], [3]. As remarked in Appendix A of [2], and remark 4 of [3], the methods used in this classification result also apply to (8.1). Our procedure will then be similar to that of the previous work of the author, [26]. In particular, our solutions will involve a modulated soliton $Q_{\frac{1}{\lambda(t)}}$ coupled to radiation, and our procedure to construct these solutions will be to find a precise relation between the radiation and the dynamics of $\lambda(t)$. To describe our main result, we define the following set of functions.

For $b > \frac{2}{3}$, let F_b denote the set of functions f such that there exists $M > 50$, and $C_{f,k} > 0$, such that

$$f \in C^\infty([M, \infty)), \quad |f^{(k)}(t)| \leqslant \frac{C_{f,k}}{t^k \log^b(t)}, \text{ for } t \geqslant M \text{ and } k \geqslant 0$$

The class of radiation profiles of our solutions can be labeled by F_b in the following way. For $f \in F_b$, we have

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi^2}\int_0^\infty \frac{(\psi \cdot f)'(t)}{t}\sin(t\xi)dt$$

(where ψ is an unimportant cutoff function defined before (11.6)), and the radiation profile v_1 is given by

$$\begin{cases} -\partial_{tt} v_1 + \partial_{rr} v_1 + \frac{1}{r}\partial_r v_1 - \frac{4v_1}{r^2} = 0 \\ v_1(0) = 0 \\ \partial_t v_1(0) = v_{1,1} \end{cases}$$

In order to describe the leading order behavior of $\lambda(t)$, we introduce the following family of functions. For $b > \frac{2}{3}$, let Λ_b denote the set of functions λ_0 for which there exists $T_{\lambda_0} > 50$ such that $\lambda_0 \in C^\infty([T_{\lambda_0}, \infty))$, and the following two conditions hold: Firstly, there exists $f \in F_b$ such that

$$\frac{\lambda_0''(t)}{\lambda_0(t)} = \frac{f'(t)}{t}, \quad t \geqslant T_{\lambda_0} \tag{8.3}$$

Secondly,

$$\lambda_0(t) > 0, \quad \frac{|\lambda_0'(t)|}{\lambda_0(t)} \leqslant \frac{C}{t\log^b(t)}, \quad t \geqslant T_{\lambda_0} \tag{8.4}$$

The condition $\frac{\lambda_0''(t)}{\lambda_0(t)} = \frac{f'(t)}{t}$, rather than simply the symbol type estimates, is imposed so as to guarantee that the radiation profile of our solution has finite energy, see (11.6). Note that the above conditions on λ_0 imply that $\lambda_0(t) \to \lambda_1 > 0$ as $t \to \infty$, despite the fact that some $\lambda_0 \in \Lambda_b$ (for $b \leqslant 1$) satisfy

$$\int_t^\infty \int_x^\infty \frac{|\lambda_0''(s)|}{\lambda_0(s)} dsdx = \infty$$

To see this, we write

$$f(t) = -\lim_{M\to\infty} \int_t^M \frac{s\lambda_0''(s)}{\lambda_0(s)} ds = -\lim_{M\to\infty} \int_t^M \frac{\frac{d}{ds}\left(\lambda_0'(s)s - \lambda_0(s)\right)}{\lambda_0(s)} ds, \quad t > T_{\lambda_0}$$

Integrating by parts and using the assumptions on $\frac{|\lambda_0'(s)|}{\lambda_0(s)}$, and the fact that $b > \frac{2}{3}$, we see that

$$\lim_{M\to\infty} \log(\lambda_0(M)) < \infty$$

Despite the fact that any $\lambda_0 \in \Lambda_b$ is asymptotically constant, we do not directly use this fact in any quantitative estimates of the terms in our ansatz, and their associated error terms. For estimating the radiation profile, we use $\frac{\lambda_0''(t)}{\lambda_0(t)} = \frac{f'(t)}{t}$, but for the entirety of the rest of the argument, we only use the symbol-type estimates on $\frac{\lambda_0'(t)}{\lambda_0(t)}$ (which can be satisfied by non-asymptotically constant λ_0). In particular, our solutions are constructed using a similar argument to the previous work of the author regarding wave maps, [26], rather than assuming apriori that $\lambda(t)$ is asymptotically constant.

Before we state our main result, we remark that, given any $f \in F_b$, there exists $T_{\lambda_0} > 50$, and a one-parameter family of $\lambda_0 \in \Lambda_b$ satisfying (8.3) and (8.4). This can be seen as follows. Given $f \in F_b$, we can first find ω satisfying

$$\omega'(t) + \omega(t)^2 = \frac{f'(t)}{t}, \quad |\omega(t)| \leqslant \frac{C}{t\log^b(t)}, \quad t \geqslant N$$

(where $N > 50$ is sufficiently large) with a fixed point argument. By inspection of this equation, $\omega \in C^\infty([N,\infty))$. Then, we can define $T_{\lambda_0} = N + 1$, and let λ_0 be given by

$$\lambda_0(t) = c\exp\left(\int_{N+1}^t \omega(s)ds\right), \quad t \geqslant N+1, \quad \text{any } c > 0$$

Then, we have (8.4) and (8.3).

An interesting feature of our solutions is that the radiation profile depends only on f (as per the formula for $v_{1,1}$ given above) which is invariant with respect to multiplying λ_0 by a constant. As we just showed, there is a one-parameter family of $\lambda_0 \in \Lambda_b$, corresponding to a given $f \in F_b$. In particular, our family of solutions includes functions of the form $Q_{\frac{1}{\lambda(t)}}(r) + v_1(t,r) + o(1)$, for a one-parameter family of possible asymptotic values of $\lambda(t)$, and the **same** v_1.

Our main result is

Theorem 8.1. *For all $b > \frac{2}{3}$ and $f \in F_b$, let λ_0 be any element of Λ_b satisfying* (8.3)*. Then, there exists $T_0 = T_0(\lambda_0)$ and a finite energy solution, u, to* (8.1)*, with the following properties.*

$$u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_1(t,r) + v_e(t,r)$$

where $\lambda(t) \in C^4([T_0,\infty))$

$$-\partial_{tt}v_1 + \partial_{rr}v_1 + \frac{1}{r}\partial_r v_1 - \frac{4}{r^2}v_1 = 0, \quad E(v_1, \partial_t v_1) < \infty$$

$$E(v_e, \partial_t v_e) < \frac{C}{\log^{4b-2}(t)}, \quad t \geqslant T_0$$

and

$$\lambda(t) = \lambda_0(t)\,(1 + e(t))$$

where, for some $\epsilon_0 > 0$*, we have*

$$|e^{(k)}(t)| \leqslant \frac{C}{t^k \log^{\epsilon_0}(t)}, \quad 4 \geqslant k \geqslant 0$$

Remark 1. The initial data for v_1 in the theorem statement is explicit in terms of $f \in F_b$, as noted above.

Remark 2. For $\frac{2}{3} < \beta < \alpha < 1$, we can let

$$f(t) = \frac{\sin(\log^{\alpha}(t))}{\log^{\beta}(t)}, \quad t \geqslant 50$$

Then, $f \in F_b$ for any $\frac{2}{3} < b < \beta$. We then carry out the procedure discussed before the main theorem, to recover a $\lambda_0 \in \Lambda_b$ satisfying (8.3) and (8.4). In this case, we have

$$\frac{\lambda_0'(t)}{\lambda_0(t)} \sim \frac{-\alpha \log^{\alpha-1}(t) \cos(\log^{\alpha}(t))}{t \log^{\beta}(t)}$$

Since $1 + \beta - \alpha < 1$, this gives rise to $\lambda_0 \in \Lambda_b$ with

$$\int_t^{\infty} \frac{|\lambda_0'(s)|}{\lambda_0(s)} ds = \infty$$

Nevertheless, as pointed out earlier in a more general context, λ_0 is asymptotically constant.

Remark 3. By choosing

$$f(t) = \frac{1}{\log^b(t)}, \quad t \geqslant 50, \quad b > \frac{2}{3}$$

we can show (see (11.6)) that

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi \log^b(\frac{1}{\xi})} + O\left(\frac{1}{\xi \log^{b+1}(\frac{1}{\xi})}\right), \quad \xi \to 0$$

which shows that we can have radiation whose initial velocity has quite a large singularity at low frequencies. In fact, the condition for the radiation to have finite energy in our setting is $\widehat{v_{1,1}}(\xi) \in L^2((0,\infty), \xi d\xi)$. The initial velocity therefore satisfies this condition only "logarithmically".

Remark 4. A more precise set of estimates on e is as follows. In terms of the parameters δ, δ_j defined later on in the paper, we have

$$|e(t)| \leqslant \frac{C}{\log^{\delta-\delta_2}(t)}, \quad |e^k(t)| \leqslant \begin{cases} \frac{C}{t^k \log^{1+\delta-\delta_2}(t)}, & k = 1,2 \\ \frac{C}{t^3 \log^{b+\delta_4}(t)}, & k = 3 \\ \frac{C}{t^4 \log^{b+\delta_5}(t)}, & k = 4 \end{cases}$$

Now, we review previous related works. As mentioned before, the work [18] established small energy global well-posedness for the $(4+1)$ dimensional Yang-Mills problem. Regarding the large energy global well posedness of the Yang-Mills equation in $4+1$ dimensions, the works of Tataru and Oh, [24], [25], [21], [22], [23], established a threshold theorem and dichotomy theorem for the $(4+1)$-dimensional Yang-Mills equation, associated to any compact, non-abelian gauge group.

As previously mentioned, our procedure in this paper is similar to that used in the previous work of the author, [26]. That work constructed infinite time blow-up solutions to the energy critical, 1-equivariant wave maps problem with $\mathbb{S}^2$ target, with a symbol class of possible asymptotic behaviors of the soliton length scale, $\lambda(t)$. The main difference in this work is that the initial data of our radiation is much more singular at low frequencies. This leads to extra technical difficulties related to the slow decay of the radiation, v_1. In addition, the constraint that the radiation has finite energy implies, in our setting, that $\lambda_0(t)$ must be asymptotically constant for large t, in contrast with [26].

The work [11] constructs finite time blow-up solutions to the same wave maps problem just mentioned, by also understanding the relation between a prescribed radiation field and the dynamics of the soliton length scale, in the context of finite time blow-up. (The problem of finite time blow-up for this wave maps equation has also been studied in the preceding works [29], [28], [14], [5], [16]). Another key reference for our work is the paper of Krieger, Schlag and Tataru [15], which constructs finite time blow-up solutions to the same equation considered here, (8.1). In our argument, we use the "distorted Fourier transform" of [15], as well as related technical information, most importantly, the transference identity of that paper. For completeness we also mention that there is an analog of [15] for the energy critical, focusing semilinear wave equation in $\mathbb{R}^{1+3}$, namely [17].

Regarding other constructions of non-scattering solutions to (8.1), the work [9] (which also applies to other energy critical wave equations) constructed two bubble solutions to (8.1). The work [4] constructed infinite time blow-up and infinite time relaxation solutions to the focusing, energy critical semilinear wave equation on $\mathbb{R}^{1+3}$. Finally, the work [1] constructed global solutions to the energy critical wave maps problem with $\mathbb{S}^2$ target associated to a codimension two manifold of data. Also, given that our result can be interpreted as some form of stability of the soliton under perturbations, we mention the work [13], which constructed a stable manifold for the quintic, focusing semilinear wave equation in $\mathbb{R}^{1+3}$, centered around the Aubin-Talentini soliton solution.

9 Notation (Yang-Mills)

We will make use of the following notation. It will be slightly more convenient for our purposes to modify the usual definition of $\langle x \rangle$ as follows.

$$\langle x \rangle = \sqrt{50^2 + x^2}$$

The elliptic part of the linear wave equation obtained by linearizing (8.1) around Q_1 is

$$-\partial_{rr}u - \frac{1}{r}\partial_r u - \frac{2}{r^2}\left(1 - 3Q_1(r)^2\right)u$$

As noted in [28], this operator can be expressed as L^*L, for

$$L(f) = -f'(r) + 2\left(\frac{1-r^2}{1+r^2}\right)\frac{f(r)}{r}$$

which has the formal adjoint on $L^2(rdr)$ given by

$$L^*(f) = f'(r) + 2\left(\frac{1-r^2}{1+r^2}\right)\frac{f(r)}{r} + \frac{f(r)}{r}$$

We denote by ϕ_0, the eigenfunction of L^*L, with eigenvalue 0.

$$\phi_0(r) = \frac{r^2}{(1+r^2)^2}$$

Note that this definition of ϕ_0 is a factor of $\sqrt{r}$ different from that of [15], because part of [15] studies the conjugation of L^*L by $\frac{1}{\sqrt{r}}$.

As mentioned above, the linear wave equation (8.2) will be important for our work, and therefore, we will make use of the Hankel transform of order 2, which will be denoted by

$$\widehat{f}(\xi) = \int_0^\infty f(r)J_2(r\xi)rdr$$

10 Summary of the proof (Yang-Mills)

As mentioned earlier, the method is similar to that used by the author in [26]. The argument can be split broadly into two steps: constructing an ansatz, and then completing this ansatz to an exact solution. These two steps are explained in more detail below.

1. Strategy for constructing the ansatz For $b > \frac{2}{3}$, we start by taking some $f \in F_b$, and $\lambda_0 \in \Lambda_b$ satisfying (8.3). Then, we let $\lambda(t)$ (which will be chosen later) be any function of the form

$$\lambda(t) = \lambda_0(t)\left(1 + e(t)\right)$$

where e is small in a C^2 sense, and consider first, $u_1(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_1(t,r)$, where v_1 solves

$$\begin{cases} -\partial_{tt}v_1 + \partial_{rr}v_1 + \frac{1}{r}\partial_r v_1 - \frac{4v_1}{r^2} = 0 \\ v_1(0) = 0 \\ \partial_t v_1(0) = v_{1,1} \end{cases}$$

and $v_{1,1}$ is yet to be chosen. We will choose $v_{1,1}$, depending on λ_0, so that the principal part of the error term of our final ansatz (which is more complicated than u_1) is orthogonal to $\phi_0\left(\frac{\cdot}{\lambda(t)}\right)$, for a choice of $\lambda(t)$ which is equal to λ_0 to leading order. In order to further describe how we choose the initial data of v_1, let us note that substituting $u = u_1 + u_2$ into (8.1) gives the equation

$$-\partial_{tt}u_2 + \partial_{rr}u_2 + \frac{1}{r}\partial_r u_2 + \frac{2u_2(t,r)\left(1-3Q_{\frac{1}{\lambda(t)}}^2(r)\right)}{r^2}$$
$$= e_1(t,r) := \partial_{tt}Q_{\frac{1}{\lambda(t)}} - \frac{6\left(1-Q_{\frac{1}{\lambda(t)}}^2\right)v_1}{r^2} + \frac{6Q_{\frac{1}{\lambda(t)}}(v_1+u_2)^2}{r^2} + \frac{2(v_1+u_2)^3}{r^2}$$

The function u_1 is not our final ansatz. However, computing the inner product of its error term with $\phi_0\left(\frac{\cdot}{\lambda(t)}\right)$ still allows us to see how to choose $v_{1,1}$. The u_2-independent terms on the right-hand side of the above equation which contribute to leading order to $\langle e(t,R\lambda(t)),\phi_0(R)\rangle_{L^2(RdR)}$ are the soliton error term $\partial_{tt}Q_{\frac{1}{\lambda(t)}}$ and the linear error term associated to v_1, which is $-\frac{6\left(1-Q_{\frac{1}{\lambda(t)}}^2\right)v_1}{r^2}$. Compared with [26], we do not need a correction analogous to the one denoted by v_1 in that paper, since $\partial_{tt}Q_{\frac{1}{\lambda(t)}} \in L^2((0,\infty),rdr)$ in this setting, and we therefore have a simpler modulation equation. We compute the inner product of the linear error term associated to v_1 in the same way done in [26], except using the Hankel transform of order 2, this time. We have

$$v_1(t,r) = \int_0^\infty J_2(r\xi)\sin(t\xi)\widehat{v_{1,1}}(\xi)d\xi$$

and

$$\int_0^\infty \frac{24R^3}{(1+R^2)^4\lambda(t)^2}J_2(R\lambda(t)\xi)dR = \frac{\xi^3\lambda(t)}{2}K_1(\xi\lambda(t))$$

(which follows from combining identities from [7]). We can therefore compute

$$\langle -\frac{6\left(1-Q_{\frac{1}{\lambda(t)}}^2\right)v_1}{r^2}\Big|_{r=R\lambda(t)},\phi_0(R)\rangle_{L^2(RdR)} = -\int_0^\infty \sin(t\xi)\widehat{v_{1,1}}(\xi)\frac{\xi^3\lambda(t)}{2}K_1(\xi\lambda(t))d\xi \quad (10.1)$$

We have an extra factor of $\xi\lambda(t)$ inside the integral, relative to the analogous integral in [26]. This leads to $\widehat{v_{1,1}}(\xi)$ being roughly a factor of $\frac{1}{\xi}$ worse at low frequencies ξ, relative to the initial velocity for the radiation in [26]. In addition to causing technical difficulties associated to very slow decay of the radiation v_1, this also constrains $\lambda_0(t)$ to asymptote to a constant for large t, in order that the radiation has finite energy. (In addition to our discussion in the introduction, see (11.6) and the discussion afterwards for more details). We also have

$$\langle \partial_{tt}Q_{\frac{1}{\lambda(t)}}\Big|_{r=R\lambda(t)},\phi_0(R)\rangle_{L^2(RdR)} = \frac{2\lambda''(t)}{3\lambda(t)}$$

The modulation equation that we use to choose $\lambda(t)$ is not simply to set the sum of these two inner products equal to zero, since we will need more to add more corrections to our ansatz. However, the leading order contribution to the modulation equation is indeed given by the sum of these two terms. Therefore, we choose the initial data of v_1 so as to make the sum of these two inner products vanish to leading order when $\lambda(t) = \lambda_0(t)$. We recall that $\frac{\lambda_0''(t)}{\lambda_0(t)} = \frac{f'(t)}{t}, \quad t \geqslant T_{\lambda_0}$, and extend this to a function $\frac{(\psi f)'(t)}{t}$ defined on $[0, \infty)$ with a cutoff ψ (whose properties are not so important, as long as $\psi(x) = 1$ for x large enough, and which is defined prior to (11.6)). Then, we let

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi^2} \int_0^\infty \frac{(\psi \cdot f)'(t)}{t} \sin(t\xi) dt$$

We have $\psi(x) = 1$ for $x \geqslant 2T_{\lambda_0}$, which gives, by the inversion of the sine transform, that

$$\int_0^\infty \sin(t\xi) \widehat{v_{1,1}}(\xi) \frac{\xi^2}{2} d\xi = \frac{2}{3} \frac{\lambda_0''(t)}{\lambda_0(t)}, \quad t \geqslant 2T_{\lambda_0}$$

This will be sufficient to allow $\lambda_0(t)$ to be a leading order solution to the eventual modulation equation for λ. In particular, in our setting, we can replace $K_1(\xi\lambda(t))$ appearing in (10.1) by $\frac{1}{\xi\lambda(t)}$ (recall that $K_1(x) = \frac{1}{x} + O(x \log(x)), \quad x \to 0$) to get the leading order behavior of the integral as a function of t.

As described earlier, the singularity of $\widehat{v_{1,1}}(\xi)$ for small ξ causes technical difficulties, in part due to the fact that v_1 has a very slow ($\frac{1}{\log^b(r)}$) decay for large r. Recall also that $Q_1(r) = \frac{1-r^2}{1+r^2}$. In particular, Q does not decay at infinity. Therefore, the quadratic and cubic nonlinear error terms involving v_1 are very far from having sufficient decay for large r, in order that the rest of our argument can be carried out. Our first correction to improve these error terms is denoted by v_2, which solves the following equation with 0 Cauchy data at infinity.

$$-\partial_{tt} v_2 + \partial_{rr} v_2 + \frac{1}{r} \partial_r v_2 - \frac{4}{r^2} v_2 = \frac{6 Q_{\frac{1}{\lambda(t)}}(r)}{r^2} v_1^2 + \frac{2}{r^2} v_1^3$$

On the other hand, v_2, only decays logarithmically better than v_1, and its nonlinear interaction with v_1 as well as its interactions with itself are not perturbative. Therefore, we successively add corrections, v_j, which solve

$$-\partial_{tt} v_j + \partial_{rr} v_j + \frac{1}{r} \partial_r v_j - \frac{4}{r^2} v_j = RHS_j(t, r)$$

where

$$\begin{aligned} RHS_j(t,r) &= \frac{6 Q_{\frac{1}{\lambda(t)}}(r)}{r^2} \left(\left(\sum_{k=1}^{j-1} v_k \right)^2 - \left(\sum_{k=1}^{j-2} v_k \right)^2 \right) + \frac{2}{r^2} \left(\left(\sum_{k=1}^{j-1} v_k \right)^3 - \left(\sum_{k=1}^{j-2} v_k \right)^3 \right) \\ &= \frac{6 Q_{\frac{1}{\lambda(t)}}}{r^2} \left(2 \sum_{k=1}^{j-2} v_k v_{j-1} + v_{j-1}^2 \right) + \frac{2}{r^2} \left(3 \left(\sum_{k=1}^{j-2} v_k \right)^2 v_{j-1} + 3 \sum_{k=1}^{j-2} v_k v_{j-1}^2 + v_{j-1}^3 \right) \end{aligned}$$

and prove that the series

$$v_s := \sum_{j=3}^\infty v_j$$

(as well as the series resulting from applying any first or second order derivative termwise) converges absolutely and uniformly on the set $\{(t,r) | t \geqslant T_1, r \geqslant 0\}$, where T_1 is some sufficiently large number. Moreover, we get that

$$\begin{aligned}&-\partial_{tt}v_s+\partial_{rr}v_s+\frac{1}{r}\partial_r v_s-\frac{4}{r^2}v_s\\&=\frac{6Q_{\frac{1}{\lambda(t)}}}{r^2}\left(2v_1\left(v_2+v_s\right)+\left(v_2+v_s\right)^2\right)+\frac{2}{r^2}\left(3v_1\left(v_2+v_s\right)^2+3v_1^2\left(v_2+v_s\right)+\left(v_2+v_s\right)^3\right)\end{aligned}$$

Let $v_c = v_1 + v_2 + v_s$. Then, the only error term of the refined ansatz

$$u_3(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r)$$

is

$$\partial_{tt}Q_{\frac{1}{\lambda(t)}} - \frac{6v_c}{r^2}\left(1 - Q^2_{\frac{1}{\lambda(t)}}(r)\right)$$

which has roughly two powers of r improved decay compared with the nonlinear error terms involving v_1.

It turns out that even this major improvement over the ansatz u_1 still does not have an error term with sufficient decay in the r variable. In order to rectify this, we introduce a length scale $g(t)$, and eliminate the portion of the u_3 error localized to the region $r \geqslant g(t)$. On one hand, we can not have $g(t)$ too small, since doing so would change the leading order behavior of the inner product of the error term of the final ansatz, which is not desired. On the other hand, we can not have $g(t)$ too large, since the whole purpose of the next set of corrections is to improve the large r decay of the error term of u_3. We therefore find an intermediate scale $g(t)$ which suffices for our purposes, and add a first correction, w_2 which improves the error term of u_3. On the other hand, there are now nonlinear interactions between w_2 and the previous corrections, which, due to the slow decay of v_1, are not perturbative. Similarly to the case with v_k, we add another series of corrections

$$w_s = \sum_{k=3}^{\infty} w_k$$

to eventually eliminate all the nonlinear error terms involving w_j and v_k. If we let $w_c = w_2 + w_s$, then, we end up with our final ansatz

$$u_5(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r) + w_c(t,r)$$

The error term of u_5 is then decomposed as $F_4 + F_5$, exactly as in [26], where F_5 is sufficiently small in sufficiently many norms so as to allow it to be eventually treated perturbatively, even though it will end up not necessarily being orthogonal to $\phi_0(\frac{\cdot}{\lambda(t)})$. We have

$$F_5(t,r) = \left(1 - \chi_{\leqslant 1}(\frac{2r}{t})\right)\left(\frac{-6\left(1 - Q^2_{\frac{1}{\lambda(t)}}(r)\right)}{r^2} w_c(t,r)\right.$$

where $\chi_{\leqslant 1}$ is a cutoff whose properties are unimportant for the purposes of this discussion. The smallness of F_5 is more precisely

$$||F_5(t, R\lambda(t))||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^3}{t^5 \log^{b-2}(t)}$$

$$||L^* L\left(F_5(t, R\lambda(t))\right)||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^5 \log^2(t)}{g(t)^2 \log^b(t) t^5}$$

where we recall that L has been defined in the notation section. We then choose $\lambda(t)$ so that the term $F_4(t,r)$ is orthogonal to $\phi_0(\frac{\cdot}{\lambda(t)})$. Unlike in [26], the principal part of the equation for $\lambda(t)$ is simply a second order ODE, rather than a Volterra equation of the second kind in the unknown λ''. Once we solve this equation for λ, we then prove that $\lambda^{(k)}$ has symbol-type estimates for $1 \leqslant k \leqslant 4$. As in [26], it is important that, in addition to being orthogonal to $\phi_0(\frac{\cdot}{\lambda(t)})$, $F_4(t,r)$ has symbol-type estimates. We have

$$\begin{aligned} F_4(t,r) = &\left(1-\chi_{\geqslant 1}(\frac{r}{g(t)})\right)\left(\partial_t^2 Q_{\frac{1}{\lambda(t)}}(r) - \frac{6v_c(t,r)}{r^2}\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)\right) \\ &+ \chi_{\leqslant 1}(\frac{2r}{t})\left(\frac{-6\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)}{r^2} w_c(t,r)\right) \end{aligned}$$

In particular, after choosing $\lambda(t)$, we get, for $0 \leqslant j, k \leqslant 2$, and $j+k \leqslant 2$,

$$\begin{aligned} |r^k \partial_r^k t^j \partial_t^j F_4(t,r)| \leqslant &C \mathbb{1}_{\{r \leqslant g(t)\}} \frac{r^2 \lambda(t)^2}{t^2 \log^b(t) (\lambda(t)^2 + r^2)^2} \\ &+ C \frac{\mathbb{1}_{\{r \leqslant \frac{t}{2}\}} \lambda(t)^2}{(\lambda(t)^2 + r^2)^2} \begin{cases} \frac{r^2 \lambda(t)^2 \log(t)}{t^2 g(t)^2 \log^b(t)}, & r \leqslant g(t) \\ \frac{\lambda(t)^2 \log(t)}{t^2 \log^b(t)}\left(\log(2+\frac{r}{g(t)}) + \frac{\log(t)}{\log^b(t)}\right), & g(t) < r < \frac{t}{2} \end{cases} \end{aligned}$$

and

$$\langle F_4(t, R\lambda(t)), \phi_0(R) \rangle_{L^2(RdR)} = 0$$

2. Completion of the ansatz to an exact solution of (8.1) To complete the ansatz u_5 to an exact solution of (8.1), we use the same approach as in [26]. In particular, we substitute $u = u_5 + v$ into (8.1), and use the "distorted Fourier transform" of [15], which we denote as $\mathcal{F}$ to recast the resulting equation for v into one for y, given by

$$\mathcal{F}(\sqrt{\cdot} v(t, \cdot \lambda(t)))(t,\xi) = \begin{bmatrix} y_0(t) \\ y_1(t, \frac{\xi}{\lambda(t)^2}) \end{bmatrix}$$

Here, we use the same convention as [15], in terms of denoting $\mathcal{F}(f)$ as a two component vector. The first entry in this vector is $\langle f, \sqrt{r}\phi_0(r)\rangle_{L^2(dr)}$. (Note that there is a notational difference between ϕ_0 in this paper and ϕ_0 in [15]). The choice of re-scaling in y_1, exactly as in [26] is explained by noting that the resulting system of equations for y_0 and y_1 takes the form

$$\begin{bmatrix} -\partial_{tt} y_0 \\ -\partial_{tt} y_1 - \omega y_1 \end{bmatrix} = F_2 + \mathcal{F}\left(\sqrt{\cdot}\left(F_3 + F_4 + F_5\right)(t, \cdot\lambda(t))\right)\left(\omega\lambda(t)^2\right)$$

where F_2 contains perturbative terms depending on y and $\partial_t y$, some of which are estimated using the transference identity of [15], and F_3 contains other linear and nonlinear error terms depending on $v(y)$. We solve this equation by finding a fixed-point of the map T given by

$$T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})(t,\omega) = \begin{bmatrix} -\int_t^\infty \int_s^\infty \left(F_{2,0} + \mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(s_1,\cdot\lambda(s_1))\right)_0\right) ds_1 ds \\ \int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}} \left(F_{2,1}(x,\omega) + \mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(x,\cdot\lambda(x))\right)_1 (\omega\lambda(x)^2)\right) dx \end{bmatrix}$$

where the subscripts i after, for example F_2, or $\mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(x,\cdot\lambda(x))\right)$ mean the $i+1$st component of the vector, for $i = 0, 1$. T is defined on a space Z, whose norm is precisely given in (12.5), but is roughly a weighted $L^\infty_t L^2_\omega$ norm of y and $\partial_t y$. Our norm on Z is roughly one derivative stronger than the analogous norm used for iteration in [26]. The most delicate terms on the right-hand side of the above equation are those involving F_4. Because of the orthogonality condition on F_4, we have

$$\mathcal{F}\left(\sqrt{\cdot}\left(F_4\right)(s_1,\cdot\lambda(s_1))\right)_0 = 0$$

On the other hand, for the second component of the vector equation above, we (just like in [26]) treat the F_4 term by integrating by parts in the x variable, using both the fact that

$$\mathcal{F}\left(\sqrt{\cdot}\left(F_4\right)(x,\cdot\lambda(x))\right)_1(\xi) \to 0, \quad \xi \to 0$$

(which follows from the orthogonality condition on F_4) and the fact that F_4 has symbol-type estimates. The other terms in the equation above can be estimated without such a delicate argument.

11 Construction of the ansatz (Yang-Mills)

Let $b > \frac{2}{3}$, $f \in F_b$, and $\lambda_0 \in \Lambda_b$ satisfying (8.3). We will have to introduce some constants and parameters to describe our setup. Let $T_0 > \exp\left(900! + 2^{-\frac{3}{2(2b-1)}}\right)(1+T_{\lambda_0})$, let $0 < \epsilon < \min\{\frac{3b-2}{1600}, \frac{2b-1}{200}, \frac{1}{900000}, \frac{b}{900}\}$, and define

$$\delta = \min\{2b-1, 3b-4\epsilon-2, 5b-8\epsilon-3\} \tag{11.1}$$

Note that $\delta > 0$, because $b > \frac{2}{3}$, and because of the constraints on ϵ. Also, $1+\delta > b$. Let

$$\delta_2 = \min\{\frac{1}{2}(\delta+1-b), \frac{\delta}{2}\} \tag{11.2}$$

Define a Banach space X to be the set of functions $e \in C^2([T_0,\infty))$ satisfying $||e||_X < \infty$, where

$$||e||_X = \sup_{t \geqslant T_0} \left(|e(t)| \log^{\delta-\delta_2}(t) + |e'(t)| t \log^{1+\delta-\delta_2}(t) + |e''(t)| t^2 \log^{1+\delta-\delta_2}(t)\right) \tag{11.3}$$

Until more precisely chosen, λ denotes any function of the form

$$\lambda(t) = \lambda_0(t) \cdot (1+e(t)), \quad e \in \overline{B}_1(0) \subset X \tag{11.4}$$

In particular, since $1+\delta-\delta_2 > b$, we have

$$\frac{|\lambda'(t)|}{\lambda(t)} \leqslant \frac{C}{t \log^b(t)}, \quad \frac{|\lambda''(t)|}{\lambda(t)} \leqslant \frac{C}{t^2 \log^b(t)}$$

For later use, let $g(t) = \lambda(t)\log^{b-2\epsilon}(t)$. By the definition of g, constraints on λ_0, and (11.4), there exists M_1 sufficiently large so that

$$\log(t) \geqslant 2|\log(g(t))|, \quad \frac{t|g'(t)|}{g(t)} \leqslant \frac{1}{900!}, \quad \frac{t}{g(t)} \geqslant 1600, \quad \text{for } t \geqslant M_1 \tag{11.5}$$

Then, we further constrain T_0 to satisfy $T_0 > \exp\left(900! + 2^{-\frac{3}{2(2b-1)}}\right)(1+T_{\lambda_0}) + M_1$.

The main result of this section is the following theorem concerning the existence of an approximate solution to (8.1).

Theorem 11.1 (Approximate solution to (8.1))**.** *For all $b > \frac{2}{3}$, $f \in F_b$, and all $\lambda_0 \in \Lambda_b$ satisfying* (8.3)*, there exists $T_3 > 0$ such that for all $T_0 > T_3$, there exists $v_{corr} \in C^2([T_0,\infty), C^2((0,\infty)))$ and $\lambda \in C^4([T_0,\infty))$, such that, if*

$$u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_{corr}(t,r)$$

then

$$E_{YM}(u,\partial_t u) < \infty, \quad -\partial_{tt}u + \partial_{rr}u + \frac{1}{r}\partial_r u + \frac{2u(1-u^2)}{r^2} = -F_4(t,r) - F_5(t,r)$$

where

$$||F_5(t,R\lambda(t))||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^3}{t^5\log^{b-2}(t)}, \quad ||L^*L\left(F_5(t,R\lambda(t))\right)||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^5\log^2(t)}{g(t)^2\log^b(t)t^5}$$

$$\langle F_4(t,R\lambda(t)), \phi_0(R)\rangle_{L^2(RdR)} = 0$$

For $0 \leqslant j,k \leqslant 2$, and $j+k \leqslant 2$,

$$\begin{aligned}|r^k\partial_r^k t^j\partial_t^j F_4(t,r)| &\leqslant C\mathbb{1}_{\{r\leqslant g(t)\}}\frac{r^2\lambda(t)^2}{t^2\log^b(t)(\lambda(t)^2+r^2)^2}\\ &+ C\frac{\mathbb{1}_{\{r\leqslant\frac{t}{2}\}}\lambda(t)^2}{(\lambda(t)^2+r^2)^2}\begin{cases}\frac{r^2\lambda(t)^2\log(t)}{t^2g(t)^2\log^b(t)}, & r\leqslant g(t)\\ \frac{\lambda(t)^2\log(t)}{t^2\log^b(t)}\left(\log(2+\frac{r}{g(t)}) + \frac{\log(t)}{\log^b(t)}\right), & g(t) < r < \frac{t}{2}\end{cases}\end{aligned}$$

λ is given by

$$\lambda(t) = \lambda_0(t)\left(1+e(t)\right)$$

$$|e(t)| \leqslant \frac{C}{\log^{\delta-\delta_2}(t)}, \quad |e^k(t)| \leqslant \begin{cases}\frac{C}{t^k\log^{1+\delta-\delta_2}(t)}, & k=1,2\\ \frac{C}{t^3\log^{b+\delta_4}(t)}, & k=3\\ \frac{C}{t^4\log^{b+\delta_5}(t)}, & k=4\end{cases}$$

where δ, δ_2 *are defined in* (11.1) *and* (11.2)*, respectively and* $\delta_4, \delta_5 > 0$*. Finally,*

$$\begin{aligned}&\Bigg(||\frac{2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))}{R^2\lambda(t)^2}||_{L^\infty_R}\\&+\sup_{R\geqslant 1}\left(\frac{|\partial_R\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2R^2}\right)\\&+\sup_{R\geqslant 1}\left(\frac{|\partial_R^2\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{R^2\lambda(t)^2}\right)\\&+\sup_{R\leqslant 1}\left(\frac{|\partial_R\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2R}\right)\\&+\sup_{R\leqslant 1}\left(\frac{|\partial_R^2\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2}\right)\Bigg)\\&\leqslant\frac{C}{t^2\log^b(t)}\end{aligned}$$

11.1 The Cauchy data for the radiation v_1

In this section, we will introduce the initial velocity for the first addition to the soliton in our ansatz, which we will denote by v_1.

Let $\psi \in C^\infty([0,\infty))$ satisfy

$$\psi(x)=\begin{cases}0, & x\leqslant T_{\lambda_0}\\ 1, & x\geqslant 2T_{\lambda_0}\end{cases}, \quad \text{and } 0\leqslant\psi(x)\leqslant 1, \quad x\geqslant 0$$

Then, $t \mapsto \psi(t)f(t)$, apriori only defined on $[T_{\lambda_0},\infty)$, extends to a smooth function on $[0,\infty)$. Note that

$$\frac{\lambda_0''(t)}{\lambda_0(t)}=\frac{(\psi\cdot f)'(t)}{t}, \quad t\geqslant 2T_{\lambda_0}$$

Finally, we define $v_{1,1}$ by specifying its Hankel transform of order 2:

$$\widehat{v_{1,1}}(\xi)=\frac{8}{3\pi\xi^2}\int_0^\infty\frac{(\psi\cdot f)'(t)}{t}\sin(t\xi)dt \tag{11.6}$$

As in [26], this definition is made so as to allow $\lambda(t)=\lambda_0(t)$ to be a leading order solution to the eventual modulation equation for λ. Now, we record some pointwise estimates on $\widehat{v_{1,1}}$ and its derivatives.

Lemma 11.2. *For* $k\geqslant 0$*, there exist constants* $C_k, C(k,N)$*, such that*

$$|\partial_\xi^k\left(\xi^2\widehat{v_{1,1}}(\xi)\right)|\leqslant\begin{cases}\frac{C_k}{\xi^{k-1}\log^b(\frac{1}{\xi})}, & \xi\leqslant\frac{1}{4}\\ \frac{C(k,N)}{\xi^{k+4N}}, & \xi>\frac{1}{4}, N\geqslant 1\end{cases}$$

Proof. We have

$$\widehat{v_{1,1}}(\xi) = \frac{8}{3\pi\xi^2}\int_0^{\frac{1}{\xi}} \frac{(\psi f)'(t)}{t} t\xi dt + \frac{8}{3\pi\xi^2}\int_0^{\frac{1}{\xi}} \frac{(\psi f)'(t)}{t}(\sin(t\xi) - t\xi)\, dt + \frac{8}{3\pi\xi^2}\int_{\frac{1}{\xi}}^{\infty} \frac{(\psi f)'(t)}{t}\sin(t\xi) dt \tag{11.7}$$

which gives

$$|\widehat{v_{1,1}}(\xi)| \leqslant \frac{C}{\xi \log^b(\frac{1}{\xi})}, \quad \xi \leqslant \frac{1}{4} \tag{11.8}$$

Note that the first term of (11.7) is where we use the condition that

$$\frac{\lambda_0''(t)}{\lambda_0(t)} = \frac{f'(t)}{t}, \quad |f(t)| \leqslant \frac{C}{\log^b(t)}, \quad t \geqslant T_{\lambda_0}$$

This condition, along with $b > \frac{2}{3}$ guarantees that $\widehat{v_{1,1}} \in L^2((0,\infty), \xi d\xi)$. (We will see shortly that $\widehat{v_{1,1}}(\xi)$ is rapidly decreasing for large ξ). Since $v_{1,1}$ will end up being the initial velocity of our radiation component of the solution, v_1, this implies that v_1 has finite energy. This condition also implies that $\lambda(t)$ must asymptote to a constant as t approaches infinity, as shown in the introduction. Now we show that, regardless of how one chooses to extend $\frac{\lambda_0''(t)}{\lambda_0(t)}$ from a function defined on, say $[2T_{\lambda_0}, \infty)$ to one defined on $[0,\infty)$, and even if we didn't assume the structural condition $t\frac{\lambda_0''(t)}{\lambda_0(t)} = f'(t)$, but rather, only the symbol-type estimates,

$$\frac{|\lambda_0^{(k)}(t)|}{\lambda_0(t)} \leqslant \frac{C_k}{t^k \log^b(t)}, \quad k \geqslant 1, \quad t \geqslant T_{\lambda_0}$$

we would still need $\lambda_0(t) \to c$ to have $\widehat{v_{1,1}} \in L^2((0,\infty), \xi d\xi)$. We show this as follows. If $\widehat{v_{1,1}} \in L^2((0,\infty), \xi d\xi)$, then, $\widehat{v_{1,1}} \in L^2((0, \frac{1}{2T_{\lambda_0}}), \xi d\xi)$. Even without the structural condition $t\frac{\lambda_0''(t)}{\lambda_0(t)} = f'(t)$, the second and third terms of (11.7) are bounded above in absolute value by

$$\frac{C}{\xi \log^b(\frac{1}{\xi})}, \quad \xi \leqslant \frac{1}{2T_{\lambda_0}}$$

(Recall that $\frac{1}{2T_{\lambda_0}} \leqslant \frac{1}{4}$)). Therefore, the condition $\widehat{v_{1,1}} \in L^2((0, \frac{1}{2T_{\lambda_0}}), \xi d\xi)$ implies that the first term of (11.7) is in $L^2((0, \frac{1}{2T_{\lambda_0}}), \xi d\xi)$. Let $g(t)$ be any extension of $\frac{\lambda_0''(t)t}{\lambda_0(t)}$, which satisfies

$$g(t) = \frac{\lambda_0''(t)t}{\lambda_0(t)}, \quad t \geqslant 2T_{\lambda_0}, \quad g \in C^\infty([0,\infty))$$

If we let

$$G(x) = \frac{8}{3\pi}\left(\int_0^{2T_{\lambda_0}} g(t)dt + \int_{2T_{\lambda_0}}^x \frac{t\lambda_0''(t)}{\lambda_0(t)} dt\right), \quad x > 2T_{\lambda_0}$$

then, the first term of (11.7) is

$$\frac{G(\frac{1}{\xi})}{\xi} \in L^2((0, \frac{1}{2T_{\lambda_0}}), \xi d\xi)$$

Therefore,

$$\int_0^{\frac{1}{2T_{\lambda_0}}} \frac{|G(\frac{1}{\xi})|^2}{\xi} d\xi = \int_{\log(2T_{\lambda_0})}^{\infty} (G(e^u))^2 du < \infty, \quad u = \log(\frac{1}{\xi})$$

But,

$$\frac{d}{du}\left(G(e^u)\right)^2 = 2G(e^u)G'(e^u)e^u = 2G(e^u)\frac{8}{3\pi}\left(\frac{e^u \lambda_0''(e^u)}{\lambda_0(e^u)}\right)e^u$$

Therefore,

$$|\frac{d}{du}\left(G(e^u)\right)^2| \leqslant C|G(e^u)|\frac{1}{\log^b(e^u)} \leqslant C\left(1+\frac{1}{\log^{b-1}(e^u)}\right)\frac{1}{\log^b(e^u)} \leqslant C, \quad u \geqslant \log(2T_{\lambda_0})$$

where we used $b > \frac{2}{3} > \frac{1}{2}$, and

$$|G(x)| \leqslant C + C\int_{2T_{\lambda_0}}^x \frac{dt}{t\log^b(t)} \leqslant C\left(1+\frac{1}{\log^{b-1}(x)}\right), \quad x \geqslant 2T_{\lambda_0}$$

and we stress again that we only use the symbol-type estimates on $\frac{\lambda_0''(x)}{\lambda_0(x)}$, and *not* the structural condition $\frac{t\lambda_0''(t)}{\lambda_0(t)} = f'(t)$ for this discussion.

But, now we can conclude that $u \mapsto (G(e^u))^2$ is Lipshitz, whence, the condition

$$\int_{2T_{\lambda_0}}^{\infty} (G(e^u))^2 du < \infty$$

implies that $\lim_{u\to\infty}(G(e^u))^2 = 0$, which is to say that

$$\lim_{x\to\infty}\int_{2T_{\lambda_0}}^x \frac{t\lambda_0''(t)}{\lambda_0(t)}dt < \infty, \quad \text{or, equivalently, that } \lim_{\xi\to 0^+}\int_{2T_{\lambda_0}}^{\frac{1}{\xi}} \frac{t\lambda_0''(t)}{\lambda_0(t)}dt < \infty$$

Therefore, a necessary (but in general insufficient) condition for $\widehat{v_{1,1}} \in L^2((0,\infty), \xi d\xi)$ is that

$$\lim_{\xi\to 0}\int_{2T_{\lambda_0}}^{\frac{1}{\xi}} \frac{\lambda_0''(t)}{\lambda_0(t)} t dt < \infty$$

(In particular, the limit has to exist). Repeating the same computation done in the introduction, before the main theorem statement, we again get

$$\lim_{\xi\to 0}\log(\lambda_0(\frac{1}{\xi})) < \infty$$

which implies that, in our setting, $\lambda_0(t)$ asymptoting to a non-zero constant for large t is necessary for the radiation v_1 to have finite energy.

Continuing our estimates, for each $k \geqslant 1$, there exist constants $C_{j,k}$ such that

$$\partial_\xi^k\left((\psi f)'(\frac{\sigma}{\xi})\right) = \sum_{j=2}^{k+1} \frac{(\psi f)^{(j)}(\frac{\sigma}{\xi})\sigma^{j-1}C_{j,k}}{\xi^{k+j-1}}$$

To estimate $|\partial_\xi^k(\xi^2\widehat{v_{1,1}}(\xi))|$ for $k \geqslant 1$, in the region $\xi \leqslant \frac{1}{4}$, it suffices to consider the case $\xi \leqslant \frac{4}{T_{\lambda_0}}$, since

$$|\partial_\xi^k\left(\xi^2\widehat{v_{1,1}}(\xi)\right)| \leqslant C_k, \quad \frac{4}{T_{\lambda_0}} \leqslant \xi \leqslant \frac{1}{4}$$

In the case $T_{\lambda_0}\xi \leqslant 4$, we let $\sigma = t\xi$ in the integral defining $\widehat{v_{1,1}}$, and differentiate under the integral sign. Then, we use the support properties of ψ, and treat the integral over $\sigma \in [T_{\lambda_0}\xi, 4]$ and $(4,\infty)$ separately. Hence, for some constant C_k, whose value may change from line to line:

$$\partial_\xi^k\left(\xi^2\widehat{v_{1,1}}(\xi)\right) = \frac{8}{3\pi}\sum_{j=2}^{k+1}\frac{C_{j,k}}{\xi^{k+j-1}}\int_{T_{\lambda_0}\xi}^4 (\psi f)^{(j)}(\frac{\sigma}{\xi})\sigma^{j-1}d\sigma + \text{Err}(\xi) \tag{11.9}$$

where

$$\begin{aligned}|Err(\xi)| &\leqslant C\sum_{j=2}^{k+1}\int_{T_{\lambda_0}\xi}^4 \frac{C_{j,k}C_j\sigma d\sigma}{\log^b(\frac{\sigma}{\xi})\xi^{k-1}} + \sum_{j=2}^{k+1}\frac{C_{j,k}}{\xi^{k-1}\log^b(\frac{1}{\xi})}\\ &\leqslant \frac{C_k}{\xi^{k-1}\log^b(\frac{1}{\xi})}, \quad \xi \leqslant \frac{4}{T_{\lambda_0}}\end{aligned}$$

By induction, for $j \geqslant 1$,

$$\int_a^b (\psi f)^{(j)}(x)x^{j-1}dx = (-1)^{j-1}(j-1)!\sum_{q=0}^{j-1}\frac{(-1)^q}{q!}b^q(\psi f)^{(q)}(b), \quad \text{if } (\psi f)^{(n)}(a) = 0, \text{ for } n \geqslant 0$$

Using this fact, and the support properties of ψ, we return to (11.9) and (11.8) and get, for $k \geqslant 0$,

$$|\partial_\xi^k\left(\xi^2\widehat{v_{1,1}}(\xi)\right)| \leqslant \frac{C_k}{\xi^{k-1}\log^b(\frac{1}{\xi})}, \quad \xi \leqslant \frac{1}{4}$$

Finally, for $k \geqslant 1$,

$$\partial_\xi^k\left(\xi^2\widehat{v_{1,1}}(\xi)\right) = \frac{8}{3\pi}\sum_{j=2}^{k+1}\int_0^\infty \frac{(\psi f)^{(j)}(\frac{\sigma}{\xi})}{\xi^{k+j-1}}C_{j,k}\sigma^{j-2}\sin(\sigma)d\sigma$$

implies

$$|\partial_\xi^k\left(\xi^2\widehat{v_{1,1}}(\xi)\right)| \leqslant \frac{C(k,N)}{\xi^{k+4N}}, \quad \xi > \frac{1}{4}, N \geqslant 1, k \geqslant 0$$

This completes the proof of the lemma. □

11.2 Estimates on v_1

We define v_1 to be the solution to the following Cauchy problem

$$\begin{cases} -\partial_{tt}v_1 + \partial_{rr}v_1 + \frac{1}{r}\partial_r v_1 - \frac{4v_1}{r^2} = 0 \\ v_1(0) = 0 \\ \partial_t v_1(0) = v_{1,1} \end{cases}$$

Lemma 11.3. *We have the following estimates*

$$|v_1(t,r)| \leqslant \begin{cases} \frac{Cr^2}{t^2 \log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{\log^b(r)}, & r \geqslant \frac{t}{2} \end{cases} \tag{11.10}$$

For $1 \leqslant j+k$*, and* $0 \leqslant j,k \leqslant 2$*,*

$$|\partial_t^j \partial_r^k v_1(t,r)| \leqslant \begin{cases} C\frac{r^{2-k}}{t^{2+j}\log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C}{\sqrt{r}\log^b(\langle t-r\rangle)\langle t-r\rangle^{\frac{1}{2}+j+k-1}}, & r > \frac{t}{2} \end{cases} \tag{11.11}$$

Proof. From (11.6), we have

$$\frac{4(\psi\cdot f)'(t)}{3t} = \int_0^\infty \sin(t\xi)\widehat{v_{1,1}}(\xi)\xi^2 d\xi, \quad t \geqslant 0$$

Then, we consider $r \leqslant \frac{t}{2}$, and have

$$\begin{aligned} v_1(t,r) &= \int_0^\infty \sin(t\xi)J_2(r\xi)\widehat{v_{1,1}}(\xi)d\xi = \frac{r^2}{6\pi}\int_0^\pi \sin^4(\theta)\int_0^\infty \xi^2\left(\sin(\xi t_+) + \sin(\xi t_-)\right)\widehat{v_{1,1}}(\xi)d\xi d\theta \\ &= \frac{2r^2}{9\pi}\int_0^\pi \sin^4(\theta)\left(\frac{(\psi\cdot f)'(t_+)}{t_+} + \frac{(\psi\cdot f)'(t_-)}{t_-}\right)d\theta \end{aligned}$$

where

$$t_{\pm} = t \pm r\cos(\theta) \geqslant \frac{t}{2}, \text{ for } r \leqslant \frac{t}{2}$$

So,

$$|v_1(t,r)| \leqslant \frac{Cr^2}{t^2\log^b(t)}, \quad r \leqslant \frac{t}{2}$$

With the same procedure, we get, for $0 \leqslant j,k \leqslant 2$

$$|\partial_r^k \partial_t^j v_1(t,r)| \leqslant C\frac{r^{2-k}}{t^{2+j}\log^b(t)}, \quad r \leqslant \frac{t}{2}$$

Because of the singularity of $\widehat{v_{1,1}}(\xi)$ for small ξ, v_1 does not decay like $\frac{1}{\sqrt{r}}$ for large r. On the other hand, its derivatives do decay like $\frac{1}{\sqrt{r}}$ near the cone, because of their improved low frequency behavior. Using the same procedure as in [26], we get (11.10).

To establish (11.11) in the region $r \geqslant \frac{t}{2}$, we first use the same argument as in [26] to prove

$$|\partial_t^j \partial_r^k v_1(t,r)| \leqslant \frac{C}{\sqrt{r}}, \quad r \geqslant \frac{t}{2}, \quad 1 \leqslant j+k \leqslant 2 \tag{11.12}$$

Then, we get

$$\partial_t v_1(t,r) = \int_0^\infty J_2(r\xi)\xi \cos(t\xi)\widehat{v}_{1,1}(\xi) d\xi$$

Using $|J_2(x)| \leqslant Cx^2, \quad x \leqslant 1$, as well as the large x asymptotics of $J_2(x)$, we get

$$\partial_t v_1(t,r) = \text{Err}(t,r) + \frac{F(t-r)}{\sqrt{r}}$$

with

$$|\text{Err}(t,r)| \leqslant \frac{C}{r\log^b(r)}, \quad r \geqslant \frac{t}{2}$$

and

$$F(x) = \frac{-1}{2\sqrt{\pi}} \int_0^\infty \sqrt{\xi}\widehat{v_{1,1}}(\xi)\left(\cos(x\xi) + \sin(x\xi)\right) d\xi$$

Then, inserting $1 = \chi_{\leqslant 1}(\omega^2(t-r)) + 1 - \chi_{\leqslant 1}(\omega^2(t-r))$ into the integral below, where

$$\chi_{\leqslant 1}(x) \in C^\infty(\mathbb{R}), \quad 0 \leqslant \chi_{\leqslant 1}(x) \leqslant 1, \quad \chi_{\leqslant 1}(x) = \begin{cases} 1, & x \leqslant \frac{1}{2} \\ 0, & x \geqslant 1 \end{cases} \tag{11.13}$$

we get

$$\begin{aligned} |\int_0^\infty \cos(|t-r|\xi)\sqrt{\xi}\widehat{v_{1,1}}(\xi)d\xi| &= |\int_{-\infty}^\infty \cos(|t-r|\omega^2)\widehat{v_{1,1}}(\omega^2)\omega^2 d\omega| \\ &\leqslant \frac{C}{\sqrt{|t-r|}\log^b(|t-r|)}, |t-r| \geqslant 50 \end{aligned}$$

We then use the above result, along with (11.12) in the region $|r-t| < 50$. The $\sin$ term in the expression for F, and the other derivatives of v_1 are treated similarly. □

11.3 Estimates on v_2, the first iterate

v_2 is defined as the solution to

$$-\partial_{tt}v_2 + \partial_{rr}v_2 + \frac{1}{r}\partial_r v_2 - \frac{4}{r^2}v_2 = \frac{6Q_{\frac{1}{\lambda(t)}}(r)}{r^2}v_1^2 + \frac{2}{r^2}v_1^3 := RHS_2(t,r)$$

with 0 Cauchy data at infinity. Now, we record estimates on RHS_2, and its various derivatives.

Lemma 11.4. *For* $0 \leqslant j, k \leqslant 2$,

$$|\partial_t^j \partial_r^k RHS_2(t,r)| \leqslant \frac{Cr^{2-k}}{t^{4+j}\log^{2b}(t)}, \quad r \leqslant \frac{t}{2}$$

$$|RHS_2(t,r)| \leqslant \frac{C}{r^2\log^{2b}(r)}, \quad r \geqslant \frac{t}{2}$$

$$|\partial_t RHS_2(t,r)| + |\partial_r RHS_2(t,r)| \leqslant \frac{C}{r^{5/2}\sqrt{\langle t-r\rangle}\log^b(r)\log^b(\langle t-r\rangle)}, \quad r \geqslant \frac{t}{2}$$

$$|\partial_{tr} RHS_2(t,r)| + |\partial_t^2 RHS_2(t,r)| + |\partial_r^2 RHS_2(t,r)| \leqslant \frac{C}{r^{5/2}\log^b(r)\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)}, \; r \geqslant \frac{t}{2}$$

If $3 \leqslant j+k$ *and* $0 \leqslant j, k \leqslant 2$, *then,*

$$|\partial_t^j \partial_r^k RHS_2(t,r)| \leqslant \frac{C}{r^{5/2}\langle t-r\rangle^{\frac{1}{2}+j+k-1}\log^{2b}(\langle t-r\rangle)}, \quad r \geqslant \frac{t}{2}$$

Let $s \geqslant s_0 \geqslant T_0$, *and* $\frac{s_0}{2} \leqslant r_0 < s_0$. *Then,*

$$\begin{aligned}&||\left(\partial_r + \frac{2}{r}\right)\partial_s RHS_2(s,r)\mathbb{1}_{\leqslant 0}(r-(s-s_0+r_0))||_{L^2(rdr)}\\ &+ ||\partial_s^2 RHS_2(s,r)\mathbb{1}_{\leqslant 0}(r-(s-s_0+r_0))||_{L^2(rdr)}\\ &\leqslant \frac{C}{s^2\langle s_0-r_0\rangle\log^b(s)\log^b(\langle s_0-r_0\rangle)}\end{aligned} \tag{11.14}$$

Proof. The estimates in the lemma follow from elementary manipulations using Lemma 11.3. The only important features to note are the following. Note that, although the expression for (for instance) $\partial_{tr} RHS_2$ includes a term involving $\partial_t v_1 \partial_r v_1$, and estimates for both $\partial_t v_1$ and $\partial_r v_1$ only have a factor of $\frac{1}{\log^b(\langle t-r\rangle)}$, as opposed to a factor of $\frac{1}{\log^b(r)}$, we still obtain the stated estimates above. This is because

$$\frac{1}{\sqrt{r}\log^b(\langle t-r\rangle)} \leqslant \frac{C}{\log^b(r)\sqrt{\langle t-r\rangle}}, \quad r > \frac{t}{2}$$

which can be proven by noting that

$$x \mapsto \frac{\sqrt{x}}{\log^b(x)} \text{ is increasing for } x > e^{2b}$$

and

$$\frac{1}{r} \leqslant \frac{1}{|t-r|}, \quad r > \frac{t}{2}$$

In addition, we remark that, for any $a > 0$, there exists $C_a > 0$ such that

$$\lambda(t) \leqslant C_a t^a \tag{11.15}$$

(which follows from $\frac{|\lambda'(t)|}{\lambda(t)} \leqslant \frac{C}{t\log^b(t)}$ and $b > \frac{2}{3}$). This is used (for some fixed, sufficiently small $a > 0$) to estimate some terms involving t derivatives of $Q_{\frac{1}{\lambda(t)}}(r)$. □

We note one more useful estimate. By the definition of v_2, and L^2 isometry property of the Hankel transform of order 2, we have

$$\begin{aligned}|v_2(t,r)| &= |\int_t^\infty \int_0^\infty \sin((t-s)\xi)\widehat{RHS_2}(s,\xi)J_2(r\xi)|d\xi ds \\ &\leqslant C\int_t^\infty \left(\int_0^{\frac{1}{r}} r^2\xi^2|\widehat{RHS_2}(s,\xi)|d\xi + \int_{\frac{1}{r}}^\infty \frac{|\widehat{RHS_2}(s,\xi)|}{\sqrt{r\xi}}d\xi\right)ds \\ &\leqslant C\int_t^\infty \Bigg(||RHS_2(s)||_{L^2(rdr)}r^2\left(\int_0^{\frac{1}{r}}\xi^3 d\xi\right)^{1/2} \\ &\qquad + \frac{C}{\sqrt{r}}||RHS_2(s)||_{L^2(rdr)}\left(\int_{\frac{1}{r}}^\infty \frac{d\xi}{\xi^2}\right)^{1/2}\Bigg)ds \\ &\leqslant C\int_t^\infty ||RHS_2(s)||_{L^2(rdr)}ds\end{aligned} \tag{11.16}$$

Then, we use an 8 step procedure to estimate all quantities related to v_2:

Step 1: We use the fact that, if $v_2 = r^2 w_2$, then, w_2 solves the 6-dimensional free wave equation, with 0 Cauchy data at infinity, and $\frac{RHS_2(t,r)}{r^2}$ on the right-hand side. We then estimate v_2 in the region $r \leqslant \frac{t}{2}$ by using Duhamel's principle, and the spherical means formula for w_2.

Step 2: To estimate $\partial_r v_2$ in the region $r \leqslant \frac{t}{2}$, we first use the fact that, if $ru_2 := \left(\partial_r + \frac{2}{r}\right)v_2$, then, u_2 solves

$$-\partial_{tt}u_2 + \partial_{rr}u_2 + \frac{3}{r}\partial_r u_2 = \frac{\left(\partial_r + \frac{2}{r}\right)RHS_2(t,r)}{r}$$

with 0 Cauchy data at infinity. Then, we use the spherical means formula for u_2.

Step 3: We estimate $\partial_r^2 v_2$ in the region $r \leqslant \frac{t}{2}$ using the fact that, if $z_2 := \left(\partial_r + \frac{1}{r}\right)\left(\partial_r + \frac{2}{r}\right)v_2$, then z_2 solves

$$-\partial_{tt}z_2 + \partial_{rr}z_2 + \frac{1}{r}\partial_r z_2 = \left(\partial_r + \frac{1}{r}\right)\left(\partial_r + \frac{2}{r}\right)RHS_2(t,r)$$

with 0 Cauchy data at infinity, and using the spherical means formula for z_2.

Step 4: Differentiating the formulae for v_2, ru_2, and z_2 with respect to t, we show that, for $j = 1, 2$, $\partial_t^j v_2$ solves the same equation as v_2, except with $\partial_t^j RHS_2$ on the right-hand side, and zero Cauchy data at infinity. Then, we use the same procedure as in steps 1-3 to obtain estimates on all remaining derivatives of v_2 of the form $\partial_t^j \partial_r^k v_2$ in the region $r \leqslant \frac{t}{2}$, for $0 \leqslant j, k \leqslant 2$.

Step 5: Next, we estimate $\left(\partial_r + \frac{2}{r}\right)v_2$ in the region $r \geqslant \frac{t}{2}$, using a slightly different representation formula than what was used in Step 2. Using the Fundamental Theorem of Calculus, we then estimate v_2 in the region $r \geqslant \frac{t}{2}$.

Step 6: Similarly, we estimate $\partial_t v_2$ in the region $r \geqslant \frac{t}{2}$, using the fact that it solves the same equation as v_2, except with $\partial_t RHS_2$ on the right-hand side.

Step 7: We estimate $\partial_t^2 v_2$ and $\partial_{tr} v_2$ in the region $t > r \geqslant \frac{t}{2}$ by using a procedure based on (11.16), which also takes advantage of the finite speed of propagation. Then, we estimate $||\partial_{tt} v_2(t,\cdot)||_{L^\infty_r(\{r \geqslant \frac{t}{2}\})}$ by using the fact that $\partial_{tt} v_2$ solves the same equation as v_2, with 0 Cauchy data at infinity, except with $\partial_t^2 RHS_2$ on the right-hand side. We estimate $||\partial_{tr} v_2(t,\cdot)||_{L^\infty_r(\{r \geqslant \frac{t}{2}\})}$ similarly. Finally, we use the equation solved by v_2 to read off estimates on $\partial_r^2 v_2$ in the region $t > r \geqslant \frac{t}{2}$, and to estimate $||\partial_r^2 v_2(t,\cdot)||_{L^\infty_r(\{r \geqslant \frac{t}{2}\})}$

Step 8: We estimate $\partial_{ttr} v_2$ in the region $t > r \geqslant \frac{t}{2}$ by using the same procedure as for $\partial_{tr} v_2$. Then, we estimate $\partial_{trr} v_2$ and $\partial_{ttrr} v_2$ in the region $t > r \geqslant \frac{t}{2}$, by using the same representation formulae for $\left(\partial_r + \frac{1}{r}\right)\left(\partial_r + \frac{2}{r}\right)\partial_t^j v_2$ (for $j = 1, 2$) as was used in step 4.

Lemma 11.5. *We have the following estimates on v_2. For $0 \leqslant j, k \leqslant 2$,*

$$|\partial_t^j \partial_r^k v_2(t,r)| \leqslant C \frac{r^{2-k}}{t^{2+j} \log^{2b}(t)}, \quad r \leqslant \frac{t}{2} \tag{11.17}$$

$$|v_2(t,r)| \leqslant \frac{C}{\log^{2b}(t)}, \quad r > \frac{t}{2} \tag{11.18}$$

$$|\partial_t v_2(t,r)| + |\partial_r v_2(t,r)| \leqslant \frac{C}{r \log^{2b}(r)}, \quad r > \frac{t}{2}$$

$$|\partial_{tr} v_2(t,r)| + |\partial_r^2 v_2(t,r)| + |\partial_t^2 v_2(t,r)| \leqslant \frac{C}{t\langle t-r\rangle \log^b(t) \log^b(\langle t-r\rangle)}, \quad t > r > \frac{t}{2}$$

$$||\partial_{tr} v_2(t,r)||_{L^\infty_r(\{r \geqslant \frac{t}{2}\})} + ||\partial_t^2 v_2(t,r)||_{L^\infty_r(\{r \geqslant \frac{t}{2}\})} + ||\partial_r^2 v_2(t,r)||_{L^\infty_r(\{r \geqslant \frac{t}{2}\})} \leqslant \frac{C}{t^{3/2} \log^b(t)},$$

For $j + k \geqslant 3$ and $0 \leqslant j, k \leqslant 2$,

$$|\partial_t^j \partial_r^k v_2(t,r)| \leqslant \frac{C}{\sqrt{t}\langle t-r\rangle^{\frac{1}{2}+j+k-1} \log^{2b}(\langle t-r\rangle)}, \quad t > r > \frac{t}{2} \tag{11.19}$$

Proof. We start with the procedure outlined in step 1. To ease notation, we write $x = r\mathbf{e}_1 \in \mathbb{R}^6$, and let

$$y = \rho(\cos(\phi), \sin(\phi)\cos(\phi_2), \dots, \sin(\phi)\sin(\phi_2)\cdot \dots \cdot \sin(\phi_5)) \in \mathbb{R}^6$$

Then,

$$|v_2(t,r)| \leqslant Cr^2 \int_t^\infty \frac{1}{(s-t)^4} \int_0^{s-t} \frac{\rho^5}{\sqrt{(s-t)^2 - \rho^2}} \int_0^\pi \text{integrand } d\phi d\rho ds$$

where

$$\text{integrand} = \frac{\sin^4(\phi)}{|x+y|^2}\left(|RHS_2(s,|x+y|)|\left(1 + \frac{\rho^2}{|x+y|^2}\right)\right.$$
$$\left. + |\partial_2 RHS_2(s,|x+y|)|\rho\left(1 + \frac{\rho}{|x+y|}\right) + |\partial_2^2 RHS_2(s,|x+y|)|\rho^2\right)$$

and we note that

$$|x+y| = \sqrt{\rho^2+r^2+2r\rho\cos(\phi)}$$

By using Cauchy's residue theorem appropriately, we get, for all $\rho \neq r$,

$$\int_0^{\pi} \frac{\sin^4(\phi)d\phi}{\rho^2+r^2+2r\rho\cos(\phi)} = \frac{1}{2}\int_0^{2\pi} \frac{\sin^4(\phi)d\phi}{\rho^2+r^2+2r\rho\cos(\phi)} = \frac{-\pi\left(\min\{r,\rho\}^2 - 3\max\{r,\rho\}^2\right)}{8(\max\{\rho,r\})^4} \tag{11.20}$$

Then, we use the above estimates for RHS_2, and carry out step 1, to get (the $r \leqslant \frac{t}{2}$ part of) (11.18). Carrying out step 2, we first get

$$|\left(\partial_r + \frac{2}{r}\right) v_2| \leqslant Cr \int_t^{\infty} \frac{1}{(s-t)^2} \int_0^{s-t} \frac{\rho^3}{\sqrt{(s-t)^2-\rho^2}} \int_0^{\pi} \sin^2(\phi) \text{ integrand}_2 \, d\phi d\rho ds$$

where

$$\text{integrand}_2 = \left(\frac{|\partial_2^2 RHS_2(s,|x+y|)|\rho}{|x+y|} + \frac{|\partial_2 RHS_2(s,|x+y|)|}{|x+y|}\left(1+\frac{\rho}{|x+y|}\right) + \frac{|RHS_2(s,|x+y|)|}{|x+y|^2}\left(1+\frac{\rho}{|x+y|}\right)\right)$$

We then use (11.20) to get

$$\int_0^{\pi} \frac{\rho\sin^2(\phi)d\phi}{\sqrt{\rho^2+r^2+2r\rho\cos(\phi)}} \leqslant C\int_0^{\pi}\left(1+\frac{\sin^4(\phi)\rho^2}{\rho^2+r^2+2r\rho\cos(\phi)}\right)d\phi \leqslant C$$

which gives

$$|\left(\partial_r + \frac{2}{r}\right) v_2(t,r)| \leqslant \frac{Cr}{t^2\log^{2b}(t)}, \quad r \leqslant \frac{t}{2}$$

Step 3 is similar. Step 4 needs no further explanation. For step 5, we first consider the case $r > 2t$, and argue as in [26] to get a representation formula for $p_2(t,r) := \left(\partial_r + \frac{2}{r}\right) v_2(t,r)$ which does not involve any derivatives of $\left(\partial_r + \frac{2}{r}\right) RHS_2$:

$$p_2(t,r) = \frac{-1}{2\pi}\int_t^{\infty}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{\left(\partial_2 + \frac{2}{|x+y|}\right) RHS_2(s,|x+y|)}{|x+y|} \left(\widehat{x}\cdot(x+y)\right) d\theta d\rho ds$$

where we now regard $x = r\mathbf{e}_1 \in \mathbb{R}^2$, $y = (\rho\cos(\theta), \rho\sin(\theta))$, and we have

$$|x+y| = \sqrt{r^2+\rho^2+2r\rho\cos(\theta)}$$

Then, we treat several pieces of p_2 separately. If $p_{2,I}$ is defined by

$$p_{2,I}(t,r) := \frac{-1}{2\pi}\int_t^{\infty}\int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{\left(\partial_2 + \frac{2}{|x+y|}\right) RHS_2(s,|x+y|)}{|x+y|} \cdot \left(\widehat{x}\cdot(x+y)\right) \mathbb{1}_{\{|x+y| \leqslant \frac{s}{2}\}} d\theta d\rho ds$$

Then, we first note that the integrand in the definition of $p_{2,I}$ vanishes unless $s \geqslant \frac{2}{3}(t+r)$: If $s < \frac{2}{3}(t+r)$ and $|x+y| \leqslant \frac{s}{2}$, then, $\rho = |y| \geqslant |x| - |x+y| \geqslant r - \frac{s}{2} \geqslant \frac{-t+2r}{3} > s-t$. On the other hand, the ρ integration is constrained to the region $\rho \leqslant s-t$. So,

$$|p_{2,I}(t,r)| \leqslant C \int_{\frac{2}{3}(t+r)}^{\infty} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{1}{s^3 \log^{2b}(s)} d\theta d\rho ds \leqslant \frac{C}{r \log^{2b}(r)}$$

Then, we let $p_{2,II} = p_2 - p_{2,I}$. We have $p_{2,II} = p_{2,II,a} + p_{2,II,b} + p_{2,II,c}$, where each term will be defined shortly.

$$\begin{aligned} &p_{2,II,a}(t,r) \\ &= \frac{-1}{2\pi} \int_t^{\infty} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{\left(\partial_2 + \frac{2}{|x+y|}\right) RHS_2(s,|x+y|)}{|x+y|} \\ &\qquad \cdot (\widehat{x} \cdot (x+y)) \mathbb{1}_{\{s-t-r>|x+y|>\frac{s}{2}\}} d\theta d\rho ds \end{aligned}$$

For $|x+y|$ in the support of the characteristic functions appearing in the equation above,

$$|s - |x+y|| = s - |x+y| \geqslant s - (s-t-r) = t+r$$

Finally, the integrand vanishes unless $s > 2(t+r)$. Then, we use the estimates on RHS_2 to get

$$|p_{2,II,a}(t,r)| \leqslant C \int_{2(t+r)}^{\infty} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{1}{s^{5/2}\sqrt{r} \log^{2b}(r)} d\theta d\rho ds \leqslant \frac{C}{r \log^{2b}(r)}$$

Next,

$$\begin{aligned} &p_{2,II,b}(t,r) \\ &= \frac{-1}{2\pi} \int_t^{t+\frac{r}{8}} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{(\widehat{x} \cdot (x+y)) \left(\partial_2 + \frac{2}{|x+y|}\right) RHS_2(s,|x+y|)}{|x+y|} \\ &\qquad \cdot \mathbb{1}_{\{|x+y|>\frac{s}{2}\}} \mathbb{1}_{\{|x+y| \geqslant s-t-r\}} d\theta d\rho ds \end{aligned}$$

This time, for $|x+y|$ in the support of the characteristic functions appearing in the equation above, we have

$$|s - |x+y|| = |x+y| - s \geqslant |x| - |y| - s \geqslant r - (s-t) - s \geqslant r - 2t - \frac{r}{4} + t \geqslant \frac{r}{4} + \frac{r}{2} - t \geqslant \frac{r}{4}$$

Also, $|x+y| \geqslant |x| - |y| \geqslant r - (s-t) \geqslant \frac{7r}{8}$.

$$|p_{2,II,b}(t,r)| \leqslant C \int_t^{t+\frac{r}{8}} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{1}{r^{5/2}\sqrt{r} \log^{2b}(r)} d\theta d\rho ds \leqslant \frac{C}{r \log^{2b}(r)}$$

Finally, we define

$$\begin{aligned}
&p_{2,II,c}(t,r)\\
&= \frac{-1}{2\pi}\int_{t+\frac{r}{8}}^{\infty}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{\left(\partial_2+\frac{2}{|x+y|}\right)RHS_2(s,|x+y|)}{|x+y|}\left(\widehat{x}\cdot(x+y)\right)\\
&\qquad\cdot\mathbb{1}_{\{|x+y|>\frac{s}{2}\}}\mathbb{1}_{\{|x+y|\geqslant s-t-r\}}d\theta d\rho ds\\
&= \frac{-1}{2\pi}\int_{t+\frac{r}{8}}^{\infty}\int_{B_{s-t}(0)\cap(B_{\frac{s}{2}}(-x))^c\cap(B_{s-t-r}(-x))^c}\frac{1}{\sqrt{(s-t)^2-|y|^2}}\frac{\left(\partial_2+\frac{2}{|x+y|}\right)RHS_2(s,|x+y|)}{|x+y|}\\
&\qquad\cdot\left(\widehat{x}\cdot(x+y)\right)dA(y)ds
\end{aligned}$$

We will prove estimates on $p_{2,II,c}$ which are valid for any $r\geqslant\frac{t}{2}$ for later use, even though we assumed $r>2t$ in the very beginning of this argument. Note that the intersection of the balls in the integral is empty, unless $s\geqslant 2(t-r)$, since $\frac{s}{2}\leqslant|x+y|\leqslant s-t+r$. Then, we use polar coordinates centered at x. More precisely, we write $z=y+x=(\rho\cos(\theta),\rho\sin(\theta))$. The integrand of the s integral above is then bounded above in absolute value by

$$C\int_{|r-(s-t)|}^{s-t+r}\rho\int_0^{\theta^*}\frac{|\left(\partial_2+\frac{2}{\rho}\right)RHS_2(s,\rho)|\mathbb{1}_{\{\rho>\max\left(\frac{s}{2},s-t-r\right)\}}}{\sqrt{(s-t)^2-r^2-\rho^2+2r\rho\cos(\theta)}}d\theta d\rho$$

where

$$\theta^*=\arccos\left(\frac{\rho^2+r^2-(s-t)^2}{2r\rho}\right)$$

To get this, we first used the inequality $\frac{|(\widehat{x}\cdot(x+y))|}{|x+y|}\leqslant 1$. Then, the only θ-dependent term remaining in the integrand of the s integral in the expression for $p_{2,II,c}$ is

$$\frac{1}{\sqrt{(s-t)^2-|z-x|^2}}=\frac{1}{\sqrt{(s-t)^2-\rho^2-r^2+2r\rho\cos(\theta)}}$$

Next, we used the facts that $\cos(2\pi-\theta)=\cos(\theta)$, and the integrand is supported in the region $\rho>s-t-r$. Note also that θ^* is well-defined, for all ρ in the region of integration in the expression above, and $0\leqslant\theta^*\leqslant\pi$. Then, we note

$$\begin{aligned}
\int_0^{\theta^*}\frac{d\theta}{\sqrt{(s-t)^2-r^2-\rho^2+2r\rho\cos(\theta)}}&=\frac{1}{\sqrt{2r\rho}}\int_0^{\theta^*}\frac{d\delta}{\sqrt{\cos(\theta^*)\left(\cos(\delta)-1\right)+\sin(\theta^*)\sin(\delta)}}\\
&=\frac{1}{\sqrt{2r\rho}}f(\theta^*)
\end{aligned}$$

where we made the substitution $\delta=\theta^*-\theta$ in the first integral. We then have

$$|f(\theta^*)|\leqslant C\langle\log(\pi-\theta^*)\rangle\leqslant\frac{C}{\sqrt{\pi-\theta^*}}$$

(Although the singularity of f as θ^* approaches π is much better than $\frac{1}{\sqrt{\pi-\theta^*}}$, the above inequality suffices for our purposes, and slightly simplifies some of our estimates). Using

$$\pi-\theta^*=\arccos\left(1+\frac{(s-t)^2-(\rho+r)^2}{2\rho r}\right)$$

we get

$$|\int_0^{\theta*} \frac{d\theta}{\sqrt{(s-t)^2-r^2-\rho^2+2r\rho\cos(\theta)}}| \leqslant \frac{C}{(r\rho)^{1/4}(\rho+r-(s-t))^{1/4}(\rho+r+s-t)^{1/4}}$$

So, we have

$$\begin{aligned}&|p_{2,II,c}(t,r)|\\ &\leqslant C\int_{\max\{t+\frac{r}{8},2(t-r)\}}^{\infty}\int_{\max\{s-t-r,\frac{s}{2}\}}^{s-t+r}\rho|\left(\partial_2+\frac{2}{\rho}\right)RHS_2(s,\rho)|\\ &\qquad\int_0^{\theta*}\frac{1}{\sqrt{(s-t)^2-\rho^2-r^2+2r\rho\cos(\theta)}}d\theta d\rho ds\end{aligned}$$

and our above estimates give

$$|p_{2,II,c}(t,r)| \leqslant \frac{C}{r^{1/4}}\int_{t+\frac{r}{8}}^{\infty}\int_{s-t-r}^{s-t+r}\frac{\sqrt{\rho}}{(\rho+r-(s-t))^{1/4}}\frac{\mathbb{1}_{\{\rho\geqslant\frac{s}{2}\}}d\rho ds}{s^{5/2}\sqrt{\langle s-\rho\rangle}\log^b(\rho)\log^b(\langle s-\rho\rangle)} \tag{11.21}$$

Let

$$p_{2,II,c,i}(t,r) := \frac{1}{r^{1/4}}\int_{t+r}^{\infty}\int_{s-t-r}^{s-t+r}\frac{\sqrt{\rho}}{(\rho+r-(s-t))^{1/4}}\frac{\mathbb{1}_{\{\rho\geqslant\frac{s}{2}\}}d\rho ds}{s^{5/2}\sqrt{\langle s-\rho\rangle}\log^b(\rho)\log^b(\langle s-\rho\rangle)}$$

and

$$p_{2,II,c,ii}(t,r) := \frac{1}{r^{1/4}}\int_{t+\frac{r}{8}}^{t+r}\int_{s-t-r}^{s-t+r}\frac{\sqrt{\rho}}{(\rho+r-(s-t))^{1/4}}\frac{\mathbb{1}_{\{\rho\geqslant\frac{s}{2}\}}d\rho ds}{s^{5/2}\sqrt{\langle s-\rho\rangle}\log^b(\rho)\log^b(\langle s-\rho\rangle)}$$

Note that, in the expression for $p_{2,II,c,i}$, $s \geqslant t+r$, so that $s-t-r > 0$. Then, we consider separately two regions of the ρ integration. In the region $s-t-r \leqslant \rho \leqslant s-t-\frac{r}{2}$, we have

$$|s-\rho| = s-\rho \geqslant s-(s-t-\frac{r}{2}) = t+\frac{r}{2}$$

In the region $s-t-\frac{r}{2} \leqslant \rho \leqslant s-t+r$, we have

$$\frac{1}{(p-(s-t-r))^{1/4}} \leqslant \frac{C}{r^{1/4}}$$

So,

$$\begin{aligned}|p_{2,II,c,i}(t,r)| &\leqslant \frac{C}{r^{1/4}}\int_{t+r}^{\infty}\int_{s-t-r}^{s-t-\frac{r}{2}}\frac{\sqrt{\rho}}{(\rho+r-(s-t))^{1/4}}\frac{d\rho ds}{s^{5/2}\sqrt{r}\log^{2b}(r)}\\ &+\frac{C}{r^{1/4}}\int_{t+r}^{\infty}\int_{s-t-\frac{r}{2}}^{s-t+r}\frac{\sqrt{\rho}d\rho ds}{r^{1/4}s^{5/2}\sqrt{\langle s-\rho\rangle}\log^b(s)\log^b(\langle s-\rho\rangle)}\end{aligned}$$

Then, we use

$$\int_{t-r}^{t+\frac{r}{2}}\frac{dx}{\sqrt{\langle x\rangle}\log^b(\langle x\rangle)} \leqslant C\frac{\sqrt{\langle t-r\rangle}}{\log^b(\langle t-r\rangle)}+C\frac{\sqrt{t+r}}{\log^b(t+r)}$$

along with

$$|t-r| \leqslant r, \quad r \geqslant \frac{t}{2}$$

to get

$$|p_{2,II,c,i}(t,r)| \leqslant \frac{C}{r\log^{2b}(r)}$$

On the other hand, in the expression for $p_{2,II,c,ii}$, $s < t+r$. So, we make use of the $\mathbb{1}_{\{\rho \geqslant \frac{s}{2}\}}$ in the integrand of (11.21). Also, in this case

$$(\rho + r - (s-t))^{1/4} \geqslant \rho^{1/4}$$

This gives

$$|p_{2,II,c,ii}(t,r)| \leqslant \frac{C}{r^{1/4}} \int_{t+\frac{r}{8}}^{t+r} \int_{\frac{s}{2}}^{s-t+r} \frac{\rho^{1/4} d\rho ds}{s^{5/2}\sqrt{\langle s-\rho\rangle} \log^b(\rho) \log^b(\langle s-\rho\rangle)}$$

which can be treated in the same way as we treated $p_{2,II,c,i}$. In the very beginning of this argument, we considered the region $r > 2t$. This was so that we could estimate $p_{2,II,b}$. If $\frac{t}{2} \leqslant r \leqslant 2t$, then, we instead decompose $p_{2,II}$ as

$$p_{2,II}(t,r) = p_{2,II,a}(t,r) + p_{2,II,d}(t,r)$$

We then estimate $p_{2,II,d}$ with the identical procedure used to estimate $p_{2,II,c}$. We obtain the same final estimate for $p_{2,II,d}$ as we did for $p_{2,II,c}$. Even though we only have $s \geqslant t$ in the integral defining $p_{2,II,d}$, as opposed to $s \geqslant t + \frac{r}{8}$ for $p_{2,II,c}$, the fact that $\frac{t}{2} \leqslant r \leqslant 2t$ ensures that we do indeed get the same final estimate for $p_{2,II,d}$. In total, we finally get

$$|p_2(t,r)| \leqslant \frac{C}{r\log^{2b}(r)}, \quad r \geqslant \frac{t}{2}$$

Then, we recover v_2 from p_2:

$$v_2(t,r) = \frac{1}{r^2}\int_0^r p_2(t,x)x^2 dx$$

If $r \geqslant \frac{t}{2}$, then,

$$|v_2(t,r)| \leqslant \frac{C}{r^2}\int_0^{\frac{t}{2}} \frac{x^3 dx}{t^2 \log^{2b}(t)} + \frac{C}{r^2}\int_{\frac{t}{2}}^r \frac{x dx}{\log^{2b}(x)}$$

and we finally get

$$|v_2(t,r)| \leqslant \frac{C}{\log^{2b}(t)}, \quad r \geqslant \frac{t}{2}$$

To similarly estimate $\partial_t v_2$, we first note that, if u solves

$$-\partial_{tt}u(t,r) + \partial_{rr}u(t,r) + \frac{1}{r}\partial_r u(t,r) - \frac{4}{r^2}u(t,r) = F(t,r), \quad t \geqslant T_0, \quad r > 0$$

and $w : [T_0,\infty) \times \mathbb{R}^2$ is defined by

$$w(t, r\cos(\theta), r\sin(\theta)) = u(t,r)\cos(2\theta), \quad r > 0$$

then, w solves

$$-\partial_{tt}w + \Delta_{\mathbb{R}^2}w = \left(2\frac{x_1^2}{|x|^2} - 1\right)F(t,|x|), \quad x \neq 0$$

Now, we apply this procedure to the case $u = \partial_t v_2$, and $F = \partial_t RHS_2$. Using $u(t,r) = w(t,r,0)$, we get

$$\partial_t v_2(t,r) = \frac{-1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \text{integrand}_{dtv2}\, d\theta d\rho ds \tag{11.22}$$

where

$$\text{integrand}_{dtv2} = \partial_1 RHS_2(s, \sqrt{r^2+\rho^2+2r\rho\cos(\theta)})\left(\frac{r^2+2r\rho\cos(\theta)+\rho^2(1-2\sin^2(\theta))}{r^2+2r\rho\cos(\theta)+\rho^2}\right)$$

Because our estimates on $\partial_t RHS_2$ are just as good (in fact, slightly better in the region $r \leqslant \frac{t}{2}$) as those for $\partial_r RHS_2$, we can repeat the same procedure used to estimate p_2, to get

$$|\partial_t v_2(t,r)| \leqslant \frac{C}{r\log^{2b}(r)}, \quad r \geqslant \frac{t}{2}$$

Using (11.14), the finite speed of propagation, as well as an appropriate analog of (11.16), we get

$$|\partial_t^2 v_2(t,r)| + |\partial_{tr} v_2(t,r)| \leqslant \frac{C}{t\langle t-r\rangle \log^b(t)\log^b(\langle t-r\rangle)}, \quad t > r > \frac{t}{2}$$

We then argue as we did for $\partial_t v_2$, to get

$$\partial_t^2 v_2(t,r) = \frac{-1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \text{integrand}_{dttv2}\, d\theta d\rho ds$$

where

$$\text{integrand}_{dttv2} = \partial_1^2 RHS_2(s, \sqrt{r^2+\rho^2+2r\rho\cos(\theta)})\left(\frac{r^2+2r\rho\cos(\theta)+\rho^2(1-2\sin^2(\theta))}{r^2+2r\rho\cos(\theta)+\rho^2}\right)$$

Then, we carry out the same procedure used to estimate p_2. The difference here is that we have an extra factor of $\frac{1}{\langle t-r\rangle}$ in the pointwise estimates for $\partial_t^2 RHS_2(t,r)$, relative to those for $\partial_r RHS_2(t,r)$. This leads to

$$|\partial_t^2 v_2(t,r)| \leqslant \frac{C}{r^{3/2}\log^b(t)}, \quad r > \frac{t}{2}$$

Also, if $w_2 = \left(\partial_r + \frac{2}{r}\right)\partial_t v_2$, then, w_2 solves

$$-\partial_{tt}w_2 + \partial_{rr}w_2 + \frac{1}{r}\partial_r w_2 - \frac{1}{r^2}w_2 = \left(\partial_r + \frac{2}{r}\right)\partial_t RHS_2(t,r)$$

with 0 Cauchy data at infinity. Using the analog of the p_2 representation formula, we can repeat the same argument used for $\partial_t^2 v_2$, and use the previous estimates on $\partial_t v_2$, to get

$$|\partial_{tr} v_2(t,r)| \leqslant \frac{C}{r^{3/2}\log^b(t)}, \quad r > \frac{t}{2}$$

The rest of step 7 and step 8 need no further explanation. □

11.4 Summation of the higher iterates, v_j

We now proceed to recursively define subsequent corrections, v_j. For $j \geqslant 3$, define RHS_j by

$$\begin{aligned} RHS_j(t,r) &= \frac{6Q_{\frac{1}{\lambda(t)}}(r)}{r^2}\left(\left(\sum_{k=1}^{j-1} v_k\right)^2 - \left(\sum_{k=1}^{j-2} v_k\right)^2\right) + \frac{2}{r^2}\left(\left(\sum_{k=1}^{j-1} v_k\right)^3 - \left(\sum_{k=1}^{j-2} v_k\right)^3\right) \\ &= \frac{6Q_{\frac{1}{\lambda(t)}}}{r^2}\left(2\sum_{k=1}^{j-2} v_k v_{j-1} + v_{j-1}^2\right) + \frac{2}{r^2}\left(3\left(\sum_{k=1}^{j-2} v_k\right)^2 v_{j-1} + 3\sum_{k=1}^{j-2} v_k v_{j-1}^2 + v_{j-1}^3\right) \end{aligned} \tag{11.23}$$

Then, we let v_j be the solution to the following equation, with 0 Cauchy data at infinity

$$-\partial_{tt} v_j + \partial_{rr} v_j + \frac{1}{r}\partial_r v_j - \frac{4}{r^2} v_j = RHS_j(t,r)$$

We proceed to prove estimates on v_j by induction. Let $C_1 > 9$ be such that (11.10), (11.11), and (11.17) through (11.19) hold, with the constant $C = C_1$ on the right-hand side. Let $n > 900$ be otherwise arbitrary, and let $T_{0,n} > e^{(900!)^{1+\frac{1}{b}}}$ satisfy

$$\frac{C_1^{90n}}{\log^b(T_{0,n})} < e^{-(900!)}$$

and be otherwise arbitrary. Our goal is to show that, for a sufficiently large $T_{0,n}$, we can prove estimates on v_j (and its derivatives), valid for all $t \geqslant T_{0,n} + \exp\left(900! + 2^{-\frac{3}{2(2b-1)}}\right)(1+T_{\lambda_0}) + M_1$ and all $r \geqslant 0$, by induction. In the following estimates, we assume

$$t \geqslant \exp\left(900! + 2^{-\frac{3}{2(2b-1)}}\right)(1+T_{\lambda_0}) + M_1 + T_{0,n}$$

Let

$$D_{n,k} = \begin{cases} C_1, & k=2 \\ C_1^{nk}, & k \geqslant 3 \end{cases}$$

Suppose, for any $j \geqslant 3$, and all k with $2 \leqslant k \leqslant j-1$, that

$$|\partial_t^p \partial_r^m v_k(t,r)| \leqslant \frac{D_{n,k} r^{2-m}}{t^{2+p}\log^{bk}(t)}, \quad r \leqslant \frac{t}{2}, \quad 0 \leqslant p,m \leqslant 2 \tag{11.24}$$

$$|v_k(t,r)| \leqslant \frac{D_{n,k}}{\log^{bk}(t)}, \quad r > \frac{t}{2}$$

$$|\partial_t v_k(t,r)| + |\partial_r v_k(t,r)| \leqslant \frac{D_{n,k}}{r\log^{bk}(t)}, \quad r > \frac{t}{2}$$

$$|\partial_{rr} v_k(t,r)| + |\partial_{tt} v_k(t,r)| + |\partial_{tr} v_k(t,r)| \leqslant \frac{D_{n,k}}{t\langle t-r\rangle \log^b(\langle t-r\rangle)\log^{b(k-1)}(t)}, \quad t > r > \frac{t}{2}$$

$$||\partial_{tr}v_k(t,r)||_{L^\infty_r\{r\geqslant\frac{t}{2}\}}+||\partial_{rr}v_k(t,r)||_{L^\infty_r\{r\geqslant\frac{t}{2}\}}+||\partial_{tt}v_k(t,r)||_{L^\infty_r\{r\geqslant\frac{t}{2}\}}\leqslant\frac{D_{n,k}}{t^{3/2}\log^{b(k-1)}(t)}$$

$$|\partial_{trr}v_k(t,r)|+|\partial_{ttr}v_k(t,r)|\leqslant\frac{D_{n,k}}{\sqrt{t}\langle t-r\rangle^{5/2}\log^{2b}(\langle t-r\rangle)\log^{b(k-2)}(t)},\quad t>r>\frac{t}{2}$$

and, for $t>r>\frac{t}{2}$,

$$|\partial_{ttrr}v_k(t,r)|\leqslant\begin{cases}\frac{D_{n,k}}{\sqrt{t}\langle t-r\rangle^{7/2}\log^{2b}(\langle t-r\rangle)}, & j=3,k=2\\ \begin{cases}\frac{D_{n,k}}{\sqrt{t}\langle t-r\rangle^{7/2}\log^{2b}(\langle t-r\rangle)}, & k=2\\ \frac{D_{n,k}}{\sqrt{t}\langle t-r\rangle^{7/2}\log^{3b}(\langle t-r\rangle)\log^{b(k-3)}(t)}, & 3\leqslant k\leqslant j-1\end{cases}, & j>3\end{cases}\tag{11.25}$$

Then, for some C *independent* of n (and t) we have

$$|\partial_t^p\partial_r^m RHS_j(t,r)|\leqslant C\left(C_1^{1-n}+\frac{C_1^n}{\log^b(t)}\right)\frac{C_1^{nj}r^{2-m}}{t^{4+p}\log^{bj}(t)},\quad r\leqslant\frac{t}{2}$$

$$|RHS_j(t,r)|\leqslant\frac{C_1^{nj}}{r^2\log^{bj}(t)},\quad r>\frac{t}{2}$$

$$\begin{aligned}&|\partial_r RHS_j(t,r)|+|\partial_t RHS_j(t,r)|\\&\leqslant C\left(\frac{C_1^{nj}}{r^3\log^{bj}(t)}\left(C_1^{1-n}+\frac{C_1^n}{\log^b(t)}\right)+\frac{C_1^{nj}C_1^{1-n}}{r^{5/2}\sqrt{\langle t-r\rangle}\log^{b(j-1)}(t)\log^b(\langle t-r\rangle)}\right),\quad r>\frac{t}{2}\end{aligned}$$

$$\begin{aligned}&|\partial_{tr}RHS_j(t,r)|+|\partial_r^2 RHS_j(t,r)|+|\partial_t^2 RHS_j(t,r)|\\&\leqslant\frac{CC_1^{nj}\left(C_1^{1-n}+\frac{C_1^n}{\log^b(t)}\right)}{\log^{b(j-1)}(t)\log^b(\langle t-r\rangle)r^{5/2}\langle t-r\rangle^{3/2}}+\frac{CC_1^{nj}\left(\frac{C_1^{2-n}}{\log^b(\langle t-r\rangle)}+\frac{C_1^{3n}}{\log^{2b}(t)\log^b(\langle t-r\rangle)}\right)}{\log^{b(j-1)}(t)r^3\langle t-r\rangle\log^b(\langle t-r\rangle)},\quad t>r>\frac{t}{2}\end{aligned}$$

We will also require another estimate on $\partial_t^2 RHS_j$ and $\partial_{tr}RHS_j$, which is valid for all $r>\frac{t}{2}$:

$$\begin{aligned}&|\partial_t^2 RHS_j(t,r)|+|\partial_{tr}RHS_j(t,r)|\\&\leqslant\frac{CC_1^{nj}C_1^{1-n}}{r^{5/2}\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)\log^{b(j-1)}(t)}+\frac{CC_1^{nj}}{r^2t^{3/2}\log^{b(j-1)}(t)}\left(\frac{C_1^n}{\log^b(t)}+\frac{C_1^{1-n}}{\log^b(t)}\right)\\&+\frac{CC_1^{nj}}{r^3\langle t-r\rangle}\frac{\left(C_1^{-n+2}+\frac{C_1^{3n}}{\log^{2b}(t)}\right)}{\log^{b(j-1)}(t)\log^{2b}(\langle t-r\rangle)},\quad r>\frac{t}{2}\end{aligned}$$

$$|\partial_{ttr}RHS_j(t,r)|+|\partial_{trr}RHS_j(t,r)|\leqslant\frac{CC_1^{nj}\left(C_1^{1-n}+\frac{C_1^n}{\log^b(t)}\right)}{r^{5/2}\log^{2b}(\langle t-r\rangle)\log^{b(j-2)}(t)\langle t-r\rangle^{5/2}},\quad t>r>\frac{t}{2}$$

$$|\partial_{ttrr}RHS_j(t,r)|\leqslant\frac{CC_1^{nj}}{r^{5/2}\langle t-r\rangle^{7/2}}\frac{\left(C_1^{1-n}+\frac{C_1^n}{\log^b(t)}\right)}{\log^{3b}(\langle t-r\rangle)\log^{b(j-1)}(t)},\quad t>r>\frac{t}{2}$$

Then, we repeat the analogs of steps 1-8 used to estimate v_2, and get (for C *independent* of n (and t)):

$$|\partial_t^p \partial_r^m v_j(t,r)| \leqslant C \frac{C_1^{nj} r^{2-m}}{t^{2+p} \log^{bj}(t)} \left(C_1^{1-n} + \frac{C_1^n}{\log^b(t)} \right), \quad r \leqslant \frac{t}{2}, \quad 0 \leqslant p,m \leqslant 2$$

$$|v_j(t,r)| \leqslant \frac{C C_1^{nj} \left(C_1^{1-n} + \frac{C_1^n}{\log^b(t)} \right)}{\log^{bj}(t)}, \quad r > \frac{t}{2}$$

$$|\partial_t v_j(t,r)| + |\partial_r v_j(t,r)| \leqslant \frac{C C_1^{nj}}{r \log^{bj}(t)} \left(C_1^{1-n} + \frac{C_1^n}{\log^b(t)} \right), \quad r > \frac{t}{2}$$

$$|\partial_t^2 v_j(t,r)| + |\partial_r^2 v_j(t,r)| + |\partial_{tr} v_j(t,r)| \leqslant \frac{C C_1^{nj}}{t \langle t-r \rangle \log^{b(j-1)}(t) \log^b(\langle t-r \rangle)} \left(C_1^{2-n} + \frac{C_1^{2n}}{\log^b(t)} \right)$$
$$, \quad t > r > \frac{t}{2}$$

$$||\partial_t^2 v_j(t,r)||_{L^\infty_r\{r > \frac{t}{2}\}} + ||\partial_{tr} v_j(t,r)||_{L^\infty_r\{r > \frac{t}{2}\}} + ||\partial_r^2 v_j(t,r)||_{L^\infty_r\{r > \frac{t}{2}\}}$$
$$\leqslant \frac{C C_1^{nj}}{t^{3/2} \log^{b(j-1)}(t)} \left(\frac{C_1^{2n}}{\log^b(t)} + C_1^{2-n} \right)$$

$$|\partial_{ttr} v_j(t,r)| + |\partial_{trr} v_j(t,r)| \leqslant \frac{C C_1^{nj} \left(C_1^{2-n} + \frac{C_1^{3n}}{\log^b(t)} \right)}{\sqrt{t} \langle t-r \rangle^{5/2} \log^{b(j-2)}(t) \log^{2b}(\langle t-r \rangle)}, \quad t > r > \frac{t}{2}$$

Finally,

$$|\partial_{ttrr} v_j(t,r)| \leqslant \frac{C C_1^{nj} \left(C_1^{2-n} + \frac{C_1^{3n}}{\log^b(t)} \right)}{\sqrt{t} \langle t-r \rangle^{7/2} \log^{b(j-3)}(t) \log^{3b}(\langle t-r \rangle)}, \quad t > r > \frac{t}{2}$$

Since C is independent of n, there exists n_0 such that

$$\max\{C,1\} C_1^{90-n_0} < e^{-(900!)}$$

Now, since C is also independent of T_{0,n_0}, we can choose T_{0,n_0} to satisfy, in addition to our previous constraints at the beginning of this argument, the following inequality

$$\max\{C,1\} \frac{C_1^{90n_0}}{\log^b(T_{0,n_0})} < e^{-(900!)}$$

By mathematical induction, the above results imply that (11.24) through (11.25) are true (with $n = n_0$) for all $k \geqslant 2$, and $t \geqslant T_{0,n_0} + \exp\left(900! + 2^{-\frac{3}{2(2b-1)}}\right)(1 + T_{\lambda_0}) + M_1 := T_1$. From here on, we further restrict T_0 to satisfy $T_0 \geqslant T_1$. So, for all $j \geqslant 2$,

$$||v_j(t,r)||_{L^\infty\{r \geqslant 0, t \geqslant T_1\}} \leqslant e^{-j(900!)}$$

$$||\partial_t v_j(t,r)||_{L^\infty\{r \geqslant 0, t \geqslant T_1\}} + ||\partial_r v_j(t,r)||_{L^\infty\{r \geqslant 0, t \geqslant T_1\}} \leqslant \frac{2}{T_1} e^{-j(900!)}$$

$$||\partial_{tr} v_j(t,r)||_{L^\infty\{r\geqslant 0, t\geqslant T_1\}} + ||\partial_t^2 v_j(t,r)||_{L^\infty\{r\geqslant 0, t\geqslant T_1\}} + ||\partial_r^2 v_j(t,r)||_{L^\infty\{r\geqslant 0, t\geqslant T_1\}} \leqslant e^{-(900!)(j-1)} T_1^{3/2}$$

Therefore, the series

$$v_s := \sum_{j=3}^{\infty} v_j$$

(as well as the series resulting from applying any first or second order derivative termwise) converges absolutely and uniformly on the set $\{(t,r) | t \geqslant T_1, r \geqslant 0\}$. Moreover, using the first line of (11.23), we get, for any $N \geqslant 4$,

$$\sum_{j=3}^{N} RHS_j(t,r) = \frac{6Q_{\frac{1}{\lambda(t)}}(r)}{r^2}\left(2v_1 \sum_{k=2}^{N-1} v_k + \left(\sum_{k=2}^{N-1} v_k\right)^2\right) + \frac{2}{r^2}\left(3v_1 \left(\sum_{k=2}^{N-1} v_k\right)^2 + 3v_1^2 \sum_{k=2}^{N-1} v_k + \left(\sum_{k=2}^{N-1} v_k\right)^3\right)$$

(where the argument (t,r) of all instances of v_k, v_1 has been omitted, for clarity). Using the uniformity of the convergence of the series defining v_s, we get

$$-\partial_{tt} v_s + \partial_{rr} v_s + \frac{1}{r}\partial_r v_s - \frac{4}{r^2} v_s = \lim_{N\to\infty} \sum_{j=3}^{N} \left(-\partial_{tt} v_j + \partial_{rr} v_j + \frac{1}{r}\partial_r v_j - \frac{4}{r^2} v_j\right)$$

$$= \lim_{N\to\infty} \sum_{j=3}^{N} RHS_j(t,r)$$

$$= \frac{6Q_{\frac{1}{\lambda(t)}}}{r^2}\left(2v_1(v_2+v_s) + (v_2+v_s)^2\right) + \frac{2}{r^2}\left(3v_1(v_2+v_s)^2 + 3v_1^2(v_2+v_s) + (v_2+v_s)^3\right)$$

11.5 Improvement of the large r behavior of the remaining error terms

Let $v_c = v_1 + v_2 + v_s$. Despite the major improvement of the decay of the error terms accomplished via the resummation of the v_k above, we will still need to improve the decay of the error terms which result from substituting $Q_{\frac{1}{\lambda(t)}} + v_c$ into (8.1). The soliton error term, and the linear error term resulting from v_c, namely

$$\partial_{tt} Q_{\frac{1}{\lambda(t)}} - \frac{6v_c}{r^2}\left(1 - Q_{\frac{1}{\lambda(t)}}^2(r)\right)$$

both contribute to leading order in the modulation equation. Therefore, any improvement of these error terms should not change their leading order inner product with the appropriately re-scaled zero eigenfunction ϕ_0. Keeping this in mind, we let

$$WRHS_2(t,r) := \chi_{\geqslant 1}\left(\frac{r}{g(t)}\right)\left(\partial_{tt} Q_{\frac{1}{\lambda(t)}} - \frac{6v_c}{r^2}\left(1 - Q_{\frac{1}{\lambda(t)}}^2(r)\right)\right), \quad \text{where } g(t) = \lambda(t)\log^{b-2\epsilon}(t)$$

where

$$\chi_{\geqslant 1}(x) \in C^\infty(\mathbb{R}), \quad 0 \leqslant \chi_{\geqslant 1}(x) \leqslant 1, \quad \chi_{\geqslant 1}(x) = \begin{cases} 0, & x \leqslant \frac{1}{2} \\ 1, & x \geqslant 1 \end{cases}$$

and define w_2 to be the solution to the following equation, with 0 Cauchy data at infinity

$$-\partial_{tt}w_2+\partial_{rr}w_2+\frac{1}{r}\partial_r w_2-\frac{4}{r^2}w_2=WRHS_2(t,r)$$

Now, we will prove estimates on w_2. Note that $WRHS_2$ depends on λ''. Because our only assumptions on λ do not include any more regularity than $\lambda\in C^2([T_0,\infty))$, we can not consider $\partial_t WRHS_2(t,r)$ until we first choose a specific λ by solving the modulation equation, and then prove that this λ has higher than C^2 regularity. In terms of estimating w_2, this means that we can not differentiate $WRHS_2$ in time, at this point. Therefore, we will first record a set of preliminary estimates on derivatives of w_2, which only involve $WRHS_2$, and its r derivatives. After choosing λ, we can then obtain more optimal, final estimates on the derivatives of w_2. This is very similar to the argument used in the wave maps paper of the author, [26].

Lemma 11.6. *[Preliminary estimates on w_2] We have the following* preliminary *estimates on w_2*

$$|w_2(t,r)|\leqslant\begin{cases}\frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{(g(t)^2+r^2)t^2\log^b(t)}, & r\leqslant\frac{t}{2}\\ \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)}, & r>\frac{t}{2}\end{cases}$$

$$|\partial_r w_2(t,r)|\leqslant\begin{cases}\frac{Cr\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2}, & r\leqslant g(t)\\ \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)}, & r>g(t)\end{cases}$$

$$|\partial_t w_2(t,r)|\leqslant\frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)},\quad r>0$$

$$|\partial_r^2 w_2(t,r)|\leqslant\frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2},\quad r>0$$

$$|\partial_t^2 w_2(t,r)|\leqslant\frac{C\lambda(t)^2\log(t)}{t^2g(t)^2\log^b(t)},\quad r>0$$

$$|\partial_{tr} w_2(t,r)|\leqslant\frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2}$$

Proof. We start with the region $r\leqslant\frac{t}{2}$. Using the same procedure and notation as in step 1 for v_2, as well as the estimates for $v_c(s,|x+y|)$ in the two regions $|x+y|\leqslant\frac{s}{2}$ and $|x+y|>\frac{s}{2}$, we get

$$\begin{aligned}|w_2(t,r)|\leqslant{}& Cr^2\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^\pi\sin^4(\phi)\left(\frac{\mathbb{1}_{\{|x+y|>\frac{g(s)}{2}\}}\mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}}\lambda(s)^2}{s^2\log^b(s)(g(s)^2+|x+y|^2)^2}\right)\\&\cdot\left(1+\frac{\rho^2}{|x+y|^2}\right)d\phi d\rho ds\\&+Cr^2\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\\&\int_0^\pi\sin^4(\phi)\left(\frac{\lambda(s)^2\mathbb{1}_{\{|x+y|>\frac{g(s)}{2}\}}\mathbb{1}_{\{|x+y|>\frac{s}{2}\}}\left(1+\frac{\rho^2}{|x+y|^2}\right)}{|x+y|^4\sqrt{s}\langle s-|x+y|\rangle^{3/2}\log^b(\langle s-|x+y|\rangle)}\right)d\phi d\rho ds\end{aligned}\tag{11.26}$$

Denote the first line of (11.26) by $w_{2,I}$. Then, we consider several pieces of $w_{2,I}$ separately. Let

$$w_{2,I,a}(t,r) := r^2 \int_t^{t+\frac{r}{8}} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{\pi} \sin^4(\phi) \left(\frac{\mathbb{1}_{\{|x+y|>\frac{g(s)}{2}\}} \mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}} \lambda(s)^2}{s^2 \log^b(s)(g(s)^2+|x+y|^2)^2} \right) \cdot \left(1+\frac{\rho^2}{|x+y|^2}\right) d\phi d\rho ds$$

Recall that $|x+y| = \sqrt{\rho^2+r^2+2r\rho\cos(\phi)}$. For $w_{2,I,a}$, we have

$$\rho \leqslant s-t \leqslant \frac{r}{8} \implies |x+y| \geqslant C(r+\rho)$$

Therefore,

$$\begin{aligned} |w_{2,I,a}(t,r)| &\leqslant C \int_t^{t+\frac{r}{8}} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{r^2\lambda(s)^2}{s^2\log^b(s)(g(s)^2+r^2)^2} d\rho ds \\ &\leqslant \frac{Cr^2\lambda(t)^2}{(g(t)^2+r^2)t^2\log^b(t)} \end{aligned}$$

where we used (11.5) to conclude that

$$s \mapsto \frac{1}{s(g(s)^2+r^2)^2} \text{ is decreasing on } [T_0,\infty) \tag{11.27}$$

For the next integrals to consider, we first appropriately use Cauchy's residue theorem to conclude

$$\int_0^{2\pi} \frac{\sin^4(\phi)d\phi}{(g(s)^2+\rho^2+r^2+2r\rho\cos(\phi))^2} \leqslant \frac{C}{(g(s)^2+\rho^2+r^2)^2}$$

$$\int_0^{2\pi} \frac{\sin^4(\phi)d\phi}{(g(s)^2+r^2+\rho^2+2r\rho\cos(\phi))^3} \leqslant \frac{C}{(g(s)^2+r^2+\rho^2)^{5/2}\sqrt{g(s)^2+(\rho-r)^2}}$$

The important point in the above integrals is that the factor $\sin^4(\phi)$ vanishes when $\phi=\pi$, which is precisely when the vectors $x = r\mathbf{e}_1$ and y (defined in step 1 used to estimate v_2) are antiparallel. Therefore, we get much more decay in $\rho^2+r^2+g(s)^2$ than we would have without the $\sin$ factor. Now, we consider $w_{2,I,b}$ defined by

$$\begin{aligned} &w_{2,I,b}(t,r) \\ &= r^2 \int_{t+\frac{r}{8}}^{t+\frac{r}{8}+g(t)} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{\pi} \sin^4(\phi) \left(\frac{\mathbb{1}_{\{|x+y|>\frac{g(s)}{2}\}} \mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}} \lambda(s)^2}{s^2 \log^b(s)(g(s)^2+|x+y|^2)^2} \right) \\ &\qquad \cdot \left(1+\frac{\rho^2}{|x+y|^2}\right) d\phi d\rho ds \end{aligned}$$

which gives

$$\begin{aligned} |w_{2,I,b}(t,r)| &\leqslant Cr^2 \int_{t+\frac{r}{8}}^{t+\frac{r}{8}+g(t)} \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \frac{\lambda(s)^2}{s^2\log^b(s)g(s)(g(s)^2+r^2)^{3/2}} d\rho ds \\ &\leqslant C \frac{r^2\lambda(t)^2}{t^2\log^b(t)(g(t)^2+r^2)} \end{aligned}$$

where we again use (11.5) to justify the analog of (11.27). Next, we consider

$$w_{2,I,c}(t,r) := r^2 \int_{t+\frac{r}{8}+g(t)}^{\infty} \frac{1}{(s-t)} \int_0^{s-t} \rho \int_0^{\pi} \sin^4(\phi) \left(\frac{\mathbb{1}_{\{|x+y|>\frac{g(s)}{2}\}} \mathbb{1}_{\{|x+y|\leqslant \frac{s}{2}\}} \lambda(s)^2}{s^2 \log^b(s)(g(s)^2+|x+y|^2)^2}\right) \cdot \left(1+\frac{\rho^2}{|x+y|^2}\right) d\phi d\rho ds$$

The point of this definition is to utilize the decay of the integrand in $s-t$ in order to do the s integral. As we will show later, the difference $\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}$ decays sufficiently fast in s so as to allow an argument which does the s integral first, before the ρ integral. A similar procedure was also used in the author's work regarding wave maps[26], in the estimation of the correction denoted by v_4 in that paper. We start by noting

$$\int_0^{s-t} \frac{\rho \mathbb{1}_{\{\rho \leqslant \frac{r}{2}\}}}{(g(s)^2+\rho^2+r^2)^{3/2}\sqrt{g(s)^2+(\rho-r)^2}} d\rho \leqslant C \int_0^{s-t} \frac{\rho d\rho}{(g(s)^2+r^2+\rho^2)^{3/2}\sqrt{g(s)^2+r^2}} \leqslant \frac{C}{g(s)^2+r^2}$$

$$\int_0^{s-t} \frac{\rho \mathbb{1}_{\{\frac{r}{2}\leqslant \rho \leqslant 2r\}}}{(g(s)^2+\rho^2+r^2)^{3/2}\sqrt{g(s)^2+(\rho-r)^2}} d\rho \leqslant \frac{Cr}{(g(s)^2+r^2)^{3/2}} \int_{\frac{r}{2}}^{2r} \frac{d\rho}{\sqrt{g(s)^2+(\rho-r)^2}} \leqslant C\frac{\log(1+\frac{r}{g(s)})}{g(s)^2+r^2}$$

$$\int_0^{s-t} \frac{\rho \mathbb{1}_{\{\rho>2r\}}}{(g(s)^2+\rho^2+r^2)^{3/2}\sqrt{g(s)^2+(\rho-r)^2}} d\rho \leqslant C \int_0^{s-t} \frac{\rho d\rho}{(g(s)^2+r^2+\rho^2)^2} \leqslant \frac{C}{g(s)^2+r^2}$$

which gives

$$|w_{2,I,c}(t,r)| \leqslant \frac{Cr^2\lambda(t)^2 \log(2+\frac{r}{g(t)})\log(t)}{t^2\log^b(t)(g(t)^2+r^2)}$$

Finally, we consider

$$w_{2,I,d}(t,r) := r^2 \int_{t+\frac{r}{8}+g(t)}^{\infty} \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \int_0^{\pi} \text{integrand}_{w_{2,I,d}} d\phi d\rho ds$$

where

$$\text{integrand}_{w_{2,I,d}} = \sin^4(\phi) \left(\frac{\mathbb{1}_{\{|x+y|>\frac{g(s)}{2}\}} \mathbb{1}_{\{|x+y|\leqslant \frac{s}{2}\}} \lambda(s)^2}{s^2\log^b(s)(g(s)^2+|x+y|^2)^2}\right) \left(1+\frac{\rho^2}{|x+y|^2}\right)$$

Using again an analog of the observation (11.27), and the ϕ integrals noted above, we first do the ϕ integral. Then, we switch the order of the s and ρ and ϕ integrals, to do the s integral first:

$$|w_{2,I,d}(t,r)| \leqslant Cr^2 \int_0^{\infty} \rho \int_{\rho+t}^{\infty} \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \cdot \frac{\lambda(t)^2}{\log^b(t) t^2 (g(t)^2+r^2+\rho^2)^{3/2}\sqrt{g(t)^2+(\rho-r)^2}} ds d\rho$$

Then, we use the same method used to study the ρ integrals appearing just above our final estimate for $w_{2,I,c}$ to get

$$|w_{2,I,d}(t,r)| \leqslant \frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})}{t^2\log^b(t)(g(t)^2+r^2)}$$

In total, we get

$$|w_{2,I}(t,r)| \leqslant \frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{t^2\log^b(t)(g(t)^2+r^2)}$$

It remains to treat the second line of (11.26), which we denote as $w_{2,II}$. If $r \leqslant \frac{t}{2}$, then, as in the case of estimating v_2, we use that $s-|x+y| \geqslant s-(r+(s-t)) \geqslant t-r \geqslant \frac{t}{2}$, and we use the estimates on $v_c(s,|x+y|)$ in the region $|x+y| \geqslant \frac{s}{2}$ to get

$$|w_{2,II}(t,r)| \leqslant Cr^2\int_t^\infty \frac{(s-t)\lambda(s)^2}{s^{9/2}t^{3/2}\log^b(t)}ds \leqslant \frac{Cr^2\lambda(t)^2}{t^4\log^b(t)}, \quad r \leqslant \frac{t}{2}$$

Finally, we need to estimate $w_2(t,r)$ for $r > \frac{t}{2}$. We use the analog of (11.22) to get

$$w_2(t,r) = \frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi} WRHS_2(s,|x+y|) \cdot\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)d\theta d\rho ds \tag{11.28}$$

Again, we insert

$$1 = \mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}} + \mathbb{1}_{\{|x+y|>\frac{s}{2}\}}$$

into the integrand of the above expression, and define $w_{2,IV}$ by

$$w_{2,IV}(t,r) = \frac{-1}{2\pi}\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\mathbb{1}_{\{|x+y|>\frac{s}{2}\}}WRHS_2(s,|x+y|) \left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)d\theta d\rho ds$$

and

$$w_{2,III}(t,r) = w_2(t,r) - w_{2,IV}(t,r)$$

$$|w_{2,IV}(t,r)| \leqslant C\int_t^\infty\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\left(\frac{\lambda(s)^2}{s^4\log^b(s)}\right)d\theta d\rho ds \leqslant \frac{C\lambda(t)^2}{t^2\log^b(t)}$$

For $w_{2,III}$ in the region $r > \frac{t}{2}$, we again consider several integrals separately. We have

$$\begin{aligned}&w_{2,III,a}(t,r)\\ &= \frac{-1}{2\pi}\int_t^{t+\frac{r}{8}}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}}WRHS_2(s,|x+y|)\\ &\qquad\cdot\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)d\theta d\rho ds\end{aligned}$$

which gives, via the same reasoning as used to estimate $w_{2,I,a}$,

$$|w_{2,III,a}(t,r)| \leqslant C\int_t^{t+\frac{r}{8}}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\frac{r^2\lambda(s)^2}{s^2\log^b(s)(g(s)^2+r^2)^2}d\theta d\rho ds \leqslant \frac{C\lambda(t)^2}{t^2\log^b(t)}$$
$$, \quad r > \frac{t}{2}$$

Next, we have

$$\begin{aligned} &w_{2,III,b}(t,r)\\ &= \frac{-1}{2\pi}\int_{t+\frac{r}{8}}^{t+\frac{r}{8}+g(t)}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}}WRHS_2(s,|x+y|)\\ &\qquad\cdot\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)d\theta d\rho ds \end{aligned}$$

We use

$$\int_0^{2\pi}\frac{r^2+\rho^2+2r\rho\cos(\theta)}{(g(s)^2+r^2+\rho^2+2r\rho\cos(\theta))^2}d\theta \leqslant \frac{C}{\sqrt{g(s)^2+\rho^2+r^2}\sqrt{g(s)^2+(\rho-r)^2}}$$

to get

$$\begin{aligned} |w_{2,III,b}(t,r)| &\leqslant C\int_{t+\frac{r}{8}}^{t+\frac{r}{8}+g(t)}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\frac{\lambda(s)^2}{s^2\log^b(s)}\frac{1}{g(s)\sqrt{g(s)^2+r^2}}d\rho ds\\ &\leqslant C\int_{t+\frac{r}{8}}^{t+\frac{r}{8}+g(t)}(s-t)\frac{\lambda(s)^2}{s^2\log^b(s)g(s)\sqrt{g(s)^2+r^2}}ds \leqslant \frac{C\lambda(t)^2}{t^2\log^b(t)} \end{aligned}$$

The next integral to treat is

$$\begin{aligned} &w_{2,III,c}(t,r)\\ &= \frac{-1}{2\pi}\int_{t+\frac{r}{8}+g(t)}^{\infty}\frac{1}{(s-t)}\int_0^{s-t}\rho\mathbb{1}_{\{\frac{r}{2}\leqslant\rho\leqslant 2r\}}\int_0^{2\pi}\mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}}WRHS_2(s,|x+y|)\\ &\qquad\cdot\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)d\theta d\rho ds \end{aligned}$$

So,

$$\begin{aligned} |w_{2,III,c}(t,r)| &\leqslant C\int_{t+\frac{r}{8}+g(t)}^{\infty}\frac{1}{(s-t)}\frac{\lambda(s)^2}{s^2\log^b(s)}\int_{\frac{r}{2}}^{2r}\frac{\rho}{\sqrt{g(s)^2+(\rho-r)^2}\sqrt{g(s)^2+r^2}}d\rho ds\\ &\leqslant Cr\int_{t+\frac{r}{8}+g(t)}^{\infty}\frac{1}{s(s-t)}\frac{\log(1+\frac{r}{g(s)})\lambda(s)^2}{s\log^b(s)\sqrt{g(s)^2+r^2}}ds \leqslant \frac{C\lambda(t)^2\log(1+\frac{r}{g(t)})}{rt\log^b(t)} \end{aligned}$$

where we used the fact that $\frac{1}{(s-t)} \leqslant \frac{C}{r}$. Next, we have

$$\begin{aligned} w_{2,III,d}(t,r) &:= \frac{-1}{2\pi}\int_{t+\frac{r}{8}+g(t)}^{\infty}\frac{1}{(s-t)}\int_0^{s-t}\rho\left(\mathbb{1}_{\{\rho<\frac{r}{2}\}}+\mathbb{1}_{\{\rho>2r\}}\right)\\ &\qquad\int_0^{2\pi}\mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}}WRHS_2(s,|x+y|)\\ &\qquad\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)d\theta d\rho ds \end{aligned}$$

which gives

$$|w_{2,III,d}(t,r)| \leqslant C \int_{t+\frac{r}{8}+g(t)}^{\infty} \frac{1}{(s-t)} \int_0^{s-t} \frac{\rho\lambda(s)^2}{s^2\log^b(s)} \left(\frac{\mathbb{1}_{\{\rho<\frac{r}{2}\}} + \mathbb{1}_{\{2r<\rho\}}}{(r^2+\rho^2+g(s)^2)}\right) d\rho ds$$

$$\leqslant C \int_{t+\frac{r}{8}+g(t)}^{\infty} \frac{1}{(s-t)} \frac{\lambda(s)^2}{s^2\log^b(s)} \left(1+\log(s-t)+\log(r)\right) ds \leqslant \frac{C\lambda(t)^2}{t\log^b(t)} \frac{\log(r)}{r}, \quad r \geqslant \frac{t}{2}$$

The final integral to estimate is

$$w_{2,III,e}(t,r) := \frac{-1}{2\pi} \int_{t+\frac{r}{8}+g(t)}^{\infty} \int_0^{s-t} \rho \left(\frac{1}{\sqrt{(s-t)^2-\rho^2}} - \frac{1}{(s-t)}\right) \int_0^{2\pi} \mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}} WRHS_2(s,|x+y|) \cdot \left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right) d\theta d\rho ds$$

We again switch the order of integration, as previously done to estimate $w_{2,I,d}$. The only difference here is that we use $\frac{1}{s} \leqslant \frac{1}{\rho+t}$ for one of the factors of $\frac{1}{s}$ which appear in the integrand of $w_{2,III,e}$ once the estimates for $WRHS_2$ are substituted. This gives

$$|w_{2,III,e}(t,r)| \leqslant C \int_0^{\infty} \frac{\rho\lambda(t)^2}{(\rho+t)\log^b(t)t\sqrt{g(t)^2+\rho^2+r^2}\sqrt{g(t)^2+(\rho-r)^2}} d\rho$$

and

$$|w_{2,III,e}(t,r)| \leqslant \frac{C\lambda(t)^2\log(r)}{rt\log^b(t)}, \quad r \geqslant \frac{t}{2}$$

Combining the above gives the final pointwise estimate on w_2:

$$|w_2(t,r)| \leqslant \begin{cases} \frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{(g(t)^2+r^2)t^2\log^b(t)}, & r \leqslant \frac{t}{2} \\ \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)}, & r > \frac{t}{2} \end{cases}$$

where we used the fact that

$$\frac{\log(1+\frac{r}{g(t)})}{r} \leqslant \frac{C\log(t)}{t}, \quad r \geqslant \frac{t}{2}$$

Next, we consider the derivatives of w_2. We recall the remarks prior to the estimation of w_2, and prove a preliminary estimate on $\partial_r w_2$. Here, we will start with the case $r \leqslant g(t)$. We first note that

$$\sin^2(\phi)\frac{\rho}{\sqrt{r^2+\rho^2+2r\rho\cos(\phi)}} \leqslant \begin{cases} C, & \rho < \frac{r}{2}, \text{ or } \rho > 2r \\ \frac{C\sin^2(\frac{\phi}{2})\cos^2(\frac{\phi}{2})r}{\sqrt{(r-\rho)^2+4r\rho\cos^2(\frac{\theta}{2})}} \leqslant C, & \frac{r}{2} \leqslant \rho \leqslant 2r \end{cases}$$

We also note that

$$\int_0^{2\pi} \frac{d\phi}{(g(t)^2+\rho^2+r^2+2r\rho\cos(\phi))^2} \leqslant \frac{C}{\sqrt{g(t)^2+r^2+\rho^2}(g(t)^2+(r-\rho)^2)^{3/2}}$$

Then, we proceed as in step 2 of the estimation of $\partial_r v_2 + \frac{2}{r}v_2$, to get

$$\begin{aligned}
&|\left(\partial_r + \frac{2}{r}\right) w_2(t,r)| \\
&\leqslant Cr \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^\pi \left(\frac{\mathbb{1}_{\{|x+y|<\frac{s}{2}\}}\lambda(s)^2}{s^2\log^b(s)(g(s)^2+r^2+\rho^2+2r\rho\cos(\phi))^2}\right) d\phi d\rho ds \\
&+ Cr \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^\pi \frac{\mathbb{1}_{\{|x+y|>\frac{s}{2}\}}\lambda(s)^2}{|x+y|^4\sqrt{s}\langle s-|x+y|\rangle^{3/2}\log^b(\langle s-|x+y|\rangle)} d\phi d\rho ds
\end{aligned}$$

Denote the first line of the above expression by $q_I(t,r)$, and the second by $q_{II}(t,r)$. Then, we estimate $q_I(t,r)$ using the same procedure used for w_2. The main difference here is that we have the factor

$$\frac{1}{\sqrt{g(t)^2+r^2+\rho^2}(g(t)^2+(r-\rho)^2)^{3/2}} \text{ instead of } \frac{1}{\sqrt{g(t)^2+(r-\rho)^2}(g(t)^2+r^2+\rho^2)^{3/2}}$$

This leads to an extra factor of $\frac{r}{g(t)}$ when we estimate certain ρ integrals in the region $\frac{r}{2} \leqslant \rho \leqslant 2r$. Since we are considering the region $r \leqslant g(t)$, we end up with

$$|q_I(t,r)| \leqslant \frac{Cr\lambda(t)^2}{t^2\log^b(t)g(t)^2}\log(t), \quad r \leqslant g(t)$$

The same procedure used for $w_{2,II}$ gives

$$|q_{II}(t,r)| \leqslant \frac{Cr\lambda(t)^2}{t^4\log^b(t)}, \quad r \leqslant g(t)$$

For the region $r \geqslant g(t)$, we will differentiate our formula (11.28) directly. We emphasize again that the estimate on $\partial_r w_2$ which we will obtain now, in the region $r \geqslant g(t)$ is a preliminary estimate, and it will be improved later, once we choose $\lambda(t)$, and show that $\lambda \in C^3([T_0,\infty))$, thereby allowing us to estimate $\partial_t w_2$ and $(-\partial_t + \partial_r) w_2$. We have

$$\begin{aligned}
&\partial_r w_2(t,r) \\
&= \frac{-1}{2\pi} \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\
&\qquad \int_0^{2\pi} \partial_r \left(WRHS_2(s,|x+y|)\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)\right) \mathbb{1}_{\{|x+y|\leqslant\frac{s}{2}\}} d\theta d\rho ds \\
&- \frac{1}{2\pi} \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\
&\qquad \int_0^{2\pi} \partial_r \left(WRHS_2(s,|x+y|)\left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)\right) \mathbb{1}_{\{|x+y|>\frac{s}{2}\}} d\theta d\rho ds
\end{aligned}$$

where we again denote the first line of the right-hand side of the above expression by q_{III}, and the second by q_{IV}.

$$\begin{aligned}|q_{III}(t,r)| \leqslant C \int_t^\infty \int_0^{s-t} & \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\ & \int_0^{2\pi} \Bigg(\frac{|x+y|\lambda(s)^2}{s^2\log^b(s)(g(s)^2+|x+y|^2)^2} \frac{|r+\rho\cos(\theta)|}{|x+y|} \\ & \quad + \frac{|x+y|^2\lambda(s)^2}{s^2\log^b(s)(g(s)^2+|x+y|^2)^2} \frac{|r+\rho\cos(\theta)|\rho^2\sin^2(\theta)}{|x+y|^4} \Bigg) d\theta d\rho ds\end{aligned}$$

We then use

$$|r+\rho\cos(\theta)| = |(r-\rho)+\rho(\cos(\theta)+1)| \leqslant |r-\rho| + \rho(1+\cos(\theta))$$

$$\int_0^{2\pi} \frac{d\theta}{(g(t)^2+\rho^2+r^2+2r\rho\cos(\theta))^2} \leqslant \frac{C}{\sqrt{g(t)^2+\rho^2+r^2}(g(t)^2+(\rho-r)^2)^{3/2}}$$

$$\frac{\rho^2\sin^2(\theta)}{\rho^2+r^2+2r\rho\cos(\theta)} \leqslant C \frac{\rho^2\cos^2(\frac{\theta}{2})}{(r-\rho)^2+4r\rho\cos^2(\frac{\theta}{2})} \leqslant C$$

$$\int_0^{2\pi} \frac{(1+\cos(\theta))}{(g(s)^2+r^2+\rho^2+2r\rho\cos(\theta))^2} d\theta \leqslant \frac{C}{\sqrt{(\rho-r)^2+g(s)^2}((\rho+r)^2+g(s)^2)^{3/2}}$$

and the identical procedure used to estimate $w_{2,III}$, to get

$$|q_{III}(t,r)| \leqslant \frac{C\lambda(t)^2}{t^2\log^b(t)g(t)} \log(t), \quad r \geqslant g(t)$$

Finally,

$$\begin{aligned}|q_{IV}(t,r)| &\leqslant C \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \left(\frac{\lambda(s)^2}{s^{9/2}\langle s-|x+y|\rangle^{1/2}\log^b(\langle s-|x+y|\rangle)} \right) d\rho ds \\ &\leqslant \frac{C\lambda(t)^2}{t^{5/2}}\end{aligned}$$

Then, we use the same observation as in step 3 of estimating v_2, along with the identical argument used to estimate $\left(\partial_r + \frac{2}{r}\right) w_2$, to get

$$|\partial_r^2 w_2(t,r) + \frac{3}{r}\partial_r w_2(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2}, \quad r > 0$$

We can estimate $\partial_t^2 w_2$ by using the equation solved by w_2. It then remains to estimate $\partial_t w_2$. For this, we return to (11.28), and make the substitution $\rho = q(s-t)$.

$$w_2(t,r) = \frac{-1}{2\pi} \int_t^\infty \int_0^1 \frac{(s-t)q}{\sqrt{1-q^2}} \int_0^{2\pi} \text{integrand}_{w_{2,1}} d\theta dq ds \tag{11.29}$$

where

$$\text{integrand}_{w_{2,1}} = WRHS_2(s, \sqrt{q^2(s-t)^2 + r^2 + 2rq(s-t)\cos(\theta)}) \cdot \left(1 - \frac{2q^2(s-t)^2\sin^2(\theta)}{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)}\right)$$

Then, we can differentiate under the integral sign. The resulting integrals can be estimated with the same procedure used to estimate analogous integrals arising in the expressions for $\partial_r w_2$ and w_2. We get

$$|\partial_t w_2(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)}, \quad r > 0$$

The same procedure is used to estimate $\partial_{tr} w_2$, and this concludes the proof of the lemma □

11.6 Summation of the higher corrections, w_k

The nonlinear interactions between w_2 and v_c can not be treated perturbatively in our final argument. Therefore, we will need to define corrections w_k, in a similar manner as the corrections v_k were defined, and sum a series of the form $\sum_{k=3}^{\infty} w_k$. Because the estimates for w_j and w_2 will be of a slightly different form, we will first (define and) estimate w_3, then prove estimates on w_j for $j \geqslant 4$ by induction. We let

$$WRHS_3(t,r) = \frac{6\left(Q_{\frac{1}{\lambda(t)}} + v_c\right)}{r^2} w_2^2 + \frac{2w_2^3}{r^2} + \frac{6w_2}{r^2}\left(v_c^2 + 2v_c Q_{\frac{1}{\lambda(t)}}\right)$$

Then,

$$|WRHS_3(t,r)| \leqslant C \begin{cases} \frac{r^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{(g(t)^2+r^2)t^4\log^{2b}(t)}, & r \leqslant \frac{t}{2} \\ \frac{\lambda(t)^2\log^2(t)}{t^{5/2}r^{3/2}\log^{2b}(t)}, & r > \frac{t}{2} \end{cases}$$

$$|\partial_r WRHS_3(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)^2\log(t)}{g(t)^2t^4\log^{2b}(t)}, & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log(t)}{t^4\log^{2b}(t)g(t)}, & g(t) < r \leqslant \frac{t}{2} \\ \frac{C\lambda(t)^2\log(t)}{t^{5/2}\log^{2b}(t)r^{3/2}g(t)}, & r > \frac{t}{2} \end{cases}$$

$$|\partial_r^2 WRHS_3(t,r)| \leqslant \begin{cases} \frac{C\lambda(t)^2\log(t)}{t^4\log^{2b}(t)g(t)^2}, & r \leqslant \frac{t}{2} \\ \frac{C\lambda(t)^2\log(t)}{t^{5/2}r^{3/2}\log^{2b}(t)g(t)^2}, & r > \frac{t}{2} \end{cases}$$

Now, we estimate w_3, starting with the region $r \leqslant g(t)$. The same remarks concerning the nature of the estimates on the derivatives of w_2 apply here for w_3, and eventually for w_j.

Lemma 11.7. *[Preliminary estimates on w_3] We have the following* preliminary *estimates on w_3*

$$|w_3(t,r)| \leqslant \begin{cases} \frac{Cr^2\lambda(t)^2\log(t)}{t^2g(t)^2\log^{2b}(t)}, & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log^2(t)}{t^2\log^{2b}(t)}, & r > g(t) \end{cases}$$

$$|\partial_r w_3(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)^2\log(t)}{t^2 g(t)^2 \log^{2b}(t)}, & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log(t)}{t^2\log^{2b}(t) g(t)}, & r > g(t) \end{cases}$$

$$|\partial_r^2 w_3(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2 g(t)^2 \log^{2b}(t)}$$

$$|\partial_t w_3(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2 g(t) \log^{2b}(t)}$$

$$|\partial_t^2 w_3(t,r)| \leqslant \begin{cases} \frac{C\lambda(t)^2\log(t)}{t^2 g(t)^2 \log^{2b}(t)}, & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log^2(t)}{t^2 g(t)^2 \log^{2b}(t)}, & r > g(t) \end{cases}$$

$$|\partial_{tr} w_3(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2\log^{2b}(t) g(t)^2}$$

Proof. Using the analog of step 1 when estimating v_2, we get

$$|w_3(t,r)| \leqslant Cr^2 \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^\pi \text{integrand}_{w_3} d\phi d\rho ds$$

where

$$\text{integrand}_{w_3} = \sin^4(\phi)\left(1+\frac{\rho^2}{|x+y|^2}\right)\left(\frac{\lambda(s)^2\log(2+\frac{|x+y|}{g(s)})\log(s)}{(g(s)^2+|x+y|^2)s^4\log^{2b}(s)} + \frac{\lambda(s)^2\log(s)}{g(s)^2 s^4\log^{2b}(s)}\right)$$

We use

$$\frac{\log(2+\frac{|x+y|}{g(s)})}{g(s)^2+|x+y|^2} \leqslant \frac{C}{g(s)^2}$$

to get

$$|w_3(t,r)| \leqslant \frac{Cr^2\lambda(t)^2\log(t)}{t^2 g(t)^2\log^{2b}(t)}, \quad r \leqslant g(t)$$

Using the same procedure as in steps 2 and 3 of estimating v_2, we get

$$|\partial_r w_3(t,r)| \leqslant \frac{Cr\lambda(t)^2\log(t)}{t^2 g(t)^2\log^{2b}(t)}, \quad r \leqslant g(t)$$

and

$$|\partial_r^2 w_3(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2 g(t)^2\log^{2b}(t)}, \quad r \leqslant g(t)$$

To estimate w_3 in the region $r > g(t)$, we use the analog of (11.28), we get

$$|w_3(t,r)| \leqslant C\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \frac{\lambda(s)^2\log^2(s)}{s^4\log^{2b}(s)} d\theta d\rho ds \leqslant \frac{C\lambda(t)^2\log^2(t)}{t^2\log^{2b}(t)}, \quad r > g(t)$$

Differentiating the analog of (11.28), we get

$$|\partial_r w_3(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2\log^{2b}(t) g(t)}, \quad r > g(t)$$

We then use the observation of step 3 of the v_2 estimates to get

$$|\partial_r^2 w_3(t,r)| \leqslant \frac{C\lambda(t)^2 \log(t)}{t^2 \log^{2b}(t) g(t)^2}, \quad r \geqslant g(t)$$

Using the same procedure that was used to estimate $\partial_t w_2$, we get

$$|\partial_t w_3(t,r)| \leqslant \frac{C\lambda(t)^2 \log(t)}{t^2 g(t) \log^{2b}(t)}, \quad r > 0$$

We then read off estimates on $\partial_t^2 w_3$, based on the equation solved by w_3, and the previous estimates. This completes the proof of the lemma. □

Now, for $j \geqslant 4$, we define w_j to be the solution to the following equation, with zero Cauchy data at infinity.

$$-\partial_{tt} w_j + \partial_{rr} w_j + \frac{1}{r}\partial_r w_j - \frac{4}{r^2} w_j = WRHS_j(t,r)$$

where

$$\begin{aligned} WRHS_j(t,r) := & \frac{6\left(Q_{\frac{1}{\lambda(t)}} + v_c\right)}{r^2} \left(w_{j-1}^2 + 2\sum_{k=2}^{j-2} w_k w_{j-1}\right) \\ & + \frac{2}{r^2}\left(w_{j-1}^3 + 3w_{j-1}^2 \sum_{k=2}^{j-2} w_k + 3w_{j-1}\left(\sum_{k=2}^{j-2} w_k\right)^2\right) + \frac{6w_{j-1}}{r^2}\left(v_c^2 + 2v_c Q_{\frac{1}{\lambda(t)}}\right) \end{aligned}$$

As with v_j, we will now prove estimates on w_j by induction. Let $C_2 > 9$ be such that the estimates of lemmas 11.6 and 11.7 hold, with the constant $C = C_2$ on the right-hand side. Let $p > 900$ be otherwise arbitrary, and let $T_{0,p} > e^{(900!)^{1+\frac{1}{b}}}$ satisfy

$$\frac{C_2^{90p}}{\log^b(t)} + \frac{C_2^{90p}\lambda(t)^2 \log^2(t)}{g(t)^2 \log^b(t)} < e^{-(900!)}, \quad t \geqslant T_{0,p}$$

and be otherwise arbitrary. (We recall that $\frac{\lambda(t)}{g(t)} = \frac{1}{\log^{b-2\epsilon}(t)}$, and $b > \frac{2}{3}$, so, such a $T_{0,p}$ exists). Our goal is to show that, for a sufficiently large $T_{0,p}$, we can prove estimates on w_j (and its derivatives), valid for all $t \geqslant T_1 + T_{0,p}$ and all $r \geqslant 0$, by induction. In the following estimates, we assume $t \geqslant T_1 + T_{0,p}$. Let

$$D_{p,k} = \begin{cases} C_2, & k = 3 \\ C_2^{pk}, & k \geqslant 4 \end{cases}$$

$$q_k = \begin{cases} 1, & k = 3 \\ 2, & k \geqslant 4 \end{cases}$$

Suppose, for any $j \geqslant 4$, and all k with $3 \leqslant k \leqslant j-1$, that

$$|w_k(t,r)| \leqslant \begin{cases} \frac{D_{p,k} r^2 \lambda(t)^2 \log^{q_k}(t)}{t^2 g(t)^2 \log^{b(k-1)}(t)}, & r \leqslant g(t) \\ \frac{D_{p,k}\lambda(t)^2 \log^2(t)}{t^2 \log^{b(k-1)}(t)}, & r > g(t) \end{cases} \tag{11.30}$$

$$|\partial_r w_k(t,r)| \leqslant \begin{cases} \frac{D_{p,k} r \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(k-1)}(t)}, & r \leqslant g(t) \\ \frac{D_{p,k} \lambda(t)^2 \log^2(t)}{t^2 \log^{b(k-1)}(t) g(t)}, & r > g(t) \end{cases}$$

$$|\partial_r^2 w_k(t,r)| \leqslant \frac{D_{p,k}\lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(k-1)}(t)}, \quad r > 0$$

$$|\partial_t w_k(t,r)| \leqslant \frac{D_{p,k}\lambda(t)^2 \log^2(t)}{t^2 g(t) \log^{b(k-1)}(t)} \tag{11.31}$$

$$|\partial_t^2 w_k(t,r)| \leqslant \frac{D_{p,k}\lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(k-1)}(t)}$$

$$|\partial_{tr} w_k(t,r)| \leqslant \frac{D_{p,k}\lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(k-1)}(t)} \tag{11.32}$$

Then, for some constant C *independent* of t, p, j, we have the following estimates, for $t \geqslant T_1+T_{0,p}$.

$$|WRHS_j(t,r)| \leqslant C \begin{cases} \frac{C_2^{p(j-1)} r^2 \lambda(t)^2 \log^2(t)}{t^4 g(t)^2 \log^{b(j-1)}(t)}, & r \leqslant g(t) \\ \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^4 \log^{b(j-1)}(t)}, & g(t) < r \leqslant \frac{t}{2} \\ \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^{5/2} r^{3/2} \log^{b(j-1)}(t)} \left(1 + \frac{\lambda(t)^2 \log^2(r)}{t^2 \log^b(t)}\right), & r > \frac{t}{2} \end{cases}$$

$$|\partial_r WRHS_j(t,r)| \leqslant C \begin{cases} \frac{C_2^{p(j-1)} r \lambda(t)^2 \log^2(t)}{t^4 g(t)^2 \log^{b(j-1)}(t)}, & r \leqslant g(t) \\ \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^4 g(t) \log^{b(j-1)}(t)}, & g(t) < r \leqslant \frac{t}{2} \\ \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^3 r \log^{b(j-1)}(t) g(t)} + \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{r^2 t^{5/2} \log^{b(j-2)}(t) \sqrt{\langle t-r \rangle} \log^b(\langle t-r \rangle)}, & r > \frac{t}{2} \end{cases}$$

$$|\partial_r^2 WRHS_j(t,r)| \leqslant C \begin{cases} \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^4 g(t)^2 \log^{b(j-1)}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{r^{3/2} t^{5/2} g(t)^2 \log^{b(j-1)}(t)} \left(1 + \frac{\lambda(t)^2 \log^2(r)}{t^2 \log^b(t)}\right), & r > \frac{t}{2} \end{cases}$$

Using the same procedure used to estimate w_3, we get the following estimates, where the constant C is *independent* of t, j, p, and $t \geqslant T_1 + T_{0,p}$.

$$|w_j(t,r)| \leqslant \begin{cases} \frac{C r^2 C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(j-1)}(t)}, & r \leqslant g(t) \\ \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 \log^{b(j-1)}(t)}, & r > g(t) \end{cases}$$

$$|\partial_r w_j(t,r)| \leqslant \begin{cases} \frac{C r C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(j-1)}(t)}, & r \leqslant g(t) \\ \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t) \log^{b(j-1)}(t)}, & r > g(t) \end{cases}$$

$$|\partial_r^2 w_j(t,r)| \leqslant \begin{cases} \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(j-1)}(t)}, & r \leqslant g(t) \\ \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(j-1)}(t)}, & r > g(t) \end{cases}$$

$$|\partial_t w_j(t,r)| \leqslant \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t) \log^{b(j-1)}(t)}$$
$$|\partial_t^2 w_j(t,r)| \leqslant \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(j-1)}(t)}$$
$$|\partial_{tr} w_j(t,r)| \leqslant \frac{C C_2^{p(j-1)} \lambda(t)^2 \log^2(t)}{t^2 g(t)^2 \log^{b(j-1)}(t)}$$

Therefore, there exists $p_0 > 900$ such that $C C_2^{p_0(j-1)} \leqslant C_2^{p_0 j}$. Then, by mathematical induction, (11.30) through (11.32) are true for all $j \geqslant 3$, provided that T_{0,p_0} is chosen sufficiently large (though we have slightly better estimates on w_3 than what we supposed for the purposes of the induction argument). Therefore, the series

$$w_s := \sum_{j=3}^{\infty} w_j$$

and the series resulting from applying any first or second order derivative termwise converges absolutely and uniformly on the set $\{(t,r) | t \geqslant T_1 + T_{0,p_0}, \quad r > 0\}$. From here on, we will further restrict T_0 to satisfy $T_0 > T_1 + T_{0,p_0}$. Then, for $t \geqslant T_0$, let $w_c(t,r) := w_2(t,r) + w_s(t,r)$. Using the fact that

$$WRHS_j(t,r) = \frac{6\left(Q_{\frac{1}{\lambda(t)}} + v_c\right)}{r^2}\left(\left(\sum_{k=2}^{j-1} w_k\right)^2 - \left(\sum_{k=2}^{j-2} w_k\right)^2\right) + \frac{2}{r^2}\left(\left(\sum_{k=2}^{j-1} w_k\right)^3 - \left(\sum_{k=2}^{j-2} w_k\right)^3\right) + \frac{6}{r^2} w_{j-1}\left(v_c^2 + 2 v_c Q_{\frac{1}{\lambda(t)}}\right)$$

we proceed as in the case of v_c, to get

$$-\partial_{tt} w_c + \partial_{rr} w_c + \frac{1}{r}\partial_r w_c - \frac{4 w_c}{r^2} = \chi_{\geqslant 1}\left(\frac{r}{g(t)}\right)\left(\partial_t^2 Q_{\frac{1}{\lambda(t)}} - \frac{6 v_c}{r^2}\left(1 - Q^2_{\frac{1}{\lambda(t)}}(r)\right)\right) + \frac{6\left(Q_{\frac{1}{\lambda(t)}} + v_c\right)}{r^2} w_c^2 + \frac{2}{r^2} w_c^3 + \frac{6}{r^2} w_c\left(v_c^2 + 2 v_c Q_{\frac{1}{\lambda(t)}}\right)$$

This will be useful for us in the next section.

11.7 Choosing $\lambda(t)$

Let

$$F_4(t,r) = \left(1 - \chi_{\geqslant 1}\left(\frac{r}{g(t)}\right)\right)\left(\partial_t^2 Q_{\frac{1}{\lambda(t)}}(r) - \frac{6 v_c(t,r)}{r^2}\left(1 - Q^2_{\frac{1}{\lambda(t)}}(r)\right)\right) + \chi_{\leqslant 1}\left(\frac{2r}{t}\right)\left(\frac{-6\left(1 - Q^2_{\frac{1}{\lambda(t)}}(r)\right)}{r^2} w_c(t,r)\right) \tag{11.33}$$

$$F_5(t,r) = \left(1-\chi_{\leqslant 1}(\frac{2r}{t})\right)\left(\frac{-6\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)}{r^2}w_c(t,r)\right) \tag{11.34}$$

where we recall that $\chi_{\leqslant 1}$ was defined in (11.13). If we substitute $u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r) + w_c(t,r) + v(t,r)$ into (8.1), we get

$$\begin{aligned}
&-\partial_{tt}v + \partial_{rr}v + \frac{1}{r}\partial_r v + \frac{2}{r^2}\left(1-3Q_{\frac{1}{\lambda(t)}}(r)^2\right)v \\
&= F_4(t,r) + F_5(t,r) + \frac{2v^3}{r^2} + \frac{6\left(Q_{\frac{1}{\lambda(t)}}(r)+v_c+w_c\right)v^2}{r^2} \\
&+ \frac{6v}{r^2}\left(\left(v_c + Q_{\frac{1}{\lambda(t)}} + w_c\right)^2 - Q^2_{\frac{1}{\lambda(t)}}\right)
\end{aligned}$$

11.7.1 Estimates on F_5

We will now show that F_5 decays sufficiently quickly in sufficiently many norms, so that we do not need to include it in the modulation equation for λ. By directly substituting the estimates of the previous sections into the definition of F_5, we get: there exists $C>0$ such that for all λ satisfying (11.4), we have

$$||F_5(t,R\lambda(t))||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^3}{t^5\log^{b-2}(t)} \tag{11.35}$$

$$||L^*L\left(F_5(t,R\lambda(t))\right)||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^5\log^2(t)}{g(t)^2\log^b(t)t^5} \tag{11.36}$$

11.7.2 Solving the modulation equation

Now, we will choose $\lambda(t)$ so that

$$\langle F_4(t,R\lambda(t)),\phi_0(R)\rangle_{L^2(RdR)} = 0$$

This equation can be re-written in the form

$$\begin{aligned}
&\langle \partial_{tt}Q_{\frac{1}{\lambda(t)}} - \frac{6}{r^2}v_1\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)\Big|_{r=R\lambda(t)},\phi_0(R)\rangle_{L^2(RdR)} \\
&= \langle\frac{6}{r^2}\sum_{k=2}^{\infty}v_k\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)\Big|_{r=R\lambda(t)},\phi_0(R)\rangle_{L^2(RdR)} \\
&+ \langle\chi_{\geqslant 1}(\frac{R\lambda(t)}{g(t)})\left(\partial_{tt}Q_{\frac{1}{\lambda(t)}} - \frac{6}{r^2}v_c\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)\right)\Big|_{r=R\lambda(t)},\phi_0(R)\rangle_{L^2(RdR)} \\
&+ \langle\frac{\chi_{\leqslant 1}(\frac{2R\lambda(t)}{t})}{r^2}6\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)w_c(t,r)\Big|_{r=R\lambda(t)},\phi_0(R)\rangle_{L^2(RdR)}
\end{aligned} \tag{11.37}$$

The main result of this section is

Proposition 11.1. *There exists $T_3 > 0$ such that for all $T_0 \geqslant T_3$, there exists a solution, λ (which is of the form* (11.4)*) to* (11.37)*, for $t \geqslant T_0$. In addition, $\lambda(t) \in C^4([T_0, \infty))$, and satisfies*

$$\lambda(t) = \lambda_0(t)\,(1 + e(t))$$

where

$$|e(t)| \leqslant \frac{C}{\log^{\delta-\delta_2}(t)}, \quad |e^k(t)| \leqslant \begin{cases} \frac{C}{t^k \log^{1+\delta-\delta_2}(t)}, & k = 1, 2 \\ \frac{C}{t^3 \log^{b+\delta_4}(t)}, & k = 3 \\ \frac{C}{t^4 \log^{b+\delta_5}(t)}, & k = 4 \end{cases}$$

where δ, δ_2 are defined in (11.1), (11.2)*, respectively, and $\delta_4, \delta_5 > 0$.*

We start by computing the left-hand side of (11.37). Firstly, we have

$$\langle \partial_{tt} Q_{\frac{1}{\lambda(t)}}\Big|_{r=R\lambda(t)}, \phi_0(R)\rangle_{L^2(RdR)} = \frac{2\lambda''(t)}{3\lambda(t)}$$

Next, we start by noting that

$$\frac{6\,(1 - Q_1^2(R))}{R^2\lambda(t)^2}\phi_0(R) = \frac{24R^2}{\lambda(t)^2(1+R^2)^4}$$

Then, we use

$$\frac{24}{\lambda(t)^2}\int_0^\infty \frac{R^3 J_2(\xi R\lambda(t))dR}{(1+R^2)^4} = \frac{\xi^3\lambda(t)}{2} K_1(\xi\lambda(t))$$

(which follows from combining integral identities of [7]) to get the following. By the representation of v_1 in terms of its Hankel transform of order 2, noted in previous sections, the choice of $\widehat{v_{1,1}}$, and the sin transform inversion formula, we get

$$\langle -\frac{6}{r^2} v_1 \left(1 - Q^2_{\frac{1}{\lambda(t)}}(r)\right)\Big|_{r=R\lambda(t)}, \phi_0(R)\rangle_{L^2(RdR)} = -\frac{2\lambda_0''(t)}{3\lambda_0(t)} + E_{v_1,ip}(t,\lambda(t))$$

where

$$\begin{aligned} E_{v_1,ip}(t,\lambda(t)) &= \frac{-\lambda(t)}{2}\int_0^\infty \sin(t\xi)\widehat{v_{1,1}}(\xi)\xi^3 \left(K_1(\xi\lambda(t)) - \frac{1}{\xi\lambda(t)}\right) d\xi \\ &= \frac{\lambda(t)}{2}\int_0^\infty \frac{\cos(t\xi)}{t^3}\partial_\xi^3 \left(\widehat{v_{1,1}}(\xi)\xi^3\left(K_1(\xi\lambda(t)) - \frac{1}{\xi\lambda(t)}\right)\right) d\xi \end{aligned}$$

Using the symbol-type estimates on $\widehat{v_{1,1}}$, asymptotics of the modified Bessel function of the second kind, (11.4), and the observation (11.15) we get

$$|E_{v_1,ip}(t,\lambda(t))| \leqslant \frac{C\log(t)}{t^{5/2}}$$

(where the power of t in the denominator could be improved, but is sufficient for our purposes). On the other hand, we have

$$\sum_{k=2}^\infty |v_k(t,r)| \leqslant \begin{cases} \frac{C_1^{2n_0} r^2}{t^2 \log^{2b}(t)}, & r \leqslant \frac{t}{2} \\ \frac{C_1^{2n_0}}{\log^{2b}(t)}, & r > \frac{t}{2} \end{cases}$$

and this gives

$$|v_{sip}(t,\lambda(t))| \leqslant \frac{C}{t^2 \log^{2b}(t)}$$

where

$$v_{sip}(t,\lambda(t)) = \langle \frac{6}{r^2}\sum_{k=2}^{\infty} v_k \left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)\Big|_{r=R\lambda(t)}, \phi_0(R)\rangle_{L^2(RdR)}$$

Using our estimates from previous sections, we get

$$|lin_{ip}(t,\lambda(t))| \leqslant \frac{C}{t^2 \log^{b}(t)\log^{2(b-2\epsilon)}(t)}$$

where

$$lin_{ip}(t,\lambda(t)) := \langle \chi_{\geqslant 1}(\frac{R\lambda(t)}{g(t)}) \left(\partial_{tt}Q_{\frac{1}{\lambda(t)}} - \frac{6}{r^2}v_c\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right)\right)\Big|_{r=R\lambda(t)}, \phi_0(R)\rangle_{L^2(RdR)}$$

and we recall the definition of g: $g(t)=\lambda(t)\log^{b-2\epsilon}(t)$. Next, for $j \geqslant 3$, we use the estimates on w_j given in (11.30), to get

$$|w_{c,ip}(t,\lambda(t))| \leqslant \frac{C}{t^2}\left(\frac{1}{\log^{3b-4\epsilon-1}(t)} + \frac{1}{\log^{5b-8\epsilon-2}(t)}\right)$$

where

$$w_{c,ip}(t,\lambda(t)) = \langle \frac{\chi_{\leqslant 1}(\frac{2R\lambda(t)}{t})}{r^2} 6\left(1-Q^2_{\frac{1}{\lambda(t)}}(r)\right) w_c(t,r)\Big|_{r=R\lambda(t)}, \phi_0(R)\rangle_{L^2(RdR)}$$

Substituting $\lambda(t) = \lambda_0(t)\left(1+e(t)\right), \quad e \in \overline{B}_1(0) \subset X$ (where we recall (11.4) and (11.3)) into (11.37), we get

$$e''(t) + \frac{2\lambda_0'(t)}{\lambda_0(t)}e'(t) = \frac{3G(t,\lambda_0(t)\left(1+e(t)\right))}{2}\left(1+e(t)\right) \tag{11.38}$$

where

$$G(t,\lambda(t)) = v_{sip}(t,\lambda(t)) + lin_{ip}(t,\lambda(t)) + w_{c,ip}(t,\lambda(t)) - E_{v_1,ip}(t,\lambda(t))$$

Let $\mathcal{B} := \overline{B}_1(0) \subset X$. Our goal is to solve (11.38) for $e \in \mathcal{B}$ using a fixed point argument. So, we define T on $\mathcal{B}$ by

$$T(e)(t) = \int_t^{\infty} \frac{1}{\lambda_0(x)^2}\int_x^{\infty} \frac{3}{2}\lambda_0(s)^2 G(s,\lambda_0(s)\left(1+e(s)\right))\left(1+e(s)\right) ds dx$$

Combining our estimates above, we get

$$|G(t,\lambda_0(t)\left(1+e(t)\right))| \leqslant \frac{C}{t^2\log^{1+\delta}(t)}$$

where we recall the definition of δ in (11.1). This gives

$$|\int_x^{\infty}\frac{3}{2}\lambda_0(s)^2 G(s,\lambda_0(s)\left(1+e(s)\right))\left(1+e(s)\right)ds| \leqslant C\int_x^{\infty}\frac{\lambda_0(s)^2 ds}{s^2\log^{1+\delta}(s)}$$

Then, we integrate by parts to get

$$\int_x^\infty \frac{\lambda_0(s)^2 ds}{s^2 \log^{1+\delta}(s)} = \frac{\lambda_0(x)^2}{x\log^{1+\delta}(x)} + \int_x^\infty \frac{1}{s}\left(\frac{2\lambda_0(s)\lambda_0'(s)}{\log^{1+\delta}(s)} - \frac{(\delta+1)\lambda_0(s)^2}{s\log^{\delta+2}(s)}\right) ds$$

Therefore,

$$|\int_x^\infty \frac{\lambda_0(s)^2 ds}{s^2 \log^{1+\delta}(s)}| \leqslant C\frac{\lambda_0(x)^2}{x\log^{1+\delta}(x)} + \left(\frac{C}{\log^b(x)} + \frac{C}{\log(x)}\right)\int_x^\infty \frac{\lambda_0(s)^2 ds}{s^2\log^{1+\delta}(s)}$$

So, there exists $T_2 > T_1 + T_{0,p}$, and $C > 0$ such that, for all $x \geqslant T_2$,

$$|\int_x^\infty \frac{\lambda_0(s)^2 ds}{s^2 \log^{1+\delta}(s)}| \leqslant C\frac{\lambda_0(x)^2}{x\log^{1+\delta}(x)} \tag{11.39}$$

So, for all $T_0 \geqslant T_2$, we have

$$|T(e)'(t)| \leqslant \frac{C}{t\log^{\delta+1}(t)}, \quad t \geqslant T_0$$

$$|T(e)(t)| \leqslant C\int_t^\infty \frac{dx}{x\log^{\delta+1}(x)} \leqslant \frac{C}{\log^\delta(t)}, \quad t \geqslant T_0$$

and

$$|T(e)''(t)| \leqslant \frac{C}{t^2\log^{1+\delta}(t)}, \quad t \geqslant T_0$$

In particular, $T : \mathcal{B} \to \mathcal{B}$. Now, we will study the Lipshitz properties of T. We recall that v_{sip} depends on λ. To emphasize the dependence of v_k on λ, we will write $v_k = v_k^\lambda$. Similarly, we denote the previously defined functions RHS_k by RHS_k^λ. Our goal is to understand the Lipshitz (in e) dependence of $v_{sip}(t, \lambda_0(t)(1+e(t)))$ and $E_{v_1,ip}(t, \lambda_0(t)(1+e(t)))$, for $e \in \mathcal{B}$. For $i = 1, 2$, let $e_i \in \mathcal{B}$, and let $\lambda_i(t) = \lambda_0(t)(1+e_i(t))$. Let

$$F(r, \lambda(t)) = \frac{\left(1 - Q_1^2(\frac{r}{\lambda(t)})\right)\phi_0(\frac{r}{\lambda(t)})}{r^2\lambda(t)^2} r$$

Then,

$$v_{sip}(t, \lambda(t)) = 6\int_0^\infty \sum_{k=2}^\infty v_k^\lambda(t,r) F(r,\lambda(t)) dr$$

We start with

$$|\partial_2 F(r,\lambda(t))| \leqslant \frac{Cr^3\lambda(t)}{(r^2+\lambda(t)^2)^4}, \quad |F(r,\lambda(t))| \leqslant \frac{Cr^3\lambda(t)^2}{(r^2+\lambda(t)^2)^4}$$

To understand the Lipshitz (in e) dependence of $v_k^{\lambda_0(1+e)}$, we start by noting that $v_2^{\lambda_1} - v_2^{\lambda_2}$ solves the following equation with 0 Cauchy data at infinity.

$$\left(-\partial_{tt} + \partial_{rr} + \frac{1}{r}\partial_r - \frac{4}{r^2}\right)\left(v_2^{\lambda_1} - v_2^{\lambda_2}\right) = \frac{6v_1(t,r)^2}{r^2}\left(Q_{\frac{1}{\lambda_1}(t)}(r) - Q_{\frac{1}{\lambda_2(t)}}(r)\right)$$

There exists an absolute constant C such that, for all $e \in \mathcal{B}$ we have

$$C^{-1}\lambda_0(t) \leqslant \lambda_0(t)|\left(1+e(t)\right)| \leqslant C\lambda_0(t), \quad t \geqslant T_0$$

Using this, we get

$$|Q_{\frac{1}{\lambda_1(t)}}(r) - Q_{\frac{1}{\lambda_2(t)}}(r)| \leqslant \frac{C|\lambda_2(t)-\lambda_1(t)|\lambda_0(t)}{r^2}$$

and this gives

$$||RHS_2^{\lambda_1}(s,r) - RHS_2^{\lambda_2}(s,r)||_{L^2(rdr)} \leqslant \frac{C|\lambda_2(s)-\lambda_1(s)|\lambda_0(s)}{s^3\log^{2b}(s)}$$

Using the procedure of (11.16), and the estimate (11.39), we get

$$|v_2^{\lambda_1} - v_2^{\lambda_2}|(t,r) \leqslant \frac{C||e_1-e_2||_X}{t\log^{\delta-\delta_2}(t)}\frac{\lambda_0(t)^2}{t\log^{2b}(t)}, \quad r \geqslant 0, \quad t \geqslant T_0$$

Then, a similar induction procedure used to construct v_s shows that there exists $C_2, m_0, T_3 > T_2$ such that, for $t \geqslant T_3$,

$$|v_j^{\lambda_1} - v_j^{\lambda_2}|(t,r) \leqslant \frac{C_2^{m_0 j}||e_1-e_2||_X\lambda_0(t)^2}{t^2\log^{bj+\delta-\delta_2}(t)}, \quad j \geqslant 2$$

(The main difference between the procedure used to establish the above estimates, and that used to construct v_s is that here, we need only inductively prove estimates on $RHS_j^{\lambda_1} - RHS_j^{\lambda_2}$, and use the procedure of (11.16) to estimate $v_j^{\lambda_1} - v_j^{\lambda_2}$). The above estimates imply that, if $T_0 \geqslant T_3$ (which we will assume from now on) then, we have

$$\sum_{k=2}^{\infty}|v_k^{\lambda_1} - v_k^{\lambda_2}|(t,r) \leqslant \frac{C||e_1-e_2||_X\lambda_0(t)^2}{t^2\log^{2b}(t)\log^{\delta-\delta_2}(t)}, \quad t \geqslant T_0, \quad r \geqslant 0$$

This gives

$$|v_{sip}(t,\lambda_1(t)) - v_{sip}(t,\lambda_2(t))| \leqslant \frac{C||e_1-e_2||_X}{t^2\log^{2b}(t)\log^{\delta-\delta_2}(t)}, \quad t \geqslant T_0$$

Again using properties of the modified Bessel function of the second kind, we get

$$|E_{v_1,ip}(t,\lambda_1(t)) - E_{v_1,ip}(t,\lambda_2(t))| \leqslant C\frac{||e_1-e_2||_X\log(t)}{t^{5/2}\log^{\delta-\delta_2}(t)}$$

Next, we estimate $lin_{ip}(t,\lambda_1(t)) - lin_{ip}(t,\lambda_2(t))$. We start by noting that

$$|\chi_{\geqslant 1}(\frac{r}{g_1(t)}) - \chi_{\geqslant 1}(\frac{r}{g_2(t)})| \leqslant \frac{C||e_1-e_2||_X\mathbb{1}_{\{r \geqslant \frac{g_0(t)}{4}\}}}{\log^{\delta-\delta_2}(t)}$$

Next, we let

$$F_s(r,\lambda(t),\lambda'(t),\lambda''(t)) = \frac{\partial_{tt}Q_1(\frac{r}{\lambda(t)})\phi_0(\frac{r}{\lambda(t)})}{\lambda(t)^2}$$

Then, we use

$$\begin{aligned}&F_s(r,\lambda_1(t),\lambda_1'(t),\lambda_1''(t))-F_s(r,\lambda_2(t),\lambda_2'(t),\lambda_2''(t))\\&=\int_0^1 DF_s(r,\lambda_\sigma(t))\cdot(\lambda_1(t)-\lambda_2(t),\lambda_1'(t)-\lambda_2'(t),\lambda_1''(t)-\lambda_2''(t))\,d\sigma\end{aligned}$$

where DF_s denotes the gradient in the last three arguments of F_s, and

$$\lambda_\sigma(t)=(\sigma\lambda_1(t)+(1-\sigma)\lambda_2(t),\sigma\lambda_1'(t)+(1-\sigma)\lambda_2'(t),\sigma\lambda_1''(t)+(1-\sigma)\lambda_2''(t))$$

This gives

$$\begin{aligned}&|F_s(r,\lambda_1(t),\lambda_1'(t),\lambda_1''(t))-F_s(r,\lambda_2(t),\lambda_2'(t),\lambda_2''(t))|\\&\leqslant\frac{C||e_1-e_2||_X\lambda_0(t)^2r^4}{t^2\log^{\delta-\delta_2}(t)(r^2+\lambda_0(t)^2)^4}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\end{aligned}$$

Then, we use our estimates above, and recall that $b>\frac{2}{3}$ to conclude

$$|lin_{ip}(t,\lambda_1(t))-lin_{ip}(t,\lambda_2(t))|\leqslant\frac{C||e_1-e_2||_X}{t^2\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\frac{1}{\log^{2b-4\epsilon}(t)}$$

To study $w_{c,ip}(t,\lambda_1(t))-w_{c,ip}(t,\lambda_2(t))$, we will need to estimate $w_k^{\lambda_1}-w_k^{\lambda_2}$, where we use the same notational convention as we used for v_k. For later use, let $g_i(t)=\log^{b-2\epsilon}(t)\lambda_i(t),\quad i=0,1,2.$ We start with $k=2$. We split $WRHS_2^{\lambda_1}-WRHS_2^{\lambda_2}$ as follows. We define

$$\begin{aligned}&WRHS_{2,lip,0}(t,r)\\&:=\chi_{\geqslant1}(\frac{r}{g_2(t)})\left(\partial_t^2Q_1(\frac{r}{\lambda_1(t)})-\partial_t^2Q_1(\frac{r}{\lambda_2(t)})-\frac{6v_c^{\lambda_2}(t,r)}{r^2}\left(Q_1^2(\frac{r}{\lambda_2(t)})-Q_1^2(\frac{r}{\lambda_1(t)})\right)\right)\\&+\left(\chi_{\geqslant1}(\frac{r}{g_1(t)})-\chi_{\geqslant1}(\frac{r}{g_2(t)})\right)\left(\partial_t^2Q_{\frac{1}{\lambda_1(t)}}(r)-\frac{6v_c^{\lambda_1}(t,r)}{r^2}\left(1-Q_1^2(\frac{r}{\lambda_1(t)})\right)\right)\end{aligned}$$

and

$$WRHS_{2,lip,1}(t,r)=WRHS_2^{\lambda_1}(t,r)-WRHS_2^{\lambda_2}(t,r)-WRHS_{2,lip,0}(t,r)$$

and write $w_2^{\lambda_1}(t,r)-w_2^{\lambda_2}(t,r)=w_{2,lip,0}(t,r)+w_{2,lip,1}(t,r)$, where $w_{2,lip,j}$ solves the following equation with 0 Cauchy data at infinity.

$$-\partial_{tt}w_{2,lip,j}+\partial_{rr}w_{2,lip,j}+\frac{1}{r}\partial_rw_{2,lip,j}-\frac{4}{r^2}w_{2,lip,j}=WRHS_{2,lip,j}$$

The point of this splitting is that we will need to use a more complicated procedure to estimate $w_{2,lip,0}$, since too many logarithmic losses in estimates are insufficient for our purposes. We have

$$|WRHS_{2,lip,1}(t,r)|\leqslant\frac{C||e_1-e_2||_X\mathbb{1}_{\{r\geqslant\frac{g_0(t)}{4}\}}\lambda_0(t)^4}{t^2\log^{\delta-\delta_2}(t)\log^{2b}(t)(g_0(t)^2+r^2)^2}$$

Using the analog of (11.28), and a similar procedure used to estimate various integrals arising in the w_2 estimates above, we get

$$|w_{2,lip,1}(t,r)| \leqslant \frac{C||e_1-e_2||_X \lambda_0(t)^4 \log(t)}{t^2 \log^{\delta-\delta_2}(t) \log^{2b}(t) g_0(t)^2}, \quad r>0$$

In particular, the procedure used to estimate $w_{2,lip,1}$ does not involve any derivatives of $WRHS_{2,lip,1}$, which is why we did not need to prove any estimates on derivatives of $v_c^{\lambda_1}-v_c^{\lambda_2}$. (Note that $v_c^{\lambda_1}-v_c^{\lambda_2}$ arises in some terms of $WRHS_{2,lip,1}$). Next, we note that

$$\begin{aligned}&|\partial_r WRHS_{2,lip,0}(t,r)| + \frac{|WRHS_{2,lip,0}(t,r)|}{r}\\ &\leqslant C\mathbb{1}_{\{r \geqslant \frac{g_0(t)}{4}\}} \left(\frac{r\lambda_0(t)^2 ||e_1-e_2||_X}{t^2 \log^{\delta-\delta_2}(t)(r^2+\lambda_0(t)^2)^2}\right)\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\\ &+C\mathbb{1}_{\{r \geqslant \frac{g_0(t)}{4}\}} \frac{\lambda_0(t)^2||e_1-e_2||_X}{\log^{\delta-\delta_2}(t)(\lambda_0(t)^2+r^2)^2}\begin{cases}\frac{r}{t^2\log^b(t)}, & r\leqslant \frac{t}{2}\\ \frac{1}{\sqrt{r}\sqrt{\langle t-r\rangle}\log^b(\langle t-r\rangle)}, & r>\frac{t}{2}\end{cases}\end{aligned}$$

and

$$\begin{aligned}&|\partial_r^2 WRHS_{2,lip,0}(t,r)|\\ &\leqslant C\mathbb{1}_{\{r \geqslant \frac{g_0(t)}{4}\}} \frac{\lambda_0(t)^2 ||e_1-e_2||_X}{t^2 \log^{\delta-\delta_2}(t)(r^2+\lambda_0(t)^2)^2}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\\ &+\frac{C\mathbb{1}_{\{r \geqslant \frac{g_0(t)}{4}\}}\lambda_0(t)^2||e_1-e_2||_X}{\log^{\delta-\delta_2}(t)(\lambda_0(t)^2+r^2)^2}\begin{cases}\frac{1}{t^2\log^b(t)}, & r\leqslant \frac{t}{2}\\ \frac{1}{r^{3/2}\sqrt{\langle t-r\rangle}\log^b(\langle t-r\rangle)}, & r>\frac{t}{2}\end{cases}\\ &+C\frac{\mathbb{1}_{\{r \geqslant \frac{g_0(t)}{4}\}}\lambda_0(t)^2||e_1-e_2||_X}{\log^{\delta-\delta_2}(t)(r^2+\lambda_0(t)^2)^2}\begin{cases}\frac{1}{t^2\log^b(t)}, & r\leqslant \frac{t}{2}\\ \begin{cases}\frac{1}{\sqrt{r}\log^b(\langle t-r\rangle)\langle t-r\rangle^{3/2}}+\frac{1}{t\langle t-r\rangle\log^b(t)\log^b(\langle t-r\rangle)}, & t>r>\frac{t}{2}\\ \frac{1}{\sqrt{r}\log^b(\langle t-r\rangle)\langle t-r\rangle^{3/2}}+\frac{1}{t^{3/2}\log^b(t)}, & r>\frac{t}{2}\end{cases}\end{cases}\end{aligned}$$

(Note that we have two estimates on $\partial_r^2 v_c$, valid in overlapping regions, which is what gives rise to the form of the estimates recorded above). Using the same procedure that we used to estimate $\partial_r^k w_2$ for $k=0,1,2$, we get

$$|w_{2,lip,0}(t,r)| \leqslant \begin{cases}\frac{Cr^2\lambda_0(t)^2\log(2+\frac{r}{g_0(t)})\log(t)||e_1-e_2||_X}{t^2(g_0(t)^2+r^2)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right), & r\leqslant\frac{t}{2}\\ \frac{C\lambda_0(t)^2||e_1-e_2||_X}{t^2\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right), & r>\frac{t}{2}\end{cases}$$

$$|\partial_r w_{2,lip,0}(t,r)| \leqslant \begin{cases}\frac{Cr\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2 g_0(t)^2\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right), & r\leqslant g_0(t)\\ \frac{C\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2 g_0(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right), & r>g_0(t)\end{cases}$$

$$|\partial_r^2 w_{2,lip,0}(t,r)| \leqslant \frac{C\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2 g_0(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right), \quad r>0$$

Next, we consider $w_3^{\lambda_1} - w_3^{\lambda_2}$, and define $WRHS_{3,lip,0}$ by

$$\begin{aligned}
&WRHS_{3,lip,0}(t,r)\\
&:= \frac{6\left(Q_{\frac{1}{\lambda_1}} + v_c^{\lambda_1}\right)}{r^2} w_{2,lip,0}\left(w_2^{\lambda_1} + w_2^{\lambda_2}\right) + \frac{2}{r^2} w_{2,lip,0}\left((w_2^{\lambda_1})^2 + w_2^{\lambda_1}w_2^{\lambda_2} + (w_2^{\lambda_2})^2\right)\\
&+ \frac{6}{r^2} w_{2,lip,0}\left((v_c^{\lambda_1})^2 + 2v_c^{\lambda_1}Q_{\frac{1}{\lambda_1}}\right)
\end{aligned}$$

As before, we also define

$$WRHS_{3,lip,1}(t,r) := WRHS_3^{\lambda_1}(t,r) - WRHS_3^{\lambda_2}(t,r) - WRHS_{3,lip,0}(t,r)$$

and, for $j = 0, 1$, we let $w_{3,lip,j}$ solve the following equation with 0 Cauchy data at infinity.

$$-\partial_{tt} w_{3,lip,j} + \partial_{rr} w_{3,lip,j} + \frac{1}{r}\partial_r w_{3,lip,j} - \frac{4}{r^2} w_{3,lip,j} = WRHS_{3,lip,j}$$

Noting the similarities between $WRHS_{3,lip,0}$ and $WRHS_3$, we get

$$|WRHS_{3,lip,0}(t,r)| \leqslant \begin{cases} \frac{Cr^2\lambda_0(t)^2\log(2+\frac{r}{g_0(t)})\log(t)||e_1-e_2||_X}{(g_0(t)^2+r^2)t^4\log^b(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r \leqslant \frac{t}{2}\\ \frac{C\lambda_0(t)^2||e_1-e_2||_X}{t^{5/2}r^{3/2}\log^b(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r > \frac{t}{2}\end{cases}$$

$$|\partial_r WRHS_{3,lip,0}(t,r)| \leqslant \frac{||e_1-e_2||_X}{\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right) \cdot \begin{cases} \frac{Cr\lambda_0(t)^2\log(t)}{g_0(t)^2t^4\log^b(t)}, & r \leqslant g_0(t)\\ \frac{C\lambda_0(t)^2\log(t)}{t^4\log^b(t)g_0(t)}, & g_0(t) \leqslant r \leqslant \frac{t}{2}\\ \frac{C\lambda_0(t)^2\log(t)}{t^{5/2}\log^b(t)r^{3/2}g_0(t)}, & r > \frac{t}{2}\end{cases}$$

$$|\partial_r^2 WRHS_{3,lip,0}(t,r)| \leqslant \frac{||e_1-e_2||_X}{\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right) \cdot \begin{cases} \frac{C\lambda_0(t)^2\log(t)}{t^4\log^b(t)g_0(t)^2}, & r \leqslant \frac{t}{2}\\ \frac{C\lambda_0(t)^2\log(t)}{t^{5/2}r^{3/2}\log^b(t)g_0(t)^2}, & r > \frac{t}{2}\end{cases}$$

The same procedure used to estimate w_3 then gives

$$|w_{3,lip,0}(t,r)| \leqslant \begin{cases} \frac{Cr^2\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2g_0(t)^2\log^b(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r \leqslant g_0(t)\\ \frac{C\lambda_0(t)^2\log^2(t)||e_1-e_2||_X}{t^2\log^b(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r > g_0(t)\end{cases}$$

$$|\partial_r w_{3,lip,0}(t,r)| \leqslant \begin{cases} \frac{Cr\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2g_0(t)^2\log^b(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r \leqslant g_0(t)\\ \frac{C\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2\log^b(t)g_0(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r > g_0(t)\end{cases}$$

$$|\partial_{rr} w_{3,lip,0}(t,r)| \leqslant \frac{C\lambda_0(t)^2\log(t)||e_1-e_2||_X}{t^2g_0(t)^2\log^b(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), \quad r > 0$$

Next, we note that

$$|WRHS_{3,lip,1}(t,r)| \leqslant \frac{C||e_1-e_2||_X \lambda_0(t)^4}{\log^{\delta-\delta_2}(t)} \begin{cases} \frac{\log(t)}{t^4 \log^{3b}(t) g_0(t)^2} + \frac{\log(2+\frac{r}{g_0(t)})\log(t)}{t^4\log^{2b}(t)(g_0(t)^2+r^2)}, & r \leqslant \frac{t}{2} \\ \frac{\log(t)}{t^{5/2}\log^{3b}(t) g_0(t)^2 r^{3/2}} + \frac{\log^2(t)}{r^{3/2}t^{5/2}\log^{2b}(t)t^2}, & r > \frac{t}{2} \end{cases}$$

After applying the procedure of (11.16), we get

$$|w_{3,lip,1}(t,r)| \leqslant \frac{C||e_1-e_2||_X \lambda_0(t)^4 \log(t)}{t^2 \log^{3b}(t) \log^{\delta-\delta_2}(t) g_0(t)^2}$$

Then, we define, for $j \geqslant 4$,

$$\begin{aligned}
&WRHS_{j,lip,0}(t,r)\\
&:= \frac{6\left(Q_{\frac{1}{\lambda_2(t)}} + v_c^{\lambda_2}\right)}{r^2}\left(w_{j-1,lip,0}(w_{j-1}^{\lambda_1}+w_{j-2}^{\lambda_2}) + 2\sum_{k=2}^{j-2} w_{k,lip,0} w_{j-1}^{\lambda_1} + 2\sum_{k=2}^{j-2} w_k^{\lambda_2} w_{j-1,lip,0}\right)\\
&+ \frac{2}{r^2}\Bigg(w_{j-1,lip,0}\left((w_{j-1}^{\lambda_1})^2 + w_{j-1}^{\lambda_1}w_{j-1}^{\lambda_2} + (w_{j-1}^{\lambda_2})^2\right) + 3w_{j-1,lip,0}(w_{j-1}^{\lambda_1}+w_{j-1}^{\lambda_2})\sum_{k=2}^{j-2} w_k^{\lambda_1}\\
&+ 3(w_{j-1}^{\lambda_2})^2 \sum_{k=2}^{j-2} w_{k,lip,0} + 3w_{j-1,lip,0}\left(\sum_{k=2}^{j-2} w_k^{\lambda_1}\right)^2\\
&+3w_{j-1}^{\lambda_2}\left(\sum_{k=2}^{j-2} w_{k,lip,0}\right)\left(\sum_{q=2}^{j-2}\left(w_q^{\lambda_1}+w_q^{\lambda_2}\right)\right)\Bigg) + \frac{6w_{j-1,lip,0}}{r^2}\left((v_c^{\lambda_1})^2 + 2v_c^{\lambda_1} Q_{\frac{1}{\lambda_1}}\right)
\end{aligned}$$

and

$$WRHS_{j,lip,1}(t,r) := WRHS_j^{\lambda_1}(t,r) - WRHS_j^{\lambda_2}(t,r) - WRHS_{j,lip,0}(t,r)$$

As with previous estimates, all estimates which we will prove by induction are valid for all $t \geqslant T_0$, provided that T_0 is sufficiently large. We will no longer explicitly write this after each such estimate. By using a similar procedure used to estimate w_j by induction, we get, for some constant $C_3 \geqslant C_2$, and all $j \geqslant 4$,

$$|w_{j,lip,0}(t,r)| \leqslant \frac{||e_1-e_2||_X}{\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right)\begin{cases} \frac{C_3^j r^2 \lambda_0(t)^2 \log^2(t)}{t^2 g_0(t)^2 \log^{b(j-2)}(t)}, & r \leqslant g_0(t) \\ \frac{C_3 \lambda_0(t)^2 \log^2(t)}{t^2 \log^{b(j-2)}(t)}, & r > g_0(t) \end{cases}$$

$$|\partial_r w_{j,lip,0}(t,r)| \leqslant \begin{cases} \frac{C_3^j r \lambda_0(t)^2 \log^2(t)||e_1-e_2||_X}{t^2 g_0(t)^2 \log^{b(j-2)}(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r \leqslant g_0(t) \\ \frac{C_3^j \lambda_0(t)^2 \log^2(t)||e_1-e_2||_X}{t^2 \log^{b(j-2)}(t) g_0(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), & r > g_0(t) \end{cases}$$

$$|\partial_r^2 w_{j,lip,0}(t,r)| \leqslant \frac{C_3^j \lambda_0(t)^2 \log^2(t)||e_1-e_2||_X}{t^2 g_0(t)^2 \log^{b(j-2)}(t)\log^{\delta-\delta_2}(t)}\left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right), \quad r > 0$$

Using a procedure similar to that used to estimate $v_j^{\lambda_1} - v_j^{\lambda_2}$, we get

$$|w_{j,lip,1}(t,r)| \leqslant \frac{C_3^j ||e_1 - e_2||_X \lambda_0(t)^4 \log(t)}{t^2 \log^{bj}(t) \log^{\delta-\delta_2}(t) g_0(t)^2}$$

Finally, this gives

$$|w_{c,ip}(t,\lambda_1(t)) - w_{c,ip}(t,\lambda_2(t))| \leqslant \frac{C||e_1 - e_2||_X \lambda_0(t)^2 \log(t)}{t^2 g_0(t)^2 \log^{\delta-\delta_2}(t)} \left(\frac{1}{\log^b(t)} + \frac{1}{\log(t)}\right)$$

When combined with our previous estimates, we get

$$|G(s,\lambda_1(s)) - G(s,\lambda_2(s))| \leqslant \frac{C||e_1 - e_2||_X}{\log^{\delta-\delta_2}(t) t^2 \log^{1+\delta_3}(t)}$$

where

$$\delta_3 = \min\{2b - 4\epsilon - 1, 3b - 4\epsilon - 2\}$$

Note that $\delta_3 > 0$, by the constraints on ϵ. This implies that there exists a constant C *independent* of T_0 such that, for all $e_1, e_2 \in \mathcal{B}$, and all $t \geqslant T_0$,

$$||T(e_1) - T(e_2)||_X \leqslant \frac{C||e_1 - e_2||_X}{\log^{\delta_3}(T_0)}$$

Combined with our previous estimates of $T(e)$ (and its derivatives) for $e \in \mathcal{B}$, we see that there exists $T_4 > T_3$ such that, for all $T_0 \geqslant T_4$, T has a fixed point, say $e_0 \in \mathcal{B}$. By inspection of the definition of T, this means that $\lambda(t) = \lambda_0(t)(1 + e_0(t))$ solves (11.37). From now on, we fix $\lambda(t) = \lambda_0(t)(1 + e_0(t))$.

11.7.3 Estimating λ'''

In this section, we will show that $e_0 \in C^3([T_0,\infty))$, and estimate $e_0'''(t)$. Estimating $e_0'''(t)$ will be done in two steps, exactly as in [26]. First, we obtain a preliminary estimate on $e_0'''(t)$ by differentiating an appropriate expression for $e_0''(t)$(see (11.41) below). Once we establish this preliminary estimate, we can differentiate $WRHS_j(t,r)$ in the t variable, and this allows us to justify a different representation formula for $\partial_t w_j$ than what was used to establish (11.31). With this different representation formula for $\partial_t w_j$, we then proceed to prove an estimate on $e_0'''(t)$ which is stronger than our preliminary estimate. As a by-product of this procedure, we obtain an estimate on $\partial_t w_j$ which is much better than (11.31), in the region $r \leqslant t$.

From (11.37), e_0 solves

$$e_0''(t) + \frac{2\lambda_0'(t)}{\lambda_0(t)} e_0'(t) = \frac{3}{2} G(t,\lambda(t))(1 + e_0(t)) \tag{11.40}$$

We start with (11.40), written in the following form.

$$e_0''(t) + \frac{2\lambda_0'(t)}{\lambda_0(t)} e_0'(t) = i_0(t) e_0''(t) + \frac{3}{2} G_{rest}(t,\lambda(t))(1 + e_0(t))$$

where

$$i_0(t) = \frac{3}{2}\lambda_0(t)\int_0^\infty \chi_{\geqslant 1}(\frac{r}{g(t)})\frac{4r^2\lambda(t)}{(\lambda(t)^2+r^2)^2}\frac{\phi_0(\frac{r}{\lambda(t)})rdr}{\lambda(t)^2}(1+e_0(t))$$

$$G_{rest}(t,\lambda(t)) = G(t,\lambda(t)) - \frac{2i_0(t)e_0''(t)}{3(1+e_0(t))}$$

There exists a constant C, independent of t, T_0, such that

$$|i_0(t)| \leqslant \frac{C}{\log^{2(b-2\epsilon)}(t)}$$

So, there exists an absolute constant $T_5 > T_4$ such that, for all $T_0 > T_5$, we have (for instance) $|i_0(t)| \leqslant \frac{1}{900000}$. Then,

$$e_0''(t) = \frac{\frac{3}{2}G_{rest}(t,\lambda(t))(1+e_0(t)) - \frac{2\lambda_0'(t)}{\lambda_0(t)}e_0'(t)}{1-i_0(t)} \tag{11.41}$$

Because $v_s, w_s \in C^2([T_0,\infty)\times[0,\infty))$, the right-hand side of (11.41) is in $C^1([T_0,\infty))$. In particular, $e_0 \in C^3([T_0,\infty))$. We will now estimate the t-derivative of the right-hand side of (11.41). We have

$$v_{s,ip}(t,\lambda(t)) = \int_0^\infty \frac{6}{r^2}\sum_{k=2}^\infty v_k(t,r)\left(1-Q_1^2(\frac{r}{\lambda(t)})\right)\phi_0(\frac{r}{\lambda(t)})\frac{rdr}{\lambda(t)^2}$$

which gives

$$|\partial_t v_{s,ip}(t,\lambda(t))| \leqslant \frac{C}{t^3\log^{2b}(t)}$$

Next, we have

$$|\partial_t\left(\int_0^\infty \chi_{\geqslant 1}(\frac{r}{g(t)})\left(\frac{4r^2(\lambda'(t))^2(r^2-3\lambda(t)^2)}{(r^2+\lambda(t)^2)^3} - \frac{6v_c(t,r)}{r^2}\left(1-Q_{\frac{1}{\lambda(t)}}(r)\right)\right)\phi_0(\frac{r}{\lambda(t)})\frac{rdr}{\lambda(t)^2}\right)|$$
$$\leqslant \frac{C\lambda(t)^2}{t^3\log^b(t)g(t)^2}$$

$$|i_0'(t)| \leqslant \frac{C\lambda_0(t)^2}{tg(t)^2}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)$$

$$|\partial_t\left(\frac{i_0(t)\left(\lambda_0''(t)(1+e_0(t))+2\lambda_0'(t)e_0'(t)\right)}{(1+e_0(t))\lambda_0(t)}\right)| \leqslant \frac{C\lambda_0(t)^2}{t^3\log^b(t)g(t)^2}$$

Using the same procedure used to estimate $E_{v_1,ip}$, we get

$$|\partial_t E_{v_1,ip}(t,\lambda(t))| \leqslant \frac{C\log(t)}{t^{7/2}}$$

Using (11.31), we get

$$|\partial_t w_{c,ip}(t,\lambda(t))| \leqslant \frac{C\log(t)}{t^2\log^b(t)g(t)}$$

As mentioned before, once we obtain a preliminary estimate on e_0''', we will be able to prove a much stronger estimate on e_0''', via improving (11.31). This gives

$$|\partial_t G_{rest}(t,\lambda(t))| \leqslant \frac{C\log(t)}{t^2\log^b(t)g(t)}$$

which leads to the following *preliminary* estimate on e_0''''.

$$|e_0''''(t)| \leqslant \frac{C\log(t)}{t^2\log^b(t)g(t)} \tag{11.42}$$

Now that we have this preliminary estimate on e_0'''', we can prove a better estimate on $\partial_t w_j$. We start with $\partial_t w_2$. We get

$$\begin{aligned}|\partial_t WRHS_2(t,r)| &\leqslant C\frac{|\chi_{\geqslant 1}(\frac{r}{g(t)})|}{\log^b(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\frac{r^2\lambda(t)^2}{t^3(r^2+\lambda(t)^2)^2}\\ &+C\chi_{\geqslant 1}(\frac{r}{g(t)})\begin{cases}\frac{r^2\lambda(t)^2}{t^3\log^b(t)(r^2+\lambda(t)^2)^2}+\frac{r^2\lambda(t)^2|e_0''''(t)|}{(r^2+\lambda(t)^2)^2}, & r\leqslant\frac{t}{2}\\ \frac{\lambda(t)^2}{r^2t^2}\left(\frac{1}{\sqrt{\langle t-r\rangle}\sqrt{r}\log^b(\langle t-r\rangle)}+\frac{1}{t\log^b(t)}+t^2|e_0''''(t)|\right), & r>\frac{t}{2}\end{cases}\end{aligned}$$

$$\begin{aligned}&|\partial_{tr}WRHS_2(t,r)|\\ &\leqslant C\frac{\mathbb{1}_{\{r\geqslant\frac{g(t)}{2}\}}}{\log^b(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\frac{r\lambda(t)^2}{t^3(r^2+\lambda(t)^2)^2}\\ &+C\mathbb{1}_{\{r\geqslant\frac{g(t)}{2}\}}\begin{cases}\frac{r\lambda(t)^2}{(r^2+\lambda(t)^2)^2}\left(\frac{1}{t^3\log^b(t)}+|e_0''''(t)|\right), & r\leqslant\frac{t}{2}\\ \frac{\lambda(t)^2}{r^3t^2}\left(\frac{1}{\sqrt{\langle t-r\rangle}\sqrt{r}\log^b(\langle t-r\rangle)}+\frac{1}{t\log^b(t)}+t^2|e_0''''(t)|\right)\\ +\frac{\lambda(t)^2}{r^4}\left(\frac{1}{\sqrt{r}\log^b(\langle t-r\rangle)\langle t-r\rangle^{3/2}}+\frac{1}{t\langle t-r\rangle\log^b(\langle t-r\rangle)\log^b(t)}\right), & t>r>\frac{t}{2}\end{cases}\end{aligned}$$

$$\begin{aligned}|\partial_{trr}WRHS_2(t,r)| &\leqslant C\frac{\mathbb{1}_{\{r\geqslant\frac{g(t)}{2}\}}}{\log^b(t)}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)\frac{\lambda(t)^2}{t^3(r^2+\lambda(t)^2)^2}\\ &+C\mathbb{1}_{\{r\geqslant\frac{g(t)}{2}\}}\begin{cases}\frac{\lambda(t)^2}{(r^2+\lambda(t)^2)^2}\left(\frac{1}{t^3\log^b(t)}+|e_0''''(t)|\right), & r\leqslant\frac{t}{2}\\ \frac{\lambda(t)^2}{r^4\sqrt{t}\log^b(\langle t-r\rangle)\langle t-r\rangle^{5/2}}+\frac{\lambda(t)^2|e_0''''(t)|}{r^4}, & t>r>\frac{t}{2}\end{cases}\end{aligned}$$

Now that we have the preliminary estimate on e_0'''', we can justify the analog of step 4 for $\partial_t w_2$, and carry out the same procedure, to get

$$|\partial_t w_2(t,r)| \leqslant \begin{cases}\frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{(g(t)^2+r^2)}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(|e_0''''(x)|x^{3/2})}{t^{3/2}}\right), & r\leqslant\frac{t}{2}\\ C\left(\frac{\lambda(t)^2}{t^{5/2}\sqrt{\langle t-r\rangle}\log^b(\langle t-r\rangle)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)\lambda(t)^2}{t^{3/2}}\right)\log^2(t), & t>r>\frac{t}{2}\end{cases}$$

$$|\partial_{tr}w_2(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)^2\log(t)}{g(t)^2}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right), & r\leqslant g(t) \\ \frac{C\lambda(t)^2}{\log^b(\langle t-r\rangle)t^{5/2}\langle t-r\rangle^{3/2}}+\frac{C\lambda(t)^2\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{5/2}} & \\ +\frac{C\lambda(t)^2\log(t)}{g(t)}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right), & g(t)<r<t \end{cases}$$

$$|\partial_{trr}w_2(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{g(t)^2}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right)+\frac{C\lambda(t)^2}{t^{5/2}\log^b(\langle t-r\rangle)\langle t-r\rangle^{5/2}}$$

Now that we have the above estimates, we proceed to estimate $\partial_t w_j$ in the region $r\leqslant t$. We start with

$$|\partial_t WRHS_3(t,r)| \leqslant \begin{cases} \frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{t^2\log^b(t)(g(t)^2+r^2)}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right), & r\leqslant \frac{t}{2} \\ \frac{C\lambda(t)^2\log^2(t)}{r^{3/2}t^{1/2}\log^b(t)}\left(\frac{1}{t^{5/2}\sqrt{\langle t-r\rangle}\log^b(\langle t-r\rangle)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right), & t>r>\frac{t}{2} \end{cases}$$

$$|\partial_{tr} WRHS_3(t,r)| \leqslant \begin{cases} \frac{Cr\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right), & r\leqslant g(t) \\ \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)}\left(\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}+\frac{1}{t^3\log^b(t)}\right), & \frac{t}{2}>r>g(t) \\ \frac{C\lambda(t)^2}{r^2\log^b(t)}\left(\frac{\log^2(t)}{\log^b(\langle t-r\rangle)t^{5/2}\langle t-r\rangle^{3/2}}+\frac{\log(t)}{g(t)}\left(\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}+\frac{1}{t^3\log^b(t)}\right)\right) & \\ +\frac{C\lambda(t)^2\log(t)}{r^{5/2}t^2\log^b(t)g(t)\log^b(\langle t-r\rangle)\sqrt{\langle t-r\rangle}}, & t>r>\frac{t}{2} \end{cases}$$

$$\begin{aligned} |\partial_{trr} WRHS_3(t,r)| \leqslant & \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right) \\ & +\frac{C\lambda(t)^2}{t^2\log^b(t)t^{5/2}\log^b(\langle t-r\rangle)\langle t-r\rangle^{5/2}}, \quad r\leqslant g(t) \end{aligned}$$

$$|\partial_{trr} WRHS_3(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)^2}\left(\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}+\frac{1}{t^3\log^b(t)}\right), \quad \frac{t}{2}>r>g(t)$$

$$\begin{aligned} |\partial_{trr} WRHS_3(t,r)| \leqslant & \frac{C}{r^2\log^b(t)}\frac{\lambda(t)^2\log(t)}{g(t)^2}\left(\frac{1}{t^3\log^b(t)}+\frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right) \\ & +\frac{C\lambda(t)^2\log(t)}{r^{5/2}t^2\log^b(t)g(t)^2\log^b(\langle t-r\rangle)\sqrt{\langle t-r\rangle}} \\ & +\frac{C\lambda(t)^2\log^2(t)}{r^{5/2}t^2\log^b(t)g(t)\log^b(\langle t-r\rangle)\langle t-r\rangle^{3/2}} \\ & +\frac{C\lambda(t)^2\log^2(t)}{r^{5/2}t^2\log^b(t)\log^b(\langle t-r\rangle)\langle t-r\rangle^{5/2}}, \quad t>r>\frac{t}{2} \end{aligned}$$

Then, we use the same observation regarding $\partial_t w_3$ as we made for $\partial_t w_2$ in this section, to get

$$|\partial_t w_3(t,r)| \leqslant \begin{cases} Cr^2 \left(\frac{\lambda(t)^2 \log(t)}{t^3 \log^{2b}(t) g(t)^2} + \frac{\lambda(t)^2 \log(t) \sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{\log^b(t) g(t)^2 t^{3/2}} \right), & r \leqslant g(t) \\ \frac{C\lambda(t)^2 \log^2(t)}{\log^b(t) t^{5/2} \sqrt{\langle t-r \rangle} \log^b(\langle t-r \rangle)} + \frac{C\lambda(t)^2 \log^2(t) \sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^b(t)}, & t > r > g(t) \end{cases} \tag{11.43}$$

$$|\partial_{tr} w_3(t,r)| \leqslant \begin{cases} Cr \left(\frac{\lambda(t)^2 \log(t)}{t^3 \log^{2b}(t) g(t)^2} + \frac{\lambda(t)^2 \log(t) \sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{\log^b(t) g(t)^2 t^{3/2}} \right), & r \leqslant g(t) \\ C \left(\frac{\lambda(t)^2 \log(t)}{g(t) \log^b(t)} \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}} + \frac{\lambda(t)^2 \log^2(t)}{\log^b(t) \log^b(\langle t-r \rangle) t^{5/2} \langle t-r \rangle^{3/2}} + \frac{\lambda(t)^2 \log(t)}{t^{5/2} \sqrt{\langle t-r \rangle} \log^b(\langle t-r \rangle) \log^b(t) g(t)} \right) \\ , \quad t > r > g(t) \end{cases}$$

and

$$|\partial_{trr} w_3(t,r) \leqslant \begin{cases} C \left(\frac{\lambda(t)^2 \log(t)}{t^3 \log^{2b}(t) g(t)^2} + \frac{\lambda(t)^2 \log(t) \sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{\log^b(t) g(t)^2 t^{3/2}} \right), & r \leqslant g(t) \\ \frac{C\lambda(t)^2 \log(t)}{\log^b(t) g(t)^2} \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}} + \frac{\lambda(t)^2 \log(t)}{t^{5/2} \log^b(t) \log^b(\langle t-r \rangle) \sqrt{\langle t-r \rangle}} \left(\frac{1}{g(t)^2} + \frac{\log(t)}{g(t)\langle t-r \rangle} + \frac{\log(t)}{\langle t-r \rangle^2} \right) \\ , \quad t > r > g(t) \end{cases}$$

Using an argument similar to that used to establish (11.30), etc., we get, after a lengthy computation, that there exists $C_4 > \max 1, C_2^p$, such that, for all $j \geqslant 4$,

$$|\partial_t w_j(t,r)| \leqslant \begin{cases} \frac{C_4^j r^2 \lambda(t)^2 \log^2(t)}{g(t)^2} \left(\frac{1}{t^3 \log^{b(j-1)}(t)} + \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^{b(j-2)}(t)} \right), & r \leqslant g(t) \\ C_4^j \lambda(t)^2 \log^2(t) \left(\frac{1}{t^{5/2} \sqrt{\langle t-r \rangle} \log^b(\langle t-r \rangle) \log^{b(j-2)}(t)} + \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^{b(j-2)}(t)} \right) \\ , \quad t > r > g(t) \end{cases}$$

$$|\partial_{tr} w_j(t,r)| \leqslant \begin{cases} \frac{C_4^j r \lambda(t)^2 \log^2(t)}{g(t)^2} \left(\frac{1}{t^3 \log^{b(j-1)}(t)} + \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^{b(j-2)}(t)} \right), & r \leqslant g(t) \\ \frac{C_4^j \lambda(t)^2 \log^2(t)}{g(t)} \left(\frac{1}{t^{5/2} \sqrt{\langle t-r \rangle} \log^b(\langle t-r \rangle) \log^{b(j-2)}(t)} + \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^{b(j-2)}(t)} \right) \\ + \frac{C_4^j \lambda(t)^2 \log^2(t)}{\log^{b(j-2)}(t) \log^b(\langle t-r \rangle) t^{5/2} \langle t-r \rangle^{3/2}}, \quad t > r > g(t) \end{cases}$$

$$|\partial_{trr} w_j(t,r)| \leqslant \begin{cases} \frac{C_4^j \lambda(t)^2 \log^2(t)}{g(t)^2} \left(\frac{1}{t^3 \log^{b(j-1)}(t)} + \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^{b(j-2)}(t)} \right), & r \leqslant g(t) \\ \frac{C_4^j \lambda(t)^2 \log^2(t)}{g(t)^2} \left(\frac{1}{t^{5/2} \sqrt{\langle t-r \rangle} \log^b(\langle t-r \rangle) \log^{b(j-2)}(t)} + \frac{\sup_{x \geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2} \log^{b(j-2)}(t)} \right) \\ + \frac{C_4^j \lambda(t)^2 \log^2(t)}{t^{5/2} \log^{b(j-2)}(t) \log^b(\langle t-r \rangle) \sqrt{\langle t-r \rangle}} \left(\frac{1}{g(t)\langle t-r \rangle} + \frac{1}{\langle t-r \rangle^2} \right), \quad t > r > g(t) \end{cases}$$

Finally, this gives

$$|\partial_t w_c(t,r)| \leqslant \sum_{k=2}^{\infty} |\partial_t w_k(t,r)|$$

$$\leqslant \begin{cases} \frac{Cr^2\lambda(t)^2\log(t)}{g(t)^2}\left(\frac{1}{t^3\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right), \quad r \leqslant g(t) \\ C\lambda(t)^2\left(\log(2+\frac{r}{g(t)}) + \frac{\log(t)}{\log^b(t)}\right)\log(t)\left(\frac{1}{t^3\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right) \\ , \quad g(t) < r < \frac{t}{2} \\ \frac{C\lambda(t)^2\log^2(t)}{\log^b(t)t^{5/2}\sqrt{\langle t-r\rangle\log^b(\langle t-r\rangle)}} + C\lambda(t)^2\log^2(t)\left(\frac{1}{t^3\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right) \\ , \quad \frac{t}{2} < r < t \end{cases}$$

Using this estimate, we then get

$$|\partial_t w_{c,ip}(t,\lambda(t))| \leqslant C\left(\frac{1}{t^3\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}}\right) \cdot \left(\frac{1}{\log^{2b-4\epsilon-1}(t)} + \frac{1}{\log^{4b-8\epsilon-1}(t)} + \frac{1}{\log^{5b-8\epsilon-2}(t)}\right)$$

Let

$$\delta_3 = \min\{2b-4\epsilon-1, 4b-8\epsilon-1, 5b-8\epsilon-2, b\}$$

Combining our previous estimates gives

$$|\partial_t G_{rest}(t,\lambda(t))| \leqslant \frac{C}{t^3\log^{b+\delta_3}(t)} + \frac{C\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}\log^{\delta_3}(t)}$$

If $\delta_4 = \min\{\delta_3, 1+\delta-\delta_2\}$, then (for C *independent* of t, T_0),

$$|e_0'''(t)| \leqslant \frac{C}{t^3\log^{b+\delta_4}(t)} + \frac{C\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|)}{t^{3/2}\log^{\delta_3}(t)}, \quad t \geqslant T_0 \tag{11.44}$$

Recalling the preliminary estimate, (11.42), we see that $x \mapsto x^{3/2}|e_0'''(x)|$ is a continuous function on $[T_0,\infty)$, which decays at infinity. Therefore, for each $t \geqslant T_0$, there exists $y(t) \geqslant t$ such that

$$\sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|) = y(t)^{3/2}|e_0'''(y(t))|$$

But then, for any $t_1 > T_0$, we evaluate (11.44) at the point $t = y(t_1)$, and get

$$\begin{aligned} y(t_1)^{3/2}|e_0'''(y(t_1))| = \sup_{x\geqslant t_1}(x^{3/2}|e_0'''(x)|) &\leqslant \frac{C}{y(t_1)^{3/2}\log^{b+\delta_4}(y(t_1))} + \frac{C\sup_{x\geqslant y(t_1)}(x^{3/2}|e_0'''(x)|)}{\log^{\delta_3}(y(t_1))} \\ &\leqslant \frac{C}{t_1^{3/2}\log^{b+\delta_4}(t_1)} + \frac{C\sup_{x\geqslant t_1}(x^{3/2}|e_0'''(x)|)}{\log^{\delta_3}(t_1)} \end{aligned}$$

where we used

$$t \mapsto \sup_{x\geqslant t}(x^{3/2}|e_0'''(x)|) \text{ is decreasing, and } y(t_1) \geqslant t_1$$

Therefore, there exists $M_2 > 0, C > 0$, *independent* of T_0 such that for all $t_1 > M_2$, we have

$$|e_0'''(t_1)| \leqslant \frac{C}{t_1^3 \log^{b+\delta_4}(t_1)}$$

Since $e_0 \in C^3([T_0,\infty))$, we conclude that there exists $C > 0$ *independent* of T_0 such that

$$|e_0'''(t)| \leqslant \frac{C}{t^3 \log^{b+\delta_4}(t)}, \quad t \geqslant T_0$$

which completes the proof of the final estimate on e_0'''.

11.7.4 Estimating λ''''

By inspection of (11.41), $e_0 \in C^4([T_0,\infty))$. Our goal in this section will be to estimate e_0''''. From (11.41), we get

$$|e_0''''(t)| \leqslant C\left(|\partial_t^2 \left(G_{rest}(t,\lambda(t))(1+e_0(t))\right)| + |\partial_t^2\left(\frac{\lambda_0'(t)e_0'(t)}{\lambda_0(t)}\right)| + |i_0''(t)e_0''(t)| + |i_0'(t)e_0'''(t)|\right)$$
$$+ C|i_0'(t)|\left(|\partial_t(G_{rest}(t,\lambda(t))\left(1+e_0(t)\right))| + |\partial_t\left(\frac{\lambda_0'(t)e_0'(t)}{\lambda_0(t)}\right)| + |i_0'(t)e_0''(t)|\right)$$

Then, we note

$$|\partial_t^2 v_{sip}(t,\lambda(t))| \leqslant \frac{C}{t^4\log^{2b}(t)}$$

$$|\partial_t^2\left(\frac{-2}{3}\frac{i_0(t)\left(\lambda_0''(t)\left(1+e_0(t)\right)+2\lambda_0'(t)e_0'(t)\right)}{\left(1+e_0(t)\right)\lambda_0(t)}\right)| \leqslant \frac{C\lambda(t)^2}{g(t)^2t^4\log^b(t)}$$

$$|i_0''(t)| \leqslant \frac{C\lambda(t)^2}{g(t)^2t^2}\left(\frac{1}{\log^b(t)}+\frac{1}{\log(t)}\right)$$

$$|\partial_t^2\left(\int_0^\infty \chi_{\geqslant 1}(\frac{r}{g(t)})\left(\frac{4r^2\lambda'(t)^2(r^2-3\lambda(t)^2)}{(r^2+\lambda(t)^2)^3} - \frac{6v_c}{r^2}\left(1-Q_1^2(\frac{r}{\lambda(t)})\right)\right)\phi_0(\frac{r}{\lambda(t)})\frac{rdr}{\lambda(t)^2}\right)|$$
$$\leqslant \frac{C\lambda(t)^2}{g(t)^2t^4\log^b(t)}$$

Using a similar procedure used to estimate $E_{v_1,ip}$, we get

$$|\partial_t^2 E_{v_1,ip}(t,\lambda(t))| \leqslant \frac{C\log(t)}{t^{9/2}}$$

Finally, we have the *preliminary* estimate

$$|\partial_t^2 w_{c,ip}(t,\lambda(t))| \leqslant \frac{C\log(t)\left(1+\frac{\log(t)}{\log^b(t)}\right)}{t^2g(t)^2\log^b(t)}$$

In total, this gives the *preliminary* estimate

$$|e_0''''(t)| \leqslant \frac{C\log(t)\left(1+\frac{\log(t)}{\log^b(t)}\right)}{t^2 g(t)^2 \log^b(t)}$$

As before, we now start to record a better estimate on $\partial_t^2 w_j$ than what was previously obtained, using a procedure which is justified by the preliminary estimate. We start with

$$|\partial_t^2 WRHS_2(t,r)| \leqslant \frac{C\mathbb{1}_{\{r\geqslant \frac{g(t)}{2}\}}\lambda(t)^2}{(g^2+r^2)}\left(|e_0''''(t)| + \begin{cases}\frac{1}{t^4\log^b(t)}, & r\leqslant \frac{t}{2}\\ \frac{1}{r^{5/2}\log^b(\langle t-r\rangle)\langle t-r\rangle^{3/2}}, & t>r>\frac{t}{2}\end{cases}\right)$$

$$|\partial_{ttr} WRHS_2(t,r)| \leqslant C\begin{cases}\frac{\mathbb{1}_{\{r\geqslant \frac{g(t)}{2}\}}\lambda(t)^2}{(r^2+g(t)^2)r}\left(\frac{1}{t^4\log^b(t)} + |e_0''''(t)|\right), & r\leqslant \frac{t}{2}\\ \frac{\lambda(t)^2|e_0''''(t)|}{r^3} + \frac{\lambda(t)^2}{r^{9/2}\langle t-r\rangle^{5/2}\log^b(\langle t-r\rangle)}, & t>r>\frac{t}{2}\end{cases}$$

$$|\partial_{ttrr} WRHS_2(t,r)| \leqslant C\begin{cases}\frac{\mathbb{1}_{\{r\geqslant \frac{g(t)}{2}\}}\lambda(t)^2}{r^2(r^2+g(t)^2)}\left(\frac{1}{t^4\log^b(t)} + |e_0''''(t)|\right), & r\leqslant \frac{t}{2}\\ \frac{\lambda(t)^2|e_0''''(t)|}{r^4} + \frac{\lambda(t)^2}{r^{9/2}\langle t-r\rangle^{7/2}\log^b(\langle t-r\rangle)}, & t>r>\frac{t}{2}\end{cases}$$

which gives

$$|\partial_t^2 w_2(t,r)| \leqslant \begin{cases}\frac{Cr^2\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{(g(t)^2+r^2)}\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right), & r\leqslant \frac{t}{2}\\ C\lambda(t)^2\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right)\log^2(t) + \frac{C\lambda(t)^2}{t^{5/2}\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)}, & t>r>\frac{t}{2}\end{cases}$$

$$\begin{aligned}&|\partial_{ttr} w_2(t,r)|\\ &\leqslant \begin{cases}\frac{Cr\lambda(t)^2\log(t)}{g(t)^2}\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right), & r\leqslant g(t)\\ \frac{C\lambda(t)^2\log(t)}{g(t)}\left(\frac{1}{t^4\log^b(t)} + |e_0''''(t)|\right) + C\lambda(t)^2\left(\frac{1}{t^{5/2}\langle t-r\rangle^{5/2}\log^b(\langle t-r\rangle)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{5/2}}\right)\\ , \quad t>r>g(t)\end{cases}\end{aligned}$$

$$|\partial_{ttrr} w_2(t,r)| \leqslant \frac{C\lambda(t)^2\log(t)}{g(t)^2}\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right) + \frac{C\lambda(t)^2}{t^{5/2}\log^b(\langle t-r\rangle)\langle t-r\rangle^{7/2}}$$

$$|\partial_t^2 WRHS_3(t,r)| \leqslant C\begin{cases}\frac{\lambda(t)^2 r^2\log(2+\frac{r}{g(t)})\log(t)}{t^2\log^b(t)(g(t)^2+r^2)}\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right), & r\leqslant \frac{t}{2}\\ \frac{\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^2\log^b(t)t^{3/2}} + \frac{\lambda(t)^2\log^2(t)}{t^{9/2}\log^b(t)\log^b(\langle t-r\rangle)\langle t-r\rangle^{3/2}}, & t>r>\frac{t}{2}\end{cases}$$

$$|\partial_{ttr} WRHS_3(t,r)| \leqslant \begin{cases}\frac{Cr\lambda(t)^2\log(t)}{t^6\log^{2b}(t)g(t)^2} + \frac{C\lambda(t)^2 r\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^2\log^b(t)g(t)^2 t^{3/2}}, & r\leqslant g(t)\\ \frac{C\lambda(t)^2\log(t)}{t^2\log^b(t)g(t)}\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right), & g(t)<r<\frac{t}{2}\\ \frac{C\lambda(t)^2\log(t)}{t^{9/2}\langle t-r\rangle^{3/2}\log^b(t)\log^b(\langle t-r\rangle)g(t)} + \frac{C\lambda(t)^2}{t^5\langle t-r\rangle^2\log^{2b}(\langle t-r\rangle)}\\ +\frac{C\lambda(t)^2\log(t)}{g(t)r^2\log^b(t)}\left(\frac{1}{t^4\log^b(t)} + \frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}}\right)\\ +\frac{C\lambda(t)^2}{t^{9/2}\langle t-r\rangle^{5/2}\log^b(t)\log^b(\langle t-r\rangle)}, & \frac{t}{2}<r<t\end{cases}$$

$$|\partial_{ttrr}WRHS_3(t,r)| \leqslant C \begin{cases} \frac{\lambda(t)^2\log(t)}{t^6\log^{2b}(t)g(t)^2} + \frac{\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^2\log^b(t)g(t)^2t^{3/2}}, & r \leqslant \frac{t}{2} \\ \frac{\lambda(t)^2\log(t)}{t^{9/2}\langle t-r\rangle^{5/2}\log^b(t)\log^b(\langle t-r\rangle)g(t)} + \frac{C\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^2\log^b(t)g(t)^2t^{3/2}} \\ +\frac{C\lambda(t)^2}{t^{9/2}\langle t-r\rangle^{7/2}\log^b(t)\log^b(\langle t-r\rangle)} + \frac{C\lambda(t)^2\log(t)}{t^{9/2}g(t)^2\log^b(t)\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)} \\ +\frac{C\lambda(t)^2\log^2(t)}{t^{9/2}\langle t-r\rangle^{7/2}\log^b(\langle t-r\rangle)\log^b(t)}, & t > r > \frac{t}{2} \end{cases}$$

and these give

$$|\partial_t^2 w_3(t,r)| \leqslant \begin{cases} Cr^2\left(\frac{\lambda(t)^2\log(t)}{t^4g(t)^2\log^{2b}(t)} + \frac{\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}\log^b(t)g(t)^2}\right), & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{\log^b(t)t^{3/2}} + \frac{C\lambda(t)^2\log^2(t)}{t^{5/2}\langle t-r\rangle^{3/2}\log^b(t)\log^b(\langle t-r\rangle)}, & t > r > g(t) \end{cases}$$

$$|\partial_{ttr} w_3(t,r)| \leqslant \begin{cases} Cr\left(\frac{\lambda(t)^2\log(t)}{t^4g(t)^2\log^{2b}(t)} + \frac{\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}\log^b(t)g(t)^2}\right), & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}g(t)\log^b(t)} + \frac{C\lambda(t)^2\log(t)}{\log^b(t)g(t)t^{5/2}\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)} \\ +\frac{C\lambda(t)^2}{\log^b(t)\log^b(\langle t-r\rangle)t^{5/2}\langle t-r\rangle^{5/2}}, & t > r > g(t) \end{cases}$$

$$|\partial_{ttrr} w_3(t,r)| \leqslant \begin{cases} C\left(\frac{\lambda(t)^2\log(t)}{t^4g(t)^2\log^{2b}(t)} + \frac{\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}\log^b(t)g(t)^2}\right), & r \leqslant g(t) \\ \frac{C\lambda(t)^2\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{\log^b(g)g(t)^2t^{3/2}} + \frac{C\lambda(t)^2\log(t)}{t^{5/2}\langle t-r\rangle^{5/2}\log^b(t)\log^b(\langle t-r\rangle)g(t)} \\ +\frac{C\lambda(t)^2\log(t)}{t^{5/2}g(t)^2\log^b(t)\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)} + \frac{C\lambda(t)^2\log^2(t)}{t^{5/2}\langle t-r\rangle^{7/2}\log^b(\langle t-r\rangle)\log^b(t)}, & t > r > g(t) \end{cases}$$

Then, using an induction argument similar to that used to estimate $\partial_t w_j$, we get that there exists $C_5 > C_4 + C_2^p$ such that, for all $j \geqslant 4$,

$$|\partial_t^2 w_j(t,r)| \leqslant \begin{cases} C_5^j r^2\left(\frac{\lambda(t)^2\log^2(t)}{t^4g(t)^2\log^{b(j-1)}(t)} + \frac{\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}\log^{b(j-2)}(t)g(t)^2}\right), & r \leqslant g(t) \\ \frac{C_5^j\lambda(t)^2\log^2(t)}{\log^{b(j-2)}(t)}\frac{\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}} + \frac{C_5^j\lambda(t)^2\log^2(t)}{t^{5/2}\langle t-r\rangle^{3/2}\log^{b(j-2)}(t)\log^b(\langle t-r\rangle)}, & t > r > g(t) \end{cases}$$

$$|\partial_{ttr} w_j(t,r)| \leqslant \begin{cases} C_5^j r\left(\frac{\lambda(t)^2\log^2(t)}{t^4g(t)^2\log^{b(j-1)}(t)} + \frac{\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}\log^{b(j-2)}(t)g(t)^2}\right), & r \leqslant g(t) \\ \frac{C_5^j\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}g(t)\log^{b(j-2)}(t)} + \frac{C_5^j\lambda(t)^2\log^2(t)}{\log^{b(j-2)}(t)t^{5/2}\langle t-r\rangle^{3/2}\log^b(\langle t-r\rangle)}\left(\frac{1}{g(t)} + \frac{1}{\langle t-r\rangle}\right) \\ , \quad t > r > g(t) \end{cases}$$

$$|\partial_{ttrr} w_j(t,r)|$$
$$\leqslant \begin{cases} C_5^j\left(\frac{\lambda(t)^2\log^2(t)}{t^4g(t)^2\log^{b(j-1)}(t)} + \frac{\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{t^{3/2}g(t)^2\log^{b(j-2)}(t)}\right), & r \leqslant g(t) \\ \frac{C_5^j\lambda(t)^2\log^2(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{\log^{b)(j-2)}(t)g(t)^2t^{3/2}} + \frac{C_5^j\lambda(t)^2\log^2(t)}{t^{5/2}\langle t-r\rangle^{3/2}\log^{b(j-2)}(t)\log^b(\langle t-r\rangle)}\left(\frac{1}{\langle t-r\rangle g(t)} + \frac{1}{g(t)^2} + \frac{1}{\langle t-r\rangle^2}\right) \\ , \quad t > r > g(t) \end{cases}$$

This leads to the improved estimate

$$|\partial_t^2 w_{c,ip}(t,\lambda(t))| \leqslant \frac{C\log(t)}{t^4\log^{3b-4\epsilon}(t)} + \frac{C\log(t)\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{\log^{2b-4\epsilon}(t)t^{3/2}}$$

which, when combined with our previous estimates from this section, yields

$$|e_0''''(t)| \leqslant \frac{C}{t^4\log^{b+\delta_5}(t)} + \frac{C\sup_{x\geqslant t}(x^{3/2}|e_0''''(x)|)}{\log^{2b-4\epsilon-1}(t)t^{3/2}}$$

where

$$\delta_5 = \min\{b, 2b - 4\epsilon - 1\}$$

Repeating the argument used to estimate e_0''', we conclude that there exists $C > 0$, *independent* of T_0 such that

$$|e_0''''(t)| \leqslant \frac{C}{t^4 \log^{b+\delta_5}(t)}, \quad t \geqslant T_0$$

11.7.5 Symbol-Type estimates on F_4

Our goal will be to establish symbol-type estimates on F_4 in the region $r \leqslant \frac{t}{2}$. In order to obtain these estimates, we first have to obtain improved estimates on $\partial_r w_c$, $\partial_r^2 w_c$, and $\partial_{tr} w_c$ in the region $\frac{t}{2} > r > g(t)$. So, we start with the following lemma.

Lemma 11.8. *We have the following symbol-type estimates on* w_c. *For* $0 \leqslant j, k \leqslant 2, \quad j + k \leqslant 2,$

$$t^j r^k |\partial_r^k \partial_t^j w_c(t,r)| \leqslant \begin{cases} \frac{Cr^2 \lambda(t)^2 \log(t)}{t^2 g(t)^2 \log^b(t)}, & r \leqslant g(t) \\ \frac{C\lambda(t)^2 \log(t)}{t^2 \log^b(t)} \left(\log(2 + \frac{r}{g(t)}) + \frac{\log(t)}{\log^b(t)}\right), & g(t) < r < \frac{t}{2} \end{cases} \tag{11.45}$$

Proof. First, we recall that $w_2(t,r) = \int_t^\infty ds w_{2,s}(t,r)$, where $w_{2,s}$ solves

$$\begin{cases} -\partial_{tt} w_{2,s} + \partial_{rr} w_{2,s} + \frac{1}{r}\partial_r w_{2,s} - \frac{4}{r^2} w_{2,s} = 0 \\ w_{2,s}(s,r) = 0 \\ \partial_t w_{2,s}(s,r) = WRHS_2(s,r) \end{cases}$$

Therefore, since $t < s$, we expect to obtain better decay for $(-\partial_t + \partial_r) w_{2,s}$ than what we would have for simply $\partial_r w_{2,s}$. We compute $\partial_t w_2$ by starting with (11.29), and differentiating under the integral sign. The key point we will use to obtain our estimate is that, for a differentiable function g, and $s - t > 0, \quad 2\pi > \theta > 0$, and $q, r > 0$, we have

$$\begin{aligned} &-\partial_t \left(g(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)})\right) \\ &= g'(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)}) \frac{(s-t)q^2 + rq\cos(\theta)}{\sqrt{r^2 + (s-t)^2 q^2 + 2rq(s-t)\cos(\theta)}} \end{aligned}$$

and

$$\begin{aligned} &\partial_r \left(g(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)})\right) \\ &= g'(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)}) \frac{r + q(s-t)\cos(\theta)}{\sqrt{r^2 + (s-t)^2 q^2 + 2rq(s-t)\cos(\theta)}} \end{aligned}$$

So,

$$\begin{aligned} &(-\partial_t + \partial_r) \left(g(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)})\right) \\ &= g'(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)}) \frac{((s-t)q + r)(1 + \cos(\theta))}{\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)}} \\ &+ g'(\sqrt{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)})\,(q-1)\, \frac{(s-t)q + r\cos(\theta)}{\sqrt{r^2 + (s-t)^2 q^2 + 2rq(s-t)\cos(\theta)}} \end{aligned}$$

For ease of notation, and later use, we introduce the following vectors in $\mathbb{R}^2$:

$$x = r\mathbf{e}_1, \quad y = q(s-t)(\cos(\theta), \sin(\theta))$$

So, the above can be written as

$$\begin{aligned}
(-\partial_t + \partial_r)\left(g(|x+y|)\right) &= \left(g'(|x+y|) \cdot |x+y|\right) \frac{((s-t)q + r)\left(1+\cos(\theta)\right)}{r^2 + q^2(s-t)^2 + 2rq(s-t)\cos(\theta)} \\
&+ g'(|x+y|)\,(q-1)\,\frac{\widehat{y}\cdot(x+y)}{|x+y|} \\
&= \left(g'(|x+y|) \cdot |x+y|\right) \frac{((s-t)q + r)\left(1+\cos(\theta)\right)}{(r - q(s-t))^2 + 2rq(s-t)(1+\cos(\theta))} \\
&+ g'(|x+y|)\,(q-1)\,\frac{\widehat{y}\cdot(x+y)}{|x+y|}
\end{aligned} \tag{11.46}$$

We will apply this observation to the integrand of the representation formula for w_2. The third line of (11.46) has a gain of decay, as can be seen below. We note, for $s-t, q, r > 0$, and $0 < \theta < 2\pi$, that

$$(r-q(s-t))^2 + 2rq(s-t)(1+\cos(\theta)) > \begin{cases} Cr^2, & q(s-t) < \frac{r}{2} \\ Crq(s-t)(1+\cos(\theta)), & \frac{r}{2} < q(s-t) < 2r \\ Cq^2(s-t)^2, & 2r < q(s-t) \end{cases}$$

Therefore,

$$\frac{((s-t)q + r)\left(1+\cos(\theta)\right)}{(r-q(s-t))^2 + 2rq(s-t)(1+\cos(\theta))} \leqslant \frac{C}{\max\{r, q(s-t)\}}$$

Note that the factor $1+\cos(\theta)$ in the numerator of the above expression is crucial in obtaining this estimate. On the other hand, the fourth line of (11.46) has the factor $q-1$. The integrand of the representation formula for w_2 contains a factor of $\frac{1}{\sqrt{1-q^2}}$. Therefore, the factor $q-1$ cancels the singularity of the integrand of the representation formula for w_2, and we can obtain a gain of decay for this term, by appropriately integrating by parts in q, as will be seen below. The decomposition of (11.46), into one term which gains decay in $r + q(s-t)$, and another, which vanishes at $q=1$, would not be possible if we only applied, for example, ∂_r to w_2.

Letting $\rho = q(s-t)$, we now have $|x+y| = \sqrt{r^2+\rho^2+2r\rho\cos(\theta)}$, like before, and we get

$$
\begin{aligned}
&-\left(-\partial_t+\partial_r\right) w_2(t,r) \\
&= \frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} WRHS_2(s,|x+y|) \\
&\qquad \left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right) d\theta d\rho ds \\
&+ \frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \\
&\qquad \int_0^{2\pi} \partial_2 WRHS_2(s,|x+y|)|x+y| \cdot \left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right) \\
&\qquad \cdot \left(\frac{(r+\rho)(1+\cos(\theta))}{(r^2+\rho^2+2r\rho\cos(\theta))}\right) d\theta d\rho ds \\
&- \frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} WRHS_2(s,|x+y|)\frac{4\rho^2\sin^2(\theta)}{(s-t)(r^2+\rho^2+2r\rho\cos(\theta))} d\theta d\rho ds \\
&+ \frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} WRHS_2(s,|x+y|)\frac{2\rho^2\sin^2(\theta)}{(r^2+\rho^2+2r\rho\cos(\theta))^2} \\
&\qquad \cdot 2(r+\rho)(1+\cos(\theta)) d\theta d\rho ds \\
&- \frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \rho\frac{\sqrt{s-t-\rho}}{\sqrt{s-t+\rho}} \int_0^{2\pi} \frac{1}{(s-t)}\partial_\rho \Big(WRHS_2(s,\sqrt{r^2+\rho^2+2r\rho\cos(\theta)}) \\
&\qquad \cdot \left(1-\frac{2\rho^2\sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right)\Big) d\theta d\rho ds \\
&- \frac{1}{2\pi}\int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} WRHS_2(s,|x+y|) \\
&\qquad \cdot \frac{-4\rho\sin^2(\theta)}{(r^2+\rho^2+2r\rho\cos(\theta))}\left(\frac{\rho}{s-t}-1\right) d\theta d\rho ds
\end{aligned}
$$

We split the s integration into two regions, one with $s-t \leqslant \frac{r}{8}$, and another, with $s-t \geqslant \frac{r}{8}$. In the region with $s-t \leqslant \frac{r}{8}$, we have $\rho \leqslant s-t \leqslant \frac{r}{8}$, and can proceed roughly as we did when previously estimating $\partial_r w_2$. On the other hand, in the region $s-t \geqslant \frac{r}{8}$, we use our previous observations, and integrate by parts in the fifth line of the above expression to get

$$\begin{aligned}
&|\left(-\partial_t+\partial_r\right)w_2(t,r)| \leqslant C\int_t^{t+\frac{r}{8}}\int_0^{s-t}\frac{\rho}{(s-t)\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}|WRHS_2(s,|x+y|)|d\theta d\rho ds\\
&+C\int_t^{t+\frac{r}{8}}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\\
&\qquad\int_0^{2\pi}\left(|\partial_2 WRHS_2(s,|x+y|)|+\frac{|WRHS_2(s,|x+y|)|}{|x+y|}\right)d\theta d\rho ds\\
&+C\int_{t+\frac{r}{8}}^{\infty}\int_0^{s-t}\frac{\rho}{\sqrt{(s-t)^2-\rho^2}}\int_0^{2\pi}\Bigg(\frac{|\partial_2 WRHS_2(s,|x+y|)|\cdot|x+y|}{(r+\rho)}\\
&\qquad+\frac{|WRHS_2(s,|x+y|)|}{(r+\rho)}\Bigg)d\theta d\rho ds\\
&+C\int_{t+\frac{r}{8}}^{\infty}\int_0^{s-t}\int_0^{2\pi}\frac{1}{(s-t)}|WRHS_2(s,|x+y|)|d\theta d\rho ds
\end{aligned}$$

Finally, the third and fourth lines of the above expression shows the gain we obtain when applying $-\partial_t+\partial_r$ to w_2. Using a procedure similar to that used to estimate w_2, we get

$$|\partial_r w_2(t,r)| \leqslant \begin{cases}\frac{C\lambda(t)^2\log(2+\frac{r}{g(t)})\log(t)}{t^2\log^b(t)\sqrt{r^2+g(t)^2}}, & \frac{t}{2}>r>g(t)\\ \frac{C\lambda(t)^2\log^2(t)}{t^{5/2}\sqrt{\langle t-r\rangle}\log^b(\langle t-r\rangle)}, & t>r>\frac{t}{2}\end{cases} \tag{11.47}$$

Using our previously recorded estimate on $\partial_r w_2$ in the region $r\leqslant g(t)$, as well as our improved estimate, (11.47) for $t>r>g(t)$, we get

$$|\partial_r WRHS_3(t,r)|+\frac{|WRHS_3(t,r)|}{r} \leqslant \begin{cases}\frac{Cr\lambda(t)^2\log(t)\log(2+\frac{r}{g(t)})}{t^4\log^{2b}(t)(r^2+g(t)^2)}, & r\leqslant\frac{t}{2}\\ \frac{C\lambda(t)^2\log^2(t)}{t^{9/2}\langle t-r\rangle\log^b(t)\log^b(\langle t-r\rangle)}, & t>r>\frac{t}{2}\end{cases}$$

We then use the same procedure as above to estimate $\left(-\partial_t+\partial_r\right)w_3$. Combining this with the final estimate on $\partial_t w_3$, namely (11.43), we get

$$|\partial_r w_3(t,r)| \leqslant \begin{cases}\frac{C\lambda(t)^2\log^2(t)}{t^3\log^{2b}(t)}, & g(t)<r<\frac{t}{2}\\ \frac{C\lambda(t)^2\log^2(t)}{t^{5/2}\sqrt{\langle t-r\rangle}\log^b(t)\log^b(\langle t-r\rangle)}, & \frac{t}{2}<r<t\end{cases}$$

With the same procedure used to estimate $\partial_r w_3$, and an induction argument similar to those previously used, we get: there exists $C_6>C_4+C_2^p$ such that, for all $j\geqslant 4$, we have

$$|\partial_r w_j(t,r)| \leqslant \begin{cases}\frac{C_6^j\lambda(t)^2\log^2(t)}{t^3\log^{b(j-1)}(t)}, & g(t)<r<\frac{t}{2}\\ \frac{C_6^j\lambda(t)^2\log^2(t)}{t^{5/2}\sqrt{\langle t-r\rangle}\log^{b(j-2)}(t)\log^b(\langle t-r\rangle)}, & \frac{t}{2}<r<t\end{cases}$$

Next, we will estimate $\partial_{tr} w_j$, for $j \geqslant 2$. For this, we start with the formula

$$\begin{aligned}
&\partial_t w_2(t,r) \\
&= \frac{-1}{2\pi} \int_t^\infty \int_0^{s-t} \frac{\rho}{\sqrt{(s-t)^2-\rho^2}} \int_0^{2\pi} \partial_1 WRHS_2(s, \sqrt{r^2+\rho^2+2r\rho\cos(\theta)}) \\
&\qquad \cdot \left(1 - \frac{2\rho^2 \sin^2(\theta)}{r^2+\rho^2+2r\rho\cos(\theta)}\right) d\theta d\rho ds
\end{aligned}$$

which was used when obtaining the final estimate on λ''', and which follows from the fact that $\partial_t w_2$ solves the same equation as w_2 does, also with 0 Cauchy data at infinity, except with $\partial_t WRHS_2$ on the right-hand side. Then, we estimate $(-\partial_t + \partial_r)\, \partial_t w_2$ with the same argument used to estimate $(-\partial_t + \partial_r)\, w_2$. Then, we repeat this argument for w_j, for $j \geqslant 3$, and get

$$|\partial_{tr} w_2(t,r)| \leqslant \begin{cases} \frac{C\lambda(t)^2 \log(t) \log(2+\frac{r}{g(t)})}{\sqrt{r^2+g(t)^2 t^3 \log^b(t)}}, & g(t) < r < \frac{t}{2} \\ \frac{C\lambda(t)^2 \log^2(t)}{t^{5/2} \langle t-r\rangle^{3/2} \log^b(t)}, & \frac{t}{2} < r < t \end{cases}$$

$$|\partial_{tr} w_3(t,r)| \leqslant \begin{cases} \frac{C\lambda(t)^2 \log^2(t)}{t^4 \log^{2b}(t)}, & g(t) < r \leqslant \frac{t}{2} \\ \frac{C\lambda(t)^2 \log^2(t)}{t^{5/2}\langle t-r\rangle^{3/2} \log^b(t) \log^b(\langle t-r\rangle)}, & \frac{t}{2} \leqslant r < t \end{cases}$$

There exists $C_7 > 3\left(C_2^p + C_4 + C_5 + C_6\right)$ such that for $j \geqslant 4$, we have

$$|\partial_{tr} w_j(t,r)| \leqslant \begin{cases} \frac{C_7^j \lambda(t)^2 \log^2(t)}{t^4 \log^{b(j-1)}(t)}, & g(t) < r < \frac{t}{2} \\ \frac{C_7^j \lambda(t)^2 \log^2(t)}{t^{5/2}\langle t-r\rangle^{3/2} \log^{b(j-2)}(t) \log^b(\langle t-r\rangle)}, & \frac{t}{2} < r < t \end{cases}$$

Finally, we read off estimates on $r^2 \partial_r^2 w_j$ by inspection of the equation it solves, and our previous estimates. This gives

$$|r^2 \partial_r^2 w_2(t,r)| \leqslant \frac{C\lambda(t)^2 \log(2+\frac{r}{g(t)}) \log(t)}{t^2 \log^b(t)}, \quad g(t) \leqslant r \leqslant \frac{t}{2}$$

$$|r^2 \partial_r^2 w_3(t,r)| \leqslant \frac{C\lambda(t)^2 \log^2(t)}{t^2 \log^{2b}(t)}, \quad g(t) \leqslant r \leqslant \frac{t}{2}$$

and, for $j \geqslant 4$,

$$|r^2 \partial_r^2 w_j(t,r)| \leqslant \frac{C_7^j \lambda(t)^2 \log^2(t)}{t^2 \log^{b(j-1)}(t)}, \quad g(t) \leqslant r \leqslant \frac{t}{2}$$

Combining all of our previous estimates, we get (11.45) □

Then, we obtain, for $0 \leqslant j,k \leqslant 2$, and $j+k \leqslant 2$,

$$\begin{aligned}
|r^k \partial_r^k t^j \partial_t^j F_4(t,r)| &\leqslant C \mathbb{1}_{\{r \leqslant g(t)\}} \frac{r^2 \lambda(t)^2}{t^2 \log^b(t) (\lambda(t)^2 + r^2)^2} \\
&\quad + C \frac{\mathbb{1}_{\{r \leqslant \frac{t}{2}\}} \lambda(t)^2}{(\lambda(t)^2+r^2)^2} \begin{cases} \frac{r^2 \lambda(t)^2 \log(t)}{t^2 g(t)^2 \log^b(t)}, & r \leqslant g(t) \\ \frac{\lambda(t)^2 \log(t)}{t^2 \log^b(t)} \left(\log(2+\frac{r}{g(t)}) + \frac{\log(t)}{\log^b(t)}\right), & g(t) < r < \frac{t}{2} \end{cases}
\end{aligned} \tag{11.48}$$

12 Solving the final equation (Yang-Mills)

Substituting $u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r) + w_c(t,r) + v(t,r)$ into (8.1), we get

$$-\partial_{tt}v + \partial_{rr}v + \frac{1}{r}\partial_r v + \frac{2}{r^2}\left(1 - 3Q_{\frac{1}{\lambda(t)}}(r)^2\right)v = F_4(t,r) + F_5(t,r) + F_3(t,r) \tag{12.1}$$

where F_4 and F_5 were defined in (11.33) and (11.34), and

$$\begin{aligned} F_3(t,r) = \frac{2v(t,r)^3}{r^2} &+ \frac{6\left(Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r) + w_c(t,r)\right)}{r^2}v(t,r)^2 \\ &+ \frac{6v(t,r)}{r^2}\left(\left(v_c(t,r) + Q_{\frac{1}{\lambda(t)}}(r) + w_c(t,r)\right)^2 - Q_{\frac{1}{\lambda(t)}}(r)^2\right) \end{aligned} \tag{12.2}$$

We will (formally) derive the equation for the distorted Fourier transform (as defined in [15]) of an appropriate re-scaling of v. We will call this function y. Then, we will show that the equation for y has a (weak) solution with enough regularity to rigorously justify the inverse of each step we perform to derive its equation, thereby obtaining a (weak) solution to the original equation, (12.1). We start by defining w by

$$v(t,r) = w(t,\frac{r}{\lambda(t)})\sqrt{\frac{\lambda(t)}{r}}$$

and evaluate (12.1) at the point $(t, R\lambda(t))$. Then, we get

$$\begin{aligned} &-\partial_{tt}w + \frac{2R\lambda'(t)}{\lambda(t)}\partial_{tR}w - \frac{\lambda'(t)}{\lambda(t)}\partial_t w - R^2\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2\left(\partial_{RR}w - \frac{1}{R}\partial_R w + \frac{3}{4R^2}w\right) \\ &+ R\left(\partial_R w - \frac{w}{2R}\right)\left(\frac{\lambda''(t)}{\lambda(t)} - 2\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2\right) + \frac{1}{\lambda(t)^2}\left(\partial_{RR}w + \frac{24}{(1+R^2)^2}w - \frac{15}{4R^2}w\right) \\ &= \sqrt{R}\left(F_4(t,R\lambda(t)) + F_5(t,R\lambda(t)) + F_3(t,R\lambda(t))\right) \end{aligned} \tag{12.3}$$

We will now use the distorted Fourier transform, described in [15]. Two important notational differences are that we denote by $\widetilde{\phi}_0$ what is denoted by ϕ_0 in that paper, and we use $\mathcal{F}$ to denote the distorted Fourier transform, rather than $\widehat{\cdot}$. So, we have

$$\widetilde{\phi}_0(r) = \frac{r^{5/2}}{(1+r^2)^2} = \sqrt{r}\phi_0(r)$$

We follow the notation of [15], which regards the distorted Fourier transform of $f \in L^2((0,\infty))$ as a two-component vector $\begin{bmatrix} a \\ g(\cdot) \end{bmatrix}$, where $a = \langle f, \widetilde{\phi}_0 \rangle_{L^2(dr)}$. We will use the transference operator, $\mathcal{K}$, of [15], and recall its definition as

$$\mathcal{F}(r\partial_r u) = -2\xi\partial_\xi \mathcal{F}(u) + \mathcal{K}(\mathcal{F}(u))$$

where $-2\xi\partial_\xi$ acts only on the second component of $\mathcal{F}(u)$. Then, for y defined by

$$\mathcal{F}(w)(t,\xi) = \begin{bmatrix} y_0(t) \\ y_1(t, \frac{\xi}{\lambda(t)^2}) \end{bmatrix}$$

we evaluate (12.3) at the point $(t, \omega\lambda(t)^2)$ to get

$$\begin{bmatrix} -\partial_{tt}y_0 \\ -\partial_{tt}y_1 - \omega y_1 \end{bmatrix} = F_2 + \mathcal{F}\left(\sqrt{\cdot}\left(F_3 + F_4 + F_5\right)(t, \cdot\lambda(t))\right)\left(\omega\lambda(t)^2\right) \tag{12.4}$$

where

$$\begin{aligned}
F_2 = &-\left(\frac{-\lambda'(t)}{\lambda(t)}\begin{bmatrix} y_0'(t) \\ \partial_1 y_1(t,\omega) \end{bmatrix} - \frac{3}{4}\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2 \begin{bmatrix} y_0(t) \\ y_1(t,\omega) \end{bmatrix} - \frac{1}{2}\begin{bmatrix} y_0(t) \\ y_1(t,\omega) \end{bmatrix}\left(\frac{\lambda''(t)}{\lambda(t)} - 2\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2\right)\right.\\
&+ 2\left(\frac{\lambda'(t)}{\lambda(t)}\right)\mathcal{K}\left(\begin{bmatrix} y_0'(t) \\ \partial_1 y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)(\omega\lambda(t)^2) - 2\left(\frac{\lambda'(t)}{\lambda(t)}\right)^2 [\mathcal{K}, \xi\partial_\xi]\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)(\omega\lambda(t)^2)\\
&\left. - \left(\frac{\lambda'(t)}{\lambda(t)}\right)^2 \mathcal{K}\left(\mathcal{K}\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)\right)(\omega\lambda(t)^2) + \frac{\lambda''(t)}{\lambda(t)}\mathcal{K}\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)(\omega\lambda(t)^2)\right)
\end{aligned}$$

As in the definition of the transference operator, in the expression for F_2, the operator $\xi\partial_\xi$ appearing in the term involving $[\mathcal{K}, \xi\partial_\xi]\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)$ only acts on the second component of $\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}$. Also, the symbols $(\omega\lambda(t)^2)$ appearing after, for instance $\mathcal{K}\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)$ mean that the second component of $\mathcal{K}\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)$ is evaluated at the point $\omega\lambda(t)^2$. (Recall that the first component of $\mathcal{K}\left(\begin{bmatrix} y_0(t) \\ y_1(t, \frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)$ is a real number, rather than a function of frequency).

12.1 The Iteration Space

We will now describe the space in which we plan to solve (12.4). We let $(Z, ||\cdot||_Z)$ denote the normed vector space defined as follows. Z is the set of elements $\begin{bmatrix} y_0(t) \\ y_1(t,\omega) \end{bmatrix}$ where $y_0 : [T_0, \infty) \to \mathbb{R}$, and y_1 is an (equivalence class) of measureable functions, $y_1 : [T_0, \infty) \times (0, \infty) \to \mathbb{R}$ such that

$$\begin{gathered}
y_0(t) \in C^1_t([T_0, \infty))\\
y_1(t,\omega)\frac{t^2\log^\epsilon(t)}{\lambda(t)}\langle\omega\lambda(t)^2\rangle^{3/2}\sqrt{\rho(\omega\lambda(t)^2)} \in C^0_t([T_0,\infty), L^2(d\omega))\\
\partial_t y_1(t,\omega)\frac{t^3\log^\epsilon(t)}{\lambda(t)}\langle\omega\lambda(t)^2\rangle\sqrt{\rho(\omega\lambda(t)^2)} \in C^0_t([T_0,\infty), L^2(d\omega))
\end{gathered}$$

and $||\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}||_Z < \infty$ where

$$\begin{aligned}
||\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}||_Z = \sup_{t \geqslant T_0}&\left(\frac{t^2\log^\epsilon(t)|y_0(t)|}{\lambda(t)^2} + \frac{\log^\epsilon(t)\lambda(t)t^2||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}}{\lambda(t)^2}\right.\\
&\left.+\frac{t^3\log^\epsilon(t)|y_0'(t)|}{\lambda(t)^2} + \frac{t^3\log^\epsilon(t)\lambda(t)||\partial_t y_1(t,\omega)\langle\omega\lambda(t)^2\rangle||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}}{\lambda(t)^2}\right)
\end{aligned} \tag{12.5}$$

12.2 F_2 estimates

We will now estimate F_2, for all $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in \overline{B_1(0)} \subset Z$. We use Proposition 5.2 of [15], which states that $\mathcal{K}$ and $[\mathcal{K}, \xi\partial_\xi]$ are bounded on $L^{2,\alpha}_\rho$, for all $\alpha \in \mathbb{R}$, which, as defined in [15], has norm

$$||f||^2_{L^{2,\alpha}_\rho} = |f(0)|^2 + \int_0^\infty |f(\xi)|^2 \langle\xi\rangle^{2\alpha} \rho(\xi) d\xi$$

Then, we proceed exactly as in [26], to get

$$|F_{2,0}(x)| \leqslant \frac{C\lambda(x)^2}{x^4 \log^{b+\epsilon}(x)}$$

$$\lambda(x)||F_{2,1}(x,\omega)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{C\lambda(x)^2}{x^4 \log^{b+\epsilon}(x)}$$

$$\lambda(x)^4||\omega F_{2,1}(x,\omega)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant C\frac{\lambda(x)^3}{x^4 \log^{b+\epsilon}(x)}$$

where we define $F_{2,i}$ by

$$F_2 = \begin{bmatrix} F_{2,0} \\ F_{2,1} \end{bmatrix}$$

12.3 F_3 estimates

The main result of this section is

Proposition 12.1. *We have the following estimates*

$$\begin{aligned}
&||F_3(t, R\lambda(t))||_{L^2(RdR)} \\
&\leqslant \frac{C}{\lambda(t)^2} \Big(y_0(t)^2 + \lambda(t)^2 ||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)} + |y_0(t)|^3 \\
&\qquad + \lambda(t)^3 ||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\Big) \\
&+ C\left(|y_0(t)| + \lambda(t)||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right) \\
&\cdot ||\frac{2v_{corr}(t,R\lambda(t))Q_1(R) + v_{corr}^2(t,R\lambda(t))}{R^2\lambda(t)^2}||_{L^\infty_R}
\end{aligned} \tag{12.6}$$

and

$$
\begin{aligned}
&||L^{*}L(F_3(t,R\lambda(t)))||_{L^2(RdR)}\\
&\leqslant C\frac{\left((y_0(t))^2+\lambda(t)^2||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||^2_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)}{\lambda(t)^2}\\
&+\frac{C\left(|y_0(t)|^3+\lambda(t)^3||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||^3_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)}{\lambda(t)^2}\\
&+C\left(|y_0(t)|+\lambda(t)||y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\right)\\
&\cdot\Bigg(||\frac{2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))}{R^2\lambda(t)^2}||_{L^\infty_R}\\
&+\sup_{R\geqslant 1}\left(\frac{|\partial_R\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2R^2}\right)\\
&+\sup_{R\geqslant 1}\left(\frac{|\partial_R^2\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{R^2\lambda(t)^2}\right)\\
&+\sup_{R\leqslant 1}\left(\frac{|\partial_R\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2R}\right)\\
&+\sup_{R\leqslant 1}\left(\frac{|\partial_R^2\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2}\right)\Bigg)
\end{aligned}
\tag{12.7}
$$

In addition,

$$
\begin{aligned}
&\Bigg(||\frac{2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))}{R^2\lambda(t)^2}||_{L^\infty_R}\\
&+\sup_{R\geqslant 1}\left(\frac{|\partial_R\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2R^2}\right)\\
&+\sup_{R\geqslant 1}\left(\frac{|\partial_R^2\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{R^2\lambda(t)^2}\right)\\
&+\sup_{R\leqslant 1}\left(\frac{|\partial_R\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2R}\right)\\
&+\sup_{R\leqslant 1}\left(\frac{|\partial_R^2\left(2v_{corr}(t,R\lambda(t))Q_1(R)+v_{corr}^2(t,R\lambda(t))\right)|}{\lambda(t)^2}\right)\Bigg)\\
&\leqslant\frac{C}{t^2\log^b(t)}
\end{aligned}
\tag{12.8}
$$

Proof. Like in [26], it is convenient for us to introduce the operator L defined by

$$L(f)=-f'(r)+2\left(\frac{1-r^2}{1+r^2}\right)\frac{f(r)}{r}$$

which has the formal adjoint on $L^2(rdr)$ given by

$$L^{*}(f)=f'(r)+2\left(\frac{1-r^2}{1+r^2}\right)\frac{f(r)}{r}+\frac{f(r)}{r}$$

As noted in [28], L satisfies

$$L^*L(u)(r) = -\partial_{rr}u - \frac{1}{r}\partial_r u - \frac{2}{r^2}\left(1-3Q_1(r)^2\right)u$$

We start with a lemma similar to Lemma 5.1 of [26].

Lemma 12.1. *There exists $C>0$ such that, for all $\begin{bmatrix}\overline{y_0}\\ \overline{y_1}(\xi)\end{bmatrix}$ with $\overline{y_1}(\xi)\langle\xi\rangle^{3/2} \in L^2((0,\infty),\rho(\xi)d\xi)$, if $\overline{v}$ is given by*

$$\overline{v}(r) = \frac{1}{\sqrt{r}}\mathcal{F}^{-1}\left(\begin{bmatrix}\overline{y_0}\\ \overline{y_1}\end{bmatrix}\right)(r) = \overline{y_0}\frac{\phi_0(r)}{||\phi_0||^2_{L^2(rdr)}} + \int_0^\infty \frac{\phi(r,\xi)}{\sqrt{r}}\overline{y_1}(\xi)\rho(\xi)d\xi$$

then, $\partial_r\overline{v}$ and $\overline{v}$ admit continuous extensions to $[0,\infty)$ (which we also denote by $\partial_r\overline{v}$ and $\overline{v}$):

$$\partial_r\overline{v}, \overline{v} \in C^0([0,\infty))$$

$$\overline{v}(0) = \partial_r\overline{v}(0) = \lim_{r\to\infty}\overline{v}(r) = \lim_{r\to\infty}\partial_r\overline{v}(r) = 0$$

$$||\overline{v}||_{L^2(rdr)} \leqslant C\left(|\overline{y_0}| + ||\overline{y_1}(\xi)||_{L^2(\rho(\xi)d\xi)}\right) \tag{12.9}$$

$$||L\overline{v}||_{L^2(rdr)} = ||\sqrt{\xi}\overline{y_1}(\xi)||_{L^2(\rho(\xi)d\xi)} \tag{12.10}$$

$$||L^*L\overline{v}||_{L^2(rdr)} = ||\xi\overline{y_1}(\xi)||_{L^2(\rho(\xi)d\xi)} \tag{12.11}$$

$$\frac{|\overline{v}(r)|}{r^2\sqrt{\langle\log(r)\rangle}} \leqslant C\left(|\overline{y_0}| + ||\overline{y_1}(\xi)\langle\xi\rangle^{3/2}||_{L^2(\rho(\xi)d\xi)}\right), \quad r\leqslant 1 \tag{12.12}$$

$$|L\overline{v}(r)| \leqslant Cr\sqrt{\langle\log(r)\rangle}||\overline{y_1}(\xi)\langle\xi\rangle^{3/2}||_{L^2(\rho(\xi)d\xi)}, \quad r\leqslant 1 \tag{12.13}$$

$$|L^*L\overline{v}(r)| \leqslant C\sqrt{\langle\log(r)\rangle}||\overline{y_1}(\xi)\langle\xi\rangle^{3/2}||_{L^2(\rho(\xi)d\xi)}, \quad 0<r\leqslant 1 \tag{12.14}$$

$$||\overline{v}||_{L^\infty} \leqslant C||\overline{v}||_{\dot{H}^1_e} \tag{12.15}$$

$$||\overline{v}||_{\dot{H}^1_e} \leqslant C\left(||L(\overline{v})||_{L^2(rdr)} + ||\overline{v}||_{L^2(rdr)}\right) \tag{12.16}$$

$$||L\overline{v}||_{L^\infty} + ||L\overline{v}||_{\dot{H}^1_e} \leqslant C||L^*L\overline{v}||_{L^2(rdr)} \tag{12.17}$$

Proof. The proof of this lemma is very similar to that of Lemma 5.1 of [26], but, we will include a proof here for completeness. To establish (12.12), it suffices to estimate the following integral, using the estimates on $\phi(r,\xi)$, a, and ρ from section 4 of [15]. For $r\leqslant 1$,

$$\begin{aligned}&\left|\int_0^\infty \overline{y_1}(\xi)\frac{\phi(r,\xi)}{r^{5/2}}\frac{\rho(\xi)d\xi}{\sqrt{\langle\log(r)\rangle}}\right|\\
&\leqslant \frac{C}{\sqrt{\langle\log(r)\rangle}}\int_0^{\frac{4}{r^2}}|\overline{y_1}(\xi)|\langle\xi\rangle^{3/2}\sqrt{\rho(\xi)}\frac{\sqrt{\rho(\xi)}d\xi}{\langle\xi\rangle^{3/2}} + C\int_{\frac{4}{r^2}}^\infty \frac{|\overline{y_1}(\xi)|\cdot|a(\xi)|\rho(\xi)d\xi}{\xi^{1/4}r^{5/2}\sqrt{\langle\log(r)\rangle}}\\
&\leqslant \frac{C}{\sqrt{\langle\log(r)\rangle}}||\overline{y_1}(\xi)\langle\xi\rangle^{3/2}||_{L^2(\rho(\xi)d\xi)}\left(\int_0^{\frac{4}{r^2}}\frac{\rho(\xi)d\xi}{\langle\xi\rangle^3}\right)^{1/2} + \frac{C}{\sqrt{\langle\log(r)\rangle}}||\overline{y_1}(\xi)\langle\xi\rangle^{3/2}||_{L^2(\rho(\xi)d\xi)}\end{aligned}$$

The proofs of (12.13) and (12.14) are similar. The only new point to note is that $L(\phi_0) = 0$, which explains why there is no $\overline{y_0}$ term on the right-hand sides of these inequalities. Also, we use the fact that the functions $\widetilde{\phi_j}$ (which appear in a representation formula of $\phi(r,\xi)$) in [15] satisfy symbol-type estimates. This is not directly stated in Proposition 4.5 of [15]. However, this fact can be proven in a straightforward manner by noting that

$$\widetilde{\phi_j}(u) = u^{-j}\widetilde{f_j}(u), \quad \widetilde{f_0}(u) = \frac{u^2}{(1+u)^2}, \quad \widetilde{f_1}(u) = \frac{-u^3(2+u)}{6(1+u)^2}$$

and then using an argument by induction to estimate $\widetilde{f_j}'(u)$, given the representation formula of $\widetilde{f_j}$ in terms of $\widetilde{f_{j-1}}$ from [15]. The lemma statement regarding continuity of $\overline{v}$ and $\partial_r\overline{v}$ is proven with the Dominated convergence theorem, again using the estimates on $\phi(r,\xi)$ from section 4 of [15], and with a similar argument as in [26]. The Dominated convergence theorem also shows that $\overline{v}$ and $\partial_r\overline{v}$ extend continuously to $[0,\infty)$ with $\overline{v}(0) = \partial_r\overline{v}(0) = \lim_{r\to\infty}\overline{v}(r) = \lim_{r\to\infty}\partial_r\overline{v}(r) = 0$. The inequality (12.9) follows directly from the L^2 isometry property of $\mathcal{F}$. To prove (12.13), we first recall the conjugation of L^*L used in [15], namely $\mathcal{L}$, which satisfies

$$\mathcal{L}(u)(r) = L^*L\left(\frac{u(\cdot)}{\sqrt{\cdot}}\right)(r)\sqrt{r}$$

and for which $\phi(r,\xi)$ are eigenfunctions. Using the Dominated convergence theorem, we have, for $r > 0$,

$$\mathcal{L}(\overline{v}(\cdot)\sqrt{\cdot})(r) = \int_0^\infty \mathcal{L}(\phi(\cdot,\xi))(r)\overline{y_1}(\xi)\rho(\xi)d\xi = \int_0^\infty \xi\phi(r,\xi)\overline{y_1}(\xi)\rho(\xi)d\xi = \mathcal{F}^{-1}\left(\begin{bmatrix}0\\ \xi\overline{y_1}(\xi)\end{bmatrix}\right)(r)$$

where we again emphasize that we regard the distorted Fourier transform as a two-component vector, following [15]. We can now prove (12.11) using the L^2 isometry property of $\mathcal{F}$.

$$||\xi\overline{y_1}(\xi)||_{L^2(\rho(\xi)d\xi)} = ||\mathcal{L}(\overline{v}(\cdot)\sqrt{\cdot})(r)||_{L^2(dr)} = ||L^*L(\overline{v})||_{L^2(rdr)}$$

Continuing the proof of (12.10), we have

$$\begin{aligned}\langle\mathcal{L}\left(\overline{v}(\cdot)\sqrt{\cdot}\right)(r),\overline{v}(r)\sqrt{r}\rangle_{L^2(dr)} &= \langle\mathcal{F}^{-1}\left(\begin{bmatrix}0\\ \xi\overline{y_1}(\xi)\end{bmatrix}\right),\mathcal{F}^{-1}\left(\begin{bmatrix}\overline{y_0}\\ \overline{y_1}\end{bmatrix}\right)(r)\rangle_{L^2(dr)} \\ &= \langle\xi\overline{y_1}(\xi),\overline{y_1}(\xi)\rangle_{L^2(\rho(\xi)d\xi)} = ||\sqrt{\xi}\overline{y_1}(\xi)||^2_{L^2(\rho(\xi)d\xi)}\end{aligned} \tag{12.18}$$

On the other hand,

$$\langle\mathcal{L}\left(\overline{v}(\cdot)\sqrt{\cdot}\right)(r),\overline{v}(r)\sqrt{r}\rangle_{L^2(dr)} = \int_0^\infty L^*L(\overline{v})(r)\overline{v}(r)rdr$$

Recalling the assumptions of the lemma, and (12.9) and (12.11), which we have proven at this point, we see that

$$L^*L(\overline{v})(r)\overline{v}(r)r \in L^1((0,\infty),dr)$$

Therefore, by the Dominated convergence theorem,

$$\int_0^\infty \overline{v}(r)L^*L\overline{v}(r)rdr = \lim_{M\to\infty}\int_0^\infty \overline{v}(r)L^*L\overline{v}(r)\psi_{\leqslant 1}(\frac{r}{M})rdr$$

where

$$0 \leqslant \psi_{\leqslant 1}(x) \leqslant 1, \quad \psi_{\leqslant 1} \in C^{\infty}(\mathbb{R}), \quad \psi'_{\leqslant 1}(x) \leqslant 0, \psi_{\leqslant 1}(x) = \begin{cases} 1, & x \leqslant \frac{1}{2} \\ 0, & x \geqslant 1 \end{cases}$$

Then, we inspect integral which arises from the term ∂_r in the expression for L^*, and integrate by parts once to get, for instance, for all $M \geqslant 1$,

$$\begin{aligned}
&\int_0^{\infty} \overline{v}(r)\partial_r L\overline{v}(r)\psi_{\leqslant 1}(\frac{r}{M})rdr = -\int_0^{\infty} L\overline{v}(r)\partial_r \left(\overline{v}(r)\psi_{\leqslant 1}(\frac{r}{M})r\right) dr \\
&= -\int_0^{\infty} L(\overline{v})(r)\left(\partial_r \overline{v}(r) + \frac{\overline{v}(r)}{r}\right)\psi_{\leqslant 1}(\frac{r}{M})rdr + \int_0^{\infty} \frac{\partial_r\left((\overline{v}(r))^2\right)}{2}\frac{\psi'_{\leqslant 1}(\frac{r}{M})}{M}rdr \\
&- \int_0^{\infty} 2\left(\frac{1-r^2}{1+r^2}\right)\frac{\overline{v}(r)^2}{r}\frac{\psi'_{\leqslant 1}(\frac{r}{M})}{M}rdr
\end{aligned}$$

For the integral below, we integrate by parts again, to get

$$|\int_0^{\infty} \frac{\partial_r\left((\overline{v}(r))^2\right)}{2}\frac{\psi'_{\leqslant 1}(\frac{r}{M})}{M}rdr| = |-\int_0^{\infty} \frac{\overline{v}(r)^2}{2}\left(\frac{\psi''_{\leqslant 1}(\frac{r}{M})}{M^2} + \frac{\psi'_{\leqslant 1}(\frac{r}{M})}{Mr}\right)rdr| \leqslant \frac{C}{M^2}||\overline{v}||^2_{L^2(rdr)}$$

In total, we have

$$\int_0^{\infty} \overline{v}(r)L^*L\overline{v}(r)rdr = \lim_{M\to\infty}\int_0^{\infty} \overline{v}(r)L^*L\overline{v}(r)\psi_{\leqslant 1}(\frac{r}{M})rdr = \lim_{M\to\infty}\int_0^{\infty}(L\overline{v}(r))^2\psi_{\leqslant 1}(\frac{r}{M})rdr$$

By the Monotone convergence theorem (recall that $\psi'_{\leqslant 1}(x) \leqslant 0$) we get

$$\int_0^{\infty}(L\overline{v}(r))^2 rdr = \int_0^{\infty}\overline{v}(r)L^*L\overline{v}(r)rdr$$

Combining this with the observation (12.18), we get

$$||\sqrt{\xi}\overline{y_1}(\xi)||^2_{L^2(\rho(\xi)d\xi)} = \int_0^{\infty}(L\overline{v}(r))^2 rdr$$

which is (12.10). The inequality (12.16) is proven similarly to the analogous estimate in [26], except that we will not need to use an approximation argument. From the lemma hypotheses, and what we have established up to now, $\overline{v} \in C^1([0,\infty)) \cap L^2((0,\infty), rdr)$, and $L\overline{v} \in L^2((0,\infty), rdr)$ with $\overline{v}(0) = \lim_{r\to\infty}\overline{v}(r) = 0$. So,

$$L\overline{v}(r) = -\overline{v}'(r) + \frac{2}{r}\overline{v}(r) + \overline{v}(r)\cdot\left(\frac{-4r}{1+r^2}\right)$$

which shows that

$$-\overline{v}'(r) + \frac{2}{r}\overline{v}(r) = L\overline{v}(r) - \overline{v}(r)\cdot\left(\frac{-4r}{1+r^2}\right) \in L^2((0,\infty), rdr) \tag{12.19}$$

So, for any $M > 4$,

$$\int_{\frac{1}{M}}^{M} (-\overline{v}'(r) + \frac{2}{r}\overline{v}(r))^2 r dr = \int_{\frac{1}{M}}^{M} \left((\overline{v}'(r))^2 + \frac{4}{r^2}\overline{v}(r)^2 \right) r dr - 2\int_{\frac{1}{M}}^{M} \frac{d}{dr}((\overline{v}(r)^2) dr \tag{12.20}$$

By the Dominated convergence theorem, and the observation (12.19), we have

$$\lim_{M\to\infty} \int_{\frac{1}{M}}^{M} (-\overline{v}'(r) + \frac{2}{r}\overline{v}(r))^2 r dr = \int_0^{\infty} (-\overline{v}'(r) + \frac{2}{r}\overline{v}(r))^2 r dr$$

We then let $M \to \infty$ in (12.20) and use the Monotone convergence theorem, to get

$$\int_0^{\infty} \left((\overline{v}'(r))^2 + \frac{4}{r^2}\overline{v}(r)^2 \right) r dr = \int_0^{\infty} (-\overline{v}'(r) + \frac{2}{r}\overline{v}(r))^2 r dr \leqslant C \left(||L\overline{v}||^2_{L^2(rdr)} + ||\overline{v}||^2_{L^2(rdr)} \right)$$

where the last inequality follows from (12.19). This in particular, proves that $\overline{v} \in \dot{H}^1_e$, with the estimate (12.16). The next inequality to prove, (12.15), is proven in the same way as in [26]. Since $\overline{v} \in C^1([0,\infty)) \cap \dot{H}^1_e$, with $\overline{v}(0) = 0$, we use the Fundamental theorem of calculus for $\overline{v}^2$, to get

$$\overline{v}(r)^2 \leqslant C \int_0^r \frac{|\overline{v}(r)|}{\sqrt{r}} |\partial_r \overline{v}| \sqrt{r} dr \leqslant C||\frac{\overline{v}}{r}||_{L^2(rdr)} ||\partial_r \overline{v}||_{L^2(rdr)} \leqslant C||\overline{v}||^2_{\dot{H}^1_e}$$

The final estimate to prove is (12.17). If we have $g \in C^1([0,\infty)) \cap L^2((0,\infty), rdr)$, and $L^* g \in L^2((0,\infty), rdr)$ with $g(0) = \lim_{r\to\infty} g(r) = 0$, then, we recall the definition of L^*:

$$L^*(f)(r) = f'(r) + V(r) f(r), \quad \text{where } V(r) = \frac{2\left(\frac{1-r^2}{1+r^2}\right)}{r} + \frac{1}{r}$$

For $M > 4$,

$$\int_{\frac{1}{M}}^{M} (L^* g)^2 \, r dr = \int_{\frac{1}{M}}^{M} (g'(r))^2 r dr + \int_{\frac{1}{M}}^{M} (V(r))^2 g(r)^2 r dr + 2\int_{\frac{1}{M}}^{M} V(r) \frac{d}{dr}(g(r)^2) \frac{r}{2} dr$$

The last term in the expression above is

$$V(r) g(r)^2 r \Big|_{\frac{1}{M}}^{M} - \int_{\frac{1}{M}}^{M} (g(r))^2 \frac{d}{dr} (V(r) r) \, dr$$

Letting $M \to \infty$, we get

$$||L^* g||^2_{L^2(rdr)} \geqslant C \lim_{M\to\infty} \int_{\frac{1}{M}}^{M} \left((g'(r))^2 + \frac{g(r)^2}{r^2} \right) r dr$$

By the Monotone convergence theorem, we then get that $g \in \dot{H}^1_e$, and

$$||g||_{\dot{H}^1_e} \leqslant C||L^* g||_{L^2(rdr)}$$

Then, we can apply our previous estimate to get

$$||g||_{L^\infty} \leqslant C||g||_{\dot{H}^1_e} \leqslant C||L^* g||_{L^2(rdr)} \tag{12.21}$$

Note that $L\overline{v}$ does not quite satisfy all of the stated assumptions on g used in the preceding argument. Therefore, we define, for $M \geqslant 4$,

$$\overline{v}_M(r) = \overline{y_0}\frac{\phi_0(r)}{||\phi_0||^2_{L^2(rdr)}} + \int_0^\infty \frac{\phi(r,\xi)}{\sqrt{r}}\overline{y_1}(\xi)\rho(\xi)\chi_{\leqslant 1}(\frac{\xi}{M})d\xi$$

As in the proof of Lemma 5.1 in [26], we verify, via the Dominated convergence theorem, that $L\overline{v_M} \in C^1([0,\infty))$, and that $L\overline{v_M} \in L^2((0,\infty), rdr)$, $L^* L\overline{v_M} \in L^2((0,\infty), rdr)$ with $L(\overline{v_M})(0) = \lim_{r\to\infty} L(\overline{v_M})(r) = 0$ Therefore, we have (12.21) for $g = L\overline{v_M}$. An approximation argument then establishes (12.17). □

We recall the definition of F_3.

$$F_3(t,r) = \frac{2v(t,r)^3}{r^2} + \frac{6\left(Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r) + w_c(t,r)\right)}{r^2}v(t,r)^2 + \frac{6v(t,r)}{r^2}\left(\left(v_c(t,r) + Q_{\frac{1}{\lambda(t)}}(r) + w_c(t,r)\right)^2 - Q_{\frac{1}{\lambda(t)}}(r)^2\right)$$

where v is given in terms of our previously defined function y, by

$$v(t,r) = y_0(t)\frac{\phi_0(\frac{r}{\lambda(t)})}{||\phi_0||^2_{L^2(rdr)}} + \int_0^\infty y_1(t, \frac{\xi}{\lambda(t)^2})\phi(\frac{r}{\lambda(t)},\xi)\sqrt{\frac{\lambda(t)}{r}}\rho(\xi)d\xi$$

Now, we will record estimates on quantities associated to F_3, for any $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ satisfying $y_1(t,\omega)\langle\omega\lambda(t)^2\rangle^{3/2} \in L^2(\rho(\omega\lambda(t)^2)d\omega)$. For ease of notation, we let

$$v_{corr}(t,r) = v_c(t,r) + w_c(t,r)$$

Using the identical procedure used in [26], and the above lemma, we get (12.6) and (12.7). The estimate (12.8) follows directly from our estimates on v_c and w_c □

12.4 F_4 Estimates

Here, we will estimate certain oscillatory integrals applied to F_4, for later use. First, we will need an estimate related to the function ρ appearing in the spectral measure associated to the distorted Fourier transform of [15]. Using the same procedure as we used in [26], and the estimates on ρ following from Lemma 4.7 of [15], we get

$$\frac{\rho(\omega\lambda(t)^2)}{\rho(\omega\lambda(x)^2)} \leqslant C\max\{\frac{\lambda(t)^4}{\lambda(x)^4}, 1\} \tag{12.22}$$

For ease of notation, we define $\mathcal{F}\left(\sqrt{\cdot}F_4\left(x,\cdot\lambda(x)\right)\right)_i\left(\omega\lambda(x)^2\right)$ by

$$\mathcal{F}\left(\sqrt{\cdot}F_4\left(x,\cdot\lambda(x)\right)\right)\left(\omega\lambda(x)^2\right) = \begin{bmatrix}\mathcal{F}\left(\sqrt{\cdot}F_4\left(x,\cdot\lambda(x)\right)\right)_0\left(\omega\lambda(x)^2\right)\\ \mathcal{F}\left(\sqrt{\cdot}F_4\left(x,\cdot\lambda(x)\right)\right)_1\left(\omega\lambda(x)^2\right)\end{bmatrix}$$

Later on, we will use this notation for F_3 and F_5 as well. Because $F_4(x,R\lambda(x))$ is orthogonal to $\phi_0(R)$ in $L^2(RdR)$, we have

$$\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2) = \begin{bmatrix}0\\ \mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\end{bmatrix}$$

Now, we will prove the following lemma

Lemma 12.2. *We have the following estimates*

$$||\lambda(t)\left(\omega\lambda(t)^2\right)^{3/2}\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C\lambda(t)^2}{t^2\log^b(t)}$$

$$\begin{aligned}&||\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C\lambda(t)^2}{t^2}\left(\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2-2\epsilon}(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^b(t)}\right)\\ &||\lambda(t)\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant C\frac{\lambda(t)^2}{t^3}\left(\frac{1}{\log^b(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2-2\epsilon}(t)}\right)\end{aligned}$$

$$\begin{aligned}&||\lambda(t)\left(\omega\lambda(t)^2\right)\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C\lambda(t)^2}{t^3}\left(\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2\epsilon-2}(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^b(t)}\right)\end{aligned}$$

Proof. We start by estimating $\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)$, and use the same procedure we used in [26]. For completeness, we will write out in detail the steps where we use the orthogonality of $F_4(x,R\lambda(x))$ to $\phi_0(R)$ in $L^2(RdR)$. We have

$$\begin{aligned}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2) &= \int_0^\infty \sqrt{R}F_4(x,R\lambda(x))\phi(R,\omega\lambda(x)^2)dR\\ &= \frac{1}{\lambda(x)^{3/2}}\int_0^\infty \sqrt{u}F_4(x,u)\phi(\frac{u}{\lambda(x)},\omega\lambda(x)^2)du\end{aligned}$$

Using the descriptions of $\phi(r,\xi)$ given in [15], we divide into two regions: $r^2\xi \leqslant 4$ and $r^2\xi > 4$, and get

$$\begin{aligned}
&\frac{1}{\lambda(x)^{3/2}}\int_0^\infty \sqrt{u}F_4(x,u)\phi(\frac{u}{\lambda(x)},\omega\lambda(x)^2)du\\
&=\frac{1}{\lambda(x)^{3/2}}\int_0^{\frac{2}{\sqrt{\omega}}}\sqrt{u}F_4(x,u)\widetilde{\phi}_0(\frac{u}{\lambda(x)})du\\
&+\frac{1}{\lambda(x)^{3/2}}\int_0^{\frac{2}{\sqrt{\omega}}}\sqrt{u}F_4(x,u)\sum_{j=1}^\infty\frac{\widetilde{\phi}_j(\frac{u^2}{\lambda(x)^2})}{u^{3/2}}\lambda(x)^{3/2}(u^2\omega)^j du\\
&+\frac{2}{\lambda(x)^{3/2}}\text{Re}\left(\int_{\frac{2}{\sqrt{\omega}}}^\infty\sqrt{u}F_4(x,u)a(\omega\lambda(x)^2)\psi^+(\frac{u}{\lambda(x)},\lambda(x)^2\omega)du\right)
\end{aligned}\tag{12.23}$$

We use the orthogonality of $F_4(x,R\lambda(x))$ to $\phi_0(R)$ in $L^2(RdR)$, as part of estimating

$$I:=\frac{1}{\lambda(x)^{3/2}}\int_0^{\frac{2}{\sqrt{\omega}}}\sqrt{u}F_4(x,u)\widetilde{\phi}_0(\frac{u}{\lambda(x)})du=\frac{1}{\lambda(x)^2}\int_0^{\frac{2}{\sqrt{\omega}}}F_4(x,u)u\phi_0(\frac{u}{\lambda(x)})du$$

We split into 4 cases, depending on the range of ω.

Case 1: $\omega<\frac{16}{x^2}$. In this case, we use the orthogonality to get

$$I=\frac{1}{\lambda(x)^2}\int_0^{\frac{2}{\sqrt{\omega}}}F_4(x,u)u\phi_0(\frac{u}{\lambda(x)})du=-\frac{1}{\lambda(x)^2}\int_{\frac{2}{\sqrt{\omega}}}^\infty F_4(x,u)u\phi_0(\frac{u}{\lambda(x)})du=0$$

where we used the support conditions on $\chi_{\geqslant 1}(x)$, and the fact that $\frac{t}{g(t)}>1600$.

Case 2: $\frac{16}{x^2}<\omega\leqslant\frac{4}{g(x)^2}$. In this case, we again use the orthogonality to get

$$\begin{aligned}
|\frac{1}{\lambda(x)^2}\int_0^{\frac{2}{\sqrt{\omega}}}F_4(x,u)u\phi_0(\frac{u}{\lambda(x)})du|&=|-\frac{1}{\lambda(x)^2}\int_{\frac{2}{\sqrt{\omega}}}^\infty F_4(x,u)\phi_0(\frac{u}{\lambda(x)})udu|\\
&\leqslant\frac{C\lambda(x)^4\log(x)}{x^2\log^b(x)}\omega^2\left(\log(2+\frac{2}{\sqrt{\omega}g(x)})+\frac{\log(x)}{\log^b(x)}\right)
\end{aligned}$$

Case 3: $\frac{4}{g(x)^2}\leqslant\omega\leqslant\frac{4}{\lambda(x)^2}$. Using the orthogonality, we get

$$\begin{aligned}
|\frac{1}{\lambda(x)^2}\int_0^{\frac{2}{\sqrt{\omega}}}F_4(x,u)u\phi_0(\frac{u}{\lambda(x)})du|&=|-\frac{1}{\lambda(x)^2}\int_{\frac{2}{\sqrt{\omega}}}^\infty F_4(x,u)\phi_0(\frac{u}{\lambda(x)})udu|\\
&\leqslant\frac{C\lambda(x)^4\log(x)\left(1+\frac{\log(x)}{\log^b(x)}\right)}{x^2\log^b(x)g(x)^4}+\frac{C\omega\lambda(x)^2}{x^2\log^b(x)}
\end{aligned}$$

Case 4: $\frac{4}{\lambda(x)^2}\leqslant\omega$. Here, we do not need to use the orthogonality, and simply directly estimate

$$|\frac{1}{\lambda(x)^2}\int_0^{\frac{2}{\sqrt{\omega}}}F_4(x,u)u\phi_0(\frac{u}{\lambda(x)})du|\leqslant\frac{C}{\omega^3\lambda(x)^6x^2\log^b(x)}$$

To estimate the other two integrals of (12.23), we use the same procedure that we used in [26]. In total, we get

$$\begin{aligned}&\lambda(x)||\frac{\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)}{\omega}||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\&\leqslant \frac{C\lambda(x)^2}{x^2\log^b(x)}+\frac{C\lambda(x)^2}{x^2\log^{2\epsilon}(x)}+\frac{C\lambda(x)^2}{x^2\log^{2b-2\epsilon-1}(x)}+\frac{C\lambda(x)^2}{x^2\log^{3b-2-2\epsilon}(x)}\end{aligned} \tag{12.24}$$

which will be used later on. The first integral to estimate is

$$\begin{aligned}&\lambda(t)\left(\omega\lambda(t)^2\right)^{3/2}\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx\\&=-\lambda(t)^4\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))_1(\omega\lambda(t)^2)\sqrt{\omega}\\&-\lambda(t)^4\int_t^\infty\frac{\cos((t-x)\sqrt{\omega})}{\sqrt{\omega}}\omega\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)dx\end{aligned}$$

Using (12.10), we have

$$||\lambda(t)^4\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))_1(\omega\lambda(t)^2)\sqrt{\omega}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}=\lambda(t)^2||L(F_4(t,\cdot\lambda(t)))||_{L^2(RdR)}$$

Then, we use the symbol-type estimates on F_4, namely (11.48), to get

$$||L(F_4(t,\cdot\lambda(t)))||_{L^2(RdR)}\leqslant\frac{C}{t^2\log^b(t)}$$

which gives

$$||\lambda(t)^4\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))_1(\omega\lambda(t)^2)\sqrt{\omega}||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\leqslant\frac{C\lambda(t)^2}{t^2\log^b(t)}$$

Next, we note that

$$\begin{aligned}\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)&=\mathcal{F}(\sqrt{\cdot}\partial_x\left(F_4(x,\cdot\lambda(x))\right))_1(\omega\lambda(x)^2)\\&+\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1'(\omega\lambda(x)^2)2\lambda(x)\lambda'(x)\omega\end{aligned}$$

Then, we recall the transference identity of [15], which says

$$\mathcal{F}(R\partial_R u)(\xi)=\begin{bmatrix}0\\-2\xi\partial_\xi\mathcal{F}(u)_1\end{bmatrix}+\mathcal{K}(\mathcal{F}(u))(\xi)$$

where we write

$$\mathcal{F}(u)=\begin{bmatrix}\mathcal{F}(u)_0\\\mathcal{F}(u)_1\end{bmatrix}$$

This gives

$$\begin{aligned}&||2\lambda(x)\lambda'(x)\omega\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1'(\omega\lambda(x)^2)\sqrt{\omega}||^2_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\&\leqslant\frac{C\lambda'(x)^2}{\lambda(x)^6}\left(||\mathcal{F}(R\partial_R(\sqrt{R}F_4(x,R\lambda(x))))(\xi)||^2_{L^{2,\frac{1}{2}}_\rho}+||\mathcal{K}(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x))))(\xi)||^2_{L^{2,\frac{1}{2}}_\rho}\right)\end{aligned}$$

Then, we estimate

$$|\langle R\partial_R\left(\sqrt{R}F_4(x,R\lambda(x))\right),\widetilde{\phi_0}\rangle_{L^2(dR)}|\leqslant\frac{C}{x^2\log^b(x)}$$

Next, we again use (12.10) to get

$$\begin{aligned}&||\mathcal{F}(R\partial_R(\sqrt{R}F_4(x,R\lambda(x))))_1(\xi)\langle\xi\rangle^{1/2}||^2_{L^2(\rho(\xi))d\xi)}\\&\leqslant C||L(\sqrt{R}\partial_R(\sqrt{R}F_4(x,R\lambda(x))))||^2_{L^2(RdR)}+C||R\partial_R(\sqrt{R}F_4(x,R\lambda(x)))||^2_{L^2(dR)}\end{aligned}$$

Using (11.48), we get

$$||L(\sqrt{R}\partial_R(\sqrt{R}F_4(x,R\lambda(x))))||^2_{L^2(RdR)}\leqslant\frac{C}{x^4\log^{2b}(x)}$$

$$||R\partial_R(\sqrt{R}F_4(x,R\lambda(x)))||_{L^2(dR)}\leqslant\frac{C}{x^2\log^b(x)}$$

which gives

$$||\mathcal{F}(R\partial_R(\sqrt{R}F_4(x,R\lambda(x))))_1(\xi)\langle\xi\rangle^{1/2}||^2_{L^2(\rho(\xi)d\xi)}\leqslant\frac{C}{x^4\log^{2b}(x)}$$

Similarly

$$\begin{aligned}||\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))||^2_{L^{2,\frac{1}{2}}_{\rho}}&\leqslant C\int_0^\infty R|F_4(x,R\lambda(x))|^2dR+C\int_0^\infty|L(F_4(x,R\lambda(x)))|^2RdR\\&\leqslant\frac{C}{x^4\log^{2b}(x)}\end{aligned}$$

This finally gives

$$||2\lambda(x)\lambda'(x)\omega\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1'(\omega\lambda(x)^2)\sqrt{\omega}||^2_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\leqslant\frac{C\lambda'(x)^2}{x^4\lambda(x)^6\log^{2b}(x)}$$

We again use (12.10), and (11.48) to get

$$\begin{aligned}||\sqrt{\omega}\mathcal{F}(\sqrt{\cdot}\partial_x\left(F_4(x,\cdot\lambda(x))\right))_1(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}&\leqslant\frac{C}{\lambda(x)^2}||L(\partial_x(F_4(x,R\lambda(x))))||_{L^2(RdR)}\\&\leqslant\frac{C}{\lambda(x)^2x^3\log^b(x)}\end{aligned}$$

So, using Minkowski's inequality and (12.22), we get

$$\begin{aligned}&||\lambda(t)^4\int_t^\infty\cos((t-x)\sqrt{\omega})\sqrt{\omega}\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\lambda(t)^4\int_t^\infty\left(1+\frac{\lambda(t)^2}{\lambda(x)^2}\right)\frac{1}{\lambda(x)^2x^3\log^b(x)}dx\leqslant\frac{C\lambda(t)^2}{t^2\log^b(t)}\end{aligned}$$

where we obtained the last inequality using the same procedure that we used to treat the integral (11.39). In total, we then obtain

$$||\lambda(t)\left(\omega\lambda(t)^2\right)^{3/2}\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \leqslant \frac{C\lambda(t)^2}{t^2\log^b(t)}$$

The next integral to estimate is

$$\begin{aligned}&\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx\\&=-\lambda(t)\frac{\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))_1(\omega\lambda(t)^2)}{\omega}\\&-\lambda(t)\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x))_1(\omega\lambda(x)^2)\right)dx\end{aligned} \tag{12.25}$$

We recall

$$\begin{aligned}\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)&=\mathcal{F}(\sqrt{\cdot}\partial_x\left(F_4(x,\cdot\lambda(x))\right))_1(\omega\lambda(x)^2)\\&+\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1'(\omega\lambda(x)^2)2\lambda(x)\lambda'(x)\omega\end{aligned}$$

This time, however, since the integrand of the third line of (12.25) has a factor of $\frac{1}{\omega}$, we will directly estimate $\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1'(\xi)$, rather than using an argument based on the transference identity. We have

$$\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1'(\omega\lambda(x)^2)=IV+V$$

where

$$IV=\int_0^{\frac{2}{\sqrt{\omega}}}\partial_2\phi(\frac{u}{\lambda(x)},\omega\lambda(x)^2)\frac{\sqrt{u}}{\lambda(x)^{3/2}}F_4(x,u)du$$

$$V=\int_{\frac{2}{\sqrt{\omega}}}^\infty \partial_2\phi(\frac{u}{\lambda(x)},\omega\lambda(x)^2)\frac{\sqrt{u}}{\lambda(x)^{3/2}}F_4(x,u)du$$

then, we get

$$|IV|\leqslant\begin{cases}\frac{Cg(x)^2}{\lambda(x)^2x^2\log^b(x)}+\frac{C\log^3(x)}{x^2\log^b(x)}, & \omega<\frac{16}{x^2}\\ \frac{Cg(x)^2}{\lambda(x)^2x^2\log^b(x)}+\frac{C\log(x)\log(2+\frac{2}{\sqrt{\omega}g(x)})}{x^2\log^b(x)}\left(\log(2+\frac{2}{\sqrt{\omega}g(x)})+\frac{\log(x)}{\log^b(x)}\right), & \frac{16}{x^2}<\omega\leqslant\frac{4}{g(x)^2}\\ \frac{C}{\omega\lambda(x)^2x^2\log^b(x)}, & \frac{4}{g(x)^2}<\omega\leqslant\frac{4}{\lambda(x)^2}\\ \frac{C}{x^2\log^b(x)\lambda(x)^8\omega^4}, & \frac{4}{\lambda(x)^2}<\omega\end{cases}$$

$$|V|\leqslant\begin{cases}0, & \omega\leqslant\frac{16}{x^2}\\ \frac{C|a(\omega\lambda(x)^2)|\log(x)\left(\log(2+\frac{2}{\sqrt{\omega}g(x)})+\frac{\log(x)}{\log^b(x)}\right)}{x^2\log^b(x)}, & \frac{16}{x^2}<\omega<\frac{4}{g(x)^2}\\ \frac{C|a(\omega\lambda(x)^2)|\sqrt{g(x)}}{\omega^{3/4}\lambda(x)^2x^2\log^b(x)}, & \frac{4}{g(x)^2}<\omega\leqslant\frac{4}{\lambda(x)^2}\\ \frac{C|a(\omega\lambda(x)^2)|\sqrt{g(x)}}{\lambda(x)^2\omega^{3/4}x^2\log^b(x)}, & \frac{4}{\lambda(x)^2}<\omega\end{cases}$$

In order to estimate

$$||\frac{\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))_1(\omega\lambda(x)^2)}{\omega}||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}$$

we note that $\partial_x(F_4(x,R\lambda(x)))$ is still orthogonal to $\phi_0(R)$ in $L^2(RdR)$, and we recall our symbol type estimates on F_4, namely (11.48). These two observations, along with an inspection of the procedure used to obtain (12.24) give

$$\begin{aligned}&||\frac{\mathcal{F}(\sqrt{\cdot}\partial_x(F_4(x,\cdot\lambda(x))))_1(\omega\lambda(x)^2)}{\omega}||_{L^2(\rho(\omega\lambda(x)^2)d\omega)}\\ &\leqslant \frac{C\lambda(x)}{x^3}\left(\frac{1}{\log^{2b-2\epsilon-1}(x)}+\frac{1}{\log^{3b-2-2\epsilon}(x)}+\frac{1}{\log^{2\epsilon}(x)}+\frac{1}{\log^b(x)}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}&||\lambda(t)\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\partial_x\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C\lambda(t)^2}{t^2}\left(\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2-2\epsilon}(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^b(t)}\right)\end{aligned}$$

which, when combined with (12.24), gives

$$\begin{aligned}&||\lambda(t)\int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}}\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\ &\leqslant \frac{C\lambda(t)^2}{t^2}\left(\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2-2\epsilon}(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^b(t)}\right)\end{aligned}$$

The next integral to estimate is

$$\begin{aligned}&\lambda(t)\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x))_1(\omega\lambda(x)^2)dx\\ &=-\lambda(t)\frac{\partial_t\left(\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))_1(\omega\lambda(t)^2)\right)}{\omega}\\ &-\lambda(t)\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)dx\end{aligned}$$

First, we will estimate $\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1''(\omega\lambda(x)^2)$, which is one of the terms arising in the expression $\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)$. We have

$$\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1''(\omega\lambda(x)^2)=VI+VII$$

where

$$VI=\int_0^{\frac{2}{\sqrt{\omega}\lambda(x)}}\partial_2^2\phi(R,\omega\lambda(x)^2)\sqrt{R}F_4(x,R\lambda(x))dR$$

$$VII=\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^\infty \partial_2^2\phi(R,\omega\lambda(x)^2)\sqrt{R}F_4(x,R\lambda(x))dR$$

We estimate VI directly, using (11.48), and estimates on $\phi(r,\xi)$ from [15]. This leads to

$$\omega|VI| \leqslant \begin{cases} \frac{C\log^2(x)}{x^2\log^b(x)\lambda(x)^2}, & \omega < \frac{16}{x^2} \\ \frac{Cg(x)^2}{x^2\log^b(x)\lambda(x)^4}\left(1+\frac{\lambda(x)^2}{g(x)^2}\log(x)\log(2+\frac{2}{\sqrt{\omega}g(x)})\right), & \frac{16}{x^2} < \omega < \frac{4}{g(x)^2} \\ \frac{C}{\lambda(x)^2x^2\log^b(x)}\left(1+\frac{1}{\omega\lambda(x)^2}\right), & \frac{4}{g(x)^2} < \omega < \frac{4}{\lambda(x)^2} \\ \frac{C}{\lambda(x)^{10}\omega^4x^2\log^b(x)}, & \frac{4}{\lambda(x)^2} < \omega \end{cases}$$

On the other hand, to estimate VII, we will need to integrate by parts in R, when $\omega > \frac{4}{\lambda(x)^2}$, exactly like we needed to do in an analogous integral in [26]. In particular, we have

$$\omega|VII| = 0, \quad \omega \leqslant \frac{16}{x^2}$$

$$\omega|VII| \leqslant \frac{C|a(\omega\lambda(x)^2)|\log(x)}{\lambda(x)^2x^2\log^b(x)}\left(\log(2+\frac{2}{\sqrt{\omega}g(x)})+\frac{\log(x)}{\log^b(x)}\right), \quad \frac{16}{x^2} < \omega < \frac{4}{g(x)^2}$$

$$\omega|VII| \leqslant \frac{C|a(\omega\lambda(x)^2)|g(x)^{3/2}}{\omega^{1/4}\lambda(x)^4x^2\log^b(x)}, \quad \frac{4}{g(x)^2} < \omega < \frac{4}{\lambda(x)^2}$$

To estimate VII in the region $\omega > \frac{4}{\lambda(x)^2}$, we first use Lemma 4.7 of [15] to get, for $R^2\xi > 4$,

$$\phi(R,\xi) = 2\text{Re}\left(a(\xi)\psi^+(R,\xi)\right)$$

and, using the properties of ψ^+ and a, we get

$$\partial_2^2\phi(R,\xi) = 2\text{Re}\left(\frac{a(\xi)\cdot -R^2}{4\xi^{5/4}}e^{iR\sqrt{\xi}}\sigma(R\sqrt{\xi},R)\right)+\text{Err}$$

where

$$|Err| \leqslant C\frac{|a(\xi)|}{\xi^{9/4}}+C\frac{|a(\xi)|R}{\xi^{7/4}}, \quad R^2\xi \geqslant 4$$

Then, we integrate by parts in R for the term

$$\omega\int_{\frac{2}{\sqrt{\omega}\lambda(x)}}^{\infty} -\text{Re}\left(\frac{a(\omega\lambda(x)^2)}{2}\frac{R^2}{\omega^{5/4}\lambda(x)^{5/2}}e^{iR\lambda(x)\sqrt{\omega}}\sigma(R\sqrt{\omega}\lambda(x),R)\right)\sqrt{R}F_4(x,R\lambda(x))dR$$

which arises as part of $w\cdot VII$, exactly as we did in an analogous setting in [26]. This leads to

$$||\omega VII||^2_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{Cg(x)^3}{\lambda(x)^9x^4\log^{2b}(x)}$$

which then leads to

$$||-\lambda(t)\int_t^{\infty}\frac{\cos((t-x)\sqrt{\omega})}{\omega}\cdot 4\lambda(x)^2\lambda'(x)^2\omega^2\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1''(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}$$
$$\leqslant \frac{C\lambda(t)^2}{t^3\log^{\frac{3b}{2}+\epsilon}(t)}$$

The rest of the terms arising in the expression $\partial_x^2 \left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)$ can be treated by using the symbol-type nature of the estimates (11.48), along with the fact that $\partial_x^2(F_4(x,R\lambda(x)))$ is still orthogonal to $\phi_0(R)$ in $L^2(RdR)$. This observation leads to

$$\begin{aligned}&||-\lambda(t)\int_t^\infty \frac{\cos((t-x)\sqrt{\omega})}{\omega}\mathcal{F}(\sqrt{\cdot}\partial_x^2\left(F_4(x,\cdot\lambda(x))\right))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\frac{\lambda(t)^2}{t^3}\left(\frac{1}{\log^b(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2-2\epsilon}(t)}\right)\end{aligned}$$

and this finally gives

$$\begin{aligned}&||\lambda(t)\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)}\\&\leqslant C\frac{\lambda(t)^2}{t^3}\left(\frac{1}{\log^b(t)}+\frac{1}{\log^{2\epsilon}(t)}+\frac{1}{\log^{2b-2\epsilon-1}(t)}+\frac{1}{\log^{3b-2-2\epsilon}(t)}\right)\end{aligned}$$

The final integral to treat in this section is

$$\begin{aligned}&\lambda(t)\cdot\omega\lambda(t)^2\cdot\int_t^\infty \cos((t-x)\sqrt{\omega})\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)dx\\&=-\lambda(t)\lambda(t)^2\partial_t\left(\mathcal{F}(\sqrt{\cdot}F_4(t,\cdot\lambda(t)))_1(\omega\lambda(t)^2)\right)\\&-\lambda(t)\cdot\int_t^\infty \lambda(t)^2\cos((t-x)\sqrt{\omega})\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)dx\end{aligned}$$

Here, most terms are treated exactly as previously. We will write out in detail how to estimate the term $\lambda(x)^4\omega^2\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1''(\omega\lambda(x)^2)$, which is part of $\partial_x^2\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\omega\lambda(x)^2)\right)$, and for which we use a slightly different argument than previously. We start with

$$\begin{aligned}\lambda(x)^4\omega^2\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1''(\omega\lambda(x)^2)&=(\xi\partial_\xi)\circ(\xi\partial_\xi)\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\xi)\right)\Big|_{\xi=\omega\lambda(x)^2}\\&-\xi\partial_\xi\left(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1(\xi)\right)\Big|_{\xi=\omega\lambda(x)^2}\end{aligned}$$

Then, we use the transference identity, which leads to

$$\begin{aligned}&\lambda(x)^4\omega^2\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))_1''(\omega\lambda(x)^2)\\&=\frac{1}{4}\Big(\mathcal{F}(R\partial_R(R\partial_R(\sqrt{R}F_4(x,R\lambda(x)))))_1(\omega\lambda(x)^2)\\&-\pi_1\circ\mathcal{K}(\mathcal{F}(R\partial_R(\sqrt{R}F_4(x,R\lambda(x)))))(\omega\lambda(x)^2)\\&+2\pi_1\circ[\xi\partial_\xi,\mathcal{K}]\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x)))(\omega\lambda(x)^2)\\&-\pi_1\left(\mathcal{K}\left(\begin{bmatrix}0\\\mathcal{F}(R\partial_R(\sqrt{R}F_4(x,R\lambda(x))))_1-\pi_1\circ\mathcal{K}(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x))))\end{bmatrix}\right)\right)\Big)(\omega\lambda(x)^2)\\&+\frac{1}{2}\left(\mathcal{F}(R\partial_R(\sqrt{R}F_4(x,R\lambda(x))))_1(\omega\lambda(x)^2)-\pi_1\circ\mathcal{K}(\mathcal{F}(\sqrt{\cdot}F_4(x,\cdot\lambda(x))))(\omega\lambda(x)^2)\right)\end{aligned}$$

where $\pi_1(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}) = v_1$. Using (11.48), and the boundedness properties of $\mathcal{K}$, we get

$$||\lambda(x)^4 \omega^2 \mathcal{F}(\sqrt{\cdot} F_4(x, \cdot\lambda(x))_1''(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{C}{\lambda(x) x^2 \log^b(x)}$$

We use the same procedure as before to estimate the other terms arising from $\partial_x^2 \left(\mathcal{F}(\sqrt{\cdot} F_4(x, \cdot\lambda(x))\right)_1(\omega\lambda(x)^2)$. Finally, we note that

$$\begin{aligned}
&|\lambda(t)^2 \partial_t (\mathcal{F}(\sqrt{\cdot} F_4(t, \cdot\lambda(t)))_1)(\omega\lambda(t)^2)| \\
&\leqslant C|\frac{\partial_t (\mathcal{F}(\sqrt{\cdot} F_4(t, \cdot\lambda(t)))_1(\omega\lambda(t)^2))}{\omega}| + C\lambda(t)^3 \sqrt{\omega} |\partial_t (\mathcal{F}(\sqrt{\cdot} F_4(t, \cdot\lambda(t)))(\omega\lambda(t)^2))|
\end{aligned}$$

and both terms on the right-hand side of the above inequality have been previously estimated. In total, we then get

$$\begin{aligned}
&||\lambda(t) \left(\omega\lambda(t)^2\right) \int_t^\infty \cos((t-x)\sqrt{\omega}) \mathcal{F}(\sqrt{\cdot} F_4(x, \cdot\lambda(x)))_1(\omega\lambda(x)^2) dx||_{L^2(\rho(\omega\lambda(t)^2)d\omega)} \\
&\leqslant \frac{C\lambda(t)^2}{t^3} \left(\frac{1}{\log^{2b-2\epsilon-1}(t)} + \frac{1}{\log^{3b-2\epsilon-2}(t)} + \frac{1}{\log^{2\epsilon}(t)} + \frac{1}{\log^b(t)}\right)
\end{aligned}$$

□

12.5 F_5 Estimates

In this section, we translate our estimates (11.35) and (11.36) using (12.9) and (12.11), which gives

$$|\mathcal{F}(\sqrt{\cdot} F_5(x, \cdot\lambda(x)))_0| \leqslant \frac{C\lambda(x)^3}{x^5 \log^{b-2}(x)}$$

$$||\mathcal{F}(\sqrt{\cdot} F_5(x, \cdot\lambda(x)))_1(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{C\lambda(x)^2}{x^5 \log^{b-2}(x)}$$

$$||\omega\lambda(x)^2 \mathcal{F}(\sqrt{\cdot} F_5(x, \cdot\lambda(x)))_1(\omega\lambda(x)^2)||_{L^2(\rho(\omega\lambda(x)^2)d\omega)} \leqslant \frac{C\lambda(x)^4 \log^2(x)}{g(x)^2 \log^b(x) x^5}$$

12.6 Setup of the iteration

Define T on Z by

$$T(\begin{bmatrix} y_0 \\ y_1 \end{bmatrix})(t,\omega) = \begin{bmatrix} -\int_t^\infty \int_s^\infty \left(F_{2,0} + \mathcal{F}\left(\sqrt{\cdot}\left(F_3 + F_4 + F_5\right)(s_1, \cdot\lambda(s_1))\right)_0\right) ds_1 ds \\ \int_t^\infty \frac{\sin((t-x)\sqrt{\omega})}{\sqrt{\omega}} \left(F_{2,1}(x,\omega) + \mathcal{F}\left(\sqrt{\cdot}\left(F_3 + F_4 + F_5\right)(x, \cdot\lambda(x))\right)_1 (\omega\lambda(x)^2)\right) dx \end{bmatrix}$$

where we define $F_{2,i}$ by

$$F_2 = \begin{bmatrix} F_{2,0} \\ F_{2,1} \end{bmatrix}$$

and $\mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(x,\cdot\lambda(x))\right)_i(\omega\lambda(x)^2)$ is defined by

$$\mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(x,\cdot\lambda(x))\right)\left(\omega\lambda(x)^2\right) = \begin{bmatrix}\mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(x,\cdot\lambda(x))\right)_0(\omega\lambda(x)^2)\\ \mathcal{F}\left(\sqrt{\cdot}\left(F_3+F_4+F_5\right)(x,\cdot\lambda(x))\right)_1(\omega\lambda(x)^2)\end{bmatrix}$$

Now, we can proceed with estimating T on $\overline{B_1(0)}\subset Z$. If $\begin{bmatrix}y_0\\y_1\end{bmatrix}\in\overline{B_1(0)}\subset Z$, then, we combine all of our previous estimates to get, for a constant $C>0$, *independent* of T_0:

$$||T\left(\begin{bmatrix}y_0\\y_1\end{bmatrix}\right)||_Z \leqslant C\left(\frac{1}{\log^{b-\epsilon}(T_0)}+\frac{1}{\log^{\epsilon}(T_0)}+\frac{1}{\log^{2b-3\epsilon-1}(T_0)}+\frac{1}{\log^{3b-2-3\epsilon}(T_0)}\right), \quad \begin{bmatrix}y_0\\y_1\end{bmatrix}\in\overline{B_1(0)}\subset Z \tag{12.26}$$

Next, we will prove a Lipshitz estimate on T restricted to $\overline{B_1(0)}\subset Z$. For this, it will be useful to use the notation

$$v_y(t,R\lambda(t)) = \frac{1}{\sqrt{R}}\mathcal{F}^{-1}\left(\begin{bmatrix}y_0(t)\\y_1(t,\frac{\cdot}{\lambda(t)^2})\end{bmatrix}\right)(R)$$

where

$$y=\begin{bmatrix}y_0(t)\\y_1(t,\frac{\cdot}{\lambda(t)^2})\end{bmatrix}$$

Then, for $y,z\in\overline{B_1(0)}\subset Z$, we have

$$F_3(v_y)-F_3(v_z) = (v_y-v_z)\Bigg(\frac{6(Q_{\frac{1}{\lambda(t)}}+v_{corr})(v_y+v_z)}{r^2}+\frac{2}{r^2}(v_y^2+v_yv_z+v_z^2) +\frac{6}{r^2}\left((v_{corr}+Q_{\frac{1}{\lambda(t)}})^2-Q^2_{\frac{1}{\lambda(t)}}\right)\Bigg)$$

where, by a slight abuse of notation, we denote by $F_3(v_y)$ the expression (12.2), with $v=v_y$, and similarly for $F_3(v_z)$. This leads to

$$||\left(F_3(v_y)-F_3(v_z)\right)(x,R\lambda(x))||_{L^2(RdR)} \leqslant \frac{C\lambda(x)^2}{x^4\log^{\epsilon}(x)}||\begin{bmatrix}y_0-z_0\\y_1-z_1\end{bmatrix}||_Z\left(\frac{1}{\log^{\epsilon}(x)}+\frac{1}{\log^{b}(x)}\right)$$

$$||L^*L\left((F_3(v_y)-F_3(v_z))(t,\cdot\lambda(t))\right)||_{L^2(RdR)} \leqslant \frac{C\lambda(t)^2}{t^4\log^{\epsilon}(t)}||\begin{bmatrix}y_0-z_0\\y_1-z_1\end{bmatrix}||_Z\left(\frac{1}{\log^{\epsilon}(t)}+\frac{1}{\log^{b}(t)}\right)$$

Since F_2 depends on y linearly, we get, for some $C>0$, *independent* of T_0

$$||T\left(\begin{bmatrix}y_0\\y_1\end{bmatrix}\right)-T\left(\begin{bmatrix}z_0\\z_1\end{bmatrix}\right)||_Z \leqslant C||y-z||_Z\left(\frac{1}{\log^{\epsilon}(T_0)}+\frac{1}{\log^{b}(T_0)}\right), \quad y,z\in\overline{B_1(0)}\subset Z$$

Combining this with (12.26), we get that there exists $M>0$ such that, for all $T_0>M$, T is a strict contraction on $\overline{B_1(0)}\subset Z$. If $T_0>M$, then, by Banach's fixed point theorem, T has a fixed point, say $y_f=\begin{bmatrix}y_{f,0}\\y_{f,1}\end{bmatrix}\in\overline{B_1(0)}\subset Z$.

13 Decomposition of the solution as in Theorem 8.1 (Yang-Mills)

We define v_f by

$$v_f(t,r) := \begin{cases} \sqrt{\frac{\lambda(t)}{r}}\mathcal{F}^{-1}\left(\begin{bmatrix} y_{f,0}(t) \\ y_{f,1}(t,\frac{\cdot}{\lambda(t)^2}) \end{bmatrix}\right)\left(\frac{r}{\lambda(t)}\right), \quad r > 0 \\ 0, \quad r = 0 \end{cases}$$

and note that $v_f(t,\cdot) \in C^1([0,\infty))$, by Lemma 12.1. By the derivation of (12.4), and the regularity of elements in $\overline{B_1(0)} \subset Z$, $u(t,r) = Q_{\frac{1}{\lambda(t)}}(r) + v_c(t,r) + w_c(t,r) + v_f(t,r)$ solves (8.1). It now remains to estimate the energy of $v_c - v_1 + w_c + v_f$. For example, for v_2, we have

$$v_2(t,r) = \int_t^\infty v_{2,s}(t,r)ds$$

where $v_{2,s}$ solves

$$\begin{cases} -\partial_{tt}v_{2,s} + \partial_{rr}v_{2,s} + \frac{1}{r}\partial_r v_{2,s} - \frac{4}{r^2}v_{2,s} = 0 \\ v_{2,s}(s,r) = 0 \\ \partial_t v_{2,s}(s,r) = RHS_2(s,r) \end{cases}$$

By using

$$\left(\partial_x + \frac{2}{x}\right)J_2(x) = J_1(x)$$

and the representation formula for $v_{2,s}$ using the Hankel transform of order 2, namely

$$v_{2,s}(t,r) = \int_0^\infty J_2(r\xi)\sin((t-s)\xi)\widehat{RHS_2}(s,\xi)d\xi$$

we can justify the energy estimate

$$||\partial_t v_{2,s}(t,r)||_{L^2(rdr)} + ||\left(\partial_r + \frac{2}{r}\right)v_{2,s}(t,r)||_{L^2(rdr)} \leqslant C||RHS_2(s,r)||_{L^2(rdr)}$$

exactly as was done in [26] for the correction denoted by v_4 in that work. Then, we have

$$\int_0^\infty \left(\left(\partial_r + \frac{2}{r}\right)v_{2,s}(t,r)\right)^2 rdr = \int_0^\infty \left(\partial_r v_{2,s}(t,r)\right)^2 rdr + \int_0^\infty \frac{4v_{2,s}^2}{r^2}rdr + 2\int_0^\infty \partial_r\left(v_{2,s}^2\right)dr \tag{13.1}$$

Even though the pointwise estimates we recorded for $v_{2,s}$ do not imply that $v_{2,s}(t,r) \to 0, \quad r \to \infty$, we can prove that $\lim_{r\to\infty} v_{2,s}(t,r) = 0$ by the Dominated convergence theorem, applied to the spherical means formula for $v_{2,s}$ (for instance, the analog of the formula (11.22)). Then, the last integral in the expression above is zero, and we get

$$||\partial_t v_{2,s}(t,r)||_{L^2(rdr)} + ||v_{2,s}(t)||_{\dot{H}^1_e} \leqslant C||RHS_2(s,r)||_{L^2(rdr)}$$

Using Minkowski's inequality, we then get

$$||\partial_t v_2(t,r)||_{L^2(rdr)} + ||v_2(t)||_{\dot{H}^1_e} \leqslant C\int_t^\infty ||RHS_2(s,r)||_{L^2(rdr)}ds$$

This same procedure can be applied for v_k and all w_k. We recall that $v_c - v_1 = \sum_{k=2}^\infty v_k$. We then get

$$||\partial_t\left(v_c - v_1\right)(t,r)||_{L^2(rdr)} + ||\left(v_c - v_1\right)(t)||_{\dot{H}^1_e} \leqslant \frac{C}{\log^{2b-1}(t)}$$

and

$$||\partial_t w_c(t,r)||_{L^2(rdr)} + ||w_c(t)||_{\dot{H}^1_e} \leqslant \frac{C\lambda(t)^2}{g(t)t\log^b(t)}$$

Finally, exactly as in [26], the transference identity of [15] gives

$$\begin{aligned}||\partial_1 v_f(t,R\lambda(t))||_{L^2(RdR)} \leqslant & \frac{C|\lambda'(t)|}{\lambda(t)}||\mathcal{F}^{-1}\left(\begin{bmatrix} y_{f,0}(t) \\ y_{f,1}(t,\frac{\cdot}{\lambda(t)^2})\end{bmatrix}\right)||_{L^2(dR)} \\ &+ C||\mathcal{F}^{-1}\left(\begin{bmatrix} y'_{f,0}(t) \\ \partial_1 y_{f,1}(t,\frac{\cdot}{\lambda(t)^2})\end{bmatrix}\right)||_{L^2(dR)} \\ &+ \frac{C|\lambda'(t)|}{\lambda(t)}||\mathcal{F}^{-1}\left(\mathcal{K}\left(\begin{bmatrix} y_{f,0}(t) \\ y_{f,1}(t,\frac{\cdot}{\lambda(t)^2})\end{bmatrix}\right)\right)||_{L^2(dR)}\end{aligned}$$

Therefore, we get

$$||\partial_1 v_f(t,r)||_{L^2(rdr)} \leqslant \frac{C\lambda(t)^3}{t^3\log^\epsilon(t)}$$

$$\begin{aligned}||v_f(t,r)||_{\dot{H}^1_e} = ||v_f(t,\cdot\lambda(t))||_{\dot{H}^1_e} &\leqslant C(||L(v_f(t,R\lambda(t)))||_{L^2(RdR)} + ||v_f(t,R\lambda(t))||_{L^2(RdR)} \\ &\leqslant \frac{C\lambda(t)^2}{t^2\log^\epsilon(t)}\end{aligned}$$

Next, we use the pointwise estimates recorded in the previous sections to get

$$||v_c(t,\cdot) + w_c(t,\cdot)||_{L^\infty} \leqslant \frac{C}{\log^b(t)}$$

For v_f, we have

$$||v_f(t,\cdot)||_{L^\infty} \leqslant ||v_f(t,\cdot\lambda(t))||_{\dot{H}^1_e} \leqslant \frac{C\lambda(t)^2}{t^2\log^\epsilon(t)}$$

We also need to verify that

$$||\partial_t v_1(t,r)||_{L^2(rdr)} + ||v_1(t,\cdot)||_{\dot{H}^1_e} < \infty$$

This can be done by noting that

$$||\partial_t v_1||^2_{L^2(rdr)} + ||\left(\partial_r + \frac{2}{r}\right)v_1||^2_{L^2(rdr)} = ||\widehat{v_{1,1}}(\xi)||^2_{L^2(\xi d\xi)}$$

which can be seen, for example using the formula

$$v_1(t,r) = \int_0^\infty J_2(r\xi)\sin(t\xi)\widehat{v_{1,1}}(\xi)d\xi$$

Then, we use the same observation as in (13.1), and the fact that $b > \frac{2}{3}$, which shows that $\widehat{v_{1,1}}(\xi) \in L^2(\xi d\xi)$, to conclude

$$||\partial_t v_1(t,r)||_{L^2(rdr)} + ||v_1(t,\cdot)||_{\dot{H}^1_e} < \infty$$

Finally, we can verify that our solution has finite energy, by noting that

$$E_{YM}(u,\partial_t u) \leqslant C\left(||\partial_t u(t,r)||^2_{L^2(rdr)} + ||\partial_r u||^2_{L^2(rdr)} + \int_0^\infty \frac{r}{r^2}\left(1 - Q_{\frac{1}{\lambda(t)}}(r)^2\right)^2 dr \right.$$
$$\left. + \int_0^\infty \frac{r}{r^2}\left(v_c + w_c + v_f\right)^2 dr\right)$$

where we used the fact that

$$||v_c(t,\cdot) + w_c(t,\cdot) + v_f(t,\cdot)||_{L^\infty} \to 0, \quad \text{as } t \to \infty$$

Also, we have

$$||\partial_t\left(v_c - v_1 + v_f + w_c\right)||_{L^2(rdr)} + ||v_c - v_1 + v_f + w_c||_{\dot{H}^1_e} \leqslant \frac{C}{\log^{2b-1}(t)}$$

which finishes the verification of the energy-related statements in theorem 8.1.

References

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