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by

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Professor Constantin Teleman, Chair
Professor Ian Agol
Professor Martin Olsson

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Abstract

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This thesis is a conglomerate of several results in algebraic topology united by the common thread of taking seriously the idea that geometric considerations can be useful for proving algebraic results in the field of chromatic homotopy theory. These results include a geometric construction of equivariant elliptic cohomology at the Tate curve, abstract derivations of the dual Steenrod algebras at all primes, and geometric presentations of higher algebraic structures.

Contents

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Chapter 1

Introduction

1.1 An explanation of the title

This thesis is a conglomerate of several results in algebraic topology united by the common thread of taking seriously the vague idea that "geometric" considerations are often useful. The algebro-topological results fall more specifically within the area of chromatic homotopy theory[1], a field which is notoriously algebraic/combinatorial[2] and un-"geometric." At this point I should probably explain what I mean by "geometric." I don't mean anything terribly mathematically precise, but the following comes close: a mathematical object or technique is *geometric* if it is presented by or involves a point-set construction referencing a manifold in some way (e.g. a stably almost complex manifold-with-singularities[3] or a Fredholm operator on a vector bundle over a Lie groupoid). The central way in which these geometric considerations find application to the homotopy theoretic results at hand is by taking the philosophical stance that cohomology theories[4]—functors from a suitable category of topological spaces to the category of graded abelian groups satisfying a suitable version of the Eilenberg-Steenrod axioms—should be (when possible) conceptually viewed and handled in practice as being the presheaf of isomorphism classes of a sheaf of groupoids of geometric objects. A convenient formalism for capturing this idea is the subject of Section 2. The reason for my advocation of this philosophical stance is (mostly[5]) practical - as this thesis hopes to prove, the geometric viewpoint often leads to substantial insights into homotopy theoretic questions. Before I make this claim more precise in Section 1.3, I would like to give

[1]The next subsection contains an introduction for those not familiar with the subject.

[2]The popular techniques in the field involve things like localizations and spectral sequence calculations, not to mention the entire area has its foundations in ∞-category theory and is thus either combinatorial or abstract-homotopy-theoretic.

[3]Manifold-with-singularities deserves further explanation but that further explanation deserves to be omitted from the Introduction. See Definition 5.1.1.

[4]And by extension, spectra.

[5]There is a small part of me that takes this stance dogmatically but I try to keep that part of me to myself.

a short overview of chromatic homotopy theory.

1.2 A chromatic introduction

One of the main branches of modern algebraic topology is chromatic homotopy theory. Broadly, chromatic homotopy theory is the practice of studying the stable homotopy category via its connection to the algebraic and arithmetic geometry of (1-dimensional, commutative) formal groups. Conventionally, chromatic homotopy theory is an algebraic practice; the methods of higher algebra [21] (E_∞-rings, Bousfield localization, the Adams spectral sequence, obstruction theory) are used to flesh out what this connection says about the stable homotopy category. The first citation in any overview of chromatic homotopy theory is Quillen's paper [23], where the connection to formal groups is first solidified in what is now simply known as "Quillen's theorem[6]:" that the complex cobordism spectrum MU carries the universal formal group law on its homotopy groups. This leads to the fact that the MU-based Adams spectral sequence has E_2-page given by certain cohomology groups of the moduli stack of formal groups M_{fg}, and that the local behavior of the stable homotopy category can be calculated by localizing at various geometric points of that moduli stack. But this thesis takes a slightly different path. It still begins with Quillen's [23], but instead of building off the *results* of that paper, it builds off its *methods*. Briefly, those methods can be outlined/highlighted as follows: first, Quillen gives a geometric presentation of the cohomology theory associated to MU whose cocycles are manifolds-with-structure. Then he defines two sets of cohomology operations using the geometry of these manifold cocycles. In the modern language of higher algebra, one of these sets of operations is (a geometric presentation of) the power operations associated to the E_∞-ring structure on MU. Using a clean intersection formula Quillen then proves a certain relation between the two sets of operations, and thereby deduces that the coefficients of the canonical formal group law on MU_* generate it as a ring, from which Quillen's main theorem follows easily. Therefore, Quillen's theorem, the central/nodal result of chromatic homotopy theory is proved using a fundamentally *geometric* viewpoint, in particular a geometric, manifold presentation of MU not only as a spectrum, but also as an E_∞-ring spectrum. This thesis is guided by the goal of taking this geometric perspective on chromatic homotopy seriously and exploring the consequences.

1.3 Merits of the geometric viewpoint

As mentioned above, I hope that this thesis serves as evidence that a geometric viewpoint on chromatic homotopy theory leads to practical merits. The applications in this thesis that should be highlighted here are the construction of chromatically interesting objects

[6]Of course, since Quillen was pretty prolific, there are probably 5 or 6 theorems that name might reasonably refer to.

(cf. Sections 4.3 and 5.3) calculational tools (Section 4.2), conceptually clear alternates for algebraic calculations (cf. Sections 5.2 and 5.5), clean presentations of higher algebraic structures (cf. Sections 5.6 and 5.7).

Chapter 2

Symmetric stable functors

Definition 2.0.1. Let $\mathrm{Fin}^\times$ be the category of finite sets and isomorphisms.

Definition 2.0.2. For a finite set N write $\mathbb{R}^N$ for the topological vector space of maps $N \to \mathbb{R}$ and $\mathbb{C}^N$ for the topological vector space of maps $N \to \mathbb{C}$. Write S^N for the 1-point compactification of $\mathbb{R}^N$.

Definition 2.0.3. Let Set_* be the category of pointed sets.

Definition 2.0.4. Let C be a 1-category. Define a monoidal structure on $\mathrm{Fun}(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*)$ as follows:

$$(F \otimes G)(N, c) := \bigoplus_{M \subset N} F(M, c) \otimes G(N - M, c).$$

That is Day convolution in the Fin-variable and the pointwise tensor product of pointed sets in the C variable. Let $\delta : \Delta \to C$ be a cosimplicial object in C. Then δ^* induces a monoidal functor

$$\delta^* : \mathrm{Fun}(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*) \to \mathrm{Fun}(\mathrm{Fin}^\times \times \Delta^{\mathrm{op}}, \mathrm{Set}_*)$$

where the codomain is equipped with the same monoidal structure as the target (with C replaced by Δ). Let $\sigma \in \mathrm{Fun}(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*)$ be such that $\delta^*\sigma$ is equivalent[1] to the "sphere" S given by $(N, \Delta^k) \mapsto S(N)_k = \mathrm{Sing}_k S^N$ as a monoidal object in $\mathrm{Fun}(\mathrm{Fin}^\times \times \Delta^{\mathrm{op}}, \mathrm{Set}_*)$. Then a *symmetric stable functor* is a σ-module in $\mathrm{Fun}(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*)$, which form a category σ-mod, also called $\mathrm{SstFun}(C, \sigma, \delta)$ to emphasize the input data. This is a monoidal category with tensor product given by the following colimit in $\mathrm{Fun}(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*)$

$$F \otimes_\sigma G := \mathrm{colim}(F \otimes \sigma \otimes G \rightrightarrows F \otimes G).$$

Write $\delta^*\sigma$-mod for the category of $\delta^*\sigma$-modules in $\mathrm{Fun}(\mathrm{Fin}^\times \times \Delta^{\mathrm{op}}, \mathrm{Set}_*)$ (with a relative tensor product similar to the above). Since $\delta^*\sigma$ is monoidally equivalent to the sphere S, $\delta^*\sigma$-mod is equivalent to the category of S-modules, which is the category of symmetric spectra

[1] In the Kan-Quillen model structure on simplicial sets.

in simplicial sets (as defined in e.g. [3]). Pullback along δ therefore defines a monoidal functor

$$\mathrm{SstFun}(C, \sigma, \delta) = \sigma\text{-mod} \xrightarrow{\delta^*} \delta^*\sigma\text{-mod} \simeq \mathrm{SymSp}(\mathrm{sSet}_*),$$

which leads directly to the following lemma.

Lemma 2.0.5. *Let $E_\infty(Sp)$ be the symmetric monoidal ∞-category of E_∞-ring spectra and let $E_1(Sp)$ be the category of E_1-ring spectra. Given a triple (C, σ, δ) as above there are functors*

$$CMon(SstFun(C, \sigma, \delta)) \to E_\infty(Sp).$$

$$AssMon(SstFun(C, \sigma, \delta)) \to E_1(Sp)$$

and the first one is symmetric monoidal.

Proof. This is immediate from the well-known fact that commutative/associative symmetric ring spectra model E_∞/E_1 ring spectra (c.f. [3]) and the fact that the functor δ^* preserves colimits so that the relevant relative tensor products are preserved. □

Definition 2.0.6. For a symmetric spectrum T let uT denote the underlying spectrum.

Definition 2.0.7. Let cMan be the category of compact manifolds-with-corners (c.f. Definition 5.1.1), and set $C = \mathrm{cMan} \times \Delta^{\mathrm{op}}$. Let $\sigma \in \mathrm{Fun}(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*)$ be defined by $\sigma(N, X, \Delta^k) = \mathrm{Hom}_{\mathrm{cMan}}(X, S^N)$ and let $\delta : \Delta \to C$ be the cosimplicial object defined by $\delta(\Delta^k) = (\Delta^k_{\mathrm{top}}, \Delta^k)$. Then $\delta^*\sigma(N, \Delta^k) = \mathrm{Hom}_{\mathrm{cMan}}(\Delta^k_{\mathrm{top}}, S^N)$ is monoidally equivalent to the sphere S as required in Definition 2.0.4, so that the category $\mathrm{SstFun}(C, \sigma, \delta)$ is defined.

Chapter 3

K-theory

In this chapter I construct a completed version of twisted S^1-equivariant K-theory, with an eye towards its application in the next chapter. I will use the formalism of symmetric stable functors to construct E_∞-ring structures in certain cases. In particular the construction will specialize to the case of ordinary (untwisted an un-equivariant) K-theory and therefore present KU as an E_∞-ring spectrum.

Some Definitions and Disclaimers: In this chapter, rings, abelian groups, and Hilbert spaces are assumed to be $\mathbb{Z}/2\mathbb{Z}$-graded, unless it is stated otherwise. As is standard, the *loop group* LG of a Lie group G is the group of *smooth* maps $S^1 \to G$ with the topology of uniform convergence. C_2 is the cyclic group of order 2. All groupoids considered in this paper are *topological groupoids* (cf. Definition 3.0.2) unless otherwise stated. The following string of definitions sets the stage for the context in which these groupoids are considered.

Definition 3.0.1. The category Top is defined to be the category whose objects are topological spaces that are homotopy equivalent to a CW-complex and whose morphisms are continuous maps.

Definition 3.0.2. A *topological groupoid* $\mathbf{X} = (X_1 \rightrightarrows X_0)$ is a groupoid object in the category Top. A morphism of topological groupoids is a morphism of the corresponding diagrams in Top.

Definition 3.0.3. Let $\mathbf{X} = (X_1 \rightrightarrows X_0)$ be a topological groupoid. The *coarse quotient* $[\mathbf{X}]$ is the quotient of X_0 under the equivalence relation defined by $x \sim y$ if there is a morphism from x to y. In other words, it is the topological space of isomorphism classes of objects of $\mathbf{X}$.

Definition 3.0.4. A map of topological groupoids $\mathbf{X} \to \mathbf{Y}$ is a *local equivalence* ([13] Definition A.4) if the induced map of (discrete) groupoid-valued presheaves on Top $\mathrm{Hom}(-, \mathbf{X}) \to \mathrm{Hom}(-, \mathbf{Y})$ is an equivalence on stalks. Two topological groupoids $\mathbf{X}$ and $\mathbf{Y}$ are said to be *weakly equivalent* if there is a diagram of local equivalences $\mathbf{X} \leftarrow \mathbf{Z} \to \mathbf{Y}$. Although it will not be needed, some readers may take comfort in the fact that weakly equivalent topological groupoids present the same underlying stack on the site of topological spaces (cf. [13] Remark A.5).

Definition 3.0.5. A topological groupoid $\mathbf{X} = (X_1 \rightrightarrows X_0)$ is called a *local quotient* groupoid if there exists a countable open cover $\{U_\alpha\}$ of the coarse quotient $[\mathbf{X}]$ such that the full subgroupoid associated to each U_α is weakly equivalent (cf. Definition 3.0.4) to the quotient groupoid of a Hausdorff space M by a compact Lie group K, denoted in this paper by M/K.

3.1 S^1-completed twisted K-theory

The S^1-equivariant theory is constructed for certain *$B\mathbb{Z}$-groupoids*, which are defined as follows.

Definition 3.1.1. A *$B\mathbb{Z}$-groupoid* is a pair[1] $(\mathbf{X}, \alpha)$ consisting of a groupoid $\mathbf{X} = (X_1 \rightrightarrows X_0)$ and a $B\mathbb{Z}$-action α, i.e. an automorphism of the identity functor $\alpha : 1_{\mathbf{X}} \Rightarrow 1_{\mathbf{X}}$. A morphism ("*$B\mathbb{Z}$-equivariant map*") of $B\mathbb{Z}$-groupoids $(\mathbf{X}, \alpha) \to (\mathbf{X}', \alpha')$ is a morphism of topological groupoids $F : \mathbf{X} \to \mathbf{X}'$ such that for every $x \in X_0$, $F(\alpha(x)) = \alpha'(F(x))$. A *$B\mathbb{Z}$-subgroupoid* of a $B\mathbb{Z}$-groupoid $(\mathbf{X}, \alpha)$ is a $B\mathbb{Z}$-groupoid $(\mathbf{X}', \alpha')$ such that $\mathbf{X}'$ is a subgroupoid of $\mathbf{X}$ containing all components $\alpha(x')$, $x' \in X'_0 \subset X_0$ and α' agrees with the restriction of α to $\mathbf{X}'$. A $B\mathbb{Z}$-groupoid $(\mathbf{X}, \alpha)$ is called *trivial* if α is the trivial automorphism of the identity.

To model the 'quotient' of a $B\mathbb{Z}$-groupoid by its $B\mathbb{Z}$-action I make the following definition.

Definition 3.1.2. For a $B\mathbb{Z}$-groupoid $(\mathbf{X}, \alpha)$ define the *$B\mathbb{Z}$-quotient* $\mathbf{X}/B\mathbb{Z}$ to be groupoid whose space of objects is X_0 and whose space of morphisms is $(X_1 \times \mathbb{R})/\mathbb{Z}$, where $\mathbb{Z}$ acts as follows: if s denotes the source morphism of $\mathbf{X}$, then for $n \in \mathbb{Z}$, $n \cdot (p, r) := (p\alpha(s(p))^{-n}, r+n)$. This is functorial in the $B\mathbb{Z}$-groupoid. The $B\mathbb{Z}$-quotient of a local quotient groupoid is again a local quotient groupoid.

Example 3.1.3. The motivating example of a $B\mathbb{Z}$-groupoid is the *loop groupoid* $\mathcal{L}(M/G)$ of a global quotient M/G, which is again a global quotient: its space of objects is the subspace of $M \times G$ of pairs $\{(m, g) \in M \times G |\ gm = m\}$ and G acts by translation on the first factor and conjugation on the second. The $B\mathbb{Z}$-action is the automorphism of the identity functor whose component at the object (m, g) is g.

When M is a point this is the quotient groupoid associated to G acting on itself by conjugation, and when G is connected the $B\mathbb{Z}$-quotient admits a (possibly enlightening) second description, up to equivalence: let $\mathcal{A}(G)$ be the space of connections on the trivial

[1] To streamline notation I reserve the right to refer to a $B\mathbb{Z}$-groupoid $(\mathbf{X}, \alpha)$ by the name of the groupoid $\mathbf{X}$ and leave the $B\mathbb{Z}$-action α as a mystery to be revealed as required.

principal G-bundle over S^1 and $\mathbb{L}G$ the group of smooth bundle isomorphisms covering rigid rotations of the base S^1. Note that $\mathbb{L}G \simeq LG \rtimes S^1$. Then

$$\mathcal{L}(\mathrm{pt}/G)/B\mathbb{Z} \simeq \mathcal{A}(G)/\mathbb{L}G.$$

The map inducing the equivalence goes from right to left and is given by taking the holonomy of a connection (cf. [15] Section 2.1).

3.1.1 Central extensions and twists

This subsection describes the model for *twists* used in this paper.

Definition 3.1.4. (cf. [13] Section 2.2) A groupoid is said to be *graded* if it is equipped with a functor ϵ to pt/C_2. A *graded central extension* $\mathrm{pt}/U(1) \to \mathbf{L} \to \mathbf{X}$ of a groupoid $\mathbf{X} = (X_1 \rightrightarrows X_0)$ is a graded groupoid $\mathbf{L} = (L \rightrightarrows X_0, \epsilon)$ and a functor $P : \mathbf{L} \to \mathbf{X}$ which is the identity map on objects and is such that the induced map $P_1 : L \to X_1$ is a principal $U(1)$-bundle. The category $\mathfrak{Ext}_{\mathbf{X}}$ is defined to have objects the graded central extensions of $\mathbf{X}$, and a morphism $(\mathbf{L}_1, \epsilon_1) \to (\mathbf{L}_2, \epsilon_2)$ is the following data: a pair (M, η) of an isomorphism class of principal $U(1)$-bundle $M \to X_0$, an isomorphism of $U(1)$-bundles $t^*M \otimes L_1 \otimes s^*M^{-1} \to L_2$, and a continuous function $\eta : X_0 \to C_2$ such that $\epsilon_2 = t^*\eta\epsilon_2 s^*\eta^{-1}$.

Example 3.1.5. Let $T = U(1)^{\times r}$ be a torus. Consider the quotient groupoid $T/T = (T \times T \rightrightarrows T)$ associated to the (trivial) conjugation action of T on itself. Let τ be a homomorphism $\tau : \pi_1 T \to \Lambda := \mathrm{Hom}_{\mathrm{Grp}}(T, U(1))$. Define a $\pi_1 T$-action on $\mathfrak{t} \times T \times U(1)$ by the formula $p \cdot (X, t, e^{i\theta}) = (X + p, t, e^{i\theta}\tau(p)(t))$ and let L^τ be the quotient $(\mathfrak{t} \times T \times U(1))/\pi_1 T$. Then $L^\tau \rightrightarrows T$ is a groupoid with source and target map both given by $([X, t, e^{\theta})] \mapsto e^X$, and the evident morphism to $T \times T \rightrightarrows T$ is a graded central extension once we equip both groupoids with the trivial grading.

Remark 3.1.6. The category of graded central extensions is extremely sensitive to the groupoid presentation $\mathbf{X}$ of the underlying stack. For example, suppose that the underlying stack is equivalent to a finite CW-complex X. In the presentation of X as $X \rightrightarrows X$, all objects of $\mathfrak{Ext}_{\mathbf{X}}$ are trivial as principal $U(1)$-bundles (the identity morphisms provide a section) and are therefore determined by their grading, so that $\pi_0 \mathfrak{Ext}_{\mathbf{X}} = H^0(X; C_2)$. On the other hand, if X is presented as the groupoid $\coprod U_{ij} \rightrightarrows \coprod U_i$ associated to a good Cech cover of X, then any Cech 1-cocycle with values in line bundles defines a central extension. It follows that $\pi_0 \mathfrak{Ext}_{\mathbf{X}} = H^1(X; C_2) \times H^1(X; BU(1)) = H^1(X; C_2) \times H^3(X)$. This is in accordance with the fact that there is a general homotopy theoretic framework for twisted cohomology theories which foresees the possibility of twisting $K^*(X)$ by classes in $H^3(X)$.

Remark 3.1.7. The appearance of H^3 in the above is not a coincidence - for all groupoids $\mathbf{X}$ considered in this paper, the set of isomorphism classes of graded central extension whose grading is trivial is isomorphic to the kernel of $H^3\mathbf{X} \to H^3 X_0$ (cf. [13] Proposition 2.13).

Definition 3.1.8. In light of the above, define the category $\mathfrak{Twist}_{\mathbf{X}}$ as follows. Its objects are *twists*: a pair $(\mathbf{V}, \mathbf{L}^\tau)$ consisting of a local equivalence (cf. Definition 3.0.4) $\mathbf{V} \to \mathbf{X}$ and a graded central extension $\mathbf{L}^\tau$ of $\mathbf{V}$. A morphism of twists from $(\mathbf{V}, \mathbf{L}^\tau)$ to $(\mathbf{U}, \mathbf{L}^\sigma)$ is a local equivalence $\phi : \mathbf{V} \to \mathbf{U}$ over $\mathbf{X}$ and a morphism in $\mathfrak{Ext}_{\mathbf{V}}$ from $\mathbf{L}^\tau$ to $\phi^*\mathbf{L}^\sigma$. A map of groupoids $f : \mathbf{X} \to \mathbf{Y}$ induces a pullback functor $f^* : \mathfrak{Twist}_{\mathbf{Y}} \to \mathfrak{Twist}_{\mathbf{X}}$.

Definition 3.1.9. If $\mathbf{X}$ is a $B\mathbb{Z}$-groupoid, then the category $B\mathbb{Z}$-$\mathfrak{Twist}_{\mathbf{X}}$ of *equivariant* twists is defined to be the category $\mathfrak{Twist}_{\mathbf{X}/B\mathbb{Z}}$. Define the category $B\mathbb{Z}$-$\mathfrak{Twist}$ whose objects are pairs consisting of a $B\mathbb{Z}$-groupoid $\mathbf{X}$ and an equivariant twist $(\mathbf{V}, \mathbf{L}^\tau) \in B\mathbb{Z}$-$\mathfrak{Twist}_{\mathbf{X}}$. A morphism from $(\mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau))$ to $(\mathbf{Y}, (\mathbf{U}, \mathbf{L}^\sigma))$ is a $B\mathbb{Z}$-equivariant map $\varphi : \mathbf{X} \to \mathbf{Y}$ and a morphism from $(\mathbf{V}, \mathbf{L}^\tau)$ to $(\varphi^*\mathbf{U}, \varphi^*\mathbf{L}^\sigma)$ in $B\mathbb{Z}$-$\mathfrak{Twist}_{\mathbf{X}}$. A *homotopy* between two such morphisms is defined to be a $B\mathbb{Z}$-equivariant map $\mathbf{X} \times [0,1] \to \mathbf{Y}$ restricting to each of the $B\mathbb{Z}$-equivariant maps $\mathbf{X} \to \mathbf{Y}$ at the endpoints.

Definition 3.1.10. Let $B\mathbb{Z}$-$\mathfrak{Twist}_{\mathrm{rel}}$ be the category whose objects are triples $(\mathbf{X}, \mathbf{A}, (\mathbf{V}, \mathbf{L}^\tau))$ consisting of a $B\mathbb{Z}$-groupoid $\mathbf{X}$, a full $B\mathbb{Z}$-subgroupoid $\mathbf{A}$, and an object $(\mathbf{V}, \mathbf{L}^\tau) \in B\mathbb{Z}$-$\mathfrak{Twist}_{\mathbf{X}}$. Morphisms are the relative versions of those in $B\mathbb{Z}$-$\mathfrak{Twist}$.

Define the *product* of two objects $(\mathbf{X}, \mathbf{A}, (\mathbf{V}, \mathbf{L}^\tau))$ and $(\mathbf{X}, \mathbf{A}, (\mathbf{W}, \mathbf{L}^\sigma))$ to be the object $(\mathbf{X}, \mathbf{A}, (\mathbf{Y}, \mathbf{L}^{\tau+\sigma}))$ with Y being the pullback $\mathbf{V} \times_{\mathbf{X}} \mathbf{W}$ and $\mathbf{L}^{\tau+\sigma}$ the graded central extension of $\mathbf{V} \times_{\mathbf{X}} \mathbf{W}$ given by the tensor product (cf. [13] Definition 2.6) of the pullbacks of $\mathbf{L}^\tau$ and $\mathbf{L}^\sigma$ along the two projections.

3.1.2 The cohomology theory

In this subsection a spectrum (in fact, a symmetric stable functor) is constructed for each object $(\mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau))$ of $B\mathbb{Z}$-$\mathfrak{Twist}$. Let G_x and $\tilde{G}_x$ denote the automorphism groups of an object x in $\mathbf{X}$ and $\mathbf{X}/B\mathbb{Z}$. By Definition 3.1.2 there is an exact sequence

$$1 \to G_x \hookrightarrow \tilde{G}_x = (G_x \times \mathbb{R})/\mathbb{Z} \to S^1 \to 1.$$

where $\mathbb{Z}$ sits inside $G_x \times \mathbb{R}$ as the subgroup $\{(\alpha(x)^{-n}, n)\}$. Because G_x is compact the sequence admits a fractional right-splitting, i.e. a splitting after replacing S^1 by a finite cover $S^1_d \to S^1$. The construction of the $\vec{K}_{S^1}$-theory spectrum will require a choice of fractional splitting $\psi_x : \tilde{G}_x \leftarrow S^1_d$ but the final product will not depend on it. Note that the space of choices of ψ_x is a torsor for the group $\mathrm{Hom}(S^1, G_x)^{\mathrm{conj}}$ of conjugacy classes of homomorphisms $S^1 \to G_x$, and that a global choice may not exist over all of $\mathbf{X}/B\mathbb{Z}$. An example of this situation is the $B\mathbb{Z}$-action on $\mathbf{X} = \mathcal{L}(\mathrm{pt}/U(1)) \simeq U(1)/U(1)$ (cf. Example 4.1.1). Then $\mathbf{X}/B\mathbb{Z}$ is equivalent to a $U(1)$-gerbe over $U(1) \times (\mathrm{pt}/B\mathbb{Z})$ whose Dixmier-Douady invariant is a generator of $H^3(U(1) \times BS^1)$. Therefore a global fractional splitting—which is equivalent to a trivialization of the gerbe classified by some (nonzero) multiple of the generator—does not exist.[2]

[2] A later section contains a treatment of this example in greater detail.

Ordinary K-theory is concerned with equivalence classes of finite-dimensional vector bundles. Morally, local $\mathbb{Z}((q))$-completion corresponds to relaxing this condition to allow for infinite-dimensional bundles, provided that the image of S^1 under ψ_x acts on the fiber with finite-dimensional isotypic subspaces and that the set of irreducible characters of S^1 that appear is bounded below as a subset of $\mathbb{Z}$. The failure of ψ_x to extend over X_0 makes this ill-defined: the same bundle can meet these requirements for one choice of ψ_x and fail them for another. Luckily the naive fix—restricting to those bundles whose fibers satisfy the above property for all choices of ψ_x—turns out to work nicely.

I will follow the Freed-Hopkins-Teleman model of twisted K-theory for local quotient groupoids but adapt it to the fit the machinery of symmetric stable functors. In particular the geometric objects will be sections of bundles of Fredholm operators. For each finite set N let Cl_N be the Clifford algebra on $\mathbb{C}^N$ (cf. Definition 2.0.2). It is generated by the elements $\gamma(e_\eta)$ corresponding to the standard basis elements $e_\eta, \eta \in N$, of $\mathbb{C}^N$. Recall that Cl_N, as a graded module over itself, is the direct sum of all its irreducible graded modules[3] For a Hilbert space H let $\mathcal{B}(H)$ denote the spaces of bounded operators with the compact-open topology and let $\mathcal{K}(H)$ denote the space of compact operators with the norm topology. Let $\mathrm{Fred}^N(H)$ be the space of odd self-adjoint Fredholm operators $A \in \mathcal{B}(Cl_N \otimes H)$ that (graded) commute with the Cl_N-action and such that $A^2 + 1$ is a compact operator. The topology is defined via the inclusion

$$\mathrm{Fred}^n(H) \hookrightarrow \mathcal{B}(H) \times \mathcal{K}(H)$$
$$A \mapsto (A, A^2 + 1).$$

Definition 3.1.11. Let $(\mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau))$ be an object of $B\mathbb{Z}$-$\mathfrak{Twist}$. Recall that there is an implicit grading $\epsilon : \mathbf{L}^\tau \to \mathrm{pt}/C_2$. A *$\tau$-twisted Hilbert bundle* over $\mathbf{X}$ is a Hilbert bundle $\mathcal{H} \to \mathbf{L}^\tau$ such that for any object x in $\mathbf{L}^\tau$ the central $U(1) \in \mathrm{Aut}(x)$ acts on the fiber $\mathcal{H}_x$ by scalar multiplication, and for any morphism $f : x \to y$ the map of fibers $\mathcal{H}_x \to \mathcal{H}_y$ has degree $\epsilon(f)$.

Definition 3.1.12. Let $(\mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau))$ be an object of $B\mathbb{Z}$-$\mathfrak{Twist}$. A τ-twisted Hilbert bundle $\mathcal{H} \to \mathbf{X}$ over a groupoid over $\mathbf{X}$ is said to be *locally universal* if for every open subgroupoid $U \hookrightarrow \mathbf{X}$ and every τ-twisted Hilbert bundle $\mathcal{V}$ over U, there exists a unitary embedding of $\mathcal{V}$ into the restriction $\mathcal{H}_U$ of $\mathcal{H}$ to U. If $\mathbf{X}$ is a local quotient groupoid (cf. Definition 3.0.5) then a locally universal Hilbert bundle always exists and is unique up to unitary equivalence (cf. [13] Lemma 3.12).

Definition 3.1.13. Let $(\mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau))$ be an object of $B\mathbb{Z}$-$\mathfrak{Twist}$ and let $\mathcal{H} \to \mathbf{L}^\tau$ be a locally universal τ-twisted Hilbert bundle. For a natural number n write $\mathrm{Fred}^n(H)$ for $\mathrm{Fred}^N(H)$ with $N = \{1, ..., n\}$. Then the (uncompleted) S^1-equivariant, τ-twisted K-theory of $\mathbf{X}$ is defined in [13] (Section 3.4, A.5) to be

$$^\tau K^n_{S^1}(\mathbf{X}) := {}^\tau K^n(\mathbf{X}/B\mathbb{Z}) := \pi_k \Gamma(\mathbf{L}^\tau; \mathrm{Fred}^{n+k}(\mathcal{H})).$$

[3] There is one irreducible representation when $|N|$ is odd, and two when $|N|$ is even.

It is independent of the choice of $\mathcal{H}$ (cf. [13] Remark 3.17).

Over an object x the fiber $\mathcal{H}_x$ is isomorphic to $L^2(\tilde{G}_x) \otimes \ell^2 \otimes Cl_1$ (cf. [13] Lemma A.32). A choice of fractional splitting $\psi_x : \tilde{G}_x \leftarrow S^1_d$ gives an isomorphism $\mathcal{H}_x \simeq L^2(G_x) \otimes L^2(S^1) \otimes \ell^2 \otimes Cl_1$. An irreducible representation of S^1 is labelled by an integer $k \in \mathbb{Z}$ and has character $q \mapsto q^k$. For $F \in \mathcal{B}(\mathcal{H}_x)^{S^1}$, let $F_{\psi_x}(k)$ denote the restriction of F to the S^1-isotypic component of weight k, after splitting $\mathcal{H}_x \simeq V_{\psi_x} \otimes L^2(S^1)$ using ψ_x as above.

Definition 3.1.14. For each finite set N define the set of *ψ_x-relative q-Fredholm operators* as

$$q_{\psi_x}\mathrm{Fred}^N(\mathcal{H}_x) := \{F \in \mathcal{B}(\mathcal{H}_x)^{S^1} | F_{\psi_x}(k) \in \mathrm{Fred}^B(V_{\psi_x})^{G_x} \text{ invertible for } k << 0\}.$$

Note that in particular each F_{ψ_x} commutes with the G_x-action. As remarked earlier, this set depends on ψ_x. Define the set of *q-Fredholm operators* $q\mathrm{Fred}^N(\mathcal{H}_x)$ to be the intersection of $q_{\psi_x}\mathrm{Fred}^N(\mathcal{H}_x)$ for all choices of ψ_x (recall that the set of such choices is a torsor for $\mathrm{Hom}(S^1, G_x)^{\mathrm{conj}}$). Finally, topologize this set via the inclusion

$$q\mathrm{Fred}^N(\mathcal{H}_x) \hookrightarrow \prod_{\gamma \in \mathrm{Hom}(S^1, G_x)} \prod_{k \in \mathbb{Z}} \mathrm{Fred}^N(V_{\gamma \cdot \psi_x})$$
$$F \mapsto \prod \prod F_{\gamma \cdot \psi_x}(k).$$

Although the inclusion depends on ψ_x, the induced topology does not, because for any other splitting ψ'_x, $V_{\psi'_x}$ is unitarily equivalent to V_{ψ_x}, so changing the splitting amounts to shifting the γ index in the product.

Remark 3.1.15. This definition is capturing the more humanly comprehensible idea that a qFredholm operator on $\mathcal{H}_x$ is an operator which, under *every* decomposition $\mathcal{H}_x \simeq V_{\psi_x} \otimes L^2 S^1$, 'looks like' a Laurent series (in the monomials q^k that label S^1-representations) of Fredholm operators on V_{ψ_x}.

Example 3.1.16. (cf. Examples 4.1.1 and 3.1.5) Let $\mathbf{X}$ be the action groupoid of the trivial action of $U(1)$ on itself $U(1)/U(1) \simeq \mathcal{L}(\mathrm{pt}/U(1))$ which is a $B\mathbb{Z}$-groupoid with $B\mathbb{Z}$-action given by the automorphism whose component at an object t is the morphism determined by t. It admits a $B\mathbb{Z}$-equivariant twist for every $\tau \in \mathbb{Z} \simeq H^3_{S^1}(U(1)/U(1))$. For each such τ the twist can be presented explicitly as the pair $(\mathbf{V}, \mathbf{L}^\tau)$ where $\mathbf{V} \simeq \mathbf{X}/B\mathbb{Z}$ and $\mathbf{L}^\tau$ is described as follows. Define an action of $\pi_1 U(1) = \mathbb{Z}$ on $i\mathbb{R} \times U(1) \times U(1) \times \mathbb{R}$ by the formula (cf. Definition 4.2.4)

$$p \cdot (iX, e^{2\pi i\theta}, e^{2\pi i\phi}, r) = (i(X + p), e^{2\pi i(\theta - p\tau X + \tau p^2 r)}, e^{2\pi i(\phi + pr)}, r).$$

There is a commuting $\mathbb{Z}$-action (cf. Definition 3.1.2) defined by the formula $n \cdot (iX, e^{2\pi i\theta}, e^{2\pi i\phi}, r) = (iX, e^{2\pi i\theta}, e^{2\pi i(\phi - nX)}, r + n)$ and write L^τ for the quotient by $\pi_1 T \times \mathbb{Z}$. Define the groupoid $\mathbf{L}^\tau = (L^\tau \rightrightarrows U(1))$ by declaring the source and target map to both be $[(X, \theta, t, r)] \mapsto e^{2\pi iX}$. The set of fractional splittings at an object t is isomorphic to the set of iX such that $e^{2\pi iX} = t$

(which is a torsor for $\mathrm{Hom}(S^1, U(1)) \simeq \mathbb{Z}$) since each such lift induces an identification of the automorphism group of t with $U(1) \times U(1) \times S^1$ by translating with the $\pi_1 U(1)$-action until iX is in the fundamental domain containing 0. Fixing such a fractional splitting ψ_t, a q_{ψ_t}Fredholm operator F is represented by a Laurent series $\sum_k F(k)q^k$ where $F(k)$ is a $U(1)$-invariant Fredholm operator on $L^2U(1)$. In other words, F can be represented by a Laurent series $\sum \chi_k q^k \in R(U(1))((q)) \simeq \mathbb{Z}[t^{\pm}]((q)))$. Write $R(U(1)) \simeq \mathbb{Z}[t\pm]$. From the explicit presentation of $\mathbf{L}^\tau$ in Example 3.1.5, changing the splitting ψ_t by the generator $1 \in \mathrm{Hom}(S^1, U(1)) = \pi_1 U(1)$ sends $\sum \chi_k(t) q^k$ to $(qt)^\tau \sum \chi_k(qt) q^k$. In order for the latter to be a Laurent series, if $\mathrm{mdeg}\chi_k(t)$ denotes the most negative degree in $\chi_k(t)$, then it must be that $\mathrm{mdeg}\chi_k(t)/k \to 0$ as $k \to \infty$.

Letting x vary over the space of objects of $\mathbf{L}^\tau$ in the above construction defines, for each N, a subspace $q\mathrm{Fred}^N(\mathcal{H})$ of the bundle $\mathcal{B}(\mathcal{H} \otimes Cl_N) \times \mathcal{K}(\mathcal{H} \otimes Cl_N) \to \mathbf{L}^\tau$. In general $q\mathrm{Fred}^N(\mathcal{H}) \to \mathbf{L}^\tau$ may not be a bundle or even a fibration. Nevertheless, it is a continuous surjection and therefore has an associated sheaf of continuous sections, denoted by $q\mathcal{F}^N(\mathcal{H})$. The following is an easy consequence of the methods developed in the Appendix of [13].

Lemma 3.1.17. *For a finite set N let $N+1 := N \coprod \{s\}$ denote the union of N with a singleton. Then the map defined by sending A to $\gamma(e_s)cos(\pi t) + Asin(\pi t)$ induces an equivalence $a_s : \Gamma(\mathbf{L}^\tau; q\mathcal{F}^{N+1}(\mathcal{H})) \xrightarrow{\sim} \Omega\Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H}))$. Write $N+M$ for the disjoint union. By iterating one gets, for each finite M, an equivalence $a_M : \Gamma(\mathbf{L}^\tau; q\mathcal{F}^{N+M}(\mathcal{H})) \xrightarrow{\sim} \Omega^M\Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H}))$.*

Definition 3.1.18. Suppose M os of even cardinality. Then because of the classical periodicity of complex Clifford algebras and their representations there is a homeomorphism

$$\beta_M : \Gamma(\mathbf{L}^\tau; q\mathcal{F}^{N+M}(\mathcal{H})) \leftarrow \Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H}))$$

induced by sending A to $A \otimes I$.

Definition 3.1.19. Define the twisted $\vec{K}_{S^1}$-theory symmetric stable sheaf on $B\mathbb{Z}$-$\mathfrak{Twist}$ as follows. As a symmetric functor, define

$$\vec{\mathcal{K}}_{S^1}(N, \mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau)) := \Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H}))$$

To smoothen out the notation write c for the object $(\mathbf{X}, (\mathbf{V}, \mathbf{L}^\tau))$. The S-module structure is induced by the maps

$$\bigoplus_{M \subset N} S(M, c) \otimes \vec{\mathcal{K}}_{S^1}(N-M, c) \to \vec{\mathcal{K}}_{S^1}(N, c)$$

$$\bigoplus_{M \subset N} S^M \otimes \Gamma(\mathbf{L}^\tau; q\mathcal{F}^{N-M}(\mathcal{H})) \to \Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H}))$$

which on each summand is the transpose of the composite

$$\Gamma(\mathbf{L}^\tau; q\mathcal{F}^{N-M}(\mathcal{H})) \xrightarrow{\beta_{2M}} \Gamma(\mathbf{L}^\tau; q\mathcal{F}^{N+M}(\mathcal{H})) \xrightarrow{a_M} \Omega^M\Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H}))$$

of the maps of Lemma 3.1.17 and Definition 3.1.18. Suppose $\mathbf{A} \subset \mathbf{X}$ is a full $B\mathbb{Z}$-subgroupoid (cf. Definition 3.1.1). The pullback of $(\mathbf{V}, \mathbf{L}^\tau)$ to $\mathbf{A}$ defines a ($B\mathbb{Z}$-equivariant) twist $(\mathbf{V_A}, \mathbf{L}^\tau_\mathbf{A})$ of $\mathbf{A}$, and the pullback of $\mathcal{H}$ to $\mathbf{L}^\tau_\mathbf{A}$ is τ-twisted and locally universal(cf. [13] Corollary A.34). Therefore the restriction map $\iota^N_\mathbf{A} : \Gamma(\mathbf{L}^\tau; q\mathcal{F}^N(\mathcal{H})) \to \Gamma(\mathbf{L}^\tau_\mathbf{A}/B\mathbb{Z}; q\mathcal{F}^N(\mathcal{H}))$ is well-defined. The relative $\vec{K}_{S^1}$-theory symmetric stable functor $\vec{\mathcal{K}}_{S^1}(\mathbf{X}, \mathbf{A})$ is defined to be its pointwise homotopy fiber. Finally, define (cf. Definition 2.0.6)

$$^\tau\vec{K}^n_{S^1}(\mathbf{X}, \mathbf{A}) := \pi_{-n} u\delta^* \vec{\mathcal{K}}_{S^1}(\mathbf{X}, \mathbf{A}).$$

Of course, at this point it is not clear how much the definition depends on the twist object $(\mathbf{V}, \mathbf{L}^\tau)$, but the following lemma settles the question: the homotopy type of the underlying spectrum depends only on the isomorphism class of the twist, which is an element in $H^1(\mathbf{X}; C_2) \oplus H^3\mathbf{X}$.

Proposition 3.1.20. *For $n \in \mathbb{Z}$, the collection of assignments*

$$(\mathbf{X}, \mathbf{A}, (\mathbf{V}, \mathbf{L}^\tau)) \mapsto {}^\tau\vec{K}^n_{S^1}(\mathbf{X}, \mathbf{A})$$

forms a twisted cohomology theory. More precisely,

i) this defines a contravariant functor from $B\mathbb{Z}$-$\mathfrak{Twist}_{rel}$ to $\mathbb{Z}((q))$-mod taking local equivalences to isomorphisms and taking homotopic[4] morphisms to equal ones;

ii) there is a natural long exact sequence

$$\ldots \to {}^\tau\vec{K}^n_{S^1}(\mathbf{X}, \mathbf{A}) \to {}^\tau\vec{K}^n_{S^1}(\mathbf{X}) \to {}^\tau\vec{K}^n_{S^1}(\mathbf{A}) \to {}^\tau\vec{K}^{n+1}_{S^1}(\mathbf{X}, \mathbf{A}) \to \ldots$$

iii) (excision) if $\mathbf{Z} \subset \mathbf{A}$ is a full $B\mathbb{Z}$-subgroupoid whose closure is contained in the interior of $\mathbf{A}$, then the restriction map

$$^\tau\vec{K}^n_{S^1}(\mathbf{X}, \mathbf{A}) \longrightarrow {}^\tau\vec{K}^n_{S^1}(\mathbf{X}\backslash\mathbf{Z}, \mathbf{A}\backslash\mathbf{Z})$$

is an isomorphism;

iv) if J is an index set and $(\mathbf{X}, \mathbf{A}, (\mathbf{V}, \boldsymbol{\tau})) = \coprod_J (\mathbf{X}_j, \mathbf{A}_j, (\mathbf{V}_j, \mathbf{L}^{\tau_j}))$ is a disjoint union, then

$$^\tau\vec{K}^n_{S^1}(\mathbf{X}, \mathbf{A}) \longrightarrow \prod_J {}^{\tau_j}\vec{K}^n_{S^1}(\mathbf{X}_j, \mathbf{A}_j)$$

is an isomorphism.

Proof. This is essentially the proof given in Section 3.5 of [13] with minor changes. Functoriality is immediate from the construction of the spectrum. Write $I = [0, 1]$ for the unit interval. Homotopy invariance follows from the fact that if $\mathcal{H} \to \mathbf{L}^\tau$ is a locally universal

[4]Homotopy is defined by the standard interval object $[0, 1]$.

τ-twisted Hilbert bundle and $p : \mathbf{L}^\tau \times I \to \mathbf{L}^\tau$ is the projection then $p^*\mathcal{H} \to \mathbf{L}^\tau \times I$ is τ-twisted[5] locally universal ([13] Lemma A.32), so

$$^\tau\vec{\mathcal{K}}_{S^1}\big((\mathbf{X},\mathbf{A}) \times I\big)_n \simeq \vec{\mathcal{K}}_{S^1}(\mathbf{X},\mathbf{A})^I_n,$$

making the two restriction maps homotopic.

The fact that local equivalences are taken to isomorphisms is a consequence of *descent* (cf. [13] Lemma A.18), which states that the pullback f^* along a local equivalence $f : \mathbf{X} \to \mathbf{Y}$ induces an equivalence of categories, from groupoids over $\mathbf{Y}$ to groupoids over $\mathbf{X}$, with a natural adjoint inverse denoted f_*. Hence for any $\mathbf{P} \to \mathbf{Y}$ the natural map from $\Gamma(\mathbf{Y}, P) \to \Gamma(\mathbf{X}, f^*\mathbf{P})$ is a homeomorphism whose inverse is the composition of the natural map $\Gamma(\mathbf{X}, f^*\mathbf{P}) \to \Gamma(\mathbf{Y}, f_*f^*\mathbf{P})$ with the map on sections induced by counit $f_*f^*\mathbf{P} \to \mathbf{P}$.

The long exact sequence in *i)* is obtained from the fiber sequence

$$\ldots \to \Omega^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{A})_n \to {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{X},\mathbf{A})_n \to {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{X})_n \to {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{A})_n \to \ldots$$

The claim *iv)* about disjoint unions is immediate from the definition. It remains to prove excision. Despite some cumbersome notation, the proof use a few standard homotopy-theoretic constructions to boil things down to the following fact ([13] Lemma A.32): the pullback of a τ-twisted locally universal Hilbert bundle over a local quotient groupoid to a full subgroupoid is again locally universal. Amusingly, the use of this fact will make its appearance in a footnote. Let $\mathbf{M} = \mathbf{X}\backslash\mathbf{Z} \cup I \times \big(\mathbf{A}\backslash\mathbf{Z}\big) \cup \big(\mathbf{A}\backslash\mathbf{Z}\big) \times I \cup \mathbf{A}$ be the double mapping cylinder of $\mathbf{X}\backslash\mathbf{Z} \hookleftarrow \mathbf{A}\backslash\mathbf{Z} \hookrightarrow \mathbf{A}$. The point-set topological conditions on $\mathbf{A}$ and $\mathbf{Z}$ imply that the collapse map $c : \mathbf{M} \to \mathbf{X}$ is an equivalence. Consider the following diagram, in which each row is a fiber sequence.

$$\begin{array}{ccccc} {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{X},\mathbf{A})_n & \longrightarrow & {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{X})_n & \longrightarrow & {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{A})_n \\ \downarrow & & \downarrow & & \downarrow \\ {}^{c^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{M},\mathbf{A})_n & \longrightarrow & {}^{c^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{M})_n & \longrightarrow & {}^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{A})_n \end{array}$$

The vertical arrow on the right is a homeomorphism and the vertical arrow in the middle is a homotopy equivalence, so the vertical arrow on the left is a weak homotopy equivalence. Let $\mathbf{N}$ be the mapping cylinder of $\mathbf{X}\backslash\mathbf{Z} \hookleftarrow \mathbf{A}\backslash\mathbf{Z}$ and $h : \mathbf{N} \to \mathbf{X}$ the obvious map. Since $(\mathbf{X}\backslash\mathbf{Z}, \mathbf{A}\backslash\mathbf{Z}) = (\mathbf{X}\backslash\mathbf{Z}, \mathbf{A}\backslash\mathbf{Z})$ a similar argument shows that h induces a weak equivalence

$$^\tau\vec{\mathcal{K}}_{S^1}(\mathbf{X}\backslash\mathbf{Z}, \mathbf{A}\backslash\mathbf{Z})_n \longrightarrow {}^{h^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z})_n.$$

Thus it suffices to show that the restriction map

$$r : {}^{c^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{M},\mathbf{A})_n \longrightarrow {}^{h^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z})_n$$

[5] To be pedantic, $p^*\tau$-twisted.

is a weak equivalence.[6]

For a based topological sheaf $\mathcal{F}$ over a groupoid $\mathbf{V}$ with subgroupoid $\mathbf{A}\backslash\mathbf{Z}$ let $\Gamma(\mathbf{V}, \mathbf{A}\backslash\mathbf{Z}; \mathcal{F})$ denote the space of global sections whose restriction to $\mathbf{A}\backslash\mathbf{Z}$ is the basepoint. Recall that the relative $\vec{K}_{S^1}$-theory spectrum of a pair is by definition the homotopy fiber of a restriction map on the space of sections of a based topological sheaf. Since the relative inclusions $\mathbf{A} \hookrightarrow \mathbf{M}$ and $\mathbf{A}\backslash\mathbf{Z} \hookrightarrow \mathbf{N}$ are cofibrations[7] the corresponding restriction maps are fibrations. Hence the maps[8]

$$\Gamma(\mathbf{M}, \mathbf{A}; q\mathcal{F}^n) \longrightarrow {}^{c^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{M}, \mathbf{A})_n$$
$$\Gamma(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z}; q\mathcal{F}^n) \longrightarrow {}^{h^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z})_n$$

are inclusions of fibers into homotopy fibers and are thus homotopy equivalences.

Finally, the relative inclusion $(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z}) \hookrightarrow (\mathbf{M}, \mathbf{A})$ induces induces a homeomorphism

$$\Gamma(\mathbf{M}, \mathbf{A}; q\mathcal{F}^n) \longrightarrow \Gamma(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z}; q\mathcal{F}^n),$$

which—along with the previous two homotopy equivalences and the map of interest Ξ—fits into the following diagram.

$$\begin{array}{ccc} \Gamma(\mathbf{M}, \mathbf{A}; q\mathcal{F}^n) & \longrightarrow & \Gamma(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z}; q\mathcal{F}^n) \\ \downarrow & & \downarrow \\ {}^{c^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{M}, \mathbf{A})_n & \xrightarrow{r} & {}^{h^*\tau}\vec{\mathcal{K}}_{S^1}(\mathbf{N}, \mathbf{A}\backslash\mathbf{Z})_n \end{array}$$

It follows that r is a weak equivalence.

□

3.2 E_∞-structures

Definition 3.2.1. A *multiplicatively closed* set of twists (of $\hat{K}_{S^1}$-theory) for a $B\mathbb{Z}$-groupoid $\mathbf{X}$ is a subset S of the objects of $B\mathbb{Z}$-$\mathfrak{Twist}_{\mathbf{X}}$ that is closed under the multiplication operation of Definition 3.1.10.

Definition 3.2.2. Let T be a set of twists for a $B\mathbb{Z}$-groupoid $\mathbf{X}$. Define the spectrum ${}^T\hat{K}_{S^1}(\mathbf{X})$ as the direct sum (cf. Definition 3.1.19)

$$\bigoplus_{(\mathbf{V}, \mathbf{L}^\tau) \in T} {}^\tau\vec{K}_{S^1}(\mathbf{X}).$$

[6] Note that the twists $h^*\tau$ and $r^*c^*\tau$ are canonically isomorphic.

[7] A cofibration of topological groupoids is defined in the same was as for topological spaces. Namely one asks for the homotopy extension property for maps to topological spaces.

[8] Since the pullback of the τ-twisted locally universal Hilbert bundle over $\boldsymbol{\tau}$ to any of the groupoids mentioned above is again locally universal, in a final abuse of notation I use $q\mathcal{F}^n$ to denote the sheaf and any of its pullbacks by maps in sight.

Lemma 3.2.3. *If T is a multiplicatively closed set of twists for a $B\mathbb{Z}$-groupoid $\mathbf{X}$ then ${}^T\hat{K}_{S^1}(\mathbf{X})$ has a natural E_∞-ring structure.*

Proof. The spectrum ${}^T\hat{K}_{S^1}(\mathbf{X})$ of Definition 3.2.2 has an evident lift to a symmetric spectrum ${}^T\hat{\mathcal{K}}_{S^1}(\mathbf{X})$ via the formula (in which the finite set N is a variable)

$$ {}^T\hat{\mathcal{K}}_{S^1}(\mathbf{X})(N) = \bigoplus_{(\mathbf{V},\mathbf{L}^\tau)\in T} \vec{\mathcal{K}}_{S^1}(N,\mathbf{X},(\mathbf{V},\mathbf{L}^\tau)). $$

It remains to proved the commutative ring structure, for then the results of Hovey-Shipley-Smith (cf. [17]) apply. The required maps

$$ \bigoplus_{M\subset N} {}^T\hat{\mathcal{K}}_{S^1}(\mathbf{X})(M) \otimes {}^T\hat{\mathcal{K}}_{S^1}(\mathbf{X})(N-M)) \to {}^T\hat{\mathcal{K}}_{S^1}(\mathbf{X})(N) $$

are defined summand-by-summand as follows. First, note that each summand of the domain is of the form

$$ \Gamma(\mathbf{L}^{\tau_1}; q\mathcal{F}^M(\mathcal{H}_1)) \otimes \Gamma(\mathbf{L}^{\tau_2}; q\mathcal{F}^{N-M}(\mathcal{H}_2)). $$

The external tensor product of operators $(A,B)\mapsto A\otimes B$ along with the pullback of Fredholm sections then defines a map (cf. Definition 3.1.10)

$$ \Gamma(\mathbf{L}^{\tau_1}; q\mathcal{F}^M(\mathcal{H}_1)) \otimes \Gamma(\mathbf{L}^{\tau_2}; q\mathcal{F}^{N-M}(\mathcal{H}_2)) \to \Gamma(\mathbf{L}^{\tau_1+\tau_2}; q\mathcal{F}^M(\mathcal{H}_1\otimes\mathcal{H}_2)). $$

Finally, $\mathcal{H}_1\otimes\mathcal{H}_2$ is locally universal over $\mathbf{L}^{\tau_1+\tau_2}$, and so the codomain of that map is a summand of ${}^T\hat{\mathcal{K}}_{S^1}(\mathbf{X})(N)$. T □

Chapter 4

Elliptic Cohomology

Kitchloo and Morava ([19]) give a strikingly simple picture of elliptic cohomology at the Tate curve by studying a completed version of S^1-equivariant K-theory for spaces. Several authors (cf. [5],[19],[21]) have suggested that an equivariant version ought to be related to the work of Freed-Hopkins-Teleman ([13],[15],[14]). However, a first attempt at this runs into apparent contradictions concerning twist, degree, and cup product. Several authors (cf. [8],[16],[18]) have solved the problem over the complex numbers by interpreting the S^1-equivariant parameter as a complex variable and using holomorphicity as a technique for an analytic form of completion. This chapter gives a solution that works integrally, by using the carefully (and of course geometrically) completed model of K-theory for S^1-equivariant stacks from the previous chapter.

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4.1 Introduction

In [19], Kitchloo and Morava construct an equivariant cohomology theory for CW-spaces with an S^1-action by taking ordinary S^1-equivariant K-theory and completing the coefficient ring $K^*_{S^1}(\mathrm{pt}) \simeq \mathbb{Z}[q^{\pm}]$ in positive q-powers. For a finite CW-complex M they define

$$\vec{K}^*_{S^1}(M) := K^*_{S^1}(M) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \qquad\qquad \mathbb{Z}((q)) := \mathbb{Z}[q^{-1}][[q]].$$

They extend this to a theory on infinite complexes by taking limits over finite skeleta. It satisfies a strong localization theorem for finite complexes: the inclusion of the fixed point set $j : M^{S^1} \hookrightarrow M$ induces an isomorphism

$$j^* : \vec{K}^*_{S^1}(M) \xrightarrow{\sim} \vec{K}^*_{S^1}(M^{S^1}).$$

Recall that S^1 acts on the free loop space LM by loop rotation. Kitchloo and Morava show that the assignment

$$M \mapsto \vec{K}^*_{S^1}(LM)$$

defines a cohomology theory. The localization theory together with the formula $LM^{S^1} = M$ shows that as a cohomology theory the above is just K-theory with coefficients in $\mathbb{Z}((q))$. But their insight is that this construction naturally gives more than a cohomology theory. Recall that a multiplicative cohomology theory E is called an *elliptic cohomology theory*[1] if it is

1. weakly even: $E^2(\mathrm{pt}) \otimes_{E^0(\mathrm{pt})} E^n(\mathrm{pt}) \to E^{n+2}(\mathrm{pt})$ is a isomorphism for all n;
2. comes with the data of an elliptic curve $\mathcal{E}$ over $E^0(\mathrm{pt})$;
3. comes with an isomorphism of formal groups $\mathrm{Spf}E^0(\mathbb{CP}^\infty) \to \hat{\mathcal{E}}$, where the first object carries the group structure induced by the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ classifying the tensor product of line bundles, and the second object is the formal completion of $\mathcal{E}$ at the identity.

The key to extracting an elliptic theory is the identification of a natural complex orientation coming from the Atiyah-Bott-Shapiro *spin orientation* of the normal bundle of the fixed point locus $M \hookrightarrow LM$. The elliptic curve relevant to the Kitchloo-Morava construction is the *Tate curve*, an elliptic curve over $\mathbb{Z}((q))$. Thus

$$Ell^*_{\mathrm{Tate}}(M) \simeq \vec{K}^*_{S^1}(LM).$$

The simplicity of this method of producing a K-theoretic picture of elliptic cohomology at the Tate curve suggests that a similar approach might work in an equivariant setting. However, a few subtleties arise.

Definition 4.1.1. For a topological groupoid $\mathbf{X} = (X_1 \rightrightarrows X_0)$, define its loop groupoid $\mathcal{L}\mathbf{X}$ (sometimes called the *inertia groupoid*) the free loop space object in the (2,1)-category of topological groupoids. i.e. the pullback of the diagonal $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ along itself.. Explicitly, $\mathcal{L}\mathbf{X}$ has as its topological space of objects the set of functors $\mathrm{pt}/\mathbb{Z} \to \mathbf{X}$ topologized as a subspace of X_1 (the space of morphisms of $\mathbf{X}$) and has as its space of morphisms the natural transformations of such functors, again topologized as a subspace of X_1.

Remark 4.1.2. Note that when $\mathbf{X}$ is the topological groupoid associated to a topological space X (i.e. the groupoid $(X \rightrightarrows X)$ with only identity morphisms) this definition does not reduce to (the topological groupoid associated to) the free loop space LX; instead $\mathcal{L}\mathbf{X} = \mathbf{X}$.

The next step in the Kitchloo-Morava construction would be to define an equivariant theory like this

$$M/G \ \mapsto \ \vec{K}^*_{S^1}(\mathcal{L}(M/G)) := K^*_{S^1}(\mathcal{L}(M/G)) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)).$$

However, that does not have the desired relationship with equivariant elliptic cohomology. For example, suppose that M is a point and that G is simple and simply-connected. By the work of Freed, Hopkins, and Teleman ([14], Theorem 5) this proposed elliptic group, suitably

[1] This definition is due to Mike Hopkins in his 1994 ICM address.

twisted by an element $k \in H^4BG \simeq \mathbb{Z}$, is concentrated in degree dim$G$, where it is isomorphic to the positive energy level $k - h$ (h is the dual Coxeter number of G) representation ring of the semidirect product $LG \rtimes S^1$ of the loop group LG with S^1, the latter acting on the former by loop rotation. So if dim$G$ is odd this trivially fails the first requirement of an elliptic cohomology theory, and the cup product is zero. Even if dim$G$ is even, the presence of the dual Coxeter shift violates the expectation that k-twisted G-equivariant elliptic classes of degree zero over the point are in bijective correspondence with equivalence classes of level k positive energy representations of $LG \rtimes S^1$ (cf. [26], [27]). Over the complex numbers these issues have been resolved by several authors; the common theme among them is to consider $q = e^{2\pi i\tau}$ as a complex variable on a certain moduli space and use holomorphicity as the method of completion (as opposed to the algebraic base-change proposed above). For example, Grojnowski's delocalized equivariant elliptic theory [16] assigns to a G-space X a holomorphic sheaf on the moduli space of bundles over the elliptic curve, constructed by patching together local sections defined using equivariant singular cohomology with complex coefficients. The connection to positive energy representation theory is made by using the Kac character formula to identify characters of those representations with sections of the sheaf. Berwick-Evans and Tripathy [8] have constructed a de Rham model refining Grojnowski's theory to a holomorphic sheaf of commutative differential graded algebras. For simple and simply-connected groups Kitchloo [18], in a method most similar to the one that will be presented here, uses a version of LG-equivariant K-theory built out of positive energy representations to construct a holomorphic sheaf together with a character map from positive energy representations to sections of this sheaf.

The purpose of the present paper is to construct a new geometric model E_G of equivariant elliptic cohomology at the Tate curve that satisfies the following conditions:

1. for an arbitrary compact Lie group G the theory E_G is defined integrally, i.e. no prime p is invertible in the coefficient ring E_G^*
2. it admits twists ${}^{\tau}E_G$ by elements $\tau \in H^4BG$,
3. when the group G is trivial E_G recovers the Kitchloo-Morava theory described at the beginning of this section,
4. the construction of E_G does not reference the positive energy representation theory of loop groups in any way,
5. there is, *a fortiori*, a natural map from the category $\mathrm{Rep}^{\tau}_{\mathrm{pos}}(LG \rtimes S^1)$ of level $\tau > 0$ positive energy representations to ${}^{\tau}E_G^*(\mathrm{pt})$ which for connected groups becomes an isomorphism of $\mathbb{Z}((q))$-modules after factoring through the Grothendieck group,
6. and after tensoring with $\mathbb{C}$ it recovers, in a suitable sense, the previously defined theories of Grojnowski and Kitchloo.

As an added bonus, the construction will naturally extend to negative twists and exhibit a duality between positive and negative twist that the more imaginative reader might like to interpret as a manifestation of Serre duality on the moduli space of G-bundles over the Tate curve.

Some Definitions and Disclaimers: Just as in the previous chapter, rings, abelian groups, and Hilbert spaces are assumed to be $\mathbb{Z}/2\mathbb{Z}$-graded, unless it is stated otherwise. As is standard, the *loop group* LG of a Lie group G is the group of *smooth* maps $S^1 \to G$ with the topology of uniform convergence. C_2 is the cyclic group of order 2. All groupoids considered are *topological groupoids* (cf. Definition 3.0.2) unless otherwise stated.

4.2 Calculations

The geometric construction of a completed S^1-equivariant K-theory (along with some E_∞-structures) in the previous chapter is all well and good, but it doesn't amount to much if I can't calculate anything. In this section I show that the manifold geometry provides powerful calculational techniques: once things have been phrased in terms of geometric cocycles, things like integration, the Becker-Gottlieb transfer, and Segal induction, all mix together to provide results such as 4.2.7, 4.2.9, and 4.2.13 which are pivotal in establishing the main calculational results, 4.2.17 and 4.2.24.

Definition 4.2.1. An element $\tau \in H^3_{S^1}(G/G)$ is called a *strongly topologically regular*[2] twist if the restriction of τ to $H^3(T/T) \simeq H^1(T)^{\otimes 2} \oplus H^3(T)$ is concentrated in the first summand (cf. Remark 3.1.7) and defines a symmetric, non-degenerate, definite bilinear form on $H_1(T)$. The twist is called *positive* (*negative*) if that bilinear form is positive (negative) definite.

Let $G(1)$ denote the identity component of a compact Lie group G. In this section the groups ${}^{\tau}\vec{K}^*_{S^1}(G(1)/G)$ will be calculated. The calculation will eventually break into two subsections—according to whether the twist τ is positive or negative definite—which have different flavors. The asymmetry not mysterious: it comes precisely from the asymmetry in completing $\mathbb{Z}[q^{\pm}]$ to $\mathbb{Z}((q))$ rather than $\mathbb{Z}[[q, q^{-1}]]$. Morally, relaxing finite-dimensionality of vector bundles in *positive* powers of q doesn't amount to much at *negative* level, since the cocycles (are expected to) correspond to loop group representations, which at negative level are of *negative energy* and have bounded *above* S^1-eigenspaces. Thus the representations themselves do not produce $\vec{K}_{S^1}$-cocycles without being 'finitized' by a Fredholm operator[3] and the story collapses to match classical twisted K-theory.

Let T denote a maximal torus of G and let N be the normalizer of T. The plan is to make a preliminary calculation over T/N and then transport that to $G(1)/G$ via the natural faithful map $\omega : T/N \to G/G$. A few key technical lemmas can be stated and proved uniformly for positive and negative twists.

[2] The adverb "strongly" is there to distinguish this from the condition of topological regularity (cf. [14] 2.1), which does not require the bilinear form to be definite.

[3] This is the (rather involved) FHT-Dirac construction (cf. [14] Section V).

4.2.1 Technical setup

Lemma 4.2.2. *(The stalks of $\vec{K}^*_{S^1}$) Let G be a compact Lie group and $z \in Z(G)$ a central element. Then z defines a $B\mathbb{Z}$-action on pt/G, and the $B\mathbb{Z}$-quotient (cf. Definition 3.1.2) is of the form $pt/\tilde{G}$ where $G \to \tilde{G} \to S^1$ is a group extension. Any twist $\tau \in H^3_{S^1}(BG)$ defines a $U(1)$-central extension $\tilde{G}^\tau \to \tilde{G}$. Let G^τ be the pullback of $\tilde{G}^\tau$ to G, so that $\tilde{G}^\tau/G^\tau = S^1$. Let $\tilde{\Lambda}^\tau$ and Λ^τ be the set of τ-affine (cf. Footnote 8) weights of $\tilde{G}^\tau$ and G^τ, and let W be the Weyl group[4]. Let $\tilde{\Lambda}^\tau_{di} \subset \tilde{\Lambda}^\tau$ and $\Lambda^\tau_{di} \subset \Lambda^\tau$ be the subsets of dominant[5] integral weights. Let $\tilde{\pi} : \tilde{\Lambda}^\tau_{di}/W \to pt/G$ and $\pi : \Lambda^\tau_{di}/W \to pt/G$ be the projections. Any choice of fractional splitting $\psi : \tilde{G}^\tau \leftarrow S^1_d$ (cf. Section 2.1) induces an isomorphism $\Psi : \tilde{\Lambda}^\tau \leftarrow \Lambda^\tau \times \mathbb{Z}$. Let K^*_c denote K-theory with compact supports[6] and let ${}^{\tilde{\pi}^*\tau}K^0_\star(\tilde{\Lambda}^\tau_{di}/W) \subset {}^{\tilde{\pi}^*\tau}K^0(\tilde{\Lambda}^\tau_{di}/W)$ be the subgroup of classes whose image under $\Psi^* : {}^{\tilde{\pi}^*\tau}K^0(\tilde{\Lambda}^\tau_{di}/W) \to {}^{\pi^*\tau}K^0(\Lambda^\tau_{di}/W \times \mathbb{Z}) = {}^{\pi^*\tau}K^0(\Lambda^\tau_{di}/W)[[q, q^{-1}]]$ is contained in ${}^{\pi^*\tau}K^0_c(\Lambda^\tau_{di}/W)((q))$ for all fractional splittings ψ. Let $\tilde{V} \to \tilde{\Lambda}^\tau_{di}/W$ be the canonical vector bundle whose fiber at a point is a copy of the $\tilde{G}^\tau$-representation labelled by that point. If M is an R-module and $r \in R$, write rM for the subgroup $\{rm | m \in M\}$. Then ${}^\tau\vec{K}^1_{S^1}(pt/G) = 0$ and 'summation along the fiber' defines an isomorphism*

$$\tilde{\pi}_! : [\tilde{V}]\big({}^{\tilde{\pi}^*\tau}K^0_\star(\tilde{\Lambda}^\tau_{di}/W)\big) \to {}^\tau\vec{K}^0_{S^1}(pt/G).$$

Finally, for $M \in R^\tau(G)$ write χ_M for its character. For $\gamma \in Hom(S^1, G)$ and $\xi = \sum M_k q^k \in R^\tau(G)((q))$ define $\gamma \cdot \xi = \sum M_k \chi_{M_k}(\gamma(q)) q^k$. Then any choice of local splitting ψ induces an injection ${}^\tau\vec{K}^0_{S^1}(pt/G) \hookrightarrow R^\tau(G)((q))$ which is an isomorphism onto the subgroup $R^\tau_\star(G)((q))$ of elements ξ for which $\gamma \cdot \xi$ is in $R^\tau((q))$ for all $\gamma \in Hom(S^1, G)$.

Proof. The first claim that is not immediate from the definitions and the standard representation theory of compact Lie groups is the implicit well-definedness (and the precise definition!) of the displayed map. Let $R^\tau(G)$ be the group of τ-projective[7] representations of G. As a warmup to the precise definition, the uncompleted analog of this map is the isomorphism $\tilde{\pi}^c_! : [\tilde{V}]\big({}^{\tilde{\pi}^*\tau}K^0_c(\tilde{\Lambda}^\tau_{\mathrm{di}}/W)\big) \to {}^\tau K^0_{S^1}(\mathrm{pt}/G) = R^\tau(\tilde{G})$ defined as follows: for $\lambda \in \tilde{\Lambda}^\tau_{\mathrm{di}}/W$ let $\tilde{V}_\lambda$ denote a copy of the corresponding irreducible representation G^τ. Then a compactly supported virtual vector bundle $E \to \tilde{\Lambda}^\tau_{\mathrm{di}}/W$ is sent to the direct sum $\oplus_{\tilde{\Lambda}^\tau_{\mathrm{di}}/W} \tilde{V}_\lambda \otimes E_\lambda$.

It is immediate from the definition (cf. Definition 3.1.14 and **??**) that classes in ${}^\tau\vec{K}^0_{S^1}(\mathrm{pt}/G)$ are represented by certain possibly infinite dimensional representations of $\tilde{G}^\tau$. To describe them, choose a fractional splitting $\psi : \tilde{G}^\tau \leftarrow S^1_d$. From Definition 3.1.14 it follows that there is an injective map $i_\psi : {}^\tau\vec{K}^0_{S^1}(\mathrm{pt}/G) \to R^\tau(G)((q))$. For an element $M \in R^\tau(G)$ let χ_M denote its virtual character. Any other choice of fractional splitting ψ' is of the form $\psi'(q) = \psi(q)\gamma(q)$ for some $\gamma : S^1 \to G$. Therefore, if $\xi \in {}^\tau\vec{K}^0_{S^1}(\mathrm{pt}/G)$

[4]The Weyl groups of all extensions in sight are canonically isomorphic.

[5]I take this to mean dominant with respect to *any* choice of positive Weyl chamber.

[6]This is the reduced K-theory of the one-point compactification.

[7]That is, the subgroup of the representation ring $R(G^\tau)$ where the central $U(1)$ acts by scalar multiplication.

and $i_\psi\xi = \sum M_k q^k \in R^\tau(G)((q))$ then $i_{\psi'}\xi = \sum M_k \chi_{M_k}(\gamma(q))q^k \in R^\tau(G)((q))$. So again it follows directly from Definition 3.1.14 again that i_ψ is an isomorphism onto the subgroup $R^\tau_\star(G)((q)) \subset R^\tau(G)((q))$ of elements $\sum M_k q^k$ for which $\sum M_k \chi_{M_k}(\gamma(q))q^k$ is also in $R^\tau(G)((q))$.

Now the image $\pi_!([V]\xi)$ under the displayed map in the lemma (defined by the same formula that ends the first paragraph of this proof) certainly defines a possibly infinite dimensional representation of $\tilde{G}^\tau$. By assumption $\Psi^*\xi \in {}^{\pi^*\tau}K^0_c(\Lambda^\tau_{\mathrm{di}}/W)((q))$, so write $\Psi^*\xi = \sum E_k q^k$. Moreover, for another splitting $\psi' = \psi\gamma$ (see the previous paragraph) we also have $(\Psi')^*\xi \in {}^{\tilde{\pi}^*\tau}K^0_c(\Lambda^\tau_{\mathrm{di}}/W)((q))$. In analogy with $\tilde{V}$ let $V \to \Lambda^\tau_{\mathrm{di}}/W$ be the canonical vector bundle whose fiber at a point is a copy of the G^τ representation labeled by that point. But it is clear that $\pi^c_!([V]\Psi^*\xi) = i_\psi(\tilde{\pi}_![\tilde{V}]\xi)$, so the conditions defining ${}^{\tilde{\pi}^*\tau}K^0_\star(\tilde{\Lambda}^\tau_{\mathrm{di}}/W)$ are tautologically the conditions for $\tilde{\pi}_!$ to be well-defined. Since $\pi^c_!$ is an isomorphism (see the first paragraph of this proof), the same equation shows that $\imath_\psi \circ \tilde{\pi}_!$ is an isomorphism onto its image, which finishes the proof. □

Lemma 4.2.3. *Let N be the normalizer of a maximal torus $T \subset G$ and write $W = N/T$. Let τ be a twist in $H^3_{S^1}(T/N)$. For each $t \in T$ with stabilizer $N_t \subset N$, τ defines a group $\tilde{N}^{\tau(t)}_t$ which is a $U(1)$-central extension of an N_t-extension $\tilde{N}_t$ of S^1, and also an extension of S^1 by a $U(1)$-extension $N^{\tau(t)}$ of N_t. This is more lucidly indicated in the commutative diagram below:*

$$\begin{array}{ccccc} U(1) & \hookrightarrow & N^{\tau(t)}_t & \longrightarrow & N_t \\ & & \downarrow & & \downarrow \\ U(1) & \hookrightarrow & \tilde{N}^{\tau(t)}_t & \longrightarrow & \tilde{N}_t \\ & & \downarrow & & \downarrow \\ & & S^1 & & S^1 \end{array}$$

Restriction to the maximal torus $T \subset N_t$ defines a group $\tilde{T}^{\tau(t)}$ which is a $U(1)$-central extension of a T-extension of S^1, and as t varies these $\tilde{T}^{\tau(t)}$ assemble into a bundle of groups over T whose fiber over 1 is equal to $U(1) \times T \times S^1$.

Proof. First note that the explicit model of the $B\mathbb{Z}$-quotient $T/N/B\mathbb{Z}$ given in Definition 3.1.2 produces a bundle of groups $T \times N \times_{\mathbb{Z}} \mathbb{R} \to T$ whose fiber at $t \in T$ is an extension $N \to \tilde{N}_t \to \mathbb{R}/\mathbb{Z} = S^1$. For each point $t \in T$ with stabilizer $N_t \subset N$, pullback along the inclusion $i_t : \{t\}/N_t \to T/N$ produces a class $\tau(t) := i^*_t\tau \in H^3(\{t\}/N_t) := H^3BN_t$, i.e a central extension $U(1) \to N^{\tau(t)}_t \to N_t$. Pulling back along $T \to N_t$ produces the desired central extension of T, denoted $T^{\tau(t)} \to T$. The ability to trivialize the extensions at the fiber over $1 \in T$ is a direct consequence of the fact that all extensions of S^1 by a torus are trivializable, and all extensions of a torus by $U(1)$ are trivializable. The bundle of groups over T is presented explicitly in Definition 4.2.4 below. □

Definition 4.2.4. Let $\tau \in H^3_{S^1}(T/T)$ be a strongly topologically regular twist and let τ also denote the corresponding bilinear form on $H_1T = \pi_1 T$ (cf. Definition 4.2.1). Let Λ be the character lattice of T. Contraction with the bilinear form on H^1T defined by τ gives a map $\kappa^\tau : H_1T \to H^1T = \Lambda$. Define an action of $\pi_1 T$ on $\mathfrak{t} \times U(1) \times T \times \mathbb{R}$ by the formula (cf. Example 3.1.5)

$$p \cdot (X, e^{2\pi i\theta}, t, r) = (X + p, e^{2\pi i(\theta+\tau(p,p)r)}\kappa^\tau p(e^X)^{-1}, tp(e^{2\pi i r}), r).$$

There is a commuting $\mathbb{Z}$-action (cf. Definition 3.1.2) defined by the formula $n \cdot (X, e^{2\pi i\theta}, t, r) = (X, e^{2\pi i\theta}, te^{-nX}, r + n)$ and write L^τ for the quotient by $\pi_1 T \times \mathbb{Z}$. Define the groupoid $\mathbf{L}^\tau = (L^\tau \rightrightarrows T)$ by declaring the source and target map to both be $[(X, \theta, t, r)] \mapsto e^X$. This defines the bundle of groups over T alluded to in Lemma 4.2.3. The map $[(X, e^{2\pi i\theta}, t, r)] \mapsto [(X, t, r)]$ defines a morphism $\mathbf{L}^\tau \to T/T/B\mathbb{Z}$ which presents the $B\mathbb{Z}$-equivariant twist associated to τ.

Lemma 4.2.5. *There is a W-equivariant covering space $\pi : P_\tau \to T$ which is the bundle of affine weights[8] associated to the bundle of central extensions defined by the groupoid $\mathbf{L}^\tau$ of Definition 4.2.4. In particular, for each $t \in T$, if $T^{\tau(t)}$ denotes the automorphism group of t in the groupoid $\mathbf{L}^\tau$ and $\tilde{\Lambda}^{\tau(t)}$ is the set of affine weights of $T^{\tau(t)}$, then there is a canonical isomorphism $\pi^{-1}(t) \simeq \tilde{\Lambda}^{\tau(t)}$.*

Proof. The proof is an explicit construction of P_τ. Define an action of $\pi_1 T$ on $\mathfrak{t} \times \Lambda \times \mathbb{Z}$ by (cf. [1] 4.9.5)

$$\pi_1 T \ni p : (X, \lambda, n) \mapsto (X + p, \lambda - \kappa^\tau p, n + \tau(\kappa^\tau p, \kappa^\tau p) + \lambda(p)).$$

The desired covering map is

$$\pi : P_\tau := \mathfrak{t} \times_{\pi_1 T} (\Lambda \times \mathbb{Z}) \to T$$

$$[(X, \lambda, n))] \mapsto [X].$$

To identify $\pi^{-1}(t)$ with $\tilde{\Lambda}^{\tau(t)}$, view L^τ as the $\pi_1 T$-quotient of $(\mathfrak{t} \times U(1) \times T \times \mathbb{R})/\mathbb{Z}$. For $(t, r) \in T \times \mathbb{R}$ write $[r]$ for the corresponding element of $S^1 = \mathbb{R}/\mathbb{Z}$, and $t_r(X) = te^{-n_r X}$ where $n_r \in \mathbb{Z}$ is such that $r + n_r \in [0, 1)$. Then $(\mathfrak{t} \times U(1) \times U(1)T \times \mathbb{R})/\mathbb{Z}$ can be identified with $\mathfrak{t} \times U(1) \times T \times S^1$ via the map $[(X, e^{2\pi i\theta}, t, r)] \mapsto (x, e^{2\pi i\theta}, t_0, [r])$, and under this identification the $\pi_1 T$ action on $\mathfrak{t} \times U(1) \times T \times S^1$ becomes $p \cdot (X, e^{2\pi i\theta}, t, \phi) = (X + p, e^{2\pi i\theta+\tau(p,p)\phi}\kappa^\tau p(e^X)^{-1}, tp(e^{2\pi i\phi}), \phi)$. The associated action on the subset of $\mathfrak{t} \times \mathrm{Hom}(U(1) \times T \times S^1, U(1))$ consisting of pairs (X, f) such that f restricts to the identity character of the $U(1)$ factor is precisely the action defining P_τ. □

Definition 4.2.6. Define a partial compactification $\vec{P}_\tau$ of P_τ. Let T_∞ denote a copy of T with the trivial W-action. As a set, the partial compactification of P_τ is $\vec{P}_\tau = P_\tau \coprod T_\infty$. Let

[8] Recall that an *affine* weight of a central extension $U(1) \to G \to H$ is a weight of G which restricts to the identity character of $U(1)$, which is contained in any maximal torus.

$\vec{\pi} := \pi \coprod \mathrm{id}_T : \vec{P}_\tau \to T$ be the natural projection. The topology of $\vec{P}_\tau$ is generated by the open sets of P_τ, the sets $\vec{\pi}^{-1}(U)$ for $U \subset T$ open, and the collection of sets $\left\{P_\tau \backslash C \coprod T_\infty\right\}$ such that

1. (a 'niceness' condition) $C \subset P_\tau$ is closed and its preimage in $\mathfrak{t} \times \Lambda \times \mathbb{Z}$ has convex intersection with $\mathfrak{t} \times \{\lambda\} \times \{n\}$ for all λ and n and
2. (a condition directly related to the definition of $\vec{K}^*_{S^1}$, c.f Lemma 4.2.2) for any $t \in T$ and any lift $X \in \mathfrak{t}$ (i.e. $e^X = t$), the preimage of $C \cap \pi^{-1}(t)$ under the isomorphism $\pi^{-1}(t) \xleftarrow{\sim} \Lambda^{\tau(t)} \times \mathbb{Z}$ defined by X is a subset whose intersection with $\Lambda \times n$ is finite for all n and empty for sufficiently negative n.

Lemma 4.2.7. *(Key Lemma[9]) Let $N \subset G$ be the normalizer of T and $W = N/T$ the Weyl group. Note that T/N is a full $B\mathbb{Z}$-subgroupoid of $\mathcal{L}(pt/N) \simeq N/N$. Let τ be a strongly topologically regular (cf. Definition 4.2.1) twist in $H^3_{S^1}(T/N)$. There is a W-equivariant map $\vec{\pi} : \vec{P}_\tau \to T$, a subspace $T_\infty \subset \vec{P}_\tau$ and an isomorphism of $\mathbb{Z}((q))$-modules*

$$\alpha : \ {}^{\vec{\pi}^*\tau}K^*(\vec{P}_\tau/W, T_\infty/W) \xrightarrow{\sim} {}^{\tau}\vec{K}^*_{S^1}(T/N).$$

Remark 4.2.8. Note that the domain of the display is an ordinary twisted K-theory (not $\vec{K}_{S^1}$!) group. Its $\mathbb{Z}((q))$-module structure is described in the proof.

Proof. To prove the lemma the first order of business is to define a $\mathbb{Z}((q))$-module structure on the domain of the displayed map. To do that it suffices to specify the action of q, which is defined to act via the map on twisted K-theory induced by pullback along the shift isomorphism 'sh' defined by $\mathrm{sh}([(X, \lambda, n)]) = [(X, \lambda, n-1)]$.

The next order of business is to define the map α. Recall that P_τ is the bundle of (affine) characters corresponding to a bundle of groups over T (cf. Lemma 4.2.5). In particular, any point in $s \in P_\tau$ defines a 1-dimensional representation: namely if $\pi(s) = t$ then the character labeled by s defines a representation of the group $T^{\tau(t)}$ (cf. Lemma 4.2.3) on $\mathbb{C}$, and these assemble into a vector bundle $V \to P_\tau$. Extending by a trivial rank 1 vector bundle over T_∞ defines a vector bundle $\vec{V} \to \vec{P}_\tau$. The *proposed* definition of α is the composite

$${}^{\vec{\pi}^*\tau}K^*(\vec{P}_\tau/W, T_\infty/W) \xrightarrow{\otimes[\vec{V}]} {}^{\vec{\pi}^*\tau}K^*(\vec{P}_\tau/W, T_\infty/W) \xrightarrow{\vec{\pi}_!} {}^{\tau}\vec{K}^*_{S^1}(T/N),$$

but some proof is required to show that this is well defined. Namely we need to show that for any ξ in the image of the first map, the possibly infinite dimensional vector bundle $\vec{\pi}_!\xi$ given by 'summation along the fibers' satisfies the conditions to define a class in the target. Since $\vec{\pi}$ is a fiber bundle over an equivariantly locally contractible base, it suffices to check this pointwise in the base. So choose a point $t \in T$, with stabilizer N_t and 'local Weyl group'

[9]This is the analog of the 'Key Lemma' of [14] (Lemma 5.2).

$W_t = N_t/T$. Write $\vec{V}_t$ for the restriction of $\vec{V}$ to $\vec{\pi}^{-1}(t)$. We must show that 'summation along the fibers' produces a well-defined map

$$(\vec{\pi}_t)_! : [\vec{V}_t]\Big({}^{\vec{\pi}^*\tau(t)}K^*(\vec{\pi}^{-1}(t)/W_t, \{t\}_\infty/W_t)\Big) \to {}^{\tau(t)}\vec{K}^*_{S^1}(\{t\}/N_t).$$

Let $\tilde{\Lambda}^{\tau(t)}$ and $\Lambda^{\tau(t)}$ denote the set of $\tau(t)$-affine weights of $\tilde{N}_t$ and N_t. Note that $\vec{\pi}^{-1}(t) = \tilde{\Lambda}^{\tau(t)}\coprod\{t\}_\infty$ (cf. Lemma 4.2.6), and any choice of fractional splitting $\psi_t : \tilde{N}_t \leftarrow S^1_d$ induces an isomorphism $\Psi_t : \tilde{\Lambda}^{\tau(t)} \leftarrow \Lambda^{\tau(t)} \times \mathbb{Z}$. Given the results of Lemma 4.2.2 (note that $\tilde{\Lambda}^{\tau(t)} = \tilde{\Lambda}^{\tau(t)}_{\mathrm{di}}$ and $\Lambda^{\tau(t)} = \Lambda^{\tau(t)}_{\mathrm{di}}$), the map is zero in degree 1 and so it suffices to show that the degree 0 part of the domain of the previously displayed map is isomorphic to $[\vec{V}_t]\big({}^{\vec{\pi}^*\tau(t)}K^0_\star(\tilde{\Lambda}^{\tau(t)}/W_t)\big)$ as a subgroup of ${}^{\vec{\pi}^*\tau(t)}K^0(\tilde{\Lambda}^{\tau(t)}/W_t)$. But this is straightforward (the topology pf $\vec{P}_\tau$ was concocted for this purpose): any relative class in the domain of the previously displayed may must have support contained in a set C satisfying the two condition listed in the definition of the topology of $\vec{P}_\tau$. I claim that this support condition on classes in ${}^{\vec{\pi}^*\tau(t)}K^0(\tilde{\Lambda}^{\tau(t)}/W_t)$ is equivalent the definition of ${}^{\vec{\pi}^*\tau(t)}K^0_\star(\tilde{\Lambda}^{\tau(t)}/W_t)$. This follows immediately from a few observations: by Lemma 4.2.5 there is a canonical isomorphism $\pi^{-1}(t) = \tilde{\Lambda}^{\tau(t)}$. Thus, the set of induced isomorphisms $\Psi : \tilde{\Lambda}^{\tau(t)} \leftarrow \Lambda^{\tau(t)}$ induced by choices of fractional splitting $\psi : \tilde{N}^{\tau(t)}_t \leftarrow S^1_d$ is equal to the set of isomorphisms $\pi^{-1}(t) \to \Lambda^{\tau(t)}$ induced by choices of lifts $X \in \mathfrak{t}$, $e^X = t$.

Not only have I shown that α is well-defined, but also that it is an isomorphism at each point. By equivariant local contractibility of T/N it follows that α is locally an isomorphism, and by the excision axiom (or equivalently, the Mayer-Vietoris axiom) it follows that α is an isomorphism.

It remains to show that α is a $\mathbb{Z}((q))$-module map. That follows immediately form the definition of the q-action in the first paragraph of this proof together with the fact that the 'shift' isomorphism defined there coincides on each fiber $\pi^{-1}(t) = \tilde{\Lambda}^{\tau(t)}$ with the action of the generator of $\mathrm{Hom}(S^1, U(1))$ (recall that $\tilde{\Lambda}^{\tau(t)}$ is a subgroup of $\mathrm{Hom}(T^{\tau(t)}, U(1))$). □

Consider the natural inclusion $\omega : T/N \to G/G$. Recall that the $B\mathbb{Z}$-actions on both groupoids are the automorphisms of the identity defined by $t \mapsto t$ and $g \mapsto g$ (cf. Example 4.1.1). Hence the map is $B\mathbb{Z}$-equivariant (cf. Definition 3.1.1) and so there is a corresponding map in $\vec{K}_{S^1}$-theory

$${}^\tau\vec{K}^*_{S^1}(G/G) \xrightarrow{\omega^*} {}^{\omega^*\tau}\vec{K}^*_{S^1}(T/N).$$

I would like to define a pushforward. Let N act on $G \times T$ and $G \times G$ by the formula $n(g,k) = (gn^{-1}, nkn^{-1})$. Let G act on the quotients $G \times_N T$ and $G \times_N G$ by left translation on the left factor. The natural inclusion $G \times_N T \hookrightarrow G \times_N G$ induces a fully faithful map $i : (G \times_N T)/G \hookrightarrow (G \times_N G)/G$. Moreover, the inclusion $N \hookrightarrow G$ and the natural map $T = \{1\} \times T \to G \times_N T$ define an equivlanece $T/N \to (G \times_N T)/G$. Finally, the map $G \times_N G \to G$ defined by $[(g,k)] \mapsto gkg^{-1}$ defines a map $j : (G \times_N G)/G \to G/G$.

Definition 4.2.9. Let $\iota : N \to G$ be a map of compact Lie groups whose kernel is finite and whose image is a closed Lie subgroup. Define *Segal induction* (cf. [25] Section 2) as follows: first suppose that ι is injective. Then Segal induction is the map $\iota_! : R(N) \to R(G)$ which sends $M \in R(N)$ to the analytic index of $d + d^*$ acting on the de Rham complex of the associated virtual vector bundle $G \times_N M \to G/N$, in a chosen equivariant orthogonal structure. If ι has finite kernel K, define Segal induction to be the composite of the 'take K-invariants' map $R(N) \to R(N/K)$ and the previously defined Segal induction along the injective map $N/K \to G$.

Lemma 4.2.10. *Let $\iota : N \hookrightarrow G$ be the inclusion of a closed subgroup into a compact Lie group. Let $\iota_! : R(N) \to R(G)$ denote Segal induction (cf. Definition 4.2.9). If M is a virtual representation let χ_M denote its character. For a regular element $t \in G$ let F_t denote the set of cosets $gN \in G/N$ such that $g^{-1}tg \in N$. Then F_t is finite and for any $M \in R(N)$*

$$\chi_{\iota_! M}(t) = \sum_{gN \in F_t} \chi_M(g^{-1}tg).$$

Since regular elements are dense in G this determines $\chi_{\iota_! M}$ completely. Finally, suppose that N is the normalizer of a maximal torus $T \subset G$. Then for regular $t \in T$ the set F_t is the singleton $\{1N\}$, so $\chi_{\iota_! M}$ and χ_M agree on T.

Proof. Note that $g^{-1}tg \in N$ is equivalent to $tgN = gN$, i.e. the condition for $gN \in G/N$ to be a fixed point of the action of $t \in G$. The finiteness of F_t is then [25] Proposition 1.9. The character formula is a direct consequence of the Atiyah-Bott fixed point formula (cf. [25], end of Section 2). To prove the last statement, note that if $t \in T$, then $g^{-1}tg \in N$ implies that $g^{-1}tg = t' \in T$, and for any $s \in T$ $g^{-1}sg \in Z(t')_1$ (the identity component of the centralizer of t'). So if t is regular, $g^{-1}Tg \subset T$, so $g \in N$. Since regular elements are dense in T the character formula gives the desired equality of characters on T. □

Lemma 4.2.11. *The map $\omega : T/N \to G/G$ factors into two maps that admit pushforwards in classical twisted K-theory*

$$T/N \xrightarrow{\sim} (G \times_N T)/G \overset{i}{\hookrightarrow} (G \times_N G)/G \xrightarrow{j} G/G.$$

For any twist $\tau \in H^3(G/G)$ and any point $t \in T$ with stabilizers N_t and G_t in N and G, the composite pushforward $j_* i_*$ coincides with Segal induction ($\tau(t)$ is defined in the proof):

$$j_* i_* : \ {}^{\tau(t)}K^*(\{t\}/N_t) \simeq R^{\tau(t)}(N_t) \to R^{\tau(t)}(G_t) \simeq \ {}^{\tau(t)}K^*(\{t\}/N_t).$$

Warning 4.2.12. This is not a $B\mathbb{Z}$-equivariant factorization. The groupoid $G \times_N G/G$ admits no obvious $B\mathbb{Z}$-action for which j is $B\mathbb{Z}$ equivariant.

Proof. The map i is an embedding and so it has a normal bundle. The map j is a fiber bundle with fiber G/N and so it has a relative normal bundle. Since every vector bundle admits a (possibly twisted) Thom isomorphism in twisted K-theory (cf. [13] 3.6), both i and

j admit pushforwards i_* and j_*. It remains to identify the local behavior of the pushforward with Segal induction.

Consider a point $t \in T$ with stabilizer $G_t \subset G$. Write T_t for a maximal torus of G_t and let $N_t \subset N$ be the normalizer of T_t in G_t. Write $\tau(t)$ for the restriction of τ to $\{t\}/G_t$. A local (i.e. in an infinitesimal neighborhood of $\{t\}$) presentation of ω is the G_t-equivariant composite

$$\omega_t : G_t \times_{N_t} \mathfrak{t}_t \xhookrightarrow{i_t} G_t \times_{N_t} \mathfrak{g}_t \xrightarrow{j_t} \mathfrak{g}_t$$

defined by $[(g, X)] \mapsto [(g, X)] \mapsto \mathrm{Ad}_g(X)$. Write $\sigma(\nu_{i_t})$ for the twist of the Thom isomorphism along the normal bundle ν_{i_t} of i_t, which is the vector bundle $\nu_{i_t} : G_t \times_{N_t} (\mathfrak{t}_t \oplus \mathfrak{g}_t/\mathfrak{t}_t) \to G_t \times_{N_t} \mathfrak{t}_t$. Linearly contracting $\mathfrak{t}_t$ and $\mathfrak{g}_t$ (which is G_t-equivariant) induces the vertical isomorphisms in the following diagram (to ease the notational burden I have left various twists syntactically unspecified, they will not be referred to again)

$$\begin{array}{ccccccccc}
 & & {}^{\sigma}\vec{K}^*_{G_t}(G_t \times_{N_t} \mathfrak{t}_t) & \xrightarrow{(i_t)_*} & {}^{\sigma'}\vec{K}^*_{G_t}(G_t \times_{N_t} \mathfrak{g}_t) & \xrightarrow{(j_t)_*} & {}^{\sigma''}\vec{K}^*_{G_t}(\mathfrak{g}_t) & & \\
 & & \sim\uparrow & & \sim\uparrow & & \sim\uparrow & & \\
{}^{\tau(t)}K_{N_t}(\{t\}) & \xrightarrow{\sim} & {}^{\sigma'''}\vec{K}^*_{G_t}(G_t/N_t)) & \xrightarrow{(i'_t)_*} & {}^{\sigma^{(iv)}}\vec{K}^*_{G_t}(G_t/N_t) & \xrightarrow{(j'_t)_*} & {}^{\sigma^{(v)}}\vec{K}^*_{G_t}(\mathrm{pt}) & \xrightarrow{\sim} & {}^{\tau(t)}K_{G_t}(\{t\}).
\end{array}$$

Since the inverse of the equivariant contraction of $\mathfrak{t}_t$ is the inclusion $G_t \times_{N_t} \{0\} \hookrightarrow G_t \times_{N_t} \mathfrak{t}_t$ the vector bundle ν_{i_t} restricts to the vector bundle $G_t \times_{N_t} \mathfrak{g}_t/\mathfrak{t}_t \to G_t/N_t$, which is the tangent bundle $T(G_t/N_t)$. Hence the horizontal map $(i'_t)_*$ is multiplication by the euler class[10] of $T(G_t/N_t)$.

Now $(j'_t)_*$ is the pushforward along the G_t-equivariant map $G_t/N_t \to \mathrm{pt}$. So the composite of of the bottom row is "multiply by the euler class of G_t/N_t and pushforward along $G_t/N_t \to \mathrm{pt}$." This coincides with the Becker-Gottlieb transfer along $G_t/N_t \to \mathrm{pt}$[11]. This extends to the twisted setting: if $G^\tau \to G$ is a central extension of G inducing a central extension $H^{\iota^*\tau} \to H$ of H then Segal induction along $\iota^\tau : H^{\iota^*\tau} \hookrightarrow G^\tau$ produces a map $\iota^\tau_! : R(H^{\iota^*\tau}) \to R(G^\tau)$ which I claim restricts to a map $R^{\iota^*\tau}(H) \to R^\tau(G)$ (cf. Footnote 7). Indeed this follows from the last statement in Lemma 4.2.10 since $U(1)$ is by definition central in $H^{\iota^*\tau}$ and G^τ and hence contained in any maximal torus. □

Lemma 4.2.13. *The inclusion $\omega : T/N \to G/G$ admits a pushforward in $\vec{K}^*_{S^1}$-theory.*

Proof. The proof is an explicit construction of the map, leveraging the existence of the non-$B\mathbb{Z}$-equivariant pushforward in classical twisted K-theory provided by Lemma 4.2.11. The plan is to define ω_* on very small open sets and prove that these patch together to a globally defined map. Fix $t \in T$ with N-stabilizer N_t and G-stabilizer G_t. Fix a sufficiently small neighborhood $i_U : U \hookrightarrow G$ such that U and $\omega^{-1}(U) = U \cap T$ are locally contractible, so that $U/G \simeq \{t\}/G_T$ and $\omega^{-1}(U)/N \simeq \{t\}/N_t$. Every choice of fractional splitting ψ_N of

[10] The euler class depends on a choice of complex orientation of K-theory, I am not specifying one because shortly it will not matter.

[11] This follows directly from the definitions of the Becker-Gottlieb transfer and of the pushforward, cf. [22] for a detailed construction of the Becker-Gottlieb transfer in equivariant cohomology theories.

$N_t \to \tilde{N}_t \to S^1$ (cf. Lemma 4.2.2) induces a fractional splitting ψ_G of $G_t \to \tilde{G}_t \to S^1$. By Lemma 4.2.2 these splittings provide an identification of ${}^{i_U^*\omega^*\tau}\vec{K}^*_{S^1}(\omega^{-1}(U)/N)$ with a subgroup $R^{\omega^*\tau(t)}_\star(N_t)((q))$ of ${}^{i_U^*\omega^*\tau}K^*(\omega^{-1}(U)/N)((q))$ and an identification of ${}^{i_U^*\tau}\vec{K}^*_{S^1}(U/G)$ with a subgroup $R^{\tau(t)}_\star(G_t)((q))$ of ${}^{i_U^*\tau}K^*(U/G)((q))$. Applying the non-equivariant pushforward j_*i_* power-by-power in q defines the right most vertical map in the following diagram, whose dashed arrows indicate maps that would make the diagram commute but are yet to be proven well-defined

$$\begin{array}{ccccc} {}^{i_U^*\omega^*\tau}\vec{K}^*_{S^1}(\omega^{-1}(U)/N) & \xrightarrow[\sim]{\psi_N} & R^{\omega^*\tau(t)}_\star(N_t)((q)) & \hookrightarrow & {}^{i_U^*\omega^*\tau}K^*(\omega^{-1}(U)/N)((q)) = R^{\omega^*\tau(t)}(N_t)((q)) \\ \downarrow{\scriptstyle \omega_*(U)} & & \downarrow{\scriptstyle \omega^\star_*(U)} & & \downarrow{\scriptstyle j_*i_*(U)} \\ {}^{i_U^*\tau}\vec{K}^*_{S^1}(U/G) & \xrightarrow[\sim]{\psi_G} & R^{\tau(t)}_\star(G_t)((q)) & \hookrightarrow & {}^{i_U^*\tau}K^*(U/G)((q)) = R^{\tau(t)}(G_t)((q)). \end{array}$$

To show that $\omega^\star_*(U)$ is well defined I must show that if $\xi \in R^{\omega^*\tau(t)}_\star(N_t)((q))$ then $j_*i_*(U)\xi \in R^{\tau(t)}_\star(G_t)((q))$. By lemma 4.2.11, $j_*i_*(U)$ is, power-by-power in q, Segal induction. So an element $\xi = \sum_k M_k q^k$ is sent to $j_*i_*(U)\xi = \sum_k \iota_! M_k q^k$. By Lemma 4.2.2, ξ defines an element of $R^{\omega^*\tau(t)}_\star(N_t)((q))$ if and only if $\gamma_N \cdot \xi = \sum_k M_k \chi_{M_k}(\gamma_N(q))q^k \in R^{\omega^*\tau(t)}(N_t)((q))$ for every $\gamma_N \in \mathrm{Hom}(S^1, N_t)$. Now consider $\gamma_G \cdot (\sum_k \iota_! M_k q^k) = \sum_k \iota_! M_k \chi_{\iota_! M_k}(\gamma_G(q))q^k$ for $\gamma_G \in \mathrm{Hom}(S^1, G_t)$. By the last statement of Lemma 4.2.10, $\chi_{\iota_! M_k}(\gamma_G(q)) = \chi_{M_k}(\gamma_G(q))$ because $\gamma_G(q)$ is conjugate to the maximal torus $T \subset G_t$. Hence $\omega^\star_*(U)$ is well-defined.

Now $\omega_*(U) := \psi_G^{-1}\omega^\star_*(U)\psi_N$ is well-defined and I claim it is actually independent of ψ_N and ψ_G, as the notation suggests. Indeed, any other choice of fractional splitting of $\tilde{N}_t \to S^1$ is of the form $\psi'_N(q) = \psi(q)\gamma_N(q)$ for some $\gamma_N \in \mathrm{Hom}(S^1, N_t)$, and the induced splitting ψ'_G satisfies the same formula where γ_N is replaced by $i_t\gamma_N$, its composite with the inclusion $i_t : N_t \to G_t$. The claim follows by another application of the the formula $\chi_{\iota_! M_k}(\gamma_G(q)) = \chi_{M_k}(\gamma_G(q))$, which implies that $\omega^\star_*(U)(\gamma_N \cdot \xi) = i_t\gamma_N \cdot \omega^\star_*(U)(\xi)$.

It remains to show that the $\omega_*(U)$ patch together into a globally defined map, i.e. that it commutes with the restriction maps in $\vec{K}^*_{S^1}$-theory. Since locally contractible neighborhoods form a basis for the topology of T/N, G/G, their central extensions, and their $B\mathbb{Z}$-quotients, and all these groupoids have compact spaces of objects, it suffices to show that for an inclusion $j : V \hookrightarrow U$ of sufficiently small equivariantly locally contractible open neighborhoods the following diagram commutes

$$\begin{array}{ccc} {}^{\omega^* i_U^*\tau}\vec{K}^*_{S^1}(\omega^{-1}(U)/N) & \xrightarrow{j^*} & {}^{\omega^* i_V^*\tau}\vec{K}^*_{S^1}(\omega^{-1}(V)/N) \\ \downarrow{\scriptstyle \omega_*(U)} & & \downarrow{\scriptstyle \omega_*(V)} \\ R^{\omega^*\tau(t)}_\star(G_t)((q)) = {}^{i_U^*\tau}\vec{K}^*_{S^1}(U/G) & \xrightarrow{j^*} & {}^{i_U^*\tau}\vec{K}^*_{S^1}(V/G) \end{array}.$$

This is true power-by-power in q, since j_*i_* is a globally defined map. □

Lemma 4.2.14. *Let G be a compact Lie group, $T \subset G$ a maximal torus with normalizer N. Suppose $W = N/T$ contains a Weyl reflection r_α defined by a root α of G. Then any $M \in$*

$R(N)$ of virtual dimension 0 is in the kernel of the Segal induction map $\iota_! : R(N) \to R(G)$ (cf. Definition 4.2.9).

Proof. Recall that the normalizer of the standard maximal torus of $SU(2)$ is $\mathrm{Pin}_-(2)$. Recall that the root α induces maps $\delta_\alpha : \mathrm{Pin}_-(2) \to N$ and $\Delta_\alpha : SU(2) \to G$ with finite kernel. Hence, if r denotes the rank of G, there is a commutative diagram

$$\begin{array}{ccc} T^{r-1} \times \mathrm{Pin}_-(2) & \xrightarrow{\delta_\alpha} & N \\ \big\downarrow{\scriptstyle \iota_\alpha} & & \big\downarrow{\scriptstyle \iota} \\ T^{r-1} \times SU(2) & \xrightarrow{\Delta_\alpha} & G \end{array}$$

Write $\epsilon \in R(\mathrm{Pin}_-(2))$ for the sign representation ϵ of C_2 pulled back along the quotient $\mathrm{Pin}_-(2) \to C_2$. Since M is in the augmentation ideal of $R(N)$ by hypothesis, and $[1-\epsilon]$ generates the augmentation ideal of $T^{r-1} \times \mathrm{Pin}_-(2)$, $\delta_\alpha^* M$ is of the form $M' \otimes [1-\epsilon]$. I claim that such an element is sent to zero under Segal induction along ι_α. Indeed, it suffices to show that for any $V \in R(\mathrm{Pin}_-(2))$, $V \otimes [1-\epsilon]$ is sent to zero by Segal induction along $\mathrm{Pin}_-(2) \hookrightarrow SU(2)$. This follows immediately from the last sentence of Lemma 4.2.10 since $V \otimes [1-\epsilon]$ has trivial T-character and $SU(2)$ is connected.

Now by [22] Lemma 4.3, $(\delta_\alpha)_! \delta_\alpha^*$ is multiplication by the N-equivariant Euler characteristic of the finite set $\mathrm{coker}\delta_\alpha = W/\langle r_\alpha \rangle$. Since Segal induction is transitive, $(\Delta_\alpha)_!(\iota_\alpha)_!\delta_\alpha^* M = \iota_!(\delta_\alpha)_!\delta_\alpha^* M$. The left hand side is zero, and the right hand side is $|W/\langle r_\alpha\rangle|\iota_! M$. Since $|W/\langle r_\alpha\rangle|$ is nonzero and $R(G)$ is torsion free, $\iota_! M = 0$.

□

4.2.2 Calculation of ${}^\tau\vec{K}^*_{S^1}(G/G)$ at negative twist

Lemma 4.2.15. *Let $\tau \in H^3_{S^1}(T/N)$ be a negative twist. Equip $(\Lambda/\pi_1 T)/W$ with the trivial $B\mathbb{Z}$-action (cf. Definition 3.1.1). There is a twist $\tau' \in H^3_{S^1}((\Lambda/\pi_1 T)/W)$ (described in the proof) and an isomorphism of $\mathbb{Z}((q))$-modules*

$${}^\tau\vec{K}^*_{S^1}(T/N) \xrightarrow{\sim} {}^{\tau'}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W).$$

Proof. The plan is to invoke Lemma 4.2.7, recognize the homotopy type of $\vec{P}_\tau$ as a Thom space, and apply the Thom isomorphism.

I claim that because τ is negative, if the complement of a set $C \subset P_\tau$ contributes to the topology of $\vec{P}_\tau$ (cf. Definition 4.2.6) then the intersection of C with any connected component of P_τ[12] has compact connected components. Indeed, by Condition 1 on the sets C listed in Definition 4.2.6 if such a component were not compact, its image in T would contain a point t such that $C \cap \pi^{-1}(t)$ contains points of the form $\{([X-p, \lambda, n)]\}$ for an infinite set of $p \in \pi_1 T$ and some fixed λ and n. But now consider the isomorphism $\pi^{-1}(t) \leftarrow \Lambda^{\tau(t)} \times \mathbb{Z}$ induced by the

[12]Recall that these connected components are all homeomorphic to $\mathfrak{t}$ (cf. Lemma 4.2.5).

lift X of t. The inverse of that map sends $[(X-p,\lambda,n)]$ to $(\lambda-\kappa^\tau p, n+\tau(\kappa^\tau p,\kappa^\tau p)+\lambda(p))$. Because τ is negative and there are infinitely many choices of p, that set certainly fails (the last part of) Condition 2 listed in Definition 4.2.6.

That leads to the following: define an action of $\pi_1 T$ on $\mathfrak{t}\times\Lambda$ by $p(X,\lambda)=(X+p,\lambda-\kappa^\tau p)$. Let $(-)_+$ denote the one-point compactification and consider the spaces $\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)\times\mathbb{Z}_{\leqslant 0}\big)_+$ and $\bigvee_{\mathbb{Z}_{>0}}\big(\mathfrak{t}\times_{\pi_1 T}\Lambda\big)_+$. Fix and $l\in\Lambda$. For any $(X,\lambda)\in\mathfrak{t}\times\Lambda$ let (X_l,λ_l) be the $\pi_1 T$-translate such that λ_l that lies the fundamental domain containing l. Then the pointed map

$$\Phi_l : \big((\mathfrak{t}\times_{\pi_1 T}\Lambda)\times\mathbb{Z}_{\leqslant 0}\big)_+ \vee \bigvee_{n\in\mathbb{Z}_{>0}} \big((\mathfrak{t}\times_{\pi_1 T}\Lambda)\times\{n\}\big)_+ \to \vec{P}_\tau/T_\infty$$

$$([X,\lambda],n)\mapsto[(X_l,\lambda_l,n)]$$

is a W-equivariant homeomorphism. Hence the group ${}^{\vec{\pi}^*\tau}K^*(\vec{P}_\tau/W,T_\infty/W)$ can be computed as the $\Phi_l^*\vec{\pi}^*\tau$-twisted, reduced W-equivariant K-theory of the domain of Φ_l. As an abelian group, that can be written suggestively in terms of ${}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)_+/W\big)$ and a bookkeeping parameter presciently called q:

$$\begin{aligned}
{}^{\vec{\pi}^*\tau}K^*(\vec{P}_\tau/W,T_\infty/W) &\xrightarrow{\Phi_l^*} {}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big(((\mathfrak{t}\times_{\pi_1 T}\Lambda)\times\mathbb{Z}_{\leqslant 0})_+/W\big)\oplus\textstyle\bigoplus_{n<0}{}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)\times\{n\})_+/W\big)\\
&\simeq {}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)_+/W\big)[[q]]\oplus\bigoplus_{n<0}{}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)_+/W\big)q^{-n}\\
&\simeq {}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)_+/W\big)((q)).
\end{aligned}$$

Given that the $\mathbb{Z}((q))$-module structure of ${}^{\vec{\pi}^*\tau}K^*(\vec{P}_\tau/W,T_\infty/W)$ comes via pullback along the shift map sh, it is clear that the $\mathbb{Z}((q))$-module structure suggested by the last displayed abelian group is actually the correct one.

It now remains to define τ' and prove that there is a $\mathbb{Z}((q))$-module isomorphism

$${}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)_+/W\big)((q)) \simeq {}^{\tau'}\vec{K}_{S^1}^{*+\dim T}((\Lambda/\pi_1 T)/W).$$

Recall that a vector bundle induces a twist of the K-theory of the base of that vector bundle (cf. [13] 3.6). Note that $(\mathfrak{t}\times_{\pi_1 T}\Lambda)_+$ is the Thom space of the vector bundle $\mathfrak{t}\times_{\pi_1 T}\Lambda\to\Lambda/\pi_1 T$, and call the corresponding twist $\sigma(\mathfrak{t})$. Then applying the Thom isomorphism (cf. [13] 3.6) in W-equivariant twisted K-theory gives

$${}^{\Phi_l^*\vec{\pi}^*\tau}\widetilde{K}^*\big((\mathfrak{t}\times_{\pi_1 T}\Lambda)_+\big)((q)) \simeq {}^{\Phi_l^*\vec{\pi}^*\tau-\sigma(\mathfrak{t})}\widetilde{K}^{*-\dim\mathfrak{t}}\big(\Lambda/\pi_1 T\big)((q)).$$

Finally, write $\tau'=\Phi_l^*\vec{\pi}^*\tau-\sigma(\mathfrak{t})$. Since $(\Lambda/\pi_1 T)/W$ is a trivial $B\mathbb{Z}$-groupoid (cf. Definition 3.1.1)

$${}^{\tau'}\vec{K}_{S^1}^*((\Lambda/\pi_1 T)/W)\simeq{}^{\tau'}K^*((\Lambda/\pi_1 T)/W)\otimes_{\mathbb{Z}}\mathbb{Z}((q)).$$

□

Corollary 4.2.16. *At negative level* ${}^{\tau}\vec{K}_{S^1}^*(T/N)$ *is spanned by classes supported at single conjugacy classes.*

Proof. This is immediate from the proof of Lemma 4.2.15, since the classes in ${}^{\tau_w}\vec{K}^*_{S^1}(T/N)$ are all images of degree zero classes under a Thom isomorphism from a vector bundle whose dimension is equal to the dimension of T. □

Lemma 4.2.17. *Let G be a compact Lie group with identity component $G(1)$. Let Λ_{reg} be the regular weights of G, i.e. those on which are not fixed by a Weyl reflection $r \in W = N/T$. For negative twists $\tau \in H^3_{S^1}(G/G)$, there is a twist $\tau' \in H^3_{S^1}(\Lambda_{reg}/\pi_1 T/W)$ (described in the proof) such that ω^* induces an isomorphism*

$$\omega^* : {}^{\tau}\vec{K}^*_{S^1}(G(1)/G) \xrightarrow{\sim} {}^{\tau'}\vec{K}^*_{S^1}(\Lambda_{reg}/\pi_1 T/W) \subset {}^{\tau'}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W).$$

Proof. As usual, consider a point $t \in T$ with stabilizer $G_t \subset G$. Write $T_t(= T)$ for a maximal torus of G_t and let $N_t \subset N$ be the normalizer of T_t in G_t. Write $\tau(t)$ for the restriction of τ to $\{t\}/G_t$. Note that $\omega : T/N \to G(1)/G$ is essentially surjective. By the definition of ω (cf. Lemma 4.2.13) it suffices to work at a fixed q-power. By the same Lemma, the local behavior of ω, denoted $(\omega_t)_*$, is the G_t-equivariant Becker-Gottlieb transfer along $G_t/N_t \to \mathrm{pt}$. By [22] Lemma 4.3 the composition $(\omega_t)_*\omega_t^*$ is multiplication by the G_t-equivariant Euler characteristic $\chi(G_t/N_t) = |N_t|/|N_t| = 1$. So the map $\omega_*\omega^*$ is locally the identity. By the Mayer-Vietoris axiom (and equivariant local contractibility of all groupoids involved) it follows that it is a global isomorphism. Then, in light of Corollary 4.2.16, the image of any class under ω_* is fixed by $\omega_*\omega^*$, so $\omega_*\omega^*$ is a projection, and so must be the identity. Therefore ${}^{\tau}\vec{K}^*_{S^1}(G/G)$ is split inside ${}^{\omega^*\tau}\vec{K}^*_{S^1}(T/N)$.

It remains to identify the summand. For that it suffices to identify the kernel of ω_* with the classes supported away from regular conjugacy classes. Recall from Corollary 4.2.16 that ${}^{\omega^*\tau}\vec{K}^*_{S^1}(T/N)$ is spanned by classes supported at single conjugacy classes. Moreover, for any $t \in T$ all classes supported at $\{t\}/N_t \subset T/N$ are of the form $[V \otimes \Theta_t]$ where $V \in R^{\omega^*\tau(t)}(N_t)$ and Θ_t represents the euler class of the normal bundle of $\{t\}/N_t \hookrightarrow T/N$, which depends on a choice of complex orientation but is always a class of virtual dimension zero in $R(N_t)$. Finally, under the identification of ${}^{\omega^*\tau}\vec{K}^*_{S^1}(T/N)$ with ${}^{\tau'}\vec{K}^*_{S^1}(\Lambda/\pi_1 T/W$, the subgroup ${}^{\tau'}\vec{K}^*_{S^1}(\Lambda_{\mathrm{reg}}/\pi_1 T/W)$ corresponds to classes supported at $t \in T$ that define regular conjugacy class in G (i.e. $G_t = N_t$). Recall that $(\omega_t)_*$ is given by Segal induction, power-by-power in q (cf. the proof of Lemma 4.2.13). Clearly if $G_t = N_t$ then $(\omega_t)_*$ is the identity. If t does not define a regular conjugacy class in G, Then $W_t = N_t/T_t$ contains a Weyl reflection, and the proof is reduced to Lemma 4.2.14. □

4.2.3 Calculation of ${}^{\tau}\vec{K}^*_{S^1}(G/G)$ at positive twist

Lemma 4.2.18. *Let $\tau \in H^3_{S^1}(T/N)$ be a positive twist. Equip $(\Lambda/\pi_1 T)/W$ with the trivial $B\mathbb{Z}$-action (cf. Definition 3.1.1). There is a twist $\tau' \in H^3_{S^1}((\Lambda/\pi_1 T)/W)$ (described in the proof) and an isomorphism of $\mathbb{Z}((q))$-modules*

$${}^{\tau}\vec{K}^*_{S^1}(T/N) \xrightarrow{\sim} {}^{\tau'}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W).$$

Proof. The proof is similar to that of Lemma 4.2.15. The plan is to invoke Lemma 4.2.7, recognize the homotopy type of $\vec{P}_\tau$ as the disjointly base-pointed total space of a vector bundle over $(\Lambda/\pi_1 T)/W$, and then equivariantly contract the fibers of that vector bundle and discard the disjoint base-point.

I claim that because τ is positive, the complement of any closed, connected, convex set $C \subset P_\tau$ contributes to the topology of $\vec{P}_\tau$ (cf. Definition 4.2.6). It suffices to consider the case that C is the image of $\mathfrak{t} \times \{\lambda\} \times \{n\}$ for some (λ, n). This certainly satisfies Condition 1 listed in Definition 4.2.6. For any $t \in T$, and lift X of t, the inverse of the induced map $\pi^{-1}(t) \leftarrow \Lambda^{\tau(t)} \times \mathbb{Z}$ sends $[(X-p, \lambda, n)]$ to $(\lambda - \kappa^\tau p, n + \tau(\kappa^\tau p, \kappa^\tau p) + \lambda(p))$, whose second coordinate is a positive quadratic function of p and so certainly satisfies Condition 2.

It follows that for any bounded below $S \subset \mathbb{Z}$, the image of $\mathfrak{t} \times \{\lambda\} \times S$ in P_τ satisfies Conditions 1 and 2. This leads to the following.

Let $\vec{\mathbb{Z}}$ denote the following partial compactification of $\mathbb{Z}$: as a set $\vec{\mathbb{Z}} = \mathbb{Z} \coprod \{\infty_+\}$ and the open neighborhoods of $\{\infty_+\}$ are defined to be the complements of sets $C \in \mathbb{Z}$ which are bounded below. As in the proof of Lemma 4.2.15, define an action of $\pi_1 T$ on $\mathfrak{t} \times \Lambda$ by $p(X, \lambda) = (X + p, \lambda - \kappa^\tau p)$. Let $\left(\mathfrak{t} \times_{\pi_1 T} \Lambda \coprod \{*\}\right)$ be the evident addition of a disjoint basepoint with trivial W-action. For any $(X, \lambda) \in \mathfrak{t} \times \Lambda$ let $(X_l, \lambda_l$ be the $\pi_1 T$-translate such that λ_l that lies the fundamental domain containing l. Then the pointed map

$$\Phi_l : \left(\mathfrak{t} \times_{\pi_1 T} \Lambda^\tau \coprod \{*\}\right) \wedge \vec{\mathbb{Z}} \to \vec{P}_\tau / T_\infty$$

$$([X, \lambda], n) \mapsto [\nu_l(X), \lambda_l, n]$$

is a W-equivariant homeomorphism. Hence ${}^{\vec{\pi}^*\tau}\widetilde{K}^*(\vec{P}_\tau/W) \simeq {}^{\Phi_l^*\vec{\pi}^*\tau}K^*\left((\mathfrak{t} \times_{\pi_1 T} \Lambda^\tau)/W\right) \otimes_{\mathbb{Z}} \mathbb{Z}((q))$. The proof of the isomorphism displayed in the lemma is completed by defining $\tau' = \Phi_l^* \vec{\pi}^* \tau$, noting that $\mathfrak{t} \times_{\pi_1 T} \Lambda^\tau$ equivariantly deformation retracts onto $\Lambda/\pi_1 T$, and that (as in the proof of Lemma 4.2.15)

$${}^{\tau'}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W) \simeq {}^{\tau'}K^*((\Lambda/\pi_1 T)/W) \otimes_{\mathbb{Z}} \mathbb{Z}((q)).$$

□

Definition 4.2.19. ([1] Chapter 9) For a compact Lie group G and an element $\tau \in H^3_{S^1}(LBG)$ write $LG^\tau \rtimes S^1 \to LG \rtimes S^1$ for the associated central $U(1)$-extension, and write $\hat{R}^\tau_{\text{pos}}(LG \rtimes S^1)$ for the free $\mathbb{Z}((q))$-module consisting of positive energy, τ-projective (also know as 'level τ') representations of $LG \rtimes S^1$.

Lemma 4.2.20. *For positive twists τ there is an injection of $\mathbb{Z}((q))$-modules $\hat{R}^\tau_{pos}(LG \rtimes S^1) \hookrightarrow {}^\tau\vec{K}^0_{S^1}(G(1)/G)$.*

Proof. Let $\mathcal{H} \in \hat{R}^\tau_{\text{pos}}(LG \rtimes S^1)$ be an irreducible level τ positive energy representation. Let $S \subset \Lambda \times \mathbb{Z}$ be the weights of $T \times S^1 \subset LG \rtimes S^1$ appearing in $\mathcal{H}$. By [1] Theorem 9.3.5 this has a unique lowest weight $(\lambda, n) \in S$. The affine Weyl group $W \ltimes \pi_1 T$ acts on S, and the subgroup $\pi_1 T$ acts by the first displayed formula in the proof of Lemma 4.2.5 (cf. formula (9.3.3) in

[1]). Hence, since τ is positive, for any $t \in G$ with stabilizer G_t the infinite dimensional (projective) representation $\mathcal{H}$ produces a well-defined element of ${}^{\tau(t)}\vec{K}^*_{S^1}(\{t\}/G_t)$ (cf. Lemma 4.2.2). Then since $(G(1)/G)/B\mathbb{Z} \simeq \mathcal{A}(G)/(LG \rtimes S^1)$ ([15], Section 2.1), the construction

$$\mathcal{A}(G) \times_{(LG \rtimes S^1)} \mathcal{H}_\lambda$$

defines the required map. It is injective since the pullback along the inclusion $1 \hookrightarrow G$ detects the character of $\mathcal{H}_\lambda$ as an element of $R^{\tau(1)}(G)((q)) \simeq {}^{\tau(1)}\vec{K}^0_{S^1}(\mathrm{pt}/G)$ (cf. Lemma 4.2.2). □

Definition 4.2.21. Let $\mathbf{X}$ be a $B\mathbb{Z}$-groupoid let $\tau \in H^3_{S^1}(\mathbf{X})$ be a twist. A class $\xi \in {}^{\tau}\vec{K}^*_{S^1}\mathbf{X}$ is said to have *virtual q-dimension 0* if for any point $x \in X_0$ with stabilizer G_x, pullback along $i_x : \{x\}/G_x \to \mathbf{X}$ produces a class $i_x^*\xi \in {}^{i_x^*\tau}\vec{K}^*_{S^1}(\{x\}/G_x)$ which maps to zero under the restriction map ${}^{i_x^*\tau}\vec{K}^*_{S^1}(\{x\}/G_x) \to \vec{K}^*_{S^1}(\{x\}) = \mathbb{Z}((q))$ induced by $1 \to G_x$ (cf. Lemma 4.2.2).

Lemma 4.2.22. *Let τ be a positive twist. Identify ${}^{\tau}\vec{K}^*_{S^1}(T/N)$ with ${}^{\tau'}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W)$ as in Lemma 4.2.18. Then the kernel of the map*

$$\omega_* : \ {}^{\tau'}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W) \longrightarrow \ {}^{\tau}\vec{K}^*_{S^1}(G/G)$$

consists of the classes of virtual q-dimension 0 (cf. Definition 4.2.21).

Proof. For $t \in N$ write N_t for its stabilizer in N, G_t for its stabilizer in G, and T_t for a maximal torus of G_t. Recall from the proof of Lemma 4.2.13 that ω_* is defined power-by-power in q, and the local model at a point $t \in G$ with stabilizer $G_t \in G$ and coincides with Segal induction $\iota_! : R(N_t) \to R(G_t)$.

Now let $[V] \in {}^{\tau}\vec{K}^*_{S^1}((\Lambda/\pi_1 T)/W)$ be a class of virtual q-dimension 0. Then its support must be contained in the set of $[\lambda] \in \Lambda/\pi_1 T$ with nontrivial W-stabilizer. Over each such point its fiber is a representation of the stabilizer of virtual q-dimension 0. It suffices to assume that $[V]$ is supported at a single such point $[\lambda]$ with stabilizer W_λ. Tracing $[V]$ backwards through the construction of Lemma 4.2.18, there is a point $t \in T$ with stabilizer N_t such that the class in ${}^{\tau}\vec{K}^*_{S^1}(T/N)$ corresponding to $[V]$ is represented by a Hilbert bundle $\mathcal{V}$ whose fiber at t can be identified (by a choice of fractional splitting $\psi : \tilde{N}_t \leftarrow S^1_d$) with a Laurent series in q with coefficients in (τ-projective) virtual representations of virtual dimension 0. Since $(\omega_t)_*$ is defined power-by-power, Lemma 4.2.14 implies that $(\omega_t)_*\mathcal{V}_t$ has q-dimension zero at t. Hence $\omega_*[V]$ and $\omega^*\omega_*[V]$ have q-dimension zero. But from the free $\mathbb{Z}((q))$-basis of ${}^{\tau}\vec{K}^*_{S^1}(T/N)$ established in Lemma 4.2.18, it is evident that the only class which is of q-dimension 0 at any point is the zero class, So $\omega^*\omega_*[\mathcal{V}] = 0$. Injectivity of ω^* implies that $\omega_*[\mathcal{V}] = 0$. It remains to prove that the classes of virtual dimension zero span the kernel of ω_*. By Lemma 4.2.20, the image of ω^* contains a subspace isomorphic to $\hat{R}^{\tau}_{\mathrm{pos}}(LG \rtimes S^1)$. Since $\omega_*\omega^*$ is an isomorphism (cf. the proof of Lemma 4.2.17), the proof is completed by a counting argument using the following result. □

Theorem 4.2.23. *(cf. [1] Theorem 9.3.5, [14] Theorem 10.2) A free* $\mathbb{Z}((q))$*-basis of* $\hat{R}^{\tau}_{pos}(LG \rtimes S^1)$ *consisting of irreducibles is in one-to-one correspondence with the set of orbits of the action of the affine Weyl group* $W \ltimes \pi_1 T$ *on* Λ *defined by* τ *(cf. Lemma 4.2.5), i.e. the coarse quotient* $[(\Lambda/\pi_1 T)/W]$.

The preceding two results immediately imply the following.

Corollary 4.2.24. *For positive* τ, ${}^{\tau}\vec{K}^{*}_{S^1}(G/G) \simeq \hat{R}^{\tau}_{pos}(LG \rtimes S^1)$*, where the right hand side is viewed as a graded* $\mathbb{Z}((q))$*-module concentrated in degree zero.*

4.2.4 Reconciliation of bases

Definition 4.2.25. Let G be a compact Lie group. Let $\mathrm{Cliff}(\mathfrak{g}^*)$ denote the Clifford algebra of the vector space $\mathfrak{g}^*$ equipped with the bilinear Killing form. By integrating over S^1, the Killing form induces a bilinear form on $L\mathfrak{g}^* = C^{\infty}(S^1, \mathfrak{g}^*)$. Let $\mathrm{Cliff}(L\mathfrak{g}^*)$ denote the associated Clifford algebra, and let $\mathcal{S}^{\pm}(L\mathfrak{g}^*)$ be an irreducible $\mathbb{Z}/2$-graded representation of $\mathrm{Cliff}(L\mathfrak{g}^*)$. If $\mathcal{S}^{\pm}(0)$ is an irreducible $\mathbb{Z}/2$-graded representation of $\mathrm{Cliff}(\mathfrak{g}^*)$ then $\mathcal{S}^{\pm}(L\mathfrak{g}^*)$ may be presented as $\mathcal{S}^{\pm}(0) \otimes \bigwedge^{\mathrm{ev/odd}}(z\mathfrak{g}_{\mathbb{C}}[z])$ (cf. [14] 8.6). It admits a projective, positive energy action of LG, and the corresponding level is denoted by σ (cf. [14] 1.6, 8.8).

While it is not curious that the calculation at positive and negative twist differ[13] it is curious *how* they differ. At positive level the representation ring $\hat{R}^{\tau}_{\mathrm{pos}}(LG \rtimes S^1)$ makes an appearance as ${}^{\tau}\vec{K}^{0}_{S^1}(G/G) \simeq {}^{\tau}\vec{K}^{\dim G}_{S^1}(\Lambda^{\tau}/W)$, while at negative level, the isomorphic ring $\hat{R}^{-\tau}_{\mathrm{neg}}(LG \rtimes S^1)$ makes its appearance as ${}^{\tau+\sigma}\vec{K}^{\dim G}_{S^1}(G/G) \simeq {}^{\tau+\sigma}\vec{K}^{\dim G}_{S^1}(\Lambda^{\tau+\sigma}_{\mathrm{reg}}/W)$. Thus at positive level there is no shift in twist and all W-orbits contribute a basis element, while at negative level there is a shift by σ in the twist and only the *regular* orbits contribute basis elements. The correspondence becomes even more curious when related to the correspondence for the maximal torus, where there is no σ-discrepancy between positive and negative twists.

This is made precise in the following lemma.

Lemma 4.2.26. *Let* G *be a simple and simply-connected Lie group. Fix a positive twist* $\tau \in H^3_{S^1}(G/G)$*. There are dualities of finitely-generated free (ungraded)* $\mathbb{Z}((q))$*-modules*

$$\mathbb{D}_T : {}^{-\tau-\sigma}\vec{K}^{r}_{S^1}(T/N) \otimes {}^{\tau+\sigma}\vec{K}^{0}_{S^1}(T/N) \longrightarrow \mathbb{Z}((q)),$$

$$\mathbb{D}_G : {}^{-\tau-\sigma}\vec{K}^{r}_{S^1}(G/G) \otimes {}^{\tau}\vec{K}^{0}_{S^1}(G/G) \longrightarrow \mathbb{Z}((q)).$$

Proof. By Corollary 4.2.24, the map $\hat{R}^{\tau}_{\mathrm{pos}}(LG \rtimes S^1) \hookrightarrow {}^{\tau}\vec{K}^{0}_{S^1}(G/G)$ in Lemma 4.2.20 is an isomorphism. The content of Section 12 and 13 of [14] (cf. 12.9, Proposition 13.6) is that for each $[\mathcal{H}_\lambda] \in \hat{R}^{\tau}_{\mathrm{pos}}(LG \rtimes S^1)$ there is a G-parametrized family F_λ of Fredholm operators on $\mathcal{H}_\lambda \otimes \mathcal{S}^{\pm}$ such that the assignment $[\mathcal{H}_\lambda] \mapsto (\mathcal{H}_\lambda \otimes \mathcal{S}^{\pm}, F_\lambda)$ defines an isomorphism

[13] See the discussion at the beginning of Section 3.

$\hat{R}^\tau_{\mathrm{pos}}(LG \rtimes S^1) \hookrightarrow {}^{-\tau-\sigma}K^r_{S^1}(G/G)$. By Lemma 4.2.17 the natural map ${}^{-\tau-\sigma}K^r_{S^1}(G/G) \to {}^{-\tau-\sigma}\vec{K}^r_{S^1}(G/G)$ is an isomorphism. Therefore the span of isomorphisms of finite rank, free $\mathbb{Z}((q))$-modules

$$ {}^{-\tau-\sigma}\vec{K}^r_{S^1}(G/G) \leftarrow \hat{R}^\tau_{\mathrm{pos}}(LG \rtimes S^1) \to {}^{\tau}\vec{K}^0_{S^1}(G/G) $$

exhibits a duality $\mathbb{D}_G$ between the left and right terms, since the middle is canonically self-dual (it has a canonical basis of irreducibles).

Let $\hat{R}^\tau_{\mathrm{pos}}(LT \rtimes S^1)^{hW}$ be the $\mathbb{Z}((q))$-module of isomorphism classes of W-*equivariant*[14] τ-twisted positive energy LT representations. The same recipe as in the previous paragraph produces a span of isomorphisms of finite rank, free $\mathbb{Z}((q))$-modules

$$ {}^{-\tau-\sigma}\vec{K}^r_{S^1}(T/N) \leftarrow \hat{R}^\tau_{\mathrm{pos}}(LT \rtimes S^1)^{hW} \to {}^{\tau+\sigma}\vec{K}^0_{S^1}(T/N) $$

exhibits a duality $\mathbb{D}_T$ between the left and right terms, since the middle is again canonically self-dual. □

The reconciliation comes in the form of the following lemma.

Lemma 4.2.27. *Let G be a simple and simply-connected Lie group. Fix a positive twist $\tau \in H^3_{S^1}(G/G)$. The dualities of the previous lemma are intertwined by maps*

$$ \omega_* : {}^{-\tau-\sigma}\vec{K}^r_{S^1}(T/N) \to {}^{-\tau-\sigma}\vec{K}^r_{S^1}(G/G), $$

$$ \omega^! : {}^{\tau}\vec{K}^0_{S^1}(G/G) \to {}^{\tau+\sigma}\vec{K}^0_{S^1}(T/N). $$

The first map is the pullback along $\omega : T/N \hookrightarrow G/G$ followed by tensoring with the spinor representation $\mathcal{S}^\pm(L\mathfrak{g}^*)$ (cf. Definition 4.2.25). The second is the Becker-Gottlieb transfer along ω.

Proof. The claim is that for any $a \in {}^{-\tau-\sigma}\vec{K}^r_{S^1}(T/N)$ and $b \in {}^{\tau}\vec{K}^0_{S^1}(G/G)$, $\mathbb{D}_G(\omega_* a \otimes b) = \mathbb{D}_T(a \otimes \omega^! b)$. Since $\omega_*\omega^*$ is the identity at negative level (cf. the proof of Lemma 4.2.17), it suffices to show that for any $c \in {}^{-\tau-\sigma}\vec{K}^r_{S^1}(G/G)$ and $b \in {}^{\tau}\vec{K}^0_{S^1}(G/G)$, $\mathbb{D}_G(c \otimes b) = \mathbb{D}_T(\omega^* c \otimes \omega^! b)$.

Under the identifications of the previous lemma the restriction maps $\omega^* : {}^{-\tau-\sigma}\vec{K}^r_{S^1}(G/G) \to {}^{-\tau-\sigma}\vec{K}^r_{S^1}(T/N)$ and $\omega^* : {}^{\tau}\vec{K}^0_{S^1}(G/G) \to {}^{\tau}\vec{K}^0_{S^1}(T/N)$ both agree with the restriction of representations $\hat{R}^\tau_{\mathrm{pos}}(LG \rtimes S^1) \to \hat{R}^\tau_{\mathrm{pos}}(LT \rtimes S^1)^{hW}$. The equation $\mathbb{D}_G(c \otimes b) = \mathbb{D}_T(\omega^* c \otimes \omega^! b)$ now follows since the discrepancy between $\omega^!$ and ω^* is precisely the tensor product with $\mathcal{S}^\pm$ that is missing on the positive level side. □

[14] That is, one has the data of an intertwiner for each $w \in W$.

4.3 Constructions

In this section the completed $\vec{K}_{S^1}$-theory defined above is used to give a K-theoretic picture of equivariant elliptic cohomology at the Tate curve. Despite being the shortest section, it is in some sense the paper's centerpiece. In fact, its shortness speaks to the wonderful simplicity of the Kitchloo-Morava picture of elliptic cohomology at the Tate curve. Fix a compact Lie group G. A G-equivariant elliptic cohomology theory is defined to be:

1. a weakly even G-equivariant cohomology theory E_G,
2. an elliptic curve $\mathcal{E}$ over $E^0(\mathrm{pt}) = E^0_G(G)$,
3. a twist $\tau \in H^4(BG)$ for E_G with associated transgressed class $\mathrm{tr}(\tau) \in H^3_{S^1}(LBG)$ and central extension $LG^\tau \rtimes S^1 \to LG \rtimes S^1$
4. an isomorphism of $\mathbb{Z}((q))$-modules ${}^\tau E^0_G(\mathrm{pt}) \to \hat{R}^\tau_{\mathrm{pos}}(LG \rtimes S^1)$ (cf. Definition 4.2.19)
5. and an isomorphism of formal groups $\mathrm{Spf} E^0(\mathbb{CP}^\infty) = E^0_G(G \times \mathbb{CP}^\infty) \to \hat{\mathcal{E}}$,

4.3.1 The equivariant Kitchloo-Morava construction

Let G-Spaces$_{\mathrm{rel}}$ be the category of pairs (M, A) where M is a G-space and $A \subset M$ is G-invariant. A class $\tau \in H^4BG$ transgresses to a class $\mathrm{tr}(\tau) \in H^3_{S^1}LBG (= H^3_{S^1}(\mathcal{L}(\mathrm{pt}/G))$ that restricts to zero in $H^3\Omega BG = H^3G$ and hence defines a graded central extension (with trivial grading) $\mathbf{L}^\tau \to \mathcal{L}(\mathrm{pt}/G)/B\mathbb{Z}$ (cf. Remark 3.1.7). Write p_M for the projection $M \to \mathrm{pt}$. Recall that $\mathcal{L}(M/G)$ has a natural $B\mathbb{Z}$-action (cf. Example 4.1.1). Define a functor $J_\tau : G\text{-Spaces}_{\mathrm{rel}} \to B\mathbb{Z}\text{-}\mathfrak{Twist}_{\mathrm{rel}}$ by the formula

$$J_\tau(M, A) := (\mathcal{L}(M/G)/B\mathbb{Z}, \mathcal{L}(A/G)/B\mathbb{Z}, \mathcal{L}(p_M)^*\mathbf{L}^\tau).$$

Proposition 4.3.1. *For any compact connected Lie group G and $\tau \in H^4BG$ whose transgression to $H^3_{S^1}(LBG)$ defines a strongly topologically regular positive twist the composite functor*

$$ {}^\tau E^*_G := \vec{K}^*_{S^1} \circ J_\tau : G\text{-}Spaces_{rel} \longrightarrow \mathbb{Z}((q))\text{-}mod$$
$$(M, A) \mapsto {}^\tau E^*_G(M, A)$$

defines a G-equivariant elliptic cohomology theory at the Tate curve.

Proof. Given that $\mathcal{L}((M\backslash A)/G) \simeq \mathcal{L}(M/G)\backslash\mathcal{L}(A/G)$, the cohomology axioms follow immediately from their holding for $\vec{K}_{S^1}$. It remains to establish ellipticity as defined above. Take $\mathcal{E}$ to be the Tate curve over $\mathbb{Z}$. By hypothesis τ defines a strongly topologically regular *positive* twist, so Corollary 3.2.5 says precisely that ${}^\tau E^0_G(\mathrm{pt}) \simeq \hat{R}^\tau_{\mathrm{pos}}(LG_1 \rtimes S^1)$. □

Remark 4.3.2. When G is disconnected all the definitions still make sense; instead the restriction to connected groups is to preserve the maximum amount of correlation between

${}^\tau\vec{K}_{S^1}(G/G)$ and the positive energy representation theory of LG. Indeed, when G is disconnected one has to introduce *twisted loop groups*[15] (cf. [14] 1.5) and their representation theory, which results in an explosion of notational complexity and bookkeeping that obscures a major virtue of the Kitchloo-Morava construction, namely its simplicity.

Remark 4.3.3. Some readers might prefer if item 4 were replaced with something like the following condition: a twist $\tau \in H^4(BG)$ for E_G with its associated line bundle $\mathcal{L}_\tau \to \mathrm{Bun}_G(\mathcal{E})$ over the moduli space of principal G-bundles over $\mathcal{E}$. If G is simple and simply-connected then by [4] Theorem D, the Kac character map gives an isomorphism of $\hat{R}^\tau_{\mathrm{pos}}(LG_1 \rtimes S^1)$ with $\Gamma(\mathrm{Bun}_G(\mathcal{E}); \mathcal{L}_\tau)$. I have chosen to require only a relation to positive energy representation theory so as not to have to invoke the Kac character theory to prove ellipticity and to treat all connected Lie groups uniformly.

Remark 4.3.4. At negative level the construction is still plagued by the problems mentioned in the introduction of this paper and so does not produce an elliptic cohomology theory. That is no surprise, since the completion that defines $\vec{K}_{S^1}$ favors *positive* q-powers. On the other hand, the functor ${}^\tau E^*_G$ makes perfect sense for negative τ and still produces a G-equivariant cohomology theory, just not an elliptic one (at least as per the 5 specifications listed above). This motivates the following section.

4.3.2 Duality in E_G

When G is simple and simply-connected, Lemma 4.2.26 immediately implies the following.

Lemma 4.3.5. *Let G be a simple and simply-connected Lie group of dimension d and let σ be the twist associated to the positive energy spin representation of the adjoint representation of LG (cf. Definition 4.2.25). There is a natural duality pairing*

$$ {}^{-\tau-\sigma}E^d_G(pt) \otimes {}^\tau E^0_G(pt) \longrightarrow {}^{-\sigma}E^d_G(pt) \simeq \mathbb{Z}((q)). $$

4.3.3 Comparisons

When G is connected, Grojnowski [16] has constructed the seminal "delocalised equivariant elliptic cohomology" over the complex numbers as follows (cf. also [8] 4.1). The theory takes values in holomorphic sheaves over $\mathbb{H} \times \mathfrak{t}_\mathbb{C}/(W \ltimes \pi_1 T^2)$. Recall that every $\tau \in \mathbb{H}$ defines an isomorphism $\mathbb{C} \simeq \mathbb{R} \times \mathbb{R}$. Write ϕ for the composite (cf. [8] Display (16))

$$ \mathbb{H} \times \mathfrak{t}_\mathbb{C} \to \mathbb{H} \times \mathfrak{t} \times \mathfrak{t} \xrightarrow{id \times \exp} \mathbb{H} \times T \times T \xrightarrow{\mathrm{pr}} T \times T, $$

where the first map is the one induced by viewing $\mathbb{H}$ as a subspace of the space of $\mathbb{R}$-bases of $\mathbb{C}$ via $\tau \mapsto (1, \tau)$. For a G-space X and a point $a \in \mathbb{H} \times \mathfrak{t}_\mathbb{C}$ write X^a for the common fixed point set of the two components of $\phi(a)$, $Z(a) \subset G$ for their common centralizer, and W_a

[15] The 'twisted' here is unrelated to the 'twisted' in 'twisted K-theory.'

for the Weyl group of $Z(a)$ (which is a subgroup of the Weyl group of G). Write $H^*_T(-;\mathbb{C})$ for equivariant singular cohomology with complex coefficients and mod 2 grading and $\mathcal{O}$ for the sheaf of holomorphic functions on both $\mathfrak{t}_\mathbb{C}$ and $\mathbb{H}\times\mathfrak{t}_\mathbb{C}$. Let U be a small neighborhood of $0\in\mathfrak{t}_\mathbb{C}$. Since $H^*_T(\mathrm{pt};\mathbb{C})$ is canonically isomorphic to $\mathrm{Sym}(\mathfrak{t}^\vee_\mathbb{C}[-2])$ it acts on $\mathcal{O}(U)$. Since $H^*_{Z(a)}(\mathrm{pt};\mathbb{C})\simeq H^*_T(\mathrm{pt};\mathbb{C})^{W_a}$ the latter acts on $\mathcal{O}(U_a)^{W_a}$. Note that the second component of a acts on the second factor of $\mathbb{H}\times\mathfrak{t}_\mathbb{C}$ by translation. The action of W and $\pi_1 T$ on $\mathfrak{t}$ combine to give a $W\ltimes\pi_1T^2$ action on $\mathbb{H}\times\mathfrak{t}\times\mathfrak{t}$ that is trivial on the first factor. Pulling back along the map 'canonical' gives an action of $W\ltimes\pi_1T^2$ on $\mathbb{H}\times\mathfrak{t}_\mathbb{C}$. Let U_a be a sufficiently small $W\ltimes\pi_1T^2$-invariant neighborhood of a. For $\tau\in H^3_{S^1}(G/G)$, the associated quadratic form on $H_1(T)$ (cf. Definition 4.2.1) defines an element of $H^2(T\times T)$, and let $\mathcal{L}^\tau$ denote the associated 'Looijenga' line bundle. Define the sheaf ${}^\tau(Ell^{\mathrm{Groj}}_G)^*X$ at U_a to be

$$(Ell^{\mathrm{Groj}}_G)^*X(U_a) := (a^{-1})^*\Big(H^*_{Z(a)}(X^a;\mathbb{C})\otimes_{H^*_{Z(a)}(\mathrm{pt};\mathbb{C})} a^*(\mathcal{O}(U_a)\otimes\mathcal{L}(U_a))^{W_a}\Big)$$

The evident restriction maps for $W\ltimes\pi_1T^2$-invariant subsets $U_{a'}\subset U_a$ centered around $a'\in U_a$ are isomorphisms by the localization theorem in equivariant cohomology and the fact that fixed point sets can only shrink locally (i.e. if g does not stabilize x then it does not stabilize a neighborhood of x).

Lemma 4.3.6. *Let G be a connected Lie group, $k\in H^4BG$ such that $tr(k)\in H^3_{S^1}(G/G)$ is a strongly topologically regular, positive twist, and let X be a G-space. Then Grojnowski's sheaves ${}^k(Ell^{Groj}_G)^*X$ can be recovered from the presheaf of groups on $\mathcal{L}(X/G)$ defined by $U\mapsto {}^{tr(k)}\vec{K}^*_{S^1}(U)\otimes\mathbb{C}$.*

Proof. It suffices to show that the groups $H^*_{Z(a)}(X^a;\mathbb{C})$ and the Looijenga line bundle $\mathcal{L}^k$ can be recovered from the $\vec{K}^*_{S^1}(-)\otimes\mathbb{C}$ presheaf. For $a\in\mathbb{H}\times\mathfrak{t}_\mathbb{C}$ write $\phi_1(a)$ and $\phi_2(a)$ for the two components of $\phi(a)$, $Z(a_1)$ and $Z(a_2)$ for their centralizers in G, and X^{a_1} and X^{a_2} for their fixed point sets in X. Regard ordinary cohomology as $\mathbb{Z}/2$-graded. By the completion theorem in twisted K-theory ([**?**] Theorem 3.9)

$$H^*_{Z(a)}(X^a;\mathbb{C})\simeq K^*_{Z(a_1)}(X^{a_1})^\wedge_{a_2}.$$

Recall that $\mathcal{L}(X/G)$ is a full subgroupoid of $(X\times G)/G$ (where G acts diagonally by the given action on X and by conjugation on G) on those objects (x,g) such that $gx=x$. Note that $X^{a_1}/Z(a_1)\simeq(X^{a_1}\times\{a_1\})/Z(a_1)$ is a subgroupoid of $\mathcal{L}(X/G)$. Fractional splittings $\psi_{x,g}: G_{x,g}\leftarrow S^1_d$ can always be extended in the X-direction since the $B\mathbb{Z}$-action only depends on the second factor. Hence there is a fractional splitting ψ_{a_1} that gives an isomorphism

$${}^{\mathrm{tr}(k)}\vec{K}^*_{S^1}(X^{a_1}/Z(a_1))\simeq {}^{\mathrm{tr}(k)}K^*(X^{a_1}/Z(a_1))((q)).$$

The twist $\mathrm{tr}(k)$, which is pulled back from $\mathcal{L}(\mathrm{pt}/G)$ defines a trivial X^a-parametrized family of $U(1)$-central extensions of $Z(a)$ (cf. Lemma 4.2.3). Restricting to the fiber over $a_2\in Z(a)$ defines a trivial $U(1)$-principle bundle over X^a, whose associated trivial line bundle is denoted

$\mathcal{L}^{\mathrm{tr}(k)_a}$ and whose fiber will be denoted by $\mathbb{C}(a)$. Extract a single q-power from the right side of the previous display, complexify, and and apply the completion theorem in twisted K-theory ([?] Theorem 3.9) to obtain

$$^{\mathrm{tr}(k)}K^*(X^{a_1}/Z(a_1)) \otimes \mathbb{C}^{\widehat{}}_{a_2} \simeq H^*_{Z(a)}(X^a; \mathcal{L}^{\mathrm{tr}(k)_a}) \simeq H^*_{Z(a)}(X^a; \mathbb{C}(a)).$$

To finish the proof it suffices to show that the vector spaces $\mathbb{C}(a)$ assemble into the (W-equivariant) Looijenga line bundle $\mathcal{L}^k$ over $T \times T$ as a (or really $\phi(a)$) varies. First, since k is a class in H^4BG, it produces a Weyl-equivariant class on $T \times T$, and its transgression $\mathrm{tr}(k)$ produces a Weyl-equivariant twist. Furthermore, $\mathbb{C}(a)$ is the fiber over the morphism $\phi(a) \in T \times T \subset (G/G/B\mathbb{Z})_1$ of the line bundle associated to the $U(1)$-extension defined by the twist $\mathrm{tr}(k)$. That is, the $U(1)$-central extension defined by $\mathrm{tr}(k)$ coincides with the $U(1)$-bundle associated to the Looijenga line bundle $\mathcal{L}^k$ defined by k. □

When G is a torus or simple and simply-connected, Kitchloo [18] has constructed an equivariant elliptic cohomology theory over the complex numbers by defining certain $W \ltimes \pi_1 T^2$-equivariant holomorphic sheaves ${}^k\mathcal{K}^*_{LG \rtimes S^1} LX$ on $\mathbb{H} \times \mathfrak{t}_{\mathbb{C}}$ as follows. Let $\tau \in H^4BG \simeq \mathbb{Z}$ be an element whose transgression in $H^3_{S^1}(G/G)$ defines a strongly topologically regular, positive twist. Let $\mathcal{F}^G_\tau$ be the space of Fredholm operators on a Hilbert space $\mathcal{H}_\tau$ which is the direct sum of countably many copies of each irreducible level τ positive energy representation of LG. Let $\mathbb{H}$ be the upper half plane. For a finite G-CW space X, define the sheaves ${}^\tau\mathcal{K}^*_{S^1 \times T} LX$ on $\mathbb{H} \times \mathfrak{t}_{\mathbb{C}}$ by sheafifying the presheaves

$$U \mapsto \pi_{*\mathrm{mod}2}\mathrm{Maps}(U \times LX, \mathcal{F}^G_k)^T.$$

Then the ${}^\tau\mathcal{K}^*_{LG \rtimes S^1} LX$ are defined to be the sheaves of $\mathcal{O}(\mathbb{H} \times \mathfrak{t}_{\mathbb{C}})$-modules whose stalks at a point (h, a) are the following inverse limits over finite $S^1 \times T$-CW subspaces of LX:

$$\varprojlim_{Y \subset LX} ({}^\tau\mathcal{K}^*_{S^1 \times T} Y)_{(h,a)} \otimes_{R(S^1 \times T)} \mathcal{O}(\mathbb{H} \times \mathfrak{t}_{\mathbb{C}})_{(h,a)}.$$

Lemma 4.3.7. *Let G be a torus or simple and simply-connected Lie group, $0 < \tau \in H^4BG$ and X a G-space. Then Kitchloo's sheaves ${}^k\mathcal{K}^*_{LG \rtimes S^1} LX$ can be recovered from the the presheaf of groups on $\mathcal{L}(X/G)$ defined by $U \mapsto {}^{tr(k)}\vec{K}^*_{S^1}(U) \otimes \mathbb{C}$*

Proof. Similarly to the above, let $Y^a \subset Y$ denote the common fixed point set for $\phi_1(a)$, $\phi_2(a)$, and S^1 in $S^1 \times T$. By Theorem 3.3 of [18], the stalks defined above are isomorphic to the groups

$$\varprojlim_{Y \subset LX} ({}^\tau K^*_{S^1 \times T} Y^a) \otimes_{R(S^1 \times T)} \mathcal{O}(\mathbb{H} \times \mathfrak{t}_{\mathbb{C}})_{(h,a)}.$$

The proof is now complete since the classical twisted K-theory groups $({}^\tau K^*_{S^1 \times T} Y^a) \simeq {}^\tau K^*_T(Y^a)[q^{\pm}]$ can certainly be recovered from the indicated $\vec{K}^*_{S^1}$-theory presheaf as in the proof of Lemma 4.3.6. □

4.4 Example: $U(1)$

The groups ${}^\tau\vec{K}^*_{S^1}(U(1)/U(1))$ can be calculated directly. Note that the twist τ is an element of $H^3_{U(1)}U(1) \simeq \mathbb{Z}$. For twists transgressed from $H^4BU(1)$, i.e. classes in $2\mathbb{Z}$, the calculation is due to Constantin Teleman (unpublished notes). Here is an approach that covers all strongly topologically regular twists $\tau \neq 0$.

The general idea us to apply Mayer-Vietoris to the usual decomposition of $U(1)$ into two overlapping arcs U and V which are equivariantly contractible and whose intersection has equivariantly contractible connected components. The automorphism group of each of these four components is a $U(1)$-central extension of a $U(1)$-extension of S^1 (cf. Example 3.1.5, Lemma 4.2.3), which is non-canonically isomorphic to $U(1) \times U(1) \times S^1$. Applying Lemma 4.2.2 and Bott periodicity the Mayer-Vietoris sequence is a 6-term hexagon

$$\begin{array}{ccccc}
 & R_\star(U(1))((q))^{\oplus 2} & \xrightarrow{i^* - M^\tau} & R_\star(U(1))((q))^{\oplus 2} & \\
\nearrow & & & & \searrow \\
{}^\tau\vec{K}^0_{S^1}(U(1)/U(1)) & & & & {}^\tau\vec{K}^1_{S^1}(U(1)/U(1)) \\
\nwarrow & & & & \swarrow \\
 & 0 & \longleftarrow & 0 &
\end{array}$$

exhibiting the desired groups as the kernel and cokernel of the top horizontal map. Write $R(U(1)) \simeq \mathbb{Z}[t^\pm]$. From the explicit presentation of the twist $\mathcal{L}^\tau$ in Example 3.1.16 the identifications of the four $\vec{K}_{S^1}$-theory stalks with $R_\star(U(1))((q))$ can be chosen such that 3 of the restriction maps are the identity and the fourth sends an element $\sum \chi_k(t)q^k$ to $(qt)^\tau \sum \chi_k(qt)q^k$. Hence the map in question is

$$(i^* - M^\tau)\begin{bmatrix} f(t,q) \\ g(t,q) \end{bmatrix} = \begin{bmatrix} f(t,q) - g(t,q) \\ g(t,q) - (tq)^\tau g(qt,q) \end{bmatrix} \in \mathrm{End}\big(\mathbb{Z}[t^\pm]((q))\big)$$

When $\tau > 0$ there is no cokernel, and the kernel has a basis

$$f(t,q) = g(t,q) = \sum_{k\in\mathbb{Z}} t^{j+\tau k} q^{\tau\frac{k(k-1)}{2} - kj} \quad j = 0, ..., n-1$$

Some readers may notice that at $\tau = 1$ this is the character of the basic level 1 positive energy representation of $LU(1)$ up to a factor of the (shifted) partition function $\prod_{k>0}(1-q^k)^{-1}$ (cf. [1] Chapter 14 Section 1).

Chapter 5

Bordism theories

5.1 Introduction and technical setup

In this chapter I will use Bullett's manifolds-with-singularities ([10]) to prove a number of results in cobordism theory. The most basic application is the exhibition of E_∞-structures on various spectra via, of course, the formalism of symmetric stable functors. More advanced is the result that certain bordism spectra of manifolds-with-singularities together with their associated filtrations based on the "depth" of singularities coincide *as filtered spectra* with certain higher algebraic pushouts (which come with their own algebraically defined filtration). Applications of these considerations include a short and conceptual[1] calculation of the dual Steenrod algebras $(H\mathbb{F}_p)_*H\mathbb{F}_p$ at all primes, a solution to Bullett's conjecture, a non-existence result on E_∞ complex orientations, some identifications of framed bordism classes, and an approach to the nilpotence theorem.

Definition 5.1.1. (cf. [9] Def. 1.4) A manifold-with-(unlabelled)-n-corners M of dimension d is a "manifold" modeled on the spaces $\mathbb{R}^d$ and $U_j = \{x \in \mathbb{R}^d | x_1, ..., x_j \geqslant 0\}$ for $j = 1, ..., n$. The strata of each of the model spaces induce strata in M. The stratum of codimension j is called the j-corners. The complement of the $j+1$-corners inside the j-corners is called the *smooth j-corners of M*. As the name suggests, it is a smooth manifold. If the only nonempty strata are of codimension 0 and 1 then M is a classical manifold-with-boundary. For $j > 1$, the j-corners is a subset of a j-fold self intersection of the 1-corners. More precisely, at each point p in the smooth j-corners there is a neighborhood U such that the intersection of the 1-corners with U can be written as the union of j hyperplanes whose j-fold intersection is the intersection of U with the j-corners. Thus, locally along a j-fold self intersection there are j well-defined components of the 1-corners which meet there, but this may not be possible globally. However this defines a Σ_j-bundle over the smooth j-corners which is called the *face-labelling bundle*, and whose fiber over p is the set of hyperplanes in U mentioned above.

[1] In particular the calculation does not make use of any particular presentation of that Hopf algebra nor any reference to the Steenrod operations.

A manifold-with-*labelled*-corners is a manifold-with-corners Q together with a *labelling* of the smooth 1-corners, which is a decomposition into disjoint smooth open subsets $d_1Q, ..., d_qQ$ such that the following holds: for each point p in the smooth j-corners, the fiber of the face-labelling bundle acquires a possibly-multivalued map to $\{1, 2, ..., q\}$ and this map is required to be an isomorphism (and therefore singly-valued). Note that if Q admits a valid labelling then $q \geqslant r'$ where r' is the maximal r for which the smooth r-corners of Q are nonempty.

Remark 5.1.2. Consider the solid square. Its interior is the smooth 0-corners, its boundary is the 1-corners. The complement of the 4 colloquial-corners in the boundary is the smooth 1-corners—it is a union of 4 lines. The 2-corners coincides with the smooth 2-corners, and is the union of the 4 corner points. Labelling the 4 components of the smooth 1-corners 1-through-4 exhibits the solid square as a manifold-with-labelled-corners.

Consider the usual drawing of a teardrop. Its interior is the smooth 0-corners, diffeomorphic to an open disk. Its boundary is the 1-corners, its smooth 1-corners are the complement of the tip point in the 1-corners and is diffeomorphic to a line, and the 2-corners is that tip point. It admits no valid labelling since the 2-corners is a self intersection of the smooth 1-corners and so no labelling can distinguish the two local parts of the 1-corners intersecting there.

Note that the teardrop is the gluing of a solid triangle and a solid semicircle along their intersection in a line segment, all three of of which are manifolds-with-labelled-singularities, but the gluing does not respect the labelling.

5.2 The dual Steenrod algebra

All discrete rings, modules, etc are implicitly graded and are also coconnective in cohomological grading. All rings are $\mathbb{F}_2$-algebras and are commutative.

5.2.1 Real orientations

Definition 5.2.1. Let E be a complex oriented homotopy ring spectrum. A *real orientation* of E consists of the data of

1. a class $x \in E^1BC_2$ such that x restricts to a generator of E^1S^1 along the inclusion of the bottom cell of BC_2.

Remark 5.2.2. The prime examples of a real orientable spectrum are the Eilenberg-Maclane spectrum $H\mathbb{F}_2$ and unoriented bordism MO.

Remark 5.2.3. Note that data of the class x is equivalent to a factorization of the unit map through the bottom cell $\mathbb{S} \to \Sigma^{-1}BC_2$.

Remark 5.2.4. Just as a complex oriented homotopy ring spectrum E exhibits a formal group law $E^*BC_2 = E^*[[c]]$ on a generator c of degree 2, a real oriented homotopy ring spectrum exhibits a formal group law $E^*BC_2 = E^*[[x]]$ on a generator x of degree 1 (the same x as in

Definition 5.2.1). In what follows, all formal group laws will be implicitly of the latter kind. Moreover an FGL $F(x, y)$ over a ring R will often be considered implicitly as the R-algebra map

$$F : R[[x]] \to R[[x, y]]$$

determined by $x \mapsto F(x, y)$

5.2.2 FGL automorphisms

Definition 5.2.5. Let $\mathrm{CRings}_{\leqslant 0}$ denote the category of graded commutative coconnective rings.

Definition 5.2.6. Recall that a morphism between two formal group laws $F_A(x, y)$ and $F_B(x, y)$ over a ring R is[2] a power series $\phi(x) \in R[[x]]$ such that $F_B(\phi(x), \phi(y)) = \phi(F_A(x, y))$. This already implies that the constant term of ϕ is zero.

Remark 5.2.7. As one might expect, the FGL associated to any real orientation of $H\mathbb{F}_2$ is the additive FGL $F_+(x, y) = x + y$.

Definition 5.2.8. Let $\mathbb{F}_2\text{-}\mathrm{Alg}_{\leqslant 0}$ denote the category of (graded commutative coconnective) $\mathbb{F}_2$-algebras.

Definition 5.2.9. Let F be an FGL over a ring R. The *automorphism groupoid* $\mathrm{Aut}\, F$ is the functor

$$\mathrm{Aut}\, F : \mathbb{F}_2\text{-}\mathrm{Alg}_{\leqslant 0} \to \mathrm{Groupoids}$$

defined as follows: the set of objects of the groupoid $\mathrm{Aut}\, F(S)$ is the set of ring maps $R \to S$ and the set $\mathrm{Aut}\, F(S)(f, g)$ of morphisms from f to g is the set of R-algebra maps $\phi : R[[x]] \to R[[x]]$ such that (cf. Remark 5.2.4)

$$f_* F \circ \phi = (\phi \otimes \phi) \circ g_* F$$

along with the normalization condition

$$\phi(x) = x + \sum_{i=1}^{\infty} a_i x^{i+1}.$$

The composition of morphisms is the composition of R-algebra maps.

Remark 5.2.10. When $F = x + y$ is the additive FGL one finds that $\phi(x + y) = \phi(x) + \phi(y)$ which implies that

$$\phi(x) = x + \sum_{i=1}^{\infty} a_i x^{2^i}.$$

[2]Recall the conventions that x and y both have degree 1 and R is a graded, commutative, coconnective $\mathbb{F}_2$-algebra.

Definition 5.2.11. Let F be an FGL over R. A *coordinate transformation* is and R-algebra map $\phi : R[[x]] \to R[[x]]$ such that

$$\phi(x) = x + \sum_{i=1}^{\infty} d_i x^{i+1}.$$

One says that ϕ *transforms* F into the FGL $\phi^* F := (\phi^{-1} \otimes \phi^{-1}) \circ F \circ \phi$. Note that coordinate transformations are also covariant. If $f : R \to S$ is a ring map then $f_* \phi$ is the S-algebra map $S[[x]] \to S[[x]]$ determined by setting $f_* \phi(x)$ to be the power series gotten by applying f to the coefficients of $\phi(x)$.

5.2.3 The derivation

Definition 5.2.12. Define the category MT as follows. Its objects are triples consisting of

1. a (contravariant) functor $X \mapsto E^*(X)$ from the category of finite CW complexes to the category of graded abelian groups,
2. a natural isomorphism from $E^{*+1}(\Sigma X)$ to $E^*(X)$, and
3. an associative "multiplication map" $E^*(X) \otimes E^{*'}(Y) \to E^{*+*'}(X \times Y)$.

Note that with the diagonal map $X \to X \times X$ the data of 3. makes $E^* X$ into a graded ring. Morphisms $T : E \to F$ are natural transformations of functors that commute with the data in 2. and 3. The component of a morphism T at a CW complex X is denoted T_X.

Remark 5.2.13. The first examples of objects of MT are those that are induced by multiplicative cohomology theories, i.e. homotopy ring spectra. In fact, MT is an acronym for *multiplicative theory*, and they are meant to capture multiplicative cohomology theories (MCTs) without the exactness axiom, which is the only Eilenber-Steenrod axiom which is not preserved under tensor product (cf. Definition 5.5.9).

Remark 5.2.14. Although the objects of MT are defined as functors out of finite spectra, when working with a fixed object one can often enlarge the domain quite a bit—namely to those spectra for which the relevant $\lim^1$-term vanishes. For all objects of MT considered in this paper (which either come from mod p oriented spectra or are tensored from them (cf. Lemma 5.2.17)), that includes the non-finite spectra BC_2 and MO.

Definition 5.2.15. (cf. Remark 5.2.13) For an object E of MT and an E^*-algebra with unit $u : E^* \to R$ let $E_u R$ be the object of MT defined by the formula

$$E_u R^* X = E^* X \otimes_{E^*} R.$$

When the map u is understood I often abbreviate $E_u R$ to ER.

Lemma 5.2.16. *(Evaluation at BC_2) Let E be a real-oriented homotopy ring spectrum with $E^*BC_2 \simeq E^*[[x]]$ and FGL F_E. Let R be an E^*-algebra with unit map $u : E^* \to R$. Then for every $T \in Hom_{MT}(E, E_uR)$ there is a unique $\phi_T \in \operatorname{Aut} F_E(R)(T_{pt}, u)$ such that $\phi_T(x) = (T_{BC_2}(x)$.*

Proof. Let T be an element of $\mathrm{Hom}_{MT}(E, ER)$. To streamline notation, let f denote the ring map $T_{\mathrm{pt}} : E^* \to R$. First note that uniquely determines a morphism $\tau \in \mathrm{Hom}_{MT}(E_fR, E_uR)$ by the diagram

$$E^*X \otimes_{E^*} R_f \xrightarrow{T\otimes\mathrm{id}} E^*X \otimes_{E^*} R_u \otimes_{E^*} R_f \xrightarrow{\mu_R} E^*X \otimes_{E^*} R_u.$$

Note that the restriction of τ along the map $E \to E_uR$ induced by u recovers T. Now consider the diagram

$$\begin{array}{ccc} E_fR^*BC_2 & \xrightarrow{\mu^*} & E_fR^*BC_2^{\times 2} \\ \downarrow{\scriptstyle \tau_{BC_2}} & & \downarrow{\scriptstyle \tau_{BC_2^{\times 2}}} \\ E_uR^*BC_2 & \xrightarrow{\mu^*} & E_uR^*BC_2^{\times 2} \end{array}$$

which commutes by naturality of τ. Write ϕ for the map $R[[x]]$ induced by τ_{BC_2}. By multiplicativity of τ, one finds that the diagram above gives rise to the equation

$$f_*F_E \circ \phi = (\phi \otimes \phi) \circ u_*F_E.$$

In particular $\phi(x) = \sum_{i=1}^{\infty} a_i x^i$ (cf. Definition 5.2.9). The leading coefficient a_1 is forced to be 1 by considering the pullback along $S^1 \to BC_2$. The proof is completed by noting that ϕ is uniquely determined by $\phi(x)$, which is equal to $T_{BC_2}(x)$. □

Lemma 5.2.17. *(Quillen functor for MO) Let x be any choice of universal real orientation class in MO^*BC_2, with corresponding FGL F. Let $M^* \subset MO^*$ be the subring generated by the coefficients of F and write F_{M^*} for the corresponding FGL over M^*. Then for every ring A there is a functor $\gamma : \operatorname{Aut} F_{M^*}(A) \to MT$ (cf. Definitions 5.2.9 and 5.2.12) which on objects sends $f : M^* \to A$ to the theory $\gamma(f)^*X := MO^*X \otimes_{M^*} A$.*

Proof. The functor γ has been defined on objects, so it remains to specify it on morphisms. Let $f, g : M^* \to A$ bet two objects and let ϕ be a morphism in $\operatorname{Aut} F_{M^*}(A)(f, g)$. When we need to distinguish between the two M^*-module structures on A we will write A_f and A_g. Let x_f and x_g denote the classes in $\gamma(f)^*BC_2$ and $\gamma(g)^*BC_2$ which are the image x under the canonical maps $MO \to \gamma(f)$ and $MO \to \gamma(f)$. We will construct (functorially) a transformation $\gamma_\phi : \gamma(f) \to \gamma(g)$ such that $\gamma_\phi(x_f) = \phi(x_g)$.

Now MO_*MO is free over MO_*. In fact there is a canonical isomorphism of MO_*MO with $MO_*[a_1, a_2, ...]$ with $|a_i| = i$ (cf. Remark 5.2.24). Using the map $MO \simeq \mathbb{S} \otimes MO \to MO \otimes MO$ we get a morphism in MT

$$MO^*X \to (MO \otimes MO)^*X \simeq MO^*X \otimes_{MO^*} MO_*MO \to MO^*X \otimes_{\mathbb{F}_2} \mathbb{F}_2[a_i].$$

By Remark 5.2.24 when $X = BC_2$ the image of x under that map is

$$x \mapsto x \otimes 1 + x^2 \otimes a_1 + x^3 \otimes a_2 + ...$$

The morphism ϕ determines an $\mathbb{F}_2$-algebra map $\Phi : \mathbb{F}_2[a_i] \to A$ sending a_i to the coefficient of x^{i+1} in $\phi(x)$. Composing with the previous display gives another morphism $T_\phi : MO \to \gamma(g)$ in MT

$$MO^*X \to MO^*X \otimes_{\mathbb{F}_2} \mathbb{F}_2[a_i] \xrightarrow{\mathrm{id}\otimes\Phi} MO^*X \otimes_{\mathbb{F}_2} A \to MO^*X \otimes_{M^*} A_g = \gamma(g)^*X$$

with the property that when $X = \mathrm{pt}$ the induced map $MO^* \xrightarrow{(T_\phi)_{\mathrm{pt}}} A$ is equal f when restricted to M^*. Indeed, by Remark 5.2.24 T_ϕ determines a morphism ϕ_{T_ϕ} in $\mathrm{Aut}\, F_{M^*}(A)((T_\phi)_{\mathrm{pt}}, g$ such that $\phi_{T_\phi}(x) = \phi(x)$. On the other hand, ϕ was by definition a morphism from f to g. So the FGLs $T_\phi(\mathrm{pt})_*F$ and f_*F are identical. Since the coefficients of F generate M^* the maps f and $T_\phi(\mathrm{pt})$ must agree on M^*. It follows that T_ϕ descends to a morphism $\gamma_\phi : \gamma(f) \to \gamma(g)$ which can be written explicitly as the following composition, using the multiplication μ_f on A as an M^*-algebra via f

$$\begin{array}{ccc} \gamma(f)^*X = MO^*X \otimes_{M^*} A_f & \xrightarrow{T_\phi \otimes id} & \gamma(g)^*X \otimes_{M^*} A_f \\ & \swarrow \sim & \\ MO^*X \otimes_{M^*} A_g \otimes_{M^*} A_f & \xrightarrow{\mu_f} & MO^*X \otimes_{M^*} A_g = \gamma(g)^*X \end{array}.$$

Functoriality—the claim that $\gamma(\phi \circ \psi) = \gamma(\phi) \circ \gamma(\psi)$—is proved by noting that $\gamma(\phi)$ is uniquely characterized by the properties of being A-linear, multiplicative, and its behavior at $X = BC_2$, i.e. sending x_f to $\phi(x_g)$. Indeed, note that $\gamma(\phi)$ is characterized by its restriction along the surjection $MO^*X \otimes_{\mathbb{F}_2} A \to MO^*X \otimes_{M^*} A_f$. By A-linearity that in turn is determined by restriction along $MO^*X \to MO^*X \otimes_{\mathbb{F}_2} A$ (which coincides with the composite of the first two maps that make up T_ϕ). By Remark 5.2.24 the latter is determined by its behavior at $X = BC_2$. □

Lemma 5.2.18. *(Kill the FGL to get a summand) Let x be any choice of universal real class in MO^*BC_2, with corresponding FGL F. Let $M^* \subset MO^*$ be the subring generated by the coefficients of F and let F_{M^*} be the restriction of F to M^*. Set $N^* = MO^* \otimes_{M^*} \mathbb{F}_2$. Then $MN^*X := MO^*X \otimes_{M^*} \mathbb{F}_2$ is a cohomology theory and a summand of MO^*X, and there is a ring isomorphism $MO^* \simeq N^* \otimes_{\mathbb{F}_2} M^*$*

Proof. Because real line bundles are their own tensor inverse, we we know that F and F_{M^*} have vanishing 2-series and are therefore isomorphic to the additive FGL. That means there is a morphism between the objects $\mathrm{id} : M^* \to M^*$ and $p : M^* \to \mathbb{F}_2 \to M^*$ in $\mathrm{Aut}\, F_{M^*}(M^*)$.

Let ϕ be such an isomorphism. The functor γ of Lemma 5.2.17 provides an isomorphism $\gamma(\phi)$ between $\gamma(\mathrm{id})$ and $\gamma(p)$ in MT. But $\gamma(\mathrm{id}) \simeq MO$ so we have

$$MO^*X \simeq \gamma(\mathrm{id})^*X \xrightarrow{\simeq} \gamma(p)^*X = MO^*X \otimes_{M^*} \mathbb{F}_2 \otimes_{\mathbb{F}_2} M^*.$$

Therefore $MN^*X := MO^*X \otimes_{M^*} \mathbb{F}_2$ is a summand of a cohomology theory (namely MO) and hence a cohomology theory itself. The ring isomorphism at the end of the lemma statement is the displayed diagram when $X = \mathrm{pt}$. □

Lemma 5.2.19. *(Quillen functor for summands) Let E be a real oriented homotopy ring spectrum which is a summand of MO. Let x be any choice of universal class in E^*BC_2, with corresponding FGL F. Let $M^* \subset E^* \subset MO^*$ be the subring generated by the coefficients of F and let F_{M^*} be the restriction of F to M^*. Then for every ring A there is a functor $\gamma : \mathrm{Aut}\, F_{M^*}(A) \to MT$ which on objects sends $f : M^* \to A$ to $X \mapsto E^*X \otimes_{M^*} A$.*

Proof. The proof is nearly identical to that of Lemma 5.2.17 except that the first displayed morphism in MT, now induced by the map $E \simeq \mathbb{S} \otimes E \to E \otimes E \to E \otimes MO$ becomes

$$E^*X \to (E \otimes MO)^*X \simeq E^*X \otimes_{\mathbb{F}_2} \mathbb{F}_2[a_i]$$

and in the rest of the proof every instance of MO is replaced with E. □

Lemma 5.2.20. *$H\mathbb{F}_2$ is a summand of MO.*

Proof. Fix notation as in the Lemma 5.2.17. Invoke Lemma 5.2.18 to get a summand MN of MO and a decomposition $MO^* \simeq N^* \otimes M^*$. MN is real oriented via its map from MO and the corresponding FGL F_{N^*} is by construction the additive one. Perform a coordinate transformation (cf. Definition 5.2.11) so that the new FGL F'_{N^*} is not the additive one. Let $M_1^* \subset N^*$ be the subring generated by the coefficients of F'_{N^*}. Since MN^* is a summand of MO we can invoke Lemmas 5.5.13 and 5.2.18 to obtain a summand MN_1 of MN, and then again (after another coordinate transformation as above) to obtain a summand MN_2 of MN_1, and so on. Now, since coordinate transformations are covariant (cf. Definition 5.2.11) and MO^* is finitely generated in each degree, the process can be reordered such that for each fixed k, B^k stabilizes after a finite number of steps. So after a transfinite process we arrive at a summand MB^*, and it must be that every coordinate transformation is an automorphism of the additive FGL over B^*. Considering coordinate transformations of the form $\phi(x) = x + bx^n$ shows that every element of B^* must be in degree $1 - 2^k$ for some $k \geqslant 0$. Moreover we have a decomposition $MO^* \simeq B^* \otimes G^*$ where $1 \otimes G^*$ contains all those elements that were coefficients of some FGL that was used along the way. Let $z = [\mathbb{RP}^2] \neq 0 \in MO^{-2}$. Since $MO^{-1} = 0$ we know that z decomposes as $1 \otimes w \in B^* \otimes G^*$. Let b be an element of minimal negative degree in B^* for all possible transfinite processes. Then $(b \otimes 1)z = b \otimes w$ is not in degree $1 - 2^k$ for any k and hence not in B^*, so we can re-run the transfinite process above but ensuring that we use the coordinate transformation $x + (b \otimes w)x^{2l}$ to get a new decomposition $MO^* \simeq B_1^* \otimes G_1^*$ with $(b \otimes 1)z$ in G_1^*. That is, under the composite of ring isomorphisms

$$B^* \otimes G^* \xrightarrow{\sim} MO^* \xrightarrow{\sim} B_1^* \otimes G_1^*$$

$b \otimes w$ gets sent to $1 \otimes g$. On the other hand $b \otimes 1$ must be sent to $c \otimes 1$ by minimality of the degree of b and covariance of coordinate transformations, and since $(b \otimes w)(b \otimes 1) = (b^2 \otimes w) = 0 \otimes w = 0$, we find that $c \otimes g = 0$ which is a contradiction. So B^* has no negative degree elements and is therefore concentrated in degree zero, which means that MB is $H\mathbb{F}_2$. □

Theorem 5.2.21. *The cogroupoid object* $\mathbb{F}_2 \rightrightarrows (H\mathbb{F}_2)_* H\mathbb{F}_2$ *corepresents the groupoid valued functor* $\operatorname{Aut} F_+$.

Proof. Let $H = H\mathbb{F}_2$ and let R be an $\mathbb{F}_2$-algebra. Since every module over $\mathbb{F}_2$ is flat, note that $\operatorname{Hom}_{MT}(H, HR)$ is precisely the subset of HR^0H consisting of homotopy ring maps. Applying the universal coefficient theorem provides an isomorphism of sets, natural in R

$$\operatorname{Hom}_{MT}(H, HR) \simeq \operatorname{Hom}_{\mathbb{F}_2\text{-Alg}_{\geqslant 0}}(H_*H, R).$$

Upgrade it isomorphism of groups by simply defining the group structure on the left to be the one induced via this isomorphism by the group structure on the right hand side, which comes the coproduct of H_*H.

Let u be the unit map of R. By Lemma 5.2.16 evaluation at BC_2 defines a group homomorphism

$$\zeta^H : \operatorname{Hom}_{MT}(H, HR) \to \operatorname{Aut} F_H(R)(u, u)$$

that is natural in R. By Remark 5.2.24 the analogous map

$$\zeta^{MO} : \operatorname{Hom}_{MT}(Mp, MA) \to \operatorname{Aut} F_{Mp}(A)(u, u)$$

is injective for an MO^*-algebra with unit u. Hence the analogous map ζ^E is injective for any summand of E of MO, and H is such a summand by Lemma 5.2.20. The map ζ^E has an inverse for any summand of MO, given by sending ϕ to T_ϕ defined in the proof of Lemma 5.2.17. Finally, the map is one of groups because the inverse $\phi \mapsto T_\phi$ is, since the group structure on the left comes from $\mathbb{S} \otimes H \to H \otimes H$ and the group structure on the right comes from $\mathbb{S} \otimes MO \to MO \otimes MO$.

Finally, since the groupoid $\operatorname{Aut} F_H(R)$ has one object when R is an $\mathbb{F}_2$-algebra and zero objects otherwise, combining the two natural group isomorphisms above finishes the proof. □

Remark 5.2.22. Note that we have arrived at a *coordinate-free* calculation of $(H\mathbb{F}_2)_* H\mathbb{F}_2$: we have identified it (as a Hopf algebra) to be whatever Hopf algebra corepresents $\operatorname{Aut} F_+$. Giving a presentation by generators and relations (e.g. (duals to) Steenrod operations) is a simple but tedious algebraic problem. For the sake of completeness I will give the usual description.

Definition 5.2.23. Define the Hopf algebra $\mathcal{A}$ as follows.

$$\mathcal{A} := \mathbb{F}_2[\xi_1, \xi_2, ...] \; |\xi_i| = 2(p^k - 1),$$

$$\Delta(\xi_i) = \sum_{l=1}^{n} \xi_{i-l}^{2^l} \otimes \xi_l,$$

It is a straightforward algebra exercise to check that the cogroupoid object $\mathbb{F}_2 \rightrightarrows \mathcal{A}$ corepresents the automorphism groupoid of the additive FGL Aut F_+.

Remark 5.2.24. In the above I have used some key facts about MO which reference to this remark. There are two reasons why I won't include their proofs. Firstly, they are well known enough that many readers will know the proofs or know where to find them. Secondly, a later section of this thesis covers the odd primary dual Steenrod algebras and does contain the proofs of the odd primary analogs of all the relevant facts. Those proofs are similar but strictly harder, so the interested reader can tease out the $p = 2$ versions from those if they really feel like it (they can also email me).

5.3 Shaun Bullett's mod p bordism spectra

5.3.1 mod p orientations

Definition 5.3.1. Let E be a complex oriented homotopy ring spectrum. A *mod p orientation* of E consists of the data of

1. a complex orientation of E with universal chern class $c_E \in E^2BU(1)$
2. a class $e \in E^1BC_p$ such that if $i : BC_p \to BU(1)$ denotes the inclusion and $x = i^*c_E$ then $E^*BC_p \simeq E^*[[x, e]]/(e^2)$ and e restricts to a generator of E^1S^1.

Remark 5.3.2. The prime example of a mod p orientable spectrum is the Eilenberg-Maclane spectrum $H\mathbb{F}_p$.

Remark 5.3.3. Note that the data of the class c_E is equivalent to a homotopy class of factorization of the unit map $\mathbb{S} \to E$ through the bottom cell $\mathbb{S} \to \Sigma^{-2}BU(1)$. Similarly the class e in item 2 is equivalent to a factorization of the unit map through the bottom cell $\mathbb{S} \to \Sigma^{-1}BC_p$.

Remark 5.3.4. Let E be a mod p oriented homotopy ring spectrum with formal group law F (coming from the complex orientation). Then the p series $[p]_F(x)$ is zero, because the pullback $i^* : E^*\mathbb{CP}^\infty \to E^*BC_p$ kills the p series but is also injective. In particular $p = 0 \in \pi_0E$.

5.3.2 $\mathbb{F}_p$-formal group laws

Definition 5.3.5. Let $\mathrm{CRings}_{\leqslant 0}$ denote the category of graded commutative coconnective rings.

Definition 5.3.6. Recall that a morphism between two formal group laws $F_A(x, y)$ and $F_B(x, y)$ over a ring R is a power series $\phi(x) \in R[[x]]$ such that $F_B(\phi(x), \phi(y)) = \phi(F_A(x, y))$. This already implies that the constant term of ϕ is zero.

Definition 5.3.7. (cf. [9] 1.8) An $\mathbb{F}_p$-*formal group law* (or $\mathbb{F}_p$-FGL for short), denoted by $\mathbf{F}$ or (F_1, F_2), over a (co-connective graded commutative) $\mathbb{F}_p$-algebra[3] R is a pair of power series $F_1, F_2 \in R[[x_1, x_2, e_1, e_2]]$ $|x_i| = 2$ and $|e_i| = 1$, with F_1 and F_2 of homogeneous degree 1 and 2, such that the following hold: write $\xi_i = (x_i, e_i)$ and $\mathbf{F}(\xi_1, \xi_2) = (F_1(\xi_1, \xi_2), F_2(\xi_1, \xi_2))$. Then

1. $\mathbf{F}(0, \xi) = F(\xi, 0) = 0$ (identity),
2. $\mathbf{F}(\xi_1, \mathbf{F}(\xi_2, \xi_3)) = \mathbf{F}(\mathbf{F}(\xi_1, \xi_2), \xi_3)$ (associativity),
3. $\mathbf{F}(\xi_1, \xi_2) = \mathbf{F}(\xi_2, \xi_1)$ (commutativity),
4. the p-fold iterate $\mathbf{F}(\xi, \mathbf{F}(\xi, ..., \xi)...)$ is zero (p-series is zero),
5. F_2 is independent of e_1 and e_2 (ordinary formal group law).

In other words, an $\mathbb{F}_p$-formal group law over R is an R-algebra map

$$\mathbf{F} : R[[x, e]] \to R[[x, e]] \otimes_R R[[x, e]]$$

determined by $\mathbf{F}(e) = F_1$ and $\mathbf{F}(x) = F_2$, and each of the 5 conditions above corresponds to a commutative diagram involving the map $\mathbf{F}$ and the augmentation $R[[x, e]] \to R$ (1), iterates of $\mathbf{F}$ (2,4), the swap map of $R[[x, e]] \otimes_R R[[x, e]]$ (3), and the inclusion $R[[x]] \to R[[x, e]]$ (5).

Finally note that $\mathbb{F}_p$-FGLs are covariant under ring maps: if $f : R \to S$ is a ring map, then $f_*\mathbf{F}$ is the $\mathbb{F}_p$-FGL over S presented by the pair (f_*F_1, f_*F_2), where f_*F_i is the element of $S[[x_1, x_2, e_1, e_2]]$ gotten by applying f to the coefficients of F_i.

Remark 5.3.8. The prime example of an $\mathbb{F}_p$-FGL is $\mathbf{F}_+$, the *additive* $\mathbb{F}_p$-FGL (which exists over any $\mathbb{F}_p$-algebra R). It is presented by the pair $(F_1, F_2) = (e_1 + e_2, x_1 + x_2)$. Indeed, the definition of an $\mathbb{F}_p$-FGL is due to Bullett, first appearing in [10] (Definition 3.4.1) in which he states that the 5 conditions are directly motivated by properties of the structure present on $H\mathbb{F}_p^* BC_p$.

[3]Note that Condition 4 forces the equation $p = 0$ in R, so the requirement that R be an $\mathbb{F}_p$-algebra is redundant and only there for emphasis.

Just as every complex oriented ring spectrum determines a formal group law, so does every mod p oriented ring spectrum determine an $\mathbb{F}_p$-FGL.

Lemma 5.3.9. *Let E be a mod p oriented ring spectrum with $E^*BC_p \simeq E^*[[x, e]]$. Let $\mu : BC_p^{\times 2} \to BC_p$ be the multiplication map. Then the pair $(F_1, F_2) = (\mu^* e, \mu^* x)$ defines an $\mathbb{F}_p$-FGL $\mathbf{F}_E$ over E^*.*

Proof. Note that $\mu^* e$ and $\mu^* x$ are elements of $E^*BC_p^{\times 2} \simeq E^*[[x_1, x_2, e_1, e_2]]$ of homogeneous degree 1 and 2. The 5 conditions required to be an $\mathbb{F}_p$-FGL follow from corresponding properties of the multiplication map μ and the inclusion $BC_p \to BU(1)$. □

Remark 5.3.10. As one might expect, the $\mathbb{F}_p$-FGL associated to any mod p orientation of $H\mathbb{F}_p$ is the additive $\mathbb{F}_p$-FGL $\mathbf{F}_+$ (cf. 5.3.8).

As in the case of ordinary formal group laws, it is clear that there is a ring carrying a universal formal group law—it is the quotient of a big old free graded-commutative algebra on generators representing the coefficients of F_1 and F_2 modulo the relations imposed by the five conditions above. The difficulty is calculating what that ring really looks like, just like in the case of ordinary formal group laws and Lazard's theorem.

Definition 5.3.11. Define the mod p Lazard ring L_p to be the ring carrying the universal $\mathbb{F}_p$-formal group law.

Theorem 5.3.12. *(Bullet [9] 1.10)*

$$L_p \simeq \mathbb{F}_p[a_p, b_r, s_r] \ \ |a_p| = 2p, \ |b_r| = 2r, \ |s_r| = 2r + 1, \ p, r > 0, \ r \neq p^k - 1.$$

5.3.3 mod p bordism spectra

In this section I recall Bullett's construction of some mod p bordism spectra. See [9] and Chapter 4 of [10] for details.

Definition 5.3.13. An ${}^nV_\infty$-manifold is a manifold-with-j-corners for $0 \leqslant j \leqslant n$, together with a MU-structure on the interior of the codimension 0 stratum, a free C_p action on the 1-corners preserving the MU-structure induced there, and such that on the j-corners the C_p action combines to a free $C_p^j \rtimes \Sigma_j$ action on the face-labelling bundle (the semidirect product is the one associated to the permutation action of Σ_j on C_p^j).

Definition 5.3.14. (cf. [9] Definition 1.3) Define the spectrum ${}^nV_\infty$ to be the bordism theory of ${}^nV_\infty$-manifolds. More precisely, ${}^nV_\infty$ is the stable homotopy type representing the following cohomology theory on manifolds: for a manifold X, the group ${}^nV_\infty^k(X)$ is the set of *cobordism* (defined shortly) classes of $\dim X - k$-dimensional ${}^nV_\infty$-manifolds Q with an ${}^nV_\infty$-oriented, proper map $f : Q \to X$, which means that Q is presented as a submanifold of $\mathbb{R}^\infty \times X$ with a MU-structure on its normal bundle (on the interior of the zero-corners) and f is the projection to X. Two such data (Q, f) and (Q', f') are *cobordant* if there is a dim$X-k+1$-dimensional ${}^nV_\infty$-manifold R with an ${}^nV_\infty$-oriented, proper map $f: R \to X \times \mathbb{R}$ that is transverse to $X \times \{0\}$ and $X \times \{1\}$ and whose pullbacks over those submanifolds are identified with (Q, f) and (Q', f').

Definition 5.3.15. Define V_∞ as the colimit of $MU \simeq {}^0V_\infty \to {}^1V_\infty \to {}^2V_\infty \to ...$ where the connecting maps are the maps induced by regarding an ${}^nV_\infty$-manifold as an ${}^{n+1}V_\infty$-manifold with empty $(n+1)$-corners.

Lemma 5.3.16. *V_∞ is mod p oriented (cf. Definition 5.3.1).*

Proof. Since V_∞ is canonically complex-oriented by construction (it is an MU-algebra), it suffices to factor the unit map of V_∞ through the map $\mathbb{S} = \Sigma^{-1}S^1 \to \Sigma^{-1}BC_p$, which will define the required universal class $e \in V^1_\infty BC_p$ (cf. Remark 5.3.3). Since the unit map $\mathbb{S} \to V_\infty$ factors as $\mathbb{S} \to^1 V_\infty$ it suffices to factor the former through $\mathbb{S} \to \Sigma^{-1}BC_p$. Let $\sim$ be the equivalence relation on ∂D^{2n} defined by the standard action of C_p on odd spheres. Then $D^{2n}/\sim$ defines a ${}^1V_\infty$-manifold, and the map $D^{2n}/\sim \hookrightarrow S^{2n+1}/C_p$ defines a class $e_n \in {}^1V^1_\infty(S^{2n+1}/C_p)$. The desired factorization comes from the class $e \in {}^1V^1_\infty(BC_p)$ defined as the limit of the e_n. □

Definition 5.3.17. (cf. [9] page 14) Let $(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n}$ be the homotopy quotient of $(MU \otimes BC_p)^{\otimes_{MU} n}$ by the permutation action of Σ_n. Geometrically $(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n}$ is the spectrum associated to the bordism theory of MU-manifolds with free a $C_p^n \rtimes \Sigma_n$-action such that C_p^n preserves the MU-structure and Σ_n acts by the sign representation it.

Definition 5.3.18. (cf. [9] page 11) Let $C_p^{\star n}$ be the 'unravelled' n-fold join of C_p, presented as the subset of $(C_p \times \mathbb{R})^n$ consisting of those $(z_1, t_1, ..., z_n, t_n)$ such that the t_i are nonnegative and sum to 1. That admits a natural action of $C_p^n \rtimes \Sigma_n$ considered as the group of $n \times n$ permutation matrices with values in $\mathbb{F}_p \simeq C_p$. Note that $C_p^{\star n}$ is the ${}^{n-1}V_\infty$-manifold defined by $C_p^n \times \Delta^{n-1}_{\text{top}}$ together with the C_p-action that over the ith face of $\Delta^{n-1}_{\text{top}}$ acts by translation on the ith factor of C_p^n.

Remark 5.3.19. In the language of [9], $C_p^{\star n}$ is the ${}^nV_\infty$-manifold obtained by 'cutting along the singularity strata' of the n-fold join C_p^{*n}.

Lemma 5.3.20. *(cf. [9] Proposition 2.1) In the defining filtration of V_∞ in Definition 5.3.15, the successive quotients are given by*

$$
{}^nV_\infty/\ {}^{n-1}V_\infty \simeq \Sigma^n(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n}.
$$

The quotient map is geometrically presented as the map that sends an ${}^nV_\infty$-manifold to the MU-manifold with free a $C_p^n \rtimes \Sigma_n$-action (cf. Definition 5.3.24) defined by its n-corner's face-labelling bundle. Moreover, the attaching map

$$
\Sigma^{n-1}(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n} \to\ {}^{n-1}V_\infty
$$

whose cofiber is ${}^nV_\infty$ *is presented geometrically as the map that sends an* MU*-manifold* P *with a free* $C_p^n \rtimes \Sigma_n$*-action*[4] *to the* ${}^{n-1}V_\infty$*-manifold given by the* $C_p^{\star n}$*-bundle* $P \times_{C_p^n \times \Sigma_n} C_p^{\star n}$ *(cf. Definition 5.3.18)*

Proof. This is the proof given in [9]. It suffices to show that the geometrically defined maps

$$\ldots \to \Sigma^{n-1}(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n} \to {}^{n-1}V_\infty \to {}^nV_\infty \to \Sigma^n(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n} \to \ldots$$

form a cofiber sequence, which is equivalent to the statement that the corresponding maps of collections of manifolds(-with-singularities) form an exact sequence up to bordism (the base manifold X plays no real role except to clutter the notation). The composite of the left two maps is zero since the ${}^{n-1}V_\infty$-manifold $P \times_{C_p^n \rtimes \Sigma_n} C_p^{\star n}$ is null as an ${}^nV_\infty$-manifold (since the cone on $C_p^{\star n}$ is naturally an ${}^nV_\infty$-manifold). Moreover if an ${}^{n-1}V_\infty$-manifold M is null as an ${}^nV_\infty$-manifold, then any such nullbordism Q exhibits M as bordant to a neighborhood of the n-corners of Q, which is in the image of the first map. The composite of the second two maps is clearly zero since an ${}^{n-1}V_\infty$-manifold has empty n-corners. Finally, if M is an ${}^nV_\infty$-manifold whose n-corners' face-labelling bundle is null in $(MU \otimes BC_p)^{\otimes_{MU} n}_{h\Sigma_n}$, then gluing such a nullbordism onto the n-corners of M at one end of the product of M with an interval produces an ${}^nV_\infty$-manifold that exhibits M as bordant to the image of the inclusion ${}^{n-1}V_\infty \to {}^nV_\infty$. □

Definition 5.3.21. An ${}^nV_1$-manifold is an ${}^nV_\infty$-manifold Q together with a decomposition[5] of the 1-corners into labelled faces $d_1Q, \ldots, d_qQ$ (whose r-fold intersections are of codimension r) such that this labelling trivializes all the face-labelling bundles (which implies that $q \geqslant n'$ where n' is the minimal k for which Q is a ${}^kV_\infty$-manifold). The associated bordism theory ${}^nV_1$ is defined analogously to Definition 5.3.14, and V_1 is defined as the colimit of $MU \simeq {}^0V_1 \to {}^1V_1 \to \ldots$ where the connecting maps are induced by viewing an ${}^nV_1$-manifold as an ${}^{n+1}V_1$-manifold with empty $(n+1)$-corners and the same labelling.

Lemma 5.3.22. V_1 *is mod* p *oriented (cf. Definition 5.3.1).*

Proof. See the proof of Lemma 5.3.16. □

Remark 5.3.23. In [9] (Definition 2.4) Bullett defines an ${}^nW$-manifold to be an ${}^nV_\infty$-manifold with a decomposition of the 1-faces into n labelled sectors $d_1M, \ldots, d_nM$, such that the labelling simultaneously trivializes all face-labelling bundles. He defines spectra ${}^nW$ and a colimit spectrum ${}^\infty W = \mathrm{colim}_n\, {}^nW$, where the bonding maps ${}^nW \to {}^{n+1}W$ are those induced by viewing an ${}^nW$-manifold as an ${}^{n+1}W$-manifold with $d_{n+1}M = \varnothing$. Although every ${}^nV_1$-manifold admits the structure of an ${}^nW$-manifold by taking the minimal labelled decomposition, the connecting maps in the colimit are different and produce different spectra. In fact we will see that V_1 admits a ring structure while Bullett proves that ${}^\infty W$ does not (cf. [9] page 24).

[4] The action interacts appropriately with the MU-structure (cf. Definition 5.3.24).

[5] This means in particular that the labelled faces are disjoint after the 2-corners have been removed.

Remark 5.3.24. In the following it will be convenient to note that the spectrum $MU \otimes BC_p^{\otimes n}$ represents the bordism theory of MU-manifolds with a free action of C_p^n.

Lemma 5.3.25. *In the defining filtration of V_1 in Definition 5.3.21, the successive quotients are given by*

$$^{n}V_1/\ ^{n-1}V_1 \simeq \Sigma^n MU \otimes BC_p^{\otimes n}.$$

The quotient map is geometrically presented as the map that sends an $^{n}V_1$-manifold to the MU-manifold with free a C_p^n-action (cf. Remark 5.3.24) defined by its n-corner's (trivial) face-labelling bundle. Moreover, the attaching map

$$\Sigma^{n-1} MU \otimes BC_p^{\otimes n} \to\ ^{n-1}V_1$$

whose cofiber is $^{n}V_1$ is presented geometrically as the map that sends a MU-manifold P with free a C_p^n-action to the $^{n-1}V_1$-manifold given by the $C_p^{\star n}$-bundle $P \times_{C_p^n} C_p^{\star n}$ (cf. Definition 5.3.18).

Proof. The proof is identical to that of Lemma 5.3.20 once it is noted that $C_p^{\star n}$ is in fact an $^{n-1}V_1$-manifold because the 1-corners of $C_p^n \times \Delta_{\text{top}}^{n-1}$ are naturally labelled. □

Lemma 5.3.26. *The mod p homology of V_∞ and V_1 agree with the mod p homology of their associated graded spectra (see Lemmas 5.3.20 and 5.3.25). In particular, $(H\mathbb{F}_p)_* V_1$ is the free associative $(H\mathbb{F}_p)_* MU$-algebra on the vector space $(H\mathbb{F}_p)_* \Sigma BC_p$.*

Proof. It suffices to show that the geometrically defined attaching maps in Lemmas 5.3.20 and 5.3.25 are null after tensoring with $H\mathbb{F}_p$. Observe that the action of the diagonal C_p in $C_p^{\times n}$ acts freely on $P \times_{C_p^n \times \Sigma_n} C_p^{\star n}$ (resp. $P \times_{C_p^n} C_p^{\star n}$) by its natural action on the right-hand factor, preserving the $^{n-1}V_\infty$-structure. So the quotient by that action is an $^{n-1}V_\infty$- (resp. $^{n-1}V_1$-) manifold. Recall that the transfer map $BC_p \to \mathbb{S}$ admits the geometric presentation as the map that sends a framed manifold with a C_p-bundle to the total space of that bundle. Hence the two attaching maps in question factor as

$$\Sigma^{n-1}(MU \otimes BC_p)_{h\Sigma_n}^{\otimes_{MU} n} \to\ ^{n-1}V_\infty \otimes BC_p \to\ ^{n-1}V_\infty$$

$$\Sigma^{n-1} MU \otimes BC_p^{\otimes n} \to\ ^{n-1}V_1 \otimes BC_p \to\ ^{n-1}V_1$$

where the second maps are the transfer map (suitably tensored). Since the transfer map is null after tensoring with $H\mathbb{F}_p$ the first sentence of the lemma is proved. The second sentence of the lemma about the mod p homology of V_1 follows immediately from the identification of the associated graded in Lemma 5.3.25.

□

I conclude the section with an easy observation which will serve as a precursor to the much more refined Lemma 5.6.6.

Lemma 5.3.27. *Both V_∞ and V_1 are naturally homotopy ring spectra, and the natural map $V_1 \to V_\infty$ is a homotopy ring map, all modulo phantom maps.*

Proof. The cartesian product of an ${}^nV_\infty$- and and ${}^mV_\infty$-manifold is an ${}^{n+m}V_\infty$ manifold. The cartesian product of an ${}^nV_1$- and and ${}^mV_1$-manifold Q and R is also naturally an ${}^{n+m}V_1$-manifold by the decomposition of the 1-corners of $Q \times R$ into $d_1Q \times R, ..., d_qQ \times R, Q \times d_1R, ..., Q \times d_rR$. Hence the natural map $V_1^*X \to V_\infty^*X$ is a map of multiplicative cohomology theories. Lifting this structure via Brown representability gives the representing spectra a ring structure which is respected by the map between them. Since I have used Brown representability, everything is modulo phantom maps. □

Remark 5.3.28. The ring structure on V_∞ is homotopy commutative (up to phantom maps), since the cartesian product of V_∞-manifolds is symmetric up to cobordism. The same is *not* true for V_1. The labelling of the 1-corners on a cartesian product (described above) is clearly sensitive to the order of the factors in that cartesian product.

Remark 5.3.29. Although it is probably not hard to rule out the possibility of phantom maps in the situation above, the much-refined Lemma 5.6.6 makes it a moot point.

5.4 A solution to Bullett's conjecture

Definition 5.4.1. Let R_* denote the image in mod p homology of the canonical map $V_1 \to V_\infty$ induced by viewing a V_1-manifold as a V_∞-manifold. It is a subalgebra of $(H\mathbb{F}_p)_*V_\infty$. Let R^* be the degree-wise $\mathbb{F}_p$-linear dual of R_*. It is a quotient co-algebra of $H\mathbb{F}_p^*V_\infty$.

Lemma 5.4.2. *R_* is isomorphic to the free commutative $(H\mathbb{F}_p)_*MU$-algebra on the $\mathbb{F}_p$-module $(H\mathbb{F}_p)_*\Sigma BC_p$.*

Proof. The map $i : V_1 \to V_\infty$ respects the filtrations defined in Definitions 5.3.15 and 5.3.21. On the successive quotients it is the canonical quotient $i_k : MU \otimes (BC_p)^k \to (MU \otimes (BC_p)^k)_{h\Sigma_k}$. Let A_n^* denote the antisymmetric part of $H\mathbb{F}_p^*(MU \otimes (BC_p)^k)$. On mod p cohomology, the image of i_k^* is $H\mathbb{F}_p^*MU \otimes A_n^*$ (cf. [9] Proposition 2.10). Combining that with Lemma 5.3.26 finishes the proof. □

Definition 5.4.3. By [9] Proposition 2.14, R^* is a free module over the Steenrod algebra, so the formula $V^*X := \mathrm{Hom}_{\mathcal{A}_p}(R^*, H\mathbb{F}_p^*X)$ defines a cohomology theory. The *universal mod p oriented ring spectrum* V (cf. [9] Corollary 3.3) is the spectrum associated to that cohomology theory. By construction, V is a summand of V_∞, as presented by the canonical inclusion $V^*X \hookrightarrow \mathrm{Hom}_{\mathcal{A}_p}(H\mathbb{F}_p^*V_\infty, H\mathbb{F}_p^*X)$, and $V_* \simeq L_p$ (cf. Definition 5.3.11) by Corollary 3.3 of [9].

Remark 5.4.4. By construction, V^*BC_p carries the universal mod p formal group law, and the set of homotopy ring maps $V \to R$ is in bijection with the set of mod p orientations of R.

Lemma 5.4.5. *The mod p cohomology of V_1 and V_∞ are free over the Steenrod algebra.*

Proof. It suffices to show that V_1 and V_∞ are sums of shifts of $H\mathbb{F}_p$. By a theorem of Rourke (cf. [9] Theorem 1.1, [24]) a multiplicative cohomology theory E^*X is (represented by) a direct sum of shifts of $H\mathbb{F}_p$ if $E^0(\mathrm{pt}) = \mathbb{F}_p$ and $E^{>0}(\mathrm{pt}) = 0$ and there is a map of cohomology theories $E^*X \to H\mathbb{F}_p^*X$ which is surjective when $X = S^{2n+1}/C_p$ for sufficiently large n (note that those are finite subcomplexes of BC_p).

Consider the morphisms $V_1 \to H\mathbb{F}_p$ and $V_\infty \to H\mathbb{F}_p$ defined by sending a V_1- or V_∞-oriented map $Q \to X$ to the Poincare dual of the associated homology class. Write $H\mathbb{F}_p^*BC_p \simeq \mathbb{F}_p[e, x]$ with $|e| = 1$ and $|x| = 2$. Let $i_n : S^{2n+1}/C_p \to BC_p$ be the inclusion. Then the surjectivity condition in Rourke's theorem for the two maps in question reduces to the condition that for sufficiently large n there are V_1-(and hence V_∞-) manifolds $Q_{e,n} \to S^{2n+1}/C_p$ and $Q_{x,n} \to S^{2n+1}/C_p$ representing the Poincare duals of i_n^*x and i_n^*e.

The Poincare dual of i_n^*x is represented by $Q_{x,n} := S^{2n-1}/C_p \to S^{2n+1}/C_p$. Let $Q_{e,n}$ be the quotient of D^{2n} by the action of C_p on its boundary S^{2n-1}. Then the Poincare dual of i_n^*e is represented by the map $Q_{e,n} \to S^{2n+1}/C_p$ induced by including D^{2n} as a half an equator in S^{2n+1}. Finally, $Q_{x,n}$ is a MU-manifold, and $Q_{e,n}$ is a V_1-manifold (in fact, a ${}^1V_1$ manifold) so the proof is complete. □

Remark 5.4.6. There is a different proof of the above (see [11]), which uses formal groups and some defining universal properties of V_1 to show that it is an $H\mathbb{F}_p$-algebra in the stable homotopy category, and one derives the mod p (dual) Steenrod algebra at the same time.

Theorem 5.4.7. *(Bullett's conjecture) The spectrum V admits a geometric presentation as the bordism theory of V_∞-manifolds which admit a labelling (cf. Definition 5.1.1) up to cobordism.*

Proof. First I claim that the image of $i_X : V_1^*X \to V_\infty^*X$ coincides with V^*X (cf. Definition 5.4.3), so that i factors as the projection onto a summand of V_1 (which must be equivalent to V) and the inclusion of the summand V into V_∞. Indeed, the mod p cohomology of V_1 and V_∞ are free over the Steenrod algebra by Lemma 5.4.5, so i_X can be written as

$$V_1^*X \simeq \mathrm{Hom}_{\mathcal{A}_p}(H\mathbb{F}_p^*V_1, H\mathbb{F}_p^*X) \xrightarrow{i^*\circ(-)} \mathrm{Hom}_{\mathcal{A}_p}(H\mathbb{F}_p^*V_\infty, H\mathbb{F}_p^*X) \simeq V_\infty^*(X)$$

so its image is $\mathrm{Hom}_{\mathcal{A}_p}(R^*, H\mathbb{F}_p^*X)$, which is the definition of V^*X. So V^*X is the image of V_1^*X in V_∞^*X, and a V_∞-oriented map $[Q \to X] \in V_\infty^*X$ is in that image precisely when it admits a V_1-structure up to cobordism, i.e. when it is cobordant to a $Q' \to X$ that admits a labelling. □

Remark 5.4.8. Bullett frames his conjecture on page 24 of [9], where he suggests that V is the image of the natural map ${}^\infty W \to V_\infty$ (cf. Remark 5.3.23). That map has the same image as the map $V_1 \to V_\infty$ (Compare Lemma 5.4.2 and Proposition 2.13 of [9], and cf. Remark 5.3.23).

5.5 The odd p dual Steenrod algebras

Much of this is joint with Tim Campion ([11]).

Definition 5.5.1. Let $\mathbb{F}_p\text{-Alg}_{\leqslant 0}$ denote the category of (graded commutative coconnective) $\mathbb{F}_p$-algebras.

Definition 5.5.2. Let $\mathbf{F}$ be an $\mathbb{F}_p$-FGL over a ring R (cf. Definition 5.3.7). The *automorphism groupoid* $\operatorname{Aut}\mathbf{F}$ is the functor

$$\operatorname{Aut}\mathbf{F} : \mathbb{F}_p\text{-Alg}_{\leqslant 0} \to \text{Groupoids}$$

defined as follows: the set of objects of the groupoid $\operatorname{Aut}\mathbf{F}(S)$ is the set of ring maps $R \to S$ and the set $\operatorname{Aut}\mathbf{F}(S)(f, g)$ of morphisms from f to g is the set of R-algebra maps $\phi : R[[x, e]] \to R[[x, e]]$ such that

$$f_*\mathbf{F} \circ \phi = (\phi \otimes \phi) \circ g_*\mathbf{F}$$

along with the condition that $\phi(x)$ depends only on x. Together with Condition 4 of Definition 5.3.7 that forces the constant term of $\phi(x)$ to be zero (cf. Definition 5.3.6) which in turn forces the constant term of $\phi(e)$ to be zero as well. Finally one imposes the normalization conditions on leading coefficients:

$$\phi(x) = x + \sum_{i=1}^{\infty} d_i x^{i+1}$$

$$\phi(e) = e + \sum_{i=1}^{\infty} b_i e x^i + a_i x^i$$

The composition of morphisms is the composition of R-algebra maps.

Remark 5.5.3. When $F_1 = e_1 + e_2$ the formula for $\phi(e)$ simplifies. Namely the b_i in the formula above vanish and $a(x) = \sum a_i x^i$ is a morphism from F_2 to the additive formal group law.

Recall that the additive $\mathbb{F}_p$-FGL is presented by the power series $\mathbf{F} = (e_1 + e_2, x_1 + x_2)$ (cf. Example 5.3.8). In some sense that's is all there is:

Theorem 5.5.4. *(c.f. [10] 3.5.1) All $\mathbb{F}_p$-FGLs are isomorphic to the additive one. More precisely, let $\mathbf{F}$ be an $\mathbb{F}_p$-FGL over a ring R. Let $f : R \to R$ be any map that factors through the unit $\mathbb{F}_p \to R$, so that $f_*\mathbf{F} = \mathbf{F}_+$. Then there is a morphism $\phi \in \operatorname{Aut}\mathbf{F}(id, f)$.*

Definition 5.5.5. Let $\mathbf{F}$ be an $\mathbb{F}_p$-FGL over R. A *coordinate transformation* is and R-algebra map $\phi : R[[x, e]] \to R[[x, e]]$ such that

$$\phi(x) = x + \sum_{i=1}^{\infty} d_i x^{i+1}$$

$$\phi(e) = e + \sum_{i=1}^{\infty} b_i e x^i + a_i x^i.$$

One says that ϕ *transforms* $\mathbf{F}$ into the $\mathbb{F}_p$-FGL $\phi^*\mathbf{F} := (\phi^{-1} \otimes \phi^{-1}) \circ \mathbf{F} \circ \phi$. Note that coorinate transformations are also covariant. If $f : R \to S$ is a ring map then $f_*\phi$ is the S-algebra map $S[[x,e]] \to S[[x,e]]$ determined by setting $f_*\phi(x)$ and $f_*\phi(e)$ to be the power series gotten by applying f to the coefficients of $\phi(x)$ and $\phi(e)$.

Definition 5.5.6. Define the category MT as follows. Its objects are triples consisting of

1. a (contravariant) functor $X \mapsto E^*(X)$ from the category of finite CW complexes to the category of graded abelian groups,
2. a natural isomorphism from $E^{*+1}(\Sigma X)$ to $E^*(X)$, and
3. an associative "multiplication map" $E^*(X) \otimes E^{*'}(Y) \to E^{*+*'}(X \times Y)$.

Note that with the diagonal map $X \to X \times X$ the data of 3. makes E^*X into a graded ring. Morphisms $T : E \to F$ are natural transformations of functors that commute with the data in 2. and 3. The component of a morphism T at a CW complex X is denoted T_X.

Remark 5.5.7. The first examples of objects of MT are those that are induced by multiplicative cohomology theories, i.e. homotopy ring spectra. In fact, MT is an acronym for *multiplicative theory*, and they are meant to capture multiplicative cohomology theories (MCTs) without the exactness axiom, which is the only Eilenber-Steenrod axiom which is not preserved under tensor product (cf. Definition 5.5.9).

Remark 5.5.8. Although the objects of MT are defined as functors out of finite spectra, when working with a fixed object one can often enlarge the domain quite a bit—namely to those spectra for which the relevant $\lim^1$-term vanishes. For all objects of MT considered in this paper (which either come from mod p oriented spectra or are tensored from them (cf. Lemma 5.5.11)), that includes the non-finite spectra BC_p, $BU(1)$, and MU.

Definition 5.5.9. (cf. Remark 5.5.7) For an object E of MT and an E^*-algebra with unit $u : E^* \to R$ let E_uR be the object of MT defined by the formula

$$E_uR^*X = E^*X \otimes_{E^*} R.$$

When the map u is understood I often abbreviate E_uR to ER.

Lemma 5.5.10. *(Evaluation at BC_p) Let E be a mod p-oriented homotopy ring spectrum with $E^*BC_p \simeq E^*[[x,e]]$ and $\mathbb{F}_p$-FGL $\mathbf{F}_E$. Let R be an E^*-algebra with unit map u : $E^* \to R$. Then for every $T \in Hom_{MT}(E, E_uR)$ (cf. Definition 5.5.9) there is a unique $\phi_T \in \operatorname{Aut} \mathbf{F}_E(R)(T_{pt}, u)$ such that $\phi_T(x) = (T_{BC_p}(x)$ and $\phi_T(e) = T_{BC_p}(e)$.*

Proof. Let T be an element of $\mathrm{Hom}_{MT}(E, ER)$. To streamline notation, let f denote the ring map $T_{\mathrm{pt}} : E^* \to R$. First note that uniquely determines a morphism $\tau \in \mathrm{Hom}_{MT}(E_f R, E_u R)$ by the diagram

$$E^*X \otimes_{E^*} R_f \xrightarrow{T\otimes\mathrm{id}} E^*X \otimes_{E^*} R_u \otimes_{E^*} R_f \xrightarrow{\mu_R} E^*X \otimes_{E^*} R_u.$$

Note that the restriction of τ along the map $E \to E_u R$ induced by u recovers T. Now consider the diagram

$$\begin{array}{ccc} E_f R^* BC_p & \xrightarrow{\mu^*} & E_f R^* BC_p^{\times 2} \\ \big\downarrow{\scriptstyle \tau_{BC_p}} & & \big\downarrow{\scriptstyle \tau_{BC_p^{\times 2}}} \\ E_u R^* BC_p & \xrightarrow{\mu^*} & E_u R^* BC_p^{\times 2} \end{array}$$

which commutes by naturality of τ. Write ϕ for the map $R[[x, e]]$ induced by τ_{BC_p}. By multiplicativity of τ, one finds that the diagram above gives rise to the equation

$$f_* \mathbf{F}_E \circ \phi = (\phi \otimes \phi) \circ u_* \mathbf{F}_E.$$

Consider the diagram above but with BC_p replaced by $BU(1)$. Since $x = i^* c_E$ (cf. Definition 5.3.1) that diagram includes into the diagram above, and we find that $\phi(x)$ depends only on x. Combining that with the displayed equation implies that $\phi(x) = \sum_{i=1}^{\infty} a_i x^i$ (cf. Definition 5.5.2). The leading coefficient a_1 is forced to be 1 by considering the pullback along $S^2 \to BU(1)$. The proof is completed by noting that ϕ is uniquely determined by $\phi(x)$ and $\phi(e)$ and those are equal to $T_{BC_p}(x)$ and $T_{BC_p}(e)$. □

Lemma 5.5.11. *(Quillen functor for V_1) Let e and x be any choice of universal mod p classes in $V_1^* BC_p$, with corresponding $\mathbb{F}_p$-FGL $\mathbf{F}$. Let $M^* \subset V_1^*$ be the subring generated by the coefficients of $\mathbf{F}$ and write $\mathbf{F}_{M^*}$ for the corresponding $\mathbb{F}_p$-FGL over M^*. Then for every ring A there is a functor $\gamma : \mathrm{Aut}\,\mathbf{F}_{M^*}(A) \to MT$ (cf. Definitions 5.5.2 and 5.5.6) which on objects sends $f : M^* \to A$ to the theory $\gamma(f)^* X := V_1^* X \otimes_{M^*} A$.*

Proof. The functor γ has been defined on objects, so it remains to specify it on morphisms. Let $f, g : M^* \to A$ bet two objects and let ϕ be a morphism in $\mathrm{Aut}\,\mathbf{F}_{M^*}(A)(f, g)$. When we need to distinguish between the two M^*-module structures on A we will write A_f and A_g. Let e_f, x_f and e_g, x_g denote the classes in $\gamma(f)^* BC_p$ and $\gamma(g)^* BC_p$ which are the image of e and x under the canonical maps $V_1 \to \gamma(f)$ and $V_1 \to \gamma(f)$. We will construct (functorially) a transformation $\gamma_\phi : \gamma(f) \to \gamma(g)$ such that $\gamma_\phi(e_f) = \phi(e_g)$ and $\gamma_\phi(x_f) = \phi(x_g)$.

Now $(V_1)_* V_1$ is free over $(V_1)_*$ (c.f. Section 2). Moreover there is a canonical map to $(V_1)_* MU \otimes_{\mathbb{F}_p} \mathbb{F}_p[a_1, a_2, ..., b_1, b_2, ...]$ with $|a_i| = 2i - 1$ and $|b_i| = 2i$.Write $c \in V_1^2 BU(1)$ for the (canonical) complex orientation of V_1. Then we can write $(V_1)_* MU \simeq (V_1)_*[d_1, d_2, ...]$ with $|d_i| = 2i$ being the image under $\Sigma^{-2} BU(1) \to MU$ of the dual of c^{i+1}. So together with the map $V_1 \simeq \mathbb{S} \otimes V_1 \to V_1 \otimes V_1$ we get a morphism in MT

$$V_1^* X \to (V_1 \otimes V_1)^* X \simeq V_1^* X \otimes_{V_1^*} V_1^* V_1 \to V_1^* X \otimes_{\mathbb{F}_p} \mathbb{F}_p[a_i, b_i, d_i].$$

By Lemma 5.5.23 when $X = BC_p$ the images of e and x under that map are

$$e \mapsto e \otimes 1 + x \otimes a_1 + ex \otimes b_1 + x^2 \otimes a_2 + ex^2 \otimes b_2 + x^3 \otimes a_3 + ex^3 \otimes b_3 + ...$$

$$x \mapsto x \otimes 1 + x^2 \otimes d_1 + x^3 \otimes d_2 + ...$$

The morphism ϕ determines an $\mathbb{F}_p$-algebra map

$$\Phi : \mathbb{F}_p[a_i, b_i, d_i] \to A$$

sending a_i to the coefficient of x^i in $\phi(e)$, b_i to the coefficient of ex^i in $\phi(e)$, and d_i the coefficient of x^{i+1} in $\phi(x)$. Composing with the previous display gives another morphism $T_\phi : V_1 \to \gamma(g)$ in MT

$$V_1^*X \to V_1^*X \otimes_{\mathbb{F}_p} \mathbb{F}_p[a_i, b_i, d_i] \xrightarrow{\mathrm{id}\otimes\Phi} V_1^*X \otimes_{\mathbb{F}_p} A \to V_1^*X \otimes_{M^*} A_g = \gamma(g)^*X$$

with the property that when $X = \mathrm{pt}$ the induced map $V_1^* \xrightarrow{(T_\phi)_{\mathrm{pt}}} A$ is equal f when restricted to M^*. Indeed, by Lemma 5.5.10 T_ϕ determines a morphism ϕ_{T_ϕ} in $\mathrm{Aut}\,\mathbf{F}_{M^*}(A)((T_\phi)_{\mathrm{pt}}, g$ such that $\phi_{T_\phi}(x) = \phi(x)$ and $\phi_{T_\phi}(e) = \phi(e)$. On the other hand, ϕ was by definition a morphism from f to g. So the $\mathbb{F}_p$-FGLs $T_\phi(\mathrm{pt})_*\mathbf{F}$ and $f_*\mathbf{F}$ are identical. Since the coefficients of $\mathbf{F}$ generate M^* the maps f and $T_\phi(\mathrm{pt})$ must agree on M^*. It follows that T_ϕ descends to a morphism $\gamma_\phi : \gamma(f) \to \gamma(g)$ which can be written explicitly as the following composition, using the multiplication μ_f on A as an M^*-algebra via f

$$\begin{array}{ccc} \gamma(f)^*X = V_1^*X \otimes_{M^*} A_f & \xrightarrow{T_\phi \otimes id} & \gamma(g)^*X \otimes_{M^*} A_f \\ & \swarrow \sim & \\ V_1^*X \otimes_{M^*} A_g \otimes_{M^*} A_f & \xrightarrow{\mu_f} & V_1^*X \otimes_{M^*} A_g = \gamma(g)^*X \end{array}.$$

Functoriality—the claim that $\gamma(\phi \circ \psi) = \gamma(\phi) \circ \gamma(\psi)$—is proved by noting that $\gamma(\phi)$ is uniquely characterized by the properties of being A-linear, multiplicative, and its behavior at $X = BC_p$, i.e. sending (e_f, x_f) to $(\phi(e_g), \phi(x_g))$. Indeed, note that $\gamma(\phi)$ is characterized by its restriction along the surjection $V_1^*X \otimes_{\mathbb{F}_p} A \to V_1^*X \otimes_{M^*} A_f$. By A-linearity that in turn is determined by restriction along $V_1^*X \to V_1^*X \otimes_{\mathbb{F}_p} A$ (which coincides with the composite of the first two maps that make up T_ϕ). By Corollary 5.5.20 the latter is determined by its behavior at $X = BC_p$. □

Lemma 5.5.12. *(Kill the FGL to get a summand) Let e and x be any choice of universal mod p classes in $V_1^*BC_p$, with corresponding $\mathbb{F}_p$-FGL $\mathbf{F}$. Let $M^* \subset V_1^*$ be the subring generated by the coefficients of $\mathbf{F}$ and let $\mathbf{F}_{M^*}$ be the restriction of $\mathbf{F}$ to M^*. Set $N^* = V_1^* \otimes_{M^*} \mathbb{F}_p$. Then $MN^*X := V_1^*X \otimes_{M^*} \mathbb{F}_p$ is a cohomology theory and a summand of V_1^*X, and there is a ring isomorphism $V_1^* \simeq N^* \otimes_{\mathbb{F}_p} M^*$*

Proof. By Theorem 5.5.4 we know that $\mathbf{F}$ and $\mathbf{F}_{M^*}$ are isomorphic to the additive $\mathbb{F}_p$-FGL. That means there is a morphism between the objects $\mathrm{id} : M^* \to M^*$ and $p : M^* \to \mathbb{F}_p \to M^*$ in $\mathrm{Aut}\,\mathbf{F}_{M^*}(M^*)$. Let ϕ be such an isomorphism. The functor γ of Lemma 5.5.11 provides an isomorphism $\gamma(\phi)$ between $\gamma(\mathrm{id})$ and $\gamma(p)$ in MT. But $\gamma(\mathrm{id}) \simeq V_1$ so we have

$$V_1^* X \simeq \gamma(\mathrm{id})^* X \xrightarrow{\simeq} \gamma(p)^* X = V_1^* X \otimes_{M^*} \mathbb{F}_p \otimes_{\mathbb{F}_p} M^*.$$

Therefore $MN^* X := V_1^* X \otimes_{M^*} \mathbb{F}_p$ is a summand of a cohomology theory (namely V_1) and hence a cohomology theory itself. The ring isomorphism at the end of the lemma statement is the displayed diagram when $X = \mathrm{pt}$. □

Lemma 5.5.13. *(Quillen functor for summands) Let E be a mod p oriented homotopy ring spectrum which is a summand of V_1. Let e, x be any choice of universal class in $E^* BC_p$, with corresponding F_p-formal group law $\mathbf{F}$. Let $M^* \subset E^* \subset V_1^*$ be the subring generated by the coefficients of $\mathbf{F}$ and let $\mathbf{F}_{M^*}$ be the restriction of $\mathbf{F}$ to M^*. Then for every ring A there is a functor $\gamma : \mathrm{Aut}\,\mathbf{F}_{M^*}(A) \to MT$ which on objects sends $f : M^* \to A$ to $X \mapsto E^* X \otimes_{M^*} A$.*

Proof. The proof is nearly identical to that of Lemma 5.5.11 except that the first displayed morphism in MT, now induced by the map $E \simeq \mathbb{S} \otimes E \to E \otimes E \to E \otimes V_1$ becomes

$$E^* X \to (E \otimes V_1)^* X \simeq E^* X \otimes_{E_*} E_* MU[a_1, a_2, ..., b_1, b_2, ...]$$

and in the rest of the proof every instance of V_1 is replaced with E. □

Lemma 5.5.14. *$H\mathbb{F}_p$ is a summand of V_1.*

Proof. Fix notation as in the Lemma 5.5.11. Invoke Lemma 5.5.12 to get a summand MN of V_1 and a decomposition $V_1^* \simeq N^* \otimes M^*$. MN is mod p oriented via its map from V_1 and the corresponding $\mathbb{F}_p$-FGL $\mathbf{F}_{N^*}$ is by construction the additive one. Perform a coordinate transformation (cf. Definition 5.5.5) so that the new $\mathbb{F}_p$-FGL $\mathbf{F}'_{N^*}$ is not the additive one. Let $M_1^* \subset N^*$ be the subring generated by the coefficients of $\mathbf{F}'_{N^*}$. Since MN^* is a summand of V_1 we can invoke Lemmas 5.5.13 and 5.5.12 to obtain a summand MN_1 of MN, and then again (after another coordinate transformation as above) to obtain a summand MN_2 of MN_1, and so on. Now, since coordinate transformations are covariant (cf. Definition 5.5.5) and V_1^* is finitely generated in each degree, the process can be reordered such that for each fixed k, B^k stabilizes after a finite number of steps. So after a transfinite process we arrive at a summand MB^*, and it must be that every coordinate transformation is an automorphism of the additive $\mathbb{F}_p$-FGL over B^*. Considering coordinate transformations of the form $\phi(x) = x + bx^n$, $\phi(e) = e$ shows that every element of B^{2*} must be in degree $2 - 2p^k$ for some $k \geqslant 0$. Considering coordinate transformations of the form $\phi(x) = x$, $\phi(e) = e + bx^n$ shows that every element of B^{2*+1} must be in degree $1 - 2p^k$ for some $k \geqslant 0$. Moreover the square of any negative degree element must be zero: if it is odd that is clear from graded-commutativity, otherwise it would be in degree not of the form $2p^k - 2$. Moreover we have a decomposition $V_1^* \simeq B^* \otimes G^*$ where $1 \otimes G^*$ contains all those elements that were coefficients of some $\mathbb{F}_p$-FGL that was used along the way. Let $z = [D^2/\sim] \neq 0 \in V_1^{-2}$ (cf. Lemma **??**). Let b be an element of minimal negative degree in B^* for all possible transfinite processes. Then $(b \otimes 1)z$ is in even degree but not of the form $2p^k - 2$ for any k and hence not in B^*, so we can re-run the transfinite process above but ensuring that we use the coordinate transformation $x + (b \otimes 1)zx^{2l}$ to get a new decomposition $V_1^* \simeq B_1^* \otimes G_1^*$ with $(b \otimes 1)z$ in G_1^*. That is, under the composite of ring isomorphisms

$$B^* \otimes G^* \xrightarrow{\sim} V_1^* \xrightarrow{\sim} B_1^* \otimes G_1^*$$

$(b \otimes 1)z$ gets sent to $1 \otimes g$. On the other hand $b \otimes 1$ must be sent to $c \otimes 1$ by minimality of the degree of b, and since $(b \otimes 1)z(b \otimes 1) = (b^2 \otimes 1)z = 0z = 0$, we find that $c \otimes g = 0$ which is a contradiction. So B^* is concentrated in degree zero, which means that MB is $H\mathbb{F}_p$. □

Theorem 5.5.15. *The cogroupoid object $\mathbb{F}_p \rightrightarrows (H\mathbb{F}_p)_* H\mathbb{F}_p$ corepresents the groupoid valued functor* $\operatorname{Aut} \mathbf{F}_{\mathbb{F}_p}$.

Proof. Let $H = H\mathbb{F}_p$ and let R be an $\mathbb{F}_p$-algebra. Since every module over $\mathbb{F}_p$ is flat, note that $\operatorname{Hom}_{MT}(H, HR)$ is precisely the subset of HR^0H consisting of homotopy ring maps. Applying the universal coefficient theorem provides an isomorphism of sets, natural in R

$$\operatorname{Hom}_{MT}(H, HR) \simeq \operatorname{Hom}_{\mathbb{F}_p\text{-Alg}_{\geqslant 0}}(H_*H, R).$$

It is an isomorphism of groups because the group structure on both sides comes from the map $\mathbb{S} \otimes H \otimes H \to H \otimes H \otimes H$ induced by the unit map $\mathbb{S} \to H$.

Let u be the unit map of R. By Lemma 5.5.10 evaluation at BC_p defines a group homomorphism

$$\zeta^H : \operatorname{Hom}_{MT}(H, HR) \to \operatorname{Aut} \mathbf{F}_H(R)(u, u)$$

that is natural in R. By Lemma 5.5.20 the analogous map

$$\zeta^{V_1} : \operatorname{Hom}_{MT}(V_1, MA) \to \operatorname{Aut} \mathbf{F}_{V_1}(A)(u, u)$$

is injective for a V_1^*-algebra with unit u. Hence the analogous map ζ^E is injective for any summand of E of V_1, and H is such a summand by Lemma 5.5.14. The map ζ^E has an inverse for any summand of V_1, given by sending ϕ to T_ϕ defined in the proof of Lemma 5.5.11. Finally, the map is one of groups because the inverse $\phi \mapsto T_\phi$ is, since the group structure on the left comes from $\mathbb{S} \otimes H \to H \otimes H$ and the group structure on the right comes from $\mathbb{S} \otimes V_1 \to V_1 \otimes V_1$.

Finally, since the groupoid $\operatorname{Aut} \mathbf{F}_H(R)$ has one object when R is an $\mathbb{F}_p$-algebra and zero objects otherwise, combining the two natural group isomorphisms above finishes the proof. □

Remark 5.5.16. Note that we have arrived at a *coordinate-free* calculation of $(H\mathbb{F}_p)_* H\mathbb{F}_p$: we have identified it (as a Hopf algebra) to be whatever Hopf algebra corepresents $\operatorname{Aut} \mathbf{F}_{\mathbb{F}_p}$. Giving a presentation by generators and relations (e.g. (duals to) Steenrod operations) is a simple but tedious algebraic problem. For the sake of completeness I will give the usual description.

Definition 5.5.17. Define the Hopf algebra $\mathcal{A}_p$ as follows.

$$\mathcal{A}_p := \mathbb{F}_p[\xi_i, \tau_j| \ |\xi_i| = 2(p^k - 1), \ |\tau_j| = 2p^k - 1, i = 1, 2, ..., j = 0, 1, ...],$$

$$\Delta(\xi_i) = \sum_{l=1}^{n} \xi_{i-l}^{p^l} \otimes \xi_l,$$

$$\Delta(\tau_j) = \tau_j \otimes 1 + \sum_{l=1}^{n} \xi_{j-l}^{p^l} \otimes \tau_l.$$

As mentioned, it is straightforward, tedious, and purely algebraic to check that the cogroupoid object $\mathbb{F}_p \rightrightarrows \mathcal{A}_p$ corepresents the automorphism groupoid of the additive $\mathbb{F}_p$-FGL (cf. Remark 5.3.8) $\operatorname{Aut} \mathbf{F}_+$.

5.5.1 Required technical facts about V_1

Lemma 5.5.18. *Let E be a mod p oriented homotopy ring spectrum. Write $E^*BC_p \simeq E^*[[e, x]]$, $|e| = 1, |x| = 2$. There is an E_*-algebra isomorphism*

$$E_*V_1 \simeq Free_{AssAlg_{E_*}}(\widetilde{E}_*\Sigma BC_p) \otimes_{E_*} E_*MU$$

$$\simeq E_*\langle u, \Sigma e, \Sigma x, \Sigma ex, \Sigma x^2, ...\rangle \otimes_{E_*} E_*[d_1, d_2, ...],$$

$$|u| = 1, \ |\Sigma ex^i| = 2i + 2, \ |\Sigma x^i| = 2i + 1, \ |d_i| = 2i.$$

Proof. This follows directly from Lemma 5.3.26. □

Corollary 5.5.19. *Let E be a mod p oriented ring spectrum. A morphism $T : V_1 \to E$ in MT is determined by its behavior at BC_p,*

$$T_{BC_p} : V_1^*BC_p \to E^*BC_p.$$

Proof. First note that since $E^*BU(1)$ is a summand of E^*BC_p, the behavior of T at BC_p determines it at $BU(1)$. By the Thom isomorphism and the splitting principle for complex vector bundles (together with the multiplicativity of T) that determines T at MU (cf. also Remark 5.5.8). Let t be an element in E^0V_1 representing T. Let X be a finite spectrum and $\xi : X \to V_1$ a map representing some $\xi \in V_1^0X$. Since X is finite, a factors through some filtration step $i_k : {}^kV_1 \to V_1$ (cf. Definition 5.3.15). Hence $T(\xi) \in E^0X$ is determined by the restriction $i_k^*t \in E^0 \, {}^kV_1$. By the proof of Lemma 5.5.18, pullback along the attaching map $a_k : \Sigma^{-1}MU \otimes (\Sigma BC_p)^{k+1} \to {}^kV_1$ is an injection in E^*-cohomology. So $T(\xi)$ is determined by $a_k^*i_k^*t$. But if $\alpha_k \in V_1^0(\Sigma^{-1}MU \otimes (\Sigma BC_p)^{k+1})$ is the class associated to a_k, then $a_k^*i_k^*t$ is $T(\alpha_k)$ (cf. again Remark 5.5.8), which has been determined. □

Corollary 5.5.20. *Let E be a mod p oriented ring spectrum, $R \subset E^*$ a subring and A an R-algebra. Let EA be the object of MT defined by $X \mapsto E^*X \otimes_R A$. Then a morphism $T : V_1 \to EA$ in MT is determined by its behavior at BC_p: $T_{BC_p} : V_1^*BC_p \to EA^*BC_p$.*

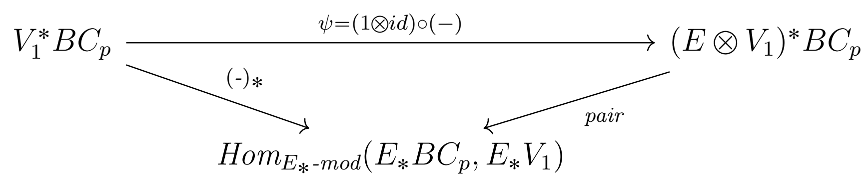
Proof. The proof of Lemma 5.5.19 applies mutatis mutandis. □

Corollary 5.5.21. *Let E be a mod p oriented homotopy ring spectrum. There is an E_*-algebra map*

$$E_*V_1 \to E_*[a_i, b_i, d_i] \ i = 1, 2, ... \ |a_i| = 2i - 1, \ |b_i| = 2i, \ |d_i| = 2i.$$

Proof. Given Lemma 5.5.18 this is an abelianization map. It sends u to a_1, Σex^i to b_{i+1} and Σx^i to a_{i+1}. □

Lemma 5.5.22. *Let E be a mod p oriented homotopy ring spectrum. There is a commutative diagram*



in which 'pair' is an isomorphism.

Proof. The commuting diagram is a special case of Lemma 6.2 page 59 of [2]. Since E is mod p oriented the previous lemmas imply that the relevant modules are sufficiently free, and so the universal coefficient theorem guarantees that 'pair' is an isomorphism. □

Lemma 5.5.23. *In the diagram above,*

$$\psi(e) = e \otimes 1 + x \otimes a_1 + ex \otimes b_1 + x^2 \otimes a_2 + ex^2 \otimes b_2 + x^3 \otimes a_3 + ex^3 \otimes b_3 + ...$$

$$\psi(x) = x \otimes 1 + x^2 \otimes d_1 + x^3 \otimes d_2 + ...$$

Proof. To calculate the effect of $e : \Sigma^{-1}BC_p \to V_1$ on E-homology, note that it factors through the first stage of the filtration $Z_1 \to V_1$. Furthermore, Z_1 sits in a cofiber sequence $MU \to Z_1 \to MU \otimes \Sigma_+BC_p$. The composite $\Sigma^{-1}BC_p \to Z_1 \to MU \otimes \Sigma_+BC_p$ has the following geometric presentation. First approximate $\Sigma^{-1}BC_p$ by $\Sigma^{-1}S^{2n+1}/C_p$. Then e is geometrically presented as the inclusion $D^{2n}/\sim \hookrightarrow S^{2n+1}/C_p$ (cf. Lemma **??**). The second map in the composite above sends a Z_1-manifold to its boundary (singularity), so the composite is geometrically presented as $S^{2n-1}/C_p \hookrightarrow S^{2n+1}/C_p$. But that is the geometric presentation of the class $x \in Z_1^*BC_p$ and hence also $x \in V_1^*BC_p$. Let $\alpha_i, \beta_i \in E_*BC_p$ be dual to ex^{i-1} and x^i. We find that $e_* : E_{*+1}BC_p \to E_*V_1$ is given by

$$e_* : E_{*+1}\{1, \alpha_1, \beta_1, \alpha_2, ...\} \to E_*\langle u, \Sigma e, \Sigma x, \Sigma ex, \Sigma x^2, ...\rangle \otimes_{E_*} E_*[d_1, d_2, ...]$$

$$1 \mapsto 0$$

$$\alpha_i \mapsto \Sigma ex^{i-2}$$

$$\beta_i \mapsto \Sigma x^{i-1}.$$

To calculate the effect of the map $x : \Sigma^{-2}BC_p \to W_1$ on E-homology, note that it factors through the canonical orientation $MU \to V_1$. Moreover the classes d_i introduced above are the standard generators defined as the images under $MU(1) \to MU$ of the classes dual to $c^{i+1} \in E^*MU(1)$ (recall that if i denotes the inclusion of BC_p into $BU(1)$ then $x = i^*c$). Then we find that $x_* : E_*BC_p \to E_{*-2}W_1$ is given by

$$\alpha_i \mapsto 0$$

$$\beta_i \mapsto d_{i-1}.$$

With the intent of applying Lemma 5.5.22, note that since the α_i and β_i were defined to be dual to monomials in e and x the images of $1, e, x, ex, x^2, ... \in (E \otimes V_1)^*BC_p \simeq E_*V_1[[e, x]]$ under the map *pair* are given as follows: for $z = 1, e, x, ex, x^2, ...$, $pair(z)$ takes a nonzero value on exactly one of the E_*-module generators a_i, b_i, which are recorded below

$$pair(e)(a_1) = 1$$

$$pair(ex^i)(a_{i+1}) = 1$$

$$pair(x^i)(b_i) = 1.$$

Finally, an application of Lemma 5.5.22 finishes the proof. □

5.5.2 Other algebras of operations

In work in progress I aim to use the formalism of symmetric functors (note the absence of "stable") to get similar derivations of algebras of *unstable* cohomology operations (which correspond to FGL *endomorphisms* including power operations such as the Dyler-Lashof operations.

5.6 E_∞-structures, etc

I will now define two sequences of objects ${}^1F_\infty$, ${}^1F_1$,${}^2F_\infty$, ${}^2F_1$,...,${}^nF_\infty$, ${}^nF_1$,...,F_∞, F_1 in the category SstFun(C, σ, δ) just defined. The underlying homotopy type of $\delta^*\ {}^nF_\infty$ will be ${}^nV_\infty$ and the underlying homotopy type of $\delta^*\ {}^nF_1$ will be ${}^nV_1$. F_∞ will be a commutative monoid, F_1 will be an associative monoid, and under Lemma 2.0.5 those two will provide E_∞- and E_1- lifts of the bordism spectra V_∞ and V_1.

Definition 5.6.1. For a finite set N and a positive number m let $U(mN)$ be the group of unitary transformations of the m-fold direct sum of the complex vector space of maps $N \to \mathbb{C}$ with its obvious hermitian inner product structure. Let BU_N be the colimit of $BU(N) \to BU(2N) \to ...$ and let BO_N be defined analogously with $\mathbb{C}$ replaced by $\mathbb{R}^2$ and "unitary" by "orthogonal."

Definition 5.6.2. Define $F_\infty \in \mathrm{SstFun}(C, \sigma, \delta)$ (cf. Def 2.0.7) by setting $F_\infty(N, X, \Delta^k)$ to be the set of k-simplices of the simplicial set associated to the following groupoid: its objects are submanifold-with-corners $Q \hookrightarrow X \times (\mathbb{C}^N)^\infty$ of dimension $\dim X - |N|$ such that the composite with the projection to X is a proper submersion. Furthermore, Q (and hence all the fibers over points in X) is equipped with the structure of a V_∞-manifold where the MU-structure is defined by a map to BU_N which lifts the map to BO_N classifying the vertical normal bundle in $(\mathbb{R}^{2N})^\infty$. The morphisms in the groupoid are diffeomorphisms commuting with the inclusions into $X \times (\mathbb{C}^N)^\infty$. The submersion property ensures that F_∞ is indeed functorial under pullbacks along maps $Y \to X$ in Man. For every finite set M there is a map

$$F_\infty(N, X, \Delta^k) \to F_\infty(N + M, X \times S^M, \Delta^k)$$

which simply pushes forward an element $Q \hookrightarrow X \times (\mathbb{C}^N)^\infty$ along the inclusion of $X \times (\mathbb{C}^N)^\infty$ into $X \times S^M \times (\mathbb{C}^{N+M})^\infty$ at the basepoint of S^M and enlarges the stable normal structure along the evident map $BU_N \to BU_{N+M}$. The maps s_M give rise to a σ-module structure via the composite

$$\mathrm{Hom}_{\mathrm{Man}}(X, S^M) \otimes F_\infty(N, X, \Delta^k) \xrightarrow{\mathrm{id} \otimes s_M} \mathrm{Hom}_{\mathrm{Man}}(X, S^M) \otimes F_\infty(N + M, X \times S^M, \Delta^k)$$

$$\to F_\infty(N + M, X \times X, \Delta^k) \xrightarrow{\quad \mathrm{diag}^* \quad} F_\infty(N + M, X, \Delta^k)$$

where the diagonal map sends $f \otimes Q$ to f^*Q.

Definition 5.6.3. The definitions of ${}^nF_\infty, F_1$, and ${}^nF_1 \in \mathrm{SstFun}(C, \sigma, \delta)$ are almost word-for-word identical to Definition 5.6.2, the only difference being that the one instance of "V_∞" is replaced by the evident symbol denoting the type of manifold allowed. In particular the functors with subscript "1" have objects that come with the additional data of the labelling (cf. Definition 5.3.21).

Lemma 5.6.4. *There are equivalences $u\delta^* F_\infty \simeq V_\infty$ (and $u\delta^* \, {}^nF_\infty \simeq {}^nV_\infty$) and $u\delta^* F_1 \simeq V_1$ (and $u\delta^* \, {}^nF_1 \simeq {}^nV_1$).*

Proof. Let $\mathbf{j}$ denote the set $\{1, 2, \ldots, j\}$. Then if F is a symmetric stable functor, the spectrum $u\delta^* F$ (cf. Definition 2.0.6) is presented by the sequential spectrum whose j-th term is $|F(\mathbf{j}, \Delta^\bullet_{\mathrm{sm}}, \Delta^\bullet)|$. I will only give the proof of the equivalence $u\delta^* F_\infty \simeq V_\infty$ since the others are completely analogous.

For any finite set N the assignment $X \mapsto |F_\infty(N, X \times \Delta^\bullet_{\mathrm{sm}}, \Delta^\bullet)|$ is an $\mathbb{R}$-invariant[6] sheaf on Man (because V_∞ manifolds can be glued along isomorphisms). Therefore by 4.3.1.2 of [3]

$$u\delta^* F^j_\infty(X) = \pi_0 \mathrm{Map}(X, \Sigma^j u\delta^* F_\infty) \simeq \pi_0 |F_\infty(\mathbf{j}, X \times \Delta^\bullet_{\mathrm{sm}}, \Delta^\bullet)|.$$

[6] Indeed it is the $\mathbb{R}$-invariantization of the sheaf of (nerves of) groupoids $X \mapsto F_\infty(N, X, \Delta^\bullet)$ on Man. $\mathbb{R}$-invariantization preserves sheaves by the main result of [7].

The latter is the set of cobordism classes of V_∞-manifolds of dim$X - j$ with a V_∞-oriented proper map to X which is also a submersion. On the other hand, $V_\infty^j(X)$ is (by Definition 5.3.14) the set of cobordism classes of V_∞-manifolds of dimension dim$X-j$ with a V_∞-oriented proper map to X. So there is a natural morphism of cohomology theories $u\delta^* F_\infty^*(X) \to V_\infty^*(X)$ induced by the inclusion of those V_∞-oriented proper maps $Q \to X$ which are also submersions. But when $X = \mathrm{pt}$ the submersion property is automatic so the inclusion is an isomorphism and hence the induced map of spectra $u\delta^* F_\infty \to V_\infty$ induces an isomorphism on homotopy groups. □

Remark 5.6.5. Note that we have implicitly given a presentation of MU as a commutative symmetric stable funtor (satisfying the sheaf condition) by simply disallowing all singularities (i.e. considering the filtration 0 part of V_∞ or V_1). Let F_{MU} denote the corresponding commutative symmetric stable functor and $F_{MU} \to F_\infty$ and $F_{MU} \to V_1$ the obvious inclusions

Lemma 5.6.6. *F_∞ is a commutative monoid in SstFun(C, σ, δ) and F_1 is an associative monoid. Hence V_∞ and V_1 are E_∞- and E_1-ring spectra. In fact they are E_∞ and E_1 MU-algebras.*

Proof. Note that the second sentence of the lemma follows from the first because of Lemma 2.0.5 and Lemma 5.6.4, and that the third sentence follows from the second because of Remark 5.6.5. So it suffices to prove the first sentence.

First I will show that both F_∞ and F_1 are associative monoids in SstFun(C, σ, δ). To that end, first note that the Cartesian product of V_∞- and V_1-manifolds makes F_∞ and F_1 associative monoids in Fun$(\mathrm{Fin}^\times \times C^{\mathrm{op}}, \mathrm{Set}_*)$. More precisely, the product sends a pair of elements $Q \in F_\infty(N, X, \Delta^k)$ and $R \in F_\infty(M, X, \Delta^k)$ to the element of $F_\infty(N + M, X, \Delta^k)$ given by the pullback along the diagonal map $X \to X \times X$ of the composite

$$Q \times R \hookrightarrow X \times (\mathbb{C}^N)^\infty \times X \times (\mathbb{C}^M)^\infty \xrightarrow{\sim} X \times X \times (\mathbb{C}^{N+M})^\infty.$$

So it suffices to show that this product is compatible with the σ-module structure, i.e. that it descends to an associative product for the σ-relative tensor product. For that it suffices to show that the two maps $F_\infty \otimes S \otimes F_\infty \rightrightarrows F_\infty \otimes F_\infty$ become equal after composing with the product $F_\infty \otimes F_\infty \to F_\infty$ (and same for F_1). That follows from the fact that the cartesian product is associative and that pullbacks and cartesian products commute.

Finally F_∞ is a commutative monoid because the V_∞-manifolds $(\mathrm{diag} \times f_{kl})^*(Q \times R)$ and $(\mathrm{diag} \times f_{kl})^*(R \times Q)$ agree since they are identical as submanifolds-with-corners-with-C_p-action on the 1-corners, and their MU-structures agree because both pairs of maps $(BU_N \times BU_M \to BU_{N+M}$ and $BU_M \times BU_N \to BU_{N+M})$ and $(\mathbb{C}^N \times \mathbb{C}^M \to \mathbb{C}^{N+M}$ and $\mathbb{C}^M \times \mathbb{C}^N \to \mathbb{C}^{N+M})$ are related by the twist map of their domains. □

5.7 The geometry of structured pushouts

This subsection requires some familiarity with the theory of pushouts in E_1- and E_∞-ring spectra. Really all the reader needs to believe is that these pushouts exist and have the evident universal properties. A detailed treatment can of course be found in [21].

What I will show is that certain structured (E_1- and E_∞) pushouts (which are naturally filtered spectra) agree as filtered spectra with geometrically defined spectra with filtration-by-singularities (as in the definition of Bullet's mod p bordism spectra, cf. 5.3.15).

5.7.1 mod p bordism spectra

In this section I will prove that the spectra V_∞ and V_1 (cf. Definitions 5.3.15 and 5.3.21) agree as filtered spectra with certain E_∞ and E_1-pushouts.

Definition 5.7.1. Let $\mathrm{tr} : BC_p \to \mathbb{S}$ be the transfer map, presented geometrically as the map that sends a framed manifold with a map to BC_p to the total space of the associated C_p-bundle (which is again a framed manifold). Let tr also denote the induced map $MU \otimes BC_p \to MU \otimes \mathbb{S}$, which has the same geometric presentation but with 'framed' replaced by 'stably almost complex.'

The following is a precise definition of "the E_∞ quotient of MU by tr."

Definition 5.7.2. Define the E_∞ MU-algebra $MU /\!\!/_\infty \mathrm{tr}$ as the following pushout in the category of E_∞ MU-algebras:

$$\begin{array}{ccc} \mathrm{Free}(BC_p) & \xrightarrow{\mathrm{tr}} & MU \\ \downarrow{\scriptstyle 0} & & \downarrow \\ MU & \longrightarrow & MU /\!\!/_\infty \mathrm{tr} \end{array} .$$

Remark 5.7.3. This means (by definition) that the space of E_∞ MU-algebra maps $MU /\!\!/_\infty \mathrm{tr} \to R$ is naturally equivalent to the space of nullhomotopies $MU \otimes \to MU \to R$.

Lemma 5.7.4. *$MU /\!\!/_\infty tr$ admits a filtration*

$$MU \simeq MU /\!\!/_\infty tr[0] \to MU /\!\!/_\infty tr[2] \to \ldots \to MU /\!\!/_\infty tr$$

by MU-modules with successive quotients given by

$$MU /\!\!/_\infty tr[k] / MU /\!\!/_\infty tr[k-1] \simeq (MU \otimes \Sigma BC_p)^{\otimes_{MU} k}_{h\Sigma_k}.$$

Proof. To get the desired filtration, lift to the category of filtered E_∞ MU-algebras and form the pushout of

$$\begin{array}{ccc} \mathrm{Free}(BC_p[2]) & \xrightarrow{\mathrm{tr}} & MU \\ \downarrow{\scriptstyle 0} & & \downarrow \\ MU & \longrightarrow & MU /\!\!/_\infty \mathrm{tr}^{\mathrm{fil}} \end{array} .$$

To analyze the successive quotients, apply the associated graded functor to present $\mathrm{gr}MU/\!\!/_{\infty}\mathrm{tr}^{\mathrm{fil}}$ as the pushout

$$\begin{array}{ccc} \mathrm{Free}(BC_p[2]) & \xrightarrow{\mathrm{gr}(\mathrm{tr})} & MU \\ \downarrow{\scriptstyle 0} & & \downarrow \\ MU & \longrightarrow & \mathrm{gr}MU/\!\!/_{\infty}\mathrm{tr}^{\mathrm{fil}} \end{array}.$$

Since $\mathrm{gr}(\mathrm{tr}) = 0$, that pushout is $\mathrm{Free}(\Sigma BC_p[2])$, from which the successive quotients are read off immediately. □

In exactly the same way we can define the E_1-quotient of MU by the transfer map.

Definition 5.7.5. Define the E_1 MU-algebra $MU/\!\!/_1\mathrm{tr}$ by declaring the following diagram to be a pushout in the category of E_1 MU-algebras:

$$\begin{array}{ccc} \mathrm{Free}(BC_p) & \xrightarrow{\mathrm{tr}} & MU \\ \downarrow{\scriptstyle 0} & & \downarrow \\ MU & \longrightarrow & MU/\!\!/_1\mathrm{tr} \end{array}.$$

Lemma 5.7.6. *$MU/\!\!/_1tr$ admits a filtration*

$$MU \simeq MU/\!\!/_1tr[0] \to MU/\!\!/_1tr[2] \to \ldots \to MU/\!\!/_1tr$$

by MU-modules with successive quotients given by

$$MU/\!\!/_1tr[k]/MU/\!\!/_1tr[k-1] \simeq MU \otimes (\Sigma BC_p)^{\otimes k}.$$

Proof. The proof is identical to the E_∞ case above except that one works in the category of E_1 MU-algebras. □

Lemma 5.7.7. *There are filtered equivalences*

$$MU/\!\!/_{\infty}tr \xrightarrow{\sim} V_\infty$$

$$MU/\!\!/_1tr \xrightarrow{\sim} V_1$$

of E_∞ and E_1 ring spectra, respectively.

Proof. I will prove the first equivalence; the second is proved completely analogously. First note that by Lemma 5.6.6 V_∞ is a filtered E_∞-ring equipped with an E_∞ map from MU (presented by the map of symmetric stable functors induced by the inclusion of those V_∞-manifolds with no singularities) and hence a filtered E_∞ MU-algebra. Moreover by Lemma 5.3.20 the first filtration step ${}^1V_\infty$ is identified with the cofiber of the transfer map $MU \otimes BC_p \to MU$, and since the unit map $MU \to V_\infty$ factors through ${}^1V_\infty$, the composite

$$MU \otimes BC_p \to MU \to V_\infty$$

has a nullhomotopy. Equivalently, the unit of V_∞ factors through the cone of the transfer. By the universal property of (the filtered version of, cf. Lemma 5.7.4) $MU/\!\!/_\infty\mathrm{tr}$, a factorization of the unit of V_∞ through a filtered map from the cone of the transfer i.e. a commutative diagram

$$\begin{array}{ccc} MU & \longrightarrow & \mathrm{cone}(\mathrm{tr}) \\ \downarrow & & \downarrow \\ MU & \longrightarrow & {}^1V_\infty \end{array},$$

we get a corresponding filtered E_∞ MU-algebra map

$$\phi : MU/\!\!/_\infty\mathrm{tr} \to V_\infty.$$

Now consider the induced E_∞-map on associated graded rings

$$\mathrm{gr}\phi : \mathrm{gr}MU/\!\!/_\infty\mathrm{tr} \to \mathrm{gr}V_\infty.$$

Applying Lemmas 5.3.20 and 5.7.4 to identify the associated graded spectra we get a filtered E_∞ map

$$\mathrm{gr}\phi : \mathrm{Free}_{E_\infty\text{-}MU}(\Sigma BC_p) \to \mathrm{Free}_{E_\infty\text{-}MU}(\Sigma BC_p).$$

To check whether that is an equivalence it suffices to check whether the filtration 1 component $\mathrm{gr}_1\phi : \Sigma BC_p \to \Sigma BC_p$ is an equivalence. But that is clear since ϕ was freely determined by its filtration 1 component, which was chosen to be an equivalence between the cone of the transfer and ${}^1V_\infty$. □

5.7.2 Ravenel's $X(n)$

Recall that Ravenel defines a filtration of MU by spectra $X(n)$: $\mathbb{S} = X(1) \to X(2) \to \ldots \to X(\infty) = MU$ and that these spectra feature prominently in the celebrated nilpotence theorem of Devinatz, Hopkins, and Smith.

Definition 5.7.8. Recall that $X(n+1)$ may be presented as the Thom spectrum of the map $\Omega SU(n+1) \to BU \to BGL_1\mathbb{S}$. Consider the map $\Omega SU(n+1) \to \Omega(SU(n+1)/SU(n)) = \Omega S^{2n+1}$. The James filtration J_kS^{2n} of the latter induces a filtration $F_k\Omega SU(n+1)$ in the domain which, after applying the Thom spectrum functor, induces the *DHS filtration* of $X(n+1)$ by spectra I will denote by F_k. Note that the filtration is multiplicative because the James filtration is.

Definition 5.7.9. Let $\Sigma^{-2}\mathbb{CP}^n \to X(n)$ be the canonical partial orientation. Let $\chi_n : S^{2n-1} \to X(n)$ be the precomposite of that with the attaching map of the top cell of $\Sigma^{-2}\mathbb{CP}^{n+1}$.

Definition 5.7.10. An $X(n)$-manifold is a manifold whose stable normal bundle is equipped with a lift along the map $\Omega SU(n) \subset \Omega SU \simeq BU \to BO$.

Remark 5.7.11. Note that χ_n is represented by the $X(n)$-manifold whose underlying manifold is S^{2n-1} and whose stable normal bundle is equipped with the lift to $\Omega SU(n)$ given by the map $S^{2n-1} \to \mathbb{CP}^{n-1} \to \Omega SU(n)$ which is the composite of the standard quotient map and the map that sends a line in $\mathbb{C}^n$ to the loop of special unitary matrices given by rotation around that line[7]. That indeed lifts the (trivializable) stable normal bundle of S^{2n-1} since the composite with $\Omega SU(n) \to BU$ is null. Write χ_n^k for the k-fold Cartesian product.

Definition 5.7.12. An $X(n)^k$-manifold Q is a manifold-with-labelled-j-corners for $0 \leqslant j \leqslant k$, together with a $X(n)$-structure on the interior of the codimension 0 stratum, an identification of each face d_iQ of the 1-corners (as $X(n)$-manifolds) with $N_i \times \chi_n$ for some $X(n)$-manifold $N_1(i)$, and such that on the j-corners the identifications combines to an identification (again, as $X(n)$-manifolds) of each face d_JQ the j-corners with $N_J \times \chi_n^j$ for some $X(n)$-manifold N_j, which will be called the *leftovers* of the j-corner. The associated bordism theory gives a geometric presentation of a spectrum that will be called $X(n)^k$.

Definition 5.7.13. An $X(n)^k$-manifold is naturally an $X(n)^{k+1}$-manifold, which induces maps $X(n) \to X(n)^1 \to X(n)^2 \to ....$ Define the colimit to be $X(n)^\infty$, presented geometrically as the bordism theory of $X(n)^k$-manifolds for all k at once, which are called $X(n)^\infty$-manifolds.

Remark 5.7.14. The cartesian product of an $X(n)^k$-manifold and an $X(n)^l$-manifold is an $X(n)^{k+j}$, which equips $X(n)^\infty$ with a natural ring spectrum; in fact it lifts to an E_1-ring structure (cf. [L] Section 2.2.2). This will not be needed here.

Definition 5.7.15. Consider the standard topological k-simplex $\Delta^k_{\text{top}} \subset \mathbb{R}^{k+1}$. Since the faces of the k-simplex are ordered the product $\Delta^k_{\text{top}} \times \chi_n^{k+1}$ is naturally an $X(n)^k$-manifold[8]. Note that the product $[0,1] \times \Delta^k_{\text{top}} \times \chi_n^{k+1}$ is naturally an $X(n)^{k+1}$-manifold (by Remark 5.7.14) and the composite of the projection to the first factor and the map $[0,1] \to \mathbb{R}$, $x \mapsto x - 1/2$ exhibits the unravelled $(k+1)$-fold join as nullbordant in $X(n)^{k+1}$-manifolds.

Lemma 5.7.16. *In the defining filtration of $X(n)^\infty$ in Definition 5.7.13, the successive quotients are given by*

$$X(n)^k / X(n)^{k-1} \simeq \Sigma^{2nk} X(n)$$

The quotient map is geometrically presented as the map that sends an $X(n)^k$-manifold Q its k-corners' leftovers N_k (c.f. Definition 5.7.12). Moreover, the attaching map

$$\Sigma^{2nk-1} X(n) \to \ X(n)^{k-1}$$

[7] One normalizes by the value at some point to get matrices with unit determinant.

[8] On the ith face one identifies χ_n^{k+1} with $N_i \times \chi_n$ where N_i is the product of the k factors of χ_n^{k+1} excluding the ith factor.

whose cofiber is $X(n)^k$ is presented geometrically as the map that sends an $X(n)$-manifold P to the ${}^{k-1}X(n)^\infty$-manifold given by the product $P \times \Delta^{k-1}_{top} \times \chi_n^k$ with the unravelled k-fold join (cf. Definition 5.7.15).

Proof. (cf. [L] Lemma 2.1.15) It suffices to show that the geometrically defined maps

$$\ldots \to \Sigma^{2nk-1}X(n) \to X(n)^{k-1} \to X(n)^k \to \Sigma^{2nk}X(n) \to \ldots$$

form a cofiber sequence, which is equivalent to the statement that the corresponding maps of collections of manifolds(-with-singularities) form an exact sequence up to bordism. The composite of the left two maps is zero since the $X(n)^{k-1}$-manifold $\Delta^{k-1}_{\text{top}} \times \chi_n^k$ is null as an $X(n)^k$-manifold because the cone on the unravelled join is (cf. Definition 5.7.15). If an $X(n)^{k-1}$-manifold Q is null as an $X(n)^k$-manifold, then any such nullbordism M exhibits Q as bordant to the k-corners of M, which is in the image of the first map. The composite of the second two maps is clearly zero since an $X(n)^{k-1}$-manifold has empty k-corners. Finally, if Q is an $X(n)^k$-manifold whose k-corners' leftover N_k is null, then gluing such a nullbordism into Q eliminates the k-corners and produces an $X(n)^{k-1}$-manifold, exhibiting M as bordant to the image of the inclusion ${}^{k-1}X(n)^\infty \to X(n)^k$. □

Definition 5.7.17. Here is an inductive definition of $X(n)^k$-manifolds that may be more geometrically intuitive. An $X(n)^1$-manifold Q in the above definition is an $X(n)$-manifold with boundary identified with $N_1 \times \chi_n$. Since χ_n is presented by S^{2n-1}, one can glue in a copy of $N_1 \times D^{2n}$ to obtain a smooth manifold Q' with a "singular" $X(n)$-structure[9]. Call such a manifold an $X(n)^1_{\text{filled}}$-manifold. Note that the filled version of the unravelled join $\Delta^1_{\text{top}} \times \chi_n^2$ has underlying manifold the usual join $\chi_n * \chi_n \simeq S^{4n-1}$. Then using the attaching maps of the filtration above one can inductively view an $X(n)^k$-manifold Q as an $X(n)^{k-1}_{\text{filled}}$-manifold with boundary identified with $N_k \times \chi_n^{*k}$ as an $X(n)^{k-1}_{\text{filled}}$-manifold. Again χ_n^{*k} is a sphere, so gluing in $N_k \times D^{2nk}$ produces an underlying smooth manifold Q' with "up-to-k-fold singularities" in its $X(n)$-structure, an $X(n)^k_{\text{filled}}$-manifold.

Note that an $X(n)^k_{\text{filled}}$-manifold is in particular a stratified manifold. Cutting along those "singularity strata" recovers the $X(n)^k$-manifold.

Here is a geometric presentation of the DHS filtration.

Lemma 5.7.18. *The DHS filtration admits a geometric presentation: there is a "filling" map inducing a filtered equivalence $X(n)^\infty \to X(n+1)$.*

Proof. The first step is to define the map. It will morally come down to the fact that in the "filled" picture of Definition 5.7.17 the $X(n)$-structures can be extended into the interior of the disks if one allows them to become $X(n+1)$-structures.

Fix a nullhomotopy of the composite $S^{2n-1} \to \Omega SU(n) \hookrightarrow F_1\Omega SU(n+1)$, which exists because the first map is homotopic to the adjoint of the map $S^{2n} \to SU(n)$ classifying

[9] One leaves the $X(n)$-structure in the interior of the disk undefined.

$F_1\Omega SU(n+1)$ as an $\Omega SU(n)$-bundle. That nullhomotopy exhibits the cone[10] on χ_n as an F_1-manifold-with-boundary. Call it z_1. Replacing D^{2n} with z_1 in the "filling" procedure of Remark 5.7.17 shows that every $X(n)^1$-manifold naturally produces an F_1-manifold with the $X(n)^1$-manifold as a subset on which the F_1-structure specializes to an $F_0 = X(n)$-structure. In particular the $X(n)^1_{\text{filled}}$-manifold $\chi_n * \chi_n$ (see Definition 5.7.17) is naturally an F_1-manifold, with F_1-structure classified by a map $S^{4n-1} \to F_1\Omega SU(n+1)$.

Towards induction, suppose that z_{k-1} is an F_{k-1}-manifold-with-boundary whose underlying manifold is $D^{2n(k-1)}$ and whose boundary is the $X(n)^{k-2}_{\text{filled}}$-manifold $\chi_n^{*(k-1)}$, in the sense that the F_{k-1} structure specializes to the $F_0 = X(n)$-structure of the $X(n)^{k-2}_{\text{filled}}$-manifold $\chi_n^{*(k-1)}$ on the locus where the latter is defined. Then the filling procedure of Definition 5.7.17 can be done to every $X(n)^{k-1}$-manifold to produce an F_{k-1}-manifold with the $X(n)^{k-1}$-manifold as a subset on which the F_{k-1}-structure specializes to an $F_0 = X(n)$-structure. Next I claim that $z_k := z_{k-1} * S^{2n-1}$ is naturally an F_k-manifold. Indeed, the underlying manifold is $D^{2nk} = D^{2n(k-1)} * S^{2n-1}$. To specify the map out of a join to F_k it is equivalent to specify maps $D^{2n(k-1)} \to F_k$, $S^{2n-1} \to F_k$ and a homotopy between the two induced maps $D^{2n(k-1)} \times S^{2n-1} \to F_k$ (using the projections to each factor). Then the maps are $D^{2n(k-1)} \to F_{k-1} \to F_k$, $S^{2n-1} \to F_1 \to F_k$, and the homotopy between the two corresponds to the pushforward along $F_1 \to F_k$ of the nullhomotopy of $S^{2n-1} \to F_1$ chosen to define z_1.

By induction on k the "filling" map $X(n)^\infty \to X(n+1)$ is defined and is clearly a map of filtered spectra (an $X(n)^k$-manifold "fills" to an F_k-manifold). Therefore it is an equivalence if it induces an equivalence of associated graded spectra. Note that the kth graded piece on both sides are indeed equivalent—that of $X(n)^\infty$ is identified in Lemma 5.7.16 and that of $X(n+1)$ is the Thom spectrum of the trivial map from $S^{2nk} = J_kS^{2n}/J_{k-1}S^{2n} \to BGL_1(X(n))$. I claim that the map on associated graded spectra can be presented by the identity map, which will finish the proof. First, the map

$$F_k \to \Sigma^{2nk}X(n)$$

has the following geometric presentation: it takes an F_k-manifold Q, composes the structure map $Q \to F_k\Omega SU(n+1)$ with the $\Omega SU(n)$-fibration $F_k\Omega SU(n+1) \to J_kS^{2n}$ and takes the transverse intersection of Q and a nondegenerate point in $J_kS^{2nk}\backslash J_{k-1}S^{2nk}$. That transverse intersection is indeed an $X(n)$-manifold (since its stable normal bundle is equipped with a map to the fiber $\Omega SU(n)$) and is of codimension $2nk$ in Q (since the point is of local codimension $2nk$ in J_kS^{2n}). The map $F_{k-1} \to F_k$ has the evident the geometric presentation: "extension of structure group". Now consider the following diagram. Having given geometric presentations of all maps—with the candidate geometric presentation of the bottom map being the identity—it suffices to show that the diagram commutes up to bordism, i.e.

[10] Recall that the underyling manifold is D^{2n} and naturally has an $X(n)^1_{\text{filled}}$-manifold (c.f. Definition 5.7.17).

produces a map of cofiber sequences.

$$\begin{array}{ccc} X(n)^{k-1} & \longrightarrow & F_{k-1} \\ \downarrow & & \downarrow \\ X(n)^k & \longrightarrow & F_k X(n+1) \\ \downarrow & & \downarrow \\ \Sigma^{2nk} X(n) & \longrightarrow & \Sigma^{2nk} X(n) \end{array}$$

The top square clearly commutes even without the need to correct by a bordism. It remains to compare the two ways to traverse the bottom square. The way involving the left vertical map sends an $X(n)^k$ manifold to its k-corners leftovers N_k (c.f. 5.7.16). The other way takes an $X(n)^k$-manifold Q, glues in the chosen z_j to get some Q', and then takes the transverse intersection of Q' with a point in $S^{2nk} \subset J_k S^{2n}$. Now, the map classifying the stable normal bundle of Q' factors through $F_{k-1}\Omega SU(n+1) \to F_k \Omega SU(n+1)$ except in a neighborhood of the glued in $N_k \times z_k$, since all other pieces that are glued in (and all of the original Q) have stable normal bundles admitting such factorizations. The F_k structure on $N_k \times z_k$ is given by the map

$$N_k \times z_k \to \Omega SU(n) \times F_k \to F_k$$

which is the composite of the cartesian product of the $X(n)$-structure of N_k and the F_k structure of z_k followed by the $\Omega SU(n)$-action on F_k. It follows that the composite with $F_k \to J_k S^{2n}$ factors through the projection $N_k \times z_k \to z_k$, and the map $D^{2nk} = z_k \to F_k \to J_k S^{2n} \to S^{2nk}$ is a degree ± 1 covering in a neighborhood of some nondegenerate point by construction (it is clear for $k = 1$ and true for general k by induction). So the transverse intersection at that point will indeed be bordant to N_k.

□

Remark 5.7.19. It is fair to say that $X(n)^\infty$ is a geometric incarnation of $X(n)/\!\!/_1\chi_n$, the E_1-$X(n)$-algebra quotient of $X(n)$ by χ_n. In that sense, Lemma 5.7.18 is a geometric incarnation of [6] Corollary 13, which identifies $X(n)/\!\!/_1\chi_n$ with $X(n+1)$.

5.7.3 Nilpotence

The the celebrated nilpotence theorem of Devinatz, Hopkins, and Smith [12] reads as follows.

Theorem 5.7.20. *For any homotopy associative ring spectrum R, the kernel of the Hurewicz map $R_* \to MU_*R$ consists of nilpotent elements.*

The proof makes crucial use of the filtration of MU by the $X(n)$ and in turn the DHS filtration of each $X(n)$ (cf. Definition 5.7.8). The key is to analyze the attaching maps of the DHS filtration, which is the heart of the proof of the nilpotence theorem. That analysis is originally done p-locally and rather algebraically phrased in terms of Bousfield classes. In work in progress I hope to use Lemma 5.7.18 to analyze the DHS attaching maps geometrically and arrive at a new proof of the nilpotence theorem.

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