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Publication Date
2004-07-01

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Working Paper No. 04-011

July, 2004

Giannini Foundation of Agricultural Economics

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ABSTRACT

When applied to time series processes containing occasional level shifts, the log-periodogram (GPH) estimator often erroneously finds long memory. For a stationary short-memory process with a slowly varying level, I show that the GPH estimator is substantially biased, and I derive an approximation to this bias. The asymptotic bias lies on the (0,1) interval, and its exact value depends on the ratio of the expected number of level shifts to a user-defined bandwidth parameter. Using this result, I formulate the Modified GPH estimator, which has a markedly lower bias. I illustrate this new estimator via applications to soybean prices and stock market volatility.

Key Words: structural break, fractional integration, mean shift.

1. INTRODUCTION

The study of long memory in time series processes dates back at least to Hurst (1951), and Granger and Joyeux (1980) and Granger (1981) introduced it to econometrics. Long-memory models prove useful in economics as a parsimonious way of modeling highly persistent yet mean-reverting processes. They apply to stock price volatility, commodity prices, interest rates, aggregate output, and numerous other series. Recently, Diebold and Inoue (2001), Liu (2000), Granger and Hyung (1999), Granger and Ding (1996), Lobato and Savin (1998), Hidalgo and Robinson (1996), Breidt and Hsu (2002), and others have suggested that the apparent long memory in many time series is an illusion generated by occasional level shifts. If this suggestion is correct, then a few rare shocks induce the observed persistence, while most shocks dissipate quickly. In contrast, all shocks are equally persistent in a long memory model. Thus, distinguishing between long memory and level shifts could dramatically improve policy analysis and forecasting performance.

Despite much anecdotal evidence that implicates level shifts as the cause of long memory, the properties of long-memory tests in the context of level shifts are not well understood. In this paper, I show formally that a popular test for long memory is substantially biased when applied to short memory processes with slowly varying means. This bias leads to the erroneous conclusion that these processes have long memory. I derive an approximation to the bias and show that it depends only on the ratio of the expected number of level shifts to a user-defined bandwidth parameter. This result illuminates the connection between long memory and level shifts, and it leads directly to a simple method for bias reduction.

A long-memory process is defined by an unbounded power spectrum at frequency zero. The most common long memory model is the fractionally integrated process, for which the elasticity of the spectrum at low frequencies measures the fractional order of integration, *d*. This feature motivates the popular log-periodogram, or GPH, regression (Geweke and Porter-Hudak, 1983). Specifically, the GPH estimate of *d* is the slope coefficient in a regression of the log periodogram on two times the log frequency.

Figure 1 illustrates why estimators such as GPH may erroneously indicate long memory when applied to level-shift processes. The upper panel shows the power spectra of a fractionally integrated long-memory process and a short-memory process with occasional level shifts. A sequence of Bernoulli trials determines the timing of the level shifts, and a draw from a distribution with finite variance determines each new level. The curves cover frequencies, ω, between zero and $2\pi/\sqrt{1000}$, which is the range recommended by a common rule of thumb for GPH regression with a sample of 1000 observations. For all but the very lowest frequencies, the spectrum of the level-shift process closely corresponds to that of the fractionally integrated process.

The lower panel of Figure 1 shows the log spectrum evaluated at the log of the first 32 Fourier frequencies, i.e., $\omega_j = 2\pi j/T$, $j = 1,2,\ldots,32$, $T = 1000$. These points represent the actual observations that would be used in a GPH regression based on the aforementioned rule of thumb. I did not incorporate sample error into these graphs, but it is already apparent that $d = 0.6$ fits this level-shift process better than the true value of zero. The fact that the spectrum of the level-shift process is flat at frequency zero only becomes evident at the very lowest Fourier frequencies. To obtain a GPH estimate near the true value of zero, one would need to focus on the far left portion of the spectrum, either by including a very low number of frequencies in the regression or by increasing the sample size *T*.

Figure 1 demonstrates the dominant source of bias in the GPH estimator. The short-memory components create bias because the estimator includes parts of the spectrum away from zero. A second source of bias is induced by the use of the log periodogram, which is a biased estimator of the log spectrum. In Section 3, I demonstrate that the first source of bias clearly dominates the second, even in small samples. Further, I show that, when the shifts are rare relative to the sample size, the dominant component of the GPH bias converges to a value on the interval (0,1) as the sample size grows. This result applies to a general class of mean-plus-noise processes, which I present in Section 2.

In Section 4, I use the asymptotic bias result to propose the Modified GPH estimator, which has smaller bias than the GPH estimator. I derive the asymptotic properties of the proposed method and illustrate it with applications to the relative price of soybeans to soybean oil and to volatility in the S&P 500. Section 5 provides concluding remarks, and an appendix contains proofs of all theorems.

2. A GENERAL MEAN-PLUS-NOISE PROCESS

This section presents a general mean-plus-noise (MN) process with short memory, which can be written as

$$y_t = \mu_t + \varepsilon_t \qquad (1)$$

$$\mu_t = (1-p)\mu_{t-1} + \sqrt{p}\eta_t\,, \qquad (2)$$

where ε_t and η_t denote short memory random variables with finite nonzero variance. Without loss of generality, I assume that ε_t and η_t have mean zero. I also assume that ε_t

and η_s and are independent of each other for all t and s. The parameter p determines the persistence of the level component, μ_t. If p is small, then the level varies slowly. To prevent the variance of μ_t from blowing up as p goes to zero, I scale the innovation in (2) by $\sqrt{p}$. Assuming Gaussianity, the GPH estimator is consistent and asymptotically normal when applied to this process (Hurvich et al., 1998). However, I show in Section 3 that the estimator is substantially biased when p is small, even for large T.

The MN process with small p and a fractionally integrated process with $d < 1$ are both highly dependent mean-reverting processes. Thus, they compete as potential model specifications for persistent mean-reverting data. However, the application of GPH regression to non-mean-reverting processes with level shifts has received some attention. Granger and Hyung (1999) and Diebold and Inoue (2001) studied the random level-shift process of Chen and Tiao (1990). Diebold and Inoue (2001) also studied the stochastic permanent breaks (STOPBREAK) model of Engle and Smith (1999).

The MN process in (1) and (2) places few restrictions on how the level component evolves. For example, if η_t is Gaussian, the level evolves continuously and the process is a linear ARMA process with autoregressive and moving average roots that almost cancel out. However, the MN process also encompasses nonlinear models with discrete level-shifts. Two prominent examples are stationary random level shifts (Chen and Tiao, 1990) and Markov switching (Hamilton, 1989), each of which I discuss in more detail below.

2.1 Random Level Shifts

Consider the following random-level-shift (RLS) specification for μ_t:

$$\mu_t = (1 - s_t)\mu_{t-1} + s_t \xi_t, \tag{3}$$

where s_t ~ *iid* Bernoulli(p) and ξ_t denotes a short-memory process with mean zero and variance σ_ξ^2. Each period, the level either equals its previous value or is drawn from some distribution with finite variance. To see that (2) encompasses (3), rewrite (3) as $\mu_t = (1-p)\mu_{t-1} + \sqrt{p}\eta_t$, where $\sqrt{p}\eta_t \equiv s_t\xi_t + (p - s_t)\mu_{t-1}$ and $E(\eta_t^2) = (2-p)\sigma_\xi^2$. Note that the variance of μ_t equals σ_ξ^2, and thus it does not blow up as p goes to zero.

The stationary RLS process in (3) resembles the random-level-shift process of Chen and Tiao (1990). The only difference is that Chen and Tiao specify the level as $\mu_t = \mu_{t-1} + s_t\xi_t$, which is not mean reverting. As Chen and Tiao (1990, pg. 85) state, this model "provides a convenient framework to assess the performance of a standard time series method on series with level shifts." They study ARIMA approximations to the RLS process and apply the model to variety store sales. McCulloch and Tsay (1993) use the Gibbs sampler to estimate a RLS model of retail gasoline prices. Such simulation methods are necessary to make inference with this model because integrating over the 2^T possible sequences of the state process $\{s_t\}$ is infeasible, even for moderate sample sizes.

Chib (1998) and Timmermann (2001) analyze a hidden Markov process that is observationally equivalent to the RLS process. However, they condition on the states, μ_t, and treat them as parameters to be estimated. They do not assume that the timing of the breaks is fixed, unlike the conventional deterministic break-point analysis (e.g., Bai and Perron, 1998), which conditions on both the timing (s_t) and the level (ξ_t) of the breaks. When one conditions on s_t or ξ_t, the moments of y_t are time varying and it is unclear what the spectrum represents or if it exists. For this reason, I treat s_t and ξ_t as realizations from a stationary stochastic process to obtain a well-defined spectrum.

2.2 Markov Switching

In a Markov-switching process (Hamilton, 1989), the level switches between a finite number of discrete values. Since Hamilton's seminal paper, Markov-switching models have been applied extensively in economics and finance. For a two-state model, the level equation is

$$\mu_t = (1 - s_t) m_0 + s_t m_1 , \tag{4}$$

where m_0 and m_1 denote finite constants and the state variable $s_t \in \{0,1\}$ evolves according to a Markov chain with the transition matrix

$$P = \begin{bmatrix} 1 - p_0 & p_1 \\ p_0 & 1 - p_1 \end{bmatrix}.$$

If the parameters p_0 and p_1 are small, then level shifts are rare.

Following Hamilton (1994, pg. 684), we can express s_t as an AR(1) process

$$s_t = p_0 + (1 - p_0 - p_1) s_{t-1} + \sqrt{p_0 + p_1}\ \nu_t \tag{5}$$

where $E(\nu_t | s_{t-1}) = 0$ and

$$E(\nu_t^2) = \frac{p_0 p_1 (2 - p_0 - p_1)}{(p_0 + p_1)^2} \equiv \sigma_\nu^2 .$$

Combining (4) and (5) yields an AR(1) representation for μ_t

$$\mu_t = p_0 m_1 + p_1 m_0 + (1 - p_0 - p_1) \mu_{t-1} + (m_1 - m_0) \sqrt{p_0 + p_1}\ \nu_t . \tag{6}$$

Without loss of generality, we can set $p_0 m_1 + p_1 m_0 = 0$ and it follows that the MN process in (2) encompasses (6). This illustration can easily be extended to allow for more than two states. Thus, the MN specification incorporates models with discrete as well as continuous state space.

3. BIAS IN THE GPH ESTIMATOR

Given the independence of ε_t and η_s for all t and s, the spectrum of the MN process at some frequency ω is

$$f(\omega) = f_\varepsilon(\omega) + f_\mu(\omega) = f_\varepsilon(\omega) + \frac{p}{p^2 + (1-p)(2-2\cos(\omega))} f_\eta(\omega), \qquad (7)$$

where $f_\varepsilon(\omega)$, $f_\mu(\omega)$, and $f_\eta(\omega)$ denote the spectra of ε_t, μ_t, and η_t, respectively. The order of integration, d, can be computed from the elasticity of the spectrum at frequencies arbitrarily close to zero, i.e.,

$$\begin{aligned} d &= -0.5 \lim_{\omega \to 0} \frac{\partial \log f}{\partial \log \omega} \\ &= -0.5 \lim_{\omega \to 0} \frac{\omega}{f(\omega)} \left(f'_\varepsilon(\omega) - \frac{p2(1-p)\sin(\omega) f_\eta(\omega)}{\left(p^2 + (1-p)(2-2\cos(\omega))\right)^2} + \frac{p f'_\eta(\omega)}{p^2 + (1-p)(2-2\cos(\omega))} \right) \\ &= 0. \end{aligned}$$

Thus, as for any short-memory process, the correct value of d equals zero.

The GPH estimate of d equals the least squares coefficient from a regression of the log periodogram on $X_j \equiv -\log\left(2 - 2\cos(\omega_j)\right) \approx -\log \omega_j^2$ for j = 1, 2, …, J where $\omega_j = 2\pi j / T$ and $J < T$. For this estimator to be consistent, it must be that $J \to \infty$ as $T \to \infty$. However, because long memory reveals itself in the properties of the spectrum at low frequencies, J must be small relative to T; that is, a necessary condition for consistency is that $J/T \to 0$ as $T \to \infty$. A popular rule of thumb is $J = T^{1/2}$, as recommended originally by Geweke and Porter-Hudak (1983).

The GPH estimate is

$$\hat{d} = d_* + \frac{\sum_{j=1}^{J} (X_j - \overline{X}) \log(\hat{f}_j / f_j)}{\sum_{j=1}^{J} (X_j - \overline{X})^2},$$

where $\hat{f}_j$ denotes the periodogram evaluated at ω_j, f_j denotes the spectrum evaluated at ω_j, and

$$d_* \equiv \frac{\sum_{j=1}^{J}(X_j - \overline{X})\log f_j}{\sum_{j=1}^{J}(X_j - \overline{X})^2}. \tag{8}$$

For the MN process, the true value of d equals zero and the bias of the GPH estimator is

$$bias(\hat{d}) = d_* + \frac{\sum_{j=1}^{J}(X_j - \overline{X})E\left(\log(\hat{f}_j / f_j)\right)}{\sum_{j=1}^{J}(X_j - \overline{X})^2}. \tag{9}$$

The first term, d_*, represents the bias induced by the short-memory components of the time series. The second term arises because the log periodogram is a biased estimator of the log spectrum.

Hurvich et al. (1998) proved that, for Gaussian long memory processes, the second term in (9) is $O(\log^3 J / J)$ and therefore negligible. Deo and Hurvich (2001) obtain similar asymptotic results for the GPH estimator in a partially non-Gaussian stochastic volatility model. However they require the long memory component of the model to be Gaussian. For fully non-Gaussian processes, such as the RLS process in (3), existing theoretical results require that the periodogram ordinates be pooled across frequencies before running the log periodogram regression (Velasco, 2000). This pooling enables the second term to be proved to be negligible. Thus, for non-Gaussian processes like RLS or Markov-switching, current theoretical results do not allow formal treatment of the second component in (9). Nonetheless, the simulations in Section 3.1 suggest that this component is unimportant.

3.1 Illustrating the GPH Bias

To demonstrate the GPH bias, I simulate data from the stationary RLS process for various parameter settings and apply the GPH estimator using the rule-of-thumb value $J = T^{1/2}$. I present the results from GPH estimation in Table 1. The rows labeled GPH contain the mean values of $\hat{d}$ over 1000 Monte Carlo trials, and the rows labeled *Exact* contain the values d_* computed from the population spectrum as in (8). The sample size, T, ranges from 1000 to 10,000. This range corresponds to the samples sizes that typically arise in economics and finance with data measured at weekly or daily frequencies.

For p less than 0.05, the GPH estimator is substantially biased. In almost all cases, however, the exact value d_* closely corresponds to the average GPH estimate. This proximity shows that d_* dominates the GPH bias, and it indicates that we can ignore the contribution of the second term in (9). Akiakloglou et al. (1993) demonstrated the same phenomenon for stationary AR(1) processes.

The only case in Table 1 where the average GPH estimate deviates from d_* is when $T = 1000$ and p is very close to zero. In this case, some of the Monte Carlo realizations contain no level shifts, causing $\hat{d}$ to be close to zero for those realizations. However, conditional on there being at least one break in a sample, the average GPH estimate is close to d_*. This bisection leads to a bimodal distribution for $\hat{d}$, a feature that Diebold and Inoue (2001) also documented. This bimodal property results from the discontinuity in the spectrum of the process at the point where the probability of a break equals zero.

There are several other notable features in Table 1. First, as T increases, the average GPH estimate approaches zero. This convergence is not surprising, given that the dominant term in the bias, d_*, is $O(J^2/T^2)$ and therefore converges to zero as $T \to \infty$

(see Hurvich et al., 1998, Lemma 1). Second, the standard errors monotonically decrease in T in all cases. Third, the average estimate of d increases in σ_ξ^2, the size of the level shifts. This association arises because larger shifts increase the importance of the persistent μ_t term relative to the *iid* ε_t term. However, the effect of shift size on the GPH bias diminishes as T increases. This diminution is consistent with Theorem 1 below, where I show that shift size does not matter asymptotically.

3.2 Asymptotic GPH Bias

Hurvich et al. (1998, Lemma 1) showed that d_* converges pointwise to zero as T increases, i.e., $d_* \rightarrow 0$ as we move from left to right along the rows in Table 1. For large p, this convergence occurs quickly. However, d_* can be far away from zero when p is small, even for large T. When $p = 0$, the value of d_* is identically zero for all T. Thus, the pointwise limit of d_* provides a satisfactory approximation when $p = 0$ and when p is large, but we need a better approximation when p lies in a local neighborhood of zero.

This problem parallels that of estimating the largest root in an autoregression when that root is near unity. Both cases involve an estimator that exhibits substantial bias in the neighborhood of a point of discontinuity. Influential work by Phillips (1988) and Cavanagh et al. (1995) showed that by specifying the autoregressive parameter as lying in a local neighborhood of unity, a better large-sample approximation to the distribution of the least squares estimator could be obtained.

Diebold and Inoue (2001) used a similar technique to analyze a Markov-switching process with rare shifts. They specified the switching probability within a local neighborhood of zero and showed that the variance of partial sums is of the same order of magnitude as the variance of partial sums of a fractionally integrated process. However, this result does not admit a particular order of magnitude for the local neighborhood of zero that contains p. Thus, it implies that a Markov-switching process can be approximated by an integrated process of any nonnegative order, depending the chosen neighborhood. Breidt and Hsu (2002) provided similar results for the RLS process.

In Theorem 1 below, I show that the appropriate choice of neighborhood size depends critically on J, the number of terms in the GPH regression. This dependence on J emanates from the denominator of the second term in the spectrum (7), which is $p^2 + (1-p)(2-2\cos(\omega)) \approx p^2 + \omega^2$ for small ω and p. By setting p to the same order of magnitude as ω, I isolate the dominant component of the spectrum. This isolation produces a good approximation to d_* for small values of p. Specifically, it is appropriate to set $p_T = cJ/T$, where c is a positive constant. Because $\text{var}(T^{-1}\sum_{t=1}^{T}\mu_t) = \sigma_\eta^2 / p_T T$, this condition implies that $\text{var}(T^{-1}\sum_{t=1}^{T}\mu_t) = O(J^{-1})$. The following theorem formalizes these arguments and obtains the asymptotic bias of the GPH estimator.

Theorem 1: Consider the MN process in (1) and (2) and suppose that $f_\varepsilon(\omega) < \tilde{B}_0 < \infty$, $f_\eta(\omega) < \tilde{B}_0 < \infty$, and $| f_\eta'(\omega) | < \tilde{B}_1 < \infty$ for all ω in a neighborhood of zero. Assume that $JT^{-1}\log(J) \to 0$ and let $p_T = cJ/T$ where $c > 0$. Then

$$\lim_{J,T\to\infty} d_* = 1 - 0.25\Phi\left(-\left(2\pi/c\right)^2, 2, 0.5\right),$$

where $\Phi(x,s,a)$ denotes the Lerch transcendent function.

Theorem 1 provides an approximation to the bias in the GPH estimator when a general mean-plus-noise process generates the data. This asymptotic bias is a function of the Lerch transcendent function, also known as the Lerch zeta function, which is a generalization of the Riemann zeta function (Gradshteyn and Ryzhik, 1980, pg 1072). The only parameter in the asymptotic bias is $c = p_T T / J$. This result indicates that the reduction in bias from decreasing *J* asymptotically equals the decrease in bias from a proportionate increase in *p* or *T*. Figure 2 plots the asymptotic bias and shows that it ranges between zero and one and monotonically decreases in *c*. This graph emphasizes the fact that the GPH bias depends both on the properties of the data and on the specification of the estimator. One cannot say that a level-shift process appears to be fractionally integrated of some order *d* without reference to the bandwidth *J*.

Theorem 1 implies that d_* does not converge to zero uniformly in $p \in [0, 1]$, i.e., $\sup_{p_0, p_1 \in [0,1]} d_*$ does not converge to zero as $T \to \infty$. In other words, for every *T* there exists a value of *p* such that $d_* > 0$, despite the fact that d_* converges pointwise to zero. Thus, traveling from left to right along the rows of Table 1, one sees d_* decreasing towards zero, but there exists a path through the table in a southeast direction for which d_* does not converge to zero.

4. MODIFIED GPH REGRESSION

As shown in Section 3, the GPH statistic may erroneously indicate long memory when the MN process generates the data. In this section, I use the result in Theorem 1 to suggest a simple modification to the GPH estimator that reduces bias when the data generating process contains level shifts. An important advantage of the Modified GPH estimator is its simplicity. It can be implemented easily by adding an extra regressor to the GPH regression. This straightforwardness makes it a useful diagnostic tool to signal whether a fully specified model with level shifts could outperform a long-memory model.

The asymptotic bias in Theorem 1 derives from the fact that, when p is small, the dominant component of the spectrum at low frequencies is $-\log(p^2+\omega^2)$ plus a constant. This dominant term is nonlinear in log(ω), so adding $-\log(p^2+\omega^2)$ as an extra regressor in the GPH regression would reduce the bias caused by level shifts. However, this strategy is infeasible because p is unknown. I create a feasible estimator by setting $p_T = kJ/T$ for some constant $k > 0$ and running the regression

$$\log \hat{f}_j = \alpha + dX_j + \beta Z_{kj} + \hat{u}_j ,$$

where

$$Z_{kj} = -\log\left(\frac{(kJ)^2}{T^2} + \omega_j^2\right),$$

and $X_j = -\log\left(2 - 2\cos(\omega_j)\right)$ as before.

The Modified GPH estimator is

$$\hat{d}^k = d_*^k + \left(\tilde{X}' M_Z \tilde{X}\right)^{-1} \tilde{X}' M_Z \log\left(\hat{f}/f\right),$$

where $\tilde{X} \equiv X - \overline{X}$, $M_Z = I - \tilde{Z}_k(\tilde{Z}_k'\tilde{Z}_k)^{-1}\tilde{Z}_k'$, $\tilde{Z}_k \equiv Z_k - \overline{Z}_k$, $\overline{X} = J^{-1}\sum_{j=1}^{J} X_j$, $\overline{Z}_k = J^{-1}\sum_{j=1}^{J} Z_{kj}$, and d_*^k denotes the estimator computed from the spectrum rather than the periodogram. Next, I derive the asymptotic properties of the Modified GPH estimator.

4.1 Asymptotic Properties of the Modified GPH Estimator

The MN process includes Gaussian processes as a special case. However many useful models, such as RLS or Markov switching are non-Gaussian. As discussed in Section 3, current theoretical results do not allow formal treatment of the second component of the bias for log periodogram regression with non-Gaussian data. Thus, as in Theorem 1, I focus on the dominant component of the bias, d_*^k. I derive an approximation to d_*^k for the potentially non-Gaussian MN process.

Theorem 2: Consider the MN process in (1) and (2) and suppose that $f_\varepsilon(\omega) < \widetilde{B}_0 < \infty$, $f_\eta(\omega) < \widetilde{B}_0 < \infty$, and $|f'_\eta(\omega)| < \widetilde{B}_1 < \infty$ for all ω in a neighborhood of zero. Assume that $JT^{-1}\log(J) \to 0$ and let $p_T = cJ/T$ where $c > 0$. Then

$$\lim_{J,T\to\infty} d_*^k = \frac{1}{v_k}\left(1 - 0.25\Phi\left(-(2\pi/c)^2, 2, 0.5\right) - r_k h_k\right),$$

where

$$h_k \equiv 1 - \frac{ck}{4\pi^2}\tan^{-1}\left(\frac{2\pi}{c}\right)\tan^{-1}\left(\frac{2\pi}{k}\right)$$

$$-\frac{k+c}{4\pi}\left(\mathrm{Im}\left(\mathrm{Li}_2\left(\frac{k+2\pi i}{c+k}\right)\right) + \frac{1}{2}\tan^{-1}\left(\frac{2\pi}{k}\right)\log\left(\frac{c^2+4\pi^2}{(k+c)^2}\right)\right)$$

$$+\frac{|k-c|}{4\pi}\left(\mathrm{Im}\left(\mathrm{Li}_2\left(\frac{\min(k,c)+2\pi i}{-|k-c|}\right)\right) + \frac{1}{2}\tan^{-1}\left(\frac{2\pi}{\min(k,c)}\right)\log\left(\frac{\max(k^2,c^2)+4\pi^2}{(k+c)^2}\right)\right)$$

$$r_k \equiv \frac{1 - 0.25\Phi(-(2\pi/k)^2, 2, 0.5)}{1 - \left(\frac{k}{2\pi}\tan^{-1}\left(\frac{2\pi}{k}\right)\right)^2 - \frac{k}{2\pi}\left(\mathrm{Im}\left(\mathrm{Li}_2\left(\frac{1}{2} + \frac{\pi i}{k}\right)\right) + \frac{1}{2}\tan^{-1}\left(\frac{2\pi}{k}\right)\log\left(\frac{1}{4} + \frac{\pi^2}{k^2}\right)\right)},$$

$$v_k \equiv 1 - r_k\left(1 - 0.25\Phi(-(2\pi/k)^2, 2, 0.5)\right),$$

$\Phi(x,s,a)$ is the Lerch transcendent function, $\mathrm{Im}(x+iy) \equiv y$, and Li_2 is the dilogarithm.

The asymptotic bias of the Modified GPH estimator is a function of *c* and *k*. Figure 3 plots this asymptotic bias for various *k*, along with the asymptotic bias of the GPH estimator for comparison. The asymptotic bias equals zero when $k = c$, so there exists a value of *k* that completely eliminates bias. For $k > c$, the bias is positive and for $k < c$ the bias is negative. There are some other notable features of the asymptotic bias. First, the absolute bias of the Modified GPH estimator is less than the GPH bias for all *k*. Second, as level shifts become more frequent, i.e., as $c \to \infty$, the asymptotic bias goes to zero for all k. Third, the asymptotic bias increases in k, and it converges to the GPH bias in Theorem 1 as $k \to \infty$.

The curves in Figure 3 indicate that the Modified GPH estimator can markedly reduce the bias in the GPH estimator due to occasional level shifts. However, such bias reduction only becomes useful if the requisite loss in precision is acceptable. To address this issue, I derive the asymptotic properties of the Modified GPH estimator under the alternative model of Gaussian long memory.

Theorem 3: Consider the fractionally integrated process $y_t = (1-L)^{-d} u_t$, where $\{u_t\}$ is a stationary short-memory process and $d \in (-0.5, 0.5)$. Suppose that $f_u'(0) = 0$, $| f_u''(\omega) | < \tilde{B}_2 < \infty$, and $| f_u'''(\omega) | < \tilde{B}_3 < \infty$ for all ω in a neighborhood of zero. Assume y_t is Gaussian and that $JT^{-1} \log(J) \to 0$. Then

$$E\left(\hat{d}^k - d\right) = b_k \frac{-2\pi^2}{9} \frac{f_u''(0)}{f_u(0)} \frac{J^2}{T^2} + o(J^2/T^2) + O(\log^3 J/J)$$

$$\operatorname{var}\left(\hat{d}^k\right) = \frac{\pi^2}{24 J v_k} + o(J^{-1}),$$

where $b_k = \frac{1}{v_k}\left(1 - r_k\left(\left(\frac{3k^2}{8\pi^2} + 1\right) - \frac{3k}{4\pi}\left(\frac{k^2}{4\pi^2} + 1\right)\tan^{-1}\left(\frac{2\pi}{k}\right)\right)\right)$ and r_k and v_k are as defined in Theorem 2.

Corollary 3: Consider the process in Theorem 3 and assume also that $J = o(T^{4/5})$ and $\log^2 T = o(J)$. Then $J^{1/2}(\hat{d}^k - d) \xrightarrow{d} N(0, \pi^2/24 v_k)$.

Except for the scale factors b_k and v_k, the asymptotic bias and variance expressions in Theorem 3 are the same as those of Hurvich et al. (1998) for the GPH estimator. The bias factor b_k takes the values –0.65, –0.41, –0.28, –0.21, and –0.15 for k = 1, 2, 3, 4, and 5 respectively. Thus, the bias of the Modified GPH estimator is much smaller than the GPH bias. This bias reduction arises because the extra term in Modified GPH regression picks up some of the curvature in the log spectrum that causes bias in the GPH estimator. The variance factor v_k takes the values 0.17, 0.26, 0.31, 0.35, and 0.37 for k = 1, 2, 3, 4, and 5, respectively. Thus, for a given J, the variance of the Modified GPH estimator is larger than the GPH variance. However, a suitable choice of J mitigates this efficiency loss.

I simulate the performance of the Modified GPH estimator in two settings; the RLS process and a fractionally integrated process. I illustrate the performance of the estimator across different values of J for one set of parameter values. In Section 4.2, I give results for a range of parameter values, sample sizes, and methods for choosing J.

Figure 4 shows the performance of the Modified GPH estimator as a function of J when applied to a RLS process with T = 5000 and p = 0.02. Comparing panels A and B, we see that the asymptotic bias from Theorem 2 closely corresponds to the actual bias. For all values of k, this bias is markedly lower than for the GPH estimator. Panel E also reveals the lower bias of the Modified GPH estimator. It shows that a standard t-test based on the Modified GPH estimate erroneously rejects the null hypothesis that d = 0 less often than the GPH estimator. The size of this test increases in k because the estimator bias increases in k. Panel C shows that RMSE for the Modified GPH estimator is minimized for larger values of J than for the GPH estimator. The minimum RMSE values are similar across the values of k and are less than those of the GPH estimator.

Panels D, E, and F illustrate that the variance and asymptotic normality results in Theorem 3 and its corollary also apply to the RLS process. Panel D shows that the estimated standard error of $\hat{d}^k$ closely corresponds to the actual standard error. I measure the actual standard error as the standard deviation of $\hat{d}^k$ across the Monte Carlo draws. The estimated standard error is computed as $(\pi/\sqrt{6})(\tilde{X}'M_Z\tilde{X})^{-1/2}$, which has a limiting value of $\pi/\sqrt{24Jv_k}$, from Theorem 3. As *J* increases towards 200, the estimated standard error becomes slightly biased downwards and the ratio of the estimated standard error to actual standard error decreases towards 0.85. However, for $J < 60$, the ratio exceeds 0.95.

Panels E and F show that a standard *t*-test rejects the null hypothesis that $d = 0$ with a similar frequency to a hypothetical test that assumes normality. The rejection frequency of this hypothetical test equals the probability that the estimate exceeds the one-sided 5% critical value, assuming that the estimator is normally distributed. The normal approximation appears to be adequate, especially for small values of *J*.

To assess the efficiency loss from the Modified GPH estimator, I simulate from a fractionally integrated process with $d = 0.3$, where the innovations follow an AR(1) process with autoregressive parameter 0.4. I present the results in Figure 5 for various *J*. Excluding the long memory component, this process exhibits less dependence than the RLS process in Figure 4 so the RMSE in Figure 5 is minimized for greater values of *J* than in Figure 4. Panels D, E, and F of Figure 5 corroborate the theoretical variance and asymptotic normality results in Theorem 3 and its corollary.

Panels A and B of Figure 5 show that the bias of the GPH estimator exceeds the Modified GPH bias for all *k*, as predicted by Theorem 3. However, for large *J* the asymptotic bias overestimates the actual bias because the second order terms in the bias expression become non-negligible. Furthermore, the asymptotic bias overestimates the actual bias by more for the Modified GPH estimator than for the GPH estimator, which implies that the asymptotic results overstate the finite sample RMSE efficiency loss of the Modified GPH estimator. I study this point further in Section 4.2.

4.2 Choosing *J* and *k*

From Theorem 3, the mean squared error of $\hat{d}^k$ for Gaussian long memory is

$$\text{MSE}\left(\hat{d}^k\right) = b_k^2 \frac{4\pi^4}{81}\left(\frac{f_u''(0)}{f_u(0)}\right)^2 \frac{J^4}{T^4} + \frac{\pi^2}{24Jv_k} + o(J^4T^{-4}) + O(JT^{-2}\log^3 J) + o(J^{-1})\,.$$

The value of *J* that minimizes MSE is

$$J = \left(v_k b_k^2\right)^{-1/5}\left(\frac{27}{128\pi^4}\right)^{1/5}\left(\frac{f_u''(0)}{f_u(0)}\right)^{-2/5} T^{4/5}, \tag{10}$$

ignoring the remainder terms and assuming that $f_u''(0) \neq 0$. The MSE-optimal value of *J* in (10) equals that for the GPH estimator (see Hurvich et al., 1998), except for the scale factor $(v_k b_k^2)^{-1/5}$. This scale factor takes the values 1.69, 1.88, 2.09, 2.33, and 2.58 for *k* = 1, 2, 3, 4, and 5 respectively. Thus for example, if *k* = 3, the MSE-optimal choice of *J* is approximately double the MSE-optimal choice for the GPH estimator. Hurvich and Deo (1999) propose a consistent estimator for the ratio $f_u''(0)/f_u(0)$, which enables plug-in selection of the MSE-optimal *J*. Their estimator is the coefficient on $0.5\omega_j^2$ in a regression of the log periodogram on X_j and $0.5\omega_j^2$.

If *J* equals its MSE-optimal value, then for Gaussian long memory

$$\text{MSE}\left(\hat{d}^k\right) = \left(\frac{|\,b_k\,|}{v_k^2}\right)^{2/5} \text{MSE}(\hat{d})\,, \tag{11}$$

excluding the remainder terms. Thus, the root mean square error (RMSE) of the Modified GPH estimator equals $(|\,b_k\,|/v_k^2)^{1/5}$ times the RMSE of the GPH estimator when *J* is chosen to be MSE optimal. This scale factor takes the values 1.86, 1.44, 1.24, 1.11, and

1.02 for k = 1, 2, 3, 4, and 5, respectively. Thus, there is negligible asymptotic efficiency loss when $k = 5$.

The asymptotic MSE in (11) does not apply to the MN process for small p. In this case, as shown in Theorem 2, the bias is O(1) and dominates the variance for all J. The Modified GPH estimator often possesses a smaller RMSE than the GPH estimator in this case because the extra term in the log periodogram regression mitigates bias. Suppose we choose J as in (10), implying that J increases in k. Because the local to zero parameter c equals pT/J, a larger value of J implies a smaller value of c. Figure 6 presents the asymptotic bias of the Modified GPH estimator from Theorem 2 assuming that J is chosen as in (10). Recall that Figure 3 shows the asymptotic bias for the case when J is fixed across values of k. Figure 6 is the same as Figure 3, except with the curves stretched horizontally to reflect decreasing values of c as k increases. Figure 6 reveals negligible bias reduction for $k = 5$. However, the bias reduces substantially when $k < 5$.

Given a value of J, choosing $k = c$ implies that the asymptotic bias of the Modified GPH estimator equals zero. However, c cannot be efficiently estimated because it defines a shrinking neighborhood around zero and thus larger samples bring little information about it. If one were ignorant about the value of c, then k could be chosen to minimize average bias over all possible values of c. To this end, I numerically integrate under the absolute value of the asymptotic bias curves in Theorem 2 and find that average bias is minimized at $k = 3.16$; it is decreasing in k for $0 < k < 3.16$ and increasing in k for $3.16 < k < \infty$. Because the asymptotic bias is almost identical for $k = 3$ as for $k = 3.16$, I recommend rounding to the nearest integer and setting $k = 3$.

For a given choice of *J*, choosing $k = 3$ minimizes average bias if the process contains rare level shifts. If we choose MSE-optimal *J*, then $k = 3$ implies a 24% higher asymptotic RMSE than for the GPH estimator if the true process contains long memory (see equation 11). However, in simulations below I show that the efficiency loss can be much less than 24% in finite samples. In fact, Modified GPH regression possesses a lower RMSE than GPH regression in some long-memory cases.

I simulate the performance of the Modified GPH estimator for various parameter settings. I use both the rule-of-thumb value of $J = T^{1/2}$ and the plug-in method of Hurvich and Deo (1999) to select *J*. Results for the RLS process are presented in Table 2 and results for a fractionally integrated process are contained in Table 3. For plug-in selection of *J*, the results for the Modified GPH estimator with $k = 5$ closely match those for the GPH estimator. This correspondence is consistent with the asymptotic bias curves in Figure 6 and the similarity between the asymptotic RMSE's of each estimator. The Modified GPH estimator with $k = 1$ can possess substantially negative bias, which leads to high RMSE values in many cases.

For the RLS process, setting $k = 3$ results in the lowest RMSE when $p > 0.02$ and *J* is chosen using the plug-in method. For example, if $T = 10{,}000$ and $p = 0.05$, the RMSE when $k = 3$ improves by 35% over the GPH estimator. The RMSE improves by 22% over GPH when $p = 0.02$ and by 20% when $p = 0.1$ for this same sample size. Size distortion also reduces substantially relative to the GPH estimator in these cases.

In the plug-in method, *J* increases with *k* according to the relationship in equation (10). This feature results in reduced RMSE for the Modified GPH estimator over the GPH estimator in many cases, reinforcing the results in Figure 4. The only cases where

RMSE for plug-in selection of J exceeds that for rule-of-thumb selection occur for RLS when p = 0.01 and for p = 0.02, which corroborates the findings of Hurvich and Deo (1999), who state that the plug-in method works well unless the spectrum is too peaked near zero frequency.

For the long-memory process with p = 0.8, the Modified GPH estimator with k = 3 outperforms the GPH estimator. It possesses a smaller bias and RMSE. Thus, the Modified GPH estimator has the power to correct bias caused by pure autoregressive processes. When the short memory component is less persistent (p = 0 and p = 0.4), the RMSE of the Modified GPH estimator slightly exceeds that for the GPH estimator. In summary, the Modified GPH estimator with k =3 and J selected using the plug-in method performs well in most settings.

4.3 Applications

To illustrate the Modified GPH estimator, I apply it to the weekly relative price of soybeans to soybean oil and to daily volatility in the S&P 500. The soybean price data span January 1, 1953 to June 30, 2001 and contain the average weekly soybean price in Central Illinois and the average weekly soybean oil price in Decatur, Illinois. There are a total of 2455 observations. Given that soybean oil derives from soybeans, the prices of these two commodities should possess a common trend, which implies that the ratio of their prices should be mean reverting. Panel A of Figure 7 plots the log relative price series and indicates that it is mean reverting with strong positive dependence. This structure suggests that potential candidate models for the relative price include long memory and short memory with level shifts.

Table 4 presents the estimated values of d from the GPH and Modified GPH estimators. The estimated value of d for the soybean data equals 0.79 when the plug-in method is used to select J and 0.84 when $J = T^{1/2}$. These values are significantly different from zero. The Modified GPH estimates are substantially smaller than the GPH estimates for both methods of bandwidth selection. The estimate of d equals 0.16 when $J = T^{1/2}$ and $k = 3$, and it equals 0.29 when $k = 5$. These estimates are insignificantly different from zero; when $k = 1$, the estimated value of d is also insignificant. Thus, a short-memory model with level shifts is a viable alternative to long memory for these data.

Liu (2000), Granger and Hyung (1999), Lobato and Savin (1998), and others cite financial market volatility as one setting where long memory and level shifts provide competing model specifications. I apply the Modified GPH estimator to absolute daily returns on the S&P 500. The data are plotted in Panel B of Figure 7. The sample period is January 1, 1961 to July 31, 2002 and returns are measured as the log price change. Table 4 indicates that a short-memory model with level shifts is not a viable alternative to long memory for this series. In fact, the Modified GPH estimates exceed the GPH estimates for all values of J and k. The GPH estimates are 0.33 and 0.38 for the two methods of choosing J, while the Modified GPH estimates range from 0.39 to 0.65. This result is not sensitive to the measure of volatility; using squared returns and the log of absolute returns leads to same conclusion. Thus, long memory in volatility of S&P 500 returns appears not to be illusory.

5. CONCLUSION

This paper addresses the illusion of fractional integration, or long memory, in time series containing level shifts. I focus on the log periodogram (GPH) estimator, which is used liberally in empirical work. When applied to a short-memory mean-plus-noise process, the GPH estimator is biased and often erroneously indicates the presence of long memory. I derive a large sample approximation to this bias and use it to formulate a new estimator that has markedly smaller bias. I illustrate the Modified GPH estimator with applications to the relative price of soybeans to soybean oil and to stock market volatility.

The Modified GPH estimator requires choosing a value for a nuisance parameter k. This parameter proxies for a local-to-zero parameter that cannot be well estimated from the data. I recommend setting $k = 3$, which minimizes average absolute bias across all possible values of the true parameter c. For a given bandwidth J, this recommendation leads to positive bias in the Modified GPH estimator if $c < 3$ and negative bias if $c > 3$. Despite this tradeoff, the Modified GPH estimator with $k = 3$ exhibits less absolute bias than the original GPH estimator for all values of c (see Figures 3 and 6). Thus, although it does not completely eliminate bias due to level shifts, the Modified GPH estimator with $k = 3$ significantly reduces bias relative to the GPH estimator.

The Modified GPH estimator suggests whether a short-memory model with level shifts should be considered as an alternative to long memory. It is based on the spectrum, which represents the linear dependence properties of a time series. However, a process with discrete level shifts possesses a nonlinear dependence structure because the innovations that define break points are much more persistent than other innovations. Models that capture this nonlinearity will generate more accurate inference about the features of the data than can be achieved with estimators such as Modified GPH.

Specifying models that identify the persistent innovations in a time series is nontrivial, especially given that each of these shocks may have a different origin. They may arise from a political event, a weather event, a war, a new technology, an earnings announcement, or a government policy change, to name a few possibilities. Most Markov-switching models and the particular STOPBREAK model in Engle and Smith (1999) take an agnostic approach and focus only on the time series characteristics of the data when identifying break points. However, Filardo (1994) and Filardo and Gordon (1998) estimate Markov-switching models that use observed data to aid in identifying break points. Further research in this vein will improve model performance and enable better discrimination between models with occasional persistent shocks and linear long-memory models.

ACKNOWLEDGEMENTS

The author is a member of the Giannini Foundation of Agricultural Economics. He thanks Oscar Jorda, Prasad Naik, Doug Steigerwald, the editor, an associate editor, and an anonymous referee for suggestions and comments that improved the quality of this paper.

APPENDIX

Proof of Theorem 1

Recall that $p_T = cJ/T$. We can decompose $\log(f_j)$ as

$$\log f_j = \log\left(f_\varepsilon(\omega_j) + \frac{p_T}{p_T^2 + (1-p_T)(2-2\cos(\omega_j))} f_\eta(\omega_j) \right)$$

$$= -\log\left(1 + p_T^{-2}\omega_j^2\right) + \log\left(f_\varepsilon(\omega_j)\left(1 + p_T^{-2}\omega_j^2\right) + \frac{p_T\left(1 + p_T^{-2}\omega_j^2\right)}{p_T^2 + (1-p_T)(2-2\cos(\omega_j))} f_\eta(\omega_j) \right)$$

$$= -\log\left(1 + (2\pi/c)^2 (j/J)^2\right) + \log\left(1 + D_{jT}\right) + \log(f_\eta(\tilde{\omega}_J)/p_T)$$

where $\tilde{\omega}_J = \arg\min_{1\le j\le J} f_\eta(\omega_j)$, and

$$D_{jT} = \frac{p_T}{f_\eta(\tilde{\omega}_J)} f_\varepsilon(\omega_j)\left(1 + p_T^{-2}\omega_j^2\right) + \frac{\left(1 + p_T^{-2}\omega_j^2\right) f_\eta(\omega_j)}{f_\eta(\tilde{\omega}_J)\left(1 + p_T^{-2}(1-p_T)(2-2\cos(\omega_j))\right)} - 1.$$

Using the fact that, for all *j*, $2 - 2\cos(\omega_j) = \omega_j^2 - \omega_j^3 \sin(\overline{\omega}_j)/3 \le \omega_j^2$ for some $0 \le \overline{\omega}_j \le \omega_j$, we have

$$D_{jT} = \frac{p_T}{f_\eta(\tilde{\omega}_J)} f_\varepsilon(\omega_j)\left(1 + p_T^{-2}\omega_j^2\right) + \frac{f_\eta(\omega_j)\left(1 + p_T^{-2}\omega_j^2\right) - f_\eta(\tilde{\omega}_J)\left(1 + p_T^{-2}(1-p_T)(2-2\cos(\omega_j))\right)}{f_\eta(\tilde{\omega}_J)\left(1 + p_T^{-2}(1-p_T)(2-2\cos(\omega_j))\right)}$$

$$= \left(1 + (2\pi/c)^2 (j/J)^2\right)\frac{p_T f_\varepsilon(\omega_j)}{f_\eta(\tilde{\omega}_J)} + \left(1 + (2\pi/c)^2 (j/J)^2\right)\frac{f_\eta(\omega_j) - f_\eta(\tilde{\omega}_J)}{f_\eta(\tilde{\omega}_J)\left(1 + p_T^{-2}(1-p_T)(2-2\cos(\omega_j))\right)}$$

$$+ \frac{p_T^{-1}(2-2\cos(\omega_j)) + \frac{1}{3} p_T^{-2}\omega_j^3 \sin(\overline{\omega}_j)}{1 + p_T^{-2}(1-p_T)(2-2\cos(\omega_j))}.$$

Note that $D_{jT} \ge 0$ and, because $\omega_j < \pi$, we have $0 \le \sin(\overline{\omega}_j) \le 1$, which implies

$$D_{jT} \le \left(1 + (2\pi/c)^2 (j/J)^2\right)\frac{p_T f_\varepsilon(\omega_j) + f_\eta(\omega_j) - f_\eta(\tilde{\omega}_J)}{f_\eta(\tilde{\omega}_J)} + p_T^{-1}\omega_j^2 + \frac{1}{3} p_T^{-2}\omega_j^3$$

$$= \left(1 + \left(\frac{2\pi}{c}\right)^2 \left(\frac{j}{J}\right)^2\right)\frac{cJT^{-1} f_\varepsilon(\omega_j) + f_\eta(\omega_j) - f_\eta(\tilde{\omega}_J)}{f_\eta(\tilde{\omega}_J)} + \frac{(2\pi j)^2}{cJT} + \frac{(2\pi j)^3}{3(cJT)^2}.$$

Now

$$d_* = \frac{\sum_{j=1}^{J}(X_j - \overline{X})\log f_j}{\sum_{j=1}^{J}(X_j - \overline{X})^2}$$

$$= \frac{-\sum_{j=1}^{J}(X_j - \overline{X})\log(1+(2\pi/c)^2(j/J)^2)}{\sum_{j=1}^{J}(X_j - \overline{X})^2} + \frac{\sum_{j=1}^{J}(X_j - \overline{X})\log\left(1+D_{jT}\right)}{\sum_{j=1}^{J}(X_j - \overline{X})^2}. \qquad \text{(A1)}$$

The following results from Hurvich and Beltrao (1994):

$$\left|X_j - \overline{X}\right| = O(\log J) = O(\log T), \qquad \sum_{j=1}^{J}(X_j - \overline{X})^2 = 4J(1+o(1)),$$

$$X_j - \overline{X} = -2\log j + 2J^{-1}\sum_{k=1}^{J}\log k + O(J^2/T^2),$$

and the formula $J^{-1}\sum_{k=1}^{J}\log k = \log J - 1 + o(1)$ imply that

$$\frac{-\sum_{j=1}^{J}(X_j - \overline{X})\log\left(1+(2\pi/c)^2(j/J)^2\right)}{\sum_{j=1}^{J}(X_j - \overline{X})^2} = \frac{\sum_{j=1}^{J}(1+\log(j/J)+o(1))\log\left(1+(2\pi/c)^2(j/J)^2\right)}{2J(1+o(1))}$$

$$\to 0.5\int_0^1(1+\log(x))\log(1+(2\pi/c)^2x^2)dx$$

$$= 1 - 0.25\Phi(-(2\pi/c)^2,2,0.5),$$

where $\Phi(x,s,a)$ denotes the Lerch transcendent function (Gradshteyn and Ryzhik, 1980, pg 1072).

For the second term in (A1), note that $\left|f_\eta(\omega_j) - f_\eta(\tilde{\omega}_J)\right| \le \tilde{B}_1\left|\omega_j - \tilde{\omega}_J\right| = O(JT^{-1})$ for all j = 1, 2, …, J. Then, using $\left|\log(1+x)\right| \le x$ for $x \ge 0$ yields

$$\left|\frac{\sum_{j=1}^{J}(X_j - \overline{X})\log\left(1+D_{jT}\right)}{\sum_{j=1}^{J}(X_j - \overline{X})^2}\right| \le \frac{\sum_{j=1}^{J}\left|X_j - \overline{X}\right|\left(\left(1+\left(\frac{2\pi j}{cJ}\right)^2\right)\frac{cJT^{-1}f_\varepsilon(\omega_j) + f_\eta(\omega_j) - f_\eta(\tilde{\omega}_J)}{f_\eta(\tilde{\omega}_J)} + \frac{(2\pi j)^2}{cJT} + \frac{(2\pi j)^3}{3(cJT)^2}\right)}{\sum_{j=1}^{J}(X_j - \overline{X})^2}$$

$$= O\left(J^{-1}\log(J)\right)\sum_{j=1}^{J}\left(\left(1+\left(\frac{2\pi j}{cJ}\right)^2\right)\frac{cJT^{-1}f_\varepsilon(\omega_j)}{f_\eta(\tilde{\omega}_J)} + \left(1+\left(\frac{2\pi j}{cJ}\right)^2\right)O\left(\frac{J}{T}\right)\right.$$

$$\left. + \frac{(2\pi)^2}{c}\frac{j^2}{JT} + \frac{(2\pi)^3}{3c^2}\frac{j^3}{J^2T^2}\right)$$

$$= O\left(JT^{-1}\log(J)\right) + O\left(JT^{-1}\log(J)\right) + O\left(JT^{-1}\log(J)\right) + O\left(JT^{-2}\log(J)\right)$$

$$= O\left(JT^{-1}\log(J)\right).$$

Thus, under the assumption $JT^{-1}\log(J) \to 0$, we have $d_* \to 1 - 0.25\Phi(-(2\pi/c)^2,2,0.5)$ as $T \to \infty$.

Proof of Theorem 2:

We can write

$$d_*^k = \frac{\sum_{j=1}^{J}(X_j - \overline{X})\log f_j - \left(\sum_{j=1}^{J}(X_j-\overline{X})(Z_{kj}-\overline{Z}_k)\right)\left(\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\log f_j\right)\Big/\left(\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)^2\right)}{\sum_{j=1}^{J}(X_j-\overline{X})^2 - \left(\sum_{j=1}^{J}(X_j-\overline{X})(Z_{kj}-\overline{Z}_k)\right)^2\Big/\left(\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)^2\right)}.$$

From the proof of Theorem 1, we have

$$\sum_{j=1}^{J}(X_j-\overline{X})(Z_{kj}-\overline{Z}_k) = 4\left(1-0.25\Phi(-(2\pi/k)^2,2,0.5)\right)J(1+o(1)),$$

$$\sum_{j=1}^{J}(X_j-\overline{X})\log f_j = 4\left(1-0.25\Phi(-(2\pi/c)^2,2,0.5)\right)J(1+o(1)),$$

$$\sum_{j=1}^{J}(X_j-\overline{X})^2 = 4J(1+o(1)),$$

where $\Phi(x,s,a)$ denotes the Lerch transcendent function (Gradshteyn and Ryzhik, 1980, pg 1072).

Note that

$$\overline{Z}_k = -J^{-1}\sum_{j=1}^{J}\log\left(1+(2\pi/k)^2(j/J)^2\right) - 2\log(kJ/T)$$

$$= -(1+o(1))\int_0^1\log(1+(2\pi/k)^2x^2)dx - 2\log(kJ/T)$$

$$= -(1+o(1))\left(\log(1+(2\pi/k)^2)+(k/\pi)\tan^{-1}(2\pi/k)-1\right) - 2\log(kJ/T).$$

Thus,

$$J^{-1}\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)^2 \to \int_0^1\left(\log\left(\frac{1+(2\pi/k)^2x^2}{1+(2\pi/k)^2}\right)-(k/\pi)\tan^{-1}(2\pi/k)+1\right)^2 dx$$

$$= 4-\left(\frac{k}{\pi}\tan^{-1}\left(\frac{2\pi}{k}\right)\right)^2 - \frac{2k}{\pi}\left(\operatorname{Im}\left(\operatorname{Li}_2\left(\frac{1}{2}+\frac{\pi i}{k}\right)\right)-\frac{1}{2}\tan^{-1}\left(\frac{2\pi}{k}\right)\log\left(\frac{1}{4}+\frac{\pi^2}{k^2}\right)\right),$$

where $\operatorname{Im}(x+iy) \equiv y$, and Li_2 signifies the dilogarithm.

Similarly for $\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\log f_j$ we have

$$J^{-1}\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\log f_j \to \int_0^1\left(\log\left(\frac{1+(2\pi/k)^2x^2}{1+(2\pi/k)^2}\right)-(k/\pi)\tan^{-1}(2\pi/k)+1\right)\log\left(1+(2\pi/c)^2x^2\right)dx$$

$$= 4-\frac{ck}{\pi^2}\tan^{-1}\left(\frac{2\pi}{c}\right)\tan^{-1}\left(\frac{2\pi}{k}\right)$$

$$-\frac{k+c}{\pi}\left(\mathrm{Im}\left(\mathrm{Li}_2\left(\frac{k+2\pi i}{c+k}\right)\right)+\frac{1}{2}\tan^{-1}\left(\frac{2\pi}{k}\right)\log\left(\frac{c^2+4\pi^2}{(k+c)^2}\right)\right)$$

$$+\frac{|k-c|}{\pi}\left(\mathrm{Im}\left(\mathrm{Li}_2\left(\frac{\min(k,c)+2\pi i}{-|k-c|}\right)\right)+\frac{1}{2}\tan^{-1}\left(\frac{2\pi}{\min(k,c)}\right)\log\left(\frac{\max(k^2,c^2)+4\pi^2}{(k+c)^2}\right)\right)$$

$$\equiv 4h_k .$$

Define

$$r_k \equiv \frac{1-0.25\Phi(-(2\pi/k)^2,2,0.5)}{1-\left(\frac{k}{2\pi}\tan^{-1}\left(\frac{2\pi}{k}\right)\right)^2-\frac{k}{2\pi}\left(\mathrm{Im}\left(\mathrm{Li}_2\left(\frac{1}{2}+\frac{\pi i}{k}\right)\right)-\frac{1}{2}\tan^{-1}\left(\frac{2\pi}{k}\right)\log\left(\frac{1}{4}+\frac{\pi^2}{k^2}\right)\right)},$$

$$v_k \equiv 1-r_k\left(1-0.25\Phi(-(2\pi/k)^2,2,0.5)\right),$$

and the result follows.

Proof of Theorem 3:

The log spectrum of the fractionally integrated process is $\log f_j = dX_j + \log f_{uj}$. Thus the Modified GPH estimator is

$$\hat{d}^k = d + \left(\tilde{X}'M_Z\tilde{X}\right)^{-1}\tilde{X}'M_Z \log f_u + \left(\tilde{X}'M_Z\tilde{X}\right)^{-1}\tilde{X}'M_Z \log\left(\hat{f}/f\right), \qquad \text{(A2)}$$

where $\tilde{X} = X - \overline{X}$, $M_Z = I - \tilde{Z}_k(\tilde{Z}_k'\tilde{Z}_k)^{-1}\tilde{Z}_k'$, and $\tilde{Z}_k = Z_k - \overline{Z}_k$.

Consider the second term on the right-hand-side of (A2). We have

$$\tilde{X}'M_Z \log f_u = \sum_{j=1}^{J}(X_j - \overline{X})\log f_{uj} - r_k(1+o(1))\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\log f_{uj} .$$

where r_k is as defined in the proof of Theorem 2. From Hurvich et al. (1998), a second order expansion of f_u around ω=0 yields

$$\log f_{uj} = \log f_u(0) + \frac{\omega_j^2}{2}\frac{f_u''(0)}{f_u(0)} + \frac{\omega_j^3}{6}K_j ,$$

where K_j is bounded uniformly in j for sufficiently large T. Given this, Hurvich et al. (1998) show that

$$\sum_{j=1}^{J}(X_j - \overline{X})\log f_{uj} = \frac{-8\pi^2}{9}\frac{f_u''(0)}{f_u(0)}\frac{J^3}{T^2} + o(J^3/T^2),$$

where f_u'' denotes the second derivative of f_u. Similarly,

$$\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\log f_{uj} = \sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\left(\frac{\omega_j^2}{2}\frac{f_u''(0)}{f_u(0)}+\frac{\omega_j^3}{6}K_j\right)$$

$$=\frac{2\pi^2 f_u''(0)}{f_u(0)}\frac{J^3}{T^2}\left(J^{-1}\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)(j/J)^2\right)+O\left(T^{-3}\sum_{j=1}^{J}j^3\right),$$

where I used the fact that $|Z_{kj}-\overline{Z}_k| = O(1)$ uniformly in j.

Now, using arguments from the proof of Theorem 2, we have

$$J^{-1}\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)(j/J)^2 = \int_0^1\left(\log\left(\frac{1+(2\pi/k)^2x^2}{1+(2\pi/k)^2}\right)-2\big((k/2\pi)\tan^{-1}(2\pi/k)-1\big)\right)x^2\,dx+o(1)$$

$$=\left(-\frac{4}{9}\left(\frac{3k^2}{8\pi^2}+1\right)+\frac{k}{3\pi}\left(\frac{k^2}{4\pi^2}+2\right)\tan^{-1}\left(\frac{2\pi}{k}\right)\right)+o(1)\,.$$

Thus

$$\sum_{j=1}^{J}(Z_{kj}-\overline{Z}_k)\log f_{uj} = \frac{-8\pi^2}{9}\frac{f_u''(0)}{f_u(0)}\frac{J^3}{T^2}\left(\left(\frac{3k^2}{8\pi^2}+1\right)-\frac{3k}{4\pi}\left(\frac{k^2}{4\pi^2}+2\right)\tan^{-1}\left(\frac{2\pi}{k}\right)\right)+o(J^3/T^2)\,.$$

Then the second term on the right-hand-side of (A2) is

$$\left(\tilde{X}'M_Z\tilde{X}\right)^{-1}\tilde{X}'M_Z\log f_u = -b_k\frac{2\pi^2}{9}\frac{f_u''(0)}{f_u(0)}\frac{J^2}{T^2}+o(J^2/T^2)$$

where $b_k = \dfrac{1}{v_k}\left(1-r_k\left(\left(\dfrac{3k^2}{8\pi^2}+1\right)-\dfrac{3k}{4\pi}\left(\dfrac{k^2}{4\pi^2}+1\right)\tan^{-1}\left(\dfrac{2\pi}{k}\right)\right)\right)$, and v_k is as defined in the proof of Theorem 2.

For the last term in (A2), I use the proof of Lemma 8 of Hurvich et al. (1998). Their proof goes through if their $a_j = X_j - \overline{X}$ is replaced by $\left((X_j-\overline{X})-r_k(1+o(1))(Z_{kj}-\overline{Z}_k)\right)$ and their $2S_{xx}$ is replaced by $4Jv_k(1+o(1))$. These replacements are valid because the substituted terms are of the same order of magnitude as their replacements. It follows that $\left(\tilde{X}'M_Z\tilde{X}\right)^{-1}\tilde{X}'M_Z E\left(\log(\hat{f}/f)\right)=O\left(\log^3 J/J\right)$. Thus, the bias is

$$E\left(\hat{d}^m-d\right)=-b_k\frac{2\pi^2}{9}\frac{f_u''(0)}{f_u(0)}\frac{J^2}{T^2}+o(J^2/T^2)+O(\log^3 J/J)\,.$$

For the variance, I use the proof of Theorem 1 of Hurvich et al. (1998). Replacing their a_j by $\left((X_j-\overline{X})-r_k(1+o(1))(Z_{kj}-\overline{Z}_k)\right)$ and their $2S_{xx}$ by $4Jv_k(1+o(1))$ leads to

$$\operatorname{var}\left(\hat{d}^{k}\right) = \left(\tilde{X}' M_Z \tilde{X}\right)^{-2} \operatorname{var}\left(\tilde{X}' M_Z \log\left(\hat{f}/f\right)\right)$$

$$= \frac{\pi^2}{24 J v_k} + o(J^{-1}) .$$

Proof of Corollary 3:

This result follows directly from Theorems 2 and 3 above and Theorem 2 of Hurvich et al. (1998), with their a_j replaced by $\left((X_j - \overline{X}) - r_k\left(1 + o(1)\right)(Z_{kj} - \overline{Z}_k)\right)$ and their $2S_{xx}$ replaced by $4Jv_k(1+o(1))$.

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Table 1: Mean Values of $\hat{d}$ for a Random Level Shift Process

		$\sigma_\xi^2 = 1$			$\sigma_\xi^2 = 3$		
		***T*=1000**	***T*=5000**	***T*=10,000**	***T*=1000**	***T*=5000**	***T*=10,000**
***p* = 0.25**	**GPH ($\hat{d}$)**	0.053	0.012	0.005	0.059	0.012	0.006
	Std. Error	(0.137)	(0.083)	(0.072)	(0.136)	(0.082)	(0.070)
	***Exact* (*d**)**	*0.050*	*0.010*	*0.005*	*0.055*	*0.011*	*0.006*
***p* = 0.05**	**GPH ($\hat{d}$)**	0.420	0.196	0.127	0.448	0.202	0.130
	Std. Error	(0.156)	(0.090)	(0.071)	(0.159)	(0.090)	(0.072)
	***Exact* (*d**)**	*0.434*	*0.198*	*0.124*	*0.463*	*0.204*	*0.128*
***p* = 0.01**	**GPH ($\hat{d}$)**	0.684	0.626	0.548	0.790	0.656	0.563
	Std. Error	(0.152)	(0.097)	(0.085)	(0.152)	(0.099)	(0.085)
	***Exact* (*d**)**	*0.717*	*0.635*	*0.550*	*0.814*	*0.663*	*0.565*
***p* = 0.005**	**GPH ($\hat{d}$)**	0.634	0.735	0.697	0.769	0.788	0.726
	Std. Error	(0.205)	(0.095)	(0.080)	(0.199)	(0.097)	(0.081)
	***Exact* (*d**)**	*0.695*	*0.747*	*0.702*	*0.830*	*0.796*	*0.730*

Note: The rows labeled GPH give the average of the GPH estimate across 1000 realizations of size T from the RLS process $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1 - s_t)\mu_{t-1} + s_t \xi_t$, $s_t \sim iid$ Bernoulli(p), $\xi_t \sim iid(0, \sigma_\xi^2)$, and $\varepsilon_t \sim N(0,1)$. The GPH statistic is computed with $J = T^{1/2}$. The rows labeled Std. Error give the standard deviation of the GPH estimates across the 1000 realizations. The asymptotic standard errors for Gaussian processes are 0.114, 0.076, and 0.064 for samples of size 1000, 5000 and 10,000 respectively (see Hurvich et al., 1998). The rows labeled Exact give the GPH estimate computed using the log spectrum in place of the log periodogram.

Table 2: Properties of Modified GPH Estimator for RLS Process

T	*p*	Plug-in selection of *J*				$J = T^{1/2}$			
		GPH	*k* = 1	*k* = 3	*k* = 5	GPH	*k* = 1	*k* = 3	*k* = 5
					Bias				
1000	0.01	0.49	0.66	0.56	0.51	0.72	0.69	0.74	0.74
	0.02	0.52	0.48	0.54	0.55	0.64	0.29	0.49	0.53
	0.05	0.34	–0.02	0.24	0.35	0.42	–0.13	0.12	0.18
	0.10	0.17	–0.16	0.05	0.16	0.22	–0.19	–0.02	0.02
10,000	0.01	0.59	0.47	0.58	0.61	0.55	0.01	0.24	0.30
	0.02	0.40	0.05	0.30	0.40	0.36	-0.12	0.06	0.11
	0.05	0.15	–0.10	0.05	0.12	0.12	-0.09	-0.02	0.00
	0.10	0.06	–0.07	–0.01	0.03	0.04	-0.04	-0.01	-0.01
					RMSE				
1000	0.01	0.51	0.74	0.58	0.54	0.71	0.91	0.81	0.80
	0.02	0.54	0.66	0.58	0.57	0.66	0.63	0.60	0.61
	0.05	0.40	0.50	0.38	0.41	0.45	0.53	0.34	0.33
	0.10	0.28	0.52	0.31	0.29	0.26	0.56	0.31	0.27
10,000	0.01	0.60	0.52	0.59	0.62	0.56	0.22	0.29	0.33
	0.02	0.41	0.18	0.32	0.41	0.36	0.24	0.16	0.17
	0.05	0.17	0.17	0.11	0.15	0.14	0.22	0.14	0.12
	0.10	0.10	0.14	0.08	0.09	0.08	0.20	0.14	0.12
					Rejection Frequency				
1000	0.01	0.98	0.83	0.94	0.98	1.00	0.42	0.76	0.84
	0.02	0.96	0.61	0.86	0.95	0.99	0.17	0.50	0.62
	0.05	0.67	0.22	0.44	0.66	0.90	0.27	0.10	0.17
	0.10	0.41	0.14	0.27	0.39	0.51	0.01	0.03	0.04
10,000	0.01	1.00	0.88	0.99	0.99	1.00	0.06	0.56	0.76
	0.02	0.99	0.20	0.89	0.99	1.00	0.01	0.12	0.23
	0.05	0.67	0.00	0.20	0.55	0.58	0.01	0.03	0.04
	0.10	0.34	0.01	0.07	0.19	0.14	0.02	0.04	0.04

Notes: (i) Data generating process: $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1 - s_t)\mu_{t-1} + s_t\xi_t$, $s_t \sim iid$ Bernoulli(*p*), $\xi_t \sim iidN(0,1)$, $\varepsilon_t \sim iidN(0,1)$.

(ii) The elements in the table are averages across 1000 Monte Carlo realizations. The plug-in method was used with $L = 0.1T^{6/7}$ frequencies in first stage regression.

Table 3: Properties of Modified GPH Estimator for Fractionally Integrated Process

		Plug-in selection of *J*				$J = T^{1/2}$			
T	***p***	**GPH**	***k* = 1**	***k* = 3**	***k* = 5**	**GPH**	***k* = 1**	***k* = 3**	***k* = 5**
		Bias							
1000	0	–0.03	–0.10	–0.05	–0.02	0.01	0.01	0.01	0.01
	0.4	0.01	–0.14	–0.05	–0.00	0.02	0.00	0.00	0.00
	0.8	0.14	–0.19	0.01	0.11	0.10	–0.09	–0.02	–0.00
10,000	0	–0.01	–0.02	–0.01	–0.01	0.00	0.00	0.00	0.00
	0.4	0.01	–0.04	–0.02	–0.00	0.01	0.01	0.01	0.01
	0.8	0.05	–0.08	–0.02	0.03	0.01	0.00	0.00	0.00
		RMSE							
1000	0	0.13	0.34	0.19	0.13	0.14	0.52	0.30	0.27
	0.4	0.14	0.32	0.18	0.14	0.14	0.50	0.30	0.27
	0.8	0.26	0.40	0.26	0.26	0.17	0.53	0.31	0.27
10,000	0	0.04	0.08	0.05	0.04	0.07	0.21	0.14	0.13
	0.4	0.04	0.09	0.05	0.04	0.07	0.20	0.14	0.12
	0.8	0.11	0.11	0.08	0.10	0.07	0.21	0.14	0.13
		Rejection Frequency							
1000	0	0.79	0.48	0.69	0.80	0.75	0.14	0.27	0.33
	0.4	0.79	0.37	0.69	0.80	0.75	0.13	0.27	0.31
	0.8	0.78	0.27	0.58	0.73	0.89	0.10	0.23	0.29
10,000	0	0.99	0.93	0.99	1.00	0.99	0.46	0.74	0.81
	0.4	1.00	0.93	0.99	1.00	0.99	0.47	0.75	0.82
	0.8	1.00	0.79	0.97	0.99	0.99	0.48	0.71	0.80

Notes: (i) Data generating process: $y_t = (1-L)^{-d} u_t$, where d=0.3, $u_t = p\, u_{t-1} + \varepsilon_t$, and $\varepsilon_t \sim iid\ N(0,1)$.

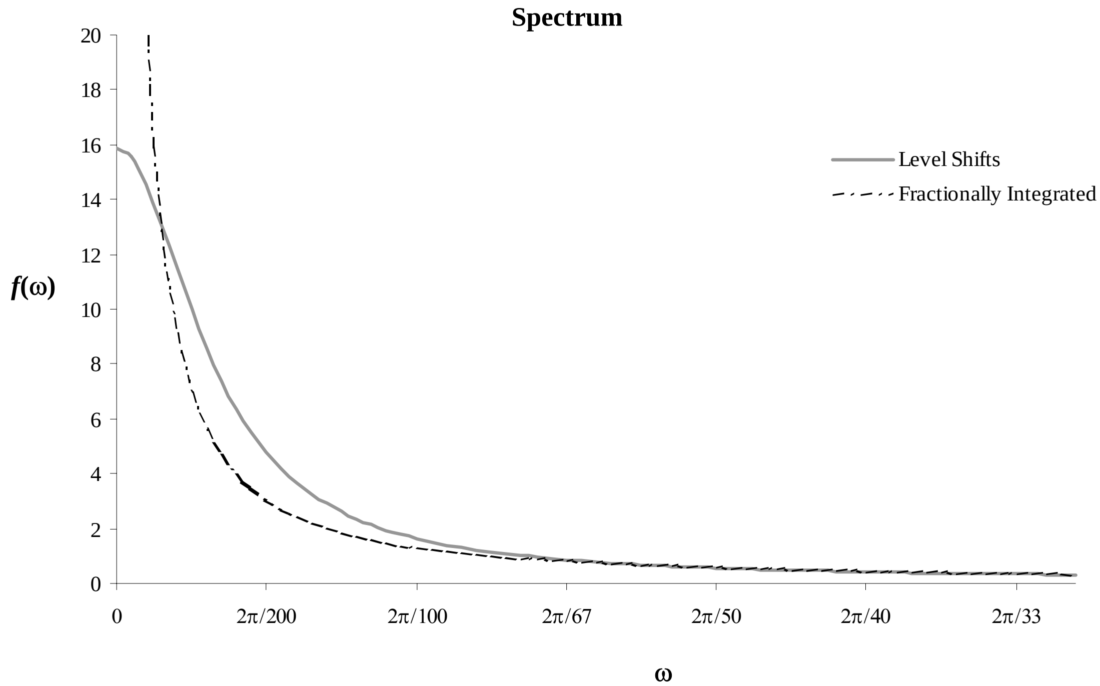
(ii) The elements in the table are averages across 1000 Monte Carlo realizations. The plug-in method was used with $L = 0.1T^{6/7}$ frequencies in first stage regression.

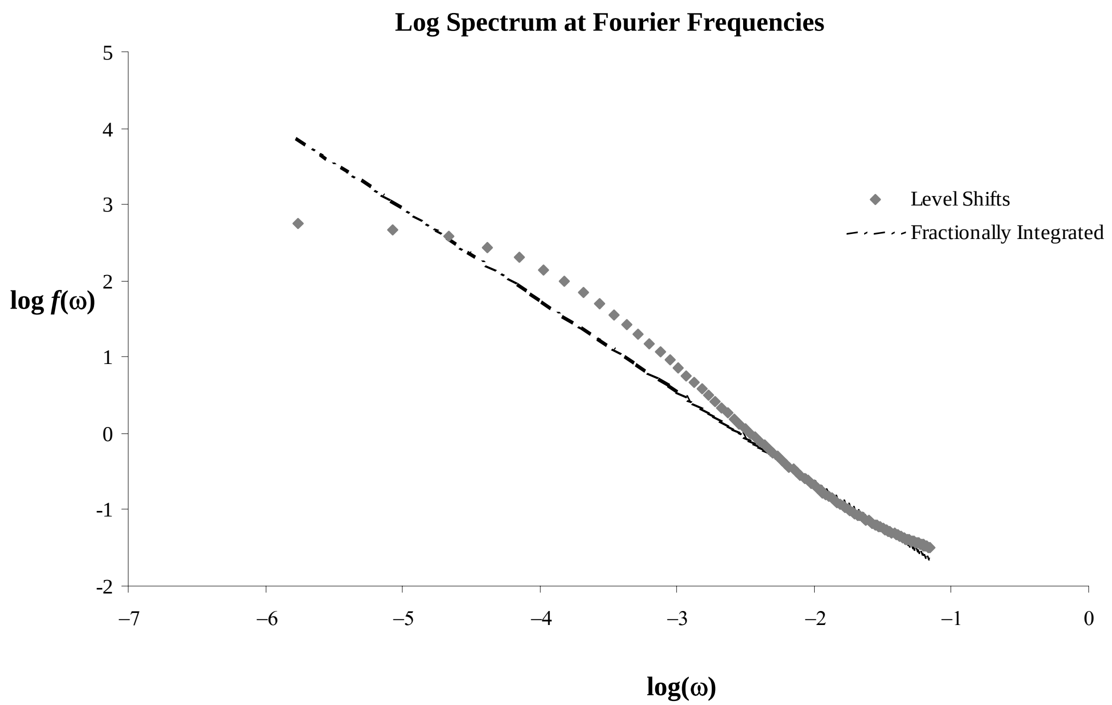
Table 4: Estimates of the Long-Memory Parameter

	GPH	Modified GPH		
		***k* = 1**	***k* = 3**	***k* = 5**
		Relative price of soybeans to soybean oil		
Plug-in	0.79*	0.35	0.70*	0.72*
	(0.09)	(0.24)	(0.14)	(0.11)
$J = T^{1/2} = 49$	0.84*	–0.29	0.16	0.29
	(0.10)	(0.35)	(0.22)	(0.20)
		Absolute daily returns on S&P 500		
Plug-in	0.33*	0.47*	0.42*	0.39*
	(0.03)	(0.06)	(0.04)	(0.03)
$J = T^{1/2} = 102$	0.38*	0.65*	0.55*	0.52*
	(0.07)	(0.20)	(0.14)	(0.12)

Note: The cells contain estimates of *d*, the long-memory parameter, with standard errors below each estimate in parentheses. A * indicates significance at 5%, using standard normal critical values and a one-sided alternative. The soybean data span January 1, 1953 to June 30, 2001 and contain the average weekly soybean price in Central Illinois and the average weekly soybean oil price in Decatur, Illinois. There are a total of 2455 observations. The stock market data span January 1, 1961 to July 31, 2002 and contain the absolute daily returns on the S&P 500 stock index. There are a total of 10463 observations. The plug-in method was used with $L = 0.1T^{6/7}$ frequencies in first stage regression. For the GPH estimator, the estimated plug-in values of *J* are 67 for soybeans and 657 for the S&P 500. The plug-in values of *J* for the Modified GPH estimator equal the scale factors in equation (10) multiplied by 67 (for soybeans) and 657 (for the S&P 500).

Figure 1: Spectra of Level Shift and Fractionally Integrated Processes





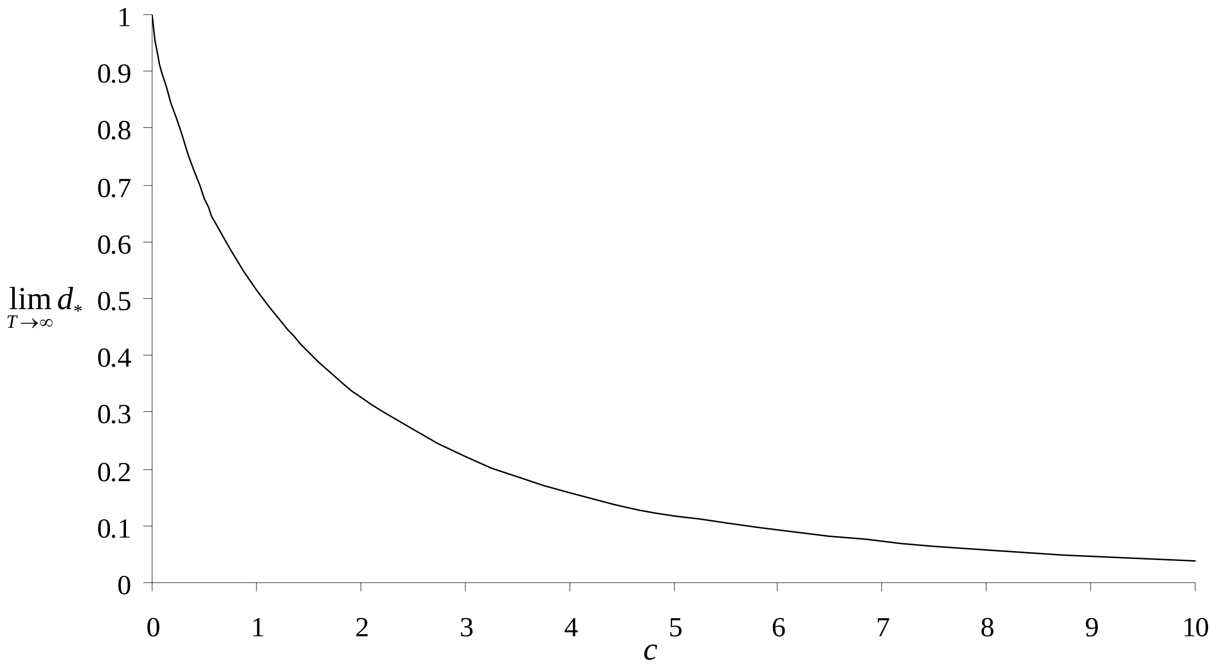
Notes:

(i) Data generating process (level shifts): $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1 - s_t)\mu_{t-1} + s_t\xi_t$, $\varepsilon_t \sim iid\,(0, 1)$, $s_t \sim iid$,Bernoulli(p), p = 0.1, $\xi_t \sim iid(0,1)$.

(ii) Data generating process (FI): $y_t = (1 - L)^{-0.6} u_t$, $u_t \sim iid\,(0, 0.3)$.

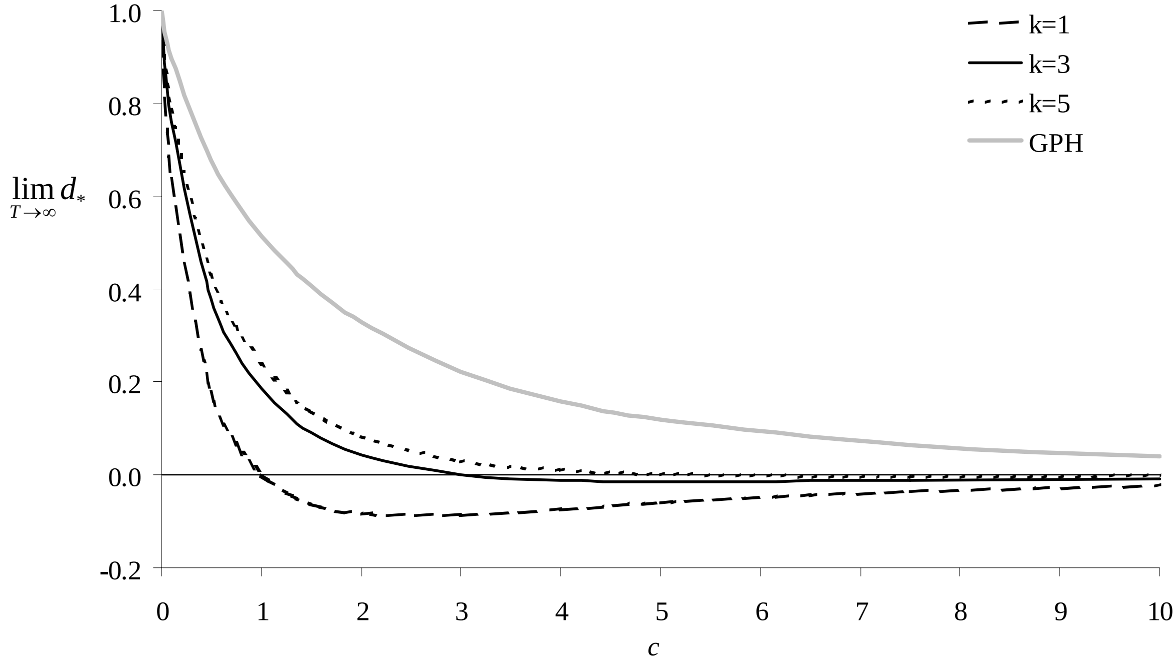
(iii) The lower panel shows a scatter plot of the log spectrum against the log of the Fourier frequencies $\omega = 2\pi j/T$, for j = 1, 2, …, $T^{1/2}$, and T = 1000.

Figure 2: Asymptotic Bias of the GPH Estimator



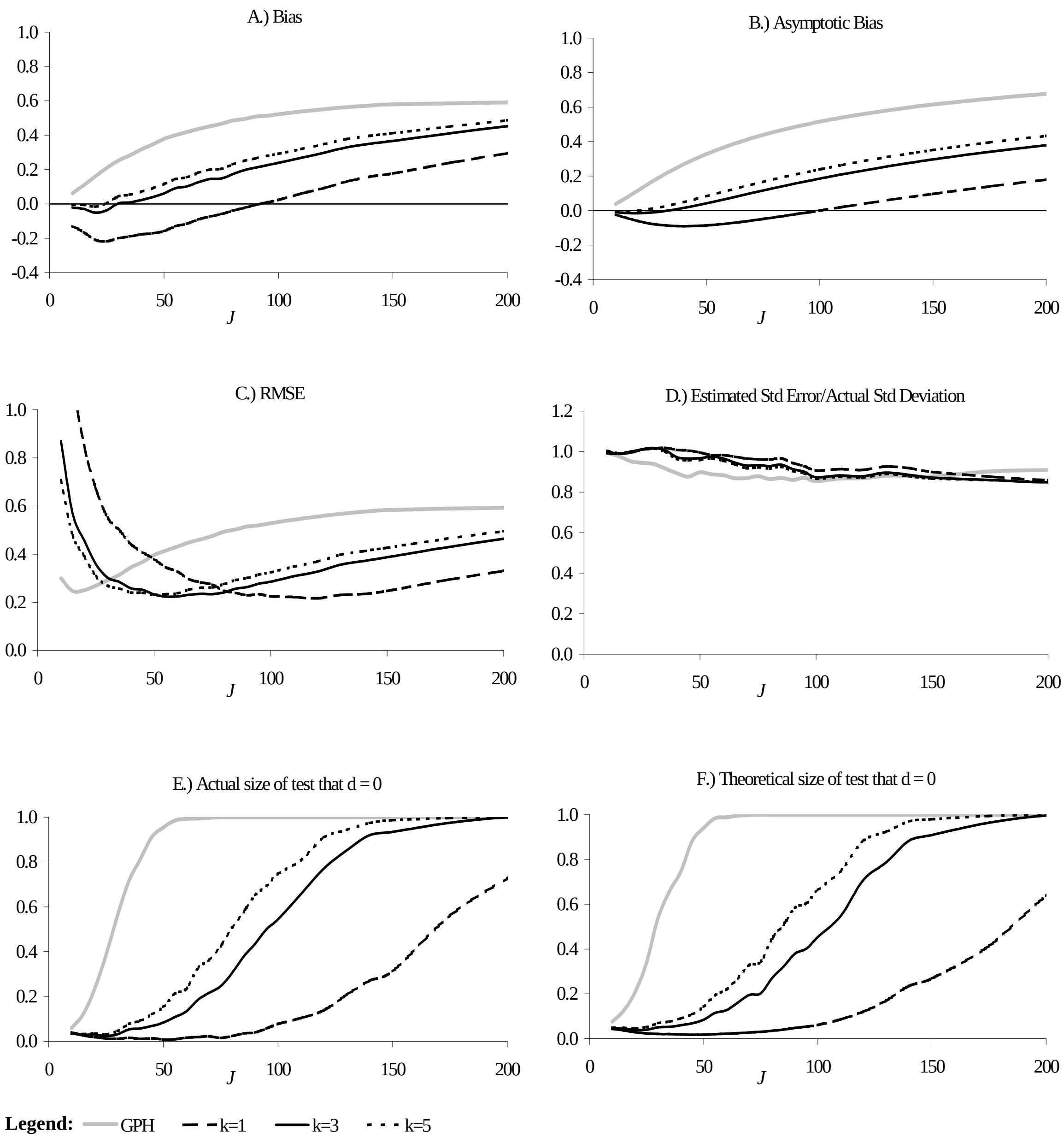
Note: Applies to data generated by $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1-p_T)\mu_{t-1} + \sqrt{p_T}\eta_t$, η_t ~ short memory, ε_t ~ short memory, and t = 1,2,...,T. The bandwidth (number of frequencies) in the GPH regression is $J = p_T T / c$.

Figure 3: Asymptotic Bias of the Modified GPH Estimator



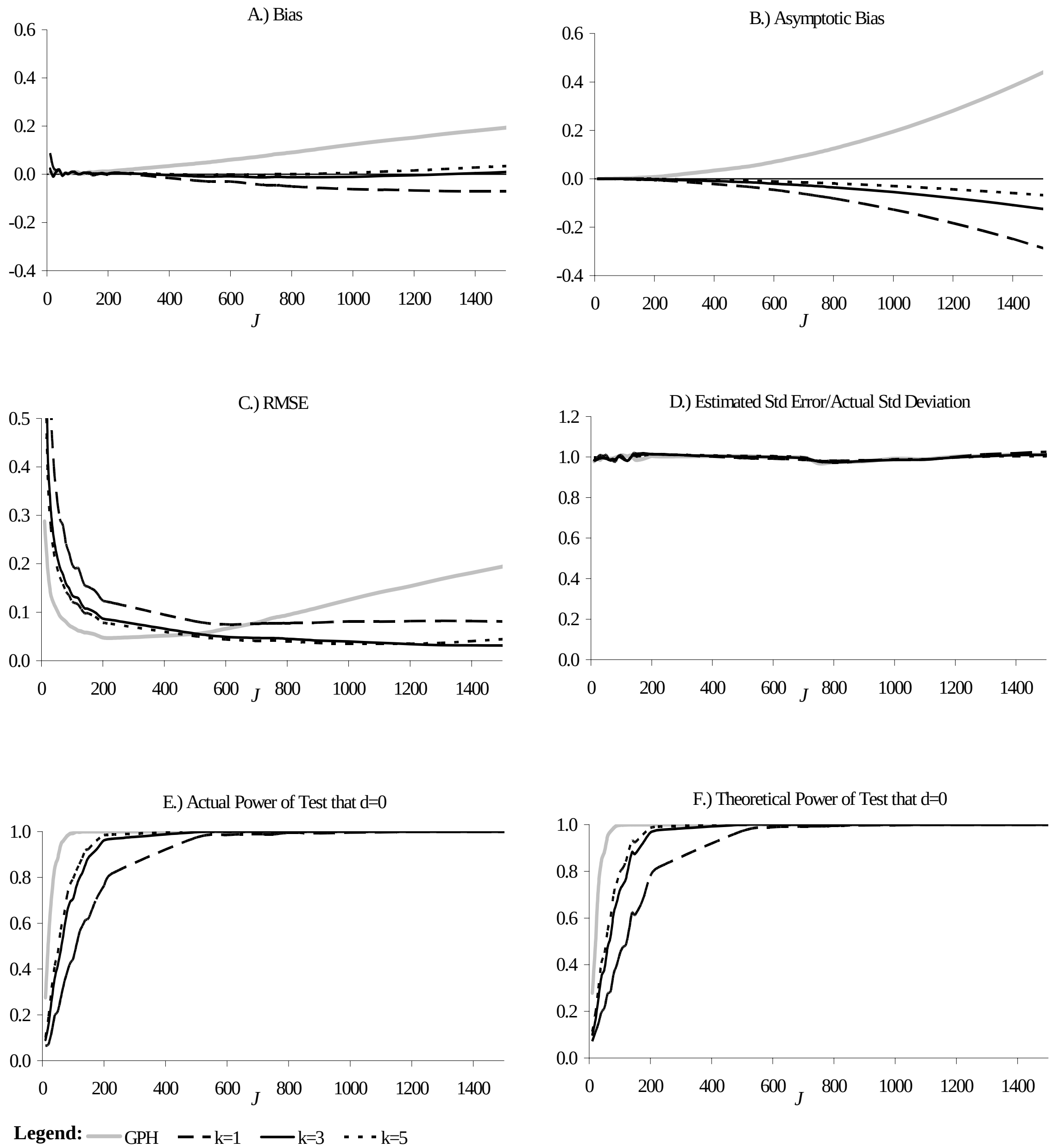
Note: Applies to data generated by $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1 - p_T)\mu_{t-1} + \sqrt{p_T}\eta_t$, η_t ~ short memory, ε_t ~ short memory, and t = 1,2,...,T. The bandwidth (number of frequencies) in the GPH regression is $J = p_T T / c$.

Figure 4: Performance of Modified GPH Estimator for a RLS Process



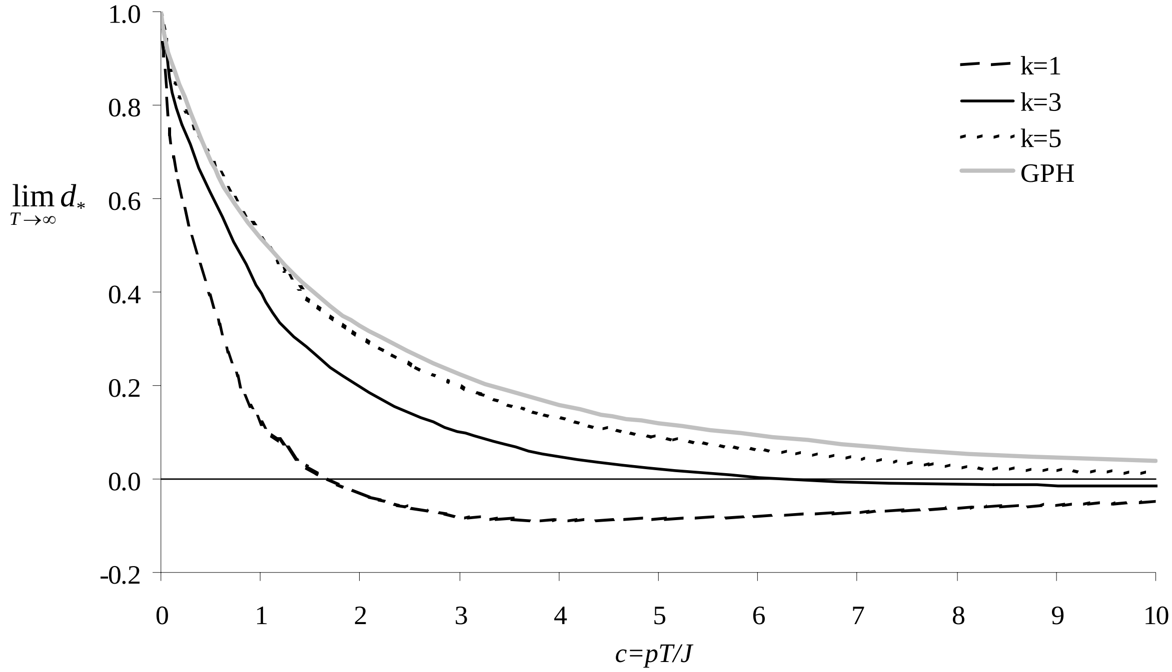
Note: Data generating process: $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1 - s_t)\mu_{t-1} + s_t \xi_t$, $s_t \sim$ *iid* Bernoulli(p), $\xi_t \sim iidN(0,1)$, $\varepsilon_t \sim iidN(0,1)$, p = 0.02, T = 5000. The curves are generated from 1000 Monte Carlo draws as follows: A.) average estimate of d across draws, B.) from Theorem 2, C.) computed from average and variance of estimates of d across draws, D.) Estimated standard error equals $(\pi / \sqrt{6})(\tilde{X}' M_Z \tilde{X})^{-1/2}$, actual standard deviation equals standard deviation of estimate of d across draws, E.) proportion of rejections of null hypothesis that d=0 against d>0, nominal size=5%, F.) size computed assuming that estimate of d is normally distributed with mean and variance given by average and variance of estimates of d across draws, nominal size=5%.

Figure 5: Performance of Modified GPH Estimator for a FI Process



Note: Data generating process: $y_t = (1-L)^{-d} u_t$, where d=0.3, u_t = 0.4u_{t-1} + ε_t, ε_t ~ *iid* N(0,1), and T=5000. The curves are generated from 1000 Monte Carlo draws as follows: A.) average estimate of d across draws, B.) from Theorem 2, C.) computed from average and variance of estimates of d across draws, D.) Estimated standard error equals $(\pi/\sqrt{6})(\tilde{X}'M_Z\tilde{X})^{-1/2}$, actual standard deviation equals standard deviation of estimate of d across draws, E.) proportion of rejections of null hypothesis that d=0 against d>0, nominal size=5%, F.) power computed assuming that estimate of d is normally distributed with mean and variance given by average and variance of estimates of d across draws, nominal size=5%.

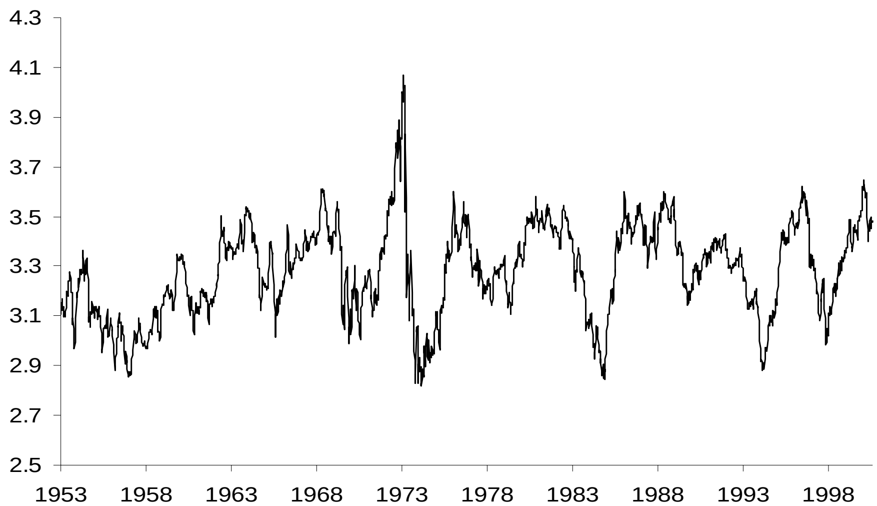
Figure 6: Asymptotic Bias of Modified GPH Estimator for MSE Optimal *J*



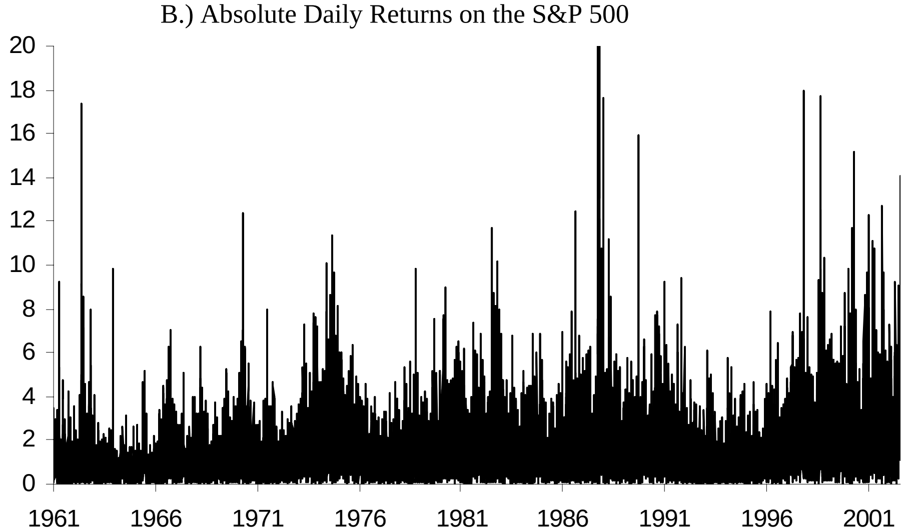
Note: Applies to data generated by $y_t = \mu_t + \varepsilon_t$, $\mu_t = (1 - p_T)\mu_{t-1} + \sqrt{p_T}\eta_t$, η_t ~ short memory, ε_t ~ short memory, and t = 1,2,...,T. The bandwidth (number of frequencies) in the GPH regression is $J = p_T T / c$.

Figure 7: Soybean and S&P 500 Time Series

A.) Log Relative Price of Soybeans and Soybean Oil



B.) Absolute Daily Returns on the S&P 500



Notes: (i) The soybean data span January 1, 1953 to June 30, 2001 and contain the average weekly soybean price in Central Illinois and the average weekly soybean oil price in Decatur, Illinois. There are a total of 2455 observations and the units of measurement are cents per bushel for soybeans and cents per pound for soybean oil.
(ii) The S&P 500 data span January 1, 1961 to July 31, 2002 and comprise the absolute daily returns on the S&P 500 stock index. There are a total of 10463 observations.