ALL λ-SEPARABLE FRISCH DEMANDS AND CORRESPONDING CARDINAL UTILITY FUNCTIONS

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Abstract. Frisch demands depend on prices and a multiplier λ associated with the consumer's budget constraint. The case in which demands or expenditures are separable in λ is the case of greatest empirical interest, since in this case latent variable methods can be adopted to control for consumer wealth when estimating demands.

Subject only to standard, modest, regularity conditions, we provide a complete characterization of all Frisch demand systems and of the utility functions that rationalize these demand systems when either quantities demanded or consumption expenditures is separable in λ.

Quantities demanded are λ-separable if and only if the rationalizing utility function is additively separable in these quantities. In contrast, expenditures are λ-separable if and only if marginal utilities for these expenditures belong to one of two simple parametric families. With n goods, the first family has $2n$ parameters, and corresponds to Houthakker's "direct addilog" utility function. The second family has $3n$ parameters and is new. It corresponds to a family of utility functions which have Stone-Geary utility as a limiting case.

1. Introduction

Frisch demand systems are demands written as a function of prices and a multiplier λ on the consumer's budget constraint. This paper provides a complete characterization of an important class of Frisch demand systems, where some function of either quantities demanded or expenditures is additively separable in λ, and where the demand system can be rationalized by the maximization of a well-behaved ("valid") utility function.

The case in which quantities demanded are λ-separable implies only one important restriction on the rationalizing utility functions: these utility functions must be additively separable. It follows that any λ-separable Frisch demand system must be separable, with demands for

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any particular good depending only on λ and on own-price. This rules out any "specific substitution" effects (Theil 1975), but permits unrestricted Hicks-Allen substitution or complementarity.

The second case, in which expenditures are λ-separable, is much more restrictive. Not only must the rationalizing cardinal utility function be additively separable, but the cardinal utility function must belong to a particular simple parametric family which generalizes the Stone-Geary and Constant Elasticity of Substitution systems: if there are n goods, there can be no more than $3n$ independent parameters in both the utility function and in the Frisch demand system.

When estimating a demand system one is typically concerned with measuring two things: demand responses to changes in price, and demand responses to changes in resources. A conclusion one might draw from the analysis of this paper is that λ-separable demand systems are useless for estimating specific substitution effects. However, for some applications this disadvantage may be offset by the very flexible Engel curve behavior they generate.

The class of demand systems we describe includes all of the rationalizable Frisch demand systems which have been estimated in the literature. Finally, this class includes a new and interesting demand system which generalizes the Stone-Geary linear expenditure system in such a way as to permit quite flexible Engel curve behavior.

Frisch demand systems arise very naturally from the first order conditions of the standard consumer's problem. Because they're so natural it's hard to know how early their introduction was, but certainly by 1930 Frisch (Frisch 2011) was making use of what he would later (Frisch 1959) call "want-independent" demands that depend only on λ and own-price. James Heckman and Thomas MaCurdy led a modern revival of the use of these demands (James J. Heckman 1974; James J Heckman 1976; Heckman and MaCurdy 1980; MaCurdy 1981), calling them λ-constant demands. Martin Browning seems to have given the demand system the name "Frisch" in the nineteen eighties (Browning 2005). The literature has generally assumed that Frisch demands are λ-separable, at least in part because when one considers life-cycle demand then one can use latent variable methods to estimate or to control for variation in consumers' permanent income (e.g., James J. Heckman 1974; James J Heckman 1976; MaCurdy 1983; Attfield and Browning 1985; Browning, Deaton, and Irish 1985; Blundell, Browning, and Meghir 1994; Hayashi, Altonji, and Kotlikoff 1996; Blundell, Pistaferri, and Saporta-Eksten 2016).

But Frisch demands are not invariant to monotonic transformations of the consumer's utility function; thus, for any particular (invariant)

Marshallian demand system there exists a equivalence class of Frisch demands, and within this class almost none of the demands will be additively-separable in λ. This has two important and perhaps under-appreciated consequences. The first is that if a Frisch demand system is λ-separable it will remain so only under linear transformations of a particular *cardinal* utility function. This paper uses this fact to derive the particular cardinal utility functions which correspond to particular λ-separable Frisch demand systems. The second is that λ-separability imposes a quite different structure on demand and utility than does the analogous property of income being additively separable in a Marshallian demand system (Lewbel 1987), and in some important ways λ-separability is much less restrictive.

Thus, this paper makes two contributions. The first, and more fundamental, is to identify all the cardinal utility functions which are consistent with λ-separable Frisch demand functions. The second is to show that—even though these particular utility functions are quite special—the resulting demand systems are in fact quite flexible. They allow for quite unrestricted Hicks-Allen substitution and complementary, and their implied income elasticities and corresponding Engel curve behavior are greatly improve on any of the Marshallian demand structures that are typically observed in the literature.

2. Properties of Frisch Demands

We begin with some properties which hold for any Frisch demand system. Among these are some which are well-known in the literature, and others which are less well understood.

2.1. **Preliminaries.** Let $\mathcal{U}_n$ be the set of strictly increasing, strictly concave, twice-continuously differentiable functions mapping $\mathbb{R}^n_+$ into $\mathbb{R}$, and call $\mathcal{U}_n$ the set of *valid* utility functions over $\mathbb{R}^n_+$.

For a consumer with a utility function $U \in \mathcal{U}_n$ with a total budget $\bar{x} > 0$ facing prices $p \in \mathbb{R}^n$, a Lagrangian formulation of the *consumer's problem* is to solve $\max_{c \in \mathbb{R}^n_+} U(c) + \lambda(\bar{x} - p^\top c)$, with λ the Lagrange multiplier.

Frisch demands map the product of positive quantity λ and n prices into n quantities demanded. We say that

Condition 1. *An n-vector of Frisch demands $f(p, \lambda)$ is* rationalized *by U if there exists a $U \in \mathcal{U}_n$ such that*

$$\frac{\partial U}{\partial c_i} \equiv u_i(f(p\lambda)) = p_i \lambda \tag{1}$$

for all $p\lambda$, $\lambda > 0$ in any open subset of $\mathbb{R}^n_+$ for $i = 1, \ldots, n$.

Similarly, we say a given f is *rationalizable* if there exists a $U \in \mathcal{U}_n$ which rationalizes f.

Condition 1 basically requires that demands be interior solutions to the problem of maximizing some valid utility function subject to a budget constraint. If a consumer has a utility function U, and solutions to that consumer's problem are characterized by the first order conditions (1), then these demands will also be solutions to this consumer's problem.

Remark 1. *Condition 1 implies some properties of Frisch demands, already well-known in the literature. These include:*

(1) *f is continuously differentiable.*
(2) *The matrix $F = [f_{ij}]$ is symmetric and negative-definite.*
(3) *Demands f are strictly decreasing in λ.*
(4) *$f_{ii} < 0$.*
(5) *$f(p, \lambda)$ is positive homogeneous of degree zero in $(p, 1/\lambda)$.*
(6) *Frisch expenditures $x_i(p, \lambda) \equiv p_i f_i(p, \lambda)$ are positive homogeneous of degree one in $(p, 1/\lambda)$.*

An immediate consequence of property 5 (homogeneous of degree zero) is that Frisch demands $f(p, \lambda) \equiv f(p\lambda, 1)$. Thus (in a minor abuse of notation) these demands can be written simply as $f(p\lambda)$, which depends only on n arguments.

2.2. **Transformations and Translations.** Marshallian demands are, of course, invariant to monotonic transformations of the utility function. This is *not* true of Frisch demands. If a utility function $U \in \mathcal{U}_n$ rationalizes a Frisch demand system $f(p, \lambda)$, then any monotonic transformation of this utility function $M(U) \in \mathcal{U}_n$ will rationalize some other Frisch demand system $f^M(p, \lambda)$, with the two demand functions related by the identity $f^M(p, \lambda M'(U(f(p, \lambda)))) \equiv f(p, \lambda)$. Thus, corresponding to any particular Marshallian demand system is a class of equivalent Frisch demands.

While monotonic transformations of utility yield different Frisch demand systems, each of these different Frisch demand systems still corresponds to a single Marshallian demand system $c(p, x)$. Now consider a different sort of operation involving the *translation* of the demand system. In particular, let the function $d(p)$ be continuously differentiable and homogeneous degree zero. Then for any rationalizable Frisch demand $f(p, \lambda)$, a translated demand $\tilde{f}(p, \lambda) = f(p, \lambda) + d(p)$ is also rationalizable, but by some other utility function $\tilde{U} \in \mathcal{U}_n$ which is *not* a monotonic transformation of U.

2.3. **λ-separability.** Now we turn our attention from general properties of Frisch demands to the special properties of those demands when they satisfy a certain separability condition that we call λ-separability. We consider two alternative notions: first that quantities are λ-separable; second, that expenditures are λ-separable.[1]

We begin with the case in which quantities are λ-separable:

Condition 2. *The Frisch demand for good i, f_i is λ-separable if there exist functions (ϕ_i, a_i, b_i) such that*

$$\phi_i(f_i(p\lambda)) = a_i(p) + b_i(\lambda), \tag{2}$$

with ϕ_i strictly increasing and continuously differentiable; and a_i either non-constant or zero.

Demands are *separable* (not just λ-separable) if they depend only on own-price; that is, in addition to (2) we have (in another abuse of notation) $a_i(p) = a_i(p_i)$.[2] Our condition generalizes the notion of quantities being separable in Browning, Deaton, and Irish (1985), which require that the functions ϕ_i be the identity function (i.e., that $f_i(p\lambda) = a_i(p) + b_i(\lambda)$).

Note that since any rationalizable Frisch demand is continuously differentiable (by Remark 1), then assuming ϕ_i continuously differentiable is enough to guarantee that any valid f satisfying (2) will have a_i and b_i continuously differentiable.

Alternatively, we may require that *expenditures* be λ-separable:

Condition 3. *The Frisch expenditures on good i, $x_i(p, \lambda) \equiv p_i f_i(p\lambda)$ are λ-separable if there exist functions (ϕ_i, a_i, b_i) such that*

$$\phi_i(x_i(p, \lambda)) = a_i(p) + b_i(\lambda), \tag{3}$$

with ϕ_i continuously differentiable and a_i either non-constant or zero.

As with demands, we say that expenditures are separable if, in addition to being λ-separable, the $a_i(p)$ in Condition 3 depends only on own-price p_i; i.e., $a_i(p) = a_i(p_i)$.

Note that while rationalizability is a property of the entire system of demands and expenditures, (λ-) separability is a property of a particular good. In particular it's possible that some but not all demands or expenditures are (λ-) separable.

1. These notions generalize what Browning, Deaton, and Irish (1985) called, respectively, "Case 1" and "Case 2" demands

2. Frisch (1959) calls this property “want independence,” to distinguish it from the much stronger property of independence of Marshallian demands.

3. Demands and utilities when demands are λ-separable

We now observe that, for any $i = 1, \ldots, n$, equations (1) and Condition 2 form a system of functional equations in four unknown functions ($u_i : \mathbb{R}^n_+ \to \mathbb{R}, \phi_i : \mathbb{R} \to \mathbb{R}, a_i : \mathbb{R}^n_+ \to \mathbb{R}, b_i : \mathbb{R} \to \mathbb{R}$). Finding a solution to this system allows us to construct a pair (f, U), with f λ-separable and rationalized by some $U \in \mathcal{U}_n$.

Both of these systems have many solutions, and it's not very challenging to find one. We instead follow the example of Gorman (1953) or Lewbel (1987) and describe the set of all possible solutions to one or the other of these systems of functional equations.

The key to characterizing the set of all possible solutions involves noticing that by combining Condition 1 and Condition 2 we obtain

$$\phi_i(f_i(p\lambda)) = a_i(p) + b_i(\lambda).$$

If we define $k(z) = \phi_i(f_i(e^z))$, $g(y) = a_i(e^y)$, and $h(z) = a_i(e^z)$, then by the change of variables $z = \log\lambda$ and $y = \log p$ we have

$$k(z + y) = g(y) + h(z), \tag{4}$$

which is known as Pexider's equation (Aczél and Dhombres 1989). This single functional equation has the remarkable property that a solution determines all three of the functions k, g, h; and so our proof exploits what is known of these solutions along with basic properties of all valid demands to give a complete characterization of the set of valid λ-separable demands in Theorem 1.

This brings us to our main result pertaining to λ-separable quantities.

Theorem 1. *If Frisch demands satisfy Condition 1 then demand for any good i also satisfying Condition 2 takes the form*

$$\phi_i(f_i(\lambda p)) = \alpha_i - \beta_i \log p_i - \beta_i \log\lambda$$

while the marginal utility of good i takes the form

$$u_i(c) = \exp\left(\alpha_i - \phi_i(c_i)/\beta_i\right) \tag{5}$$

for some constants α_i, positive constants β_i, and ϕ_i some strictly increasing, continuously differentiable function.

Proof. The function b_i must be decreasing by Remark 1 and Condition 2, while the function a_i can be non-constant or zero by assumption. This gives us two cases to consider. Suppose $a_i(p) = 0$ for all p. Then Condition 1 implies that $u_i(\phi_i^{-1}(b_i(\lambda))) = \lambda p_i$ for all $p_i > 0$, which admits no solution, as the right-hand side is increasing in p_i while the left-hand side is not.

Thus $a_i(p)$ must be non-constant. Then we have a system of equations

$$\log u_i(\phi_i^{-1}(a_i(p) + b_i(\lambda))) = \log \lambda + \log p_i$$

for all $i = 1, \dots, n$. Note that we do *not* impose separability, allowing $a_i(p)$ to depend on all prices.

But this takes the form of Pexider's equation. The following lemma is just an immediate consequence of a result from Aczél and Dhombres (1989) (Corollary 10, p. 43):

Lemma 1. *The general solutions of (4) in the class of functions $g, h, k : \mathbb{R}^n \to \mathbb{R}^m$, where at least one of g, h, k is continuous at a point, are given by the matrix equations*

$$g(x) = Cx + A, \quad h(y) = Cy + B, \quad k(z) = Cz + A + B,$$

where A and B are arbitrary constants in $\mathbb{R}^m$ and where C is an arbitrary $m \times n$ matrix.

Applying this lemma implies that any possible solutions must take the form $b(\lambda) = C \log \lambda + A$; $a(p) = C \log p + B$; and $\log u(c) = C \log \phi(c)$, where C is an $n \times n$ constant matrix; A and B are $n \times 1$ vectors; and where $\phi(c) = (\phi_i(c_i))_{i=1}^n$.

This implies that $u_i(c) = \exp\left(\alpha_i - \sum_{j=1}^n \beta_{ij}\phi_i(c_j)\right)$. This appears to allow for specific substitution effects. But the cross-partial u_{ij} and u_{ji} must be equal, which requires

$$u_i \beta_{ij} \phi_i'(c_i) = \beta_{ji} \phi_j'(c_j) u_j$$

for all c_i and c_j, which can only hold if $\beta_{ij} = \beta_{ji} = 0$ for all $j \neq i$. Concavity of the utility function then implies that $\phi_i(c_i)/\beta_i$ must be decreasing, so that $\beta_i > 0$. $\square$

This form of utility is extremely flexible, because each ϕ_i can be an arbitrary increasing, continuously differentiable function. The only restriction (though it's an important one) is that there can be no specific substitution effects in this family of demand systems (with the consequence that no inferior goods are allowed). (5) can be used to describe *any* valid additively-separable marginal utility function, with the corresponding Frisch demands of Theorem 1. We summarize this point in the following corollary of Theorem 1.

Corollary 1. *Let $f(\lambda p)$ be rationalized by a utility function $U \in \mathcal{U}_n$, with marginal utility function u. Then f_i is λ-separable if and only if $u_{ij} = 0$ for all $j \neq i$.*

Thus, having quantities λ-separable is equivalent to having the utility function additively separable. However, as noted in the corollary, if *some* demands are λ-separable then this is equivalent to having only the corresponding subutility of those goods being additively separable.

4. Demands and utilities when expenditures are λ-separable

When expenditures (rather than quantities) are λ-separable the Pexider equation will no longer generally characterize demands. However, exploiting the fact that expenditures must be linearly homogeneous, it turns out that one can write any rationalizable λ-separable expenditures in the form

$$k(p+\lambda) = g(\lambda)\ell(p) + h(p),$$

which is called the *generalized* Pexider equation. This gives us a single functional equation in two variables, which can be solved for the four functions g, h, k, and ℓ. Exploiting this allows us to describe all rationalizable Frisch demands and utilities when expenditures are λ-separable:

Theorem 2. *If expenditures for some good i satisfy Condition 1 and Condition 3 with ϕ_i increasing; $a_i(p)$ either non-constant or zero, and continuous at a point; and with b_i continuous at a point, then transformation functions ϕ_i, Frisch demands f_i and rationalizing marginal utility u_i must satisfy one of the following two cases for positive constants α_i, β_i, and σ_i:*

1. *(Addilog utility): $\phi_i(x_i) = \log(x_i)$; $f_i(p,\lambda) = (\alpha_i/(\lambda p_i))^{\beta_i}$; and $u_i(c) = \alpha_i c_i^{-1/\beta_i}$.*
2. *(New utility): $\phi_i(x_i) = x_i^{\sigma_i}$; $f_i(p,\lambda) = [(\beta_i/(\lambda p_i))^{\sigma_i} + \alpha_i]^{1/\sigma_i}$; and $u_i(c) = \beta_i\left(c_i^{\sigma_i} - \alpha_i\right)^{-1/\sigma_i}$.*

Proof. First, Lemma 2 establishes that (ϕ_i, a_i, b_i) in (3) are all either logarithmic or positive homogeneous of some degree $(-)\sigma_i$.

In the logarithmic case the logarithm of demand for good i can be written $\log f_i(\lambda p) = [-\log p_i + a_i(p)] + b_i(\lambda)$, which not only has expenditures λ-separable, but also quantities λ-separable, thus satisfying the conditions of Theorem 1 with $\phi_i = \log$, yielding the result that $\log(f_i(\lambda p)) = \tilde{\alpha}_i - \beta_i \log(p_i\lambda)$. Let $c_i = f_i(\lambda p)$ and solve for $p_i\lambda$, obtaining $p_i\lambda = \alpha_i c_i^{-1/\beta_i}$, where $\alpha_i = e^{\tilde{\alpha}_i}$ must be positive.

In the homogeneous case, we have

$$(p_i f_i(\lambda p))^{\sigma_i} = a_i(p) + b_i(\lambda),$$

or

$$f_i(\lambda p)^{\sigma_i} = p_i^{-\sigma_i} a_i(p) + p_i^{-\sigma_i} b_i(\lambda).$$

This takes the form of the generalized Pexider equation (7), with $x = \log \lambda$ and y the vector $\log p$, when the vector-valued function $k(x+y) = [f_i(\exp(x+y))^{\sigma_i}]$, $h(y) = [a_i(\exp(y))e^{-\sigma_i y_i}]$, $g(x) = [b_i(\exp(x))]$, and $\ell(y) = [e^{-\sigma_i y_i}]$. Now, we seek to apply Proposition 1, which gives solutions to the system of functional equations 7–13. Part of this system is the function $\varphi(y)$. Using our knowledge that $\ell_i(y) = e^{-\sigma_i y_i}$ and (10), it follows that in this equation the function $\varphi(y) = \ell(y) = [e^{-\sigma_i y_i}]$. Now, consulting the different possible cases of Proposition 1 we see that with this solution of φ the only cases that can apply are the cases indicated by 2a and 2bii. The former implies that $f_i(\lambda p)^{\sigma_i}$ is a constant, so that (to be consistent with the properties of Frisch demands) $\sigma_i = 0$. But then the function ϕ_i isn't increasing, and the only solutions that are relevant to our problem are the solutions 2bii. These imply that $k(z) = Cz + B$, with C and B constant matrices, and $\varphi(y) = \exp(Cy)$. Thus C is a diagonal matrix, with diagonal elements $-\sigma_i$. Using equations (11) and (12) we obtain $k(x+y) = (Cx+B)e^{Cy} + [h(y) - h(0)\varphi(y)]$; then using our definition of $h(y)$ in terms of p gives us $f_i(\lambda p)^{\sigma_i} = \alpha_i/(p_i\lambda)^{\sigma_i} + \beta_i$. Noting that $u_i(c) = p_i\lambda$ and solving for this gives us the solution for marginal utilities. □

4.1. **Rationalizing Utility Functions.** The labels of the different cases in Theorem 2 indicate names for the rationalizing utility function U having marginal utilities $u_i(c)$; for example, "Addilog" utility (Houthakker 1960) is the utility function $U(c) = \sum_{i=1}^n \alpha_i \beta_i \frac{c^{1-1/\beta_i} - 1}{\beta_i - 1}$. The Addilog system generalizes the Constant Elasticity of Substitution (CES) system (take $\beta_i = \beta$) and the Cobb-Douglas system is a limiting case (take $\beta_i \to 1$, applying L'Hôpital's rule). Finally, the "New" case gives what is, to the best of my knowledge, a marginal utility function which has not previously appeared in the literature. This case gives demands which are not linear in parameters, which may limit its usefulness in applied empirical work. However, when $\sigma_i = 1$ one obtains the Stone-Geary utility function, which suggests that it could be used to explore the behavior of Engel curves, perhaps exploiting a Box-Cox approach to estimation.

Since our characterization implies that a rationalizing utility function satisfying Condition 1 and Condition 3 will itself be additively separable, it is also possible to use Theorem 2 to construct a 'mongrel' utility function which could generate any rationalizable expenditures, possibly combining goods with quite different demand functions.

Theorem 3. *Let $\{S_0, S_1, S_2\}$ be a partition of the index set $\{1, \ldots, n\}$ such that expenditures on a good $i \in S_s$ satisfy case s of Theorem 2 for $s = 1, 2$, and where S_0 is the set of demands which satisfy none of the cases of Theorem 2. Then expenditures are rationalized by a monotonic transformation of $U \in \mathcal{U}_n$ if and only if*

(6)
$$U(c) = U^{(0)}(c^{(0)}) + \sum_{i \in S_1} \frac{\alpha_i \beta_i}{\beta_i - 1}\left(c_i^{1-1/\beta_i} - 1\right) + \sum_{i \in S_2} \beta_i \int_0^{c_i} (c^{\sigma_i} - \alpha_i)^{-1/\sigma_i}\, dc,$$

for some positive constants $(\alpha_i, \beta_i, \sigma_i)_{i=1}^n$ and for some $U^{(0)} \in \mathcal{U}_m$, where m is the cardinality of S_0 and where $c^{(0)}$ is a vector of the goods with indices in S_0.

Proof. Necessity is trivial. For sufficiency, write utility as the sum of three components $U(c) = U^{(0)}(c^{(0)}) + U^{(1)}(c^{(1)}) + U^{(2)}(c^{(2)})$, using a notation similar to that defined for S_0. Because U is valid by assumption, it is strictly increasing, strictly concave, and continuously twice differentiable. Each of the components $U^{(1)}$ and $U^{(2)}$ share these properties by Theorem 2, and $U^{(0)}$ inherits these properties from U. $\square$

Thus, this 'mongrel' utility function can combine or simultaneously rationalize several different kinds of demand systems.

5. Discussion

5.1. **Separability.**

As we've shown, any λ-separable system (whether in quantities or expenditures) is also separable in the stronger sense that demand for some good i depends only on λ and own-price p_i. In a Frisch demand system, this implies that for such goods there is no "specific substitution effect" (Theil 1975), and that the Frischian substitution matrix $F = [f_{ij}]$ is diagonal.

This does not, of course imply that the Slutsky substitution matrix S is diagonal, or that there cannot be Hicks-Allen substitution or complementarity. The two substitution matrices are related by

$$S = F + \eta\left(\frac{\partial c}{\partial x}\right)\left(\frac{\partial c}{\partial x}\right)^\top,$$

where the vector $\partial c/\partial x$ gives the Engel effects of the change in Marshallian demand in response to a change in total expenditures, and where $\eta = \lambda/(x\partial\lambda/\partial x)$ is what Frisch called "money flexibility". But neither F nor η is invariant to monotonic transformations of a rationalizing utility function.

In particular (and unlike the Marshallian case) if there exists a separable Frisch demand system that is rationalized by a utility function U, then it will not generally also be rationalized by $M(U)$ (where $M : \mathbb{R} \rightarrow \mathbb{R}$ is any monotonic transformation). Instead, $M(U)$ will validate a *different* Frisch demand system f^M, and the system f^M will only be separable if M is linear. Thus, having F diagonal imposes much weaker restrictions on demand than would having S diagonal.

5.2. **Price Elasticities.** What can be said about the price response of λ-separable Frisch demands? There are two cases we're interested in; the price response of Frisch demands, and the price response of the corresponding Marshallian demands. We consider these in turn.

First, because of the separability of the Frisch demands it's apparent that there are no *cross*-price substitution effects, since $\partial f_i/\partial p_j = 0$ for any $i \neq j$. This is what's meant when we say that these demands feature no "specific substitution" effects. As for own-price response, we need to consider the quantities-separable case separately from the expenditures-separable case.

For quantities separable, we have no restrictions aside from those implied by the first order condition $u_i(f_i(p_i, \lambda)) = p_i\lambda$ (but note that because of the separability in this case marginal utility and f_i depend only on c_i and p_i, respectively). This implies that $\partial f_i/\partial p_i = \lambda/u_{ii}$, and that the price elasticity be equal to $([u_{ii}c_i/u_i]^{-1}$; that is, that the own-price elasticity for good i is equal to the elasticity of marginal utility of good i (note the resemblance to the reciprocal of the usual measure of relative risk aversion). Since U is strictly concave, it follows that this elasticity must be negative, but is otherwise unrestricted.

The expenditures-separable case inherits the connection between price response and utility functions observed in the quantities separable case, but from Theorem 2 we know that the form of the marginal utility function is much more restricted. In particular, for the addilog case we obtain an own price elasticity of $-\beta_i$: this can take on any negative value, but is constant for all levels of income or λ. For the "new" utility case the elasticity will be given by $\alpha_i/c_i^{\sigma_i} - 1$; since we must have $c_i \geq \alpha_i^{1/\sigma_i}$ it follows that the price elasticity is constrained to lie between zero (at the lowest possible levels of income) and minus one (as income goes to plus infinity).

6. Conclusion

An enormous literature relies on specifications of Frisch demand functions which are λ-separable.[3] This paper is the first to give a comprehensive description of the consequences of assuming this sort of separability, building on an incomplete description given by Browning, Deaton, and Irish (1985). We give a complete characterization of all the λ-separable Frisch demand functions and the cardinal utility functions that rationalize these demand functions.

The results are striking. First, λ-separability of a demand system turns to be equivalent to having an additively-separable utility function. Second, λ-separability of an *expenditure* system is equivalent to having an additively-separable utility function belonging to one of two parametric families. The first family is the addilog family of Houthakker (1960), having two parameters for every good. This has as special cases HARA, CES, and Cobb-Douglas utilities. The second family has not been previously described, but has three parameters for every good, with the Stone-Geary system as a special case.

Because (some monotonic transformation of) any utility functions consistent with λ-separable Frisch demands or expenditures are additively separable, λ-separability implies that there can be no specific substitution effects between goods. However, Hicks-Allen substitution and complementarity is comparatively unrestricted.

While these utility functions may be rather poorly suited for modeling cross-price demand responses, they appear to be very well suited to modeling Engel curve behavior. This provides a very sharp contrast to the usual sorts of Marshallian analysis, in which demand systems (Stone-Geary, CES, AIDS, Rotterdam, Translog) which can be rationalized by valid utility functions require linear Engel curves. In contrast, the parametric utility functions consistent with λ-separable Frisch expenditures feature extremely flexible Engel curves; the only important restriction is that inferior goods are not allowed.

7. Appendix

In this appendix, we first supply a lemma pertaining to the homogeneity of λ-separable expenditure systems, and then provide some necessary results on the solutions to Pexider equations.

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3. A somewhat arbitrary selection of examples includes Heckman and MaCurdy (1980), Browning, Deaton, and Irish (1985), Altonji, Hayashi, and Kotlikoff (1992), Blundell, Browning, and Meghir (1994), Blundell (1998), Baxter, Jermann, and King (1998), Pistaferri (2003), Hayashi and Prescott (2008), Ham and Reilly (2013), and Blundell, Pistaferri, and Saporta-Eksten (2016).

7.1. **Homogeneity.**

Lemma 2. *If demand for good i satisfies Condition 1 and Condition 3, then the functions ϕ_i, a_i and b_i are either all logarithmic or ϕ_i and a_i are both positive homogeneous of some degree σ_i, while b_i is positive homogeneous of degree $-\sigma_i$.*

Proof. From Remark 1 expenditures x_i are homogeneous of degree one in $(p, 1/\lambda)$. Exploiting Condition 3 then implies that

$$x_i = \phi_i^{-1}\left(a_i(p) + b_i(\lambda)\right)$$

is similarly homogeneous of degree one. The function ϕ_i must then either be homogeneous of degree σ_i, with $\phi_i(x_i) = x_i^{\sigma_i}$, or else $\phi_i(x_i) = \log(x_i)$. In either case Frisch quantities can be written as

$$c_i = f_i(p\lambda) = \frac{1}{p_i}\phi_i^{-1}\left(a_i(p) + b_i(\lambda)\right) - d_i(p)$$

for some function d_i homogeneous of degree zero.

We consider the power and logarithmic cases in turn.

First suppose that $\phi_i(x) = x^{\sigma_i}$. Then the sum $a_i + b_i$ must also be homogeneous of degree σ_i in $(p, 1/\lambda)$, and the individual functions a_i and b_i respectively either homogeneous of degree σ_i and $-\sigma_i$ or else the zero function. It follows that $f_i(p,r) = \phi_i^{-1}\left(a_i(p)/p_i^{\sigma_i} + b_i(\lambda)/p_i^{\sigma_i}\right) - d_i(p)$, and that $a_i(p)/p_i^{\sigma_i}$ and $b_i(\lambda)/p_i^{\sigma_i}$ are either zero or positive homogeneous of degree zero, so that

$$(a_i(p\theta) + b_i(\lambda/\theta)) = \theta^{\sigma_i}(a_i(p) + b_i(\lambda)) = \theta^{\sigma_i}a_i(p) + \theta^{\sigma_i}b_i(\lambda).$$

for any positive scalar θ. Differentiating this with respect to $1/\lambda$ establishes that b_i' is homogeneous of degree $\sigma_i - 1$, so that b_i is homogeneous of degree σ_i (by Euler's theorem of positive homogeneous functions). A similar argument involving the gradient with respect to p establishes the same for a_i.

For the logarithmic case, $\phi(x_i) = \log(x_i) = \log(p_i) + \log(c_i)$ implies that

$$f_i(p,\lambda) + d_i(p) = \exp\left(a_i(p) + b_i(\lambda) - \log(p_i)\right)$$

which must be positive homogeneous of degree zero in $(p, 1/\lambda)$. This implies that for any $\theta > 0$

$$a_i(\theta p) + b_i(\lambda/\theta) - \log(\theta p_i) = a_i(p) + b_i(\lambda) - \log(p_i),$$

which in turn implies that

$$a_i(\theta p) + b_i(\lambda/\theta) = a_i(p) + b_i(\lambda) + \log(\theta),$$

implying that both a_i and b_i are linear in logs of (p,λ). $\square$

7.2. **Generalized Pexider Equation Applied to Vector Spaces.** We now introduce our main tool for solving the functional equations implied by separability and rationalizability; this tool is an application of what is called the generalized Pexider equation, when the domain of application is limited to real vector spaces.

Consider the generalized Pexider equation

$$k(x+y) = g(x)l(y) + h(y) \tag{7}$$

where

$$g(x) = \frac{k(x) - h(0)}{l(0)} \tag{8}$$

$$\varphi(y) = \frac{l(y)}{l(0)} \tag{9}$$

$$\psi(y) = h(y) - h(0)\frac{l(y)}{l(0)} \tag{10}$$

$$k(x+y) = k(x)\varphi(y) + \psi(y) \tag{11}$$

$$\kappa(x) = k(x) - k(0) \tag{12}$$

$$\kappa(x+y) = \kappa(x)\varphi(y) + \kappa(y). \tag{13}$$

Next we give statements of two related lemmata. The first is just a statement of the solution of the well-known functional equation of Cauchy applied to real vector spaces; the second is a statement of the solution to what is sometimes called Cauchy's exponential equation, again for real vector spaces.

Lemma 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, with f continuous at a point. Then if*

$$f(x+y) = f(x) + f(y) \tag{14}$$

then $f(x) = Cx$ for some constant $m \times n$ matrix C.

Also

Lemma 4. *Let $h : \mathbb{R}^n \to \mathbb{R}^m$. If*

$$h(x+y) = h(x)h(y)$$

then either $h(x) = 0$ or $h(x){=}e^{f(x)}$, where f is an arbitrary solution to Cauchy's equation (14).

Corollary 2. *Any solution to the functional equation of Lemma 4 which is continuous and non-constant is of the form*

$$h(x) = \exp(Cx),$$

where C is a constant matrix and the exp *operator is element by element.*

The following is just a restatement of Theorem 15.1 of Aczél and Dhombres (1989), and describes all solutions to the generalized Pexider equation (7) over the general domain of Abelian groupoids.

Theorem 4. *For any x, y in an Abelian groupoid, solutions to (7) will satisfy one of:*

(1) *If $\varphi(x) = 1$ for all x, then $\kappa(x)$ is an arbitrary function; $\psi(x) = \kappa(x)$; and $k(x) = \kappa(x) + B$. Or;*

(2) *if $\varphi(x_0) \neq 0$ for some x_0, then we have $C = \frac{\kappa(x_0)}{\varphi(x_0)-1}$; and $\kappa(x) = C[\varphi(x) - 1]$; and two sub-cases:*
  - (a) *$C = 0$; $\kappa(x) = 0$; $\varphi(x)$ arbitrary; $k(x) = B$; $\psi(y) = B(1 - \varphi(y))$; or*
  - (b) *$C \neq 0$; $k(x) = C\varphi(x) + B$; $\psi(x) = B(1 - \varphi(x))$; where $\varphi(x)$ satisfies $\varphi(x + y) = \varphi(x)\varphi(y)$ (Cauchy's exponential equation); and where $\kappa(x)$ satisfies $\kappa(x + y) = \kappa(x) + \kappa(y)$ (Cauchy's equation).*

If we restrict the domain under consideration to a real vector space, then we can give explicit solutions to (7), as follows:

Proposition 1. *For any $x, y \in \mathbb{R}^n$, solutions to (7) will satisfy one of:*

(1) *If $\varphi(x) = 1$ for all x, then $\kappa(x) = \psi(x) = Cx$ and $k(x) = Cx + B$, where $B \in \mathbb{R}^m$. Or;*

(2) *if $\varphi(x_0) \neq 0$ for some x_0, then we have $C = \frac{\kappa(x_0)}{\varphi(x_0)-1}$; and $\kappa(x) = C[\varphi(x) - 1]$; and two sub-cases:*
  - (a) *$C = 0$; $\kappa(x) = 0$; $\varphi(x)$ arbitrary; $k(x) = B$; $\psi(y) = B(1 - \varphi(y))$; or*
  - (b) *$C \neq 0$; $k(x) = C\varphi(x) + B$; $\psi(x) = B(1 - \varphi(x))$; $\kappa(x) = Cx$; and one of:*
    - (i) *$\varphi(x) = 0$;*
    - (ii) *$\varphi(x) = \exp(Ax)$; or*
    - (iii) *$\varphi(x) = \exp(f(x))$, f nowhere continuous.*

Proof. Just a specialization of Theorem 4 to the case in which domain is a real vector space, which then allows subsequent application of Lemma 4 and Lemma 3. □

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