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Interdisciplinary Essays in Economics and Operations Management

A dissertation submitted in partial satisfaction of the

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in Management

by

Georgios Georgiadis

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ABSTRACT OF DISSERTATION

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Professor Christopher Siu Tang, Chair

In this dissertation I present three papers, each as an individual chapter. The first two papers are in the field of economics, while the third paper is in the field of operations management.

In the first paper, titled "Projects and Team Dynamics", I study the dynamic collaboration of a team on a project that progresses gradually over time and generates a payoff upon completion. The main result is that members of a larger team work harder than members of a smaller team if and only if the project is sufficiently far from completion. In contrast, as the project gets close to completion, the aggregate effort of a larger team can become less than that of a smaller team due to aggravated free-riding. This result has three implications for the organization of partnerships and when a manager recruits agents into a team to undertake a project on her behalf. First, given a fixed budget, larger teams are preferable the

longer the project is. Second, the manager can benefit from dynamically decreasing the team size as the project approaches completion. Third, asymmetric compensation is preferable if the project is sufficiently short.

The second paper titled "Project Design with Limited Commitment and Teams" studies the interaction between a group of agents who exert costly effort over time to complete a project, and a manager who chooses its objectives. The manager can commit to the requirements only when the project is sufficiently close to completion. This is common in projects that involve *design* or *quality* objectives which are hard to define far in advance. The main result is that the manager has incentives to extend the project as it progresses: she is time-inconsistent. This result has three implications. First, the manager will choose a larger project if she has less commitment power. Second, if the agents receive a fraction of the project's worth upon its completion, then the manager should delegate the decision rights over the project size to the agents unless she has sufficient commitment power. Third, cultivating an insider culture so that the agents act in the interest of the entire team may aggravate the manager's commitment problem and lower profits.

The third paper titled "The Retail Planning Problem Under Demand Uncertainty" studies the problem faced by a retailer who chooses suppliers, and determines the production, distribution and inventory planning for products with uncertain demand in order to minimize total expected costs. This problem is often faced by large retail chains that carry private label products. We formulate this problem as a convex mixed integer program and show that it is strongly NP-hard. We determine a lower bound by applying a Lagrangean relaxation and show that this bound outperforms the standard convex programming relaxation, while being computationally efficient. We then develop heuristics to generate feasible solutions. Our computational results indicate that our convex programming heuristic yields feasible solutions that are close to optimal with an average suboptimality gap at 3.4%. Finally, we develop managerial insights for practitioners facing this problem.

The dissertation of Georgios Georgiadis is approved.

Simon Adrian Board

Steven A Lippman

Kumar Rajaram

Christopher Siu Tang, Committee Chair

University of California, Los Angeles,

2013

Dedicated to my parents, with all my love.

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Chapter 1

Projects and Team Dynamics

1.1 Introduction

Teamwork is central in the organization of firms and partnerships. Between 1987 and 1996, the use of employee participation teams nearly doubled from 37% to 66% among Fortune 1000 corporations (Lawler, Mohrman and Benson (2001)).[1] Despite the theoretical predictions that effort and group size should be inversely related (Olson (1965), Andreoni (1988), Bonatti and Hörner (2011) and others), empirical studies commonly find that organizing workers into teams or providing group incentives has increased productivity in both manufacturing and service firms. Hamilton, Nickerson and Owan (2003) find that the adoption of teamwork and group incentives improved worker productivity for apparel production, as is the case for Continental airlines (Knez and Simester (2001)), steel finishing lines (Boning, Ichniowski and Shaw (2007)), and call centers (Batt (1999)).

To explain why the adoption of teamwork often leads to increased productivity in organizations in spite of the free-rider problem, scholars have argued that teams benefit from various motivational forces such as mutual monitoring (Alchian and Demsetz (1972)), complementary skills (Lazear (1998)), peer pressure to achieve a group norm (Kandel and Lazear

[1]Since the late 1990s, team use seems to have reached a plateau, but it's a relatively high plateau (Lazear and Shaw (2007)).

(1992)), *warm-glow* (Andreoni (1990)), and non-pecuniary benefits such as more engaging work and social interaction.

I develop a tractable framework to study the team problem faced by a group of agents who collaborate to complete a project. The primary focus is on how the agents' incentives depend on the team composition and on how far the project is from completion. Using this framework, I examine (i) how the agents should organize into a partnership, and (ii) how a manager who recruits agents into a team to carry out a project on her behalf, should determine the team composition as well as the agents' compensation scheme.

The key features of the model are that the project progresses gradually and stochastically towards completion at a rate that depends on the agents' costly effort, and it generates a payoff when it is completed. Many applications fall within this framework. For instance, consider new product development, where a group of individuals collaborate on the design and manufacture of the product: features are gradually incorporated into the project, and it starts generating a revenue stream after it is released to the market. Start-up companies also share these dynamics: their evolution is uncertain, and they (predominantly) generate value for the stakeholders when they are acquired by a larger corporation or they become public. Similarly, these features are common in many consulting, marketing, as well as construction projects.

Outline of the Results

A Markov Perfect equilibrium (hereafter MPE) is characterized by a system of ordinary differential equations subject to a set of boundary conditions. By examining how the geometry of the solution depends on the parameters of the problem, I obtain insights about how the agents' incentives to exert effort at different stages of the project depend on the team composition (*i.e.*, the team size, as well as each agent's reward and patience level), and on the degree of uncertainty associated with the evolution of the project.[2] A key result is that agents increase their effort as the project progresses. Intuitively, because they discount time and they are compensated upon completion, they have stronger incentives the closer the project is to completion. This result was first shown by Yildirim (2006) and Kessing (2007) who studied similar models, and its implication is that efforts are strategic complements in this model. This is because by increasing his effort level, an agent brings the project closer to completion, which incentivizes others to also increase their effort.

The main result is that members of a larger team work harder than members of a smaller team - both individually and on aggregate - if and only if the project is sufficiently far from completion.[3] Intuitively, by increasing the size of the team, agents obtain stronger incentives to free-ride. However, because the total progress that needs to be carried out is fixed, the agents benefit from the ability to complete the project quicker, which increases the present discounted value of their reward, and consequently strengthens their incentives. I shall refer to these forces as the *free-riding* and the *encouragement effect*, respectively.[4] Because the marginal cost of effort is increasing and agents work harder the closer the project is to completion, their incentives to free-ride, and consequently the free-riding effect, becomes stronger as the project progresses. On the other hand, the benefit of being able to complete the project faster in a bigger team is smaller the less progress remains, and hence the encouragement effect becomes weaker as the project progresses. Therefore, the encouragement effect dominates the free-riding effect, and consequently members of a larger

[2]A similar approach is used by Cao (2010), who studies a continuous-time version of the patent race of Harris and Vickers (1985).

[3]This result holds both if the project is a public good so that each agent's reward is independent of the team size, and if the project generates a fixed payoff that is shared among the team members so that doubling the team size halves each agent's reward.

[4]The latter is reminiscent to the encouragement effect in Bolton and Harris (1999), which reveals that more experimentation by the other team members in the future increases each agent's present incentives to experiment.

team have stronger incentives than those of a smaller team, if and only if the project is sufficiently far from completion.

This result has two implications for the organization of partnerships. First, if the project is a public good so that each agent's reward is independent of the group size, then expanding the partnership *ad-infinitum* is optimal. On the other hand, if the project generates a payoff upon completion that is shared among the team members, then agents prefer to expand the partnership only if the project is sufficiently *long*.[5]

I then introduce a manager who is the residual claimant of a project, and she recruits a group of agents to undertake it on her behalf. Her objective is to determine how large a team to employ and how to compensate its members. A key result is that the optimal symmetric scheme compensates the agents only upon completion of the project. The intuition is that by backloading payments (compared to rewarding the agents for reaching intermediate milestones), the manager can provide the same incentives at the early stages of the project, while providing stronger incentives when the project is close to completion.[6]

These results have three implications for team recruiting. First, the optimal team size increases in the length of the project. To see the intuition behind this result, recall that a larger team works harder relative to a smaller one if and only if the project is sufficiently far from completion. Because the team size is chosen before the agents begin to work, the benefit from a larger team working harder while the project is far from completion outweighs the loss from working less when it is close to completion only if it is sufficiently long.

[5] A project is referred to as *long* if the expected amount of progress necessary to complete it is large.

[6] If asymmetric rewards are permitted, then compensating the agents for reaching intermediate milestones can be beneficial, as it effectively enables the manager to dynamically change the team size as the project progresses.

Second, a manager can benefit from dynamically decreasing the size of the team as the project gets close to completion. The intuition is that she prefers a larger team while the project is far from completion since free-riding is not a major concern, while she prefers a smaller team when the project gets close to completion. With two agents, this can be implemented using an asymmetric compensation scheme in which one agent receives a reward as soon as the project hits a pre-specified intermediate milestone and no further compensation so that he stops working, while the second agent is rewarded only when the project is completed.

Finally, with two (identical) agents, the manager is better off compensating them asymmetrically if the project is sufficiently short. Intuitively, asymmetric compensation mitigates the free-rider problem as the agent who receives the larger reward can *rely* less on the other agent (to exert effort).

Related Literature

First, this paper is related to the moral hazard in teams literature (Holmström (1982), Ma, Moore and Turnbull (1988), Bagnoli and Lipman (1989), Legros and Matthews (1993) and others). These papers focus on the free-rider problem, which arises when each agent must share the benefit of his effort with the other members of the team, and they explore ways to restore efficiency.

Most closely related to this paper is the literature on dynamic contribution games, and in particular the papers that study *threshold* or *discrete* public goods. The general theme of these games is that a group of agents interact repeatedly, and in every period (or moment), each agent chooses his contribution (or effort) to a joint project at a personal cost. Contributions accumulate until they reach a certain threshold, at which point each agent receives a payment (that is independent of his individual contributions) and the game ends.

Admati and Perry (1991) consider a setting in which two agents take turns in contributing to a public good, and they characterize an equilibrium where agents make small contributions at a time, each conditional on the previous contributions of the other agent. Their main takeaway is that contributing little by little over multiple periods helps mitigate the free-rider problem. Marx and Matthews (2000) consider a simultaneous action $n-$player game in which agents receive flow payoffs while the project is in progress in addition to a lump-sum upon completion. They show that multiple contribution periods can achieve a higher provision level of the public good compared to what is achievable in the single period game provided that there is a discrete payoff jump upon completion of the project.[7] Lockwood and Thomas (2002), Compte and Jehiel (2004) and Matthews (2012) also study related contribution games.

Yildirim (2006) and Kessing (2007) show that if the project generates a payoff only upon completion, then contributions are strategic complements even when there are no complementarities in the agents' production function.[8,9] Yildirim (2006) also examines how each agent's (but not the team's aggregate) effort level depends on the team size, and he establishes a result similar to the one in this paper: members of a larger team work harder at the early stages of the project provided that it is sufficiently long, while they work less when it

[7]Duffy, Ochs and Vesterlund (2007) experimentally test this prediction. They find that contributions in the repeated game are higher than in the static game, but the increase does not depend crucially on the existence of a discrete payoff jump upon completion.

[8]The model in this paper is essentially a stochastic version of the one studied by Kessing (2007). On the other hand, the model by Yildirim (2006) differs in that the project comprises of multiple discrete stages, and in every period, the current stage is completed as long as at least one agent exerts effort (which is binary). Consequently, equilibrium strategies are mixed, and higher effort is the interpretation that each agent is more likely to exert effort.

[9]If agents only receive flow payoffs while the project is in progress, then Fershtman and Nitzan (1991) show that contributions become strategic substitutes. In the intermediate case in which agents receive a mix of flow payoffs and a payoff upon completion, then Battaglini, Nunnari and Palfrey (2012) show this game typically has a continuum of equilibria, some of which exhibit strategic complementarity and some strategic substitutability.

is close to completion.

I make the following contributions to this literature. First, I propose a tractable framework to analyze the dynamic problem faced by a group of agents who collaborate over time to complete a project. This framework can be useful for addressing an array of other questions related to dynamic moral hazard problems that involve completing a project. Second, I generate several insights for the organization of partnerships where agents must determine how large a partnership to form, and for team recruiting where the manager must determine how large a team to employ and how to compensate its members.[10] Third, contrary to previous literature, while mutual monitoring, peer pressure, synergies, and *warm-glow* are helpful for explaining the benefits of teamwork, I show that they are actually not necessary when the team's objective is to complete a project.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the equilibria of the game, and establishes some basic results. Section 4 examines how the size of the team influences the agents' incentives, and the implications of this result for the organization of partnerships. Section 5 studies the problem faced by a manager who recruits agents into a team to undertake a project, and she must determine the team composition and how to compensate the team members. Finally, Section 6 concludes. Appendices A.1 and A.2 contain a discussion of non-Markovian strategies and a robustness test of the main result, respectively. All proofs are provided in Appendix B.

[10]The latter problem is also studied by Rahmani, Roels and Karmarkar (2013). However, their analysis focuses on the contractual relationship between the members of a two-agent team (one of whom is the residual claimant of the project).

1.2 The Model

A team of n agents collaborate to complete a project. Time $t \in [0, \infty)$ is continuous. The project starts at some initial state $q_0 < 0$, its state q_t evolves according to a stochastic process, and it is completed at the first time τ such that q_t hits the completion state which is normalized to 0. Note that $|q_0|$ is interpreted as the length of the project. Agent $i \in \{1, .., n\}$ is risk neutral, discounts time at rate $r > 0$, and receives a pre-specified reward $V_i > 0$ upon completing the project. An incomplete project has zero value, and each agent's outside option equals 0.[11] At time t each agent observes the state of the project q_t, and exerts costly effort to influence the drift of the stochastic process

$$dq_t = \left(\sum_{i=1}^{n} a_{i,t} \right) dt + \sigma dW_t \,,$$

where $a_{i,t}$ denotes the effort level of agent i at time t, $\sigma > 0$ captures the degree of uncertainty associated with the evolution of the project, and W_t is a standard Brownian motion.[12,13] Efforts are unobservable and each agent's flow cost of exerting effort a is given by $c(a) = \frac{\lambda}{p+1} a^{p+1}$, where $\lambda > 0$ and $p \geq 1$.[14]

At every moment t, agent i observes the state of the project q_t, and chooses his effort strategy

[11]See Remark 3 for a discussion of the case in which each agent has a strictly positive outside option.

[12]For simplicity, I assume that the variance of the stochastic process (*i.e.*, σ) does not depend either on q_t or on the agents' effort levels. First, if σ is a continuously differentiable function of q_t with range $[\underline{\sigma}, \bar{\sigma}]$, where $0 < \underline{\sigma} < \bar{\sigma} < \infty$ and $\sigma'(\cdot) < \infty$, then it is straightforward to show that all of the results hold. Second, note that q_t will hit the completion state at some finite time with probability 1 even if no agent ever exerts any effort. This can be avoided if effort influences both the drift and the diffusion of the stochastic process such that $dq_t = 0$ if $a_{i,t} = 0$ for all i. While the analysis of this case is intractable, numerical examples with $dq_t = \left(\sum_{i=1}^{n} a_{i,t}\right) dt + \sigma \left(\sum_{i=1}^{n} a_{i,t}\right)^{1/2} dW_t$ suggest that the main result (*i.e.*, Theorem 2) and its implications continue to hold.

[13]I assume that efforts are perfect substitutes. To capture the notion that agents are more productive when working in teams (due to complementary skills), one can consider the case in which the project evolves according to $dq_t = \left(\sum_{i=1}^{n} a_{i,t}^{1/\gamma}\right)^{\gamma} dt + \sigma dW_t$, where $\gamma \geq 1$. The main result (*i.e.*, Theorem 2), and its implications continue to hold.

[14]The case in which $p \in (0, 1]$ is discussed in Remark 1, while the case in which effort costs are linear is discussed in Appendix A.2.

$A_{i,t} = \{a_{i,s}\}_{s\geq t}$ to maximize his expected discounted payoff while taking into account the effort strategies $A_{-i,t} = \{a_{-i,s}\}_{s\geq t}$ of the other team members. As such, for a given set of strategies, his expected discounted payoff function satisfies

$$J_i(q_t) = \mathbb{E}_\tau \left[e^{-r(\tau-t)} V_i - \int_t^\tau e^{-r(s-t)} c(a_{i,s})\, ds \right], \tag{1.1}$$

where the expectation is taken with respect to τ: the random variable that denotes the completion time of the project.

Assuming that $J_i(\cdot)$ is twice differentiable for all i, and using standard arguments (Dixit (1999)), one can derive the Hamilton-Jacobi-Bellman (hereafter HJB) equation for the expected discounted payoff function of agent i:

$$rJ_i(q_t) = -c(a_{i,t}) + \left(\sum_{j=1}^n a_{j,t} \right) J_i'(q_t) + \frac{\sigma^2}{2} J_i''(q_t) \tag{1.2}$$

defined on $(-\infty, 0]$ subject to the value-matching conditions

$$\lim_{q\to-\infty} J_i(q) = 0 \quad \text{and} \quad J_i(0) = V_i\,. \tag{1.3}$$

(1.2) asserts that agent i's flow payoff is equal to his flow cost of effort, plus his marginal benefit from bringing the project closer to completion times the aggregate effort of the team, plus a term that captures the sensitivity of his payoff to the volatility of the project. To interpret (1.3), observe that as $q \to -\infty$, the expected time until the project is completed so that agent i collects his reward diverges to ∞, and because $r > 0$, his expected discounted payoff asymptotes to 0. On the other hand, because he receives his reward and exerts no further effort after the project is completed, $J_i(0) = V_i$.[15]

[15]By noting that $J_i(q) \in [0, V_i]$ for all q and i, it follows that the transversality condition

Finally, observe that V_i and λ are isomorphic. Therefore, without loss of generality, for the remainder of this paper I normalize $\lambda = 1$.[16]

1.3 Results

1.3.1 Markov Perfect Equilibrium

I assume that agents play Markovian strategies, so that at every moment, each agent chooses his effort level as a function of the current state of the project. Therefore, given q, agent i chooses his effort level $a_i(q)$ such that

$$a_i(q) \in \arg\max_{a_i} \left\{ a_i J_i'(q) - c(a_i) \right\} .$$

The first-order condition for agent i's problem is $J_i'(q) = c'(a_i)$: each agent chooses his effort level such that the marginal cost of effort is equal to the marginal benefit associated with bringing the project closer to completion. Because $c'(0) = 0$ and $c(\cdot)$ is strictly increasing, given any q there exists a unique non-negative effort level $a_i(q)$ that satisfies the first-order condition as long as $J_i'(q) \geq 0$. Suppose for now that $J_i'(q) \geq 0$ for all q, and let $f(\cdot) = c'^{-1}(\cdot)$.[17] Then $a_i(q) = f(J_i'(q))$, and by substituting this into (1.2), the expected discounted payoff for agent i satisfies

$$rJ_i(q) = -c(f(J_i'(q))) + \left[\sum_{j=1}^{n} f\left(J_j'(q)\right) \right] J_i'(q) + \frac{\sigma^2}{2} J_i''(q) \tag{1.4}$$

$\lim_{t\to\infty} \mathbb{E}\left[e^{-rt} J_i(q_t)\right] = 0$ of the verification theorem (p. 123 in Chang (2004)) is satisfied, thus ensuring that a solution to the system of HJB equations (1.2) subject to (1.3) is indeed optimal for (1.1).

[16] To verify this, let $\tilde{J}_i(q) = \frac{J_i(q)}{\lambda}$, substitute this into (1.2), and observe that λ cancels out. Using (1.3), observe that $\tilde{J}_i(\cdot)$ satisfies $\lim_{q\to-\infty} \tilde{J}_i(q) = 0$ and $\tilde{J}_i(0) = \frac{V_i}{\lambda}$.

[17] Theorem 1 establishes that in fact $J_i'(q) > 0$ for all q, which implies that the first-order always binds. The interpretation is that each agent is strictly better off the closer the project is to completion.

subject to (1.3).

A MPE is characterized by the system of nonlinear ordinary differential equations defined by (1.4) subject to (1.3) for all $i \in \{1, .., n\}$. As a result, to show that a MPE exists, it suffices to show that a solution to the system of differential equations exists, and $J_i'(q) \geq 0$ for all i and q. The MPE will be unique if the system of differential equations has exactly one solution, because every MPE must satisfy (1.4) subject to (1.3), and the first-order condition is both necessary and sufficient.

Theorem 1. *A Markov Perfect equilibrium (MPE) for the game defined by (1.1) exists. For each agent i, the expected discounted payoff function $J_i(q)$ is infinitely differentiable on $(-\infty, 0]$, and it satisfies:*

(i) $0 < J_i(q) \leq V_i$ *for all q.*

(ii) $J_i'(q) > 0$ *for all q, and hence the equilibrium effort $a_i(q) > 0$ for all q.*

(iii) $J_i''(q) > 0$ *for all q, and hence $a_i'(q) > 0$ for all q.*

(iv) *If agents are symmetric (i.e., $V_i = V_j$ for all $i \neq j$), then the MPE is symmetric.*

(v) *Finally, the equilibrium is unique with n symmetric agents or 2 asymmetric agents.*[18]

$J_i'(q) > 0$ implies that each agent is strictly better off, the closer the project is to completion. Because $c'(0) = 0$ (*i.e.*, the marginal cost of *a little* effort is negligible), each agent exerts a strictly positive amount of effort at every state of the project: $a_i(q) > 0$ for all q.[19]

The facts that agents are impatient, they incur the cost of effort at the time effort is exerted, and they are compensated upon completing the project implies that they have stronger incentives the closer the project is to completion: $a_i'(q) > 0$ for all q. The implication of

[18]To simplify notation, if the agents are symmetric, then the subscript i is interchanged with the subscript n to denote the team size throughout the remainder of this paper.

[19]If $c'(0) > 0$, then there exists a quitting threshold Q_q, such that agent i exerts 0 effort on $(-\infty, Q_q]$, while he exerts strictly positive effort on $(Q_q, 0]$, and his effort increases in q.

this result is that efforts are strategic complements. This is because a higher effort level by one agent brings the project closer to completion, which incentivizes the other agents to also raise their effort. This result was first shown by Yildirim (2006) and Kessing (2007), and it is in contrast to static models (Holmström (1982)), dynamic models in which the agents receive flow payoffs while the project is in progress (Fershtman and Nitzan (1991)), as well as dynamic models in which the project can be completed instantaneously (Bonatti and Hörner (2011)) where efforts are strategic substitutes.

A natural question that arises is whether agents can increase their expected discounted payoff by adopting non-Markovian strategies, so that their effort at t depends on the entire evolution path of the project $\{q_s\}_{s\leq t}$. While a formal analysis of this case is beyond the scope of this paper, following Sannikov and Skrzypacz (2007), I conjecture that there does not exist a symmetric Public Perfect equilibrium in which agents can achieve a higher expected discounted payoff than the MPE at any state of the project. See Appendix A.1 for details.

Remark 1. Recall that I have assumed that $p \geq 1$; *i.e.*, effort costs are at least quadratic. If $p \in (0,1)$, then a MPE exists as long as $\frac{\sigma^2}{4}\int_0^\infty \frac{s\,ds}{r\sum_{i=1}^n V_i + ns^{\frac{p+1}{p}}} > \sum_{i=1}^n V_i$.[20] In this case, statements (i)-(iv) of Theorem 1 continue to hold, and the MPE is unique if the agents are symmetric. By noting that none of the subsequent proofs use that $p \geq 1$, it follows that all comparative statics hold for any $p > 0$.

Remark 2. The model assumes that the project is never canceled. If there is an exogenous *cancellation* state $Q_C < q_0$ such that the project is canceled (and the agents receive payoff 0) at the first time that q_t hits Q_C, then effort needs no longer be increasing in q. Instead,

[20]This condition is satisfied if $\sum_{i=1}^n V_i$, r and n are sufficiently small, or if σ and p are sufficiently large. If $p \geq 1$, then it is satisfied for any choice of the other parameters.

it increases in q only if Q_c is sufficiently small (*i.e.*, close to $-\infty$), it is U-shaped if Q_C is in some medium range, and it decreases in q if Q_C is sufficiently large. Intuitively, this is because agents have incentives to work harder when the state of the project is near the cancellation state to avoid hitting it, and these incentives are stronger, the larger Q_C is.

Remark 3. Recall that agents are assumed to have outside option 0. Consider a team of symmetric agents, each of whom has a positive outside option $0 \leq u < J_n(q_0)$. In this case, there exists an *abandonment* state $Q_A < q_0$ satisfying the smooth-pasting condition $\frac{\partial}{\partial q} J_n(q, Q_A)\Big|_{q=Q_A} = 0$ such that they abandon the project at the first moment q hits Q_A. In this case, effort is increasing in progress.[21] Note that if $u = 0$, then $Q_A = -\infty$.

1.3.2 Comparative Statics

This section establishes some comparative statics, which are helpful for understanding how the agents' incentives depend on the parameters of the problem. To examine the effect of each parameter to the agents' incentives, I consider two symmetric teams that differ in exactly one attribute: their members' rewards V_i, patience levels r_i, or the volatility of the project σ.[22]

Proposition 1. *Consider two teams comprising of symmetric agents.*

(i) *If $V_1 < V_2$, then other things equal, $a_1(q) < a_2(q)$ for all q.*

(ii) *If $r_1 > r_2$, then other things equal, there exists an interior threshold Θ_r such that $a_1(q) \leq a_2(q)$ if and only if $q \leq \Theta_r$.*

(iii) *If $\sigma_1 > \sigma_2$, then other things equal, there exist interior thresholds $\Theta_{\sigma,1} \leq \Theta_{\sigma,2}$ such that $a_1(q) \geq a_2(q)$ if $q \leq \Theta_{\sigma,1}$ and $a_1(q) \leq a_2(q)$ if $q \geq \Theta_{\sigma,2}$.*

[21] Here, $J_n(\cdot, Q_A)$ denotes each agent's expected discounted payoff conditional on the abandonment state Q_A.

[22] Since the teams differ in a single parameter (*i.e.*, their reward V_i in statement (i)), abusing notation, I let $a_i(\cdot)$ denote each agent's effort strategy corresponding to the parameter with subscript i.

The intuition behind statement (i) is straightforward. If the agents receive a bigger reward, then they always work harder in equilibrium.

Statement (ii) asserts that a team of less patient agents works harder relative to a team of more patient agents if and only if the project is sufficiently close to completion. Intuitively, less patient agents have more to gain from an earlier completion (provided that the project is sufficiently close to completion). However, bringing the completion time forward requires them to exert more effort, whose costs are incurred at the time that effort is exerted, whereas the reward is only collected upon completion of the project. Therefore, the benefit from bringing the completion time forward (by exerting more effort) outweighs its cost only when the project is sufficiently close to completion.

Finally, statement (iii) asserts that incentives become stronger in the volatility of the project σ when it is far from completion, while the opposite is true when it gets close to completion.[23] To see the intuition behind this result, note that as the volatility increases, it becomes more likely that the project will be completed either earlier than expected (*upside*), or later than expected (*downside*). If the project is sufficiently far from completion, then $J_i(q) \simeq 0$ so that the downside is negligible, while $J_i''(q) > 0$ implies that the upside is not (negligible), and consequently $a_1(q) \geq a_2(q)$. On the other hand, because the completion time of the project is non-negative, the upside diminishes as the project approaches completion. Therefore, when the project is sufficiently close to completion (*i.e.*, $q \geq \Theta_{\sigma,2}$), the downside is bigger than the upside so that $a_1(q) \leq a_2(q)$.

[23] A limitation of this result is that it does not guarantee that $\Theta_{\sigma,1} = \Theta_{\sigma,2}$, which implies that it does not provide any prediction about how the agents' effort depends on σ when $q \in [\Theta_{\sigma,1}, \Theta_{\sigma,2}]$. However, numerical analysis indicates that in fact $\Theta_{\sigma,1} = \Theta_{\sigma,2}$.

1.3.3 First-Best Outcome

To obtain a benchmark for the agents' equilibrium effort levels, I compare them to the first-best outcome, where at every moment, each agent chooses his effort level to maximize the team's, as opposed to his individual expected discounted payoff. I focus on the symmetric case, and denote by $\hat{J}_n(q)$ and $\hat{a}_n(q)$ the first-best expected discounted payoff and effort level of each member of an n-person team, respectively. The first-order condition for each agent's effort level satisfies $\hat{a}_n(q) \in \arg\max_a \left\{ an\hat{J}_n'(q) - c(a) \right\}$, and substituting the first order condition into (1.2) yields

$$r\hat{J}_n(q) = -c\left(f\left(n\hat{J}_n'(q)\right)\right) + nf\left(n\hat{J}_n'(q)\right)\hat{J}_n'(q) + \frac{\sigma^2}{2}\hat{J}_n''(q)$$

subject to the boundary conditions (1.3). It is straight-forward to show that the properties established in Theorem 1 apply for $\hat{J}_n(q)$ and $\hat{a}_n(q)$. In particular, the system of first-best ODE subject to the boundary conditions (1.3) has a unique solution, $\hat{J}_n'(q) > 0$ for all q so that the first order condition always binds, and $\hat{J}_n''(q) > 0$ for all q, which implies that $\hat{a}_n'(q) > 0$; *i.e.*, similar to the MPE, the first-best effort level increases with progress.

Proposition 2. *In a team of $n \geq 2$ agents, $\hat{a}_n(q) > a_n(q)$ and $\hat{J}_n(q) > J_n(q)$ for all q.*

This result is not surprising: due to the free-rider problem, in the MPE, each agent exerts strictly less effort and he is strictly worse off at every state of the project as compared to the case in which agents behave collectively by choosing their effort level at every moment to maximize the team's expected discounted payoff.

1.4 Team Dynamics

1.4.1 The Effect of Team Size to the Agents' Incentives

When examining the relationship between the agents' incentives and the size of the team, it is important to consider how each agent's reward depends on the team size. I consider the two extreme cases: the *public good allocation* scheme, wherein each agent receives a reward V upon completing the project, which does not depend on the size of the team, and the *budget allocation* scheme, wherein each agent receives a reward $\frac{V}{n}$ upon completing the project.

With n symmetric agents, each agent's expected discounted payoff function satisfies

$$rJ_n(q) = -c\left(f\left(J_n'(q)\right)\right) + nf\left(J_n'(q)\right)J_n'(q) + \frac{\sigma^2}{2}J_n''(q)$$

subject to $\lim_{q\to-\infty} J_n(q) = 0$ and $J_n(0) = V_n$, where $V_n = V$ or $V_n = \frac{V}{n}$ under the public good or the budget allocation scheme, respectively.

Theorem 2. *Consider two teams comprising of n and $m > n$ identical agents. Under both allocation schemes, there exist thresholds $\Theta_{n,m}$ and $\Phi_{n,m}$ such that*
*(**A**) $a_m(q) \geq a_n(q)$ if and only if $q \leq \Theta_{n,m}$; and*
*(**B**) $ma_m(q) \geq na_n(q)$ if and only if $q \leq \Phi_{n,m}$.*

Statement (A) asserts that under both allocation schemes, members of a larger team work harder than members of a smaller team if and only if the project is sufficiently far from completion.[24] Figure 1 illustrates an example. To understand the intuition behind this result, note that by increasing the size of the team, two forces influence the agents' incentives: First,

[24] A similar result is established by Yildirim (2006). However, the comparative static applies only to each agent's individual effort in his model rather than to the team's aggregate effort as well.

agents obtain stronger incentives to free-ride. To see why, consider an agent's dilemma at time t to (unilaterally) reduce his effort by a *small* amount ε for a *short* interval Δ. By doing so, he saves approximately $\varepsilon c'(a(q_t))\Delta$ in cost of effort, but at $t+\Delta$, the project is (on expectation) $\varepsilon\Delta$ father from completion (compared to the scenario in which he does not reduce his effort). In equilibrium, this agent will carry out only $\frac{1}{n}$ of that *lost* progress, which implies that the benefit from shirking increases in the size of the team. On the other hand, because the total progress that needs to be carried out is fixed, increasing the team size (and holding strategies fixed) implies that the project will (on expectation) be completed sooner. This increases the present discounted value of each agent's reward (*i.e.*, $\mathbb{E}_\tau[e^{-r\tau}]$), which it turn strengthens his incentives. I shall refer to these forces as the *free-riding* and the *encouragement effect*, respectively, and the intuition will follow from examining how the magnitude of these effects changes as the project progresses.

First, let us consider the free-riding effect, and recall that agents work harder, the closer the project is to completion. By noting that the marginal cost of effort is increasing, it follows that an agent's gain from free-riding, which is proportional to $c'(a(q_t))$, increases in q_t. Therefore, the free-riding effect becomes stronger as the project progresses. In addition, recall that effort vanishes as $q \to -\infty$ and $c'(0)=0$, which implies that the free-riding effect is negligible when the project is sufficiently far from completion.

To understand how the magnitude of the encouragement effect changes as the project progresses, it is simpler to consider the deterministic case in which $\sigma = 0$. Let τ denote the completion time when the team comprises of n agents, and note that each agent's marginal benefit of bringing the completion time forward is $-\frac{d}{d\tau}V_n e^{-r\tau} = rV_n e^{-r\tau}$. Doubling the team size and holding strategies fixed halves the completion time, and each agent's respective marginal benefit now becomes $rV_{2n}e^{-\frac{1}{2}r\tau}$. Therefore, the magnitude of the encouragement effect can be measured by the ratio of the marginal benefits: $\frac{V_{2n}}{V_n} e^{\frac{r\tau}{2}}$. Under the public good allocation (in which case $\frac{V_{2n}}{V_n} = 1$), this ratio increases in τ and it is always greater than 1, which implies that the benefit of a bigger team being able to complete the project sooner decreases as the project progresses, and it becomes negligible as the project nears completion. On the other hand, under the budget allocation (in which case $\frac{V_m}{V_n} = \frac{1}{2}$), while the ratio in consideration still increases in τ, it is greater than 1 only if τ is sufficiently large. Therefore, the encouragement effect is *positive* when the project is sufficiently far from completion, it becomes weaker as the project progresses, and it is *negative* when the project is close to completion.

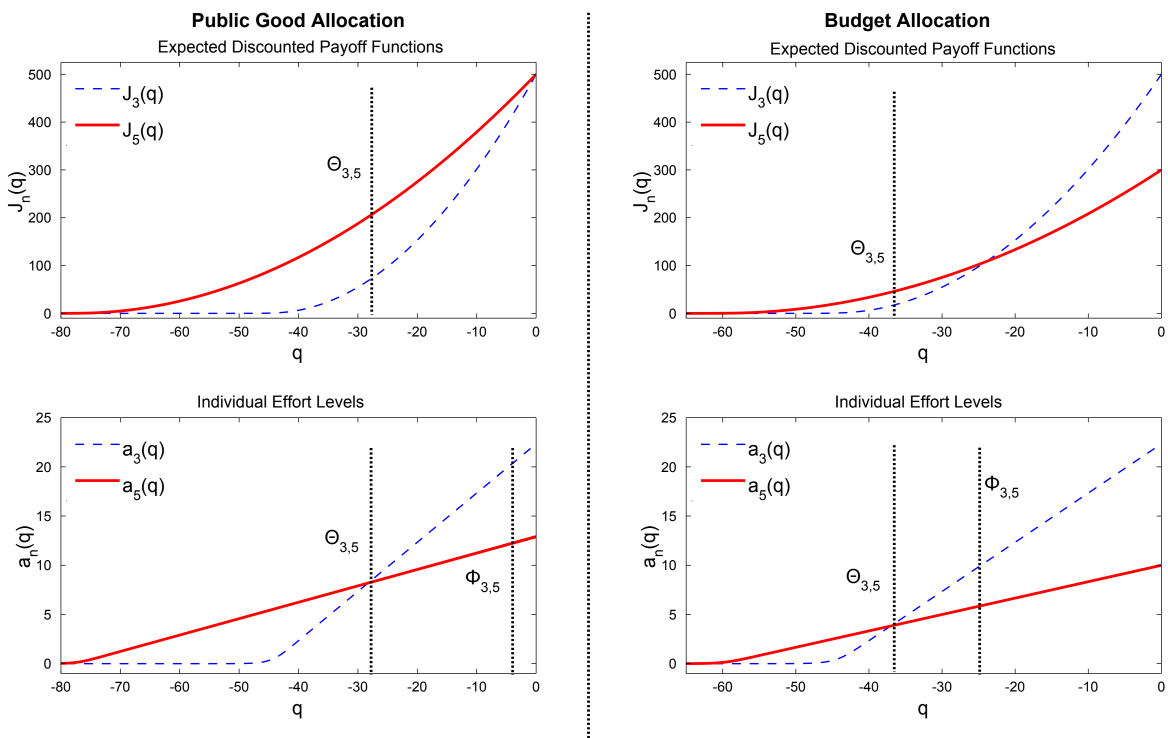

Figure 1.1: **Illustration of Theorem 2.** The upper panels illustrate each agent's expected discounted payoff under public good (left) and budget (right) allocation for two different team sizes: $n = 3$ and 5. The lower panels illustrate each agent's equilibrium effort. In both cases, there exists an interior threshold $\Theta_{3,5}$ such that each member of the larger team exerts more effort relative to each member of the smaller team if and only if $q \leq \Theta_{3,5}$. Similarly, there exists a threshold $\Phi_{3,5}$ such that the total aggregate of the larger team is greater than that of the smaller team if and only if $q \leq \Phi_{3,5}$.

Therefore, under both allocation schemes, the encouragement effect dominates the free-riding effect, and consequently, members of a larger team work harder than those of a smaller team if and only if the project is sufficiently far from completion.

Statement (B) shows that under both allocation schemes, the aggregate effort exerted by the larger team is greater than that of the smaller team if and only if the project is sufficiently far from completion. When the project is far from completion such that $q \leq \Theta_{n,m}$, it is straightforward that the aggregate effort of the larger team exceeds that of the smaller team by statement (A). The perhaps surprising aspect of this result is that the free-riding effect can become so aggravated when the project is near completion, that not only each member of the larger team exerts less effort relative to each member of the smaller team, but also the aggregate effort of the larger team becomes less than that of the smaller team.

By using the same proof technique, one can show that under both allocation schemes, the first-best aggregate effort increases in the team size at every state of the project. This strengthens the intuition that statement (B) is a consequence of the free-riding effect becoming overwhelmingly stronger than the encouragement effect when the project is close to completion.

Theorem 2 reaches an opposite conclusion relative to earlier results in the moral hazard in teams and the public good contribution literatures that establish an inverse relationship between individual effort (or contribution) and team size (Holmström (1982), Andreoni (1988), Bonatti and Hörner (2011), and others).[25] The key difference is that efforts are strategic complements in the model studied in this paper, so that as the team size increases,

[25] While Bonatti and Hörner (2011) focus on the uncertainty pertaining to the feasibility of the project, their result that the aggregate effort of the team decreases in its size (equation 7) continues to hold as $\bar{p} \to 1$, in which case the project is known to be feasible.

in addition to the free-rider problem becoming aggravated (which is consistent with previous findings), the agents also benefit from the ability to complete the project sooner.

Note that the thresholds of Theorem 2 need not always be interior. In particular, it is possible that $\Theta_{n,m} = -\infty$ under budget allocation, which would imply that each member of the smaller team always works harder than each member of the larger team. However, numerical analysis indicates that $\Theta_{n,m}$ is always interior under both allocation schemes. On the other hand, $\Phi_{n,m}$ is guaranteed to be interior only under budget allocation if effort costs are quadratic, while one can find examples in which $\Phi_{n,m}$ is interior as well as examples in which $\Phi_{n,m} = 0$ otherwise. Numerical analysis indicates that the most important parameter that determines whether $\Phi_{n,m}$ is interior is the convexity of the effort costs, and it is interior as long as effort costs are not too convex (*i.e.*, p is sufficiently small). This is intuitive, because more convex effort costs favor the larger team more. In addition, under public good allocation, for $\Phi_{n,m}$ to be interior, it is also necessary that n and m are sufficiently small. Intuitively, this because the *size of the pie* increases in the team size under this scheme, which (again) favors the larger team.[26]

1.4.2 Partnership Formation

Now let us examine the problem faced by a group of agents who organize into a partnership. Suppose that teams are formed sequentially, and the agents who have already committed to join, decide whether to admit another member.[27] Admission to the team is costless, and no agent will begin to work until the team composition has been finalized.

Proposition 3. *Suppose that n identical agents have committed to join a team.*

[26] The case in which effort costs are linear is examined in Appendix A.2, and an analogous result to Theorem 2 is shown: members of an $(n+1)$-member team have stronger incentives relative to those of an n-member team as long as n is sufficiently small.

[27] Note that because the equilibrium is symmetric, the team members will be in agreement with respect to whether they should admit a new member.

(i) *Under public good allocation, the team is always better off admitting another member.*

(ii) *Under budget allocation, there exists a threshold T_n such that the team is better off admitting another member if and only if the project length $|q_0| \geq T_n$.*

By adding another member to the team, each agent will need to exert less effort to complete the project, which implies that his total expected discounted effort cost will decrease. If his reward upon completing the project does not depend on the team size, as is the case under public good allocation, then expanding the partnership *ad-infinitum* is optimal.

On the other hand, if each agent who has committed to join must surrender part of his reward in order to expand the team (*i.e.*, under budget allocation), then he will do so only if the gain from being able to complete the project sooner in a bigger group is sufficiently large to offset the decrease in his net payoff upon completing the project. This is true only if the project is sufficiently long. This result is illustrated in the top panels of Figure 1.

1.5 Manager's Problem

1.5.1 The Model with a Manager

The manager is risk neutral, she discounts time at the same rate $r > 0$ as the agents, and her outside option is normalized to 0. She is the residual claimant of a project, and she hires a group of agents to undertake it on her behalf, which has length $|q_0|$, and generates a payoff $U > 0$ upon completion. To incentivize the agents, the manager designates a set of milestones $q_0 < Q_1 < .. < Q_K = 0$ (where $K \in \mathbb{N}$), and for every $k \in \{1, .., K\}$ she allocates non-negative payments $\{V_{i,k}\}_{i=1}^{n}$ that are due upon reaching milestone Q_k *for the first time*. She then makes a take-it-or-leave-it offer to a group of agents, and once the team composition has been finalized, the agents begin to work.[28,29] For each k, the manager disburses the payments $\{V_{i,k}\}_{i=1}^n$ as soon as the project hits Q_k for the first time. As soon as the project is completed, the manager collects her payoff U, she disburses the final payments $\{V_{i,K}\}_{i=1}^n$ to the agents, and the game ends.

The manager's problem entails choosing the team size and the agents' contracts to maximize her expected discounted profit at time 0 (*i.e.*, at q_0) subject to the agents' incentive compatibility constraints.

1.5.2 A Preliminary Result

I begin by considering the case in which the manager compensates the agents only upon completing the project, and I show in Theorem 3 that her problem is well-defined and it satisfies some desirable properties. Then I explain how this result extends to the case in which the manager also rewards the agents for reaching intermediate milestones.

Given the team size n and the agents' compensations $\{V_i\}_{i=1}^n$ that are due upon completion of the project, the manager's expected discounted profit function can be written as

$$F(q) = \left(U - \sum_{i=1}^{n} V_i \right) \mathbb{E}_\tau \left[e^{-r\tau} \,|\, q \right] ,$$

where the expectation is taken with respect to the project's completion time τ, which depends on the agents' strategies and the stochastic evolution of the project.[30] By using the first order condition for each agent's equilibrium effort level as determined in Section 3, the manager's

[28] The details of each offer is public information, so that without loss of generality, I can assume that offers will be made such that every one is accepted.

[29] It is important to acknowledge that the manager's contracting space is limited. While a contract a-la-Sannikov (2008) or one in which the agents' payoffs may also depend on time is more desirable, such analysis is not tractable using the present model, and is therefore left for future research.

[30] Note that the subscript k is dropped when $K = 1$ (in which case $Q_1 = 0$).

expected discounted profit at any given state of the project satisfies

$$rF(q) = \left[\sum_{i=1}^{n} f\left(J_i'(q)\right)\right] F'(q) + \frac{\sigma^2}{2} F''(q) \tag{1.5}$$

defined on $(-\infty, 0]$ subject to the boundary conditions

$$\lim_{q \to -\infty} F(q) = 0 \quad \text{and} \quad F(0) = U - \sum_{i=1}^{n} V_i \,. \tag{1.6}$$

The interpretation of these conditions is similar to (1.3). As the state of the project diverges to $-\infty$, its expected completion time diverges to ∞, and because $r > 0$, the manager's expected discounted profit diminishes to 0. On the other hand, the manager's profit is realized when the project is completed, and it equals her payoff U less the payments $\sum_{i=1}^{n} V_i$ disbursed to the agents.[31]

Theorem 3. *Given $(n, \{V_i\}_{i=1}^{n})$, a solution to the manager's problem defined by (1.5) subject to the boundary conditions (1.6) and the agents' problem as defined in Theorem 1 exists, and it has the following properties:*

(i) *$F(q) > 0$ and $F'(q) > 0$ for all q.*

(ii) *$F(\cdot)$ is infinitely differentiable on $(-\infty, 0]$.*

(iii) *Finally, $F(\cdot)$ is unique if the team comprises of n symmetric or 2 asymmetric agents.*

Now let us discuss how Theorems 1 and 3 extend to the case in which the manager rewards the agents upon reaching intermediate milestones. Recall that she can designate a set of milestones, and attach rewards to each milestone that are due as soon as the project reaches the respective milestone for the first time. Let $J_{i,k}(\cdot)$ denote agent i's expected discounted payoff given that the project has reached $k-1$ milestones, which is defined on $(-\infty, Q_k]$, and note that it satisfies (1.4) subject to $\lim_{q \to -\infty} J_{i,k}(q) = 0$ and $J_{i,k}(Q_k) = V_{i,k} + J_{i,k+1}(Q_k)$, where $J_{i,K+1}(0) = 0$. The second boundary condition states that upon reaching milestone

[31] Because the manager's outside option is equal to 0, without loss of generality, I can restrict attention to the case in which the payments $\{V_i\}_{i=1}^{n}$ are chosen such that $\sum_{i=1}^{n} V_i \leq U$.

k, agent i receives the reward attached to that milestone, plus the continuation value from future rewards. Starting with $J_{i,K}(\cdot)$, it is straightforward that it satisfies the properties of Theorem 1, and in particular, that $J_{i,K}(Q_{k-1})$ is unique (as long as $n = 2$ or compensation is symmetric) so that the boundary condition of $J_{i,K-1}(\cdot)$ at Q_{K-1} is well-defined. Proceeding backwards, it follows that for every k, $J_{i,k}(\cdot)$ satisfies the properties established in Theorem 1.

To examine the manager's problem, let $F_k(\cdot)$ denote her expected discounted profit given that the project has reached $k-1$ milestones, which is defined on $(-\infty, Q_k]$, and note that it satisfies (1.5) subject to $\lim_{q\to-\infty} F_k(q) = 0$ and $F_k(Q_k) = F_{k+1}(Q_k) - \sum_{i=1}^{n} V_{i,k}$, where $F_{K+1}(Q_k) = U$. The second boundary condition states that upon reaching milestone k, the manager receives the continuation value of the project, less the payments that she disburses to the agents for reaching this milestone. Again starting with $k = K$ and proceeding backwards, it is straightforward that $F_k(\cdot)$ satisfies the properties established in Theorem 3 for all k.

1.5.3 Contracting Problem

Theorem 4. *The optimal symmetric scheme compensates the agents only upon completion of the project.*

To prove this result, I consider an arbitrary set of milestones and arbitrary rewards attached to each milestone, and I construct an alternative scheme that rewards the agents only upon completing the project and renders the manager better off. Intuitively, because rewards are sunk (in terms of incentivizing the agents) after they are disbursed, by backloading payments, the manager can provide the same incentives at the early stages of the project, while providing stronger incentives when the project is close to completion.

The value of Theorem 4 lies in that it reduces the infinite-dimensional problem of determining the team size, the number of milestones, the set of milestones, and the rewards attached to each milestone into a two-dimensional problem, in which the manager only needs to determine the team size and her budget $B = \sum_{i=1}^{n} V_i$ for compensating the agents.

The restriction that the manager compensates the agents symmetrically is not without loss of generality. As shown in Remark 4, an asymmetric scheme that rewards the agents upon reaching (different) intermediate milestones may be desirable, because it enables the manager to dynamically decrease the team size as the project progresses, which in turn mitigates free-riding. However, because individuals value fairness in pay (Lazear (1989) and Baron and Kreps (1999)), it is of interest to examine the symmetric case. The following Proposition examines how the manager should determine her budget.

Proposition 4. *Suppose that the manager employs n identical agents and she compensates them symmetrically. Then her optimal budget B increases in the projects length $|q_0|$.*

Contemplating an increase in her budget, the manager trades off a decrease in her net profit $U-B$ and an increase in the project's present discounted value $\mathbb{E}_\tau\left[e^{-r\tau} \mid q_0\right]$. Because a longer project takes (on average) a larger amount of time to be completed, a decrease in her net profit has a smaller effect on her expected discounted profit at time 0 the longer the project is. Therefore, the benefit from raising the agents' compensations outweighs the decrease in her net profit if and only if the project is sufficiently long, and the desired comparative static follows by applying the Monotonicity Theorem of Milgrom and Shannon (1994).

Proposition 5. *Suppose the manager has a fixed budget B to (symmetrically) compensate a group of identical agents. For any $m > n$, there exists a threshold $T_{n,m}$ such that she is better off employing an m-member team instead of an n-member one if and only if the length of the project $|q_0| \geq T_{n,m}$.*

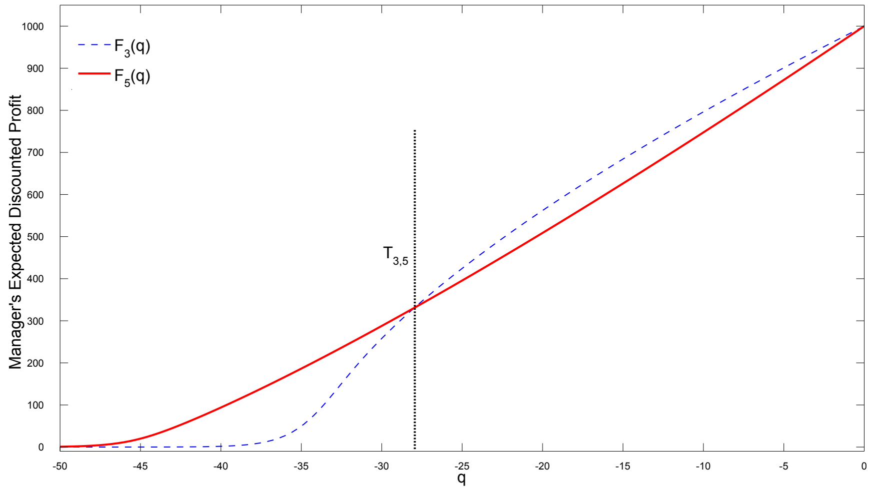

Figure 1.2: **Illustration of Proposition 5.** Given a fixed budget, the manager's expected discounted profit is higher if she recruits a 5-member team relative to a 3-member team if and only if the initial state of the project q_0 is to the left of the threshold $-T_{3,5}$; or equivalently, if and only if $|q_0| \geq T_{3,5}$.

Given a fixed budget, the manager's objective is to choose the team size to minimize the expected completion time of the project. This is equivalent to maximizing the aggregate effort of the team along the evolution path of the project. Hence, the intuition behind this result follows from statement (B) of Theorem 2. If the project is short, then on expectation, the aggregate effort of the smaller team will be greater than that of the larger team due to the free-riding effect (on average) dominating the encouragement effect. The opposite is true if the project is long. Figure 2 illustrates an example.

Applying the Monotonicity Theorem of Milgrom and Shannon (1994) leads one to the following Corollary.

Corollary 1. *Given a fixed budget to (symmetrically) compensate a group of identical agents, the manager's optimal team size n increases in the length of the project $|q_0|$.*

The take-away from Proposition 5 (and Corollary 1) is that a larger team is more desirable while the project is far from completion, whereas a smaller team becomes preferable when the project gets close to completion. Therefore, it seems desirable to construct a scheme that dynamically decreases the team size as the project progresses. Suppose that the manager employs two identical agents on a fixed budget B, and she designates a *retirement state* R, such that one of the agents is permanently retired (*i.e.*, he stops exerting effort) at the first time that the state of the project hits R. From that point onwards, the other agent continues to work alone. Both agents are compensated only upon completion of the project, and the payments (say V_1 and V_2) are chosen such that the agents are indifferent with respect to who will retire at R; *i.e.*, their expected discounted payoffs are equal at $q_t = R$.[32,33]

Proposition 6. *Suppose the manager employs two identical agents with quadratic effort costs. Consider the retirement scheme described above, where the retirement state R is chosen such that $|R| \leq \min\{|q_0|, T_{1,2}\}$ and $T_{1,2}$ is taken from Proposition 5. There exists a threshold $\Theta_R > |R|$ such that the manager is better off implementing this retirement scheme relative to allowing both agents to work together until the project is completed if and only if its length $|q_0| < \Theta_R$.*

First, note that after one agent retires, the other will exert first-best effort until the project is completed. Because the manager's budget is fixed, this retirement scheme is preferable only if it increases the expected aggregate effort of the team along the evolution path of the project. A key part of the proof involves showing that agents have weaker incentives before one of them is retired as compared to the case in which they always work together (*i.e.*, when a retirement scheme is not used). Therefore, the benefit from having one agent exert first-best effort *after* one of them retires outweighs the loss from the two agents exerting less effort *before* one of them retires (relative to the case in which they always work together)

[32]It is shown that (i) such a pair $\{V_1, V_2\}$ exists, and (ii) the agent who will retire at R (say agent 2) receives a smaller payment that the other agent; *i.e.*, $V_2 < V_1$.

[33]Note that this is one of many possible *retirement schemes*. A more elaborate analysis of dynamic team size management is beyond the scope of this paper, and it is left for future research.

only if the project is sufficiently short. Hence, this retirement scheme is preferable if and only if $|q_0| < \Theta_R$.

Remark 4. This result implies that an asymmetric scheme that rewards the agents upon reaching (different) intermediate milestones can do better than the best symmetric one if the project is sufficiently short. Observe that the retirement scheme proposed above can be implemented using the following (asymmetric) *rewards-for-milestones* scheme. Let $Q_1 = R$ (where $|R| \leq \min\{|q_0|, T_{1,2}\}$), and suppose that agent 1 receives V as soon as the project is completed, while he receives no intermediate rewards. On the other hand, agent 2 receives the expected discounted value of $B - V$ upon hitting R for the first time (*i.e.*, $(B-V)\,\mathbb{E}_\tau\left[e^{-r\tau}|R\right]$), and he receives no further compensation, so that he effectively retires at that point. From Proposition 6 we know that there exists a *budget split* V and a threshold Θ_R such that this scheme is preferable if $|q_0| < \Theta_R$.

We know that an asymmetric compensation scheme may be beneficial, because it enables the manager to dynamically decrease the team size as the project gets close to completion. In this case, the asymmetry arises from the fact that agents are compensated upon reaching different milestones. The following result shows that an asymmetric scheme may be preferable even if the manager compensates the (identical) agents upon reaching the same milestone; namely, upon completing the project.

Proposition 7. *Suppose the manager has a fixed budget $B > 0$, and she employs two identical agents with quadratic effort costs whom she compensates upon completion of the project. Then for all $\epsilon \in \left(0, \frac{B}{2}\right]$, there exists a threshold T_ϵ such that the manager is better off compensating the two agents asymmetrically such that $V_1 = \frac{B}{2} + \epsilon$ and $V_2 = \frac{B}{2} - \epsilon$ instead of symmetrically, if and only if the length of the project $|q_0| \leq T_\epsilon$.*

To see the intuition behind this result, note that $\epsilon = \frac{B}{2}$ is equivalent to the case in which the manager employs a single agent, and from Proposition 5 we know that there exists a threshold $T_{1,2}$ such that the manager is better off employing one agent instead of two if and only if $|q_0| \leq T_{1,2}$. The intermediate cases in which $\epsilon \in \left(0, \frac{B}{2}\right)$ can be thought of as if the manager employs a *full-time agent* and a *part-time* one. Part of the proof involves showing that the aggregate effort under an asymmetric scheme is larger compared to a symmetric one if and only if the project is sufficiently close to completion. Intuitively, this is because the full-time agent cannot free-ride on the other agent as much. By noting that the manager's objective is to allocate her budget so as to maximize the agents' expected aggregate effort along the evolution path of the project, it follows that this is best done by allocating it asymmetrically between the agents if the project is sufficiently short.

1.6 Concluding Remarks

This paper studies the dynamic collaboration of a team on a project that gradually progresses towards completion. The main result is that members of a larger team work harder than those of a smaller team if and only if the project is sufficiently far from completion. On the other hand, when the project is close to completion, free-riding becomes so severe that a larger team may on aggregate exert less effort than a smaller team. I then examine the implications of this result for the organization of partnerships where agents must determine how large a partnership to form, and for team recruiting where the manager must determine how large a team to employ and how to compensate its members.

This paper opens several opportunities for future research. First, Georgiadis, Lippman and Tang (2012) consider the case in which the project length is endogenous. Motivated by projects that involve *design* or *quality* objectives which are often difficult to define in advance, they examine how the manager's optimal project length depends on her ability to commit to a given project length in advance. Second, this paper provides several testable predictions that lend themselves to empirical or experimental investigation; in particular, Ederer, Georgiadis and Nunnari test the predictions of Theorem 2 using laboratory experiments. Third, one may consider the case in which the state of the project can only be observed imperfectly, in which case the agents would need to update their beliefs about how close the project is to completion over time, and base their strategies on those beliefs. Optimal contracting for incentivizing a group of agents to undertake a project is an issue that deserves further exploration; for example, by incorporating time into the agents' contracts and using an approach in the mold of Sannikov (2008). Finally, from an applied perspective, it might be interesting to examine how a project can be split into subprojects.

1.7 Additional Results

1.7.1 Equilibria with Non-Markovian Strategies

I have assumed that agents' strategies are Markovian so that at every moment, each agent's effort is a function of only the current state of the project q_t. This raises the question whether agents can increase their expected discounted payoff by adopting non-Markovian strategies, so that their effort depends on the entire evolution path of the project $\{q_s\}_{s\leq t}$. Sannikov and Skrzypacz (2007) study a related model in which agents can change their actions only at times $t = 0, \Delta, 2\Delta, ..$, where $\Delta > 0$ (but *small*), and the information structure is similar; *i.e.*, the state variable evolves according to a diffusion process whose drift is influenced by the agents' actions. They show that the payoffs from the best symmetric Public Perfect equilibrium (hereafter PPE) converge to the payoffs corresponding to the MPE as $\Delta \to 0$ (see their Proposition 5).

A natural discrete-time analog of the model considered in this paper is one in which at $t \in \{0, \Delta, 2\Delta, ..\}$ each agent chooses his effort level $a_{i,t}$ at cost $c(a_{i,t})\Delta$, and at $t + \Delta$ the state of the project is equal to $q_{t+\Delta} = q_t + \left(\sum_{i=1}^{n} a_{i,t}\right)\Delta + \epsilon_{t+\Delta}$, where $\epsilon_{t+\Delta} \sim N(0, \sigma^2\Delta)$. In light of the similarities between this model and the model in Section VI of Sannikov and Skrzypacz (2007), it is reasonable to conjecture that in the continuous-time game, there does not exist a PPE in which agents can achieve a higher expected discounted payoff than the MPE at any state of the project. However, because a rigorous proof is difficult for the continuous-time game and the focus of this paper is on team formation, a formal analysis of non-Markovian PPE of this game is left for future work.

Nevertheless, it is useful to present some intuition. Following Abreu, Pearce and Stacchetti (1986), an optimal PPE involves a collusive regime and a punishment regime, and in every period, the decision whether to remain in the collusive regime or to switch is guided by the outcome in that period alone. In the context of this model, at $t+\Delta$, each agent will base his decision on $\frac{q_{t+\Delta}-q_t}{\Delta}$. As Δ decreases, two forces influence the scope of cooperation. First, the gain from a deviation in a single period decreases, which helps cooperation. On the other hand, because $\mathbb{V}\left(\frac{q_{t+\Delta}-q_t}{\Delta}\right)=\frac{\sigma^2}{\Delta}$, the agents must decide whether to switch to the punishment regime by observing noisier information, which increases the probability of type I errors (*i.e.*, triggering a punishment when no deviation has occurred), thus hurting cooperation. As Sannikov and Skrzypacz (2007) show, the latter force becomes overwhelmingly stronger than the former as $\Delta \to 0$, thus eradicating any gains from cooperation.

1.7.2 Linear Effort costs

The assumption that effort costs are convex affords tractability as it allows for comparative statics despite the fact that the underlying system of HJB equations does not have a closed-form solution. However, convex effort costs also favor larger teams. Therefore, it is useful to examine how the comparative statics with respect to the team size extend to the case in which effort costs are linear; *i.e.*, $c(a)=a$. In this case, the marginal value of effort is equal to $J_i'(q)-1$, so that agents find it optimal to exert the largest possible effort level if $J_i'(q)\geq 1$, and the smallest possible effort level otherwise. As a result, it is necessary to impose bounds on the minimum and maximum effort that each agent can exert at any moment. Let us assume that $a\in[0,1]$. Moreover, suppose that agents are symmetric, and $\sigma=0$ so that the project evolves deterministically.[34] By using (1.2) subject to (1.3) and the corresponding first order condition, it follows that a unique *project-completing* MPE exists if $q_0\geq\psi_n$, where $\psi_n=\frac{n}{r}\ln\left(\frac{n}{rV_n+1}\right)$, it is symmetric, and each agent's discounted payoff and

[34]While the corresponding HJB equation can be solved analytically if effort costs are linear, the solution is too complex to obtain the desired comparative statics if $\sigma>0$.

effort strategy satisfies

$$J_n(q) = \left[-\frac{1}{r} + \left(V_n + \frac{1}{r} \right) e^{\frac{rq}{n}} \right] \mathbf{1}_{\{q \geq \psi_n\}} \text{ and } a_n(q) = \mathbf{1}_{\{q \geq \psi_n\}},$$

respectively.[35] Observe that agents have stronger incentives the closer the project is to completion, as evidenced by the facts that $J_n''(q) \geq 0$ for all q, and $a_n(q) = 1$ if and only if $q \geq \psi_n$. To investigate how the agents' incentives depend on the team size, one needs to examine how ψ_n depends on n. This threshold decreases in the team size n under both allocation schemes (*i.e.*, both if $V_n = V$ and $V_n = \frac{V}{n}$ for some $V > 0$) if and only if n is sufficiently small. This implies that members of an $(n+1) -$ member team have stronger incentives relative to those of an $n -$ member team as long as n is sufficiently small.

If agents maximize the team's rather than their individual discounted payoff, then the first-best threshold $\hat{\psi}_n = \frac{n}{r} \ln\left(\frac{1}{rV+1}\right)$, and it is straightforward to show that it decreases in n under both allocation schemes. Therefore, similar to the case in which effort costs are convex, members of a larger team always have stronger incentives than those of a smaller one.

1.8 Proofs

Proof of Theorem 1. This proof is organized in 7 parts. I first show that a MPE for the game defined by (1.1) exists. Next I show that properties (i) thru (iii) hold, and that the value functions are infinitely differentiable. Then I show that with symmetric agents, the equilibrium is also symmetric. Finally, I show that the solution to the system of boundary value ODE is unique when the game comprises of n symmetric, or 2 asymmetric agents.

[35] Note that if $q_0 \in \left[\frac{n}{r} \ln\left(\frac{n}{rV+1}\right), \frac{1}{r} \ln\left(\frac{1}{rV+1}\right) \right)$, then there exists another equilibrium in which no agent exerts any effort and the project is never completed.

Part I: *Existence of a MPE.*

To show that a MPE exists, it suffices to show that a solution satisfying the system of ordinary nonlinear differential equations defined by (1.4) subject to the boundary conditions (1.3) for all $i = 1, .., n$ exists.

To begin, fix some arbitrary $N \in \mathbb{N}$ and rewrite (1.4) and (1.3) as

$$J''_{i,N}(q) = \frac{2}{\sigma^2}\left[rJ_{i,N}(q) + c\left(f\left(J'_{i,N}(q)\right)\right) - \left(\sum_{j=1}^{n} f\left(J'_{j,N}(q)\right)\right) J'_{i,N}(q)\right] \quad (1.7)$$
$$\text{subject to } J_{i,N}(-N) = 0 \text{ and } J_{i,N}(0) = V_i$$

for all i. Let $g_i(J_N, J'_N)$ denote the the RHS of (1.7), where J_N and J'_N are vectors whose i^{th} row corresponds to $J_{i,N}(q)$ and $J'_{i,N}(q)$, respectively , and note that $g_i(\cdot,\cdot)$ is continuous. Now fix some arbitrary $K > 0$, and define a new function

$$g_{i,K}(J_N, J'_N) = \max\left\{\min\left\{g_i(J_N, J'_N), K\right\}, -K\right\}.$$

Note that $g_{i,K}(\cdot,\cdot)$ is continuous and bounded. Therefore, by Lemma 4 in Hartman (1960), there exists a solution to $J''_{i,N,K} = g_{i,K}\left(J_{i,N,K}, J'_{i,N,K}\right)$ subject to $J_{i,N,K}(-N) = 0$ and $J_{i,N,K}(0) = V_i$ for all i. This Lemma, which is due to Scorza-Dragoni (1935), states:

Let $g(q, J, J')$ be a continuous and bounded (vector-valued) function for $\alpha \leq q \leq \beta$ and arbitrary (J, J'). Then, for arbitrary q_α and q_β, the system of differential equations $J'' = g(q, J, J')$ has at least one solution $J = J(q)$ satisfying $J(\alpha) = q_\alpha$ and $J(\beta) = q_\beta$.

The next part of the proof involves showing that there exists a $\bar{K}$ such that $g_{i,K}\left(J_{i,N,K}(q), J'_{i,N,K}(q)\right) \in \left(-\bar{K}, \bar{K}\right)$ for all i, K and q, which will imply that the solution $J_{i,N,\bar{K}}(\cdot)$ satisfies (1.7) for all i. The final step involves showing that a solution exists when $N \to \infty$, so that the first boundary condition in (1.7) is replaced by $\lim_{q \to -\infty} J_i(q) = 0$.

First, I show that $0 \leq J_{i,N,K}(q) \leq V_i$ and $J'_{i,N,K}(q) \geq 0$ for all i and q. Because $J_{i,N,K}(0) > J_{i,N,K}(-N) = 0$, either $J_{i,N,K}(\cdot)$ is increasing, or it has an interior global extreme point. If the former is true, then the desired inequality holds. Suppose the latter is true and let z^* denote such interior global extreme point. By noting that $J_{i,N,K}(\cdot)$ is at least twice differentiable, $J'_{i,N,K}(z^*) = 0$, and hence $J''_{i,N,K}(z^*) = \max\left\{\min\left\{\frac{2r}{\sigma^2} J_{i,N,K}(z^*),\, K\right\},\, -K\right\}$. Suppose z^* is a global maximum. Then $J''_{i,N,K}(z^*) \leq 0 \Longrightarrow J_{i,N,K}(z^*) \leq 0$, which contradicts the fact that $J_{i,N,K}(0) > 0$. Now suppose that z^* is a global minimum. Then $J''_{i,N,K}(z^*) \geq 0 \Longrightarrow J_{i,N,K}(z^*) \geq 0$. Therefore either $J_{i,N,K}(\cdot)$ is increasing, or it has an interior global minimum z^* such that $J_{i,N,K}(z^*) \geq 0$. As a result, $0 \leq J_{i,N,K}(q) \leq V_i$ for all i and q.

Next, let us focus on $J'_{i,N,K}(\cdot)$. Suppose that there exists a z^{**} such that $J'_{i,N,K}(z^{**}) < 0$. Because $J_{i,N,K}(-N) = 0$, either $J_{i,N,K}(\cdot)$ is decreasing on $[-N, z^{**}]$, or it has a local maximum $\bar{z} \in (-N, z^{**})$. If the former is true, then $J'_{i,N,K}(z^{**}) < 0$ implies that $J_{i,N,K}(q) < 0$ for some $q \in (-N, z^{**}]$, which is a contradiction because $J_{i,N,K}(q) \geq 0$ for all q. So the latter must be true. Then $J'_{i,N,K}(\bar{z}) = 0$ implies that $J''_{i,N,K}(\bar{z}) = \max\left\{\min\left\{\frac{2r}{\sigma^2} J_{i,N,K}(\bar{z}),\, K\right\},\, -K\right\}$. However, because $\bar{z}$ is a maximum, $J''_{i,N,K}(\bar{z}) \leq 0$, and together with the fact that $J_{i,N,K}(q) \geq 0$ for all q, this implies that $J_{i,N,K}(q) = 0$ for all $q \in [-N, z^{**})$. But since $J'_{i,N,K}(z^{**}) < 0$, it follows that $J_{i,N,K}(q) < 0$ for some q in the neighborhood of z^{**}, which is a contradiction. Therefore, it must be the case that $J'_{i,N,K}(q) \geq 0$ for all i and q.

The next step involves establishing that there exists an $\bar{A}$, independent of N and K, such that $J'_{i,N,K}(q) < \bar{A}$ for all i and q. First, let $S_{N,K}(q) = \sum_{i=1}^{n} J_{i,N,K}(q)$. By summing $J''_{i,N,K} = g_{i,K}\left(J_{i,N,K}, J'_{i,N,K}\right)$ over i, using that (i) $0 \leq J_{i,N,K}(q) \leq V_i$ and $0 \leq J'_{i,N,K}(q) \leq S'_{N,K}(q)$ for all i and q, (ii) $f(x) = x^{1/p}$, and (iii) $c(x) \leq x\, c'(x)$ for all $x \geq 0$, and letting $\Gamma = r\sum_{i=1}^{n} V_i$, we have that for all q

$$\begin{aligned}
\left|S''_{N,K}(q)\right| &\leq \frac{2}{\sigma^2}\sum_{i=1}^{n}\left[rJ_{i,N,K}(q) + c\left(f\left(J'_{i,N,K}(q)\right)\right) + \left[\sum_{j=1}^{n} f\left(J'_{j,N,K}(q)\right)\right] J'_{i,N,K}(q)\right] \\
&\leq \frac{2}{\sigma^2}\left[\Gamma + \sum_{i=1}^{n} c'\left(c'^{-1}\left(J'_{i,N,K}(q)\right)\right) c'^{-1}\left(J'_{i,N,K}(q)\right) + S'_{N,K}(q)\sum_{j=1}^{n} f\left(J'_{j,N,K}(q)\right)\right] \\
&\leq \frac{4}{\sigma^2}\left[\Gamma + nS'_{N,K}(q)\, f\left(S'_{N,K}(q)\right)\right] = \frac{4}{\sigma^2}\left[\Gamma + n\left(S'_{N,K}(q)\right)^{\frac{p+1}{p}}\right].
\end{aligned}$$

By noting that $S_{N,K}(0) = \sum_{i=1}^{n} V_i$, $S_{N,K}(-N) = 0$, and applying the mean value theorem, it follows that there exists a $z^* \in [-N, 0]$ such that $S'_{N,K}(z^*) = \frac{\sum_{i=1}^{n} V_i}{N}$. It follows that for all $z \in [-N, 0]$

$$\sum_{i=1}^{n} V_i > \int_{z^*}^{z} S'_{N,K}(q)\, dq \geq \frac{\sigma^2}{4}\int_{z^*}^{z} S'_{N,K}(q)\frac{S''_{N,K}(q)}{\Gamma + n\left(S'_{N,K}(q)\right)^{\frac{p+1}{p}}} dq \geq \frac{\sigma^2}{4}\int_{0}^{S'_N(z)} \frac{s}{\Gamma + ns^{\frac{p+1}{p}}} ds\,,$$

where I let $s = S'_{N,K}(q)$ and used that $S'_{N,K}(q)\, S''_{N,K}(q)\, dq = S'_{N,K}(q)\, dS'_{N,K}(q)$. It suffices to show that there exists a $\bar{A} < \infty$ such that $\frac{\sigma^2}{4}\int_0^{\bar{A}} \frac{s}{\Gamma + ns^{\frac{p+1}{p}}} ds = \sum_{i=1}^{n} V_i$. This will imply that $S'_{N,K}(q) < \bar{A}$, and consequently $J'_{i,N,K}(q) \leq \bar{A}$ for all $q \in [-N, 0]$. To show that such $\bar{A}$ exists, it suffices to show that $\int_0^{\infty} \frac{s}{\Gamma + ns^{\frac{p+1}{p}}} ds = \infty$. First, observe that if $p = 1$, then $\int_0^{\infty} \frac{s}{\Gamma + ns^2} ds = \frac{1}{2n}\ln\left(\Gamma + ns^2\right)|_0^{\infty} = \infty$. By noting that $\frac{s}{\Gamma + ns^2}$ is bounded for all $s \in [0, 1]$, $\frac{s}{\Gamma + ns^{\frac{p+1}{p}}} > \frac{s}{\Gamma + ns^2}$ for all $s > 1$ and $p > 1$, and $\int_0^{\infty} \frac{s}{\Gamma + ns^2} ds = \infty$, integrating both sides over $[0, \infty]$ yields the desired inequality.

Because $\bar{A}$ is independent of both N and K, this implies that $J'_{i,N,K}(q) \in \left[0, \bar{A}\right]$ for all $q \in [-N, 0]$, $N \in \mathbb{N}$ and $K > 0$. In addition, we know that $J_{i,N,K}(q) \in [0, V_i]$ for all $q \in [-N, 0]$, $N \in \mathbb{N}$ and $K > 0$. Now let $\bar{K} = \max_i\left\{\frac{2}{\sigma^2}\left[rV_i + c\left(f\left(\bar{A}\right)\right)\right]\right\}$, and observe that a solution to $J''_{i,N,\bar{K}} = g_{i,\bar{K}}\left(J_{N,\bar{K}}, J'_{N,\bar{K}}\right)$ subject to $J_{i,N,\bar{K}}(-N) = 0$ and $J_{i,N,\bar{K}}(0) = V_i$ for all i exists, and $g_{i,\bar{K}}\left(J_{N,\bar{K}}(q), J'_{N,\bar{K}}(q)\right) = g\left(J_{i,N,\bar{K}}(q), J'_{N,\bar{K}}(q)\right)$ for all i and $q \in [-N, 0]$. Therefore, $J_{i,N,\bar{K}}(\cdot)$ solves (1.7) for all i.

To show that a solution for (1.7) exists at the limit as $N \to \infty$, I use the Arzela-Ascoli theorem, which states:

> Consider a sequence of real-valued continuous functions $(f_n)_{n\in\mathbb{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence (f_{n_k}) that converges uniformly.

Recall that $0 \leq J_{i,N}(q) \leq V_i$ and that there exists a constant $\bar{A}$ such that $0 \leq J'_{i,N}(q) \leq \bar{A}$ on $[-N, 0]$ for all i and $N > 0$. Hence the sequences $\{J_{i,N}(\cdot)\}$ and $\left\{J'_{i,N}(\cdot)\right\}$ are uniformly bounded and equicontinuous on $[-N, 0]$. By applying the Arzela-Ascoli theorem to a sequence of intervals $[-N, 0]$ and letting $N \to \infty$, it follows that the system of ODE defined by (1.4) has at least one solution satisfying the boundary conditions (1.3) for all i.

Part II: $J_i(q) > 0$ *for all* q *and* i.

By the boundary conditions we have that $\lim_{q\to-\infty} J_i(q) = 0$ and $J_i(0) = V_i > 0$. Suppose that there exists an interior z^* that minimizes $J_i(\cdot)$ on $(-\infty, 0]$. Clearly $z^* < 0$. Then $J'_i(z^*) = 0$ and $J''_i(z^*) \geq 0$, which by applying (1.4) imply that

$$rJ_i(z^*) = \frac{\sigma^2}{2} J''_i(z^*) \geq 0\,.$$

Because $\lim_{q\to-\infty} J_i(q) = 0$, it follows that $J_i(z^*) = 0$. Next, let $z^{**} = \arg\max_{q\leq z^*}\{J_i(q)\}$. If z^{**} is on the boundary of the desired domain, then $J_i(q) = 0$ for all $q \leq z^*$. Suppose that z^{**} is interior. Then $J'_i(z^{**}) = 0$ and $J''_i(z^{**}) \leq 0$ imply that $J_i(z^{**}) \leq 0$, so that $J_i(q) = J'_i(q) = 0$ for all $q < z^*$.

Using (1.4) we have that

$$\left|J_i''(q)\right| \;\leq\; \frac{2r}{\sigma^2}\left|J_i(q)\right| + \frac{2}{\sigma^2}(n+1)\, f\left(\bar{A}\right)\left|J_i'(q)\right| \,,$$

where this bound follows from part I of the proof. Now let $h_i(q) = |J_i(q)| + |J_i'(q)|$, and observe that $h_i(q) = 0$ for all $q < z^*$, $h_i(q) \geq 0$ for all q, and

$$h_i'(q) \leq |J_i'(q)| + |J_i''(q)| \leq \frac{2r}{\sigma^2}|J_i(q)| + \frac{2}{\sigma^2}\left[f_i\left(\bar{A}\right) + \sum_{j=1}^{n} f_j\left(\bar{A}\right) + \frac{\sigma^2}{2}\right]|J_i'(q)| \leq C\, h_i(q) \,,$$

where $C = \frac{2}{\sigma^2}\max\left\{r,\, (n+1)\, f\left(\bar{A}\right) + \frac{\sigma^2}{2}\right\}$. Fix some $\hat{z} < z^*$, and applying the differential form of Grönwall's inequality yields $h_i(q) \leq h_i(\hat{z}) \exp\left(\int_{\hat{z}}^{q} C dx\right)$ for all q. Because (i) $h_i(\hat{z}) = 0$, (ii) $\exp\left(\int_{z^*}^{q} C dx\right) < \infty$ for all q, and (iii) $h_i(q) \geq 0$ for all q, this inequality implies that $J_i(q) = 0$ for all q. However this contradicts the fact that $J_i(0) = V_i > 0$. As a result, $J_i(\cdot)$ cannot have an interior minimum, and there cannot exist a $z^* > -\infty$ such that $J_i(q) = 0$ for all $q \leq z^*$. Hence $J_i(q) > 0$ for all q.

Part III: $J_i'(q) > 0$ *for all* q *and* i.

Pick a K such that $J_i(0) < J_i(K) < V_i$. Such K is guaranteed to exist, because $J_i(\cdot)$ is continuous and $J_i(0) > 0 = \lim_{q\to-\infty} J_i(q)$. Then by the mean-value theorem there exists a $z^* \in (K, 0)$ such that $J_i'(z^*) = \frac{J_i(0)-J_i(K)}{-K} = \frac{V_i - J_i(K)}{-K} > 0$. Suppose that there exists a $z^{**} \leq 0$ such that $J_i'(z^{**}) \leq 0$. Then by the intermediate value theorem, there exists a $\bar{z}$ between z^* and z^{**} such that $J_i'(\bar{z}) = 0$, which using (1.4) and part II implies that $rJ_i(\bar{z}) = \frac{\sigma^2}{2} J_i''(\bar{z}) > 0$ (*i.e.*, $\bar{z}$ is a local minimum). Consider the interval $(-\infty, \bar{z}]$. Because $\lim_{q\to-\infty} J_i(q) = 0$, $J_i(\bar{z}) > 0$ and $J_i''(\bar{z}) > 0$, there exists an interior local maximum $\hat{z} < \bar{z}$. Since $\hat{z}$ is interior, it must be the case that $J_i'(\hat{z}) = 0$ and $J_i''(\hat{z}) \leq 0$, which using (1.4) implies that $J_i(\hat{z}) \leq 0$. However this contradicts the fact that $J_i(q) > 0$ for all q. As a result there there cannot exist a $\bar{z} \leq 0$ such that $J_i'(\bar{z}) \leq 0$. Together with part II, this proves properties (i) and (ii).

Part IV: *$J_i(q)$ is infinitely differentiable on* $(-\infty, 0]$ for all i.

By noting that $\lim_{q\to-\infty} J_i(q) = \lim_{q\to-\infty} J_i'(q) = 0$ for all i, and by twice integrating both sides of (1.7) over the interval $(-\infty, q]$, we have that

$$J_i(q) = \int_{-\infty}^{q} \int_{-\infty}^{y} \frac{2r}{\sigma^2} J_i(z) + \frac{2}{\sigma^2} \left[c\left(f\left(J_i'(z)\right)\right) - \left(\sum_{j=1}^{n} f\left(J_j'(z)\right) \right) J_i'(z) \right] dz\, dy\,.$$

Recall that $c(a) = \frac{a^{p+1}}{p+1}$, $f(x) = x^{1/p}$, and $J_i'(q) > 0$ for all q. Since $J_i(q)$ and $J_i'(q)$ satisfy (1.4) subject to the boundary conditions (1.3) for all i, $J_i(q)$ and $J_i'(q)$ are continuous for all i. As a result, the function under the integral is continuous and infinitely differentiable in $J_i(z)$ and $J_i'(z)$ for all i. Because $J_i(q)$ is differentiable twice more than the function under the integral, the desired result follows by induction.

Part V: *$J_i''(q) > 0$ and $a_i'(q) > 0$ for all q and i.*

I have thus far established that for all q, $J_i(q) > 0$ and $J_i'(q) > 0$. By applying the envelope theorem to (1.4) we have that

$$rJ_i'(q) = \left[f\left(J_i'(q)\right) + A_{-i}(q)\right] J_i''(q) + \frac{\sigma^2}{2} J_i'''(q)\,, \tag{1.8}$$

where $A_{-i}(q) = \sum_{j\neq i}^{n} f\left(J_j'(q)\right)$. Choose some finite $z^* \leq 0$, and let $z^{**} = \arg\max\left\{J_i'(q) : q \leq z\right\}$. By part III, $J_i'(z^{**}) > 0$. Because $\lim_{q\to-\infty} J_i'(q) = 0$, either $z^{**} = z$, or z^{**} is interior. Suppose z^{**} is interior. Then $J_i''(z^{**}) = 0$ and $J_i'''(z^{**}) \leq 0$, which using (1.8) implies that $J_i'(z^{**}) \leq 0$. However this contradicts the fact that $J_i'(z^{**}) > 0$, and therefore $J_i'(\cdot)$ does not have an interior maximum on $(-\infty, z]$ for any $z \leq 0$. Therefore $z^{**} = z$, and hence $J_i'(\cdot)$ is strictly increasing; *i.e.*, $J_i''(q) > 0$ for all q. By differentiating $a_i(q)$ we have that

$$\frac{d}{dq}a_i(q) = \frac{d}{dq}c'^{-1}\left(J_i'(q)\right) = \frac{J_i''(q)}{c''\left(c'^{-1}\left(J_i'(q)\right)\right)} > 0\,.$$

Part VI: *When the agents are symmetric, the MPE is also symmetric.*

Suppose agents are symmetric; *i.e.*, they have identical effort costs, patience levels, and they receive the same reward upon completing the project. In any MPE, $\{J_i(\cdot)\}_{i=1}^n$ must satisfy (1.4) subject to (1.3). Arbitrarily pick two agents i and j, and let $\Delta(q) = J_i(q) - J_j(q)$. Observe that $\Delta(\cdot)$ is smooth, and $\lim_{q\to-\infty}\Delta(q) = \Delta(0) = 0$. Therefore either $\Delta(\cdot) \equiv 0$ on $(-\infty, 0]$, which implies that $J_i(\cdot) \equiv J_j(\cdot)$ on $(-\infty, 0]$ for all $i \neq j$ and hence the equilibrium is symmetric, or $\Delta(\cdot)$ has at least one interior global extreme point. Suppose the latter is true, and denote this extreme point by z^*. Then and by using (1.4) and the fact that $\Delta'(z^*) = 0$, we have $r\Delta(z^*) = \frac{\sigma^2}{2}\Delta''(z^*)$. Suppose that z^* is a maximum. Then $\Delta''(z^*) \leq 0$, which implies that $\Delta(z^*) \leq 0$. However, because $\Delta(0) = 0$ and z^* is assumed to be a maximum, $\Delta(z^*) = 0$. Next, suppose that z^* is a minimum. Then $\Delta''(z^*) \geq 0$, which implies that $\Delta(z^*) \geq 0$. However, because $\Delta(0) = 0$ and z^* is assumed to be a minimum, $\Delta(z^*) = 0$. Therefore it must be the case that $\Delta(\cdot) \equiv 0$ on $(-\infty, 0]$. Since i and $j \neq i$ were chosen arbitrarily, $J_i(\cdot) \equiv J_j(\cdot)$ on $(-\infty, 0]$ for all $i \neq j$, which implies that the equilibrium is symmetric.

Part VII: *The system of ordinary nonlinear differential equations defined by (1.4) for (a) $n \in \mathbb{N}$ symmetric agents, and (b) 2 asymmetric agents, has at most one solution satisfying the boundary conditions (1.3).*

Case (A): I first prove uniqueness for n symmetric agents. From Part VI of the proof, we know that if agents are symmetric, then the MPE is symmetric. Therefore to facilitate exposition, I drop the notation for the i^{th} agent. Any solution $J(\cdot)$ must satisfy

$$rJ(q) = -c(f(J'(q))) + nf(J'(q))\,J'(q) + \frac{\sigma^2}{2}J''(q) \text{ subject to } \lim_{q\to-\infty} J(q) = 0 \text{ and } J(0) = V.$$

Suppose that there exist 2 functions $J_A(q)$, $J_B(q)$ that satisfy the above boundary value problem. Then define $D(q) = J_A(q) - J_B(q)$, and note that $D(\cdot)$ is smooth and $\lim_{q\to-\infty} D(q) = D(0) = 0$. Hence either $D(\cdot) \equiv 0$ in which case the proof for (a) is complete, or $D(\cdot)$ has an interior global extreme point z^*. Suppose the latter is true. Then $D'(z^*) = 0$, which implies that $rD(z^*) = \frac{\sigma^2}{2}D''(z^*)$. Suppose that z^* is a global maximum. Then $D''(z^*) \leq 0$ implies that $D(z^*) \leq 0$, and $D(0) = 0$ implies that $D(z^*) = 0$ and $D(q) \leq 0$ for all q. Hence either $D(\cdot) \equiv 0$ or z^* is a global minimum. Suppose the latter is true. Then $D''(z^*) \geq 0$ implies that $D(z^*) \geq 0$, and $D(0) = 0$ implies that $D(z^*) = 0$ and $D(q) \geq 0$ for all q. Therefore it must be the case that $D(\cdot) \equiv 0$ and the proof for (a) is complete.

Case (b): Now consider (1.4) for the case with 2 asymmetric agents. Any solution $J_1(q)$ and $J_2(q)$ must satisfy

$$\begin{aligned} J_1''(q) &= -\frac{2}{\sigma^2}\frac{p}{p+1}\left[J_1'(q)\right]^{\frac{p+1}{p}} - \frac{2}{\sigma^2}J_1'(q)\left[J_2'(q)\right]^{\frac{1}{p}} + \frac{2r}{\sigma^2}J_1(q) \quad \text{and} \\ J_2''(q) &= -\frac{2}{\sigma^2}\frac{p}{p+1}\left[J_2'(q)\right]^{\frac{p+1}{p}} - \frac{2}{\sigma^2}\left[J_1'(q)\right]^{\frac{1}{p}}J_2'(q) + \frac{2r}{\sigma^2}J_2 \end{aligned}$$

subject to $J_i(0) = V_i$ and $\lim_{q\to-\infty} J_i(q) = 0$ for all $i = \{1, 2\}$.

To show that there exists a unique solution to the above system of ODE, I shall use Theorem 5 in Hartman (1960), which states

Let $g(q, J, J')$ be defined on $D(T, R)$ (*i.e.*, be a continuous vector value function on $[-N, 0]$ such that $|J_i(q)| \leq V_i$ for all q and i) and possess continuous partial derivatives with respect to the components of J and J'. Let $F(q, J, J')$ and $G(q, J, J')$ denote the Jacobian matrices

$$F(q, J, J') = \frac{\partial g}{\partial J} \quad \text{and} \quad G(q, J, J') = \frac{\partial g}{\partial J'}$$

and suppose that $4F - GG^T \succeq 0$. Then (1.7) has at most one solution satisfying $J_i(0) = V_i$ and $J_i(-N) = 0$ for all $i \in \{1, 2\}$.

Clearly in this case $g(q, J, J')$ is continuous, $|J_i(q)|$ is bounded for all q and i, and possesses continuous partial derivatives with respect to J_i and J'_i for all i. Therefore it suffices to show that $4F - GG^T \succeq 0$ holds for all $N > 0$ and by letting $N \to \infty$ conclude that the above system of 2 ODE has a unique solution on $(-\infty, 0]$. We have

$$F(q) = \frac{2}{\sigma^2}\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \quad \text{and} \quad G(q) = -\frac{2}{\sigma^2}\begin{bmatrix} [J'_1(q)]^{1/p} + [J'_2(q)]^{1/p} & \frac{1}{p}\frac{J'_1(q)}{J'_2(q)}[J'_2(q)]^{1/p} \\ \frac{1}{p}\frac{J'_2(q)}{J'_1(q)}[J'_1(q)]^{1/p_1} & [J'_1(q)]^{1/p} + [J'_2(q)]^{1/p} \end{bmatrix}.$$

To facilitate exposition, let us denote (only for this proof) $\alpha = [J'_1(q)]^{1/p}$, $\beta = [J'_2(q)]^{1/p}$ and $\kappa = \frac{J'_1(q)}{J'_2(q)}$. By noting that $J'_i(q) > 0$ for all q and i, it follows that $0 < \kappa < \infty$. Then

$$4F - GG^T \;=\; \frac{2}{\sigma^2}\begin{bmatrix} 4r + (\alpha+\beta)^2 + \left(\frac{\kappa\beta}{p}\right)^2 & (\alpha+\beta)\left(\frac{\alpha}{\kappa p} + \frac{\kappa\beta}{p}\right) \\ (\alpha+\beta)\left(\frac{\alpha}{\kappa p} + \frac{\kappa\beta}{p}\right) & 4r + (\alpha+\beta)^2 + \left(\frac{\alpha}{\kappa p}\right)^2 \end{bmatrix}.$$

To check that the above matrix is positive definite I use Sylvester's criterion. First note that $4r_1 + (\alpha+\beta)^2 + \left(\frac{\kappa\beta}{p}\right)^2 > 0$. The determinant of the above matrix is equal to

$$16r^2 + 8(\alpha+\beta)^2 r + 4r\left[\left(\frac{\alpha}{\kappa p}\right)^2 + \left(\frac{\kappa\beta}{p}\right)^2\right] + (\alpha+\beta)^2\left[(\alpha+\beta)^2 - \frac{\alpha\beta}{p^2}\right].$$

Clearly the first three terms are positive. By noting that $p \geq 1$ and using the property that $J_i'(q) > 0$ for all q, it follows that the last term is also positive. Hence the determinant of the above matrix is positive and uniqueness follows by applying Theorem 5 from Hartman (1960) for any given $N > 0$ and letting $N \to \infty$.

In light of the fact that $J_i'(q) > 0$ for all q, it follows that the first-order condition for each agent's best response always binds. As a result, any MPE must satisfy the system of ODE defined by (1.4) subject to (1.3). Since this system of ODE has a unique solution with n symmetric or 2 asymmetric agents, it follows that in these two cases, the dynamic game defined by (1.1) has a unique MPE.

□

Proof of Proposition 1. This proof is organized in 4 parts. To begin, let $J_i(\cdot)$ denote the expected discounted payoff of each member of an n-person team with parameters $\{r_i, c_i(\cdot), V_i\}$ who undertakes a project with volatility σ.

Proof for property (i): First, pick $\alpha < 1$ and V such that $V_1 = \alpha V_2 < V_2 = V$, and let $r = r_1 = r_2$. Let $D_V(q) = J_1(q) - J_2(q)$, and note that it is smooth, and $D_V(0) = (\alpha - 1)V < 0 = \lim_{q \to -\infty} D_V(q) = 0$. Suppose that $D_V(\cdot)$ has some interior extreme point, which I denote by z^*. Then $D_V'(z^*) = 0$, and by using (1.4) we have

$$rD_V(z^*) = \frac{\sigma^2}{2} D_V''(z^*) .$$

Suppose that z^* is a global minimum. Then $D_V''(z^*) \geq 0 \Longrightarrow D_V(z^*) \geq 0$, which contradicts the fact that $D_V(0) < 0$. So z^* must be a global maximum. Then $D_V''(z^*) \leq 0 \Longrightarrow D_V(z^*) \leq 0$, which contradicts the fact that z^* is interior. Hence $D_V(\cdot)$ cannot have any interior extreme points, and thus it must be decreasing for all q; *i.e.*, $D_V'(q) \leq 0$ for all q and $D_V'(q) < 0$ for at least some q.

The next step involves showing that in fact, $D'_V(q) < 0$ for all q. Suppose that there exists a z such that $D'_V(z) = 0$. If $D_V(z) = 0$, then $\lim_{q \to -\infty} D_V(q) = 0$, any interior maximum on $(-\infty, z]$ must satisfy $D_V(z) \leq 0$, and any interior minimum must satisfy $D_V(z) \geq 0$. It follows that $D_V(q) = D'_V(q) = 0$ for all $q < z$. So suppose that $D_V(z) < 0$, and let $\hat{z} = \arg\min_{q \leq z} \{D_V(q)\}$. Clearly, $\hat{z} > -\infty$. Second, to show that $\hat{z} < z$, suppose that the contrary is true; *i.e.*, $\hat{z} = z$. Then $D'_V(z) = 0$, $D_V(z) < 0$, and (1.4) imply that $D''_V(z) < 0$, which contradicts the assumption that $\hat{z}$ is a minimum. Hence $\hat{z}$ is interior, so that $D'_V(z) = 0$ and $D''_V(z) \geq 0$, which together with (1.4) imply that $D_V(z) \geq 0$. However, this contradicts the assumption that $D_V(z) < 0$. Therefore, $D_V(z) = 0$, and it follows that $D_V(q) = D'_V(q) = 0$ for all $q < z$. Next, let $M(q) = [J_1(q) - J_2(q)] + [J'_1(q) - J'_2(q)]$, and note that $M(q) \leq 0$ for all q, $M(0) < 0$, and $M(q) = 0$ for all $q < z$. By applying the differential form of Grönwall's inequality, it follows that $M(q) = 0$ for all q, which contradicts the fact that $M(0) < 0$. Hence, I conclude that there does not exist a z such that $D'_V(z) = 0$. Therefore, $D'_V(q) < 0$ for all q, which implies that $a_1(q) < a_2(q)$ for all q.

Proof for property (ii): First pick $\delta > 1$ and r such that $r_1 = \delta r > r = r_2$. Next, define $D_r(q) = J_1(q) - J_2(q)$. By noting that $\lim_{q \to -\infty} D_r(q) = D_r(0) = 0$, observe that either $D_r(\cdot) \equiv 0$, or $D_r(\cdot)$ has at least one interior extreme point. Suppose $D_r(\cdot) \equiv 0$. Then $D'_r(\cdot) \equiv D''_r(\cdot) \equiv 0$, and using (1.4) we have that $\delta J_1(\cdot) \equiv J_2(\cdot)$. However this is a contradiction, because $J_1(\cdot) \equiv J_2(\cdot)$, and $\delta > 1$. Therefore $D_r(\cdot)$ must have at least one interior extreme point, which I denote by z^*. By noting that $D'_r(z^*) = 0$ and using (1.4), we have that

$$r\left[\delta J_1(z^*) - J_2(z^*)\right] = \frac{\sigma^2}{2} D''_r(z^*) \ .$$

Suppose that z^* is a global maximum. Then $D''_r(z^*) \leq 0$, and hence $\delta J_1(z^*) - J_2(z^*) \leq 0$.

However because $J_i(\cdot) > 0$ and $\delta > 1$, this implies that $D_r(z^*) < 0 = D_r(0)$, which contradicts the assumption that z^* is a global maximum. Therefore, z^* must be a global minimum, and $D_r(q) \leq 0$ for all q.

I next show that $D_r(\cdot)$ is single-troughed. Suppose it is not. Then I can find an interior local minimum z^* followed by an interior a local maximum $\bar{z} > z^*$. Since $\bar{z}$ is an interior maximum, $D_r'(\bar{z}) = 0$ and $D_r''(\bar{z}) \leq 0$, and from (1.4) it follows that $\delta J_1(\bar{z}) \leq J_2(\bar{z})$. Because z^* is an interior minimum, $D_r''(z^*) \geq 0$ implies that $\delta J_1(z^*) \geq J_2(z^*) \Rightarrow -\delta J_1(z^*) \leq -J_2(z^*)$, and by using $\delta J_1(\bar{z}) \leq J_2(\bar{z})$, we have that $0 < \delta [J_1(\bar{z}) - J_1(z^*)] \leq J_2(\bar{z}) - J_2(z^*)$, where the first inequality follows from Theorem 1 (iii) and the fact that $\bar{z} > z^*$. By assumption $D_r(\bar{z}) > D_r(z^*)$, which implies that $J_2(\bar{z}) - J_2(z^*) < J_1(\bar{z}) - J_1(z^*)$, so that

$$\delta [J_1(\bar{z}) - J_1(z^*)] \leq J_2(\bar{z}) - J_2(z^*) < J_1(\bar{z}) - J_1(z^*) ,$$

which contradicts the facts that $\delta > 1$ and $J_1(\bar{z}) - J_1(z^*) > 0$. Hence $D_r(\cdot)$ must be single-troughed. Because $\lim_{q \to -\infty} D_r(q) = D_r(0) = 0$, there exists a $\Theta_r < 0$ such that $D_r'(q) \leq 0$ if and only if $q \leq \Theta_r$. Finally, because $c_1(\cdot) \equiv c_2(\cdot) \Rightarrow f_1(\cdot) \equiv f_2(\cdot)$, it follows that $a_1(q) \leq a_2(q)$ if and only if $D_r'(q) \leq 0$, or equivalently, if and only if $q \leq \Theta_r$.

Proof for property (iii): First pick $\alpha > 1$ and σ such that $\sigma_1^2 = \alpha\sigma_2^2 > \sigma_2^2 = \sigma^2$. Let $J_1(\cdot)$ and $J_2(\cdot)$ denote each agent's expected discounted payoff associated with σ_1 and σ_2, respectively. Moreover let $D_\sigma(q) = J_1(q) - J_2(q)$ and observe that $\lim_{q \to -\infty} D_\sigma(q) = D_\sigma(0) = 0$. So either $D_\sigma(\cdot) \equiv 0$ on $(-\infty, 0]$, or $D_\sigma(\cdot)$ has some interior global extreme point. Suppose that $D_\sigma(\cdot) \equiv 0$ on $(-\infty, 0]$. This implies that $D_\sigma(q) = D_\sigma'(q) = D_\sigma''(q) = 0$

for all q, and using (1.4) it follows that for all q

$$rD_{\sigma}(q) = \frac{\sigma^2}{2}\left[\alpha D''_{\sigma}(q) + (\alpha - 1) J''_2(q)\right] \Longrightarrow J''_2(q) = 0\,.$$

However this contradicts Theorem 1 (iii), which implies that $D_{\sigma}(\cdot)$ hast at least one interior global extreme point, denoted by z^*. Then $D'_{\sigma}(z^*) = 0$, and using (1.4) yields $rD_{\sigma}(z^*) = \frac{\sigma^2}{2}\left[\alpha D''_{\sigma}(z^*) + (\alpha - 1) J''_2(z^*)\right]$. Suppose that z^* is a global minimum. Then $D''_{\sigma}(z^*) \geq 0$, $\alpha > 1$, and $J''_2(z^*) > 0$ imply that $D_{\sigma}(z^*) > 0$. However, this contradicts the fact that $D_{\sigma}(0) = 0$. Therefore z^* must be a maximum. This implies that there exist interior thresholds $\Theta_{\sigma,1} \leq \Theta_{\sigma,2}$ such that $D_{\sigma}(\cdot)$ is increasing on $(-\infty, \Theta_{\sigma,1}]$ and decreasing on $[\Theta_{\sigma,2}, 0]$.[36] Finally, because $a_1(q) \geq a_2(q)$ if and only if $D'_{\sigma}(q) \geq 0$, the desired result follows.

□

Proof of Proposition 2. This proof is organized in 3 parts. I first show that the desired relationships hold with weak inequality. Then I show that they in fact hold with strict inequality.

Part I: $\hat{a}(q) \geq a(q)$ for all q.

Note that $c(a) = \frac{a^{p+1}}{p+1}$ implies that $f(x) = x^{1/p}$ and $c(f(x)) = \frac{x^{\frac{p+1}{p}}}{p+1}$. As a result (1.4) and the first-best HJB equation can be written as

$$\begin{aligned} rJ(q) &= \left(n - \frac{1}{p+1}\right)\left[J'(q)\right]^{\frac{p+1}{p}} + \frac{\sigma^2}{2} J''(q) \text{ and} \\ r\hat{J}(q) &= \frac{p}{p+1}\left[n\hat{J}'(q)\right]^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\hat{J}''(q)\ , \end{aligned}$$

respectively, where the subscript for the i^{th} agent has been suppressed since the equilibria are symmetric. Note that the equilibrium effort level of each agent is given by $f(J'(q))$,

[36] Unfortunately, it is not possible to prove that $D_{\sigma}(\cdot)$ does not have any local extrema so that $\Theta_{\sigma,1} = \Theta_{\sigma,2}$, which would imply that $a_1(q) \geq a_2(q)$ if and only if $q \leq \Theta_{\sigma,i}$.

while the first-best effort level of each agent is given by $f\left(n\hat{J}'(q)\right)$. Because $f(\cdot)$ is strictly increasing, it suffices to show that $n\hat{J}'(q) \geq J'(q)$ for all q. Let $\alpha = \left[\frac{np}{np+(n-1)}\right]^p n$, and note that $\alpha|_{n=1} = 1$, $\alpha \leq n$ and α is strictly increasing in n for all $p > 0$ and $n \geq 2$, which implies that $1 < \alpha \leq n$ for all $p > 0$ and $n \geq 2$. Because $\hat{J}'(q) > 0$ and $J'(q) > 0$ for all q, it suffices to show that $\alpha\hat{J}'(q) \geq J'(q)$ for all q. Now define $\Delta_\alpha(q) = \alpha\hat{J}(q) - J(q)$ and note that $\Delta_\alpha(\cdot)$ is smooth, $\lim_{q\to-\infty}\Delta_\alpha(q) = 0$, and $\Delta_\alpha(0) = (\alpha-1)V > 0$. So either $\Delta_\alpha(\cdot)$ is increasing on $(-\infty, 0]$ or it has at least one interior global extreme point. If the former is true, then the desired inequality holds. Now suppose the latter is true and let us denote this extreme point by z^*. Using that $\alpha\hat{J}'(z^*) = J'(z^*)$, (1.4) and the first-best HJB equation, we have that

$$
\begin{aligned}
r\Delta_\alpha(z^*) &= \left[\frac{\alpha p}{p+1}\left(\frac{n}{\alpha}\right)^{\frac{p+1}{p}} - n + \frac{1}{p+1}\right][J'(q)]^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\Delta''_\alpha(z^*) \\
\Longrightarrow r\Delta_\alpha(z^*) &= \frac{\sigma^2}{2}\Delta''_\alpha(z^*)\ .
\end{aligned}
$$

[37] Suppose that z^* is a global maximum. Then $\Delta''_\alpha(z^*) \leq 0$ implies that $\Delta_\alpha(z^*) \leq 0$, contradicting the fact that $\Delta_\alpha(0) > 0$. Therefore, z^* must be a minimum. Then $\Delta''_\alpha(z^*) \geq 0$ implies that $\Delta_\alpha(z^*) \geq 0$, contradicting the facts that $\lim_{q\to-\infty}\Delta_\alpha(q) = 0$ and that z^* is interior. Therefore $\Delta_\alpha(\cdot)$ cannot have any interior extreme points, which implies that $\Delta_\alpha(\cdot)$ is increasing on $(-\infty, 0]$.

Part II: $\hat{J}(q) \geq J(q)$ for all q.

Let us define $\Delta_1(q) = \hat{J}(q) - J(q)$ and note that $\Delta_1(\cdot)$ is smooth, and $\lim_{q\to-\infty}\Delta_1(q) = \Delta_1(0) = 0$. Therefore either $\Delta_1(\cdot) \equiv 0$, or $\Delta_1(\cdot)$ has at least one local interior extreme point. If the former is true, then $\Delta'_1(q) = \Delta''_1(q) = 0$ for all q. Then using (1.4) and the first-best HJB equation, it follows that $\frac{1}{p+1}\left[pn^{\frac{p+1}{p}} - n(p+1) + 1\right][J'(q)]^{\frac{p+1}{p}} = 0$, which contradicts

[37] Note that the constant α has been chosen such that the term in brackets equals 0 when $\alpha\hat{J}'(z^*) = J'(z^*)$.

the facts that $J'(z^*) > 0$ and $\left[pn^{\frac{p+1}{p}} - n(p+1) + 1\right] > 0$ for all $n \geq 2$ and $p > 0$. Therefore it must be the case that $\Delta_1(\cdot)$ has an interior extreme point, which we denote by z^*. Using that $\hat{J}'(z^*) = J'(z^*)$, (1.4) and the first-best HJB equation, we have that

$$r\Delta_1(z^*) = \frac{pn^{\frac{p+1}{p}} - n(p+1) + 1}{p+1}\left[J'(z^*)\right]^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\Delta_1''(z^*) .$$

Suppose that z^* is a minimum. Then $\Delta_1''(z^*) \geq 0$ and $\left[pn^{\frac{p+1}{p}} - n(p+1) + 1\right] > 0$ implies that $\Delta_1(z^*) > 0$, which in turn implies that $\Delta_1(q) \geq 0$, or equivalently $\hat{J}(q) \geq J(q)$ for all q.

Part III: $\hat{a}(q) > a(q)$ and $\hat{J}(q) > J(q)$ for all q.

Recall that in proving existence of a MPE in Theorem 1 (Part I), I obtained a bound $|J''(q)| \leq C[|J(q)| + |J'(q)|]$ for all q, where $C > 0$ is a constants. Using an analogous approach, one can obtain a similar bound for $\left|\hat{J}''(q)\right|$; *i.e.*, $\left|\hat{J}''(q)\right| \leq \hat{C}\left[\left|\hat{J}(q)\right| + \left|\hat{J}'(q)\right|\right]$ for all q.

Suppose that there exists a $z \leq 0$ such that $\Delta_\alpha'(z) = 0$. Because $r\Delta_\alpha(z) = \frac{\sigma^2}{2}\Delta_\alpha''(z)$, using the same argument used to establish Proposition 1 (ii), it follows that z must be a minimum such that $\Delta_\alpha(z) = 0$, and $\Delta_\alpha(q) = 0$ for all $q \leq z$. The last equality implies that $\Delta_\alpha'(q) = 0$ for all $q < z$. Now define $M_\alpha(q) = \alpha\left[\hat{J}(q) + \hat{J}'(q)\right] - [J(q) + J'(q)]$, and note by parts I and II that $M_\alpha(q) \geq 0$ for all q. Also $M_\alpha(q) = 0$ for all $q < z$, and there exists a constant $C_\alpha > 0$ such that $M_\alpha'(q) \leq C_\alpha \cdot M_\alpha(q)$ for all q. By applying the differential form of Grönwall's inequality, it follows that $M_\alpha(q) = 0$ for all q. However this contradicts the facts that $\alpha\hat{J}(0) - J(0) > 0$ and $\alpha\hat{J}'(0) \geq J'(0)$. Therefore there does not exist a z such that $\Delta_\alpha'(z) = 0$, so that $\alpha\hat{J}'(q) > J'(q)$ for all q, which implies that $\hat{a}(q) > a(q)$ for all q.

To show that $\hat{J}(q) > J(q)$ for all q, I use the same approach as above. First note that if there exists a $\hat{z} < 0$ such that $\Delta_1(\hat{z}) = 0$, then $\Delta_1(q) = 0$ for all $q \leq \hat{z}$. Then by defining $M(q) = \left[\hat{J}(q) + \hat{J}'(q)\right] - [J(q) + J'(q)]$, and by using the fact that $M(q) > 0$ for at least some q, and the differential form of Grönwall's inequality, the desired result follows. The details are omitted.

□

Proof of Theorem 2. This proof is organized in 4 parts.

Proof for (A) under Public Good Allocation:

To begin let us define $D_{n,m}(q) = J_m(q) - J_n(q)$, and note that $D_{n,m}(q)$ is smooth, and $D_{n,m}(0) = \lim_{q\to-\infty} D_{n,m}(q) = 0$. Therefore, either $D_{n,m}(\cdot) \equiv 0$, or it has an interior extreme point. Suppose the former is true. Then $D_{n,m}(\cdot) \equiv D'_{n,m}(\cdot) \equiv D''_{n,m}(\cdot) \equiv 0$ together with (1.4) implies that $f(J'_n(q))\, J'_n(q) = 0$ for all q. However, this contradicts Theorem 1 (ii), so that $D_{n,m}(\cdot)$ must have an interior extreme point, which I denote by z^*. Then $D'_{n,m}(z^*) = 0 \Rightarrow J'_m(z^*) = J'_n(z^*)$, and $D''_{n,m}(z^*) \geq 0$. By using (1.4) we have

$$rD_{n,m}(z^*) = \frac{\sigma^2}{2} D''_{n,m}(z^*) + (m-n)\, f(J'_n(z^*))\, J'_n(z^*) > 0 = rD_{n,m}(0) \ ,$$

which implies that z^* is either a global maximum, or a local extreme point satisfying $D_{n,m}(z^*) \geq 0$. Therefore, $J_m(q) \geq J_n(q)$ (*i.e.*, $D_{n,m}(q) \geq 0$) for all q.

I now show that $D_{n,m}(q)$ is single-peaked. Suppose it is not. Then there must exist a local maximum z^* followed by a local minimum $\bar{z} > z^*$. Clearly, $D_{n,m}(\bar{z}) < D_{n,m}(z^*)$, $D'_{n,m}(\bar{z}) = D'_{n,m}(z^*) = 0$, $D''_{n,m}(\bar{z}) \geq 0 \geq D''_{n,m}(z^*)$, and by Theorem 1 (iii), $J'_n(\bar{z}) > J'_n(z^*)$.

By using (1.4), at $\bar{z}$ we have

$$\begin{aligned} rD_{n,m}(\bar{z}) &= \frac{\sigma^2}{2}D''_{n,m}(\bar{z}) + (m-n) f(J'_m(\bar{z})) J'_m(\bar{z}) \\ &> \frac{\sigma^2}{2}D''_{n,m}(z^*) + (m-n) f(J'_m(z^*)) J'_m(z^*) = rD_{n,m}(z^*) \ , \end{aligned}$$

which contradicts the assumption that z^* is a local maximum and $\bar{z}$ is a local minimum. By noting that $D_{n,m}(\cdot)$ cannot be strictly increasing or strictly decreasing due to the boundary conditions, it follows that $D_{n,m}(\cdot)$ is single-peaked; *i.e.*, there exists a $\Theta_{n,m} \leq 0$ such that $J'_m(q) \geq J'_n(q)$ (because $D'_{n,m}(q) \geq 0$), and consequently $a_m(q) > a_n(q)$, if and only if $q \leq \Theta_{n,m}$.

Proof for (A) under Budget Allocation

Recall that under the public good allocation scheme, we had the boundary condition $D_{n,m}(0) = 0$. This condition is now replaced by $D_{n,m}(0) = \frac{V}{m} - \frac{V}{n} < 0$. Therefore, $D_{n,m}(\cdot)$ is either decreasing, or it has at least one extreme point. Using similar arguments as above, it follows that any extreme point z^* is a global maximum and $D_{n,m}(\cdot)$ may be at most single-peaked. Hence either $D_{n,m}(\cdot)$ is decreasing in which case $\Theta_{n,m} = -\infty$, or there exists an interior $\Theta_{n,m}$ such that $a_m(q) \geq a_n(q)$ if and only if $q \leq \Theta_{n,m}$. The details are omitted.

Proof for (B) under Public Good Allocation:

Note that $c(a) = \frac{a^{p+1}}{p+1}$ implies that $f(x) = x^{1/p}$ and $c(f(x)) = \frac{x^{\frac{p+1}{p}}}{p+1}$. As a result, (1.4) can be written for an $n-$member team as

$$rJ_n(q) = \left(n - \frac{1}{p+1}\right)(J'_n(q))^{\frac{p+1}{p}} + \frac{\sigma^2}{2}J''_n(q) \ . \tag{1.9}$$

To compare the total effort of the team at every state of the project, we need to compare $mf\left(J_m'\left(q\right)\right)$ and $nf\left(J_n'\left(q\right)\right)$, or equivalently $\left(m^pJ_m'\left(q\right)\right)^{1/p}$ and $\left(n^pJ_n'\left(q\right)\right)^{1/p}$. Define $\bar{D}_{n,m}\left(q\right) = m^pJ_m\left(q\right) - n^pJ_n\left(q\right)$, and observe that $\bar{D}_{n,m}'\left(q\right) \geq 0 \iff ma_m\left(q\right) \geq na_n\left(q\right)$. Note that $\bar{D}_{n,m}\left(0\right) = \left(m^p - n^p\right)V > 0$ and $\lim_{q\to-\infty}\bar{D}_{n,m}\left(q\right) = 0$. As a result, either $\bar{D}_{n,m}\left(q\right)$ is increasing for all q, which implies that $ma_m\left(q\right) \geq na_n\left(q\right)$ for all q and hence $\Phi_{n,m} = 0$, or $\bar{D}_{n,m}\left(q\right)$ has an interior extreme point z^*. Suppose the latter is true. Then $\bar{D}_{n,m}'\left(z^*\right) = 0$ implies that $J_m'\left(z^*\right) = \left(\frac{n}{m}\right)^p J_n'\left(z^*\right)$. Multiplying both sides of (1.9) by m^p and n^p for $J_m\left(\cdot\right)$ and $J_n\left(\cdot\right)$, respectively, and subtracting the two quantities yields

$$r\bar{D}_{n,m}\left(z^*\right) = -\frac{n^p}{p+1}\left[\frac{n}{m} - 1\right]\left(J_n'\left(z^*\right)\right)^{\frac{p+1}{p}} + \frac{\sigma^2}{2}\bar{D}_{n,m}''\left(z^*\right),$$

and observe that the first term in the RHS is strictly positive. Now suppose z^* is a global minimum. Then $\bar{D}_{n,m}''\left(z^*\right) \geq 0$, which implies that $\bar{D}_{n,m}\left(z^*\right) > 0$, but this contradicts the facts that $\lim_{q\to-\infty}\bar{D}_{n,m}\left(q\right) = 0$ and z^* is interior. Hence z^* must be a global maximum or a local extreme point satisfying $\bar{D}_{n,m}\left(z^*\right) \geq 0$.

To complete the proof for this case, I now show that $\bar{D}_{n,m}\left(\cdot\right)$ can be at most single-peaked. Suppose that the contrary is true. Then there exists a local maximum z^* followed by a local minimum $\bar{z} > z^*$. Because $\bar{D}_{n,m}'\left(z^*\right) = \bar{D}_{n,m}'\left(\bar{z}\right) = 0$, $\bar{D}_{n,m}''\left(\bar{z}\right) \geq 0 \geq \bar{D}_{n,m}''\left(z^*\right)$, and by Theorem 1 (iii) $J_n'\left(z^*\right) < J_n'\left(\bar{z}\right)$, it follows that $\bar{D}_{n,m}\left(z^*\right) < \bar{D}_{n,m}\left(\bar{z}\right)$. However, this contradicts the facts that z^* is a local maximum and $\bar{z}$ is a local minimum, which implies that $\bar{D}_{n,m}\left(\cdot\right)$ is either strictly increasing in which case $\Phi_{n,m} = 0$, or it has a global interior maximum and no other local extreme points, in which case there exists an interior $\Phi_{n,m}$ such that $m\,a_m\left(q\right) \geq n\,a_n\left(q\right)$ if and only if $q \leq \Phi_{n,m}$.

Proof for (B) under Budget Allocation

The only difference compared to the proof under public good allocation is the boundary condition at 0; *i.e.*, $\bar{D}_{n,m}(0) = m^p J_m(0) - n^p J_n(0) = (m^{p-1} - n^{p-1}) V > 0$ (recall $p \geq 1$). As a result, the same proof applies. Note that if $p = 1$ (*i.e.*, effort costs are quadratic), then $\bar{D}_{n,m}(0) = 0$ and hence $\Phi_{n,m}$ is interior.

□

Proof of Proposition 3. Let us first consider the statement under public good allocation. In the proof for statement (A) of Theorem 2, I showed that $D_{n,n+1}(q) = J_{n+1}(q) - J_n(q) \geq 0$ for all q. This implies that $J_{n+1}(q_0) \geq J_n(q_0)$ for all $q_0 \leq 0$.

Now consider the statement under budget allocation. In the proof for statement (A) of Theorem 2, I showed that $D_{n,n+1}(\cdot) = J_{n+1}(\cdot) - J_n(\cdot)$ is either decreasing, or it has exactly one extreme point which must be a maximum. Moreover, $\lim_{q \to -\infty} D_{n,n+1}(q) = 0$ and $D_{n,n+1}(0) < 0$. This implies that there exists a threshold T_n (may be $-\infty$) such that $J_{n+1}(q_0) \geq J_n(q_0)$ if and only if $q_0 \leq -T_n$, or equivalently if and only if $|q_0| \geq T_n$.

□

Proof of Theorem 3. This proof is organized in 5 parts. I first show that a solution to (1.5) subject to the boundary conditions (1.6) exists. Then I show that properties (i) and (ii) hold. Finally, I show that the solution to the above boundary value problem is unique. The proofs resemble those in Theorem 1 closely.

Part I: Existence of a solution.

First note that $J_i(\cdot)$ depends only on V_i for all i and not on $F(\cdot)$, so for given V_i I can solve $F(\cdot)$ by taking $J_i(\cdot)$ as given for all i. I shall use a similar approach as that used to prove existence for $J_i(\cdot)$. Note that (1.5) and (1.6) can be re-written as

$$F_N''(q) = \frac{2r}{\sigma^2}F_N(q) + \frac{2}{\sigma^2}\left[\sum_{i=1}^{n} f\left(J_i'(q)\right)\right]F_N'(q) \qquad (1.10)$$
$$\text{subject to } F_N(-N) = 0 \text{ and } F_N(0) = F_0\,,$$

where $F_0 = U - \sum_{i=1}^{n} V_i > 0$. Let $h(F_N, F_N')$ denote the RHS of (1.10), and observe that $h(\cdot,\cdot)$ is continuous. Now fix some arbitrary $K > 0$ and define a new function

$$h_K(F_N, F_N') = \max\left\{\min\left\{h(F_N, F_N')\,,\,K\right\},\,-K\right\}\,.$$

Note that $h_K(\cdot,\cdot)$ is continuous and bounded, so that by the Scorza-Dragoni Lemma (see Lemma 4 in Hartman (1960)), there exists a solution to $F_{N,K}'' = h_K\left(F_{N,K}, F_{N,K}'\right)$ $[-N,0]$ subject to $F_{N,K}(-N) = 0$ and $F_{N,K}(0) = F_0$. The next part of the proof involves showing that there exists some $\bar{K}$ such that $h_K\left(F_{N,K}, F_{N,K}'\right) \in \left[-\bar{K}, \bar{K}\right]$ for all K on $[-N,0]$, which will imply that the solution $F_{N,\bar{K}}''(\cdot)$ satisfies (1.10). The final step involves showing that a solution exists when $N \to \infty$, so that a solution to (1.5) subject to (1.6) exists.

By part I of Theorem 1, there exists an $\bar{A}$ such that $|J_i'(q)| \leq \bar{A}$ for all q, and it is straightforward to show that $F_{N,K}(q) \in [0\,, F_0]$ and $F_{N,K}'(q) \geq 0$ for all q. As a result, letting $\Omega = nf\left(\bar{A}\right)$, a bound for $\left|F_{N,K}''(q)\right|$ can be obtained by

$$\left|F_{N,K}''(q)\right| \leq \frac{2r}{\sigma^2}F_0 + \frac{2}{\sigma^2}\Omega F_{N,K}'(q)\,.$$

By noting that $F_N(0) > 0$ and using the mean-value theorem, it follows that there exists a

$z^* \in [-N, 0]$ such that $F_N'(z^*) = \frac{F_0}{N}$. Hence, for all $z \in [-N, 0]$

$$F_0 \;\; > \;\; \left|\int_{z^*}^{z} F_N'(q)\,dq\right| \geq \frac{\sigma^2}{2}\left|\int_{z^*}^{z} F_N'(q)\,\frac{F_N''(q)}{rF_0 + \Omega F_N'(q)}dq\right| \geq \frac{\sigma^2}{2}\left|\int_{0}^{F_N'(z)} \frac{s}{rF_0 + \Omega s}ds\right|,$$

where I let $s = F_N'(q)$ and used that $F_N'(q)\,F_N''(q) = F_N'(q)\,dF_N'(q)$. The fact that $\int_0^\infty \frac{s}{rF_0+\Omega s}ds = \infty$ implies that there exists a $\bar{B} < \infty$ such that $\frac{\sigma^2}{2}\left|\int_0^{\bar{B}} \frac{s}{rF_0+\Omega s}ds\right| = F_0$. This implies that $F_N'(q) \leq \bar{B}$ for all $q \in [-N, 0]$.

Because $\bar{B}$ is independent of both N and K, $F_{N,K}'(q) \in \left[0,\, \bar{B}\right]$ for all $q \in [-N, 0]$, $N \in \mathbb{N}$, and $K > 0$. In addition, we now that $F_{N,K}(q) \in [0\,, F_0]$ for all $q \in [-N, 0]$, $N \in \mathbb{N}$, and $K > 0$. Now let $\bar{K} = \frac{2r}{\sigma^2}F_0 + \frac{2}{\sigma^2}\Omega\bar{B}$, and observe that a solution to $F_{N,\bar{K}}'' = h_{\bar{K}}\left(F_{N,\bar{K}}, F_{N,\bar{K}}'\right)$ subject to $F_{N,\bar{K}}(-N) = 0$ and $F_{N,\bar{K}}(0) = F_0$ exists, and $h_{\bar{K}}\left(F_{N,\bar{K}}(q)\,, F_{N,\bar{K}}'(q)\right) = h\left(F_{N,\bar{K}}(q)\,, F_{N,\bar{K}}'(q)\right)$ for all $q \in [-N, 0]$. Therefore, $F_{N,\bar{K}}(\cdot)$ solves (1.10).

To show that a solution for (1.10) as $N \to \infty$ exists, recall that there exists a constant $\bar{B}$ such that $|F_N'(q)| \leq \bar{B}$ on $[-N, 0]$ for all $N \in \mathbb{N}$. Hence the sequences $\{F_N(\cdot)\}$ and $\{F_N'(\cdot)\}$ are uniformly bounded and equicontinuous on $[-N, 0]$. By applying the Arzela-Ascoli theorem to a sequence of intervals $[-N, 0]$ and letting $N \to \infty$, it follows that the system of ODE defined by (1.5) subject to (1.6) has at least one solution.

Part II: $F(q) > 0$ for all q.

First note that $\lim_{q\to-\infty} F(q) = 0$ and $F(0) > 0$. Let $z^* = \arg\min_{q\leq 0}\{F(q)\}$. Clearly, $z^* < 0$. If $z^* = -\infty$, then together with the fact that $\lim_{q\to-\infty} F(q) = 0$, this implies that $F(q) > 0$ for all q, which proves the desired statement. So suppose that z^* is interior. Then $F'(z^*) = 0$ and $F''(z^*) \geq 0$, which using (1.5) implies that $F(z^*) \geq 0$. Because $\lim_{q\to-\infty} F(q) = 0$, it follows that $F(z^*) = 0$. Now suppose that $F(q) \neq 0$ for at least some

q. Then there exists some $\bar{z}$ such that $F'(\bar{z}) = 0$, which using (1.5) implies that $rF(\bar{z}) = \frac{\sigma^2}{2} F''(\bar{z})$. By noting that any maximum must satisfy $F''(\bar{z}) \leq 0 \Longrightarrow F(\bar{z}) \leq 0$, while any minimum must satisfy $F''(\bar{z}) \geq 0 \Longrightarrow F(\bar{z}) \geq 0$, it follows that if there exists a z^* such that $F(z^*) = 0$, then $F(q) = 0$ and $F'(q) = 0$ for all $q < z^*$. By applying the differential form of Grönwall's inequality to $|F(q)| + |F'(q)|$ and using that $|F''(q)| \leq \frac{2r}{\sigma^2} |F(q)| + \frac{2nf(\bar{A})}{\sigma^2} |F'(q)|$, it follows that $F(q) = 0$ for all q. However this contradicts the fact that $F(0) > 0$. Hence $F(\cdot)$ cannot have an interior minimum, and there cannot exist an interior z^* such that $F(z^*) = 0$. Hence $F(q) > 0$ for all q.

Part III: $F'(q) > 0$ for all q.

Because $F(\cdot)$ is continuous and $\lim_{q \to -\infty} F(q) = 0 < F(0)$, there exists a $-\infty < \Lambda < 0$ such that $F(\Lambda) < F(0)$, and by the mean-value theorem, there exists a $z^* \in (\Lambda, 0)$ such that $F'(z^*) = \frac{F(0) - F(\Lambda)}{-\Lambda} > 0$. Suppose that there exists a z^{**} such that $F'(z^{**}) \leq 0$. Then by the intermediate value theorem, there exists a $\bar{z}$ between z^* and z^{**} such that $F'(\bar{z}) = 0$. Using (1.5) and the fact that $F(q) > 0$ for all q, it follows that $rF(\bar{z}) = \frac{\sigma^2}{2} F''(\bar{z}) > 0$; *i.e.*, $\bar{z}$ is a minimum. Because $\bar{z}$ is interior, $\lim_{q \to -\infty} F(q) = 0$, and $F(\bar{z}) > 0$, there exists an interior local maximum $\hat{z} < \bar{z}$, so that $F'(\hat{z}) = 0$ and $F''(\hat{z}) \leq 0$. Using (1.5), it follows that $F(\hat{z}) \leq 0$, which contradicts the fact that $F(q) > 0$ for all q. Therefore, $F'(q) > 0$ for all q.

Part IV: $F(q)$ is infinitely differentiable on $(-\infty, 0]$.

By noting that $\lim_{q \to -\infty} F(q) = \lim_{q \to -\infty} F'(q) = 0$, and by twice integrating both sides of (1.5) over the interval $(-\infty, q]$, we have that

$$F(q) = \int_{-\infty}^{q} \int_{-\infty}^{y} \frac{2r}{\sigma^2} F(z) + \frac{2}{\sigma^2} \left[\sum_{i=1}^{n} f(J_i'(z)) \right] F'(z) \, dz \, dy \, .$$

Recall that $f(x) = x^{1/p}$ and $J_i'(q) > 0$ for all q. Since a solution $F(\cdot)$ satisfying (1.5) subject to the boundary conditions (1.6) exists, $F(q)$ and $F'(q)$ are continuous. As a result, the function under the integral is continuous and infinitely differentiable in $F(z)$, $F'(z)$ and $J_i'(z)$ for all i. By noting that $F(q)$ is differentiable twice more than the function under the integral, using Theorem 1 (iv), and proceeding by induction, property (iii) is proven.

Part V: Uniqueness of a solution.

Because $F(\cdot)$ is a function of $J_i'(\cdot)$ for all i, and Theorem 1 established that $J_i(\cdot)$ is unique if the team comprises of n symmetric, or 2 asymmetric agents, I focus only on these cases only. Suppose that there exist two solutions that solve (1.5) subject to the initial conditions (1.6), denoted by $F_1(\cdot)$ and $F_2(\cdot)$, respectively. Let $\Delta F(q) = F_1(q) - F_2(q)$, and note that $\Delta F(0) = \lim_{q \to -\infty} \Delta F(q) = 0$, and $\Delta F(\cdot)$ is smooth. Also observe that either $\Delta F(\cdot) \equiv 0$, or $\Delta F(\cdot)$ has a global extreme point. Suppose the latter is true and letting z^* be such extreme point, we have that $\Delta F'(z^*) = 0$. Using (1.5) and the facts that $\Delta F''(z^*) \geq 0$ if z^* is a minimum and $\Delta F''(z^*) \leq 0$ if z^* is a maximum, it follows that $\Delta F(q) = 0$ for all q. Hence $F_1(\cdot) \equiv F_2(\cdot)$ and the proof is complete.

□

Proof of Theorem 4. To prove this result, first fix a set of arbitrary milestones $Q_1 < .. < Q_K = 0$ where K is arbitrary but finite, and assume that the manager allocates budget $w_k > 0$ for compensating the agents upon reaching milestone k for the first time. Now consider the following compensation schemes. Let $B = \sum_{k=1}^{K} w_k$. Under *scheme (a)*, each agent is paid $\frac{B}{n}$ upon completion of the project and receives no intermediate compensation while the project is in progress. Under *scheme (b)*, each agent is paid $\frac{w_k}{n\mathbb{E}_{\tau_k}[e^{r\tau_k} \mid Q_i]}$ when q_t hits Q_k for the first time, where τ_k denotes the random time to completion given that the current state of the project is Q_k. I shall show that the manager is always better off using scheme (a) relative to scheme (b).

Some remarks are in order. First, note that scheme (b) ensures that the expected total cost for compensating each agent equals $\frac{B}{n}$ to facilitate comparison between the two schemes. Second, observe that while the expected total cost for compensating the agents is the same under the two schemes, the associated variance is zero under scheme (a), while it is strictly positive under scheme (b) due to the stochastic evolution of the project. Therefore, if the manager is credit constrained or ambiguity / risk averse, then scheme (a) is favored even more. Third, since the manager values the project at U, without loss of generality, I can restrict attention to allocations $\{w_k\}_{k=1}^{K}$ such that $\sum_{k=1}^{K} w_k = B < U$.

This proof is organized in 3 parts. In **part I**, I introduce the necessary functions (*i.e.*, ODEs) that will be necessary for the proof. In **part II**, I show that each agent exerts higher effort under scheme (a) relative to scheme (b). Finally, in **part III**, I show that the manager's expected discounted profit is higher under scheme (a) relative to scheme (b) for any choice of Q_k's and w_k's.

Part I: To begin, I introduce the expected discounted payoff and discount rate functions (*i.e.*, ODEs) that will be necessary for the proof. Under *scheme (a)*, given the current state q, each agent's expected discounted payoff satisfies

$$rJ(q) = -c(f(J'(q))) + nf(J'(q))\,J'(q) + \frac{\sigma^2}{2}J''(q) \;\text{ subject to }\; \lim_{q\to-\infty} J(q) = 0 \text{ and } J(0) = \frac{B}{n}.$$

On the other hand, under *scheme (b)*, given the current state q and that $k-1$ milestones have been reached, each agent's expected discounted payoff, which is denoted by $J_k(q)$, satisfies

$$rJ_k(q) = -c(f(J_k'(q))) + nf(J_k'(q))\,J_k'(q) + \frac{\sigma^2}{2}J_k''(q) \quad \text{on } (-\infty, Q_k]$$

subject to

$$\lim_{q\to-\infty} J_k(q) = 0 \text{ and } J_k(Q_k) = \frac{w_k}{n\mathbb{E}_{\tau_k}\left[e^{r\tau_k} \,|\, Q_k\right]} + J_{k+1}(Q_k)\ ,$$

where $J_{K+1}(Q_K) = 0$.[38] The second boundary condition states that upon reaching milestone Q_k for the first time, each agent is paid $\frac{w_k}{n\mathbb{E}_{\tau_k}[e^{r\tau_k} \,|\, Q_k]}$, and he receives the continuation value $J_{k+1}(Q_k)$ from future progress. Eventually upon reaching the K^{th} milestone, the project is completed so that each agent is paid $\frac{w_K}{n}$, and receives no continuation value. Note that due to the stochastic evolution of the project, even after the k^{th} milestone has been reached for the fist time, the state of the project may drift below Q_k. Therefore, the first boundary condition ensures that as $q \to -\infty$, the expected time until the project is completed so that each agent collects his reward diverges to ∞, which together with the fact that $r > 0$, implies that his expected discounted payoff asymptotes to 0. Using the same approach as used in Theorem 1, it is straightforward to show that for each k, $J_k(\cdot)$ exists, it is unique, smooth, strictly positive, strictly increasing and strictly convex on its domain.

Next, let us denote the expected *discount rate* until the project is completed under scheme (a), given the current state q, by $T(q) = \mathbb{E}_\tau\left[e^{-r\tau} \,|\, q\right]$. Using the same approach as used to derive the manager's HJB equation, it follows that

$$rT(q) = nf(J'(q))\,T'(q) + \frac{\sigma^2}{2}T''(q) \text{ subject to } \lim_{q\to-\infty} T(q) = 0 \text{ and } T(0) = 1\,.$$

The first boundary condition states that as $q \to -\infty$, the expected time until the project is completed diverges to ∞, so that $\lim_{q\to-\infty} T(q) = 0$. On the other hand, when the project is completed so that $q = 0$, then $\tau = 0$ with probability 1, which implies that $T(0) = 1$.

[38] Since this proof considers a fixed team size n, we use to subscript k to denote that $k-1$ milestones have been reached.

Next, let us consider scheme (b). Similarly, we denote the expected *discount rate* until the project is completed, given the current state q and that $k-1$ milestones have been reached, by $T_k(q) = \mathbb{E}_{\tau_k}\left[e^{-r\tau_k} \,|\, q\right]$. Then, it follows that

$$rT_k(q) = nf\left(J_k'(q)\right) T_k'(q) + \frac{\sigma^2}{2} T_k''(q) \quad \text{on } (-\infty, Q_k]$$

subject to

$$\lim_{q \to -\infty} T_k(q) = 0 \ , \ T_k(Q_k) = T_{k+1}(Q_k) \text{ for all } k \leq n \, ,$$

where $T_{K+1}(Q_K) = 1$. The first boundary condition has the same interpretation as above. The second boundary condition ensures value matching; *i.e.*, that upon reaching milestone k for the first time, $T_k(Q_k) = T_{k+1}(Q_k)$. Using the same approach as used in Theorem 3, it is straightforward to show that $T(\cdot)$ and for each k, $T_k(\cdot)$ exists, it is unique, smooth, strictly positive, and strictly increasing on its domain.

Note that by Jensen's inequality, $\frac{1}{\mathbb{E}_{\tau_k}[e^{r\tau_k}]} \leq \mathbb{E}_{\tau_k}\left[e^{-r\tau_k}\right]$.[39] Therefore, using this inequality, and the second boundary condition for $J_k(\cdot)$, it follows that $J_k(Q_k) \leq \frac{w_k}{n} T_k(Q_k) + J_{k+1}(Q_k)$.

Part II: The next step of the proof is to show that for any k, $J(Q_k) \geq J_k(Q_k)$, and as a consequence of Proposition 1 (i), $J'(q) \geq J_k'(q)$ for all $q \leq Q_k$. This will imply that agents exert higher effort under scheme (a) at every state of the project. To proceed, let us define $\Delta_k(q) = J(q) - J_k(q) - \frac{1}{n}\left(\sum_{i=1}^{k-1} w_i\right) T_k(q)$ on $(-\infty, Q_k]$ for all k, and note that $\lim_{q \to -\infty} \Delta_k(q) = 0$ and $\Delta_k(\cdot)$ is smooth.

[39]Because $e^{\pm rt}$ is convex, it follows that $e^{r\mathbb{E}\tau_k} \leq \mathbb{E}\left[e^{r\tau_k}\right]$ and $e^{-r\mathbb{E}\tau_k} \leq \mathbb{E}\left[e^{-r\tau_k}\right]$. The second inequality can be re-written as $\frac{1}{\mathbb{E}\left[e^{-r\tau_k}\right]} \leq e^{r\mathbb{E}\tau_k}$, so that $\frac{1}{\mathbb{E}_{\tau_k}[e^{r\tau_k}]} \leq \mathbb{E}_{\tau_k}\left[e^{-r\tau_k}\right]$.

First, I consider on the case in which $k = K$, and then I proceed by backward induction. Noting that $\Delta_K(Q_K) = 0$ (where $Q_K = 0$), either $\Delta_K(\cdot) \equiv 0$ on $(-\infty, Q_K]$, or $\Delta_K(\cdot)$ has some interior global extreme point z. If the former is true, then $\Delta_K(q) = 0$ for all $q \le Q_K$, so that $J(Q_K) \ge J_K(Q_K)$. Now suppose that the latter is true. Then $\Delta'_K(z) = 0$ so that

$$\begin{aligned} r\Delta_K(z) &= -c(f(J'(z))) + nf(J'(z))J'(z) + c(f(J'_K(z))) - nf(J'_K(z))J'_K(z) \\ &\quad - \left(\sum_{i=1}^{m-1} w_i\right) f(J'_K(z))T'_K(z) + \frac{\sigma^2}{2}\Delta''_K(z) . \end{aligned}$$

Because $\Delta'_K(z) = 0$ implies that $\left(\sum_{i=1}^{k-1} w_i\right) T'_K(z) = n[J'(z) - J'_K(z)]$, the above equation can be re-written as

$$\begin{aligned} r\Delta_K(z) &= c(f(J'_K(z))) - c(f(J'(z))) + nf(J'(z))J'(z) - nf(J'_K(z))J'(z) + \frac{\sigma^2}{2}\Delta''_K(z) \\ &= \left\{\frac{[J'_K(z)]^{\frac{p+1}{p}} - [J'(z)]^{\frac{p+1}{p}}}{p+1} + n[J'(z)]^{\frac{p+1}{p}} - n[J'_K(z)]^{\frac{1}{p}}J'(z)\right\} + \frac{\sigma^2}{2}\Delta''_K(z) . \end{aligned}$$

To show that the term in brackets is strictly positive, note that $J(Q_K) > J_K(Q_K)$ so that $J'(z) > J'_K(z)$ by Proposition 1 (i), and $J'_K(z) > 0$. Therefore, let $x = \frac{J'_K(z)}{J'(z)}$, where $x < 1$, and observe that the term in brackets is non-negative if and only if

$$\begin{aligned} n(p+1)[J'(z)]^{\frac{p+1}{p}} - [J'(z)]^{\frac{p+1}{p}} &\ge n(p+1)[J'_K(z)]^{\frac{1}{p}}J'(z) - [J'_K(z)]^{\frac{p+1}{p}} \\ \Longrightarrow n(p+1) - 1 &\ge n(p+1)x^{\frac{1}{p}} - x^{\frac{p+1}{p}} . \end{aligned}$$

Because the RHS is strictly increasing in x, and it converges to the LHS as $x \to 1$, I conclude that the above inequality holds.

Suppose that z is a global minimum. Then $\Delta''_K(z) \ge 0$ together with the fact that the term in brackets is strictly positive implies that $\Delta_K(z) > 0$. Therefore, any interior global minimum must satisfy $\Delta_K(z) \geq 0$, which in turn implies that $\Delta_K(q) \geq 0$ for all q. As a result, $\Delta_K(Q_{K-1}) \geq 0$ or equivalently $J(Q_{K-1}) \geq J_K(Q_{K-1}) + \frac{1}{n}\left(\sum_{i=1}^{K-1} w_i\right) T_K(Q_{K-1})$.

Now consider $\Delta_{K-1}(\cdot)$, and note that $\lim_{q\to-\infty} \Delta_{K-1}(q) = 0$. By using the last inequality, that $J_{K-1}(Q_{K-1}) \leq \frac{w_{K-1}}{n} T_{K-1}(Q_{K-1}) + J_K(Q_{K-1})$, and $T_{K-1}(Q_{K-1}) = T_K(Q_{K-1})$, it follows that

$$\Delta_{K-1}(Q_{K-1}) = J(Q_{K-1}) - J_{K-1}(Q_{K-1}) - \frac{1}{n}\left(\sum_{i=1}^{K-2} w_i\right) T_{K-1}(Q_{K-1}) \geq 0\,.$$

Therefore, either $\Delta_{K-1}(\cdot)$ is increasing on $(-\infty, Q_{K-1}]$, or it has some interior global extreme point $z < Q_{K-1}$ such that $\Delta'_{K-1}(z) = 0$. If the former is true, then $\Delta_{K-1}(Q_{K-2}) \geq 0$. If the latter is true, then by applying the same technique as above we can again conclude that $\Delta_{K-1}(Q_{K-2}) \geq 0$.

Proceeding inductively, it follows that for all $k \in \{2, .., K\}$, $\Delta_k(Q_{k-1}) \geq 0$ or equivalently $J(Q_{k-1}) \geq J_k(Q_{k-1}) + \frac{1}{n}\left(\sum_{i=1}^{k-1} w_i\right) T_k(Q_{k-1})$ and using that $J_{k-1}(Q_{k-1}) \leq \frac{w_{k-1}}{n} T_k(Q_{k-1}) + J_k(Q_{k-1})$, it follows that $J(Q_{k-1}) \geq J_{k-1}(Q_{k-1})$. Finally, by using Proposition 1 (i), it follows that for all k, $J'(q) \geq J'_k(q)$ for all $q \leq Q_k$.

Part III: Given a fixed expected budget B, the manager's objective is to maximize $\mathbb{E}_\tau\left[e^{-r\tau} \mid q_0\right]$ or equivalently $T(q_0)$, where τ denotes the completion time of the project, and it depends on the agents' strategies, which themselves depend on the set of milestones $\{Q_k\}_{k=1}^K$ and payments $\{w_k\}_{k=1}^K$. Since $q_0 < Q_1 < .. < Q_K$, it suffices to show that $T(q_0) \geq T_1(q_0)$ in order to conclude that given any arbitrary choice of $\{Q_k, w_k\}_{k=1}^K$, the manager is better off compensating the agents only upon completing the project relative to also rewarding them for reaching intermediate milestones.

Define $D_k(q) = T(q) - T_k(q)$ on $(-\infty, Q_k]$ for all $k \in \{1,..,K\}$, and note that $D_k(\cdot)$ is smooth and $\lim_{q\to-\infty} D_k(q) = 0$. Let us begin with the case in which $k = K$. Note that $D_K(Q_K) = 0$ (where $Q_K = 0$). So either $D_K(\cdot) \equiv 0$ on $(-\infty, Q_K]$, or $D_K(\cdot)$ has an interior global extreme point $\bar{z} < Q_K$. Suppose that $\bar{z}$ is a global minimum. Then $D_K'(\bar{z}) = 0$ so that

$$rD_K(\bar{z}) = n\left[J'(\bar{z}) - J_K'(\bar{z})\right]T'(\bar{z}) + \frac{\sigma^2}{2}D_K''(\bar{z}) \; .$$

Recall that $J'(q) \geq J_k'(q)$ for all $q \leq Q_k$ from part II. Since $\bar{z}$ is assumed to be a minimum, it must be true that $D_K''(\bar{z}) \geq 0$, which implies that that $D_K(\bar{z}) \geq 0$. Therefore, any interior global minimum must satisfy $D_K(\bar{z}) \geq 0$, which implies that $D_K(q) \geq 0$ for all $q \leq Q_K$. As a result, $T(Q_{K-1}) \geq T_K(Q_{K-1}) = T_{K-1}(Q_{K-1})$.

Next, consider $D_{K-1}(\cdot)$, recall that $\lim_{q\to-\infty} D_{K-1}(q) = 0$, and note that the above inequality implies that $D_{K-1}(Q_{K-1}) \geq 0$. By using the same technique as above, it follows that $T(Q_{K-2}) \geq T_{K-1}(Q_{K-2}) = T_{K-2}(Q_{K-2})$, and proceeding inductively we obtain that $D_1(q) \geq 0$ for all $q \leq Q_1$ so that $T(q_0) \geq T_1(q_0)$.

□

Proof of Proposition 4. In preparation, I establish a Lemma that ensures that the single-crossing property of Milgrom and Shannon (1994) is satisfied.

Lemma 1. *Suppose the manager employs n identical agents, each of whom receives $\frac{B}{n}$ upon completion. Then for all $\delta \in (0, U - B)$, there exists a threshold T_δ such that she is better off increasing each agent's reward by $\frac{\delta}{n}$ so that each agent receives $\frac{B+\delta}{n}$ if and only if the length of the project $|q_0| \geq T_\delta$.*

Proof of Lemma 1. Consider 2 teams each comprising of n symmetric agents. Upon completion of the project, each member of the first team receives a reward $\frac{B}{n}$, while each member of the second team receives a reward $\frac{B+\delta}{n}$, where $\delta > 0$. Let us denote each agent's expected discounted payoff and equilibrium effort level of the two teams given q by $\{J_0(q), a_0(q)\}$ and $\{J_\delta(q), a_\delta(q)\}$, respectively. From Proposition 1 (i) we know that $a_\delta(q) > a_0(q)$ for all q; *i.e.*, each agent's effort level is strictly increasing in his compensation. Abusing notation, let us denote the manager's expected discounted profit given q for the two cases by $F_B(q)$ and $F_{B+\delta}(q)$, respectively. Now let $\Delta_V(\cdot) = F_B(\cdot) - F_{B+\delta}(\cdot)$, and observe that $\lim_{q\to-\infty}\Delta_V(q) = 0 < \delta = \Delta_V(0)$. Because $\Delta_V(\cdot)$ is smooth, it is either increasing on $(-\infty, 0]$, or it has an interior global extreme point. Suppose the latter is true and denote that extreme point by $\bar{z}$. By using (1.5), it follows that

$$r\Delta_V(\bar{z}) = n\left[a_B(\bar{z}) - a_{B+\delta}(\bar{z})\right] F_B'(\bar{z}) + \frac{\sigma^2}{2}\Delta_V''(\bar{z}) \ .$$

Because $F_B'(\bar{z}) > 0$, $a_B(\bar{z}) < a_{B+\delta}(\bar{z})$, $\Delta_V(0) > 0$, and $\bar{z}$ is interior, it follows that $\bar{z}$ must be a global minimum. By noting that any local maximum $\hat{z}$ must satisfy $\Delta_V(\hat{z}) \le 0$, it follows that $\Delta_V(\cdot)$ is either increasing on $(-\infty, 0]$, or it crosses 0 exactly once from below. Therefore there exists a T_δ such that $\Delta_V(q_0) \le 0$ if and only if $q_0 \le -T_\delta$, or equivalently, the manager is better off increasing each agent's reward by $\frac{\delta}{n}$ if and only if $|q_0| \ge T_\delta$. By noting that $T_\delta = -\infty$ if $\Delta_V(\cdot)$ is increasing on $(-\infty, 0]$, the proof is complete.

□

Other things equal, the manager chooses her budget $B \in [0, U]$ to maximize her expected discounted profit at q_0; *i.e.*, she chooses $B(|q_0|) = \arg\max_{B\in[0,U]}\{F_n(q_0; B)\}$. By noting that the necessary conditions for the Monotonicity Theorem (*i.e.*, Theorem 4) of Milgrom and Shannon (1994) to hold are satisfied, it follows that the manager's optimal budget $B(|q_0|)$ is (weakly) increasing in the project length $|q_0|$.

□

Proof of Proposition 5. Let us denote the manager's expected discounted profit when she employs n (symmetric) agents by $F_n(\cdot)$, and note that $\lim_{q\to-\infty} F_n(q) = 0$ and $F_n(0) = U - V > 0$ for all n. Now let us define $\Delta_{n,m}(\cdot) = F_m(\cdot) - F_n(\cdot)$ and note that $\Delta_{n,m}(\cdot)$ is smooth and $\lim_{q\to-\infty} \Delta_{n,m}(q) = \Delta_{n,m}(0) = 0$. It suffices to show that for all n and m there exists a $T_{n,m} \leq 0$ such that $F_m(q_0) \geq F_n(q_0)$ if and only if $q_0 \leq T_{n,m}$. Note that either $\Delta_{n,m}(\cdot) \equiv 0$, or $\Delta_{n,m}(\cdot)$ has at least one global extreme point. Suppose that the former is true. Then $\Delta_{n,m}(q) = \Delta'_{n,m}(q) = \Delta''_{n,m}(q) = 0$ for all q, together with (1.5), implies that $[A_m(q) - A_n(q)]\, F'_n(q) = 0$ for all q, where $A_n(\cdot) \equiv n a_n(\cdot)$. However, this is a contradiction, because $A_m(q) > A_n(q)$ for at least some q by Theorem 2 (B), and $F'_n(q) > 0$ for all q by Theorem 3 (i). Therefore, $\Delta_{n,m}(\cdot)$ has at least one global extreme point, which I denote by $\bar{z}$. By using that $\Delta'_{n,m}(\bar{z}) = 0$ and (1.5), we have that

$$r\Delta_{n,m}(\bar{z}) = [A_m(\bar{z}) - A_n(\bar{z})]\, F'_n(\bar{z}) + \frac{\sigma^2}{2}\Delta''_{n,m}(\bar{z}) \; .$$

Recall that $F'_n(\bar{z}) > 0$, and from Theorem 2 (B) that for each n and m there exists an (interior) threshold $\Phi_{n,m}$ such that $A_m(q) \geq A_n(q)$ if and only if $q \leq \Phi_{n,m}$. It follows that $\bar{z}$ is a global maximum if $\bar{z} \leq \Phi_{n,m}$, while it is a global minimum if $\bar{z} \geq \Phi_{n,m}$. Next observe that if $\bar{z} \leq \Phi_{n,m}$ then any local minimum must satisfy $\Delta_{n,m}(\bar{z}) \geq 0$, while if $\bar{z} \geq \Phi_{n,m}$ then any local maximum must satisfy $\Delta_{n,m}(\bar{z}) \leq 0$. Therefore either one of the following three cases must be true: (i) $\Delta_{n,m}(\cdot) \geq 0$ on $(-\infty, 0]$, or (ii) $\Delta_{n,m}(\cdot) \leq 0$ on $(-\infty, 0]$, or (iii) $\Delta_{n,m}(\cdot)$ crosses 0 exactly once from above. Therefore there exists a $T_{n,m}$ such that $\Delta_{n,m}(q_0) \geq 0$ if and only if $q_0 \leq -T_{n,m}$, or equivalently the manager is better off employing $m > n$ rather than n agents if and only if $|q_0| \geq T_{n,m}$. By noting that $T_{n,m} = 0$ under case (i), and $T_{n,m} = \infty$ under case (ii), the proof is complete.

□

Proof of Corollary 1. Other things equal, the manager chooses the team size $n \in \mathbb{N}$ to maximize her expected discounted profit at q_0; *i.e.*, she chooses $n(|q_0|) = \arg\max_{n\in\mathbb{N}} \{F_n(q_0)\}$. By noting that the necessary conditions for the Monotonicity Theorem (*i.e.*, Theorem 4) of Milgrom and Shannon (1994) to hold are satisfied, it follows that the optimal team size $n(|q_0|)$ is (weakly) increasing in the project length $|q_0|$.

□

Proof of Proposition 6. This proof is organized in 2 parts.

Part I: *Agents' Problem*

(a) *Formulation of the Agents' Problem*

To begin, fix the manager's budget $B < U$ and the retirement state R. Then denote by $\bar{J}(\cdot)$ each agent's expected discounted payoff when both agents carry out the project to completion together. Let us assume by convention that as soon as the project hits R for the first time, agent 2 will retire, and agent 1 will carry out the remainder of the project on his own. Upon completion of the project, agent i receives V_i, where $V_1 + V_2 = B$. The V_i's will be chosen such that $J_1(R) = J_2(R)$; *i.e.*, the agents have the same expected discounted payoff when the project hits R for the first time. This will ensure that the agents' strategies before agent 2 retires are identical (which makes the analysis tractable). Therefore, denote by $J_R(\cdot)$ the expected discounted payoff of each agent before agent 2 has retired. Note that $\bar{J}(\cdot)$ and $J_i(\cdot)$ are defined on $(-\infty, 0]$, while $J_R(\cdot)$ is defined on $(-\infty, R]$.

Using (1.4), $\bar{J}(\cdot)$ satisfies

$$r\bar{J}(q) = -c\left(f\left(\bar{J}'(q)\right)\right) + 2f\left(\bar{J}'(q)\right)\bar{J}'(q) + \frac{\sigma^2}{2}\bar{J}''(q) \text{ s.t. } \lim_{q\to-\infty}\bar{J}(q) = 0 \text{ and } \bar{J}(0) = \frac{B}{2}.$$

Because the state of the project q can drift back below R after agent 2 has retired, $J_1(\cdot)$ and

$J_2(\cdot)$ satisfy

$$\begin{aligned} rJ_1(q) &= -c(f(J_1'(q))) + f(J_1'(q))J_1'(q) + \frac{\sigma^2}{2}J_1''(q) \text{ s.t. } \lim_{q\to-\infty} J_1(q) = 0 \text{ and } J_1(0) = V_1 \text{ , and} \\ rJ_2(q) &= f(J_1'(q))J_2'(q) + \frac{\sigma^2}{2}J_2''(q) \text{ s.t. } \lim_{q\to-\infty} J_2(q) = 0 \text{ and } J_2(0) = B - V_1 \end{aligned}$$

on $(-\infty, 0]$, respectively. Observe that after agent 2 retires, his expected discounted payoff depends on the effort of agent 1 and on his net payoff V_2 upon completion of the project. By using the same approach as used to prove Proposition 1 (i), it follows that $J_1(\cdot)$ $\{J_2(\cdot)\}$ increases $\{$decreases$\}$ in V_1, and $J_1(\cdot)$ and $J_2(\cdot)$ depend continuously on V_1. Moreover, $J_1(R) > J_2(R) = 0$ if $V_1 = B$, and it is straightforward to show that $J_1(R) < J_2(R)$ if $V_1 = \frac{B}{2}$. Therefore, by the intermediate value theorem, there exists a $V_1 > \frac{B}{2}$ such that $J_1(R) = J_2(R)$.

Next let us consider $J_R(\cdot)$. Using (1.4), $J_R(\cdot)$ satisfies

$$rJ_R(q) = -c(f(J_R'(q))) + 2f(J_R'(q))J_R'(q) + \frac{\sigma^2}{2}J_R''(q) \text{ s.t. } \lim_{q\to-\infty} J_R(q) = 0 \text{ and } J_R(R) = J_1(R) \text{ ,}$$

where the second condition ensures value matching at $q = R$. Because $J_1(\cdot)$ and $J_2(\cdot)$ are *pinned down* independently of $J_R(\cdot)$, the above boundary conditions completely characterize $J_R(\cdot)$.

(b) *Show that $J_R(R) \leq \bar{J}(R)$, and hence $J_R'(q) \leq \bar{J}'(q)$ for all $q \leq R$.*

Let $D(q) = J_1(q) + J_2(q) - 2\bar{J}(q)$, note that $\lim_{q\to-\infty} D(q) = D(0) = 0$, and $D(\cdot)$ is smooth. Therefore either $D(\cdot) \equiv 0$ on $(-\infty, 0]$, or $D(\cdot)$ has at least one interior extreme point. Suppose the latter is true, and let us denote this extreme point by $\hat{z}$. Then $D'(\hat{z}) = 0$

so that

$$\begin{aligned} rD(\hat{z}) &= -c\left(f\left(J_1'(\hat{z})\right)\right) + 2c\left(f\left(\bar{J}'(\hat{z})\right)\right) + 2\left[f\left(J_1'(\hat{z})\right) - 2f\left(\bar{J}'(\hat{z})\right)\right]\bar{J}'(\hat{z}) + \frac{\sigma^2}{2}D''(\hat{z}) \\ \Rightarrow rD(\hat{z}) &= -\frac{1}{2}\left\{2\left[\bar{J}'(\hat{z})\right]^2 + \left[J_1'(\hat{z}) - 2\bar{J}'(\hat{z})\right]^2\right\} + \frac{\sigma^2}{2}D''(\hat{z})\,. \end{aligned}$$

Suppose that $\hat{z}$ is a maximum. Then $D''(\hat{z}) \leq 0$, and because the first term in the RHS is strictly negative, it follows that $D(\hat{z}) < 0$. This implies that any local maximum $\hat{z}$ must satisfy $D(\hat{z}) \leq 0$, which leads me to conclude that $D(q) \leq 0$ for all q. Moreover, because the inequality is strict, note that it cannot be case that $D(\cdot) \equiv 0$ on $(-\infty, 0]$. Because $J_R(R) = J_1(R) = J_2(R)$, the result implies that $J_R(R) \leq \bar{J}(R)$. Finally, by applying Proposition 1 (i), it follows that $J_R'(q) \leq \bar{J}'(q)$ for all $q \leq R$.

Part II: *Manager's Problem*

(a) *Formulation of the Manager's Problem*

To begin, denote by $\bar{F}(\cdot)$ the manager's expected discounted profit when both agents carry out the project to completion together. Denote by $F_1(\cdot)$ the manager's expected discounted profit when one agent carries out the project alone (*i.e.*, after agent 2 has retired). Denote by $F_R(\cdot)$ the manager's expected discounted profit taking into account that agent 2 will retire at the first time that the state of the project hits R. Note that $\bar{F}(\cdot)$ and $F_1(\cdot)$ are defined on $(-\infty, 0]$, while $F_R(\cdot)$ is defined on $(-\infty, R]$. Using (1.5), $\bar{F}(\cdot)$ and $F_1(\cdot)$ satisfy

$$\begin{aligned} r\bar{F}(q) &= 2f\left(\bar{J}'(q)\right)\bar{F}'(q) + \frac{\sigma^2}{2}\bar{F}''(q) \text{ s.t. } \lim_{q\to-\infty}\bar{F}(q) = 0 \text{ and } \bar{F}(0) = U - B\text{ , and} \\ rF_1(q) &= f\left(J_1'(q)\right)F_1'(q) + \frac{\sigma^2}{2}F_1''(q) \text{ s.t. } \lim_{q\to-\infty}F_1(q) = 0 \text{ and } F_1(0) = U - B\,, \end{aligned}$$

respectively. Finally, the manager's expected discounted profit before one agent is retired satisfies

$$rF_R(q) = 2f\left(J_R'(q)\right)F_R'(q) + \frac{\sigma^2}{2}F_R''(q) \text{ s.t. } \lim_{q\to-\infty} F_R(q) = 0 \text{ and } F_R(R) = F_1(R) ,$$

where the second condition ensures value matching at $q = R$. Because $F_1(\cdot)$ is determined independently of $F_R(\cdot)$, these boundary conditions completely characterize $F_R(\cdot)$.

(b) *Show that* there exists a $\Theta_R \leq R$ *such that* $F_R(q_0) \geq \bar{F}(q_0)$ *if and only if* $\Theta_R \leq q_0 < R$. First, let $\Delta_1(q) = F_1(q) - \bar{F}(q)$, and note that $\lim_{q\to-\infty}\Delta_1(q) = \Delta_1(0) = 0$, and that $\Delta_1(\cdot)$ is smooth. As a result, either $\Delta_1(\cdot) \equiv 0$ on $(-\infty, 0]$, or it has at least one interior extreme point. Suppose that the latter is true, and let us denote such extreme point by z^*. Then $\Delta_1'(z^*) = 0$, which implies that

$$r\Delta_1(z^*) = \left[f\left(J_1'(z^*)\right) - 2f\left(\bar{J}'(z^*)\right)\right]\bar{F}'(z^*) + \frac{\sigma^2}{2}\Delta_1''(z^*) .$$

It is straightforward to prove a result analogous to Theorem 2 (B): that there exists a threshold Φ such that $f\left(J_1'(z^*)\right) \leq 2f\left(\bar{J}'(z^*)\right)$ if and only if $z^* \leq \Phi$. As a result $\Delta_1(z^*) \leq 0$ if $z^* \leq \Phi$, while $\Delta_1(z^*) \geq 0$ if $z^* \geq \Phi$. It follows that $\Delta_1(\cdot)$ crosses 0 at most once from below.

Next, define $\Delta_R(q) = F_R(q) - \bar{F}(q)$ on $(-\infty, R]$. Note that $\lim_{q\to-\infty}\Delta_R(q) = 0$, $\Delta_R(R) = \Delta_1(R)$, and $\Delta_R(\cdot)$ is smooth, where the second equality follows from the value matching condition $F_R(R) = F_1(R)$. Because $\Delta_1(\cdot)$ crosses 0 at most once from below, depending on the choice of the retirement point R, it may be the case that $\Delta_1(R) \lesseqgtr 0$.

Suppose $\Delta_1(R) \geq 0$. Then either $\Delta_R(\cdot)$ increases in $(-\infty, R]$, or it has at least one interior extreme point. Suppose the latter is true, and let us denote such extreme point by $\bar{z}$. Then

$\Delta_R'(\bar{z}) = 0$ implies that

$$r\Delta_R(\bar{z}) = 2\left[f\left(J_R'(\bar{z})\right) - f\left(\bar{J}'(\bar{z})\right)\right]\bar{F}'(\bar{z}) + \frac{\sigma^2}{2}\Delta_R''(\bar{z}) \, .$$

Recall from part I (c) of this proof that $J_R'(q) \leq \bar{J}'(q)$ for all $q \leq R$, which implies that $f\left(J_R'(\bar{z})\right) \leq f\left(\bar{J}'(\bar{z})\right)$. It follows that $\bar{z}$ must satisfy $\Delta_R(\bar{z}) \leq 0$. Because $\Delta_1(R) \geq 0$, it follows that there exists a threshold $\Theta_R < R$ such that $\Delta_1(q_0) \geq 0$ if and only if $\Theta_R \leq q_0 < R$. If $\Delta_1(R) < 0$, the same analysis yields that $\Delta_R(\cdot)$ decreases in $(-\infty, R]$, and hence $\Delta_1(q_0) \leq 0$ for all $q_0 \leq R$.

(c) *Conclusion of the Proof*

I have shown that as long as R is chosen such that $F_1(R) \geq \bar{F}(R)$ (so that $\Delta_1(R) \geq 0$), there exists a threshold $\Theta_R < R$ such that $F_R(q_0) \geq \bar{F}(q_0)$ for all $q_0 \in [\Theta_R, R]$. The last relationship implies that as long as the the length of the project $|R| < |q_0| \leq |\Theta_R|$, the manager is better off implementing the proposed retirement scheme relative to allowing both agents to carry out the project to completion together. Finally, the requirement that R is chosen such that $F_1(R) \geq \bar{F}(R)$ is equivalent to the requirement that if the project length were $|q_0| = |R|$, and the manager did not use a dynamic team size management scheme, she would be better off employing one instead of two agents.

□

Proof of Proposition 7. In preparation, I first establish two Lemmas.

Lemma 2. *Consider a project undertaken by two identical agents who differ only in their final rewards such that $V_1 > V_2$. Also, suppose that effort costs are quadratic. Then $\frac{d}{dq}\left[a_1(q) - a_2(q)\right] \geq 0$ for all q.*

Proof of Lemma 2. Observe that when effort costs are quadratic, then $a_i(q) = J_i'(q)$, so it suffices to show that $D_J'(\cdot) = J_1'(\cdot) - J_2'(\cdot)$ is (weakly) increasing on $(-\infty, 0]$. First note that $\lim_{q\to-\infty} D_J'(q) = 0$, and from Proposition 1 (i), it follows that $D_J'(q) > 0$ for all q. Fix $z \leq 0$, and let $\bar{z} = \arg\max\{D_J'(q) : q \leq z\}$. Clearly, $\bar{z} > -\infty$. Suppose that $\bar{z}$ is interior. Then $D_J''(\bar{z}) = 0$ and $D_J'''(\bar{z}) \leq 0$, and by using (1.8) we have that $rD_J'(\bar{z}) = \frac{\sigma^2}{2} D_J'''(\bar{z}) \leq 0$. However, this contradicts the fact that $D_J'(\bar{z}) > 0$, which implies that $\bar{z} = z$. Since z was chosen arbitrarily, this implies that $D_J'(\cdot)$ is (weakly) increasing on $(-\infty, 0]$.

□

Lemma 3. *Consider a project undertaken by two identical agents, and suppose that effort costs are quadratic. Consider the following two scenarios for the agents' compensation: (i)* $V_1 = V_2 = \frac{B}{2}$*, and (ii)* $V_1 = \frac{B}{2} + \epsilon > \frac{B}{2} - \epsilon = V_2$*. Then for all* $\epsilon \in \left(0, \frac{B}{2}\right]$ *there exists a* $\Theta_\epsilon < 0$ *such that the aggregate effort of the team is larger under asymmetric rewards (*i.e., *under scenario (ii)) if and only if* $q \geq \Theta_\epsilon$*.*

Proof of Lemma 3. First let us denote the expected discounted payoff function of the agents under asymmetric compensation by $J_1(q)$ and $J_2(q)$, respectively, and let us denote the expected discounted payoff function of the agents under symmetric compensation by $J_S(q)$. Because effort costs are quadratic, $a_i(q) = J_i'(q)$. Observe that we are interested in comparing $2a_S(q)$ and $a_1(q) + a_2(q)$, or equivalently $2J_S'(q)$ and $J_1'(q) + J_2'(q)$ on $(-\infty, 0]$. Let us define $M(q) = 2J_S(q) - J_1(q) - J_2(q)$. By noting that $\lim_{q\to-\infty} M(q) = M(0) = 0$ and $M(\cdot)$ is smooth on $(-\infty, 0]$, it follows that either $M(\cdot) \equiv 0$, or it has at least one interior global extreme point. Suppose the latter is true and let us denote that extreme point by z^*. By using (1.4), and the facts that $f(x) = x$ and $c(f(x)) = \frac{x^2}{2}$, it follows that

$$rM(z^*) = \frac{1}{2}\left[6\left(J_S'(z^*)\right)^2 - 2\left(J_1'(z^*) + J_2'(z^*)\right)^2 + \left(J_1'(z^*)\right)^2 - \left(J_2'(z^*)\right)^2\right] + \frac{\sigma^2}{2} M''(z^*) \ .$$

Because z^* is an extreme point, $M'(z^*) = 0$ implies that $J_S'(z^*) = \frac{J_1(z^*)+J_2(z^*)}{2}$. By substituting into the above equality and simplifying the terms, we have

$$rM(z^*) \;=\; \frac{1}{4}\left[J_1'(z^*) - J_2'(z^*)\right]^2 + \frac{\sigma^2}{2}M''(z^*) \,.$$

Suppose that z^* is a global interior minimum. Then the facts that $M''(z^*) \geq 0$ and $J_1'(z^*) > J_2'(z^*)$ (which follows from Proposition 1 (i)), imply that $M(z^*) > 0$. However, this contradicts the fact that $M(0) = 0$, which implies that z^* must be a maximum and $M(q) \geq 0$ for all q. Moreover, because $J_1(z^*) > J_2(z^*)$, note that it cannot be the case that $M(\cdot) \equiv 0$.

Now suppose that $M(\cdot)$ has more than one extreme points. Then there must exist a local maximum z^* followed by a local minimum $\bar{z} > z^*$. This implies that $M''(z^*) \leq 0 \leq M''(\bar{z})$, and by Lemma 2, $0 \leq J_1'(z^*) - J_2'(z^*) \leq J_1'(\bar{z}) - J_2'(\bar{z})$. These equalities imply that $M(z^*) \leq M(\bar{z})$, which contradicts the assumption that z^* is a maximum and $\bar{z}$ is a minimum. Hence $M(\cdot)$ has a global maximum on $(-\infty, 0]$ and no other local extreme points. Therefore there exists a $\Theta_\epsilon < 0$ such that $M'(q) \geq 0$ if and only if $q \leq \Theta_\epsilon$.

□

To begin, let us denote the manager's expected discounted profit by $F_0(q)$ and $F_\epsilon(q)$ under the symmetric (*i.e.*, $\left(\frac{B}{2}, \frac{B}{2}\right)$) and the asymmetric (*i.e.*, $\left(\frac{B}{2}+\epsilon, \frac{B}{2}-\epsilon\right)$) compensation scheme, respectively. Moreover, let us denote the expected discounted payoff of each agent by $J_S(\cdot)$, $J_1(\cdot)$, and $J_2(\cdot)$, where the subscripts follow the convention from Lemma 3. Next, let $\Delta_\epsilon(q) = F_0(q) - F_\epsilon(q)$, and observe that $\lim_{q\to-\infty}\Delta_\epsilon(q) = \Delta_\epsilon(0) = 0$. Therefore, either $\Delta_\epsilon(\cdot) \equiv 0$, or $\Delta_\epsilon(\cdot)$ has at least one interior global extreme point. Suppose the latter is true, and let us denote that extreme point by $\bar{z}$. By using (1.5) and the fact that $\Delta_\epsilon'(\bar{z}) = 0$, it follows that

$$r\Delta_\epsilon(\bar{z}) = [2J_S'(\bar{z}) - J_1'(\bar{z}) - J_2'(\bar{z})]\, F_0'(\bar{z}) + \frac{\sigma^2}{2}\Delta_\epsilon''(\bar{z}) \ .$$

From Lemma 2, we know that there exists a threshold Θ_ϵ such that $2J_S'(q) \geq J_1'(q) + J_2'(q)$ if and only if $q \leq \Theta_\epsilon$, and from Theorem 3 (ii) that $F_0'(q) > 0$ for all q. It follows that $\bar{z}$ is a global maximum if $\bar{z} \leq \Theta_\epsilon$, while it is a global minimum if $\bar{z} \geq \Theta_\epsilon$. Moreover, any local extreme point $\bar{z} \leq \Theta_\epsilon$ must satisfy $\Delta_\epsilon(\bar{z}) \geq 0$, while any local extreme point $\bar{z} \geq \Theta_\epsilon$ must satisfy $\Delta_\epsilon(\bar{z}) \leq 0$. Moreover, because $2J_S'(q) > J_1'(q) + J_2'(q)$ for at least some q, and $F_0'(q) > 0$ for all q, it cannot be the case that $\Delta_\epsilon(\cdot) \equiv 0$. Therefore, either one of the following three cases must be true: (i) $\Delta_\epsilon(\cdot) \geq 0$ on $(-\infty, 0]$, (ii) $\Delta_\epsilon(\cdot) \leq 0$ on $(-\infty, 0]$, or (iii) $\Delta_\epsilon(\cdot)$ crosses 0 exactly once from above. Hence, there exists a T_ϵ such that $F_0(q_0) \geq F_\epsilon(q_0)$ if and only if $q_0 \leq -T_\epsilon$, or equivalently if and only if $|q_0| \geq T_\epsilon$.

□

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Chapter 2

Project Design with Limited Commitment and Teams

2.1 Introduction

A key component of a project, such as the development of a new product, is choosing the features that must be included before the decision maker deems the product ready to market. Naturally, which features are to be included must be communicated to the relevant stakeholders. When choosing these features, the decision maker must balance the added value derived from a *bigger* or a more complex project (*i.e.*, one that contains more features) against the additional cost associated with designing and implementing the additional features. Such costs include not only engineering inputs but also the implicit cost associated with delayed cash flow.

Issues regarding technological uncertainty come to the forefront: will the engineering team be able to implement the desired new features, and if so, will the associated cost and delay be acceptable. Stumbling blocks, setbacks, and surprises are almost certain to enter the path of a new product introduction. Consequently, it may not be possible to contract either on various aspects of the new features or on the time when the new product will be introduced, at least not at the early stages of the project. Even the decision maker herself is not yet certain about the exact specifications and appearance of the final output.

Anecdotal evidence from the development of Apple's first generation iPod indicates that Steve Jobs kept changing the requirements of the iPod as it progressed. This suggests that committing to a set of features/requirements early on was infeasible in the development of a new product as innovative as the iPod back in 2001 (Wired Magazine (2004)). Similarly, consider the process of designing a new car. If it were possible to describe in advance what the design must look like for management to give its approval, then there would be far fewer delays as the new car makes its way to production and design would be relatively easy. However, as the final product takes shape, the decision maker can better guide the design team to fulfill her objectives.

What we have in mind about the incontractibility of the project requirements was eloquently posed by Tirole (1999)

> In practice, the parties are unlikely to be able to describe precisely the specifics of an innovation in an ex ante contract, given that the research process is precisely concerned with finding out these specifics, although they are able to describe it ex post.

More generally, such incontractibilities arise in projects that involve significant novelty in quality or design. This also applies to many innovation projects where it is difficult to describe in sufficient detail for the purpose of a contract until the project is close to completion.

We develop a tractable model to study the interaction between a group of individuals who exert costly effort over time to complete a project and a manager who chooses its requirements. When the choice of the project requirements is endogenous, we characterize the Markov Perfect equilibrium and show that it is unique. Then, we investigate how the manager's optimal choice of the requirements depends on her ability to commit earlier or later on. To model the manager's limited ability to commit, we assume that given the current state q of the project, she can commit to any $Q \in [q, q+y]$, where $y \geq 0$ captures her commitment power, and y is common knowledge.[1] Therefore, the manager can commit to a project size $Q^* > y$ only after the agents have made sufficient progress such that $q \geq Q^* - y$.size of the project is now endogenous.

The main result is that the manager's incentives propel her to extend the project as it progresses; for example, by introducing additional requirements. The intuition behind this result is as follows. Because agents are impatient, they incur the cost of effort at the time they exert it, and they get compensated upon completion of the project, in equilibrium, they increase their effort as the project progresses. On the other hand, the manager chooses the project size by trading off the marginal benefit of a larger project against the marginal cost associated with having to wait longer for a larger project to be completed. Of course, due to discounting, a larger project induces a larger opportunity cost of waiting. However, because the agents increase their effort, this marginal cost decreases as the project progresses, while the respective marginal value is independent of the progress made on the project. Because the project size will be chosen such that the two marginal values are equal, it follows that the manager's optimal project size increases as the project progresses. This result has three implications.

First, the project size that the manager will eventually choose decreases in her commitment power.[2] If the manager has sufficiently large commitment power, she will commit to her optimal project size at time 0. On the other hand, if she has *less* commitment power, then

[1]Here, Q is a one-dimensional parameter that captures the project requirements, or equivalently, the project size.

[2]We assume that the manager's commitment power y is given exogenously as it depends on the nature of the project. For example, projects that contain a significant innovation or design component are associated with a relatively small commitment power, since the requirements are difficult to unambiguously define *ex-ante*. On the other hand, in construction projects, the requirements can typically be specified in advance so that they are associated with a relatively large commitment power.

at time 0 she can either commit to a small project, or she can wait to commit until the project is at an advanced stage so that she can commit to her optimal project size. We show that the manager always finds the latter option preferable. However, once such an advanced stage has been reached, her optimal project size is larger than it was at time 0. Consequently, the less commitment power the manager has, the bigger a project she will choose. Furthermore, she will commit to her optimal project size at a later state.[3]

Second, due to the agents' rational expectations, anticipating that the manager will choose a larger project if she has less commitment power, the agents respond by decreasing their effort; this renders her worse off (see Proposition 2). Therefore, if the agents receive a share of the project's worth upon completion (*i.e.*, an equity contract), the manager might delegate the decision rights over the project size to the agents. Intuitively, note that the smaller is the manager's commitment power, the larger a project she will eventually choose, which implies that without delegating, her *ex-ante* discounted profit decreases in her commitment power. On the other hand, the agents would choose a project smaller than is optimal for the manager, but they are time-consistent, which implies that the manager's discounted profit is independent of the agents' commitment power. The upshot is that there exists an interior threshold such that the manager should delegate the decision rights over the project size to the agents unless she has sufficient commitment power (see Proposition 3).

The third implication is related to the organizational culture that the manager should cultivate within the team. Motivated by the concept of *insiders* (who act in the best interest of the team) and *outsiders* (who act in their own best interest) introduced by Akerlof and Kranton (2000), we characterize a continuum of (non-Markovian) Public Perfect equilibria

[3] An additional source of inability to commit to specific requirements is an asymmetry in the bargaining power of the parties involved. For example, if a project is undertaken in-house where the manager can significantly influence the team members' career paths and contracts are typically implicit, the manager is less likely to be able to commit relative to the case in which the project is outsourced and contract is explicit.

where agents choose their effort to maximize a convex combination of their individual and the entire team's discounted payoff. The weight that the agents place on maximizing the team's payoff can be interpreted as the team's *cooperativeness*. In Proposition 4, we examine the degree to which the manager should influence it, for example, by selecting the team members or encouraging interaction among them. Given a fixed project size, a fully cooperative environment where the agents place all the weight on maximizing the entire team's payoff is first-best. However, if the size of the project is endogenous, then Proposition 5 shows that a fully cooperative environment is profit-maximizing only if the manager has sufficient commitment power. Otherwise, the degree of cooperativeness that maximizes her discounted profit is interior, and both her profit and the degree of cooperativeness increase in her commitment power. Intuitively, by cultivating a lower degree of cooperativeness, the manager can mitigate her *ex-post* incentives to extend the project, which are inversely related to her commitment power.

To test the robustness of the main results, we examine four extensions. First, we study synergies, where the team's total effort is greater than the sum of the individual efforts. Second, we consider the case in which each agent receives a fixed lump-sum payment upon completion that it independent of the project size. Third, we consider a scenario in which the interaction between the manager and the agents is persistent in that a new project is initiated as soon as the previous one is completed. Finally, we consider the case in which, in addition to a lump-sum payment upon completion of the project, the agents receive flow payments while the project is ongoing. In all four extensions, we find that (i) the manager has incentives to extend the project as it progresses, (ii) she should delegate the decision rights over the project size to the agents unless she has sufficient commitment power, and (iii) the cooperation level that maximizes her *ex-ante* discounted profit increases in her commitment power.

Related Literature

First and foremost, this paper is related to the moral hazard in teams literature - in particular to the papers that study dynamic contribution games. Admati and Perry (1991) and Marx and Matthews (2000) examine how the incentives to contribute to a public good evolve over time, and establish conditions under which the project is completed. Kessing (2007) shows that in contrast to the case in which the project generates flow payments while it is in progress considered by Fershtman and Nitzan (1991), efforts are strategic complements when the agents receive a payoff only upon completion of the project. Georgiadis (2012) examines how the incentives to contribute to a public good depend on the team composition, and he focuses on how a manager should choose the team composition and the agents' compensation scheme. A feature common to most of the papers in this stream of literature is that the size of the project is given exogenously. However, in new product design, the choice of the objectives of any given project is a central part of the problem. Our contribution to this literature is to endogenize the size of the project, and to examine how this choice depends on who has the decision rights and on the extent of the decision maker's commitment power.

A second strand of related literature is that on incomplete contracting. In particular, our paper is closely related to the papers that study how *ex-ante* contracting limitations generate incentives to renegotiate the initial contract *ex-post* (Grossman and Hart (1986), Hart and Moore (1990), Aghion and Tirole (1994), Tirole (1999), and Al-Najjar, Anderlini and Felli (2006)). A subset of this literature focuses on situations wherein the involved parties have asymmetric information. Here, ratchet effects have been shown to arise in principal-agent models in which the principal learns about the agent's ability over time, and the agent reduces his effort to manipulate the principal's beliefs about his ability (Freixas, Guesnerie and Tirole (1985) and Laffont and Tirole (1988)). In another thread of this strand are papers that consider the case in which the agent is better informed than the principal, or he has better access to valuable information. The common result is that delegating the decision rights to the agent is beneficial as long as the he is sufficiently better informed and the incentive conflict is not too large (Aghion and Tirole (1997) and Dessein (2002)). In our model, however, all parties have full and symmetric information, so that ratchet effects and the incentives to delegate the decision rights to the agents arise purely out of moral hazard.

Finally, this paper is related to the literatures on corporate culture (Kreps (1990)) and social identity (Tajfel and Turner (1979)). The game that we analyze, contains a continuum of non-Markovian equilibria in addition to the unique Markov equilibrium. We couple this fact with the concepts introduced by Kreps (1990) and Akerlof and Kranton (2000) to examine which equilibrium will be played. Moreover, experiments in social identity theory have demonstrated that it is surprisingly easy to influence subjects' behavior as insiders, who act in their group's best interest, or outsiders, who act in their own best interest (Akerlof and Kranton (2005)). Consequently, our result can be employed by a manager who seeks to influence the team's corporate culture (which we term *cooperativeness*) when the choice of the project size is endogenous.

This paper is organized as follows. In Section 2 we introduce the model, and we analyze the agents' as well as the manager's problem. In Section 3 we study the manager's optimal choice of the project requirements as a function of her commitment power, and we examine her option to delegate the decision rights over the requirements to the agents. Section 4 focuses on how the manager's optimal choice of the requirements depends on the agents' cooperation level, and we examine how the manager should influence it. Section 5 concludes. In Appendix A we extend our model to test the robustness of our results. All proofs are provided in Appendix B.

2.2 The Model

A group of n identical agents contracts with a manager to undertake a project. The agents exert (costly) effort over time to complete the project, they receive a lump-sum compensation upon completing the project, and they are protected by limited liability.[4] The manager has the authority to choose the size of the project. A project of size $Q \geq 0$ generates a payoff equal to Q upon completion. This payoff is split between the parties as follows: each agent receives $\frac{\beta Q}{n}$, and the manager receives $(1-\beta)\,Q$.[5] Time $t \in [0,\infty)$ is continuous; all parties are risk neutral and discount time at rate $r > 0$. The project starts at state $q_0 = 0$. At every moment t, each agent observes the state q_t of the project, and exerts costly effort to influence the process

$$dq_t = \left(\sum_{i=1}^{n} a_{i,t} \right) dt \,,$$

where $a_{i,t}$ denotes the effort level of agent i at time t.[6,7] To avoid trivializing the problem, we assume that efforts are not contractible. Each agent's flow cost of exerting effort level a is $\frac{\lambda}{2} a^2$, where $\lambda > 0$, while his outside option is equal to 0. The project is completed at the

[4]For the sake of tractability, we assume that the agents are compensated only upon completing the project. However, in Appendix A.4 we consider the case in which, in addition to a lump-sum payment upon completion of the project, they receive a per unit of time compensation while the project is ongoing. We find that all the main results of this paper continue to hold.

If the agents have unlimited liability, then the manager can achieve first-best by *selling* the project to the agents for a price that makes their participation constraint bind.

[5]We assume that β is independent of Q; otherwise, the assumption that the manager has limited ability to commit to a project size would be violated. However, we defer a detailed justification until after we have formalized what we mean by limited commitment power in Section 3.1.2.

Note that this is essentially an equity contract. In Appendix A.2 we consider the case in which each agent receives a flat payment upon completion of the project that is independent of the project size Q, and we find that such a contract aggravates the manager's commitment problem.

[6]Efforts are perfect substitutes in the base model. In Appendix A.1, we examine the case in which they are complementary, and we show that all results continue to hold.

[7]The assumption that the project progresses deterministically is made for the sake of tractability. Georgiadis (2012) analyzes a similar model in which the project progresses stochastically and the project size is given exogenously. The insights regarding the Markov Perfect equilibrium are essentially identical in the two cases. However, a closed-form characterization of the equilibrium and of the optimal project size cannot be obtained if the project progresses stochastically.

first time τ such that $q_\tau = Q$.

In this Section, we consider differentiable Markov Perfect equilibria (hereafter MPE) such that at any time t, agent i observes the state q_t of the project, and chooses his effort strategy $\{a_{i,s}\}_{s\geq t}$ to maximize his expected discounted payoff while accounting for the effort strategies of the other team members.[8]

In the remainder of this Section, we study the agents' problem, and we determine the manager's discounted profit given a fixed project size Q. We endogenize the choice of Q in Section 3.

2.2.1 Agents' Problem

Given a project of size Q and the current state q_t of the project, agent i's expected discounted payoff function satisfies

$$\Pi_{i,t}\left(q\,;Q\right) = \max_{\{a_{i,s}\}_{s\geq t}} \left[e^{-r(\tau-t)}\frac{\beta Q}{n} - \int_t^\tau e^{-r(s-t)}\frac{\lambda}{2}a_{i,s}^2 ds \mid \{a_{-i,s}\}_{s\geq t}\,,\, Q \right]\,, \tag{2.1}$$

where τ denotes the completion time of the project and it depends on the agents' strategies. Note that the first term captures the agent's net payoff upon completion of the project, while the second term captures his discounted cost of effort for the remaining duration of the project. Because payoffs depend solely on the state of the project (*i.e.*, q) and not on the time t, this problem is stationary; hence the subscript t can be dropped. Using standard arguments (Dixit (1999)), one can derive the Hamilton-Jacobi-Bellman equation for the

[8] Besides the MPE in this Section, there also exist Public Perfect equilibria with history dependent strategies. Such equilibria are characterized in Section 4.

expected discounted payoff function for agent i

$$r\Pi_i\left(q\,;Q\right)=\max_{a_i}\left\{-\frac{\lambda}{2}a_i^2+\left(\sum_{j=1}^{n}a_j\right)\Pi_i'\left(q\,;Q\right)\right\}$$

subject to the boundary conditions

$$\Pi_i\left(q\,;Q\right)\geq 0 \text{ for all } q \quad \text{and} \quad \Pi_i\left(Q\,;Q\right)=\frac{\beta Q}{n}\,.$$

The first boundary condition captures the fact that each agent's discounted payoff must be non-negative because he has the option to exert no effort and incur no effort cost, thus guaranteeing himself a payoff of 0. The second boundary condition states that upon completing the project, each agent receives his reward and exerts no further effort. The following Proposition characterizes the MPE for this game.

Proposition 8. *For any given project size Q, there exists a Markov Perfect equilibrium (MPE) for the game defined by (2.1). This equilibrium is symmetric, and each agent's effort strategy is given by* [9]

$$a\left(q\,;Q\right)=\frac{r}{2n-1}\left[q-C\left(Q\right)\right]^{+}\ \text{, where } C\left(Q\right)=Q-\sqrt{\frac{2\beta Q}{r\lambda}\frac{2n-1}{n}}.$$

In equilibrium, each agent's expected discounted payoff is given by

$$\Pi\left(q\,;Q\right)=\frac{r\lambda}{2}\frac{\left(\left[q-C\left(Q\right)\right]^{+}\right)^{2}}{2n-1}\,.$$

If $Q<\frac{2\beta}{r\lambda}$, then this equilibrium is unique, and the project is completed in finite time. Otherwise, there also exists an equilibrium in which no agent ever exerts any effort and the

[9] To simplify notation, because the equilibrium is symmetric and unique, the subscript i is dropped throughout the remainder of this paper. Moreover, $[\cdot]^{+}=\max\{\cdot,0\}$.

project is never completed.[10]

Observe that the agents exert no effort if $C(Q) \geq 0$, or equivalently if $Q \geq \frac{2\beta}{r\lambda}\frac{2n-1}{n}$, in which case the project is never completed. Intuitively, if the project is too large, then the discounted cost of effort to complete it is larger than the discounted net payoff. As a result, the agents are better off abandoning the project altogether. On the other hand, if $C(Q) < 0$ (*i.e.*, if $Q < \frac{2\beta}{r\lambda}\frac{2n-1}{n}$), then each agent's effort level increases in the state of the project q. Intuitively, this is due to the facts that agents are impatient and they incur the cost of effort at the time it is exerted, while they are compensated only when the project is completed. As a result, the closer the project is to completion, the stronger are their incentives to exert effort.

While the MPE need not be unique, it turns out that the manager will always choose the size of the project such that the equilibrium is unique when the project size Q is endogenous (see Remark 1 in Section 3).

2.2.2 Manager's Problem

We now introduce the manager's problem. Given a project of size Q and the agents' belief $\tilde{Q}$ about the manager's choice of the project size, the manager's discounted profit can be written as $W\left(q\,;Q,\tilde{Q}\right) = \left[e^{-r\tau}\left(1-\beta\right)Q\,|\,Q,\tilde{Q}\right]$, where the project's completion time τ depends on the agents' strategies which in turn depend on the agents' belief $\tilde{Q}$. Note that the manager's discounted payoff depends on the agents' belief about the manager's choice of the project size because this belief influences their effort strategy. However, in equilibrium beliefs must be correct; *i.e.*, $Q = \tilde{Q}$. Using standard arguments, one can derive the HJB

[10]If $Q \geq \frac{2\beta}{r\lambda}\frac{2n-1}{n}$, then the two MPE coincide, and the project is never completed.

equation for the manager's expected discounted profit

$$rW\left(q\,;Q,\tilde{Q}\right)=\left[n\,a\left(q\,;\tilde{Q}\right)\right]W'\left(q\,;Q,\tilde{Q}\right)$$

subject to the boundary conditions

$$W\left(q\,;Q,\tilde{Q}\right)\geq 0 \text{ for all } q \quad\text{and}\quad W\left(Q\,;Q,\tilde{Q}\right)=(1-\beta)\,Q\,.$$

To interpret these boundary conditions, note that manager's discounted profit is non-negative at every state of the project, because she does not incur any cost or disburse any payments to the agents while the project is in progress.[11] On the other hand, she receives her net profit $(1-\beta)\,Q$, and the game ends as soon as the state of the project hits Q for the first time. It is straightforward to show that this ordinary differential equation has the following unique solution

$$W\left(q\,;Q,\tilde{Q}\right)=(1-\beta)\,Q\left(\frac{\left[q-C\left(\tilde{Q}\right)\right]^{+}}{Q-C\left(\tilde{Q}\right)}\right)^{\frac{2n-1}{n}}\,,. \tag{2.2}$$

Note that $(1-\beta)\,Q$ represents the manager's net profit upon completion of the project, while the next term can be interpreted as an *effective discount rate* that captures the completion time of the project, which depends on the agents' strategies characterized in Proposition 1 and their belief about the project size.

2.3 Project Choice and the Commitment Problem

In this Section we endogenize the project size Q. The manager has full decision rights over the choice of the project size, but she may not be able to commit to a specific Q until the

[11]In Appendix A.4 we consider the case in which the manager compensates the agents per unit of time while the project is in progress, and we find that all results continue to hold.

project is sufficiently close to that state. Formally, we assume that given the current state of the project q, the manager can only commit to a project size in the interval $[q,\, q+y]$, where $y \geq 0$ is common knowledge. We think of it as capturing the describability (or verifiability) of the project requirements.

The extreme case $y = \infty$ represents the situation in which the requirements are perfectly describable. Therefore, when $y = \infty$, the manager can (and will) commit to her optimal project size at time 0. On the other hand, if $y = 0$, then the requirements are completely indescribable, and the manager only knows that the project is complete *when she sees it*. In this case, at every moment the manager observes the current state of the project q, and she decides whether it is *good enough* (in which case the size of the project will be $Q = q$), or whether to let the agents continue to work and re-evaluate the completion decision an instant later. Therefore, y can be interpreted as the manager's commitment power, where a larger y indicates greater commitment power.

For example, y is likely to be large in a construction project where the requirements are relatively standardized and easy to define. On the other hand, in a project that involves a significant innovation or quality component, such as the development of Apple's first-generation iPhone, y is likely to be small, because the manager cannot contract on the requirements of the final product until the project is at an advanced stage. Similarly, y is typically small in design-related projects such as automotive design or the commissioning of a sculpture, as the requirements are difficult to describe.

The main result of this paper is that the manager is time-inconsistent with respect to her optimal choice of the project size: she is inclined to introduce additional requirements as the project progresses. The implication of this result is that if she has less commitment power (*i.e.*, if y is smaller), then she will (eventually) select a larger project. The agents, anticipating that she will choose a larger project if she has less commitment power, decrease their effort rendering her (*ex-ante*) worse off. We show that the manager benefits from delegating the decision rights over the project size to the agents unless her commitment power is sufficiently large, in which case the agents will choose a smaller project but their preferences are time-consistent.

2.3.1 Optimal Project Size

To examine the manager's optimal project size, we first consider the case in which she has full commitment power (*i.e.*, $y = \infty$), so that she can commit to any project size before the agents begin to work. Second, we consider the opposite extreme case in which she has no commitment power (*i.e.*, $y = 0$), so that at every moment she observes the current state of the project q and decides whether to complete the project immediately, or to let the agents continue to work on the project and re-evaluate her option to complete the project a moment later. Finally, we consider the case in which she has intermediate commitment power (*i.e.*, $0 < y < \infty$), and we examine how her optimal project size depends on y.

2.3.1.1 Full Commitment Power ($y = \infty$)

If the manager has full commitment power, then she can commit to a project size before the agents begin to work. Therefore, at time 0 with $q = 0$, the manager leads a Stackelberg game in which she chooses the project size that maximizes her discounted profit and the agents follow by adopting the equilibrium strategy characterized in Proposition 1. As a result, her optimal project size with full commitment (FC) satisfies $Q_{FC}^{M} \in \arg\max_{Q} W(0\,; Q, Q)$.[12] Noting from (2.2) that $W(0\,; Q, Q)$ is concave in Q, and differentiating it with respect to Q

[12] Because the manager leads the agents in a Stackelberg game, given any choice Q, the agents will choose their strategy based on that Q, and the agent's belief $\tilde{Q}$ will coincide with Q *ex-ante*.

we have:

$$Q_{FC}^{M}=\frac{\beta}{r\lambda}\frac{2n-1}{2n}\left(\frac{4n}{4n-1}\right)^{2}.$$

Note that the concavity of her discounted profit function implies that she commits to Q_{FC}^{M} at $q=0$ for any commitment power $y\geq Q_{FC}^{M}$.

2.3.1.2 No Commitment Power ($y=0$)

On the other hand, if the manager has no commitment power, then at every moment she observes the current state of the project q, and she decides whether to stop work and collect the net profit $(1-\beta)\,Q$ or to let the agents continue working and re-evaluate her decision to complete the project a moment later. In this case, the manager and the agents engage in a simultaneous-action game, where the manager chooses Q to maximize her discounted payoff given the agents' beliefs $\tilde{Q}$ and their strategies, and the agents form their beliefs by anticipating the manager's choice Q. Therefore, her optimal project size with no commitment (NC) satisfies $Q_{NC}^{M}\in\arg\max_{Q}\left\{W\left(q\,;Q,\tilde{Q}\right)\right\}$, where in equilibrium beliefs must be correct; *i.e.*, $Q=\tilde{Q}$. By solving $\left.\frac{\partial W\left(q\,;Q,\tilde{Q}\right)}{\partial Q}\right|_{Q=\tilde{Q}}=0$, we have:

$$Q_{NC}^{M}=\frac{\beta}{r\lambda}\frac{2n}{2n-1}.$$

Observe that if $y=0$, then she will choose a strictly larger project relative to the case in which she has full commitment power: $Q_{NC}^{M}>Q_{FC}^{M}$. We shall discuss the intuition behind this result in Section 3.1.3, where we determine the manager's optimal project size for intermediate levels of commitment power.[13]

[13]Note that Q_{FC}^{M} increases in the team size n while Q_{NC}^{M} decreases in n. By examining the effort strategies from Proposition 1, it follows that given any Q, there exists some interior threshold φ such that the total effort of the team (and consequently the manager's discounted profit) increases in the team size if and only if the project is sufficiently far from completion; *i.e.*, $q\leq\varphi$. The reader is referred to Georgiadis (2012) for a detailed analysis and discussion of this result. The upshot is that if $y=\infty$ so that she can choose the size of the project at $q=0$, then her optimal project size increases in n. On the other hand, suppose that $y=0$,

Conceptually, this commitment problem could be resolved by allowing β to be contingent on the project size. In particular, suppose that the manager can fix β, and let $\hat{\beta}(Q)$ equal β if $Q = Q_{FC}^{M}$, and 1 otherwise. Then, her optimal project size is equal to Q_{FC}^{M} regardless of her commitment power because any other project size will yield her a net payoff of 0. However, this implicitly assumes that Q_{FC}^{M} is contractible at $q = 0$, which is clearly not true for any $y < Q_{FC}^{M}$. Therefore, we rule out this possibility by assuming that β is independent of Q.

2.3.1.3 Partial Commitment Power $(0 < y < \infty)$

Recall that the manager's optimal project size is equal to Q_{FC}^{M} for all $y \geq Q_{FC}^{M}$, and it is equal to Q_{NC}^{M} if $y = 0$. To determine her optimal project size when $y \in \left(0, Q_{FC}^{M}\right)$, we solve an auxiliary problem, and we show that there is a one-to-one correspondence between this auxiliary problem and the original problem.

Suppose that the manager can credibly commit to her optimal project size as soon as the project hits x. In this case, the manager leads a Stackelberg game, where she chooses Q_{x}^{M} to maximize her discounted profit at x, so that $Q_{x}^{M} \in \arg\max_{Q \geq x} \{W(x\,; Q, Q)\}$, and the agents follow by choosing their strategies based on Q_{x}^{M}. We then show that for all $y \in \left(0, Q_{FC}^{M}\right)$, there exists a unique $x(y) \in \left(0, Q_{NC}^{M}\right)$, such that the manager will commit to the project size $Q_{x(y)}^{M}$ as soon as the project hits $x(y)$ for the first time.

Proposition 9. *Suppose that given the current state q, the manager can commit to any*

and observe from Proposition 1 that the rate at which agents increase their effort as the project progresses decreases in the team size; *i.e.*, $\frac{\partial}{\partial q} a(q\,; Q) = \frac{r}{2n-1} \downarrow$ in n. By noting that the manager's incentives to extend the project are driven by the agents working harder as the project progresses, it follows that in a larger team the manager has weaker incentives to extend the project relative to a smaller team.

project size $Q \in [q,\, q+y]$. Then at $x(y)$ the manager will commit to $Q^M_{x(y)}$, where

$$Q^M_{x(y)} = \left(\frac{2n}{4n-1}\right)^2 \left(\sqrt{\frac{\beta}{r\lambda}\frac{2n-1}{2n}} + \sqrt{\frac{\beta}{r\lambda}\frac{2n-1}{2n} + \frac{4n-1}{4n^2}x(y)}\right)^2, \qquad (2.3)$$

$x(y)$ is the unique solution to the equation $\max\left\{Q^M_{x(y)} - y,\, 0\right\} = x(y)$, and $x(y)$ decreases in y.

Therefore, the manager's optimal project size decreases in her commitment power: $Q^M_{x(y)}$ decreases in y.[14]

The first part of this Proposition asserts that the manager has incentives to extend the project as it progresses: Q^M_x increases in x. To understand the intuition behind this result, note that the manager trades off a larger project that yields a larger net profit upon completion against having to wait longer until that profit is realized, but she ignores the additional effort cost associated with a larger project. Moreover, recall that the agents increase their effort level, and hence the manager's marginal cost associated with choosing a larger project decreases, as the project progresses. On the other hand, her marginal benefit from choosing a larger project is independent of the progress made. Since the project size will be chosen such that the two marginal values are equal, it follows that the manager's optimal project size increases as the project progresses.[15]

The implication of this result is that if the manager has less commitment power, then she

[14] If the manager has full commitment power, then $Q^M_{x(\infty)} = Q^M_{FC}$. If she has no commitment power, then the fixed point of (2.3) coincides with Q^M_{NC}.

[15] To reinforce the intuition that this is due to the agents increasing their effort as the project progresses, suppose that each agent exerts constant effort $a > 0$ throughout the duration of the project. Then given the current state q, the project will be completed in $\frac{Q-q}{na}$ units of time so that the manager's discounted profit is equal to $(1-\beta)\, Q e^{-\frac{r(Q-q)}{na}}$. Differentiating this expression with respect to Q and using the first-order condition, it follows that the manager's discounted profit is maximized at $Q = \frac{na}{r}$. Observe that the optimal project size is independent of q, which leads us to conclude that the manager's time-inconsistency arises due to the agents increasing their effort along the evolution path of the project.

will (eventually) commit to a larger project; *i.e.*, $Q^M_{x(y)}$ decreases in y. By noting that the extreme cases in which the manager has full (no) commitment power correspond to $y = 0$ ($y = \infty$), this intuition also explains why $Q^M_{NC} > Q^M_{FC}$. Figure 1 illustrates this result.

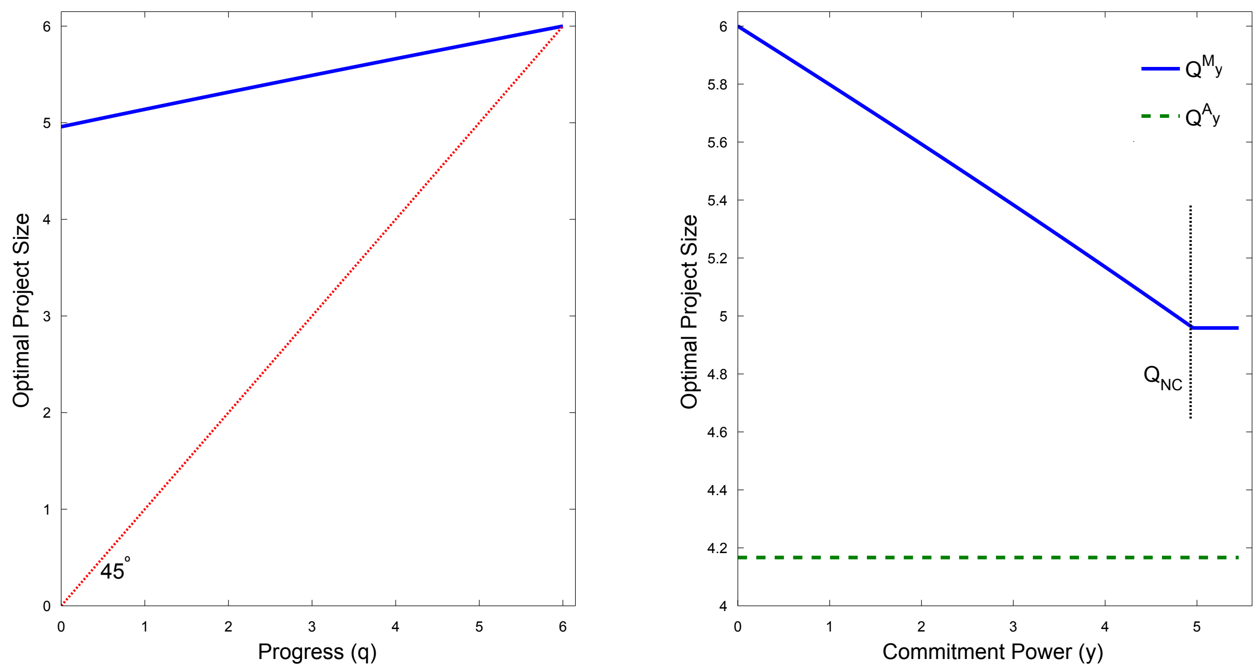

Figure 2.1: **Optimal project size** when $\beta = 0.5$, $r = 0.1$, $\lambda = 1$, and $n = 4$. The left panel illustrates the manager's incentives the extend the project as it progresses: observe that her optimal project size increases in the state of the project q, and there exists a state at which the manager is better off completing the project without further delay. The right panel illustrates that her optimal project size (solid line) decreases in her commitment power, while the agents' optimal project size (dashed line) is independent of their commitment power.

Remark 5. *Recall that (i) the MPE is unique if* $Q < \frac{2\beta}{r\lambda}$, *(ii)* $Q^M_{NC} < \frac{2\beta}{r\lambda}$ *for all* $n \geq 2$, , *and (iii)* $Q^M_{x(y)} \leq Q^M_{NC}$ *for all* y. *Therefore, the game has a unique MPE for any level of commitment power when the project size is chosen by the manager.*[16]

[16]Note that the MPE is always unique if $n = 1$. However, if $y = 0$, then $a\left(0; Q^M_{NC}\right) = 0$, which implies

It is important to emphasize that the agents internalize the manager's limited ability to commit, and they choose their effort strategy appropriately. In particular, each agent's effort increases in the manager's commitment power (*i.e.*, $a\left(q\,;Q^M_{x(y)}\right)$ increases in y) since $C(Q)$ increases in Q for all $Q > \frac{\beta}{r\lambda}\frac{2n-1}{2n}$ and $Q^M_{x(y)} > \frac{\beta}{r\lambda}\frac{2n-1}{2n}$ for all y. This implies that the manager's ability to commit induces a ratchet effect: anticipating that she will choose a larger project, the agents respond by scaling down their effort. While ratchet effects have been shown to arise in settings with asymmetric information (e.g., Freixas, Guesnerie and Tirole (1985) and Laffont and Tirole (1988)), in our model they arise under moral hazard with full and symmetric information.

Besides disincentivizing the agents from exerting effort, the manager's limited ability to commit is also detrimental to her *ex-ante* discounted profit; *i.e.*, $W\left(0\,;Q^M_{x(y)}\right)$ increases in y.[17] Thus, unable to commit sufficiently early, the manager might consider delegating the decision rights over the project size to the agents.

2.3.2 Delegating the Choice of the Project Size to the Agents

We begin by examining how the agents would select the project size. Let $Q^A \in \arg\max_Q \{\Pi(x\,;Q)\}$ denote the agents' optimal project size given the current state x.[18] Solving this maximization problem yields

$$Q^A = \frac{\beta}{r\lambda}\frac{2n-1}{2n}\,.$$

that the project is never completed in equilibrium; in fact, it is not even started.

[17] This is because $W(0\,;Q)$ is concave in Q, the manager's *ex-ante* discounted profit is maximized at Q^M_{FC}, $Q^M_{x(y)} \geq Q^M_{FC}$ for all y, and $Q^M_{x(y)}$ decreases in y.

[18] Because agents are identical and the equilibrium is symmetric, they will be in agreement with respect to the optimal project size.

First, observe that the agents' optimal project size is independent of the current state x. Intuitively, this is because they incur the cost of their effort, so that their effort cost increases together with their effort level as the project progresses. As a result, unlike the manager, their marginal cost associated with choosing a larger project does not decrease as the project evolves, so that they do not have incentives to extend or shrink the project as it progresses.

Second, observe that $Q^A < Q^M_{x(y)}$ $\forall y$; *i.e.*, the agents always prefer a smaller project than the manager.[19] This is because they incur the cost of their effort, so that their marginal cost associated with a larger project is greater than that of the manager's.

Proposition 10. *Suppose that given the current state q, the manager can commit to any project size $Q \in [q,\, q+y]$. Then the manager should delegate the choice of the project size to the agents unless she has sufficient commitment power; i.e., there exists an interior threshold* θ such that $W\left(0\,;Q^A,Q^A\right) > W\left(0\,;Q^M_{x(y)},Q^M_{x(y)}\right)$ *if and only if $y < \theta$.*

Recall that the agents' optimal project size is time-consistent, which implies that if the manager delegates the decision rights to the agents, then her *ex-ante* discounted profit is independent of when the project size is chosen. The key part of this result is that if the manager has no commitment power (*i.e.*, $y = 0$), then she is always better off delegating the decision rights over the project size to the agents. By noting that the manager's optimal project size (and hence her *ex-ante* discounted profit) increases in her commitment power, the Proposition follows.

[19] An implication of this observation, together with Remark 1, is that the equilibrium of the game is unique also when the project size is chosen by the agents.

2.4 The Benefits and Costs of Cooperation

So far, we have focused exclusively on Markov Perfect equilibria. The main feature of such equilibria is that the agents' strategies depend on the current state of the project q_t but not on its evolution path (*i.e.*, $\{q_s\}_{s\leq t}$). While the restriction to MPE is reasonable in situations where teams are large and members cannot monitor each other, there typically exist other Public Perfect equilibria (hereafter PPE) with history-dependent strategies. In this Section, we characterize a continuum of such equilibria in which at every moment, each agent chooses his effort to maximize a convex combination of his individual and the entire team's discounted payoff along the equilibrium path.

Building upon the concepts introduced in the seminal paper on *social identity* by Tajfel and Turner (1979), Akerlof and Kranton (2000) argue that depending on the work environment, employees may behave as *insiders* who act in the best interest of the organization or as *outsiders* who act in their individual best interest. Therefore, the weight that an agent places on maximizing the team's discounted payoff can be interpreted as the degree to which he feels an insider, and we shall refer to an equilibrium as *more cooperative* the more weight each agent places on maximizing the team's discounted payoff.

By noting that experiments in social identity theory have demonstrated that it is surprisingly easy to affect subjects' behavior as insiders or outsiders within a group (Akerlof and Kranton (2005)), the objective of this Section is to examine how the manager should influence the agents' cooperation level to maximize her discounted profit.[20] The main result is that if the agents play a more cooperative equilibrium, then the manager will choose a larger project,

[20]Note that given a fixed Q, all parties are better off if the agents play a more cooperative equilibrium. In particular, the extreme case in which each agent puts all the weight on maximizing the team's discounted payoff, corresponds to the efficient outcome (*i.e.*, the Samuelson-Lindahl condition). On the other hand, the case in which each agent puts all the weight on his individual discounted payoff corresponds to the MPE characterized in Section 2.

and she will have stronger incentives to extend it as it progresses. The upshot is that unless the manager has sufficient commitment power, she is better off if the agents do not play the fully cooperative equilibrium (*i.e.*, place all the weight on maximizing the team's discounted payoff). More generally, the cooperation level that maximizes the manager's (and the agents') discounted profit decreases in her commitment power. Intuitively, the manager can mitigate her incentives to extend the project by inducing the agents to play a less cooperative equilibrium, which is useful if she has small commitment power.

2.4.1 Public Perfect Equilibria

Recall from Section 2.1 that at every moment, each agent observes the current state of the project and chooses his effort to maximize his expected discounted payoff. On the other side of the spectrum, the Samuelson-Lindahl condition (which is a sufficient condition for efficiency given a fixed project size) dictates that each agent chooses his effort to maximize the discounted payoff of the entire team. One can conceive of a continuum of intermediate cases in which at every moment, each agent chooses his effort to maximize a convex combination of his individual and the entire team's discounted payoff. We model this by assuming that given the current state of the project q, each agent chooses his effort to maximize the expected discounted payoff of $k \in [1, n]$ agents; *i.e.*, he solves

$$a\left(q\,;Q,\,k\right) \in \arg\max_{a} \left\{ a\,k\,\Pi'\left(q\,;Q,\,k\right) - \frac{\lambda}{2}a^2 \right\} , \tag{2.4}$$

Note that $k = 1$ $(k = n)$ corresponds to the case in which each agent places all the weight on maximizing his individual (the team's) discounted payoff, while $k \in (1, n)$ corresponds to intermediate cooperation levels. The following Proposition establishes that for all $k \in (1, n]$ there exists a PPE in which at every moment along the equilibrium path, each agent chooses his effort by solving (2.4).[21]

[21]Note that there exist PPE where each agent's cooperation level varies as the project progresses, and the cooperation level may differ across team members. However, we restrict attention to the case in which the

Proposition 11. *For any given $k \in [1, n]$ and project size Q, there exists a Public Perfect equilibrium (PPE) in which each agent's effort strategy satisfies*

$$a\left(q\,;Q,\,k\right)=\frac{r}{2n-k}\left[q-C\left(Q\,;k\right)\right]^{+} \tag{2.5}$$

along the equilibrium path, where $C\left(Q\,;k\right)=Q-\sqrt{\frac{2\beta Q}{r\lambda}\frac{(2n-k)k}{n}}$. After any deviation from the equilibrium path, all agents revert to the Markov Perfect equilibrium (i.e., $k=1$) for the remaining duration of the game. In equilibrium, each agent's discounted payoff is given by

$$\Pi\left(q\,;Q,\,k\right)=\frac{r\lambda}{2k}\frac{\left(\left[q-C\left(Q\,;k\right)\right]^{+}\right)^{2}}{2n-k}\,,$$

and it increases in k.

The intuition behind the existence of *cooperative* PPE is as follows. First, if all agents choose their effort by solving (2.4) for some $k > 1$, then each agent is strictly better off relative to the case in which $k = 1$. Second, $k = 1$ corresponds to the Markov equilibrium, so that the threat of punishment is credible. Third, by examining the progress made until time t, each agent can infer whether all agents followed the equilibrium strategy; *i.e.*, if q_t corresponds to the progress that should occur if all agents follow (2.5). Because a deviation from the equilibrium path is detectable (and punishable) *arbitrarily* quickly, the gain from a deviation is infinitesimally small. As a result, no agent has an incentive to deviate from the strategy dictated by (2.5), so that it constitutes a PPE.[22]

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cooperation level is constraint throughout the duration of the project and identical across all agents because we interpret k as part of the organization's corporate culture which is persistent.

[22]There is a well known problem associated with defining a trigger strategy in continuous-time games. To see why, suppose that a deviation occurs at some t', and agents revert to the MPE at t''. However, because there is no *first time* after t', there always exists some $t \in (t', t'')$ such that the agents are better off reverting to the MPE at that t; i.e., subgame perfection fails. To resolve this problem, we use the concept of inertia strategies proposed by Bergin and MacLeod (1993).

We now solve the manager's problem. Using (2.5) and proceeding as in Section 2.2, it follows that given the current state of the project q, the project size Q, the agents' beliefs $\tilde{Q}$, and a cooperation level k, the manager's discounted profit satisfies

$$W\left(q\,;Q,\tilde{Q},\,k\right)=(1-\beta)\,Q\left(\frac{\left[q-C\left(\tilde{Q}\,;k\right)\right]^{+}}{Q-C\left(\tilde{Q}\,;k\right)}\right)^{\frac{2n-k}{n}}.$$

2.4.2 Optimal Project Size

In this Section we determine the manager's and the agents' optimal project size, and we examine how it depends on the agents' cooperation level. Moreover, we show that the delegation result established in Proposition 3 continues to hold within this larger class of equilibria.

By using a similar approach as in Section 3.1, it follows that the manager's optimal project size satisfies

$$Q_{FC}^{M}\left(k\right)=\frac{\beta}{r\lambda}\frac{k\left(2n-k\right)}{2n}\left(\frac{4n}{4n-k}\right)^{2}\quad\text{and}\quad Q_{NC}^{M}\left(k\right)=\frac{2\beta}{r\lambda}\frac{kn}{2n-k}$$

when $y=\infty$ (FC) and when $y=0$ (NC), respectively. Observe that for any k, the manager's optimal project size with no commitment is strictly greater than that with full commitment; *i.e.*, $Q_{NC}^{M}\left(k\right)>Q_{FC}^{M}\left(k\right)$. Also note that for any level of commitment power $y\geq Q_{FC}^{M}\left(k\right)$, the manager will commit to a project size equal to $Q_{FC}^{M}\left(k\right)$ before the agents begin to work on the project.

On the other hand, for any intermediate level of commitment power $y\in\left[0,Q_{FC}^{M}\left(k\right)\right]$, one can show that the manager will commit at $x(y)$ to her optimal project size

$$Q^M_{x(y)}(k) = \left(\frac{2n}{4n-k}\right)^2 \left(\sqrt{\frac{\beta}{r\lambda}\frac{k(2n-k)}{2n}} + \sqrt{\frac{\beta}{r\lambda}\frac{k(2n-k)}{2n} + \frac{k(4n-k)}{4n^2}x(y)}\right)^2,$$

where $x(y)$ is the unique solution to $\max\left\{Q^M_{x(y)}(k) - y,\, 0\right\} = x(y)$, and it decreases in y.

Observe that $Q^M_x(k)$ increases in x, which implies that similar to the base model analyzed in Section 3, the manager has incentives to extend the project as it progresses: given a cooperation level, the manager will commit to a larger project as her commitment power decreases.

Remark 6. *Given any y, the manager's optimal project size increases in the agents' cooperation level: $Q^M_{x(y)}(k)$ increases in k for all y.*

This is intuitive: since each agent's effort increases in k, the team can achieve more progress during any given time interval by *playing* a more cooperative equilibrium. Therefore, the marginal cost associated with choosing a larger project decreases in k, while the associated marginal benefit does not depend on k, which implies that the manager has incentives to choose a larger project if the agents play a more cooperative equilibrium. This observation is illustrated in the left panel of figure 2 for the cases in which $k = 0.8n$ and $k = 0.95n$.

Remark 7. *The manager's incentives to extend the project as it progresses become stronger in the agents' cooperation level: $\frac{\partial Q^M_x(k)}{\partial x}$ increases in k.*

To understand the intuition behind this observation, recall from Section 3 that the manager's incentives to extend the project are driven by the fact that agents ramp up their effort as the project progresses, and observe that $\frac{\partial^2 a(q\,;Q,k)}{\partial q \partial k} > 0$; *i.e.*, agents ramp up their effort faster if a more cooperative equilibrium is played. Consequently, the manager has stronger incentives to extend the project in this case.

Now let us consider the manager's option to delegate the decision rights over the project size to the agents. Given their cooperation level k, the agents will choose their optimal project size by solving $Q^A(k) \in \arg\max_Q \{\Pi(q\,; Q,\, k)\}$, which yields

$$Q^A(k) = \frac{\beta}{r\lambda}\frac{k(2n-k)}{2n}\,.$$

Observe that $Q^A(k)$ is independent of q, which implies that the agents' preferences with respect to the project size are time-consistent. Moreover, similar to the manager, the agents find it optimal to choose a larger project if they play a more cooperative equilibrium. However, for any cooperation level k and regardless of the manager's commitment power, the agent's optimal project size is smaller than that of the manager; *i.e.*, $Q^A(k) \leq Q^M_{x(y)}(k)$ for any y. By using a similar approach as in Proposition 3, we can establish the following Remark.

Remark 8. *Given cooperation level k, there exists an interior threshold θ_k such that manager should delegate the decision rights over the project size to the agents if and only if $y \leq \theta_k$.*

By numerically examining this threshold θ_k, we find that it increases in k so that delegation becomes more attractive the higher is the agents' cooperation level. This is intuitive since the manager's commitment problem becomes aggravated if a more cooperative equilibrium is played.

2.4.3 Cultivating a Cooperative Environment

Given a fixed Q, both the agents' and the manager's discounted payoff increase in the cooperation level k. However, when the choice of Q is endogenous, this need no longer be true because a higher cooperation level induces the manager to choose a larger project, and it aggravates her commitment problem. Unless the manager has sufficient commitment power, the agents anticipate this and decrease their effort, rendering all parties worse off.

Using equilibrium selection concepts introduced by Kreps (1990), we now consider the possibility that the manager can influence the agents' cooperation level (*i.e.*, the PPE that the agents will *play*) by cultivating a more cooperative environment within the team with a variety of policies such as organizing sponsored activities, encouraging interaction among the team members, and engaging the agents when making decisions, as well as with appropriate selection of those who join the team.

Taking into account her commitment power y at time 0, the manager chooses the agents' cooperation level to maximize her *ex-ante* discounted profit:

$$k_y^M \in \arg\max_{1\leq k\leq n} \left\{W\left(0\,; Q^M_{x(y)}\left(k\right), Q^M_{x(y)}\left(k\right),\, k\right)\right\} .$$

To obtain clean results, we restrict attention to the extreme cases $y=\infty$ and $y=0$, and we illustrate that a threshold result holds for intermediate levels of commitment power by using numerical examples.

Proposition 12. *Suppose that given the current state q, the manager can commit to any project size $Q \in [q,\, q+y]$, where $y \geq 0$, and suppose that at $q=0$ the manager can choose the agents' cooperation level k.*

When $y=\infty$, a fully cooperative environment is optimal: $k^M_{FC}=n$. However, when $y=0$,

$k_{NC}^{M}(0) < n$.

Choosing a higher cooperation level has two opposite effects. For any (fixed) Q, the agents work harder, which increases the manager's discounted profit. However, precisely because the agents work harder and they ramp up their effort faster as the project progresses, the manager has stronger incentives to extend the project, which harms her *ex-ante* discounted profit. Therefore, if $y = \infty$ so that she can commit to her optimal project size at $q = 0$, then she is better off fostering a fully cooperative environment within the team; *i.e.*, $k = n$. On the other hand, if $y = 0$, Proposition 5 asserts that the latter effect dominates the former, so that a less than fully cooperative environment within the team renders her (*ex-ante*) better off; *i.e.*, $k < n$.

The right panel of Figure 2 illustrates the manager's optimal cooperation level as a function of her commitment power. Observe that a fully cooperative environment is optimal if and only if the manager has sufficient commitment power; *i.e.*, there exists some interior φ such that $k_y^M = n$ if and only if $y \geq \varphi$.

If the manager delegates the decision rights over the project size to the agents, then a fully cooperative environment is optimal. Intuitively, a fully cooperative environment will induce the agents to choose a larger project, as evidenced by the fact that $Q^A(k)$ increases in k, which will in turn increase the manager's discounted profit.

Finally, we discuss the case in which the team members can themselves choose their cooperation level. Suppose that at $q = 0$, the manager and the agents engage in a simultaneous-action game, where the manager chooses her optimal project size, and the agents select their cooperation level. Because the agents' optimal cooperation level for any given project size is equal to n, it follows that the unique Nash equilibrium of this game is for the agents to

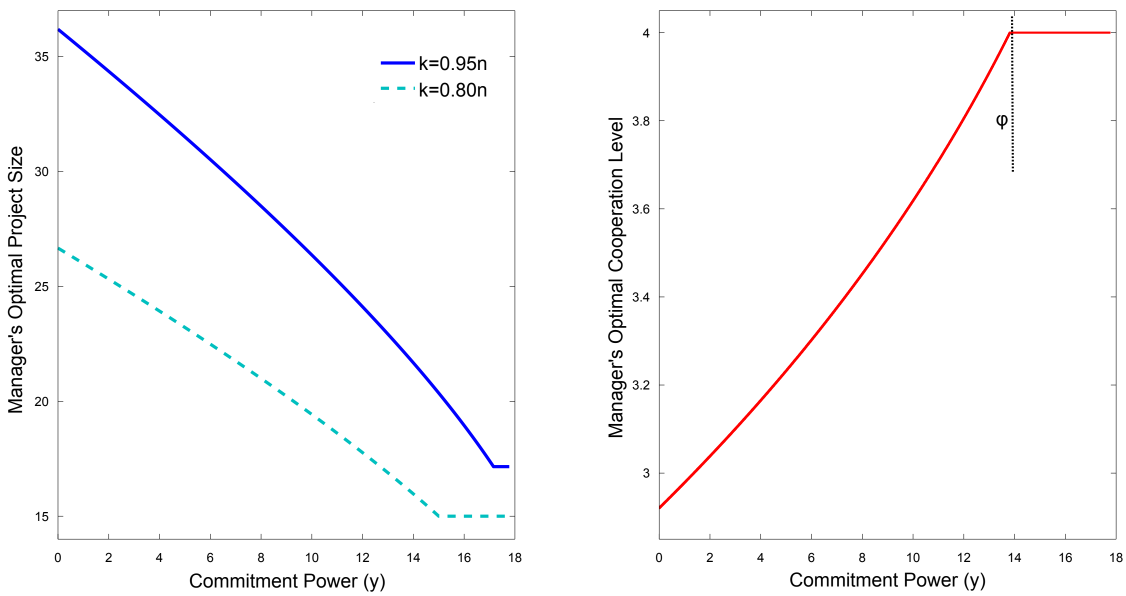

Figure 2.2: **The manager's optimal project size** (left panel) **and her optimal cooperation level** (right panel) when $\beta = 0.5$, $r = 0.1$, $\lambda = 1$, and $n = 4$. The left panel illustrates that a higher cooperation level ($k = 0.95n$ versus $0.80n$) induces the manager to choose a larger project, and it aggravates her commitment problem as evidenced by the fact that the upper line is steeper than the lower line. The right panel illustrates that the manager's optimal cooperation level increases in her commitment power such that a fully cooperative equilibrium (*i.e.*, $k = n$) is optimal if and only if she has sufficient commitment power (*i.e.*, $y \geq \varphi$).

select $k = n$ and for the manager with commitment power y to choose $Q^M_{x(y)}(n)$.

An alternative specification of the game is that at $q = 0$, the agents, knowing the manager's commitment power, lead a Stackelberg game by selecting their (verifiable) cooperation level k^A_y, and the manager follows by choosing her optimal project size $Q^A_{x(y)}\left(k^A_y\right)$. If $y = \infty$, the cooperation level that maximizes the agents' discounted payoff $k^A_{FC} = \arg\max_k \left\{\Pi\left(0\,; Q^A_{FC}(k)\,,\,k\right)\right\} = \frac{4n}{5}$. On the other hand, when $y = 0$, the agents' optimal cooperation level $k^A_{NC} = \frac{n}{2}$. While the agents' optimal cooperation level cannot be determined analytically for intermediate levels of commitment power, numerical examples suggest that k^A_y increases in y, and $k^A_y \in \left[\frac{n}{2}, \frac{4n}{5}\right]$ for all y. The key take-away from this analysis is that the agents are better off under a partially cooperative environment (*i.e.*, $k_y^A < n$ for all y) regardless of the manager's commitment power. By choosing a lower cooperation level, the agents *induce* the manager to choose a smaller project that is closer to their optimal project size. Moreover, if the manager has less commitment power, then the agents prefer an even lower cooperation level in order to mitigate the manager's incentive to extend the project.

2.5 Concluding Remarks

We develop a tractable model to study the interaction between a group of agents who collaborate over time to complete a project and a manager who chooses its size to maximize her discounted profit. A central feature of the model is that the manager has limited commitment ability. This is captured in our model by assuming that given the current state q of the project and her commitment power y, she can only commit to a project size in the interval $[q, q + y]$.

The main result is that the manager has incentives to extend the project (e.g., introduce additional requirements) as it progresses. This implies that if the manager has less commitment power, then she will (eventually) commit to a bigger project. The implication of this result is that anticipating this behavior, the agents reduce their effort, rendering the manager worse off. Consequently, the manager is better off delegating the decision rights over the project size to the agents unless she has sufficient commitment power.

In the latter part of the paper we characterize non-Markovian equilibria where each agent chooses his effort to maximize a convex combination of his and the entire team's discounted payoff. Here, we show that the manager will choose a larger project, and she will have stronger incentives if the agents place more weight on maximizing the team's payoff. In contrast to the case in which the project size is exogenous, the equilibrium in which agents place all the weight on maximizing the team's payoff maximizes the manager's discounted profit only if she has sufficient commitment power: surprisingly, it is not always beneficial for the manager to foster an insider culture within an organization if she has limited ability to commit to a particular project size early on.

In our model agents are compensated upon completion of the project. Georgiadis (2012) shows that this scheme is optimal when the project size is given exogenously. However, it is far from clear that this scheme continues to be optimal when the project size is endogenous and the manager has limited commitment power. It is possible that a more complex scheme in which the manager provides each agent with flow payments while the project is in progress (e.g., Sannikov (2008)), or she compensates the agents upon reaching pre-designated milestones can improve her discounted profit and mitigate her commitment problem. This is a promising direction for future research.

2.6 Extensions

In this Section we consider four extensions to our model to test the robustness of the main results.

2.6.1 Production Synergies

First, we consider the case in which the agents' efforts are complementary, so that at every moment, the total effort of the team is greater than the sum of the agents' individual efforts. We show that all three main results continue to hold for any degree of complementarity.

To obtain tractable results, we consider the production function proposed by Bonatti and Hörner (2011), so that the project evolves according to $dq = \left(\sum_{i=1}^{n} a_i^{1/\gamma}\right)^{\gamma} dt$, where $\gamma \geq 1$, and a larger γ indicates a stronger degree of complementarity. By assuming symmetric strategies, it follows that given the current state of the project q, cooperation level k, and the completion state Q, each agent's discounted payoff and effort strategy are given by

$$\Pi\left(q\,;Q,k,\gamma\right) = \frac{r\lambda n^{2-2\gamma}}{2k}\frac{\left(\left[q - C\left(Q\,;k,\gamma\right)\right]^{+}\right)^{2}}{2n-k} \quad \text{and} \quad a\left(q\,;Q,\,k,\gamma\right) = \frac{rn^{1-\gamma}}{2n-k}\left[q - C\left(Q\,;k,\gamma\right)\right]^{+} ,$$

respectively, where $C\left(Q\,;k,\gamma\right) = Q - \sqrt{\frac{2\beta Q}{r\lambda}\frac{n^{2\gamma-2}(2n-k)k}{n}}$.[23] Because (with other things equal) $\Pi\left(q\,;Q,k,\gamma\right)$ increases in k for all γ, it follows that $\forall k \in [1,n]$ there exists a PPE such that each agent follows the strategy dictated by $a\left(q\,;Q,\,k,\gamma\right)$, and after any deviation from the equilibrium path, all agents revert to the MPE; *i.e.*, $k = 1$. Furthermore, each agent's discounted payoff, his equilibrium effort, as well as the aggregate effort of the entire team, increase in the degree of complementarity γ.

[23] As the algebra is straightforward and similar to that used to derive Propositions 1 and 4, it is omitted here in order to streamline the exposition.

By using the agents' strategies, it follows that the manager's discounted profit satisfies

$$W\left(q\,;Q,\tilde{Q},\,k,\gamma\right) = (1-\beta)\,Q\left(\frac{\left[q - C\left(\tilde{Q}\,;k,\gamma\right)\right]^{+}}{Q - C\left(\tilde{Q}\,;k,\gamma\right)}\right)^{\frac{2n-k}{n}}.$$

To streamline the exposition, we focus on the extreme cases in which the manager has either full or no commitment power. It follows that

$$Q^{M}_{FC}\left(k,\gamma\right) = \frac{\beta}{r\lambda}\frac{k\left(2n-k\right)}{2n}\left(\frac{4n}{4n-k}\right)^{2} n^{2\gamma-2} \quad \text{and} \quad Q^{M}_{NC}\left(k,\gamma\right) = \frac{2\beta}{r\lambda}\frac{kn}{2n-k}n^{2\gamma-2}.$$

Observe that the manager's optimal project size increases in the degree of complementarity, and similar to the case analyzed in Section 4, $Q^{M}_{NC}\left(k,\gamma\right) > Q^{M}_{FC}\left(k,\gamma\right)$. Moreover, the counterpart of Proposition 2 continues to hold; *i.e.*, if the manager has less commitment power, then she will choose a bigger project.

We now examine the manager's option to delegate the choice of Q to the agents, as well as her optimal choice of the agents' cooperation level. To begin, note that the agents' optimal project size satisfies $Q^{A}\left(k,\gamma\right) = \frac{\beta}{2r\lambda}\frac{k(2n-k)}{n}n^{2\gamma-2}$. By following a similar approach as in Section 3, it follows that given γ, there exists a threshold θ_{γ} such that the manager is better off delegating the choice of the project size to the agents if and only if her commitment power $y < \theta_{\gamma}$. Similarly, the manager should cultivate a partially cooperative environment within the team (*i.e.*, $k < n$) unless she has sufficient commitment power.

2.6.2 Fixed Compensation

In the base model, we have assumed that the agents' net payoff upon completion of the project is proportional to its value. While a more valuable project will typically yield a larger net payoff to the agents - for example a bigger bonus, a salary increase, greater job security, or a larger outside option, this assumption can be thought of as an extreme case, since any incentive scheme will likely consist of a fixed component that is independent of the project size, and a performance-based component. In this Section, we consider the opposite extreme where each agent's net payoff is fixed and independent of the project size, while efforts are perfect substitutes; *i.e.*, $dq_t = \left(\sum_{i=1}^{n} a_{i,t}\right) dt$.

The take-away from this Section is that the main results continue to hold. In fact, the manager's commitment problem becomes so aggravated in this case, that the project may never be completed in equilibrium. However, this commitment problem can be mitigated so that the project is completed even if the manager has no commitment power by choosing a sufficiently low cooperation level k. Moreover, because the agents' payoff does not depend on the project size, the manager cannot benefit by delegating the choice of the project size to the agents,since their optimal project size is 0.

To begin, suppose that each agent receives a lump-sum $\frac{V}{n}$ as soon as the project is completed regardless of its size. Then given the current state of the project q, the cooperation level k, and the completion state Q, each agent's equilibrium effort is given by

$$\bar{a}\left(q\,;Q,\,k\right) = \frac{r}{2n-k}\left[q - \bar{C}\left(Q\,;k\right)\right]^{+} \quad \text{where } \bar{C}\left(Q\,;k\right) = Q - \sqrt{\frac{2V}{r\lambda}\frac{\left(2n-k\right)k}{n}}\,,$$

while the manager's discounted profit satisfies

$$\bar{W}\left(q\,;Q,\tilde{Q},\,k\right) = \left(Q - V\right)\left(\frac{\left[q - \bar{C}\left(\tilde{Q}\,;k\right)\right]^{+}}{Q - \bar{C}\left(\tilde{Q}\,;k\right)}\right)^{\frac{2n-k}{n}}.$$

Using the same approach as in Section 4, one can show that for all $k \in [1, n]$ there exists a PPE such that each agent follows the strategy dictated by $\bar{a}\left(q\,;Q,\,k\right)$ contingent on all other agents following the same strategy, and reverts to the MPE (*i.e.*, $k = 1$) after observing a deviation.

By examining the manager's optimal project size, it follows that with full and with no commitment power, we have

$$\bar{Q}_{FC}^{M}(k) = \frac{2n-k}{3n-k}V + \frac{n}{3n-k}\sqrt{\frac{2V}{r\lambda}\frac{(2n-k)\,k}{n}} \quad \text{and} \quad \bar{Q}_{NC}^{M}(k) = V + \sqrt{\frac{2V}{r\lambda}\frac{kn}{2n-k}}\,,$$

respectively. Observe that $\bar{Q}_{NC}^{M}(k) > \bar{Q}_{FC}^{M}(k)$, and by solving for $\bar{Q}_{x}^{M}(k) \in \arg\max_{Q} \bar{W}(q\,;Q,Q,\,k)$, it follows that $\bar{Q}_{x}^{M}(k)$ increases in x. Therefore, similar to the base model, the manager has incentives to extend the project as it progresses. In fact, these incentives can be so strong that the project is never completed in equilibrium. To see why, note that the project is completed only if $\bar{C}(Q\,;k) < 0$, and this inequality is true at $Q = \bar{Q}_{NC}^{M}(k)$ if and only if $\sqrt{r\lambda V} < \sqrt{\frac{2(2n-k)k}{n}} - \sqrt{\frac{2kn}{2n-k}}$. Moreover, if each agent's net payoff is independent of the project size, then delegating the choice of the project size to the agents is not beneficial, because they will choose a project of size 0.

To examine the manager's optimal choice of k, note that the last inequality is violated if $k = n$, which implies that if the agents play the fully cooperative PPE and the manager has no commitment power, then the project is never completed. Therefore, the manager can increase her discounted profit by choosing some $k < n$ such that the project is completed. On the other hand, by noting that $\sqrt{\frac{2(2n-k)k}{n}} > \sqrt{\frac{2kn}{2n-k}}\Big|_{k=1}$ for all $n \geq 2$, and observing that V is the only parameter that the manager can choose (since r and λ are given exogenously), it follows that there always exists some $V > 0$ and $k \in [1, n]$ such that the project is completed in equilibrium even if the manager has no commitment power.

Therefore, the manager's commitment problem is severely aggravated if the agents' net payoffs are independent of the project size. Intuitively, this is because the manager obtains the entire marginal benefit from a larger project (as opposed to $1-\beta$ thereof), which provides her with stronger incentives to extend the project. As a result, anticipating this behavior, the agents prefer to exert no effort and abandon the project altogether.

2.6.3 Sequential Projects

Insofar, we have assumed that the manager interacts with the agents for the duration of a single project. The main purpose of this assumption was to maintain tractability. However, because relationships between a manager and work teams are often persistent in practice, it is important to verify that the main results of this paper are robust to repeated interactions. In this Section we consider the case in which as soon as a project is completed, with probability $\alpha < 1$, the manager and the agents interact for the duration of another project, while the relationship is terminated with probability $1 - \alpha$, and each party receives its outside option which is normalized to 0.[24]

Indeed, we find that when the manager and the agents engage in sequential projects, all the main results continue to hold. Moreover, we observe that if the relationship is more persistent (*i.e.*, α is larger), then the manager has stronger incentives to delegate the choice of the project size to the agents, and her optimal cooperation level is larger.

Since the problem is stationary, the manager will choose the same project size every time. Both the agents' and the manager's problem remain unchanged, except for the boundary

[24]If $\alpha = 0$, then this case reduces to the base model. On the other hand, because the value of the project has been assumed to be linear in its size, and it generates a payoff only upon completion, if $\alpha = 1$, then both the manager and the agents would choose an arbitrarily small project, which would be completed arbitrarily quickly. Therefore, we restrict attention to the cases in which $\alpha < 1$.

conditions, which become $\hat{\Pi}(Q\,;Q) = \frac{\beta Q}{n} + \alpha\hat{\Pi}(0\,;Q)$ and $\hat{W}\left(Q\,;Q,\tilde{Q}\right) = (1-\beta)\,Q + \alpha\hat{W}\left(0\,;Q,\tilde{Q}\right)$, respectively. The interpretation of these conditions is that upon completion of each project, each party receives its net payoff from this project, plus the expected continuation value from future projects.

Unfortunately, it is no longer possible to determine the manager's optimal completion state analytically, and consequently to analyze the manager's option to delegate the choice of the project size to the agents, or to influence the agents' cooperation level. Therefore, we present a numerical example to illustrate how the main results of the paper extend to this case. Figure 3 demonstrates how the manager's optimal project size, the value of delegation, as well as the agents' cooperation level depend on her commitment power. The takeaway is that the main results of this paper continue to hold when the relationship between the manager and the agents is persistent.

2.6.4 Flow Payments while the Project is in Progress

Throughout the analysis we have maintained the assumption that the agents receive a lump-sum payment upon completing the project, but they do not receive any flow payments while the project is ongoing. Therefore, to extend the project, the manager must only incur the cost associated with having to wait longer until the project is completed. In this Section, we extend our model to consider the case in which the manager compensates each agent with a flow payment $\frac{w}{n} > 0$ per-unit of time while the project is in progress, in addition to a lump-sum payment upon completing the project.

We find that similar to the base case, the manager has incentives to extend the project as it progresses, and that she is better off delegating the decision rights to the project size to the agents unless she has sufficient commitment power. Moreover, her optimal cooperation level

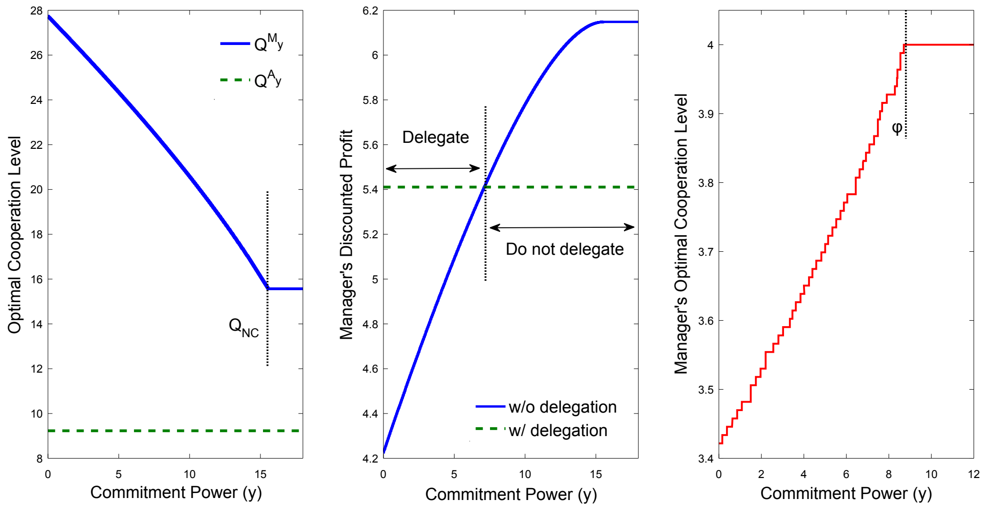

Figure 2.3: An example in which the manager interacts with the agents repeatedly when $\beta = 0.5$, $r = 0.1$, $\lambda = 1$, $n = 4$, and $\alpha = 0.25$. The left panel illustrates that her optimal project size decreases in her commitment power, while the agents' optimal project size is independent of their commitment power. The middle panel illustrates that delegating the decision rights over the project size to the agents is beneficial if and only if the manager doesn't have sufficient commitment power. Finally, the right panel illustrates that the manager's optimal cooperation level increases in her commitment power. Therefore, the main results continue to hold when the relationship between the manager and the agents is persistent.

k increases in her commitment power, and a fully cooperative equilibrium is optimal only if she has sufficient commitment power.

It is straightforward to show each agent's discounted payoff, and the manager's discounted profit satisfy the HJB equations

$$\begin{aligned} r\check{\Pi}\left(q\,;Q\right) &= \frac{w}{n} + \frac{k\left(2n-k\right)}{2\lambda}\left[\check{\Pi}\left(q\,;Q\right)\right]^2 \quad \text{s.t. } \check{\Pi}\left(Q\,;Q\right) = \beta Q \\ r\check{W}\left(q\,;Q,\tilde{Q}\right) &= -w + \left[na\left(q\,;\tilde{Q},\,k\right)\right]\check{W}'\left(q\,;Q,\tilde{Q}\right) \quad \text{s.t. } \check{W}\left(Q\,;Q,\tilde{Q}\right) = \left(1-\beta\right)Q\,, \end{aligned}$$

respectively, where $a\left(q\,;\tilde{Q},\,k\right) = \frac{k\check{\Pi}(q\,;Q)}{\lambda}$. Unfortunately, this model is analytically not tractable. Therefore, to examine how the main results extend to this case, we present a

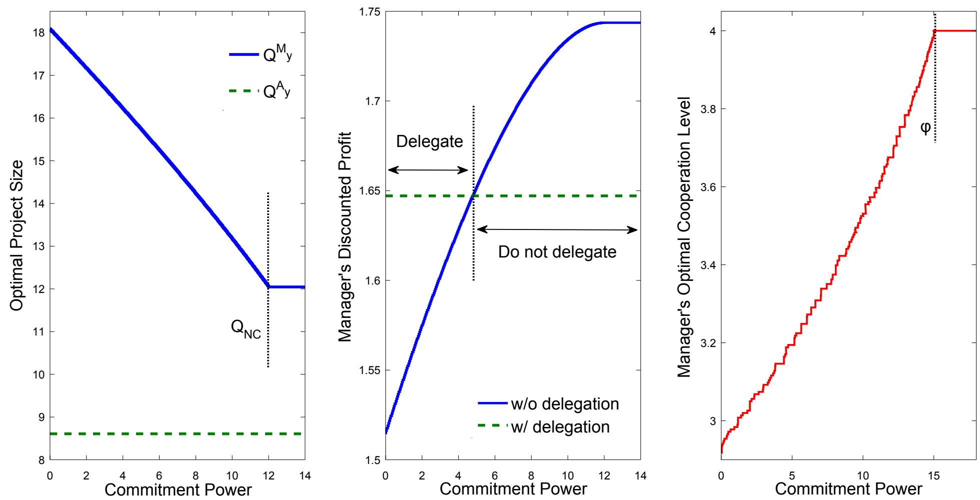

Figure 2.4: An example in which the manager compensates the agents per unit of time while the project is in progress when $\beta = 0.5$, $r = 0.1$, $\lambda = 1$, $n = 4$, and $w = 0.001$. Similar to Figure 3, this figure illustrates that the main results continue to hold in this case.

numerical illustration (see Figure 4). Observe in the left panel that the manager's optimal project size decreases in her commitment power, while the agents' optimal project size is independent of their commitment power. From the middle panel, one observes a similar pattern to Proposition 3: the manager should delegate the choice of the project size unless she has sufficient commitment power. Finally, note from the right panel that the cooperation level that maximizes the manager's discounted profit increases in her commitment power, and a fully cooperative equilibrium is optimal only if she has sufficient commitment power.

2.7 Proofs

Proof of Proposition 1. Given the project's state q, agent i chooses his effort level

$$a_i(q) \in \arg\max_{a_i} \left\{ \left(\sum_{j=1}^{n} a_j \right) \Pi_i'(q\,; Q) - \frac{\lambda}{2} a_i^2 \right\} .$$

The first order condition for agent i's problem is $\Pi_i'(q\,;Q) = \lambda a_i$, which implies that $a_i(q\,;Q) = \frac{\Pi_i'(q\,;Q)}{\lambda}$. By noting that the SOC is satisfied, and by substituting $a_i(q\,;Q)$ into the HJB equation, the expected discounted payoff for agent i satisfies

$$r\Pi_i(q\,;Q) = -\frac{\lambda}{2}\left[\frac{\Pi_i'(q\,;Q)}{\lambda}\right]^2 + \left[\sum_{j=1}^{n}\frac{\Pi_j'(q\,;Q)}{\lambda}\right]\Pi_i'(q\,;Q) \tag{2.6}$$

subject to $\Pi_i(q\,;Q) \geq 0\ \forall q$ and $\Pi_i(Q\,;Q) = \frac{\beta Q}{n}$.

To show that a MPE with differentiable strategies exists for this game, it suffices to show that a solution to (2.6) exists. To show this, we derive a symmetric solution analytically. In particular, for symmetric strategies (*i.e.*, $\Pi_i(q\,;Q) = \Pi_j(q\,;Q)\ \forall i$ and j), (2.6) can be re-written as

$$r\Pi(q\,;Q) = \frac{2n-1}{2\lambda}\left[\Pi'(q\,;Q)\right]^2\,, \tag{2.7}$$

and the solution to this differential equation satisfies

$$\Pi(q\,;Q) = \frac{r\lambda}{2}\frac{\left([q-C(Q)]^+\right)^2}{2n-1}\,,\text{ where } C(Q) = Q - \sqrt{\frac{2\beta Q}{r\lambda}\frac{2n-1}{n}}$$

is determined by the value matching condition. By using the FOC, it follows that each agent's effort strategy is given by

$$a(q\,;Q) = \frac{r}{2n-1}\left[q - C(Q)\right]\mathbf{1}_{\{q\geq C(Q)\}}\,.$$

To show that there do not exist any asymmetric solutions to (2.6) we proceed by contradiction. Fix $Q > 0$, and suppose there exist at least two agents a and b whose discounted payoff functions $\Pi_a(q\,;Q)$ and $\Pi_b(q\,;Q)$ satisfy (2.6), but $\Pi_a(q\,;Q) \neq \Pi_b(q\,;Q)$ for at least some $q < Q$. Then let $D(q) = \Pi_a(q\,;Q) - \Pi_b(q\,;Q)$, and note that $D(Q) = 0$ and $D(\cdot)$ is differentiable. Then using (2.6) we can write $2r\lambda D(q) = \left[2\sum_i \Pi_i(q\,;Q) - \Pi_a(q\,;Q) - \Pi_b(q\,;Q)\right]D'(q)$.

Moreover, because agents are impatient ($r > 0$) and the amount of effort that needs to be exerted until the project is completed diverges to infinity as $q \to -\infty$, it must be true that $\Pi_i(q\,;Q) \to 0$ as $q \to -\infty$. Therefore, $\lim_{q\to-\infty} D(q) = 0$, so if $D(q) \neq 0$ for at least some $q < Q$, then it must be the case that there exists some interior $z < Q$ such that $D(z) \neq 0$ and $D'(z) = 0$, which yields a contradiction. Hence we conclude that (2.6) cannot admit an asymmetric solution.

To show that (2.7) has a unique symmetric solution, we use a similar approach. Fix $Q > 0$, and suppose that there exist $\Pi_A(q\,;Q)$ and $\Pi_B(q\,;Q)$ that both satisfy (2.7). Then let $\Delta(q) = \Pi_A(q\,;Q) - \Pi_B(q\,;Q)$, and note that $\Delta(Q) = 0$ and $\Delta(\cdot)$ is differentiable. Therefore, (2.7) can be re-written as $2r\lambda\Delta(q) = (2n-1)\left[\Pi'_A(q\,;Q) + \Pi'_B(q\,;Q)\right]\Delta'(q)$. Moreover, $\lim_{q\to-\infty}\Delta(q) = 0$ by the same argument as above, so if $\Delta(q) \neq 0$ for at least some $q < Q$, then it must be the case that there exists some interior $z < Q$ such that $\Delta(z) \neq 0$ and $\Delta'(z) = 0$, which yields a contradiction. Therefore, there exists a unique symmetric solution to (2.6).

We have insofar shown that there exists a unique solution to (2.6), and that this solution is symmetric. Moreover, note that if $C(Q) \geq 0$ (or equivalently $Q \geq \frac{2\beta}{r\lambda}\frac{2n-1}{n}$), then the equilibrium strategy dictates that no agent ever exerts any effort, in which case the project is never completed. On the other hand, as long as $C(Q) < 0$, the strategy $a(q\,;Q)$ constitutes the unique *project-completing* MPE. Next, suppose that $C(Q) < 0 \leq C(Q)|_{n=1}$ (or equivalently $\frac{2\beta}{r\lambda} \leq Q < \frac{2\beta}{r\lambda}\frac{2n-1}{n}$), and fix all effort strategies except of that of agent i to 0. Then agent i's best response is to also exert 0 effort, since $C(Q)|_{n=1} \geq 0$; *i.e.*, he is not willing to undertake the entire project by himself. As a result, if $Q \geq \frac{2\beta}{r\lambda}$, then in addition to the *project-completing* MPE, there also exist an equilibrium in which no agent exerts any effort, and the project is never completed.

□

Proof of Proposition 2. To begin, note that for any x, $W(x\,;Q,Q)$ is strictly concave in Q, and applying the first order condition yields (2.3). It is straightforward to verify that $\frac{\partial}{\partial x}Q_x^M > 0$ and $\frac{\partial^2}{\partial x^2}Q_x^M < 0$ $\forall q > 0$. Finally, solving the fixed point $Q_Q^M = Q$ yields $Q_Q^M = \frac{\beta}{r\lambda}\frac{2n}{2n-1}$.

Next, let $g(x) = Q_x^M - x$, and observe that $g(0) = Q_0^M > 0$ and $g\left(Q_Q^M\right) = 0$. Moreover, it is easy to check that $g'(x) < 0$ on $\left[0,\, Q_Q^M\right]$, which implies that given any $y \leq Q_0^M$, there exists a unique $x(y)$ such that $g(x(y)) = y$.

Clearly, if $y \geq Q_0^M$, then the manager finds it optimal to commit to Q_0^M at $x = 0$. Therefore, $\forall y \geq 0$, there exists a unique $x(y)$ that solves $\max\left\{Q_{x(y)}^M - y,\, 0\right\} = x(y)$.

To proceed, suppose that $y < Q_0^M$, and note that $W(q\,;Q,Q)$ is strictly concave in Q for all $Q \geq q$. Given the current state of the project q, the manager can either commit to a completion state in the interval $[q,\, q+y]$, in which case her discounted payoff is equal to $\max_{q \leq Q \leq q+y} W(q\,,Q,Q)$, or she can delay committing, anticipating that she will be able to commit to some completion state $q' > q+y$ later, which will yield her a discounted payoff $W(q\,,q',q')$. Therefore, the manager will choose to commit to a completion state at q if and only if $\max_{q \leq Q \leq q+y} W(q\,,Q) \geq W(q\,,q',q')$ $\forall q' > q+y$, or equivalently if and only if $\arg\max_{Q \geq q} W(q\,,Q,Q) \leq q+y$. By noting that $Q_q^M \in \arg\max_{Q \geq q}\{W(q\,;Q,Q)\}$, it follows that the manager finds it optimal to commit to project size $Q_{x(y)}^M$ at $q = x(y)$, where $x(y)$ is the unique solution to the equation $\max\left\{Q_{x(y)}^M - y,\, 0\right\} = x(y)$, and $Q_{x(y)}^M$ is given by (2.3).

□

Proof of Proposition 3. If the project size is chosen by the agents, then they will choose $Q^A = \frac{\beta}{r\lambda}\frac{2n-1}{2n}$, and by substituting this into the manager's expected discounted profit yields $W\left(0\,;Q^A,Q^A\right) = \frac{(1-\beta)\beta}{r\lambda}\frac{2n-1}{2n}\left(\frac{1}{2}\right)^{\frac{2n-1}{n}}$.

Next, consider the case in which the completion state is chosen by the manager, and she has no commitment power (*i.e.*, $y = 0$) so that she eventually completes the project at

$Q_{NC}^M = \frac{\beta}{r\lambda}\frac{2n}{2n-1}$. By substituting this the manager's expected discounted profit we have that $W\left(0\,; Q_{NC}^M, Q_{NC}^M\right) = \frac{(1-\beta)\beta}{r\lambda}\frac{2n}{2n-1}\left(\frac{n-1}{2n-1}\right)^{\frac{2n-1}{n}}$.

Now consider the ratio $\frac{W\left(0\,;Q_{NC}^M,Q_{NC}^M\right)}{W(0\,;Q^A,Q^A)} = \left(\frac{2n}{2n-1}\right)^2\left(\frac{2n-2}{2n-1}\right)^{\frac{2n-1}{n}}$, and for the purpose of this proof, let $h(n) = \left(\frac{2n}{2n-1}\right)^2\left(\frac{2n-2}{2n-1}\right)^{\frac{2n-1}{n}}$ where $n \in \mathbb{R} \cap (1,\infty)$. Observe that $h(0) = 0$ and $\lim_{n\to\infty} h(n) = 1$. Differentiating with respect to n yields $h'(n) = \frac{4\left[(2n-1)(n-1)\ln\left(\frac{2n-2}{2n-1}\right)+n\right]}{(2n-1)^3(n-1)}\left(\frac{2n-2}{2n-1}\right)^{\frac{2n-1}{n}} > 0$ if and only if $(2n-1)(n-1)\ln\left(\frac{2n-2}{2n-1}\right)+n > 0$ or equivalently if $\ln\left(\frac{2n-2}{2n-1}\right)+\frac{n}{(2n-1)(n-1)} > 0$. Now observe that $\lim_{n\to\infty}\left[\ln\left(\frac{2n-2}{2n-1}\right)+\frac{n}{(2n-1)(n-1)}\right] = 0$, and $\frac{\partial}{\partial n}\left[\ln\left(\frac{2n-2}{2n-1}\right)+\frac{n}{(2n-1)(n-1)}\right] < 0$ $\forall n \geq 1$. This implies that $\ln\left(\frac{2n-2}{2n-1}\right)+\frac{n}{(2n-1)(n-1)} > 0$, and hence $h'(n) > 0$. By noting that $h(0) = 0$ and $\lim_{n\to\infty} h(n) = 1$, it follows that $h(n) < 1\ \forall n \in \mathbb{N}$, which implies that $W\left(0\,;Q^A,Q^A\right) > W\left(0\,;Q_{NC}^M,Q_{NC}^M\right) \forall n \geq 1$.

We have thus far established that $W\left(0\,;Q^A,Q^A\right) > W\left(0\,;Q_{NC}^M,Q_{NC}^M\right)$. Moreover, it is straightforward to verify that $W\left(0\,;Q_{FC}^M,Q_{FC}^M\right) > W\left(0\,;Q^A,Q^A\right)$; *i.e.*, the manager should not delegate the choice of Q to the agents if she has full commitment power. Because $Q_{x(y)}^M$ is strictly decreasing in y for all $y < Q_{FC}^M$, $W(0\,;Q,Q)$ is strictly concave in Q, and $Q_{FC}^M < Q_{NC}^M$, it follows that $W\left(0\,;Q_{x(y)}^M,Q_{x(y)}^M\right)$ is strictly increasing in y on $\left[0,Q_{FC}^M\right)$. By noting that $W\left(0\,;Q^A,Q^A\right)$ is independent of y, it follows that there exists some threshold $\theta < Q_{FC}^M$ such that $W\left(0\,;Q^A,Q^A\right) > W\left(0\,;Q_{x(y)}^M,Q_{x(y)}^M\right)$ if and only if $y < \theta$.

□

Proof of Proposition 4. This proof is organized as follows. First, we show that $\Pi(q\,;Q,\,k)$ is the solution to a game in which each agent chooses his effort according to (2.4), and that (2.5) is the corresponding effort strategy. Then we show that this strategy constitutes a PPE.

From (2.4), given the current state of the project q, the FOC yields $a(q\,;Q,\,k) = \frac{k\Pi'(q\,;Q,\,k)}{\lambda}$, and the SOC is always satisfied. Substituting the FOC into each agent's HJB equation yields

$$r\Pi\left(q\,;Q\,k\right)=\frac{\left(2n-k\right)k}{2r\lambda}\frac{\left[\Pi'\left(q\,;Q,\,k\right)\right]^{2}}{2}$$

subject to $\Pi\left(Q\,;Q,\,k\right)\geq 0\ \forall q$ and $\Pi\left(Q\,;Q,\,k\right)=\frac{\beta Q}{n}$.

It is straightforward to verify that $\Pi\left(q\,;Q,\,k\right)=\frac{r\lambda}{2k}\frac{\left(\left[q-C\left(Q\,;k\right)\right]^{+}\right)^{2}}{2n-k}$ solves the above HJB equation, and by using the FOC, it follows that each agent's effort satisfies (2.5). If $k=1$, then (2.5) corresponds to the Markov equilibrium.

We now show that the strategy defined above is indeed a PPE for any $1<k\leq n$. First note that any deviation from the described strategy is detectable arbitrarily quickly. Since the agents can react *quickly*, such deviation can be punished with arbitrarily small delay, so that the gains from a deviation are arbitrarily small. Second, reverting to the Markov equilibrium after a deviation is sequentially rational since the MPE is (by definition) a PPE. Third, observe that $\Pi\left(q\,;Q,\,k\right)>\Pi\left(q\,;Q,\,1\right)$ for all $k>1$ and $q\geq C\left(Q\,;k\right)$, which implies that for any given $1<k\leq n$, as long as each agent chooses his effort to maximize the expected discounted payoff of k agents, no agent has an incentive to unilaterally deviate. Finally, by applying Theorem 4 of Bergin and MacLeod (1993) it follows that there exists a Public Perfect equilibrium in which each agent follows 2.5 along the equilibrium path.

$\square$

Proof of Proposition 5. Suppose first that the manager has full commitment power. Then, her optimal project size is equal to $Q_{FC}^{M}\left(k\right)=\frac{2\beta}{r\lambda}\frac{k(2n-k)}{n}\left(\frac{2n}{4n-k}\right)^{2}$, and it follows that

$$W\left(0\,;Q_{FC}^{M}\left(k\right),Q_{FC}^{M}\left(k\right),\,k\right)=\frac{2\beta\left(1-\beta\right)}{r\lambda}\frac{k\left(2n-k\right)}{n}\left(\frac{2n}{4n-k}\right)^{2}\left(\frac{2n-k}{4n-k}\right)^{\frac{2n-k}{n}}.$$

By differentiating this with respect to k we have that $\frac{\partial}{\partial k}W\left(0\,;Q_{FC}^{M}\left(k\right),Q_{FC}^{M}\left(k\right),\,k\right)>0$ if and only if $2n\left(n-k\right)-k\left(2n-k\right)\ln\left(\frac{2n-k}{4n-k}\right)>0$. This condition holds $\forall k\in[1,n]$. Therefore, in this case the manager's optimal coordination level is $k_{FC}^{M}=n$.

Next, suppose that the manager has no commitment power, so that she eventually completes the project at $Q_{NC}^{M}(k) = \frac{2\beta}{r\lambda}\frac{kn}{2n-k}$. Then it follows that

$$W\left(0\,;Q_{NC}^{M}(k)\,,Q_{NC}^{M}(k)\,,\,k\right) = \frac{2\beta(1-\beta)}{r\lambda}\frac{kn}{(2n-k)}\left(\frac{n-k}{2n-k}\right)^{\frac{2n-k}{n}},$$

and by differentiating this with respect to k we have that $\frac{\partial}{\partial k}W\left(0\,;Q_{NC}^{M}(k)\,,Q_{NC}^{M}(k)\,,\,k\right) > 0$ if and only if $\frac{k(2n-k)}{n}(n-k)\ln\left(\frac{n-k}{2n-k}\right) - \left[k-\left(2+\sqrt{2}\right)n\right]\left[k-\left(2-\sqrt{2}\right)n\right] < 0$. Because $\lim_{k\to n}(n-k)\ln\left(\frac{n-k}{2n-k}\right) = 0$ and $\left(\left[k-\left(2+\sqrt{2}\right)n\right]\left[k-\left(2-\sqrt{2}\right)n\right]\right|_{k=n} < 0$, the last inequality is violated as $k \to n$. Therefore, $\lim_{k\to n}\frac{\partial}{\partial k}W_k\left(0\,;Q_{NC}^{M}(k)\,,Q_{NC}^{M}(k)\,,\,k\right) < 0$, so that $\arg\max_k W\left(0\,;Q_{NC}^{M}(k)\,,Q_{NC}^{M}(k)\,,\,k\right) < n$.

Therefore, we have show that with full commitment power, the manager's optimal cooperation level $k_{FC}^{M} = n$, while with no commitment power, her optimal cooperation level $k_{NC}^{M} < n$.

□

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Chapter 3

The Retail Planning Problem Under Demand Uncertainty

3.1 Introduction

Retail store chains typically carry private label merchandise. For example, department store chain Macy's carries several private label brands such as Alfani, Club Room, Hotel Collection and others. Similarly, Target, J. C. Penney and others carry their own private label brands. Other retail store chains such as GAP, H&M and Zara carry private label products exclusively. Private labels allow firms to differentiate their products from those of their competitors, enhance customer loyalty, and they typically provide higher profit margins. However, these benefits are accompanied by additional challenges. The retailer must plan the entire supply chain by selecting suppliers, and by making decisions on production, distribution and inventory at the retail (and possibly other) locations for each of these private label products in order to minimize total costs. This problem can be complicated when there is a large number of products with uncertain demand that can be sourced from various suppliers, and they are distributed across various demand zones. An example of such a supply chain is illustrated in Figure 1.

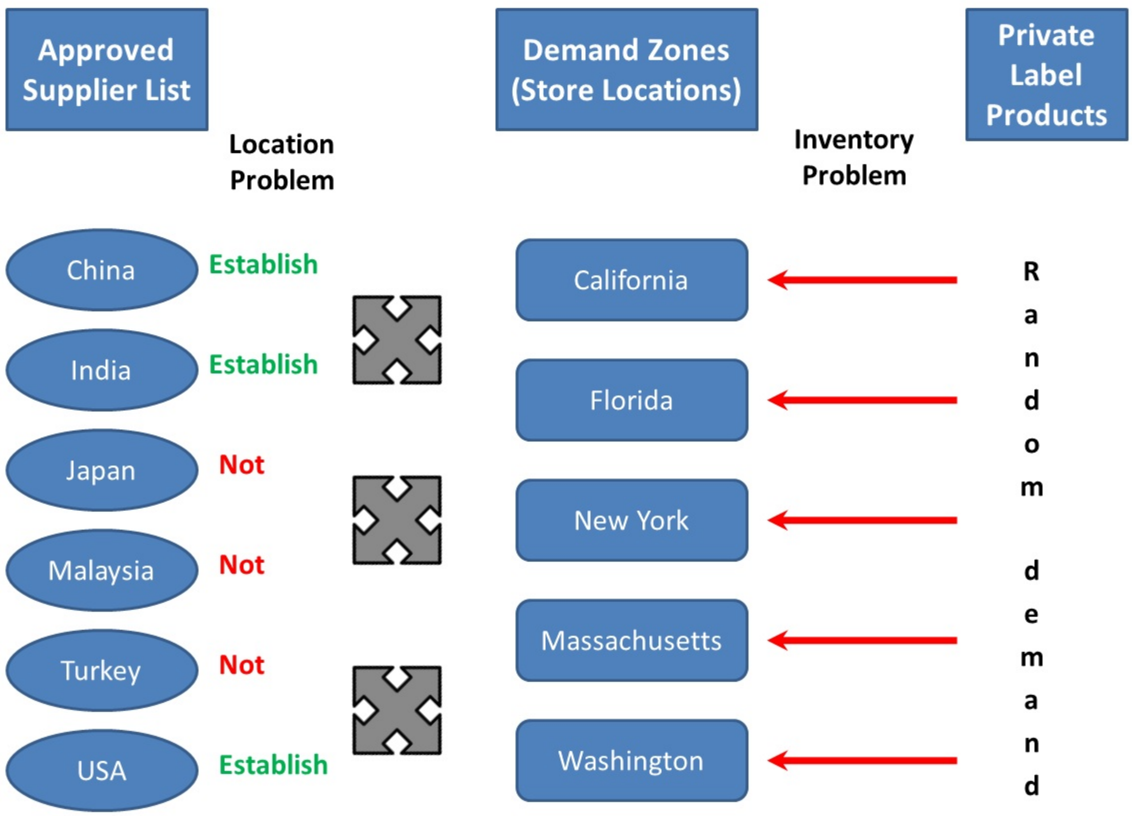

Figure 3.1: The retail supply chain for private label products.

Private label products can be produced in-house, or production can be outsourced to third party suppliers. Without loss of generality, we refer to these options as suppliers. Supplier choice entails fixed costs such as building and staffing a plant when producing in-house or negotiating, contracting and control costs when outsourcing it. Each production facility can manufacture multiple products interchangeably, and there are economies of scale in manufacturing and distribution. Demand at each zone (i.e., store or city) is stochastic and inventory is carried at every demand zone. Here, demand zones can be interpreted either as retail stores, or as distribution centers (DC's).[1] The retailer incurs overstock and understock

[1]With the latter interpretation we implicitly assume (i) that the locations of DCs and the assignment of stores to DCs are pre-determined, and (ii) that stores maintain only a minimal amount of inventory so that inventory costs at individual stores are negligible. This latter assumption is consistent with the existing literature (e.g., Shen at al. (2003)). While it is plausible that management must also determine the location of DCs and allocate stores to DCs, we leave this important problem for future research.

costs for leftover inventory and unmet demand, respectively. In this context, there are three types of decisions. First, the retailer needs to decide which suppliers to choose. Second, they need to conduct production and logistics planning. Third, inventory management decisions on how much of each product to stock at each demand zone need to be made.

We develop the retail planning problem under uncertainty to address these decisions. In this problem, we model the selection of suppliers, production, distribution and inventory decisions faced by the retailer as a nonlinear mixed integer program that minimizes total expected costs. We show that this problem is convex and strongly NP-hard. An interesting attribute of this problem is that it combines two well-known subproblems: a generalized multi-commodity facility location problem and a newsvendor problem. We exploit this structure to develop computationally efficient heuristics to generate feasible solutions. In addition, we apply a Lagrangean relaxation to obtain a lower bound, which we use to assess the quality of the feasible solutions provided by the heuristics. We show that the feasible solutions of a convex programming heuristic are close to optimal: on average within 3.4% of optimal, while in the majority of cases they are closer to optimal as evidenced by the 2.8% median suboptimality gap. Further, the performance gap of this heuristic improves with larger problem sizes, and the computational time of this heuristic scales up approximately linearly in the problem size. We also conduct robustness checks and find that the performance of this heuristic, as well as its advantage relative to the benchmark practitioner's heuristic is not sensitive to changes in the problem parameters. All these are desirable attributes for potential implementation in large-sized, real applications.

Our analysis enables us to draw several managerial insights. First, the optimal inventory level when solving the joint supplier choice, production, distribution and inventory problem is smaller than when the inventory subproblem is solved separately. This is because when solving the joint problem, the solution accounts for the fact that a larger downstream inventory level raises production quantities, which increases upstream production and distribution costs as well as the costs associated with establishing production capacity. In contrast, these costs are not considered when the inventory subproblem is solved separately, and hence result in a larger inventory level. Thus, in order to minimize total supply chain costs, one needs to adopt an integrated approach to solve the joint problem by considering the effect of downstream inventory decisions on upstream production and distribution costs. Our model provides a framework to analyze these decisions. Second, the two major costs that influence total (expected) supply chain costs are production costs and the understock costs associated with the variance in demand. Therefore retailers should focus on reducing these costs first before considering the effects of supplier capacity and contracting costs. Third, it is important to consider establishment, production, distribution and inventory costs together when choosing suppliers, because a supplier who is desirable in any one of these aspects may in fact not be the best overall choice. Our analysis provides a mechanism to integrate these aspects and pick the best set of suppliers.

Since one of the decisions considered in the retail planning problem under demand uncertainty is the establishment of production capacity by the explicit choice of suppliers, this problem can be placed in the broad category of facility location problems under uncertain demand. Aikens (1985), Drezner (1995), Owen and Daskin (1998), Snyder (2006), and Melo et al. (2009) provide extensive reviews. The problem with stochastic demand was first studied by Balachandran and Jain (1976) and Le Blanc (1977), who developed a branch and bound procedure, and a Lagrangean heuristic, respectively. This paper generalizes their models by considering multiple products, as well as incorporating economies of scale in production and distribution.

This paper can also be placed in the general area of integrated supply chain models. Shen (2007) provides a comprehensive review of this area. In particular, this paper is related to Daskin et. al. (2002) and Shen at al. (2003) who studied a location-inventory problem in a supplier - DC - retailer network. Here, the planner's problem is to determine which DCs to establish, the inventory replenishment policy at each DC, and logistics between DCs and retailers. Daskin et. al. (2002) and Shen at al. (2003) solved this problem by using a Lagrangean relaxation and a column-generation approach, respectively. Shen (2005) studied a multi-commodity extension of Daskin et. al. (2002) with economies of scale but without explicitly modeling inventory decisions and without capacity constraints. Relative to these papers, we incorporate economies of scale in both production and distribution, as well as capacity constraints at each supplier. Moreover, we explicitly model the inventory problem. Here, by using the newsvendor instead of a replenishment model to make inventory decisions, we capture features of the retail fashion industry, where lead times are long relative to product lifecycles so that inventory cannot be replenished mid-season, and unmet demand is lost, resulting in underage costs.[2] A related problem was also studied by Oszen et al. (2008) who studied a capacitated extension of Shen at al. (2003). However, unlike these papers we focus on the joint supplier choice, logistics and inventory planning problem, as opposed to the risk pooling effects from strategically locating DCs. This is because manufacturing is often outsourced to third party suppliers and contracts are volume-based, production and inventory decisions are best made simultaneously (Fisher and Rajaram (2000)).[3] Finally, in contrast to all these papers, we motivate an important problem faced by retail chains carrying private label products, propose an effective methodology to generate feasible solutions for this problem, test it on realistic data to assess its performance, and develop insights

[2]Specific lead times faced by manufacturers are reported to be seven months for Oxford shirts ordered by J.C. Penney, and five months for Benetton apparel (Iyer and Bergen (1997)).

[3]For example, leading retailers H&M and GAP outsource 100% of their manufacturing, while Zara outsources approximately 40% of its manufacturing to third party suppliers (Tokatli (2008)). Anecdotal evidence suggests that Macy's outsources all of its manufacturing.

that practitioners can use for choosing suppliers, and making production, distribution and inventory decisions.

The paper is organized as follows: In Section 2 we present the basic model formulation, in Section 3 we discuss the corresponding Lagrangean relaxation, while in Section 4 we propose heuristics. In Section 5 we present results from our numerical study. In Section 6 we summarize and provide future research directions.

3.2 Model Formulation

We formulate the retail planning problem under uncertainty as a nonlinear mixed-integer program. To provide a precise statement of this problem, we define:

Indices:

I, J, K: The set of possible suppliers, demand zones and products, respectively.

i, j, k: The subscripts for suppliers, demand zones and products, respectively.

Parameters:

f_i: Fixed annualized cost associated with choosing supplier i.

d_{ik}: Setup cost associated with producing product k at supplier i.

e_{ij}: Setup cost associated with shipping from supplier i to demand zone j.

c_{ijk}: Marginal cost to produce and ship product k from supplier i to demand zone j.

L_i, U_i: Minimum acceptable throughput and capacity of supplier i, respectively.

α_{ijk}: Units of capacity consumed by a unit of product k at supplier i that is shipped to demand zone j.

h_{jk} / p_{jk}: Per unit overstock / understock cost associated with satisfying demand for product k at demand zone j.

$\Phi_{jk}(\xi)$ / $\phi_{jk}(\xi)$: The cumulative / probability density function of the demand distribution for product k at demand zone j.

Decision variables:

z_i: $0-1$ variable that equals 1 if supplier i is chosen to supply products, and 0 otherwise.

w_{ik}: $0-1$ variable that equals 1 if product k is produced in supplier i, and 0 otherwise.

v_{ij}: $0-1$ variable that equals 1 if supplier i ships to demand zone j, and 0 otherwise.

x_{ijk}: Quantity of product k shipped from supplier i to demand zone j.

y_{jk}: Inventory level of product k carried at demand zone j.

To capture economies of scale so that per-unit production and shipping costs decrease in quantity, we approximate these costs by a setup cost d_{ik} that is incurred to initiate production for each product k at every supplier i, a setup cost e_{ij} that is incurred to ship from each supplier i to every demand zone j, and a constant marginal cost (c_{ijk}) that is incurred to produce and distribute each additional unit. While a more complex cost structure could be desirable in some applications, we employ this structure as it captures economies of scale and it permits structural analysis of the problem.

To model the inventory problem faced by the retailer we employ the newsvendor model. In contrast to Daskin et. al. (2002) and Shen at al. (2003) who use a (Q, r) replenishment model, this paper is motivated by the fashion retail industry, where merchandise is often seasonal and lead times are long relative to the season length. Consequently, the retailer cannot replenish inventory mid-season so that unmet demand is lost, while leftover demand needs to be salvaged via mark downs at the end of the season. Therefore, the standard single-period newsvendor model would seem most appropriate here. Under this model, let $S_{jk}(y)$ denote the expected overstock and understock cost associated with carrying y units of inventory for product k at demand zone j. This can be written as

$$\begin{aligned} S_{jk}(y) &= h_{jk}\int_0^y (y-\xi)\,\phi_{jk}(\xi)\,d\xi + p_{jk}\int_y^\infty (\xi-y)\,\phi_{jk}(\xi)\,d\xi \\ \Longrightarrow S_{jk}(y) &= (h_{jk}+p_{jk})\int_0^y \Phi_{jk}(\xi)\,d\xi + p_{jk}\left[E(\xi)-y\right] \end{aligned} \tag{3.1}$$

The problem of supplier selection, production, distribution, and inventory planning faced by the retailer can be expressed by the following nonlinear mixed-integer program, which we call the Retail Planning Problem (RPP):

(RPP)

$$Z_P = \min\left\{\sum_{i\in I} f_i z_i + \sum_{i\in I}\sum_{k\in K} d_{ik}w_{ik} + \sum_{i\in I}\sum_{j\in J} e_{ij}v_{ij} + \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} c_{ijk}x_{ijk} + \sum_{j\in J}\sum_{k\in K} S_{jk}(y_{jk})\right\}$$

subject to

$$\sum_{i\in I} x_{ijk} = y_{jk} \quad \forall j\in J,\, k\in K \tag{3.2}$$

$$L_i z_i \le \sum_j \sum_k \alpha_{ijk}x_{ijk} \le U_i z_i \quad \forall i\in I \tag{3.3}$$

$$\sum_{j\in J} \alpha_{ijk}x_{ijk} \le U_i w_{ik} \quad \forall i\in I\,,\, k\in K \tag{3.4}$$

$$\sum_{k \in K} \alpha_{ijk} x_{ijk} \leq U_i v_{ij} \quad \forall i \in I\,,\ j \in J \tag{3.5}$$

$$x_{ijk} \geq 0\ \ ,\ \ y_{jk} \geq 0 \quad \forall i \in I\,,\ j \in J\,,\ k \in K \tag{3.6}$$

$$w_{ik} \in \{0,\, 1\}\ \ ,\ \ v_{ij} \in \{0,1\}\ \ ,\ \ z_i \in \{0,\, 1\} \quad \forall i \in I\,,\ j \in J\,,\ k \in K \tag{3.7}$$

The objective function of the RPP consists of four terms. The first term represents the annualized fixed cost associated with securing capacity at supplier i. The second term represents the setup cost associated with production, while the third term represents the setup cost associated with distribution. The fourth term represents the corresponding (constant) marginal production and distribution costs. The fifth term represents the total expected cost associated with carrying inventory at the demand zones.

Constraint (3.2) ensures that total inventory level for each product at every demand zone equals the total quantity produced and shipped to that zone. Note that it is also a coupling constraint. Were it not for (3.2), the RPP would decompose by supplier i into a set of mixed integer linear problems, and by demand zone j and product k into a set of newsvendor problems. This observation suggests that this may be a good candidate constraint to use in any eventual decomposition of the problem. The left hand side inequality of (3.3) imposes a lower bound on the minimum allowable throughput of a supplier, if the supplier is selected. A lower bound on a supplier's throughput may be desirable in order to achieve sufficient economies of scale. The right hand side inequality of (3.3) imposes the capacity constraint (i.e., U_i) for each supplier that is selected, and it enforces that no production will take place with suppliers that are not selected. Constraint (3.4) enforces the condition that $x_{ijk} > 0$ if and only if product k is produced at supplier i (i.e., iff $w_{ik} = 1$ for some $j \in J$), while (3.5) enforces the condition that $x_{ijk} > 0$ if and only if some quantity is shipped from supplier i to demand zone j (i.e., iff $v_{ij} = 1$ for some $k \in K$). Finally (3.6) are non-negativity constraints, while (3.7) are binary constraints.

Observe that the RPP is a convex mixed integer program since it consists of a linear generalized facility location subproblem and a convex inventory planning subproblem. By noting that the Capacitated Plant Location Problem (CPLP) is strongly NP-hard (Cornuejols et. al. (1991)), it can be shown that the RPP is also strongly NP-hard.[4] Therefore, it is unlikely that real-sized problems can be solved to optimality. We verify this in our computational results. Consequently, it is desirable to develop heuristics to address this problem. The quality of these heuristics can be assessed by comparing them to a lower bound, which we establish in the next section.

3.3 Decomposition & Lower Bounds

In order to obtain a tight lower bound, we apply a Lagrangean relaxation to the RPP (see Geoffrion (1974) and Fisher (1981)). An important issue when designing a Lagrangean relaxation is deciding which constraints to relax. In making this choice, it is important to strike a suitable compromise between solving the relaxed problem efficiently and yielding a relatively tight bound. Observe that by relaxing (3.2), the problem can be decomposed into a mixed integer linear program (MILP) containing the x_{ijk}, w_{ik}, v_{ij}, and z_i variables, and into a convex program containing the y_{jk} variables. Moreover, this relaxation enables us to further decompose the MILP by supplier (i.e., by i), and the convex program by demand zone and product (i.e., by j and k) into multiple subproblems. A key attribute of this decomposition

[4]This result can be shown by reducing an instance of the RPP to the CPLP. Specifically, in this reduction, let (i) demand assume a degenerate probability distribution, (ii) the overage and underage costs to be arbitrarily large (i.e., h_{jk} and $p_{jk} \to \infty \; \forall j, k$), (iii) $d_{ik} = 0$ and $e_{ij} = 0 \; \forall i, j, k$, (iv) $L_i = 0 \; \forall i \in I$, and (v) U_i's take values from the set $\{1, .., p\}$ for any fixed $p \geq 3 \; \forall i \in I$.

is that all subproblems can be solved analytically. On the other hand, a potential concern is that this decomposition generates a relatively large number of dual multipliers: $J \times K$ of them, which we denote by λ_{jk}. Relaxing (3.2) for a given $J \times K$-matrix λ of multipliers, the Lagrangean function takes the following form:

$$(L_\lambda)$$

$$L(\lambda) = \min \sum_{i \in I} \left[f_i z_i + \sum_{k \in K} \left(d_{ik} w_{ik} + \sum_{j \in J} (c_{ijk} - \lambda_{jk}) x_{ijk} \right) + \sum_{j \in J} e_{ij} v_{ij} \right] + \sum_{j \in J} \sum_{k \in K} [\lambda_{jk} y_{jk} + S_{jk}(y_{jk})] \quad (3.8)$$

subject to (3.3), (3.4), (3.5), (3.6) and (3.7).

Note that (L_λ) decomposes by i into I independent production and distribution subproblems, and by j and k into $J \times K$ independent inventory subproblems. More specifically, (3.8) can be re-written as:

$$L(\lambda) = \sum_{i \in I} L_i^{milp}(\lambda) + \sum_{j \in J} \sum_{k \in K} L_{jk}^{cvx}(\lambda)$$

where

$$L_i^{milp}(\lambda) = \min \left\{ f_i z_i + \sum_{k \in K} \left[d_{ik} w_{ik} + \sum_{j \in J} (c_{ijk} - \lambda_{jk}) x_{ijk} \right] + \sum_{j \in J} e_{ij} v_{ij} \right\}$$

and

$$L_{jk}^{cvx}(\lambda) = \min \{ \lambda_{jk} y_{jk} + S_{jk}(y_{jk}) \}$$

Note that the Lagrangean multipliers in the production and distribution subproblems (i.e., $L_i^{milp}(\lambda)$) can be interpreted as the cost saved (or cost incurred if $\lambda_{jk} < 0$) from producing and distributing an additional unit of product k to demand zone j. On the other hand, the

Lagrangean multipliers in the inventory subproblems (i.e., $L_{jk}^{cvx}(\lambda)$) can be interpreted as the change in holding cost associated with carrying an additional unit of inventory of product k at demand zone j.

For any given set of multipliers λ, the following Proposition determines the optimal solution for (L_λ), thus providing a lower bound for the RPP.

Proposition 13. *For given set of multipliers $\lambda \in \mathbb{R}^{J\times K}$, a lower bound for the RPP is given by*

$$L(\lambda) = \sum_{i\in I} \min\left\{\min_{j\in J}\left\{e_{ij} + \min_{k\in K}\left\{d_{ik} + (c_{ijk} - \lambda_{jk})\frac{U_i}{\alpha_{ijk}}\right\}\right\} + f_i\,,\, 0\right\}$$
$$+ \sum_{j\in J}\sum_{k\in K}\left[p_{jk}\mathbb{E}_{jk}(\xi) - (p_{jk} + h_{jk})\int_0^{y_{jk}(\lambda_{jk})} \xi\phi_{jk}(\xi)\, d\xi\right], \tag{3.9}$$

where

$$y_{jk}(\lambda) = \begin{cases} \Phi_{jk}^{-1}(1) & \text{if } \lambda_{jk} \le -h_{jk} \\ \Phi_{jk}^{-1}\left(\frac{p_{jk}-\lambda_{jk}}{p_{jk}+h_{jk}}\right) & \text{if } -h_{jk} \le \lambda_{jk} \le p_{jk} \\ \Phi_{jk}^{-1}(0) & \text{if } \lambda_{jk} \ge p_{jk} \end{cases} \tag{3.10}$$

Proof. To begin, fix $\lambda \in \mathbb{R}^{J\times K}$. Let us first consider each production and distribution subproblem. To solve each subproblem, we apply the integer linearization principle by Geoffrion (1974). First, observe that if $z_i = 0$, then $L_i^{milp} = 0$. Hence the optimal solution must satisfy $L_i^{milp}(\lambda) \le 0$. As a result, we fix $z_i = 1$ and solve

$$L_i^{milp}(\lambda\,,\, z_i = 1) \triangleq \min \sum_{k\in K}\left[d_{ik}w_{ik} + \sum_{j\in J}(c_{ijk} - \lambda_{jk})\, x_{ijk}\right] + \sum_{j\in J} e_{ij}v_{ij} + f_i$$

subject to (3.3), (3.4), (3.5), (3.6), and (3.7).

Because the problem is linear, using (3.3) it can easily be shown that $x_{ijk}(\lambda) \in \left\{0, \frac{U_i}{\alpha_{ijk}}\right\}$. Using that $L_i^{milp}(\lambda) \le 0$, (3.4), (3.5) and (3.7), it follows that

$$L_i^{milp}(\lambda) = \min\left\{\min_{j\in J}\left\{e_{ij} + \min_{k\in K}\left\{d_{ik} + (c_{ijk} - \lambda_{jk})\frac{U_i}{\alpha_{ijk}}\right\}\right\} + f_i\,,\, 0\right\}$$

Next, consider each inventory subproblem. It is easy to show that this problem is convex in y_{jk} and by solving the first order condition with respect to y_{jk}, we obtain (3.10), where $\Phi_{jk}^{-1}(\bullet)$ denotes the inverse of $\Phi_{jk}(\bullet)$. Finally, by using (3.1) and $y_{jk}(\lambda_{jk})$, it is easy to show that for each $j \in J$ and $k \in K$, $L_{jk}^{cvx}(\lambda_{jk})$ can be written as

$$L_{jk}^{cvx}(\lambda_{jk}) = p_{jk}\mathbb{E}_{jk}(\xi) - (p_{jk} + h_{jk})\int_0^{y_{jk}(\lambda_{jk})} \xi\phi_{jk}(\xi)\,d\xi$$

By noting that a lower bound can be obtained by $L(\lambda) = \sum_{i\in I} L_i^{milp}(\lambda) + \sum_{j\in J}\sum_{k\in K} L_{jk}^{cvx}(\lambda)$, the proof if complete. □

Note that the Lagrangean solution will chose a supplier (i.e., set $z_i(\lambda) = 1$) if and only if the cost savings associated with producing and distributing an additional unit of product k to demand zone j exceed the fixed cost associated with choosing this supplier for at least some j and k (i.e., if and only if $-\min_{j\in J}\left\{e_{ij} + \min_{k\in K}\left\{d_{ik} + (c_{ijk} - \lambda_{jk})\frac{U_i}{\alpha_{ijk}}\right\}\right\} \geq f_i$). However, this solution may not be feasible. Thus, the purpose of this solution is more to establish the value of the objective function of (L_λ), which is a lower bound on the value of the optimal solution of the RPP. This lower bound can then be used to evaluate the quality of any feasible solution generated by heuristics for this problem. In the unlikely event that the corresponding solution is feasible for the original problem, it then solves the RPP optimally.

In the following Lemma we show that the Lagrangean problem (L_λ) does not possess the integrality property (see Geoffrion (1974)). Therefore the Lagrangean bound is likely to be strictly better than that of a convex programming relaxation (i.e., the relaxation that is obtained by replacing the binary constraints in 3.7 by the continuous interval [0, 1] for the

RPP). We confirm this in our computational results in Section 5.

Lemma 4. *The Lagrangean problem (L_λ) does not possess the integrality property.*

Proof. It suffices to show that a convex programming relaxation of the RPP where (3.7) is replaced by

$$0 \leq w_{ik} \leq 1 \, , \, 0 \leq v_{ij} \leq 1 \quad \text{and} \quad 0 \leq z_i \leq 1 \quad \forall i \in I \, , \, j \in J \, , \, k \in K$$

does not yield a solution such that the w,v, and z variables are integral. We prove this by constructing a counterexample as follows: Let $|J| = |K| = 1$, $e_{i1} = d_{i1} = L_i = 0 \ \forall i \in I$, $\alpha_{i11} = 1 \ \forall i \in I$, and $\Phi_{11}(\xi) = \xi$. To simplify exposition, in the remainder of this proof we drop the subscripts j and k. Observe that by cost minimization, $\forall i$ we will have that $z_i = \frac{x_i}{U_i}$. As a result, it suffices to show that there exists an instance of the convex programming relaxation of the RPP with optimal solution $x_i^* \notin \{0, U_i\}$ for some $i \in I$ (and hence $z_i^* \notin \{0, 1\}$. To proceed, by noting that Slater's condition is satisfied for the primal problem, we dualize (3.2) and write the Lagrangean

$$L(\nu) = \min_{0 \leq x_i \leq U_i} \left\{ \sum_{i \in I} \left(c_i + \frac{f_i}{U_i} + \nu \right) x_i + (h+p) \int_0^y \xi d\xi + \frac{p}{2} - (\nu + p) \, y \right\}$$

It is straightforward to check that for any given dual multiplier ν, the Lagrangean program assumes the following optimal solution:

$$x_i(\nu) = \begin{cases} U_i \ \text{ if } c_i + \frac{f_i}{U_i} + \nu < 0 \\ \in [0, \, U_i] \ \text{ if } c_i + \frac{f_i}{U_i} + \nu = 0 \\ 0 \quad \text{otherwise} \end{cases} \text{, and } y(\nu) = \frac{\nu + p}{h + p}$$

Observe that a solution of the form $x_i \in \{0, \, U_i\}$ will be optimal (and hence $z_i \in \{0, 1\}$) if and only if there exists a dual multiplier ν such that

$$\sum_{i \in I} U_i \mathbf{1}_{\left\{c_i + \frac{f_i}{U_i} + \nu \leq 0\right\}} = \frac{\nu + p}{h + p}$$

By noting that the RHS is a smooth function strictly increasing in ν, while the LHS is a step function decreasing in ν, it follows that there may exist at most one ν such that the above equality is satisfied. We now construct an example in which there exists no ν such that the above equality is satisfied. Letting $h = p = 1$, $|I| = 2$, $U_i = \frac{i}{2}$ and $c_i + \frac{f_i}{U_i} = \frac{i}{3}$, observe that if

$$\begin{array}{ccc} -\frac{1}{3} < \nu & & (\text{LHS}) = 0 < \frac{\nu+1}{2} = (\text{RHS}) \\ -1 < \nu \leq -\frac{1}{3} & \text{then} & (\text{LHS}) = \frac{1}{2} > \frac{\nu+1}{2} = (\text{RHS}) \\ \nu < -1 & & (\text{LHS}) = \frac{3}{2} > \frac{\nu+1}{2} = (\text{RHS}) \end{array}$$

We have thus constructed an instance for which the convex programming relaxation does not yield an optimal solution that is integral, and hence proven that the Integrality Property does not hold. □

We next consider the problem of choosing the matrix of Lagrangean multipliers λ to tighten the bound $L(\lambda)$ as much as possible. Specifically, we are interested in the tightest possible lower bound, which can be obtained by solving:

$$LB_{LR} = \max_{\lambda \in \mathbb{R}^{J \times K}} L(\lambda)$$

One way to maximize $L(\lambda)$ is by using a traditional subgradient algorithm (see Fisher (1985) for details). However this technique may be computationally intensive in our problem as we have $J \times K$ Lagrangean multipliers.

To overcome this difficulty, we exploit the structure of the dual problem to demonstrate how the optimal set of Lagrangean multipliers λ can in some cases be fully or partially determined analytically. In preparation, we establish the following Lemma.

Lemma 5. *The optimal set of Lagrangean multipliers $\lambda^* \in J \times K$ satisfy*

$$\min\left\{\min_{i\in I}\left\{c_{ijk} + \frac{\alpha_{ijk}}{U_i}\left(d_{ik} + e_{ij} + f_i\right)\right\}\ ,\ p_{jk}\right\} \leq \lambda^*_{jk} \leq p_{jk} \quad \forall j \in J \text{ and } k \in K$$

Proof. First, it is easy to check from the first line of (3.9) that $L^{milp}(\lambda)$ decreases in λ, and $L^{milp}(\lambda) = 0$ if $d_{ik} + e_{ij} + (c_{ijk} - \lambda_{jk})\frac{U_i}{\alpha_{ijk}} + f_i \geq 0\ \forall i,\ j,\ k$. By re-arranging terms, one can show that $L^{milp}(\lambda) = 0$ if $\lambda_{jk} \leq c_{ijk} + \frac{\alpha_{ijk}}{U_i}(d_{ik} + e_{ij} + f_i)\ \ \forall i,\ j,\ k$. It is also easy to verify from the second line of (3.8) that $L^{cvx}_{jk}(\lambda_{jk})$ increases in λ_{jk}, and $L^{cvx}_{jk}(\lambda_{jk}) = p_{jk}\mathbb{E}_{jk}(\xi)$ if $\lambda_{jk} \geq p_{jk}\ \forall j\ ,\ k$.
To show that $\min\left\{\min_{i\in I}\left\{c_{ijk} + \frac{\alpha_{ijk}}{U_i}(d_{ik} + e_{ij} + f_i)\right\}\ ,\ p_{jk}\right\} \leq \lambda^*_{jk} \leq p_{jk}$, first suppose that the LHS inequality is not satisfied for some $j\ ,\ k$. Then $L^{milp}_i(\lambda^*) = L^{milp}_i\left(\hat{\lambda}\right)$ and $L^{cvx}_{jk}\left(\lambda^*_{jk}\right) \leq L^{cvx}_{jk}\left(\hat{\lambda}_{jk}\right)$, where $\hat{\lambda} = \max\left\{\lambda^*,\ \min\left\{\min_{i\in I}\left\{c_{ijk} + \frac{\alpha_{ijk}}{U_i}(d_{ik} + e_{ij} + f_i)\right\}\ ,\ p_{jk}\right\}\right\}$. As a result, $L(\lambda^*) \leq L\left(\hat{\lambda}\right)$ and hence λ^* cannot be optimal. Now suppose that $\lambda^*_{jk} > p_{jk}$ for some $j,\ k$. then $L^{cvx}_{jk}\left(\lambda^*_{jk}\right) = L^{cvx}_{jk}(p_{jk})$ and $L^{milp}_i(\lambda^*) \leq L^{milp}_i\left(\lambda^*_{jk}\right)$, where λ^*_{jk} denotes the set of Lagrangean multipliers λ^*, in which the $j-k^{th}$ element has been replaced by p_{jk}. As a result $L(\lambda^*) \leq L\left(\lambda^*_{jk}\right)$, and hence λ^* cannot be optimal. This completes the proof. $\square$

This Lemma states that the optimal set of Lagrangean multipliers λ^* lies in a well-defined compact set. Observe from the left hand side expression in Lemma 2 that $\lambda^*_{jk} > 0\ \forall j\ ,\ k$. From (3.10) observe that the optimal inventory level $y_{jk}\left(\lambda^*_{jk}\right)$ is strictly smaller than the optimal inventory level that would be determined from solving the inventory subproblem separately from the supplier choice and production planning subproblem. This is a direct consequence of performing production, distribution and inventory planning in an integrated manner. The second implication of this Lemma is that the optimal solution of the Lagrangean relaxation will always satisfy $\sum_{i \in I} x_{ijk}(\lambda^*) \geq y_{jk}(\lambda^*)$, and if the set defined in Lemma 2 is a singleton for some j and k, then it is possible to partially characterize the optimal set of Lagrangean multipliers ex-ante. When these sets are singletons for all j and k, then it is possible to completely characterize λ^* ex-ante. This is established by the following Proposition.

Proposition 14. *If* $\min_{i \in I} \left\{ c_{ijk} + \frac{\alpha_{ijk}}{U_i}(d_{ik} + e_{ij} + f_i) \right\} \geq p_{jk}$, *then the optimal Lagrangean multiplier* $\lambda^*_{jk} = p_{jk}$. *If this inequality holds* $\forall j \in J$ *and* $k \in K$, *then* $\lambda^* = p$, *and* $Z_P = LB_{LR}$ *(i.e., the Lagrangean relaxation solves the RPP).*

Proof. For any j and k, if $\min_{i \in I} \left\{ c_{ijk} + \frac{\alpha_{ijk}}{U_i}(d_{ik} + e_{ij} + f_i) \right\} \geq p_{jk}$, then by Lemma 2 $\lambda^*_{jk} = p_{jk}$. If this condition holds for all j and k, then it follows that $\lambda^*_{jk} = p_{jk}$, and by substituting $\lambda^*_{jk} = p_{jk}$ into (3.8) it is easy to check that (L_λ) is feasible for RPP. This completes the proof. □

Observe that $\min_{i \in I} \left\{ c_{ijk} + \frac{\alpha_{ijk}}{U_i}(d_{ik} + e_{ij} + f_i) \right\}$ can be interpreted as the lowest marginal cost associated with establishing capacity at some supplier, producing product k, and distributing it to demand zone j. As a result, when this marginal cost exceeds the marginal underage cost, it is optimal not to produce any quantity of product k for demand zone j, and incur the expected underage cost ; i.e., set $\lambda_{jk} = p_{jk}$, which yields $y_{jk}(p_{jk}) = 0$ by applying (3.10).

By using Lemma 2 and Proposition 2 we now establish a worst-case error bound for the Lagrangean relaxation studied in this section.

Corollary 2. *The worst-case error bound for this Lagrangean relaxation satisfies*

$$\epsilon_{LR} \;\geq\; 1+\max\left\{\frac{-\sum_{j,k}\left(p_{jk}+h_{jk}\right)\int_0^{y\left(\lambda^1_{jk}\right)}\xi\phi_{jk}\left(\xi\right)d\xi}{\sum_{j,k}p_{jk}\mathbb{E}_{jk}\left(\xi\right)}\,,\right.$$
$$\left.\frac{\sum_i \min\left\{\min_{j\in J}\left\{e_{ij}+\min_{k\in K}\left\{d_{ik}+\left(c_{ijk}-p_{jk}\right)\frac{U_i}{\alpha_{ijk}}\right\}\right\},0\right\}}{\sum_{j,k}p_{jk}\mathbb{E}_{jk}\left(\xi\right)}\right\}$$

where $\epsilon_{LR}=\frac{LB_{LR}}{Z_P}$ *and* $\lambda^1_{jk}=\min\left\{\min_{i\in I}\left\{c_{ijk}+\frac{\alpha_{ijk}}{U_i}\cdot\left(d_{ik}+e_{ij}+f_i\right)\right\},\,p_{jk}\right\}\ \forall j$ *and* k. *Moreover, there exists a problem instance of the RPP such that the bound is tight (i.e.,* $\epsilon_{LR}=1$*).*

Proof. First note that the Lagrangean dual is a concave maximization problem, and recall from Lemma 2 that $\lambda^*_{jk}\geq\min\left\{\min_{i\in I}\left\{c_{ijk}+\frac{\alpha_{ijk}}{U_i}\left(d_{ik}+e_{ij}+f_i\right)\right\}\,,\,p_{jk}\right\}=\lambda^1_{jk}$. Moreover, it is easy to check that a trivial feasible solution can be obtained by setting $z_i=w_{ik}=x_{ijk}=y_{jk}=0\ \forall i,\ j,\ k$, in which case the objective function is equal to $\sum_{j,k}p_{jk}\mathbb{E}_{jk}\left(\xi\right)$. As a result, the following inequalities hold:

$$\max\left\{L\left(\lambda^1\right)\,,\,L\left(p\right)\right\}\leq LB_{LR}\leq Z_P\leq\sum_{j,k}p_{jk}\mathbb{E}_{jk}\left(\xi\right)$$

Hence $\epsilon_{LP}=\frac{LB_{LR}}{Z_P}\geq\frac{\max\left\{L\left(\lambda^1\right),L\left(p\right)\right\}}{\sum_{j,k}p_{jk}\mathbb{E}_{jk}\left(\xi\right)}$, and the result follows by substituting $L\left(\lambda^1\right)$ and $L\left(p\right)$ from (3.9). To show that there exists an instance such that this bound is tight, for every $i\in I$, pick f_i such that $\min_{j,k}\left\{d_{ik}+e_{ij}+\left(c_{ijk}-p_{jk}\right)\frac{U_i}{\alpha_{ijk}}\right\}+f_i\geq 0$. Then it is easy to check that $\epsilon_{LR}\geq 1$. Because $\epsilon_{LR}\leq 1$ by definition, we conclude that $\epsilon_{LR}=1$ in this instance. This completes the proof. □

3.4 Heuristics & Upper Bounds

In this section we develop heuristics, which can be used to obtain feasible solutions for the RPP. These heuristics can be used in conjunction with the lower bound developed in Section 3 to provide upper bounds for a branch and bound algorithm, or to generate a feasible solution for the RPP. We initially propose two intuitive heuristics. The first is a practitioner's heuristic developed based on observed practice at a large retail chain. The second is a sequential heuristic, which solves the inventory management subproblem first, and then it solves the remaining standard facility location problem by applying the well-known *Drop* procedure (Klincewicz and Luss (1986)).

These two heuristics can be used to benchmark the performance of the analytically more rigorous heuristics we develop. The first is a convex programming based heuristic, which generates a feasible solution by solving a sequence of convex programs. We also propose a simpler LP-based heuristic, which is computationally more efficient. This heuristic uses the inventory levels from the Lagrangean problem (i.e., $y(\lambda^*)$), and it generates a feasible solution by solving a sequence of linear programs. We next present these heuristics, and we evaluate their performance in Section 5.

3.4.1 Practitioner's Heuristic

This heuristic first chooses the inventory level for every product at each demand zone to equal the respective expected demand; i.e., $y_{jk} = \mu_{jk} \; \forall j \in J, k \in K$. Second, suppliers are sorted according to the ratio $R_i = \frac{f_i}{U_i}$, which captures the fixed cost per-unit of capacity associated with choosing supplier i. Third, the algorithm establishes sufficient capacity to satisfy the total inventory by choosing suppliers that have the lowest R_i. For example if $R_1 \leq R_2 ... \leq R_I$, then the algorithm will set $z_i = 1 \; \forall i \in \{1, .., n\}$ and $z_i = 0$ otherwise, where $n =$

$\min\left\{n \le I \,:\, \sum_{i=1}^{n} U_i \ge \sum_{j\in J}\sum_{k\in K} y_{jk}\right\}$. Finally, production and transportation decisions are made by solving a relaxed version of the RPP, where the fixed cost variables w_{ik} and v_{ij} are relaxed to lie in $[0,1]$. Here, a feasible solution is obtained by rounding to 1 the fractional w_{ik} and v_{ij} variables, and by re-solving the linear program with respect to $x_{ijk} \ge 0$. Note that this heuristic does not take into account the underage and overage costs due to the variation in demand as inventory levels are set to simply equal the mean demand. We denote the objective function of this heuristic by UB_{Pr}. This procedure is formalized in Algorithm 1.

Algorithm 3.1 Practitioner's Heuristic.

1: Let $R_i = \frac{f_i}{U_i}$, and sort candidate facilities such that $R_1 \le R_2 ... \le R_I$.
2: Fix $y_{jk} = \mu_{jk}$ $\forall j$ and k.
3: Let $n = \min\left\{n \le I \,:\, \sum_{i=1}^{n} U_i \ge \sum_{j\in J}\sum_{k\in K} y_{jk}\right\}$.
4: Fix $z_i = 1$ $\forall i = 1,..,n$ and $z_i = 0$ otherwise.
5: Solve the RPP with relaxed variables v_{ij}, $w_{ik} \in [0,1]$.
6: Fix to 1 any $v_{ij} > 0$ and $w_{ik} > 0$, re-solve LP, and compute objective function UB_{Pr}.

A more sophisticated version of this heuristic can be obtained by choosing the inventory levels according to the newsvendor model, and then using the same approach as described in Algorithm 1 to choose suppliers and conduct logistics planning. We call this the newsvendor-based practitioner's heuristic, and we denote its objective function by UB_{Pr-NV}.

3.4.2 Sequential Heuristic

This heuristic obtains a feasible solution for the RPP in two stages: In the first stage, it fixes the inventory level for each product at every demand zone by solving $J \times K$ newsvendor problems. This reduces the problem to a standard capacitated facility location problem with piece-wise linear costs. Then, in the second stage it uses a *Drop* heuristic - a well-known construction heuristic for facility location problems to determine which suppliers to choose. The general idea of the Drop heuristic is to start with a solution in which all candidate suppliers are chosen (i.e., $z_i = 1 \ \forall i$), iteratively deselect one supplier at a time, and solve the remaining subproblem in which the fixed cost variables w_{ik} and v_{ij} are relaxed to lie in $[0, 1]$. Then any fractional w_{ik} and v_{ij} variables are rounded to 1, and the problem is resolved with respect to the x_{ijk} variables. In each loop, the heuristic permanently deselects the supplier who provides the greatest reduction in total expected costs, and it terminates if no further cost reduction is possible. Since exactly one z_i is dropped in each loop, and at least one supplier must be selected in any feasible solution, the algorithm needs at most $I\,(I-1)$ iterations in total, and two convex programs are solved in each iteration. We denote the objective function of this heuristic by UB_{Seq}. This procedure is formalized in Algorithm 2.

Algorithm 3.2 Sequential Heuristic.

Fix $y_{jk} = y$ (newsvendor) $\forall j$ and k
Fix $z_i = 1 \ \forall i$ and $UB_{Seq} = +\infty$.
for $n = 1$ **to** I **do**
    for $m = 1$ **to** I **do**
        if $z_m = 1$ **do**
            Fix $z_i^m = z_i \ \forall i \neq m$ and $z_m^m = 0$.
            Solve the RPP with z_i^m and relaxed variables v_{ij}, $w_{ik} \in [0, 1]$.
            Fix to 1 any $v_{ij} > 0$ and $w_{ik} > 0$, and resolve RPP to find x_{ijk} variables.
            Compute objective function UB_{Seq}^m.
        end if
    end for
    if $\min_m UB_{Seq}^m < UB_{Seq}$ **do**
        $UB_{Seq} = \min_m UB_{Seq}^m$ and $z_{m^*} = 0$, where $m^* = \arg\min_m UB_{Seq}^m$.
        terminate
    end if
end for

For completeness, we also consider a variant of the sequential heuristic that fixes the inventory level for each product at every demand zone to equal the respective expected demand. We call this the *simplified* sequential heuristic, and we denote its objective function by $UB_{Seq-Simple}$.

3.4.3 Convex Programming Based Heuristic

One disadvantage of the practitioner's and the sequential heuristics is that inventory decisions are made independent of supplier selection and logistics decisions. Moreover, the Drop approach used in the sequential heuristic can be computationally intensive. Therefore, we construct a convex programming based heuristic as an alternative way to obtain a feasible solution for the RPP.

The heuristic begins by solving a relaxed RPP where the fixed cost variables z_i, w_{ik} and v_{ij} have been relaxed to lie in $[0,1]$. First, it temporarily fixes the largest fractional z_i to 1, solves the remaining (relaxed) problem, and rounds to 1 any fractional w_{ik} and v_{ij} variables. Second, it temporarily fixes the smallest fractional z_i to 0, and again it solves the remaining (relaxed) problem and rounds to 1 any fractional w_{ik} and v_{ij} variables. The algorithm then permanently fixes the z_i that yielded the lowest total expected costs, and it continues to iterate until all z_i variables have been fixed to 0 or 1. The assumption behind this approach is that the fractional value of z_i is a good indicator of the "worthiness" of choosing supplier i. Since at least one z_i is fixed in each loop, the algorithm needs at most I iterations in total, and two convex programs are solved in each iteration. We denote the objective function of this heuristic by UB_{Cvx}. This procedure is formalized in Algorithm 3.

To gauge the value of joint logistics and inventory planning, we also consider a simplified version of the convex programming heuristic, in which inventory levels are selected in advance using the solution corresponding to the lower bound from the Lagrangean relaxation (i.e., $y_{jk}(\lambda^*)$ $\forall j$ and k). Then the problem of finding a feasible solution reduces to solving a sequence of linear programs, which are easier to solve than convex programs. We denote the objective function associated with this LP-based heuristic by UB_{Lp}.

Algorithm 3.3 Convex Programming Based Heuristic.

Initiate $z_i^{min} = 0$ and $z_i^{max} = 1 \; \forall i$
while $z_i^{max} > z_i^{min}$ for some i **do**
Solve the RPP with relaxed variables v_{ij}, $w_{ik} \in [0, 1]$ and $z_i^{min} \leq z_i \leq z_i^{max}$
    if $z_i \in \{0, 1\}$ **do**
        Set $z_i^{min} = z_i^{max} = z_i$
    end if
    Let $i_{max} = \arg\max \{z_i : z_i \in (0, 1)\}$ and $i_{min} = \arg\min \{z_i : z_i \in (0, 1)\}$.
    Solve the RPP with relaxed variables v_{ij}^+, $w_{ik}^+ \in [0, 1]$, $z_i^{min} \leq z_i^+ \leq z_i^{max}$ and $z_{i_{max}}^+ = 1$.
    Fix to 1 any $v_{ij}^+ > 0$ and $w_{ik}^+ > 0$, and compute objective function UB_{CVX}^+.
    Solve the RPP with relaxed variables v_{ij}^-, $w_{ik}^- \in [0, 1]$, $z_i^{min} \leq z_i^- \leq z_i^{max}$ and $z_{i_{min}}^- = 0$.
    Fix to 1 any $v_{ij}^- > 0$ and $w_{ik}^- > 0$, and compute objective function UB_{CVX}^-.
    if $Z^+ > Z^-$ do
        $z_{i_{max}}^{min} = 1$
    else
        $z_{i_{min}}^{max} = 0$
    end if
end while
Fix to 1 any $v_{ij} > 0$ and $w_{ik} > 0$, re-solve the convex program, and compute UB_{Cvx}.

3.5 Computational Results

In this section we present a computational study to evaluate the performance of the heuristics. In addition, we investigate the key factors that drive their performance, and also examine their robustness. In addition, we use our analysis develop managerial insights about the solution of the RPP.

To test our methods across a broad range of data, we randomly generated the parameter values using a realistic set of data made available to us by a large retailer. We generated 500 random problem instances, each comprising between 5 to 20 candidate suppliers, 10 to 40 demand zones, and 1 to 25 products (i.e., $I \sim U\{5, .., 20\}$, $J \sim U\{10, .., 40\}$ and $K \sim U\{1, .., 25\}$). The parameters we used in our computational study are summarized in Table 1. To solve the optimization problems associated with the bounding techniques we propose, we used the CVX solver for Matlab (CVX (2011)) running on a computer with an

Parameters	Distribution of Values	Parameters	Distribution of Values
Fixed Cost of Choosing a Supplier	$\bar{f} \sim N(50,\, 10)$ $f_i \sim N\left(\frac{2JK}{3I}\bar{f},\, \frac{3JK}{2I}\bar{f}\right)$	Overstock Cost	$\bar{h} \sim N(5,\, 1)$ $h_{jk} \sim U\left[\frac{2}{3}\bar{h},\, \frac{3}{2}\bar{h}\right]$
Setup Cost Associated with Production	$\bar{d} \sim N(200,\, 40)$ $d_{ik} \sim U\left[0,\, \bar{d}\right]$	Understock Cost	$\bar{p} \sim N(50,\, 10)$ $p_{jk} \sim U\left[\frac{2}{3}\bar{p},\, \frac{3}{2}\bar{p}\right]$
Setup Cost Associated with Distribution	$\bar{e} \sim N(200,\, 40)$ $e_{ij} \sim U[0,\, \bar{e}]$	Mean Demand	$\bar{\mu} \sim N(20,\, 4)$ $\mu_{jk} \sim U\left[\frac{2}{3}\bar{\mu},\, \frac{3}{2}\bar{\mu}\right]$
Marginal Production and Distribution Cost	$\bar{c} \sim N(10,\, 2)$ $c_{ijk} \sim U\left[\frac{2}{3}\bar{c},\, \frac{3}{2}\bar{c}\right]$	Demand Variance	$\bar{\sigma} \sim N(5,\, 1)$ $\sigma_{jk} \sim U\left[\frac{2}{3}\bar{\sigma},\, \frac{3}{2}\bar{\sigma}\right]$
Supplier Capacity	$\bar{U} \sim N(100,\, 20)$	Weights	$\alpha_{ijk} = 1$
	$U_i \sim N\left(\frac{40JK}{I}\bar{U},\, \frac{90JK}{I}\bar{U}\right)$	Min. Throughput	$L_i = 0$

Table 3.1: Summary of Parameters used in our Computational Study.

Intel Core i7-2670QM 2.2GHz processor and 6 GB of RAM memory.

To evaluate the performance of the Lagrangean lower bound, we benchmark it against a standard convex programming relaxation, in which the integrality constraints are relaxed so that (3.7) is replaced by

$$0 \leq w_{ik} \leq 1\ ,\ 0 \leq v_{ij} \leq 1\ ,\ 0 \leq z_i \leq 1\ \ \forall i \in I\,,\, j \in J\,,\, k \in K$$

In every one of the problem instances tested, the Lagrangean relaxation generated a better lower bound than the convex programming relaxation, on average by 2.34%. This is consistent with Lemma 1, which asserts that the Lagrangean problem L_λ does not possess the Integrality Property.

To test the performance of the heuristics developed in Section 4, we evaluate the suboptimality gaps relative to the Lagrangean lower bound. Let $\Delta_x = 100\% \, \frac{UB_x - LB_{LR}}{LB_{LR}}$, where $x \in$

$\{Cvx, Lp, Seq, Seq-Simple, Pr, Pr-NV\}$ denote the convex programming based, the LP-based, the sequential, the simplified sequential, the practitioner's, and the newsvendor-based practitioner's heuristic, respectively. The average and median values, as well as the range of these metrics are illustrated in Figure 2.

In addition, to get an idea of the computational complexity of these heuristics, Table 2 reports the mean, median and maximum computational time for the problem instances tested.

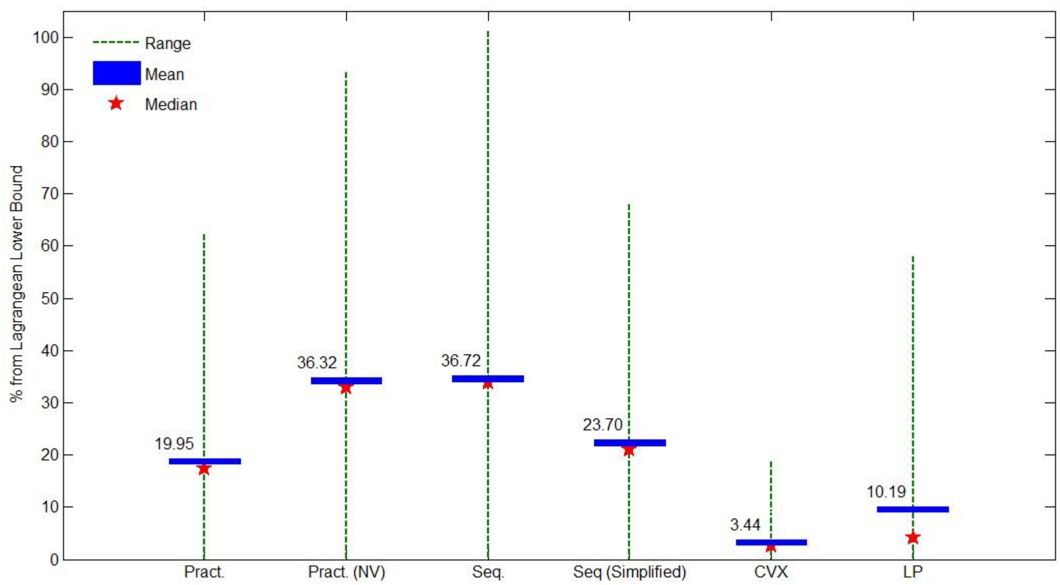

Figure 3.2: Suboptimality Gap.

	UB_{Pr}	UB_{Pr-NV}	UB_{Seq}	$UB_{Seq-Simple}$	UB_{Cvx}	UB_{Lp}
mean	2.02	2.03	80.85	187.41	175.76	105.04
median	1.68	1.72	59.48	91.00	96.78	52.48
max	9.44	10.23	463.98	1974.96	1472.18	949.64

Table 3.2: Computational Times (sec).

First observe from Figure 2 that the convex programming based heuristic unambiguously outperforms the other heuristics. In particular, it provides feasible solutions that are on average within 3.44% of optimal, and range from 0.41% to 18.76%. While the gap of the

LP-based heuristic is higher than the convex programming based heuristic on average, in the majority of cases it generates a feasible solution that is quite close to optimal as evidenced by the median gap of 4.32%. The practitioner's heuristics generate feasible solutions that are on average 19.95% and 36.32% from optimal for the standard and the newsvendor-based version, respectively. On the other hand, the suboptimality gap for the sequential heuristics is on average 36.72% and 23.7% for the standard and the simplified version, respectively.

Interestingly, with both the practitioner's and the sequential heuristic, the version in which inventory levels are set equal to the mean demand outperforms the version in which inventory levels are chosen according to the newsvendor solution. This is because the understock costs are generally larger than the overstock costs, and hence the newsvendor model leads to a larger stocking quantity than the average demand. This in turn increases production and distribution costs, as well as the fixed costs associated with establishing capacity in excess of the benefit of reducing underage costs.

In addition, the inventory levels corresponding to the solution of the convex programming based heuristic are always lower than those determined by the newsvendor solution, and they are often lower than those chosen by the LP-based heuristic. This is because the convex programming heuristic solves the joint problem in contrast to the LP-based heuristic as well as other heuristics in which the inventory levels are chosen separately from the joint problem. When one solves for the joint problem, the solution accounts for the fact that a larger downstream inventory level raises production quantities, which increases upstream production and distribution costs as well as the costs associated with establishing production capacity. In contrast, these costs are not considered when the inventory subproblem is solved separately, and hence result in a larger inventory level. The takeaway from this is that when planning the entire supply chain, it is important to consider the effect of downstream inventory decisions to the upstream production and distribution costs. Retailers often underestimate the impact of upstream costs in their urge to have a higher market share associated with higher fill rates. When such costs are adequately represented, a lower fill rate may actually be preferable in order to lower total costs. Finally, note that the cost reduction resulting from the convex programming based heuristic relative to the other heuristics is important, because retailers operate in a highly competitive environment with very low margins and even a small cost reduction can lead to a large profit increase.

Next, consider the computation time for each heuristic. Observe that both practitioner's heuristics are computationally very fast, while both sequential heuristics are quite slow. Also note that the standard sequential heuristic is computationally less intensive than its simplified counterpart. Because the standard sequential heuristic chooses the stocking quantities according to the newsvendor model, which in general are higher than the expected demand, the Drop procedure needs fewer iterations in the standard sequential heuristic. Finally, observe that the convex programming based heuristic is about as computationally intensive as the simplified sequential heuristic, but leads to much lower average gaps. Thus it clearly dominates both versions of the sequential heuristic. However, as expected, it is computationally more intensive than the LP-based heuristic.

Since the convex programming heuristic dominates the other heuristics in terms of the gap from the lower bound, we focus on this heuristic to examine (a) how the computational time scales up with the size of the problem, (b) how the suboptimality gap and its performance advantage relative to the practitioner's heuristic depend on the parameters of the problem, and (c) which parameters have the greatest impact on the total expected costs.

To conduct this analysis, we regress the computational times, the suboptimality gap of the convex programming heuristic (i.e., Δ_{Cvx}), the gap between the convex programming and the practitioner's heuristic (i.e., $100\% \frac{UB_{Cvx}-UB_{Pr}}{UB_{Pr}}$), and the total expected cost associated with the convex programming heuristic (i.e., UB_{Cvx}) of the 500 problem instances tested earlier on the size (i.e., I, J, K), and the parameters of the problem (i.e., $\bar{\mu}$, $\bar{\sigma}$, $\bar{h}$, $\bar{p}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{f}$, $\bar{U}$). Table 3 summarizes the results.[5]

	Computational Time	*Suboptimality Gap*	Cvx. vs. Pract. H.	Expected Cost
I	23.93*** (1.301)	-0.13*** (0.034)	0.32*** (0.089)	5247.77*** (1094.73)
J	7.66*** (0.651)	-0.09*** (0.016)	0.06 (0.042)	9798.92*** (517.59)
K	18.33*** (0.803)	-0.16*** (0.026)	0.15** (0.067)	19590.85*** (822.44)
$\bar{\mu}$	-0.35 (0.893)	-0.12*** (0.017)	0.25*** (0.044)	11042.93*** (543.84)
$\bar{\sigma}$	0.04 (3.732)	0.101 (0.071)	-0.12 (0.185)	48.77 (2278.22)
$\bar{h}$	7.79 (5.706)	0.12 (0.109)	0.12 (0.287)	7702.49** (3521.77)
$\bar{p}$	0.023 (0.548)	0.012 (0.0104)	0.25*** (0.027)	2048.43*** (334.45)
$\bar{c}$	6.13 (2.69)	0.055 (0.051)	-0.038 (0.134)	8629.04*** (1647.61)
$\bar{d}$	0.02 (1.108)	0.026 (0.0204)	-0.053 (0.054)	261.74 (657.6)
$\bar{e}$	-1.59 (1.099)	0.011 (0.021)	-0.007 (0.055)	1269.01* (672.77)
$\bar{f}$	-1.40** (0.551)	-0.00003*** (0.000003)	-0.00008*** (0.00001)	2.29*** (0.12)
$\bar{U}$	1.38*** (0.266)	0.0015*** (0.0001)	0.002*** (0.0003)	-37.82*** (3.35)
Intercept	-689.2*** (88.8)	8.52*** (1.469)	-36.62*** (3.851)	-830954.6*** (47333.18)
R^2	0.641	0.40	0.315	0.891

Table 3.3: Suboptimality Gap vs. Problem Parameters (Convex Programming Heuristic).

[5]The values above the parantheses denote the regression coefficients corresponding to the parameter in the left column. The values in parentheses denote standard errors. * denotes significance at 10% level, ** denotes significance at 5% level, and *** denotes significance at 1% level.

First note that the computational time of the convex programming heuristic is strongly dependent on the problem size (i.e., I, J and K), while it is insensitive to the other parameters of the problem. More interestingly, the relatively large R^2 ratio implies that the computational time of the convex programming heuristic is explained by a linear model well, which in turn suggests that the computational time scales up approximately linearly in the problem size.

From the second column, observe that the suboptimality gap decreases in the size of the problem (I, J and K), and this effect is significant at the 1% level. This finding is encouraging: it predicts that the convex programming heuristic will perform even better in larger problem instances that could be expected in some applications. The suboptimality gap increases in the capacity of the candidate suppliers ($\bar{U}$), while it decreases in the mean demand (μ) and the fixed costs associated with choosing a supplier ($\bar{f}$). The suboptimality gap also increases in the demand variance ($\bar{\sigma}$), the underage and overage costs ($\bar{p}$ and $\bar{h}$), and the production costs ($\bar{c}$, $\bar{d}$, and $\bar{e}$), but this effect is not significant at the 10% level. Finally, note that a 95% confidence interval for each regression coefficient can be obtained from (regression coeff.) $\pm$ 1.9648 (std. error).[6] Therefore, as seen in Table 3, because the values of all regression coefficients and their respective standard errors are close to zero, |(regression coeff.) $\pm$ 1.9648 (std. error)| is close to zero for all parameters. This shows that the performance of the convex programming heuristic is robust to changes in the parameters of the RPP.

The third column examines how the performance advantage of the convex programming heuristic relative to the practitioner's heuristic depends on the parameters of the problem.

[6] $\pm$1.9648 corresponds to the 2.5 and 97.5 percentile of a t-distribution with $500 - 13$ degrees of freedom.

Observe that the performance advantage of the convex programming heuristic becomes larger in the size of the problem, while it is insensitive to the cost parameters as evidenced by the small regression coefficients and the small (respective) standard errors. This, together with the finding that the value of the intercept is negative at the 1% significance level, reinforces the benefits from using the convex programming heuristic, as one could expect even larger problems with different cost parameters in certain applications.

The fourth column considers the relationship between the total expected cost of the feasible solutions generated by the convex programming heuristic, and the parameters of the problem. Predictably, the expected cost increases in the size of the problem (I, J and K), in the mean demand ($\bar{\mu}$), in the production costs ($\bar{c}$, $\bar{d}$ and $\bar{e}$), in the fixed costs associated with choosing a supplier ($\bar{f}$), as well as in the underage and overage costs ($\bar{p}$ and $\bar{h}$). On the other hand, the expected cost decreases in the capacity of the candidate suppliers ($\bar{U}$), while the effect of the demand variance ($\bar{\sigma}$) is insignificant. Therefore, our findings suggest that besides the problem size (i.e., I, J, K and $\bar{\mu}$), the two most important factors that affect the expected cost of a feasible solution are (i) the marginal production cost and (ii) the inventory underage and overage costs. The latter observation emphasizes the value of an improved demand forecast. On the other hand, the capacity of a supplier as well as the fixed contracting costs appear to have a secondary effect. This is consistent with the initiatives undertaken at several retailers to reduce the impact of production, inventory underage and overage costs (Fisher and Raman (2010)).

Since the gaps of the convex programming based heuristic are the smallest, we analyze the solutions to develop some insights about how it chooses suppliers. This could be useful for practitioners who make such decisions. We find that suppliers are chosen in increasing order of the ratio r_i, where

$$r_i = \frac{1}{U_i}\left[f_i + \frac{1}{|J|\,|K|}\sum_{j,k}\left(d_{ik} + v_{ij} + c_{ijk}\frac{\alpha_{ijk}}{U_i}\right)\right]$$

The term in brackets represents the sum of fixed establishment costs and the average production and distribution costs across products and demand zones when a supplier is fully utilized. Therefore the ratio r_i can be interpreted as the average total cost per unit of capacity at supplier i. This suggests that it is important to consider establishment, production and distribution costs together when choosing suppliers, and it is beneficial to choose suppliers with the lowest total average cost per unit of capacity.

3.6 Conclusions

We analyze a multi-product retail planning problem under demand uncertainty, in which the retailer jointly chooses suppliers, plans production and distribution, and selects inventory levels to minimize total expected costs. This problem typically arises in retail store chains carrying private label products, who need to plan the entire supply chain by making decisions with respect to (i) supplier selection for their private label products, (ii) distribution of products from suppliers to demand zones (i.e., stores or distribution centers), and (iii) the inventory levels for every product at each demand zone. This problem is formulated as a mixed integer convex program.

Since the retail planning problem is strongly NP-hard, we use a Lagrangean relaxation to obtain a lower bound, and we develop heuristics to generate feasible solutions. First we develop an analytic solution for the Lagrangean problem (Proposition 1), and we establish conditions under which the Lagrangean dual can be solved analytically (see Proposition 2).

We first develop a practitioner's and a sequential heuristic. We then propose two heuristics, which reduce the problem of generating a feasible solution to solving a sequence of convex or linear programs. To test the performance and the robustness of our methods we conduct an extensive computational study. The convex programming based heuristic and its LP-based counterpart yields feasible solutions that are on average within 3.4% and 10.2% from optimal, respectively. Sensitivity analysis suggests that the computational time of the convex programming heuristic scales up approximately linearly in the problem size, while it is stable to changes in problem parameters. Finally, these heuristics outperform both the Sequential and the Practitioner's heuristics, and the performance advantage of the convex programming based heuristic relative to the practitioner's heuristic is robust to the parameters of the problem. All these are desirable features for any eventual implementation in large sized real applications.

Several managerial insights can be drawn from this work. First, solving the more complicated joint supplier choice, production, distribution and inventory problem leads to a leaner supply chain with lower inventory levels than solving the inventory subproblem separately from the supplier choice and logistics subproblem. This highlights the importance of considering the effect of inventory decisions on upstream production and distribution costs. Our methodology provides an effective approach to solve this joint problem. Second, the major costs that influence supply chain costs across the retailer are production costs, as well as the understock and overstock costs associated with carrying inventory at the demand zones. Therefore retailers should focus on reducing these costs first before considering the effects of supplier capacity and contracting costs. Third, it is important to consider establishment, production, distribution and inventory costs together when choosing suppliers, because a supplier who is desirable in any one of these aspects may in fact *not* be the best overall choice. Our analysis provides a mechanism to integrate these aspects and pick the best set of suppliers.

This paper opens up several opportunities for future research. First, this problem could be extended to explicitly model nonlinear production and shipping costs, which is of particular interest for applications that exhibit significant economies of scale. In that case the problem formulation is a mixed integer nonlinear program that is neither convex nor concave (see Caro et al. (2012) for details about addressing a related problem in the process industry with uncertain yields). Second, our model could be extended to incorporate multiple echelons in the supply chain (i.e., wholesalers, distribution centers, etc.) and allow multiple echelons to carry inventory. Third, it may be desirable to incorporate side constraints pertaining to facilities, production and distribution (i.e., v, x, w, and z variables) as in (Geoffrion and McBride (1978)). Undoubtedly, all of these extensions would require significant, non-trivial modifications to our model. Finally, further work could be done to improve the heuristics in order to further reduce the suboptimality gap.

In conclusion, we believe the methods described in this paper provide an effective methodology to address the retail planning problem under demand uncertainty.

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