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Barcode Entropy for Symplectomorphisms Isotopic to the Identity

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BARCODE ENTROPY FOR SYMPLECTOMORPHISMS ISOTOPIC TO THE IDENTITY

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by

John Gabriel P. Pelias

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Abstract

Barcode Entropy for Symplectomorphisms Isotopic to the Identity

by

John Gabriel P. Pelias

Çineli, Ginzburg, and Gürel recently defined a new quantity, called the barcode entropy, which is calculated using barcodes of a Floer-Novikov complex, similar to barcodes arising in persistence homology and Morse theory. They were able to relate this to the classical topological entropy, a number that quantifies the complexity of the orbits of a map. This quantity is of high interest in dynamics as its positivity indicates that the orbits of a dynamical system are more likely chaotic. They were able to define barcode entropy and find a connection between this quantity and topological entropy for the case when the map in question is a Hamiltonian diffeomorphism. In this dissertation, we extend their results to the case when the map is more generally just a symplectomorphism isotopic to the identity.

The author dedicates this work to three “women”
who have shown me love in three different ways:

To my grandmother S.

To my dearest beloved J.

And, to my undergraduate *alma mater*, U.P.

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Chapter 1

Introduction

In this dissertation, we present how to extract the topological entropy of a symplectomorphism isotopic to the identity from its Floer-Novikov complex, via methods inspired by persistent homology. An important special class of symplectomorphisms isotopic to the identity are the Hamiltonian diffeomorphisms. And indeed, in [ÇGG21], Çineli, Ginzburg, and Gürel introduced for the first time a Floer-theoretic invariant called the barcode entropy $\hbar(\psi)$ of a Hamiltonian diffeomorphism ψ and demonstrated how this new invariant relates to the classical topological entropy of ψ. The barcode entropy roughly measures the exponential growth rate with respect to iterations of ψ of numbers of bars with length exceeding a threshold ϵ in the barcode of the Floer complex associated to ψ. The so-called bars are differences in the action functional associated to the Hamiltonian diffeomorphism, which in turn are energies of pseudoholomorphic cylinders. The investigation of such a barcode associated to a Floer complex is inspired by recent trends of using techniques of persistent homology to study Morse theory and Floer homology.

The definition of the barcode entropy somewhat resembles the definition of topological entropy, as the latter also roughly measures the exponential growth rate with respect to iterations of ψ of numbers of orbit segments that are distinguishable up to an accuracy threshold ϵ. While the two definitions structurally look similar, there is no *a priori* reason to believe that the barcode entropy is immediately related—even more so, equal—to the topological entropy. The former is a Floer-theoretic invariant, while the latter is a dynamical invariant that is defined for a general dynamical system even if the underlying manifold is absent of a symplectic structure. It is not even apparent why there must be a connection between topological entropy and features of a Floer complex. And yet, in [ÇGG21], Çineli, Ginzburg, and Gürel show in Theorems A and B that the two quantities bound each other under certain conditions, and, in Theorem C, that the two notions of entropy in fact coincide if the underlying symplectic manifold is a closed surface.

As the authors have indicated in their work, the only prior work that hinted towards possible connections between topological entropy and symplectic topology are those by Alves such as in [Al16-1], [Al16-2], [Al19], in which the topological entropy of a Reeb flow, symplectomorphism, or contactomorphism was related to homological growth. While this inspired the authors, it is not exactly applicable to their setting, since in the case of Hamiltonian diffeomorphisms, the Floer homology is independent of the order of iteration and hence no homological growth registers. Indeed, in [ÇGG21], the barcode entropy the authors have defined is a quantity that depends on the map and its iterations, and hence lends itself more to a connection with the more dynamical invariant that is the classical topological entropy.

In this dissertation, we give generalizations of the results that Çineli, Ginzburg, and Gürel proved in [ÇGG21]. We weaken the assumption that φ is a Hamiltonian diffeomorphism to the more general case that φ is only a symplectomorphism isotopic to the identity. Aside from the fact that symplectomorphisms isotopic to the identity are a natural next step to generalize Hamiltonian diffeomorphisms, as we will present in this work, under certain conditions, the action functional in our setting differs from that in the Hamiltonian setting by a term that involves the flux homomorphism. The fact that this is non-zero can be thought of as a measure of how far from being a Hamiltonian diffeomorphism the symplectomorphism is. However, when considering the barcode and ultimately defining the barcode entropy, the bars being action functional differences, this "flux term" disappears, and so it is reasonable to believe that analogous results can be proved in our more general setting.

Of course, the difficulty rests more on whether or not we can even define a barcode and ultimately a barcode entropy for a symplectomorphism isotopic to the identity, and indeed much of the first few stages of this work is devoted to constructing the appropriate Floer complex to which we can associate an appropriate barcode. Certain conditions must be imposed on the underlying symplectic manifold in order for our definitions to make sense. Indeed, in the entirety of this work, we assume that (M, ω) is a weakly monotone symplectic manifold. This will guarantee that no pseudo-holomorphic sphere will have negative Chern number. See, for instance, [HS]. We will denote the group of symplectomorphisms $\varphi : M \to M$ by $\mathrm{Symp}(M, \omega)$. The connected component of id_M, i.e. the subgroup of symplectomorphisms isotopic to the identity, will be denoted $\mathrm{Symp}_0(M, \omega)$. Let $\widetilde{\mathrm{Symp}}_0(M, \omega)$ denote the universal cover of $\mathrm{Symp}_0(M, \omega)$.

The raison d'être of this dissertation is to prove the following theorems, which are generalizations of the analogous theorems of Çineli, Ginzburg, and Gürel in [ÇGG21]:

Theorem (Theorem A)**.** *Let* $\varphi \in \mathrm{Symp}_0(M, \omega)$ *and* $\{\varphi_t\}$ *be a symplectic isotopy* *connecting* $\varphi_0 = \mathrm{id}_M$ *to* $\varphi_1 = \varphi$. *Then*

$$\hbar([\varphi_t]) \leq h_{\mathrm{top}}(\varphi).$$

Theorem (Theorem B)**.** *Let* $\varphi \in \mathrm{Symp}_0(M, \omega)$ *and* $\{\varphi_t\}$ *be a symplectic isotopy connecting* $\varphi_0 = \mathrm{id}_M$ *to* $\varphi_1 = \varphi$. *If* $K \subseteq M$ *is a locally maximal hyperbolic subset, then*

$$\hbar([\varphi_t]) \geq h_{\mathrm{top}}\left(\varphi|_K\right).$$

Corollary (Theorem C)**.** *If* (M, ω) *is a closed symplectic surface and* $\varphi \in \mathrm{Symp}_0(M, \omega)$ *with symplectic isotopy* $\{\varphi_t\}$, *then*

$$\hbar([\varphi_t]) = h_{\mathrm{top}}(\varphi).$$

This dissertation is organized as follows: In Chapter 2, we construct the Floer-Novikov complex of a symplectomorphism isotopic to the identity. We closely follow constructions made by Batoréo in [Bat], which are ultimately based on the constructions made by Burghelea and Haller in [BH].

In Chapter 3, we review the theory of persistent homology and barcodes. We recall the barcodes arising classically in topological data analysis and Morse theory, following for instance Polterovich, Rosen, Samvelyan, and Zhang in [PRSZ], and those arising in Floer theory. Specifically, we recall barcodes associated to Floer-type complexes as in the work of Usher and Zhang in [UZ].

In Chapter 4, we review the notion of topological entropy, and some classical results on topological entropy and the closely related notion of measure-theoretic entropy. In particular, we recall two important concepts—Yomdin's Theorem and horseshoes in hyperbolic dynamics—that will enable us to relate the classical topological entropy with barcode entropy.

Finally, in Chapter 5, we define the barcode entropy of a symplectomorphism isotopic to the identity. It generalizes the barcode entropy of a Hamiltonian diffeomorphism defined by Çineli, Ginzburg, and Gürel in [ÇGG21]. We then restate and prove Theorems A, B, and C above.

Chapter 2

Floer-Novikov Complex of a Symplectomorphism Isotopic to the Identity

In this chapter, we construct the Floer-Novikov complex associated to a symplectomorphism isotopic to the identity. We will closely follow the construction of Batoréo in [Bat], which is ultimately based on the construction of Burghelea and Haller in [BH]. Let (M, ω) be a closed connected symplectic manifold of dimension $2n$. Let $\widehat{c_1} \in \Omega^2(M)$ be a 2-form representing the first Chern class $c_1 \in H^2(M, \mathbb{Z})$. Recall that (M, ω) is *weakly monotone* if for every $A \in \pi_2(M)$,

$$3 - n \leq c_1(A) < 0 \quad \Longrightarrow \quad \omega(A) \leq 0.$$

2.1 Flux and the Deformation Lemma

Let $\varphi \in \mathrm{Symp}_0(M,\omega)$. The lift of φ in $\widetilde{\mathrm{Symp}_0}(M,\omega)$ is then the homotopy class, with fixed endpoints, of a path of symplectomorphisms $\{\varphi_t\}_{0\leq t\leq 1}$ connecting $\varphi_0 = \mathrm{id}$ to $\varphi_1 = \varphi$. We will denote such a homotopy class by $[\varphi_t]$.

Recall that there exists a homomorphism $\mathrm{Flux} : \widetilde{\mathrm{Symp}_0}(M,\omega) \to H^1(M,\mathbb{R})$, called the *flux homomorphism*, that associates to a lift $[\varphi_t] \in \widetilde{\mathrm{Symp}_0}(M,\omega)$ the cohomology class

$$\left[\int_0^1 \omega(X_t,\cdot)\mathrm{d}t\right] = \left[\int_0^1 i_{X_t}\omega\mathrm{d}t\right],$$

where X_t is the vector field associated with $\{\varphi_t\}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = X_t \circ \varphi_t.$$

See [MSa17] for a proof, for instance. Clearly, if φ is a Hamiltonian diffeomorphism, then $\mathrm{Flux}[\varphi_t] = 0$.

An important result of Banyaga (for instance in [Ban]) is that the kernel of the flux homomorphism is in fact $\mathrm{Ham}(M,\omega)$. Moreover, another standard fact is that Flux is surjective. Hence, one can think of a non-zero cohomology class in $H^1(M,\mathbb{R})$ as an obstruction to the corresponding symplectomorphism being Hamiltonian. For instance, since $H^1(S^2,\mathbb{R}) = 0$, we know that every symplectomorphism on the 2-sphere isotopic to the identity is in fact Hamiltonian. On the other hand, $H^1(\mathbb{T}^2,\mathbb{R}) = \mathbb{R}^2$, and so there is a wealth of symplectomorphisms on the 2-torus that are not Hamiltonian. For instance, whenever a symplectomorphism $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ is of the form

$$\varphi(x,y) = (x + \alpha(x,y), y + \beta(x,y)),$$

where either

$$\int_{\mathbb{T}^2} \alpha \mathrm{d}x\mathrm{d}y \neq 0 \quad \text{or} \quad \int_{\mathbb{T}^2} \beta \mathrm{d}x\mathrm{d}y \neq 0,$$

φ is not Hamiltonian. See [MSa17] for more details.

Let θ be a 1-form on M such that $[\theta] = \mathrm{Flux}[\varphi_t]$. By the Deformation Lemma 2.1 in [LO] proved by Lê and Ono, there exists an isotopy with fixed endpoints within $\widetilde{\mathrm{Symp}}_0(M, \omega)$ that deforms the path of symplectomorphisms $\{\varphi_t\}$ to a path $\{\varphi'_t\}$ such that for all $t \in [0,1]$,

$$\mathrm{Flux}[\varphi'_t] = [\theta] = \left[i_{X'_t}\omega \right],$$

where X'_t is the vector field associated to $\{\varphi'_t\}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi'_t = X'_t \circ \varphi'_t.$$

Thus, there exists a Hamiltonian $H_t : M \to \mathbb{R}$ periodic in t with period 1 such that for all $t \in [0,1]$,

$$i_{X'_t}\omega = \theta + \mathrm{d}H_t.$$

Define $\eta_t := i_{X'_t}\omega = \theta + \mathrm{d}H_t$. Define the vector fields X_{η_t}, X_θ, and X_{H_t} by

$$\begin{aligned} i_{X_{\eta_t}}\omega &= -\eta_t, \\ i_{X_\theta}\omega &= -\theta, \\ i_{X_{H_t}}\omega &= -\mathrm{d}H_t. \end{aligned}$$

Note that while X_{η_t} and X_{H_t} are time-dependent, X_θ is time-independent. Moreover,

$$X_{\eta_t} = X_\theta + X_{H_t}.$$

2.2 Loop Spaces

We will denote by $\mathcal{L}M$ the space of all smooth free loops $x : S^1 \to M$ in M. That is, $\mathcal{L}M := C^\infty(S^1, M)$. Here, we treat S^1 as $\mathbb{R}/\mathbb{Z}$. Note that for a loop $x \in \mathcal{L}M$, the tangent space $T_x(\mathcal{L}M)$ consists of vector fields $\xi : S^1 \to TM$ along the loop x. Let $I \subseteq \mathbb{R}$ be any interval. There is an obvious one-to-one correspondence

$$\begin{array}{rcl} \mathrm{Maps}(I, \mathcal{L}M) & \to & \mathrm{Maps}(I \times S^1, M) \\ \widetilde{u} & \mapsto & u \end{array}$$

between the space $\mathrm{Maps}(I, \mathcal{L}M)$ of mappings $\widetilde{u} : I \to \mathcal{L}M$, i.e. "paths" in $\mathcal{L}M$, and the space $\mathrm{Maps}(I \times S^1, M)$ of mappings $u : I \times S^1 \to M$, i.e. "cylinders" in M, given by

$$u(s,t) = (\widetilde{u}(s))(t). \tag{2.1}$$

Define the 1-form $\bar{\omega} \in \Omega^1(\mathcal{L}M)$ by

$$\bar{\omega}_x(\xi) := \int_{S^1} \omega_{x(t)} \left(\xi(t), \dot{x}(t)\right) \mathrm{d}t,$$

for each loop $x \in \mathcal{L}M$ and vector field $\xi \in T_x(\mathcal{L}M)$ along x. Then, whenever $\widetilde{u} : I \to \mathcal{L}M$ is a smooth path of loops in M,

$$\begin{aligned} \int_I \widetilde{u}^* \bar{\omega} &= \int_I \bar{\omega}_{\widetilde{u}(s)} \left(\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}s} \right) \mathrm{d}s \\ &= \int_I \int_{S^1} \omega_{(\widetilde{u}(s))(t)} \left(\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}s}, \frac{\mathrm{d}}{\mathrm{d}t} \left(\widetilde{u}(s)\right) \right) \mathrm{d}t \mathrm{d}s \\ &= \int_{I \times S^1} \omega_{u(s,t)} \left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) \mathrm{d}s \mathrm{d}t, \\ &= \int_{I \times S^1} u^* \omega \end{aligned}$$

where $u : I \times S^1 \to M$ is the map corresponding to $\widetilde{u}$ in (2.1).

Now, for each $t_0 \in S^1$, denote by $\mathrm{ev}_{t_0} : \mathcal{L}M \to M$ the evaluation map at t_0. That is, for a loop $x : S^1 \to M$,

$$\mathrm{ev}_{t_0} : x \mapsto x(t_0).$$

Then, for a path of loops $\widetilde{u} : I \to \mathcal{L}M$ with corresponding cylinder $u : I \times S^1 \to M$ according to (2.1), we have the segment $\mathrm{ev} \circ \widetilde{u} : I \to M$ in M given by

$$(\mathrm{ev}_{t_0} \circ \widetilde{u})(s) = (\widetilde{u}(s))\,(t_0) = u(s, t_0)$$

Define the 1-form $\varrho \in \Omega^1(\mathcal{L}M)$ by $\varrho = -\bar{\omega} + \mathrm{ev}_0^* \theta$. Thus, if $\widetilde{u} : I \to \mathcal{L}M$,

$$\begin{aligned} \int_I \widetilde{u}^* \varrho &= -\int_I \widetilde{u}^* \bar{\omega} + \int_I (\mathrm{ev}_0 \circ \widetilde{u})^* \theta \\ &= -\int_I u^* \omega + \int_I (u_0)^* \theta, \end{aligned}$$

where $u_0 : I \to M$ is the segment given by

$$u_0(s) := u(s, 0) = (\mathrm{ev}_0 \circ \widetilde{u})(s).$$

Thus, if $I = S^1$, so that $\tilde{u}$ represents a loop of loops (or a "torus") in M, and $[\varrho] \in H^1(\mathcal{L}M)$ and $[\widetilde{u}] \in \pi_1(\mathcal{L}M)$ are the cohomology and homotopy classes of ϱ and $\widetilde{u}$, respectively, we have

$$\langle [\varrho], [\tilde{u}] \rangle = -\int_{S^1 \times S^1} u^* \omega + \int_{S^1} (u_0)^* \theta.$$

Now, for a fixed homotopy class ζ of loops $S^1 \to M$, fix a reference loop $z : S^1 \to M$ in ζ. For each loop $x : S^1 \to M$ in ζ, there exists a homotopy

$v : [0,1] \times S^1 \to M$ that connects x to z, i.e.

$$\begin{aligned} v(0,\cdot) &= z(\cdot), \\ v(1,\cdot) &= x(\cdot). \end{aligned}$$

We shall call v a *capping* of x. In general, such cappings are cylinders. In the special case when ζ is the trivial homotopy class, the reference loop z can be chosen as a point and so the cappings become disks, leading us to the contractible case.

In order for the action functional later to be well-defined, we will need to define it on lifts of capped loops. Consider the space $\mathcal{L}_\zeta(M)$ of pairs (x, v) (which we shall call *capped loops*), where $x : S^1 \to M$ is a loop in ζ and $v : [0,1] \times S^1 \to M$ is a capping of x. Define an equivalence relation $\sim$ on this space as follows: Given $(x, v), (x', v') \in \mathcal{L}_\zeta(M)$, let $u = v\#(-v')$, the 2-torus obtained by concatenating the cylinders v and $-v'(s,\cdot) := v'(1-s,\cdot)$. We define $(x, v) \sim (x', v')$ if and only if

$$\begin{aligned} x &= x', \\ \int_{S^1\times S^1} u^*\widehat{c_1} &= 0, \qquad (2.2) \\ \int_{S^1\times S^1} u^*\omega &= \int_{S^1} u_0^*\theta, \qquad (2.3) \end{aligned}$$

An equivalence class $[x, v] := [(x, v)]$ will be called a *lift* of the capped loop (x, v), and the space of such lifts will be denoted by $\widetilde{\mathcal{L}}_\zeta M$.

2.3 Periodic Orbits and the Action Functional

Observe that the fixed points of $\varphi = \varphi_1$ are in one-to-one correspondence with the 1-periodic solutions of the differential equation

$$\dot{x}(t) = X_{\eta_t}(t, x(t)). \tag{2.4}$$

See also Lemma 3.2 of [BH]. Moreover, recall that such a periodic orbit $x(t)$ of φ is called *nondegenerate* if 1 is not an eigenvalue of the linearized return map

$$\mathrm{d}\varphi_{x(0)} : T_{x(0)}M \to T_{x(0)}M.$$

If, furthermore, for each eigenvalue λ of $\mathrm{d}\varphi_{x(0)}$, we have $|\lambda| \neq 1$, we say that φ is *hyperbolic*. Finally, recall also that φ is said to be *nondegenerate* if each of its 1-periodic orbits is nondegenerate.

Define the *action 1-form* $\alpha_{[\varphi_t]} \in \Omega^1(\mathcal{L}M)$ by

$$\left(\alpha_{[\varphi_t]}\right)_x(\xi) := \int_{S^1} \omega_{x(t)}\left(\dot{x}(t) - X_{\eta_t}(t, x(t)), \xi(t)\right) \mathrm{d}t, \tag{2.5}$$

for a loop $x : S^1 \to M$ and a tangent vector field ξ along x. Then the 1-periodic solutions of (2.4) are precisely the zeroes of $\alpha_{[\varphi_t]}$ (by non-degeneracy of ω). Note that

$$\begin{aligned}
\left(\alpha_{[\varphi_t]}\right)_x(\xi) &= \int_{S^1} \left\{\omega_{x(t)}(\dot{x}(t), \xi(t)) - i_{X_{\eta_t}}\omega_{x(t)}(\xi(t))\right\} \mathrm{d}t \\
&= \int_{S^1} \left\{\omega_{x(t)}(\dot{x}(t), \xi(t)) + (\eta_t)_{x(t)}(\xi(t))\right\} \mathrm{d}t \\
&= \int_{S^1} \left\{\omega_{x(t)}(\dot{x}(t), \xi(t)) + \theta_{x(t)}(\xi(t)) + (\mathrm{d}H_t)_{x(t)}(\xi(t))\right\} \mathrm{d}t \quad (2.6)
\end{aligned}$$

We are now in a position to define the action functional. Given $[\varphi_t] \in$

$\widetilde{\mathrm{Symp}_0}(M, \omega)$, we define $\mathcal{A}_{[\varphi_t]} : \mathcal{L}_\zeta M \to \mathbb{R}$ by

$$\mathcal{A}_{[\varphi_t]}(x, v) := -\int_{[0,1]\times S^1} v^*\omega + \int_{S^1} \left\{ \left(\int_0^1 (v_t)^*\theta \right) + H_t(x(t)) \right\} \mathrm{d}t, \tag{2.7}$$

where v is a capping of x and $v_t : [0,1] \to M$ is the segment defined by

$$v_t(s) = v(s,t).$$

Observe that if $[x,v] = [x,v'] \in \widetilde{\mathcal{L}}_\zeta M$ and $u := v\#(-v')$, then, since θ is a closed 1-form on M,

$$\begin{aligned}
\mathcal{A}_{[\varphi_t]}(x,v) - \mathcal{A}_{[\varphi_t]}(x,v') &= -\int_{S^1\times S^1} u^*\omega + \int_0^1 \left(\int_{S^1} (u_t)^*\theta \right) \mathrm{d}t \\
&= -\int_{S^1\times S^1} u^*\omega + \int_0^1 \left(\int_{S^1} (u_0)^*\theta \right) \mathrm{d}t \\
&= -\int_{S^1\times S^1} u^*\omega + \int_{S^1} (u_0)^*\theta
\end{aligned}$$

Now then, by (2.3),

$$\mathcal{A}_{[\varphi_t]}(x,v) - \mathcal{A}_{[\varphi_t]}(x,v') = -\int_{S^1\times S^1} u^*\omega + \int_{S^1} (u_0)^*\theta = 0.$$

Therefore, $\mathcal{A}_{[\varphi_t]}$ descends to a map $\widetilde{\mathcal{A}}_{[\varphi_t]} : \widetilde{\mathcal{L}}_\zeta M \to \mathbb{R}$.

Now, in order to see the critical points of the action functional and its gradient flowlines, let us compute $\mathrm{d}\tilde{\mathcal{A}}_{[\varphi_t]}$. Fix a capped loop $(x,v) \in \mathcal{L}_\zeta M$ and a vector field $\xi \in T_x(\mathcal{L}M)$. For sufficiently small $\epsilon \geq 0$, let x_ϵ be the loop in ζ obtained by "moving" x along $\epsilon\xi$, i.e. $x_\epsilon : [0,1] \to M$ is defined by

$$x_\epsilon(t) = \exp_{x(t)}(\epsilon\xi(t)),$$

where $\exp_{x(t)} : T_{x(t)}M \to M$ is the exponential map. Note that $x_0 = x$. Define

$v_\epsilon : [0,1] \times S^1 \to M$ by

$$v_\epsilon(s,t) = x_{s\epsilon}(t),$$

i.e. v_ϵ is the cylinder connecting x to x_ϵ. Then,

$$\begin{aligned}
\left(\mathrm{d}\widetilde{\mathcal{A}}_{[\varphi_t]}\right)_{[x,v]}(\xi) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\tilde{\mathcal{A}}_{[\varphi_t]}[x_\epsilon, v\# v_\epsilon] - \tilde{\mathcal{A}}_{[\varphi_t]}([x,v])\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{-\int_{[0,1]\times S^1} v_\epsilon^*\omega + \int_{S^1}\int_0^1 (v_\epsilon)_t^*\theta \mathrm{d}t\right. \\
&\qquad \left. + \int_{S^1}\left(H_t(x_\epsilon(t)) - H_t(x(t))\right)\mathrm{d}t\right\}
\end{aligned}$$

We then have

$$\begin{aligned}
-\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{[0,1]\times S^1} v_\epsilon^*\omega &= -\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{S^1}\int_0^1 \omega\left(\frac{\partial v_\epsilon}{\partial s}, \frac{\partial v_\epsilon}{\partial t}\right)\mathrm{d}s\mathrm{d}t \\
&= -\int_{S^1}\int_0^1 \lim_{\epsilon\to 0}\frac{1}{\epsilon}\omega\left(\frac{\partial}{\partial s}(\exp_{x(t)}(s\epsilon\xi(t))), \frac{\mathrm{d}x_{s\epsilon}}{\mathrm{d}t}\right)\mathrm{d}s\mathrm{d}t \\
&= -\int_{S^1}\int_0^1 \lim_{\epsilon\to 0}\frac{1}{\epsilon}\omega\left(\epsilon\xi(t), \frac{\mathrm{d}x_{s\epsilon}}{\mathrm{d}t}\right)\mathrm{d}s\mathrm{d}t \\
&= -\int_{S^1}\int_0^1 \lim_{\epsilon\to 0}\omega\left(\xi(t), \frac{\mathrm{d}x_{s\epsilon}}{\mathrm{d}t}\right)\mathrm{d}s\mathrm{d}t \\
&= -\int_{S^1}\int_0^1 \omega\left(\xi(t), \dot{x}(t)\right)\mathrm{d}s\mathrm{d}t \\
&= \int_{S^1}\omega\left(\dot{x}(t), \xi(t)\right)\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{S^1}\int_0^1 (v_\epsilon)_t^*\theta \mathrm{d}t &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^1\int_0^1 \theta\left(\frac{\mathrm{d}(v_\epsilon)_t}{\mathrm{d}s}\right)\mathrm{d}s\mathrm{d}t \\
&= \int_{S^1}\int_0^1 \lim_{\epsilon\to 0}\frac{1}{\epsilon}\theta\left(\frac{\partial v_\epsilon}{\partial s}\right)\mathrm{d}t \\
&= \int_{S^1}\int_0^1 \lim_{\epsilon\to 0}\frac{1}{\epsilon}\theta\left(\frac{\partial}{\partial s}(\exp_{x(t)}(s\epsilon\xi(t)))\right)\mathrm{d}t \\
&= \int_{S^1}\int_0^1 \lim_{\epsilon\to 0}\frac{1}{\epsilon}\theta\left(\epsilon\xi(t)\right)\mathrm{d}s\mathrm{d}t \\
&= \int_{S^1}\int_0^1 \theta\left(\xi(t)\right)\mathrm{d}s\mathrm{d}t \\
&= \int_{S^1} \theta\left(\xi(t)\right)\mathrm{d}t.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{S^1}\left(H_t(x_\epsilon(t)) - H_t(x(t))\right)\mathrm{d}t &= \int_{S^1}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(H_t(x_\epsilon(t)) - H_t(x_0(t))\right)\mathrm{d}t \\
&= \int_{S^1}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} H_t(x_\epsilon(t))\mathrm{d}t \\
&= \int_{S^1}(\mathrm{d}H_t)_{x_0(t)}\left(\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\exp_{x(t)}(\epsilon\xi(t))\right)\mathrm{d}t \\
&= \int_{S^1}(\mathrm{d}H_t)_{x(t)}\left(\xi(t)\right)\mathrm{d}t.
\end{aligned}$$

That is,

$$\left(\mathrm{d}\widetilde{\mathcal{A}}_{[\varphi_t]}\right)_{[x,v]}(\xi) = \int_{S^1}\omega\left(\dot{x}(t),\xi(t)\right)\mathrm{d}t + \int_{S^1}\theta\left(\xi(t)\right)\mathrm{d}t + \int_{S^1}(\mathrm{d}H_t)_{x(t)}\left(\xi(t)\right)\mathrm{d}t$$

and thus by (2.6),

$$\left(\mathrm{d}\tilde{\mathcal{A}}_{[\varphi_t]}\right)_{[x,v]} = \left(\alpha_{[\varphi_t]}\right)_x.$$

In particular, $[x,v]\in\widetilde{\mathcal{L}}_\zeta M$ is a critical point of the functional $\tilde{\mathcal{A}}_{[\varphi_t]}$ if and only if x is a zero of the 1-form $\alpha_{[\varphi_t]}$. Thus, there is a one-to-one correspondence between the critical points of $\tilde{\mathcal{A}}_{[\varphi_t]}$ and the set of fixed points of $\varphi = \varphi_1$. We shall denote the set of critical points of $\tilde{\mathcal{A}}_{[\varphi_t]}$ by $\operatorname{Crit}\tilde{\mathcal{A}}_{[\varphi_t]}$.

The action functional is homogeneous with respect to iterations of the symplectomorphism in the following sense:

Lemma 2.3.1. *For each $k \in \mathbb{N}$, if $[x,v]^k$ denotes the lift of the capped loop (x^k, v^k), where x^k and v^k are the k-th iterations of x and v, respectively, then*

$$\tilde{\mathcal{A}}_{[\varphi_t^k]}([x,v]^k) = k\tilde{\mathcal{A}}_{[\varphi_t]}([x,v]).$$

Proof. Note that we also take the k-th iteration of the reference loop z and hence also the homotopy class ζ. First of all, clearly,

$$\int_{v^k} \omega = k \int_v \omega.$$

Moreover, since Flux is a group homomorphism, we have $\mathrm{Flux}([\varphi_t^k]) = [k\theta] = [k\theta + \mathrm{d}H_t^{\natural k}]$, where $H_t^{\natural k}$ denotes the k-th iteration of the Hamiltonian H_t. Now,

$$\int_{S^1} \left(\int_{v_t^k} k\theta \right) \mathrm{d}t = k \int_{S^1} \left(\int_{v_t} \theta \right) \mathrm{d}t$$

and

$$\int_{S^1} H_t^{\natural k}(x^k(t))\mathrm{d}t = \int_{S^1} H_t(x(t))\mathrm{d}t.$$

Combining these immediately yields the desired result. □

Furthermore, the manner by which the attachment of "tori" to the reference loop affects the action functional is described by the following:

Lemma 2.3.2. *If $A = [a] \in \pi_1(\mathcal{L}_\zeta M)$, then*

$$\tilde{\mathcal{A}}_{[\varphi_t]}([x, v\#a]) = \tilde{\mathcal{A}}_{[\varphi_t]}([x,v]) + \langle [\varrho], A \rangle.$$

Proof. Indeed, the homotopy class A of the loop a of loops in $\mathcal{L}_\zeta M$ only affects the first two terms in (2.7). More precisely, the Hamiltonian H_t depends only on $[\varphi_t]$. Thus, by (2.3)

$$\begin{aligned}\tilde{\mathcal{A}}_{[\varphi_t]}([x, v\#a]) - \tilde{\mathcal{A}}_{[\varphi_t]}([x, v]) &= -\int_{S^1\times S^1} a^*\omega + \int_{S^1}\left(\int_0^1 (a_t)^*\theta\right)\mathrm{d}t \\ &= -\int_{S^1\times S^1} a^*\omega + \int_{S^1}\left(\int_{S^1} (a_0)^*\theta\right)\mathrm{d}t \\ &= -\int_{S^1\times S^1} a^*\omega + \int_{S^1} (a_0)^*\theta \\ &= \langle[\varrho], A\rangle,\end{aligned}$$

as desired. □

The constructions above heavily depend on the homotopy class ζ and the choice of reference loop z in ζ. Moreover, note that if φ is a Hamiltonian diffeomorphism, then $\mathrm{Flux}[\varphi_t] = 0$ and it can be arranged that $\{\varphi_t\}$ is furthermore a Hamiltonian isotopy. See for instance Theorem 10.2.5 of [MSa17] or Theorem of [Po]. Thus, in this case, the action functional becomes

$$\mathcal{A}_{[\varphi_t]}(x, v) := -\int_{[0,1]\times S^1} v^*\omega + \int_{S^1} H_t(x(t))\mathrm{d}t,$$

which coincides with the standard action functional for a Hamiltonian diffeomorphism, used for instance in the construction of the filtered Floer complex for a Hamiltonian diffeomorphism in [ÇGG21] which we intend to generalize here.

2.4 Floer Equation

We now wish to understand the (negative) gradient flowlines of the action functional. Recall that an almost complex structure $J \in \mathrm{End}(TM)$ is *compatible* with ω if ω is J-invariant and

$$\omega(X, JX) > 0$$

for any $X \in TM$. Thus, the 2-form $g_J \in \Omega^2(M)$ defined by

$$g_J(X,Y) := \omega(X, JY)$$

is a Riemannian metric on M. Denote by $\mathcal{J}(M,\omega)$ the set of all almost complex structures compatible with ω. It is a well-known theorem of Gromov that $\mathcal{J}(M,\omega)$ is non-empty and contractible. Pick any $J \in \mathcal{J}(M,\omega)$. Denote by $\widetilde{g_J}$ the Riemannian metric induced by g_J on the loop space $\mathcal{L}M$: If $\xi_1, \xi_2 \in T_x(\mathcal{L}M)$ for $x \in \mathcal{L}M$, i.e. ξ_1, ξ_2 are vector fields along the loop $x : [0,1] \to M$, then

$$\widetilde{g_J}(\xi_1, \xi_2) := \int_0^1 (g_J)_{x(t)}(\xi_1(t), \xi_2(t))\mathrm{d}t.$$

Then, for a capped loop $[x,v] \in \widetilde{\mathcal{L}}_\zeta M$ and a vector field ξ along x,

$$\begin{aligned}
(\mathrm{d}\widetilde{\mathcal{A}}_{[\varphi_t]})_{[x,v]}(\xi) &= (\alpha_{[\varphi_t]})_x(\xi) \\
&= \int_0^1 \omega_{x(t)} \left(\dot{x}(t) - X_{\eta_t}(t, x(t)), \xi(t)\right) \mathrm{d}t \\
&= \int_0^1 \omega_{x(t)} \Big(J_{x(t)}(\dot{x}(t) - X_{\eta_t}(t, x(t))), J_{x(t)}\xi(t)\Big) \,\mathrm{d}t \\
&= \int_0^1 (g_J)_{x(t)} \Big(J_{x(t)} \left(\dot{x}(t) - X_{\eta_t}(t, x(t))\right), \xi(t)\Big) \,\mathrm{d}t \\
&= (\widetilde{g_J})_x(J(\dot{x} - X_{\eta_t}), \xi)
\end{aligned}$$

Thus, the gradient of $\widetilde{\mathcal{A}}_{[\varphi_t]}$ with respect to $\widetilde{g_J}$ is the vector field $\nabla_{\widetilde{g_J}}\widetilde{\mathcal{A}}_{[\varphi_t]}$ defined by

$$\nabla_{\widetilde{g_J}}\widetilde{\mathcal{A}}_{[\varphi_t]}([x,v]) = J(\dot{x} - X_{\eta_t})$$

and hence a negative gradient flowline is a map $\widetilde{u} : \mathbb{R} \to \mathcal{L}M$ such that

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}s} = -J\left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{u}(s) - X_{\eta_t}(\widetilde{u}(s))\right).$$

Now, the one-to-one correspondence $\widetilde{u} \mapsto u$ defined in (2.1) allows us to view the gradient flowlines of $\widetilde{\mathcal{A}}_{[\varphi_t]}$ as mappings $u : \mathbb{R} \times S^1 \to M$ satisfying the *Floer equation*

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_{\eta_t}(u(s,t))\right) = 0. \tag{2.8}$$

Thus, from this point forward, we view the gradient flowlines as *J-holomorphic cylinders*, i.e. maps $u : \mathbb{R} \times S^1 \to M$ satisfying the Floer equation (2.8).

If $[x^-, v^-], [x^+, v^+] \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$, then the gradient flowlines from x^- to x^+ can be viewed as the mappings $u : \mathbb{R} \times S^1 \to M$ satisfying the Floer equation (2.8) and the boundary conditions

$$\begin{aligned} \lim_{s \to -\infty} u(s,t) &= x^-(t), \\ \lim_{s \to +\infty} u(s,t) &= x^+(t). \end{aligned}$$

Denote by $\mathcal{M}([x^-, v^-], [x^+, v^+])$ the space of all mappings $u : \mathbb{R} \times S^1 \to M$ satisfying the Floer equation and the above boundary conditions. These mappings are called the *connecting trajectories* from $[x^-, v^-]$ to $[x^+, v^+]$. More particularly, $\mathcal{M}([x^-, v^-], [x^+, v^+])$ is the space of connecting trajectories that are *stably asymptotic* to x^+ and *unstably asymptotic* to x^-.

The *energy* of a connecting trajectory $u \in \mathcal{M}([x^-, v^-], [x^+, v^+])$ is given by

$$E(u) := \int_{\mathbb{R}} \left|\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}s}\right|^2_{\widetilde{g}_J} \mathrm{d}s = \int_{\mathbb{R}} \int_{S^1} \left|\frac{\partial u}{\partial s}\right|^2_{g_J} \mathrm{d}t\mathrm{d}s = \int_{\mathbb{R}} \int_{S^1} \omega\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}\right) \mathrm{d}t\mathrm{d}s$$

A standard result is that the energy of a connecting trajectory coincides with the action difference of the capped orbits to which it is asymptotic, i.e.

$$E(u) = \widetilde{\mathcal{A}}_{[\varphi_t]}([x^-, v^-]) - \widetilde{\mathcal{A}}_{[\varphi_t]}([x^+, v^+]).$$

See for instance [BH].

Now, weak monotonicity of M guarantees that no bubbling can occur. Thus, by Propositions 4.5 and 4.6 of [BH], if the periodic orbits x^- and x^+ are non-degenerate, then the operator $\mathscr{F} : W^{1,2}(\mathbb{R} \times S^1; M) \to L^2(\mathbb{R} \times S^1; M)$ defined by

$$\mathscr{F}(u) = \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_{\eta_t}(u(s,t))\right)$$

is Fredholm and $\mathcal{M}([x^-, v^-], [x^+, v^+])$ is a smooth manifold with dimension given by

$$\dim \mathcal{M}([x^-, v^-], [x^+, v^+]) = \operatorname{ind} \mathscr{F},$$

where $\operatorname{ind} \mathscr{F}$ denotes the Fredholm index of $\mathscr{F}$. For more details, see for instance [SZ], [HS]. There is an obvious $\mathbb{R}$-action on the space of connecting trajectories by reparametrization: For each $a \in \mathbb{R}$, there is an endomorphism $u \mapsto u_a$ of $\mathcal{M}([x^-, v^-], [x^+, v^+])$, where $u_a : \mathbb{R} \times S^1 \to M$ is given by

$$u_a(s,t) = u(s-a, t).$$

Thus, the quotient space $\widehat{\mathcal{M}}([x^-, v^-], [x^+, v^+]) := \mathcal{M}([x^-, v^-], [x^+, v^+])/\mathbb{R}$ by this action, called the *moduli space of connecting trajectories*, is a manifold of dimension $\operatorname{ind} \mathscr{F} - 1$. In particular, if $\operatorname{ind} \mathscr{F} = 1$, the moduli space of connecting trajectories is a 0-dimensional manifold.

2.5 Floer-Novikov Complex of a Symplectomorphism Isotopic to the Identity

Let the ground field be $\mathbb{F} = \mathbb{F}_2 := \mathbb{Z}/2$, and consider the *universal Novikov field*

$$\Lambda := \left\{ \sum_{j \geq 0} f_j T^{a_j} : \{f_j\} \subset \mathbb{F}, \{a_j\} \subset \mathbb{R}, \text{ either } \{a_j : f_j \neq 0\} \text{ is finite or } \lim_{j \to \infty} a_j = \infty \right\}.$$

Equivalently,

$$\Lambda = \left\{ \sum_{j \geq 0} f_j T^{a_j} : \{f_j\} \subset \mathbb{F}, \{a_j\} \subset \mathbb{R}, \text{ and } \forall r \in \mathbb{R}, \{a_j : f_j \leq r\} \text{ is finite} \right\}.$$

For more details on Novikov rings, see for instance [Fa], and for Novikov's original motivation for their construction, see [No]. By viewing the complex over the Novikov field, one circumvents the difficulty arising from sequences of connecting trajectories with index difference 1 whose energy tends to ∞. Let $CF([\varphi_t])$ denote the vector space generated over Λ by $\operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$. We will treat these generators as just the orbits themselves and drop the capping. That is, we have

$$CF([\varphi_t]) = \bigoplus_{x \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}} \Lambda x.$$

The *Floer complex associated to* $[\varphi_t]$ is then defined to be the (ungraded) complex $(CF([\varphi_t]), \partial)$ where the boundary map $\partial : CF([\varphi_t]) \to CF([\varphi_t])$ is defined as follows: For each generator $x_i \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$, define

$$\partial x_i = \sum_{j \in \mathcal{I}} \sum_{[v] \in \mathcal{S}_j} f([v]) T^{-\langle [\varrho], [v] \rangle} x_j, \tag{2.9}$$

where:

(i) $\mathcal{I} = \Big\{ j : x_j \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$ for which $\dim \mathcal{M}([x_i, v_i], [x_j, v_j]) = 1 \Big\}$, where v_i are v_j are fixed cappings of x_i and x_j, respectively,

(ii) for each $j \in \mathcal{I}$, $\mathcal{S}_j$ is the set of homotopy classes of recappings of x_j, i.e.

$$\mathcal{S}_j = \{[v_j \# a] : [a] \in \pi_1(\mathcal{L}_\zeta M, z)\},$$

and

(iii) for each recapping v of x_j, $f([v])$ is the $\mathbb{Z}/2\mathbb{Z}$-count of the moduli space $\widehat{\mathcal{M}}([x_i, v_i], [x_j, v])$.

Note that the condition that $\dim \mathcal{M}([x_i, v_i], [x_j, v_j]) = 1$ and the orbits x_i and x_j be nondegenerate guarantees that $\widehat{\mathcal{M}}([x_i, v_i], [x_j, v])$ is finite for any $[v] \in \mathcal{S}_j$. See for instance Proposition 4.6 of [BH]. This proposition also asserts that for any $r \in \mathbb{R}$,

$$\{A = [a] \in \pi_1(\mathcal{L}_\zeta M) : -\langle [\varrho], A \rangle \leq r \text{ and } \mathcal{M}([x_i, v_i], [x_j, v_j \# a]) \neq \emptyset\}$$

is finite. Thus, for any $r \in \mathbb{R}$, the number of $[v] = [v_j \# a] \in \mathcal{S}_j$ such that

$$-\langle [\varrho], [v] \rangle = -\langle [\varrho], [v_j] \rangle - \langle [\varrho], [a] \rangle \leq r$$

and $f([v]) \neq 0$ is finite. This guarantees that the coefficient of each x_j in (2.9) is indeed in the universal Novikov field Λ.

To define the differential ∂ on the whole complex, we simply extend (2.9)

Λ-linearly to $CF([\varphi_t])$: For $x_i \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$ and $\lambda_i \in \Lambda$, we define

$$\partial \left(\sum_i \lambda_i x_i \right) := \sum_i \lambda_i \partial x_i.$$

We will refer to ∂ as the *Floer differential.* Under the assumption that M is weakly monotone, this defines a boundary map, making the floer complex an authentic cochain complex. See Proposition 5.4 of [BH].

Chapter 3

Persistent Homology and Barcodes

In this chapter, we recall the theory of *barcodes*. The notion of a barcode was originally introduced in topology as an algebraic and compact way to record the *persistent homology* associated to a simplicial complex. We review persistent homology and barcodes in Section 3.3, following mainly the discussion and development by Polterovich, Rosen, Samvelyan, and Zhang in [PRSZ]. As one finds in [PRSZ], one can also associate a barcode to a Morse complex, and ultimately, as shown by Usher and Zhang in [UZ], to a filtered Floer complex. This is the context that is most relevant to our work. Hence, we review some preliminaries on filtered Floer complexes as developed in [UZ] in Sections 3.1 and 3.2.

3.1 Non-Archimedean Vector Spaces and Singular Value Decompositions

As our theory involves filtered Floer complexes, let us first recall some facts about orthogonalizable non-archimedean vector spaces and singular value decompositions of maps between such spaces. We closely follow the notions developed by Usher and Zhang in [UZ].

A *valuation* ν on a field Λ is a function $\nu : \Lambda \to \mathbb{R} \cup \{\infty\}$ satisfying the following axioms:

(i) $\nu(\lambda) = \infty$ if and only if $\lambda = 0$;

(ii) $\nu(\lambda\mu) = \nu(\lambda) + \nu(\mu)$ for any $\lambda, \mu \in \Lambda$; and

(iii) $\nu(\lambda + \mu) \geq \min\{\nu(\lambda), \nu(\mu)\}$ for any $\lambda, \mu \in \Lambda$.

If Λ is the universal Novikov field, one can define a valuation $\nu : \Lambda \to \mathbb{R} \cup \{\infty\}$ as follows: Whenever $\{f_j\} \subset \mathbb{F}$ and $\{a_j\} \subset \mathbb{R}$ are sequences such that either $\{a_j : f_j \neq 0\}$ is finite or $\lim_{j\to\infty} a_j = \infty$, define

$$\nu\left(\sum_{j\geq 0} f_j T^{a_j}\right) := \min\{a_j : f_j \neq 0\}$$

Note that the conditions on the sequences $\{f_j\}$ and $\{a_j\}$ guarantee that this minimum always exists.

If Λ is a field endowed with a valuation $\nu : \Lambda \to \mathbb{R} \cup \{\infty\}$, a *non-archimedean vector space* over Λ is a pair $(C, \mathcal{A})$, where C is a vector space over Λ and $\mathcal{A} : C \to \mathbb{R} \cup \{-\infty\}$ is a function satisfying the following axioms:

(i) $\mathcal{A}(x) = -\infty$ if and only if $x = 0$;

(ii) $\mathcal{A}(\lambda x) = \mathcal{A}(x) - \nu(\lambda)$, for any $x \in C$ and $\lambda \in \Lambda$; and

(iii) $\mathcal{A}(x + y) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\}$, for any $x, y \in C$.

The function $\mathcal{A}$ above is called a *filtration* for C.

A standard fact, proved as Proposition 2.3 in [UZ], is that we actually have

$$\mathcal{A}(x + y) = \max\{\mathcal{A}(x), \mathcal{A}(y)\}$$

whenever $\mathcal{A}(x) \neq \mathcal{A}(y)$.

We define a filtration $\mathcal{A} : CF([\varphi_t]) \to \mathbb{R}$, which we will also refer to as the *action filtration*, on the Floer complex as follows: For each $\lambda \in \Lambda$ and $x \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$, we define

$$\mathcal{A}(\lambda x) := \widetilde{\mathcal{A}}_{[\varphi_t]}(x) - \nu(\lambda).$$

In particular, since ν vanishes on the ground field $\mathbb{F}$, we have $\mathcal{A}(x) = \widetilde{\mathcal{A}}_{[\varphi_t]}(x)$ for any generator $x \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$. Now, for $\lambda_i \in \Lambda$ and $x_i \in \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$, we define

$$\mathcal{A}\left(\sum_i \lambda_i x_i\right) := \max_i \mathcal{A}(\lambda_i x_i).$$

We then refer to the triple $(CF([\varphi_t]), \partial, \mathcal{A})$ as the *filtered Floer complex of* $[\varphi_t]$.

A finite ordered collection $(x_1, \ldots, x_m)$ of vectors in a non-archimedean vector space $(C, \mathcal{A})$ over a field Λ with valuation ν is said to be *orthogonal* if for all $\lambda_1, \ldots, \lambda_m \in \Lambda$,

$$\mathcal{A}\left(\sum_{i=1}^{m} \lambda_i x_i\right) = \max_{1 \leq i \leq m} \mathcal{A}(\lambda_i x_i)$$

Note that an orthogonal set of non-zero vectors in C is necessarily linearly independent over Λ. Indeed, suppose $(x_1, \ldots, x_m)$ is an orthogonal ordered set of non-zero vectors. Suppose $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that

$$\sum_{i=1}^{m} \lambda_i x_i = 0.$$

Then

$$\max_{1 \le i \le m} \mathcal{A}(\lambda_i x_i) = \mathcal{A}\left(\sum_{i=1}^{m} \lambda_i x_i\right) = -\infty.$$

Thus, for each i, $\mathcal{A}(\lambda_i x_i) = -\infty$, and so $\lambda_i x_i = 0$. Since $x_i \neq 0$, $\lambda_i = 0$ for each i.

A non-archimedean vector space is said to be *orthogonalizable* if it admits an orthogonal basis. If $(C, \mathcal{A}_C)$ and $(D, \mathcal{A}_D)$ are orthogonalizable non-archimedean vector spaces over Λ and $\partial : C \to D$ is a a Λ-linear map with rank r, a *singular value decomposition of* ∂ is a choice of orthogonal ordered bases $(y_1, \ldots, y_n)$ for C and $(x_1, \ldots x_m)$ for D such that

(i) $(y_{r+1}, \ldots, y_n)$ is an orthogonal ordered basis for $\ker \partial$;

(ii) $(x_1, \ldots, x_r)$ is an orthogonal ordered basis for $\operatorname{im} \partial$;

(iii) $\partial y_i = x_i$ for $i \in \{1, \ldots, r\}$; and

(iv) $\mathcal{A}_C(y_1) - \mathcal{A}_D(x_1) \geq \mathcal{A}_C(y_2) - \mathcal{A}_D(x_2) \geq \cdots \geq \mathcal{A}_C(y_r) - \mathcal{A}_D(x_r)$.

Usher and Zhang proved in [UZ] as Theorem 3.4 that any linear map between orthognalizable non-archimedean vector spaces over a field Λ admits a singular value decomposition. Moreover, it is not difficult to see that one can rearrange the y_i and x_i so that the inequalities above are reversed. Thus, in our situation, if the Λ-linear Floer differential $\partial : CF([\varphi_t]) \to CF([\varphi_t])$ has rank r, then there exist orthogonal ordered bases $(y_1, \ldots, y_n)$ and $(x_1, \ldots, x_n)$ for $CF([\varphi_t])$ such that

(i) $\partial y_i = 0$, for all $i = r+1, \ldots, n$;

(ii) $\partial y_i = x_i$, for all $i = 1, \ldots, r$.

(iii) $\mathcal{A}(y_1) - \mathcal{A}(x_1) \leq \mathcal{A}(y_2) - \mathcal{A}(x_2) \leq \cdots \leq \mathcal{A}(y_r) - \mathcal{A}(x_r)$

3.2 Floer-Type Complexes and Floer Packages

Let Λ be the universal Novikov field. A *Floer-type complex* over Λ is a triple $(C_*, \partial_*, \mathcal{A})$, where (C_*, ∂_*) is a chain complex over Λ and $\mathcal{A} : C_* \to \mathbb{R} \cup \{-\infty\}$ is a function such that $\left(C_k, \mathcal{A}|_{C_k}\right)$ is an orthogonalizable non-archimedean vector space over Λ for each k, and $\mathcal{A}(\partial x) \leq \mathcal{A}(x)$ for each $x \in C_k$. A *two-storey Floer-type complex* is a Floer-type complex whose chain complex is of the form

$$\cdots \longrightarrow 0 \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0 \longrightarrow \cdots$$

We will also use a convenient notion used in [ÇGG21]: A *Floer package* over Λ is a quadruple $(\mathcal{C}, \{x_i\}, \partial, \mathcal{A})$, where $\mathcal{C}$ is a finite-dimensional vector space over Λ with a prescribed set of generators $\{x_i\}$, $\partial : \mathcal{C} \to \mathcal{C}$ is a boundary map, and $\mathcal{A}$ is an action filtration on $\mathcal{C}$ such that ∂ is strictly action-decreasing. Observe that whenever $(\mathcal{C}, \mathcal{A})$ is orthogonalizable, the Floer package gives rise to a two-storey Floer-type complex, by simply taking $C_1 = C_0 = \mathcal{C}$.

In our case, we consider the Floer package $(CF([\varphi_t]), \{\gamma_i, \alpha_j\}, \partial, \mathcal{A})$, giving rise to the two-storey Floer-type complex with chain complex

$$\cdots \longrightarrow 0 \longrightarrow CF([\varphi_t]) \xrightarrow{\partial} CF([\varphi_t]) \longrightarrow 0 \longrightarrow \cdots .$$

We verify that the Floer differential is indeed action-decreasing: First, if $x_i \in \operatorname{Crit} \mathcal{A}_{[\varphi_t]}$ with

$$\partial x_i = \sum_j f_{ij} T^{\varrho(u_{ij})} x_j,$$

then

$$\mathcal{A}(\partial x_i) \leq \max_j \mathcal{A}\left(f_{ij} T^{\varrho(u_{ij})} x_j\right) = \max_j \left(\widetilde{\mathcal{A}}_{\{\varphi_t\}}(x_j) - \varrho(u_{ij})\right).$$

Observe that for each j, if v_i and v_j are cappings of x_i and x_j, respectively, then $v_i \# u_{ij}$ is a capping of x_j equivalent to v_j. Thus, for each j,

$$\begin{aligned}
\widetilde{\mathcal{A}}_{[\varphi_t]}(x_j) - \widetilde{\mathcal{A}}_{[\varphi_t]}(x_i) &= -\int_{[0,1]\times S^1} (u_{ij})^*\omega + \int_{S^1}\left(\int_0^1 (u_{ij})_t^*\theta\right) \mathrm{d}t \\
&= -\int_{\mathbb{R}\times S^1} (u_{ij})^*\omega + \int_{\mathbb{R}} (u_{ij})_0^*\theta \\
&= \varrho(u_{ij})
\end{aligned}$$

Thus

$$\mathcal{A}(\partial x_i) \leq \max_j \left(\widetilde{\mathcal{A}}_{[\varphi_t]}(x_j) - \varrho(u_{ij})\right) = \widetilde{\mathcal{A}}_{[\varphi_t]}(x_i).$$

Now, for a general $x = \sum_i \lambda_i x_i$ where $\{x_i\} \subseteq \operatorname{Crit} \widetilde{\mathcal{A}}_{[\varphi_t]}$,

$$\begin{aligned}
\mathcal{A}(\partial x) &= \mathcal{A}\left(\sum_i \lambda_i \partial x_i\right) \\
&\leq \max_i \left(\mathcal{A}(\partial x_i) - \nu(\lambda_i)\right) \\
&\leq \max_i \left(\mathcal{A}(x_i) - \nu(\lambda_i)\right) \\
&= \mathcal{A}\left(\sum_i \lambda_i x_i\right) \\
&= \mathcal{A}(x),
\end{aligned}$$

since the action values $\mathcal{A}(\lambda_i x_i)$ are pairwise distinct.

3.3 Persistent Homology and Barcodes

To a Floer-type complex, we can associate a persistence module, and henceforth, a barcode. To this end, we recall some facts about persistent homology and barcodes. One can find full details and proofs in a more general context in [PRSZ], for instance. In our paper, we will need persistent homology in the context of Floer complexes; more details and proofs can be found in [UZ]. A *persistence module* (V^*, σ^*) consists of a family $\{V^t\}_{t\in\mathbb{R}}$ of vector spaces V^t over a fixed field Λ such that each pair of vector spaces (V^s, V^t) for which $s \leq t$ is equipped with a homomorphism $\sigma^{s,t} : V^s \to V^t$ satisfying the following functoriality properties:

(i) whenever $s \leq t \leq u$, the diagram

$$\begin{array}{ccc} V^s & \xrightarrow{\sigma^{s,u}} & V^u \\ & {\scriptstyle \sigma^{s,t}} \searrow \quad \nearrow {\scriptstyle \sigma^{t,u}} & \\ & V^t & \end{array}$$

commutes; and

(ii) for any $s \in \mathbb{R}$, $\sigma^{s,s} : V^s \to V^s$ is the identity automorphism on V^s.

Given a field Λ and an interval $I \subseteq \mathbb{R}$, one can construct a canonical persistence module (Λ_I^*, π_I^*), called the *interval module for I over* Λ, as follows: Define

$$\Lambda_I^t := \begin{cases} \Lambda, & \text{if } t \in I; \\ 0, & \text{if } t \notin I. \end{cases}$$

Whenever $s \leq t$ for which $s, t \in I$ so that $\Lambda_I^s = \Lambda_I^t = \Lambda$, define $\pi_I^{s,t} : \Lambda_I^s \to \Lambda_I^t$ to be the identity automorphism on Λ. Otherwise, define $\pi_I^{s,t}$ to be the zero map.

If (V^*, σ^*) and (W^*, τ^*) are two persistence modules over a field Λ, a *morphism* $A^* : (V^*, \sigma^*) \to (W, \tau^*)$ consists of a family of Λ-linear maps $A^t : V^t \to W^t$ such that whenever $s \leq t$, the diagram

$$\begin{array}{ccc} V^s & \xrightarrow{\sigma^{s,t}} & V^t \\ {\scriptstyle A^s}\big\downarrow & & \big\downarrow{\scriptstyle A^t} \\ W^s & \xrightarrow[\tau^{s,t}]{} & W^t \end{array}$$

commutes. This allows us to speak of the category of persistence modules. In particular, if $A^* : (V^*, \sigma^*) \to (W, \tau^*)$ and $B^* : (W^*, \tau^*) \to (U^*, \rho^*)$ are morphisms of persistence modules, the *composition* $(B \circ A)^* : (V^*, \sigma^*) \to (U^*, \rho^*)$ is the morphism such that for each $t \in \mathbb{R}$,

$$(B \circ A)^t = B^t \circ A^t : V^t \to U^t.$$

Moreover, we have the *identity morphism* $\mathrm{id}^* : (V^*, \sigma^*) \to (V^*, \sigma^*)$, where for each $t \in \mathbb{R}$, $\mathrm{id}^t : V^t \to V^t$ is just the identity automorphism on V^t.

A morphism $A^* : (V^*, \sigma^*) \to (W^*, \tau^*)$ of persistence modules is an *isomorphism* if there exists a morphism $B^* : (W^*, \tau^*) \to (V^* \to \sigma^*)$ such that both $(A \circ B)^*$ and $(B \circ A)^*$ are the identity morphisms on (W^*, τ^*) and (V^*, σ^*), respectively. If there is an isomorphism $A^* : (V^*, \sigma^*) \to (W^*, \tau^*)$, we say that the two persistence modules (V^*, σ^*) and (W^*, τ^*) are *isomorphic*.

If (V^*, σ^*) and (W^*, τ^*) are persistence modules, the *direct sum* $(V^*, \sigma^*) \oplus (W^*, \tau^*)$ is the persistence module $((V \oplus W)^*, (\sigma \oplus \tau)^*)$ defined as follows: For each $t \in \mathbb{R}$, $(V \oplus W)^t := V^t \oplus W^t$ and whenever $s \leq t$, the linear map $(\sigma \oplus \tau)^{s,t} : V^s \oplus W^t \to V^t \oplus W^t$ is defined by

$$(\sigma \oplus \tau)^{s,t}(v, w) = \left(\sigma^{s,t}(v), \tau^{s,t}(w)\right).$$

A standard result on persistence modules is the following structure theorem.

For instance, see [PRSZ] for a proof.

Theorem 3.3.1 (Normal Form Theorem for Persistence Modules)**.** *Let* (V, σ) *be a persistence module over the field* Λ. *Then there exists a family* $\{I_\alpha\}$ *of intervals* $I_\alpha \subseteq \mathbb{R}$ *such that*

$$(V^*, \sigma^*) \cong \bigoplus_\alpha \left(\Lambda^*_{I_\alpha}, \pi^*_{I_\alpha}\right) \tag{3.1}$$

The intervals I_α in the above decomposition are not necessarily distinct. Thus, the family $\{I_\alpha\}$ can be viewed as a set of intervals, each counted with multiplicity. To formalize this, we use the notion of a multiset: A *multiset* M is a pair (S, μ), where S is a set and $\mu : S \to \mathbb{N} \cup \{\infty\}$ is a function, called the *multiplicity function* of M. One should think of μ as the function that counts the number of copies of an element in the multiset. A multiset (S, μ) for which S is a set of intervals $I_\alpha \subseteq \mathbb{R}$ is called a *barcode*.

Thus, for example, the Normal Form Theorem for Persistence Modules enables us to associate to any persistence module (V^*, σ^*) the barcode (S, μ), where S is the set of intervals I_α in the direct sum decomposition (3.1), and for each $I_\alpha \in S$, $\mu(I_\alpha)$ is the number of times I_α appears in the decomposition. This specific multiset of intervals is then called the *(persistent homology) barcode* of the persistence module (V^*, σ^*).

Now, returning to our context, consider a Floer-type complex $(C_*, \partial_*, \mathcal{A})$. For each $t \in \mathbb{R}$, let

$$C^t_k := \{x \in C_k : \mathcal{A}(x) \leq t\}.$$

Note that for any $x \in C^t_k$,

$$\mathcal{A}(\partial x) \leq \mathcal{A}(x) \leq t,$$

and hence $\partial x \in C^t_{k-1}$. Thus there is a subcomplex $(C^t_*, \partial^t_*, \mathcal{A})$, where $\partial^t_k : C^t_k \to C^t_{k-1}$ is the restriction of ∂_k on C^t_k, with codomain restricted to C^t_{k-1}.

Now for each $k \in \mathbb{Z}$, let $V_k^t := H_k(C_*^t, \partial_*^t)$, the k-th homology group of (C_*^t, ∂_*^t). Whenever $s \leq t$, define $\sigma_k^{s,t} : V_k^s \to V_k^t$ as follows: For $x \in \ker \partial_k^s$,

$$\sigma_k^{s,t}(x + \operatorname{im} \partial_{k+1}^s) = x + \operatorname{im} \partial_{k+1}^t. \tag{3.2}$$

To see that this is well-defined, observe that whenever $x \in \ker \partial_k^s$, $\mathcal{A}(x) \leq s$ and $\partial_k(x) = 0$. Thus, with $s \leq t$, $\mathcal{A}(x) \leq t$ as well, and hence $x \in \ker \partial_k^t$. Moreover, if $x' \in \ker \partial_k^s$ such that $x' - x \in \operatorname{im} \partial_{k+1}^s$, then $x' - x \in \operatorname{im} \partial_{k+1}$ with $\mathcal{A}(x' - x) \leq s$. Again, since $s \leq t$, $\mathcal{A}(x' - x) \leq t$ as well and hence $x' - x \in \operatorname{im} \partial_{k+1}^t$.

It is also clear that whenever $s \leq t \leq u$,

$$\begin{aligned} \sigma_k^{t,u}\left(\sigma_k^{s,t}(x + \operatorname{im} \partial_{k+1}^s)\right) &= \sigma_k^{t,u}(x + \operatorname{im} \partial_{k+1}^t) \\ &= x + \operatorname{im} \partial_{k+1}^u \\ &= \sigma_k^{s,u}(x + \operatorname{im} \partial_{k+1}^s). \end{aligned}$$

i.e. $\sigma_k^{s,u} = \sigma_k^{t,u} \circ \sigma_k^{s,t}$, and that $\sigma_k^{s,s}$ is just the identity on V_k^s. Thus, (V_k^*, σ_k^*) is a persistence module. The Normal Theorem for Persistence Modules then associates a barcode to (V_k^*, σ_k^*). Now, Usher and Zhang prove in [UZ] the following theorem that allows us to obtain this barcode via a singular value decomposition:

Theorem 3.3.2. *Let $(C_*, \partial_*, \mathcal{A})$ be a Floer-type complex over Λ. Let $k \in \mathbb{Z}$ and (V_k^*, σ_k^*) be the persistence module for which $V_k^* = H_k(C_*, \partial_*)$ and $\sigma_k^{s,t}$ is defined as in (3.2). If $((y_1, \dots, y_n), (x_1, \dots, x_m))$ is a singular value decomposition of the boundary map $\partial_{k+1} : C_{k+1} \to \ker \partial_k$ (with codomain restricted to $\ker \partial_k$) and $r = \operatorname{rank}(\partial_{k+1})$, then the barcode for (V_k^*, σ_k^*) consists precisely of:*

- *an interval $[\mathcal{A}(x_i), \mathcal{A}(y_i))$ for each $i \in \{1, \dots, r\}$ such that $\mathcal{A}(y_i) > \mathcal{A}(x_i)$; and*

- *an interval* $[\mathcal{A}(x_i), \infty)$ *for each* $i \in \{r+1, \ldots, m\}$.

Usher and Zhang then define the *degree-k verbose barcode* of $(C_*, \partial_*, \mathcal{A})$ as the multiset of elements of $\mathbb{R} \times [0, \infty]$ consisting of

(i) a pair $(\mathcal{A}(x_i), \mathcal{A}(y_i) - \mathcal{A}(x_i))$ for $i \in \{1, \ldots, r\}$; and

(ii) a pair $(\mathcal{A}(x_i), \infty)$ for $i \in \{r+1, \ldots, m\}$.

The submultiset of the verbose barcode consisting of pairs of the first type above is then called the *concise barcode* of $(C_*, \partial_*, \mathcal{A})$. Elements of the concise barcode capture the left endpoints and the lengths of the finite bars of the persistent homology barcode.

Going back to our context, we adopt conventions in [ÇGG21]. Consider the ungraded filtered Floer complex $(CF([\varphi_t]), \mathcal{A})$ with boundary map the Floer differential $\partial : CF([\varphi_t]) \to CF([\varphi_t])$. We consider the two-storey Floer-type complex

$$\cdots \longrightarrow 0 \longrightarrow CF([\varphi_t]) \xrightarrow{\partial} CF([\varphi_t]) \longrightarrow 0 \longrightarrow \cdots$$

Suppose $n = \dim_\Lambda CF([\varphi_t])$ and $r = \operatorname{rank} \partial$. Note then that if $Z := \ker \partial$, $\dim_\Lambda Z = n - r$, and since $\operatorname{im} \partial \subseteq \ker \partial$, $r \leq n - r$. Suppose that the orthogonal ordered basis $(\gamma_1, \ldots, \gamma_r, \alpha_1, \ldots, \alpha_{n-r})$ for $CF(\varphi)$ and the orthogonal ordered basis $(\eta_1, \ldots, \eta_r, \beta_1, \ldots, \beta_{n-2r})$ for Z, respectively, that form a singular value decomposition for $\partial : CF([\varphi_t]) \to Z$. That is, $(\alpha_1, \ldots, \alpha_{n-r})$ is an orthogonal ordered basis for $\ker \partial$, $(\eta_1, \ldots, \eta_r)$ is an orthogonal ordered basis for $\operatorname{im} \partial$, $\partial \gamma_i = \eta_i$ for each $i \in \{1, \ldots, r\}$, and

$$\mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1) \leq \mathcal{A}(\gamma_2) - \mathcal{A}(\eta_2) \leq \cdots \leq \mathcal{A}(\gamma_r) - \mathcal{A}(\eta_r).$$

For each $t \in \mathbb{R}$, let $C^t := \{x \in CF([\varphi_t]) : \mathcal{A}(x) \leq t\}$ and $V^t = H_1(C^t_*, \partial^t_*)$. Then, by Theorem 3.3.2, the verbose barcode for the persistence module (V^*, σ^*) is the multiset $(\mathcal{B}, \mu)$ consisting of the pairs

$$(\mathcal{A}(\eta_1), \mathcal{A}(\gamma_1) - \mathcal{A}(\eta_1)), \ldots, (\mathcal{A}(\eta_r), \mathcal{A}(\gamma_r) - \mathcal{A}(\eta_r))$$

and the pairs

$$(\mathcal{A}(\beta_1), \infty), \ldots, (\mathcal{A}(\beta_{n-2r}), \infty).$$

Definition 3.3.3. *The* barcode associated to the homotopy class $[\varphi_t]$ of a symplectic isotopy $\{\varphi_t\}$*, denoted* $\mathcal{B}([\varphi_t])$*, is the multiset consisting of the second elements of the pairs in the verbose barcode for the persistence module* (V^*, σ^*) *defined above.*

That is, $\mathcal{B}([\varphi_t])$ is the multiset of lengths of the bars in the concise barcode and ∞ counted $(n - 2r)$ times. We shall also refer to these lengths as "bars;" more precisely, the lengths will be *unpinned bars*, as opposed to the actual intervals being *pinned bars*.

Note that the number of finite bars in $\mathcal{B}([\varphi_t])$, i.e. the number of bars in the concise barcode (counting multiplicities), is the rank of ∂. Moreover, observe that

$$\dim_\Lambda HF([\varphi_t]) = \dim_\Lambda \ker \partial - \dim_\Lambda \operatorname{im} \partial = (n - r) - r = n - 2r.$$

where $HF([\varphi_t])$ denotes the homology of the ungraded complex $(CF([\varphi_t]), \partial)$. That is, the number of infinite bars (counting multiplicities) in $\mathcal{B}([\varphi_t])$ is the dimension of the total homology of the Floer complex for $[\varphi_t]$. We will also denote the total number of bars (counting multiplicities) in the barcode by $b([\varphi_t])$. Note that

$$b([\varphi_t]) = r + (n - 2r) = n - r = \dim_\Lambda \ker \partial.$$

For instance, observe that $b([\varphi_t])$ gives a lower bound for the number of critical points of the action functional $\widetilde{\mathcal{A}}_{[\varphi_t]}$:

$$b([\varphi_t]) = n - r \leq n = \dim_\Lambda CF([\varphi_t]) = \#\,\mathrm{Crit}\,\widetilde{\mathcal{A}}_{[\varphi_t]}.$$

3.4 Bottleneck Distance and Interpolating Distance

We recall in this section two notions of distance, that of the bottleneck distance between barcodes and that of the interpolating distance between Floer-type complexes. We also recall the Isometry Theorem which relates these two distances.

Recall that for $\delta > 0$, a *δ-matching* between two concise barcodes $\mathcal{B}_1$ and $\mathcal{B}_2$ is a bijection $\mu : \mathcal{C}_1 \to \mathcal{C}_2$, where $\mathcal{C}_1 \subseteq \mathcal{B}_1$ and $\mathcal{C}_2 \subseteq \mathcal{B}_2$, satisfying the following conditions:

(i) each bar from $\mathcal{B}_1$ of length greater than 2δ belongs to $\mathcal{C}_1$;

(ii) each bar from $\mathcal{B}_2$ of length greater than 2δ belongs to $\mathcal{C}_2$;

(iii) whenever $(a, b] \in \mathcal{C}_1$ and $(c, d] \in \mathcal{C}_2$ such that $\mu((a, b]) = (c, d]$, $(a, b] \subseteq (c - \delta, d + \delta]$ and $(c, d] \subseteq (a - \delta, b + \delta]$

The *bottleneck distance* between the barcodes $\mathcal{B}_1$ and $\mathcal{B}_2$ is then the infimum of all $\delta > 0$ for which there exists a δ-matching between $\mathcal{B}_1$ and $\mathcal{B}_2$.

We recall the notion of δ-quasiequivalence, as formulated by Usher and Zhang in [UZ] as their version of the more classical notion of interleaving distance in general persistent homology theory. If $\delta > 0$ and $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$ are

Floer-type complexes, then a δ-*quasiequivalence* between C_* and D_* is a quadruple (Φ, Ψ, K_C, K_D) where:

(i) $\Phi : C_* \to D_*$ and $\Psi : D_* \to C_*$ are chain maps, with $\mathcal{A}_D(\Phi(c)) \leq \mathcal{A}_C(c) + \delta$ and $\mathcal{A}_C(\Psi(d)) \leq \mathcal{A}_D(d) + \delta$ for all $c \in C_*$ and $d \in D_*$;

(ii) $K_C : C_* \to C_{*+1}$ and $K_D : D_* \to D_{*+1}$ obey the homotopy equations

$$\Psi \circ \Phi - I_{C_*} = \partial_C K_C + K_C \partial_C \quad \text{and} \quad \Phi \circ \Psi - I_{D_*} = \partial_D K_D + K_D \partial_D$$

and for all $c \in C_*$ and $d \in D_*$,

$$\mathcal{A}_C(K_C c) \leq \mathcal{A}_C(c) + 2\delta \quad \text{and} \quad \mathcal{A}_D(K_D d) \leq \mathcal{A}_D(d) + 2\delta.$$

The *quasiequivalence distance* $d_Q((C_*, \partial_C, \mathcal{A}_C), (D_*, \partial_D, \mathcal{A}_D))$ between $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$ is then given by the infimum of all $\delta \geq 0$ for which there exists a δ-equivalence between $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$. The bottleneck distance between barcodes and the quasiequivalence distance between Floer-type complexes are then related via the Stability Theorem:

Theorem 3.4.1 (Stability Theorem)**.** *For any Floer-type complexes* $(C_*, \partial_C, \mathcal{A}_C)$ *and* $(D_*, \partial_D, \mathcal{A}_D)$ *with barcodes* $\mathcal{B}_C$ *and* $\mathcal{B}_D$*, respectively,*

$$d_Q((C_*, \partial_C, \mathcal{A}_C), (D_*, \partial_D, \mathcal{A}_D)) \leq d_B(\mathcal{B}_C, \mathcal{B}_D) \leq 2 d_Q((C_*, \partial_C, \mathcal{A}_C), (D_*, \partial_D, \mathcal{A}_D)).$$

More details and the proof of the theorem can be found in [UZ]. A stronger and somewhat more complicated related notion is that of the interpolating distance. Recall that for $\delta > 0$, a δ-*interpolation* between two Floer-type complexes $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$ is a family of Floer-type complexes $(C^s_*, \partial^s_C, \mathcal{A}^s_C)$ indexed by a parameter $s \in [0, 1] \setminus S$, for some finite $S \subset (0, 1)$, such that

(i) $(C^0_*, \partial^0_C, \mathcal{A}^0_C) = (C_*, \partial_C, \mathcal{A}_C)$ and $(C^1_*, \partial^1_C, \mathcal{A}^1_C) = (D_*, \partial_D, \mathcal{A}_D)$; and

(ii) for all $s, t \in [0,1]\backslash S$, $(C^s_*, \partial^s_C, \mathcal{A}^s_C)$ and $(C^t_*, \partial^t_C, \mathcal{A}^t_C)$ are $\delta|s-t|$-quasiequivalent.

Now then the *interpolating distance* d_P between the Floer-type complexes $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$ is the infimum of the set of all $\delta \geq 0$ for which there exists a δ-interpolation between $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$. Usher and Zhang also prove in [UZ] a result that is stronger than the Stability Theorem; this is the Isometry Theorem that equates the interpolating distance and the bottleneck distance:

Theorem 3.4.2 (Isometry Theorem). *For any Floer-type complexes $(C_*, \partial_C, \mathcal{A}_C)$ and $(D_*, \partial_D, \mathcal{A}_D)$ with barcodes $\mathcal{B}_C$ and $\mathcal{B}_D$, respectively,*

$$d_B(\mathcal{B}_C, \mathcal{B}_D) = d_P((C_*, \partial_C, \mathcal{A}_C), (D_*, \partial_D, \mathcal{A}_D)).$$

This result is reminiscent of the classical Isometry Theorem of persistent homology theory, which likewise equates the *interleaving distance* between two persistence modules and the bottleneck distance between their associated barcodes.

Chapter 4

Topological Entropy

One of the primary goals of this paper is to extract the topological entropy of a symplectomorphism isotopic to the identity from the barcode associated to a symplectic isotopy connecting it to the identity. This number is of interest in dynamics as it is an attempt to quantify chaotic behavior of dynamical systems. In this chapter, we recall some facts about topological entropy in a general context. In Section 4.1, we recall how topological entropy is defined and some of its properties. In Section 4.2, we recall how measure-theoretic entropy is defined and how it is related to topological entropy via the Variational Principle. In Section 4.3, we recall special properties of topological entropy in the context of hyperbolic dynamics, and specifically in the presence of horseshoes. Finally, in Section 4.4, we recall how topological entropy is related to volume growth via Yomdin's Theorem.

Note that much of the content of this chapter are standard notions and facts from the theory of dynamical systems and ergodic theory. For more details, the reader is encouraged to consult the excellent expositions by Katok and Hasselblatt in [KH], Barreira and Pesin in [BP], Coudène in [Co], Walters in [Wa], Guckenheimer and Holmes in [GH], and Brin and Stuck in [BS].

4.1 Topological Entropy

Let (M, d) be a compact metric space and $\varphi : M \to M$ a continuous map. For each $k \in \mathbb{N}$, define the *k-shadowing metric* d_k^φ on M by

$$d_k^\varphi(x, y) := \max_{0 \le i \le k-1} d\left(\varphi^i(x), \varphi^i(y)\right).$$

It is a standard fact that this is a metric on M for each $k \in \mathbb{N}$. For each $\epsilon > 0$, set

$$B_\varphi(x, \epsilon, k) := \left\{y \in X : d_k^\varphi(x, y) < \epsilon\right\},$$

the ϵ-neighborhood of x with respect to the k-shadowing metric. Recall that a subset $E \subseteq X$ is said to be *(k, ϵ)-spanning* for X if

$$X \subseteq \bigcup_{x \in E} B_\varphi(x, \epsilon, k).$$

Denote by $S_d(\varphi, \epsilon, k)$ the *ϵ-spanning number for the metric space* (X, d_k^φ), i.e. the minimal cardinality of a (k, ϵ)-spanning set for X. Define

$$h_d(\varphi, \epsilon) := \limsup_{k \to \infty} \frac{\log S_d(\varphi, \epsilon, k)}{k}.$$

One can interpret $\log S_d(\varphi, \epsilon, k)$ as the quantity of information needed to specify a point of M to accuracy ϵ up to k iterations of φ, and hence $h_d(\varphi, \epsilon)$ can be interpreted as the asymptotic behavior (as the number of iteration steps becomes arbitrarily large) of the average quantity of information per iteration step needed to specify an orbit under φ to accuracy ϵ.

Finally, the topological entropy is obtained by also observing the asymptotic behavior of the above quantity as the accuracy threshold ϵ is made arbitrarily small. That is, set

$$h_d(\varphi) := \lim_{\epsilon \searrow 0} h_d(\varphi, \epsilon).$$

so that, $h_d(\varphi)$ roughly measures the exponential growth rate of divergence of orbits. Now, it is a standard fact that if d' is another metric on X that induces the same metric topology as d, then $h_{d'}(\varphi) = h_d(\varphi)$. We then define this common quantity that depends on the topology rather than the actual metric as the *topological entropy* of φ, and denote it by $h_{\text{top}}(\varphi)$.

There are other ways to compute topological entropy. One way is via covering numbers. Let $C_d(\varphi, \epsilon, k)$ be the *ϵ-covering number of* (X, d_k^φ), i.e. the minimal cardinality of a cover of X consisting of subsets whose diameter in the metric d_k^φ is at most ϵ. It is not difficult to see that the sequence $\{\log C_d(\varphi, \epsilon, k)\}_{k=0}^\infty$ is sub-additive, and hence

$$\lim_{k \to \infty} \frac{\log C_d(\varphi, \epsilon, k)}{k}$$

exists (or is $-\infty$). It turns out that this limit is exactly $h_d(\varphi, \epsilon)$, and hence

$$h_{\text{top}}(\varphi) = \lim_{\epsilon \searrow 0} \lim_{k \to \infty} \frac{\log C_d(\varphi, \epsilon, k)}{k}$$

Another way of computing topological entropy is via separated numbers. Let $N_d(\varphi, \epsilon, k)$ be the *ϵ-separated number of* (X, d_k^φ), i.e. the maximum cardinality of a subset $Y \subseteq X$ consisting of points which are pairwise separated by a d_k^φ-distance greater than ϵ. It turns out that

$$C_d(\varphi, 2\epsilon, k) \leq N_d(\varphi, \epsilon, k) \leq C_d(\varphi, \epsilon, k)$$

and hence it is not difficult to see that

$$h_{\text{top}}(\varphi) = \lim_{\epsilon \searrow 0} \limsup_{k \to \infty} \frac{\log N_d(\varphi, \epsilon, k)}{k}.$$

We recall some properties of topological entropy and collect them in the next theorem. See [KH] for instance, for proofs.

Theorem 4.1.1. *Let (M, d) be a compact metric space and $\varphi : M \to M$ a continuous map.*

(i) If $\varphi' : M' \to M'$ is a continuous map of a compact metric space (M', d') and $f : M \to M'$ is a homeomorphism such that $\psi = f \varphi f^{-1}$, then

$$h_{\text{top}}(\psi) = h_{\text{top}}(\varphi).$$

That is, topological entropy is an invariant of topological conjugacy.

(ii) If $Y \subseteq X$ is a closed φ-invariant subset, then

$$h_{\text{top}}\left(\varphi|_Y\right) \leq h_{\text{top}}(\varphi).$$

(iii) If $m \in \mathbb{N}$ and $X_1, \ldots, X_m \subseteq X$ are closed φ-invariant subsets such that

$$X = \bigcup_{i=1}^{m} X_i,$$

then $h_{\text{top}}(\varphi) = \max_{1 \leq i \leq m} h_{\text{top}}\left(\varphi|_{X_i}\right)$.

(iv) Suppose furthermore that φ is a homeomorphism. Then, for any $k \in \mathbb{Z}$,

$$h_{\text{top}}\left(\varphi^k\right) = |k| h_{\text{top}}(\varphi).$$

(v) If $\varphi' : M' \to M'$ is a continuous map of a compact metric space (M', d'), define $\varphi \times \psi : M \times M' \to M \times M'$ by

$$(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y)).$$

Then $h_{\text{top}}(\varphi \times \psi) = h_{\text{top}}(\varphi) + h_{\text{top}}(\psi)$.

As we mentioned previously, the positivity of topological entropy typically indicates chaotic behavior of a dynamical system. For example, it is not difficult to see that if $\varphi : X \to X$ is an isometry of a metric space X, then $h_{\text{top}}(\varphi) = 0$. Indeed, an isometry is as far from being chaotic as possible. As a special example, any isometry of the torus, and more specifically any rotation of the torus $\mathbb{R}^2/\mathbb{Z}^2$ which lifts to a translation of $\mathbb{R}^2$, has zero topological entropy. In contrast, the famous "cat map" of Arnold, which is the hyperbolic toral automorphism $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by

$$f(x + \mathbb{Z}, y + \mathbb{Z}) = (2x + y + \mathbb{Z}, x + y + \mathbb{Z}),$$

has topological entropy $\log \lambda$, where $\lambda = \dfrac{3 + \sqrt{5}}{2}$ is the greater of the two eigenvalues λ and $1/\lambda$ of the matrix

$$A_f := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

representing f. Evidently, $\lambda > 1$ and so $h_{\text{top}}(f) = \log \lambda > 0$. And indeed, Arnold's cat map is a prototypical example of a chaotic dynamical system, evidenced by the fact that its stable and unstable manifolds are dense in the torus. We must mention however that Arnold's cat map, while being a symplectomorphism, is not isotopic to the identity. As a matter of fact, there does not exist an isotopy of f to the identity consisting of just continuous maps (let alone symplectomorphisms), as f induces a map on homology, represented precisely by A_f.

4.2 Measure-Theoretic Entropy and the Variational Principle

In this section, we recall measure-theoretic entropy and how it is related to topological entropy. While topological entropy is a quantity extracted from topological dynamics, the measure-theoretic entropy is a quantity extracted from an invariant measure. While topological entropy measures

In particular, while the definition of topological entropy starts with a space endowed with a metric topology, the definition of measure-theoretic entropy begins instead with a measure space.

Let $(M, \mathcal{B}, \mu)$ be a measure space. Recall that a map $\varphi : M \to M$ is *measure-preserving*, and that the measure μ is *φ-invariant*, if

$$\mu(\varphi^{-1}(B)) = B$$

for every measurable subset $B \in \mathcal{B}$.

Recall that if $(M, \mathcal{B}, \mu)$ is a probability space and I is a finite or countable set of indices, a *measurable partition* of M is a collection of measurable subsets $\xi = \{C_\alpha \in \mathcal{B} : \alpha \in I\}$ satisfying

$$\mu\left(X \setminus \bigcup_{\alpha \in I} C_\alpha\right) = 0$$

and, for any indices $\alpha_1 \neq \alpha_2$,

$$\mu\left(C_{\alpha_1} \cap C_{\alpha_2}\right) = 0.$$

If ξ is a measurable partition of M, then the *entropy* of ξ is

$$H_\mu(\xi) := -\sum_{\alpha \in I} \mu(C_\alpha) \log \mu(C_\alpha) \in [0, +\infty),$$

where in this sum, we define $0 \log 0 := 0$.

Now we are ready to recall measure-theoretic entropy. If ξ and η are two partitions of M, define the *joint partition* as

$$\xi \vee \eta := \{X \cap Y : X \in \xi, Y \in \eta, \mu(X \cap Y) > 0\}.$$

If ξ is a measurable partition of M and $\varphi : M \to M$ is measure-preserving, set

$$\xi_{-n}^{\varphi} := \xi \vee \varphi^{-1}(\xi) \vee \cdots \vee \varphi^{-n+1}(\xi).$$

Then the *metric entropy of the transformation φ relative to the partition ξ* is

$$h_\mu(\varphi, \xi) := \lim_{n \to \infty} \frac{1}{n} H(\xi_{-n}^{\varphi}).$$

This limit always exists as an extended nonnegative real number since the sequence $\{H(\xi_{-n}^{\varphi})\}_{n=1}^{\infty}$ is sub-additive. Finally, the *(measure-theoretic) entropy of the transformation φ with respect to μ* (or the *entropy of the measure μ*) is defined to be

$$h_\mu(\varphi) := \sup\{h_\mu(\varphi, \xi) : \xi \text{ is a measurable partition with } H(\xi) < \infty\}.$$

We collect in the next theorem some standard properties of measure-theoretic entropy.

Theorem 4.2.1. *Let $(M, \mathcal{B}, \mu)$ be a probability space and $\varphi : M \to M$ a μ-preserving transformation.*

(i) If $\psi : (M', \mathcal{B}', \mu') \to (M', \mathcal{B}', \mu')$ is a μ'-preserving transformation that is a factor of $\varphi : (M, \mathcal{B}, \mu) \to (M, \mathcal{B}, \mu)$, then

$$h_{\mu'}(M') \leq h_\mu(M).$$

(ii) If $A \subset M$ is φ-invariant and $\mu(A) > 0$, then

$$h_\mu(\varphi) = \mu(A) h_{\mu_A}(\varphi) + \mu(M \setminus A) h_{\mu_{X \setminus A}}(\varphi).$$

(iii) If ν is another φ-invariant probability measure for M, then for any $p \in [0, 1]$,

$$h_{p\mu + (1-p)\nu}(\varphi) \geq p h_\mu(\varphi) + (1 - p) h_\nu(\varphi).$$

Now, let us recall a way to relate measure-theoretic entropy and topological entropy. Perhaps at first seemingly in an ironic fashion, such a connection can be found by observing the following contrast between the two notions of entropy: As we can see in the theorem above, the measure-theoretic entropy on the union of two invariant subsets is the measure-weighted sum of the measure-theoretic entropies of the two subsets. On the other hand, recall that the topological entropy on a union is the maximum of the entropies of the components, by Theorem 4.1.1.

Perhaps, a connection can be found if there is a special measure with respect to which the measure-theoretic entropy is taken. Indeed, this is what the Variational

Principle below provides. For a homeomorphism φ, let $\mathfrak{M}(\varphi)$ denote the set of all φ-invariant Borel probability measures on M.

Theorem 4.2.2 (Variational Principle)**.** *If $\varphi : M \to M$ is a homeomorphism of a compact metric space (M, d), then*

$$h_{\text{top}}(f) = \sup\{h_\mu(\varphi) : \mathfrak{M}(\varphi)\}$$

In particular, if μ is a measure of maximal entropy with respect to φ, then the entropy of μ with respect to φ is exactly the topological entropy of φ.

4.3 Entropy and Horseshoes

Topological entropy attempts to measure the orbit complexity of a dynamical system. The lore in dynamics is that positivity of the topological entropy is an indication of complexity in orbit growth and hence interesting "chaotic" dynamics. For instance, a classical example is that of the horseshoe map, originally introduced by Smale in 1967 in [Sm]. The horseshoe map is also a classical example of a dynamical system exhibiting hyperbolic behavior, and indeed a further lore in dynamics is that hyperbolic behavior typically create positive topological entropy. Recall that if $\lambda < \mu$, then a sequence of invertible linear maps $\{L_m : \mathbb{R}^n \to \mathbb{R}^n\}$ is said to *admit a* (λ, μ)*-splitting* if there exist decompositions $\mathbb{R}^n = E_m^+ \oplus E_m^-$ such that $L_m E_m^\pm = E_{m+1}^\pm$ and

$$\left\| L_m|_{E_m^-} \right\| \leq \lambda \quad \text{and} \quad \left\| L_m^{-1}\Big|_{E_{m+1}^+} \right\| \leq \frac{1}{\mu}.$$

A set $K \subseteq M$ is called *hyperbolic* for a smooth map $\varphi : M \to M$ if there exists a Riemannian metric, called a *Lyapunov metric*, in an open neighborhood U of K

and $\lambda < 1 < \mu$ such that for any point $x \in K$, the sequence of differentials

$$\left\{(D\varphi)_{\varphi^m(x)} : T_{\varphi^m(x)}M \to T_{\varphi^{m+1}(x)}M\right\}_{m\in\mathbb{Z}}$$

admits a (λ, μ)-splitting. A compact invariant subset $K \subseteq M$ of φ is said to be *locally maximal* or *basic* if there exists a neighborhood $U \supset K$, called an *isolating neighborhood*, such that K is the maximal φ-invariant subset of U, i.e.

$$K = \{x \in U : \varphi^m(x) \in U \text{ for all } m \in \mathbb{Z}\} = \bigcap_{m\in\mathbb{Z}} \varphi^m(U).$$

Smale originally defined his version of the horseshoe in the following manner: Let $\Delta = [0, 1] \times [0, 1]$, the unit square in $\mathbb{R}^2$, and let $f : \Delta \to \mathbb{R}^2$ be a diffeomorphism of Δ onto its image such that $\Delta \cap f(\Delta)$ consists of two "horizontal" rectangles Δ_0 and Δ_1 and the restriction of f to each component $\Delta^i \subseteq f^{-1}(\Delta)$ is a hyperbolic affine map that contracts in the vertical direction and expands in the horizontal direction. The maximal invariant subset

$$\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(\Delta)$$

of Δ under f then turns out to be the product of two Cantor sets (which is then itself a Cantor set). Having the maximal invariant set of a dynamical system be a Cantor set is characteristic of chaotic systems. This thus motivates us to consider dynamical systems having horseshoes as subsystems. Moreover, a standard fact about the horseshoe map is that its topological entropy is $\log 2$ (which is evidently positive).

Recall that a homeomorphism $\varphi : X \to X$ of a metric space (X, d) is called *expansive* if there exists $\delta > 0$ such that for any distinct $x, y \in M$, there exists

$m \in \mathbb{Z}$ such that

$$d(\varphi^m(x), \varphi^m(y)) \geq \delta.$$

It is known that the restriction of any diffeomorphism to a hyperbolic subset is an expansive map. For instance, see Corollary 6.4.10 of [KH].

Hyperbolicity is an interesting condition since it typically gives rise to many periodic orbits. Recall that if $\epsilon > 0$, a sequence of points $x_0, x_1, \ldots, x_{m-1}, x_m = x_0$ is a *periodic ϵ-orbit* or an *ϵ-pseudo-orbit* if $d(\varphi(x_k), x_{k+1}) < \epsilon$ for $k = 0, 1, \ldots, m-1$.

Theorem 4.3.1 (Anosov Closing Lemma)**.** *If K is a hyperbolic set for φ, then there exists an open neighborhood $V \supset K$ and C, $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ and any ϵ-pseudo-orbit $\hat{x} = (x_0, x_1, \ldots, x_k)$ in V, there exists $y_0 \in U$ such that $\varphi^k(y_0) = y_0$ and $d(\varphi^i(y_0), x_i) < C\epsilon$ for $i = 0, 1, \ldots, k-1$.*

We then say that the periodic orbit $\hat{y} := \{\varphi^i(y_0) : i \in \mathbb{Z}/k\mathbb{Z}\}$ of y *shadows* the pseudo-orbit x. Thus, in particular, near any point of a hyperbolic subset whose orbit almost returns to the point, there is an actual periodic orbit that shadows the pseudo-orbit.

Locally maximal hyperbolic sets provide a setting in which the topological entropy coincides with the exponential growth rate of periodic points. See, for instance, [KH] for a proof.

Theorem 4.3.2 (Theorem 18.5.1 in [KH])**.** *Let M be a compact Riemannian manifold, $U \subseteq M$ open, $\varphi : U \to M$ a diffeomorphism, and $K \subset U$ a compact locally maximal hyperbolic set for φ. Then*

$$h_{\text{top}}(\varphi|_K) = \limsup_{k\to\infty} \frac{\log^+ P_k(\varphi|_K)}{k},$$

where $P_k(\varphi|_K)$ is the number of periodic points of φ with period k (not necessarily *minimal), i.e. the number of fixed points of φ^k in K.*

Moreover, the topological entropy of the restriction of φ on a hyperbolic set can be made arbitrarily close to the entropy of the restriction on some locally maximal hyperbolic set. More precisely, we have the following lemma:

Lemma 4.3.3. *Let $K \subset M$ be a hyperbolic set. Then, for any $\delta > 0$, there exists a locally maximal hyperbolic set $K' \subset M$ such that $h_{\text{top}}(\varphi|_{K'}) \geq h_{\text{top}}(\varphi|_K) - \delta$.*

Proof. Let $\delta > 0$. By Katok's Approximation [Theorem 3.3 in [ACW]], for any φ-invariant ergodic probability measure μ on M, there exists a locally compact hyperbolic set K' such that

$$h_{\text{top}}(\varphi|_{K'}) > h_\mu(\varphi|_K) - \delta.$$

Now, by the Variational Principle (Theorem 4.5.3 in [KH]), $h_{\text{top}}(\varphi|_K)$ is the supremum of all measure-theoretic entropies $h_\mu(\varphi|_K)$. Thus, taking the supremum of the above inequality yields the desired inequality. □

This result will be instrumental in the proof of Theorem B.

4.4 Entropy, Volume Growth, and Yomdin's Theorem

To prove Theorem A, we will need a connection between topological entropy and the rates of growth of volumes of certain embedded submanifolds. This is provided by what we shall refer to in this work as *Yomdin's Theorem*, which Yomdin proved in [Yo]. See also the survey by Gromov in [Gr] for an excellent exposition of Yomdin's Theorem.

Theorem 4.4.1 (Yomdin's Theorem)**.** *Let M be a compact m-dimensional smooth manifold with fixed Riemannian metric. Let $\varphi : M \to M$ be a smooth map and $N \subseteq M$ a compact n-dimensional smooth submanifold. Then*

$$\limsup_{k\to\infty} \frac{\log \operatorname{vol} \operatorname{Gr}\left(\varphi^k\big|_N\right)}{k} \leq h_{\text{top}}\left(\varphi|_N\right) \leq h_{\text{top}}(\varphi),$$

where $\operatorname{Gr}\left(\varphi^k\big|_N\right) \subseteq N \times M$ *is the graph of the restriction of φ^k to N and* vol *denotes the n-dimensional Riemannian volume.*

That is, the topological entropy bounds the growth rates of the volumes of the graphs of the iterates of the map. Around a year later, Newhouse proved in [Ne] an inequality that can be considered roughly the reverse of Yomdin's inequality:

Theorem 4.4.2. *For $\ell, k \in \mathbb{N}$, let $\Sigma(k,l)$ be the collection of C^k-maps $\sigma : Q^l \to M$, where $Q^l := [0,1]^l$, the l-dimensional unit cube. Let* $\operatorname{vol}(\sigma)$ *denote the l-dimensional volume of the image of σ in M counted as many times as σ covers its image. For $n = 1, \dots, k$ and $l \leq m$, let*

$$\begin{aligned} V_{l,k}(\varphi) &:= \sup_{\sigma\in\Sigma(k,l)} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{vol}(\varphi^n \circ \sigma), \\ V(\varphi) &:= \max_l V_{l,\infty}(\varphi), \\ R(\varphi) &:= \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in M} \|D\varphi^n(x)\|. \end{aligned}$$

If φ is $C^{1+\epsilon}(M)$ for any $\epsilon > 0$, then

$$h_{\text{top}}(\varphi) \leq V(\varphi).$$

Chapter 5

Barcode Entropy

In this chapter, we define the barcode entropy of a symplectic isotopy to the identity and relate it with topological entropy. Our construction generalizes the construction of the barcode entropy of a Hamiltonian diffeomorphism by Çineli, Ginzburg, and Gürel in [ÇGG21].

5.1 Barcode Entropy of a Symplectomorphism Isotopic to the Identity

In Chapter 3, we defined the barcode $\mathcal{B}([\varphi_t])$ for a homotopy class of a symplectic isotopy. We now proceed to define the barcode entropy for $[\varphi_t]$. For each $\epsilon > 0$, define

$$b_\epsilon([\varphi_t]) := \sum \left\{\mu(\ell) : \ell \in \mathcal{B}([\varphi_t]),\, \ell > \epsilon\right\},$$

the number of bars in the barcode (counting multiplicities) whose lengths are greater than ϵ. To see how these numbers are affected by Hamiltonian perturbations, recall that Hofer defined in [Ho] a norm, which we now call the *Hofer norm*, of a Hamiltonian diffeomorphism $\psi \in \mathrm{Ham}(M,\omega)$, by

$$\|\psi\|_H := \inf\left\{\int_{S^1}(\max_M H_t - \min_M H_t) : H_t \text{ is 1-periodic and generates } \psi\right\}.$$

That this defines a norm on $\mathrm{Ham}(M,\omega)$ is indeed a nontrivial fact. The reader can consult for instance [Po] and [MSa17] for more details. Now, if two symplectomorphisms φ and φ' are Hamiltonian isotopic to each other, i.e. $\varphi = \varphi'\psi$ for some $\psi \in \mathrm{Ham}(M,\omega)$, we define the *distance* between φ and φ' to be

$$d(\varphi,\varphi') := \|\varphi^{-1}\varphi'\|_H.$$

The numbers $b_\epsilon([\varphi_t])$ are fairly stable under Hamiltonian perturbations, as expressed more precisely by the following:

Lemma 5.1.1. *If $\varphi \in \mathrm{Symp}_0(M,\omega)$, with symplectic isotopy $\{\varphi_t\}_{0\le t\le 1}$, and $\psi \in \mathrm{Ham}(M,\omega)$ such that $\|\psi\|_H < \epsilon/2$, then for every $\delta > \epsilon$,*

$$b_{\delta+\epsilon}([(\psi\circ\varphi)_t]) \le b_\delta([\varphi_t]) \le b_{\delta-\epsilon}([(\psi\circ\varphi)_t]).$$

The key idea in the proof is that a symplectomorphism φ isotopic to the identity perturbed by a Hamiltonian diffeomorphism ψ has the same flux as the original symplectomorphism φ; hence, comparing bar counts of the two barcodes is equivalent to comparing barcodes if we instead pretended that φ was a Hamiltonian diffeomorphism (as the flux entirely becomes irrelevant in the action differences).

Proof. Indeed, observe that $[\psi_t]$ being in the kernel of the flux homomorphism, we have $\text{Flux}[\psi_t \circ \varphi_t] = \text{Flux}[\varphi_t] = [\theta]$ and hence for any $(x, v) \in \mathcal{L}_\zeta M$,

$$\mathcal{A}_{[\psi_t \circ \varphi_t]}(x) = -\int_{[0,1]\times S^1} v^*\omega + \int_{S^1} \left\{ \left(\int_0^1 (v_t)^*\theta \right) + (G_t \natural H_t)(x(t)) \right\} \mathrm{d}t,$$

where $G_t : M \to \mathbb{R}$ is a 1-periodic-in-t Hamiltonian such that X_{G_t} generates the Hamiltonian isotopy $\{\psi_t\}$. Thus,

$$\mathcal{A}_{[\psi_t \circ \varphi_t]}(x) = \mathcal{A}_{[\varphi_t]}(x) + \int_{S^1} (G_t \natural H_t - H_t)(x(t))\mathrm{d}t$$

Hence, whenever we have $\mathcal{A}_{[\varphi_t]}(y_1) - \mathcal{A}_{[\varphi_t]}(x_1) > \delta$, we have

$$\begin{aligned} \mathcal{A}_{[\psi_t \circ \varphi_t]}(y_1) - \mathcal{A}_{[\psi_t \circ \varphi_t]}(x_1) &= \mathcal{A}_{[\varphi_t]}(y_1) - \mathcal{A}_{[\varphi_t]}(x_1) + \int_{S^1} (G_t \natural H_t - H_t)(y_1(t))\mathrm{d}t \\ &\quad + \int_{S^1} (G_t \natural H_t - H_t)(x_1(t))\mathrm{d}t \\ &\leq \mathcal{A}_{[\varphi_t]}(y_1) - \mathcal{A}_{[\varphi_t]}(x_1) + 2\|\psi\|_H \\ &< \mathcal{A}_{[\varphi_t]}(y_1) - \mathcal{A}_{[\varphi_t]}(x_1) + \epsilon. \end{aligned}$$

Thus, for every bar in the barcode for $[(\psi \circ \varphi)_t]$ of length greater than $\delta + \epsilon$, there exists a bar in the barcode for $[\varphi_t]$ of length greater than δ. Therefore,

$$b_{\delta+\epsilon}([(\psi \circ \varphi)_t]) \leq b_\delta([\varphi_t])$$

Likewise, observing that $\|\psi^{-1}\|_H = \|\psi\|_H$ a similar computation then shows that

$$\mathcal{A}_{[\varphi_t]}(y_1) - \mathcal{A}_{[\varphi_t]}(x_1) < \mathcal{A}_{[\psi_t \circ \varphi_t]}(y_1) - \mathcal{A}_{[\psi_t \circ \varphi_t]}(x_1) + \epsilon.$$

Thus, for every bar in the barcode for $[\varphi_t]$ of length greater than δ, there exists a bar in the barcode for $[(\psi \circ \varphi)_t]$ of length greater than $\delta - \epsilon$. Therefore,

$$b_\delta([\varphi_t]) \leq b_{\delta-\epsilon}([(\psi \circ \varphi)_t])$$

□

We are now ready to define the barcode entropy of a symplectomorphism isotopic to the identity. For this purpose, we define $\log^+ : [0, \infty) \to \mathbb{R}$ by

$$\log^+(x) := \begin{cases} \log_2 x, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Note that for any integer $m \geq 0$,

$$\log^+(m) = \log_2 \max\{m, 1\}.$$

Definition 5.1.2. *Let $\varphi \in \mathrm{Symp}_0(M, \omega)$ and $[\varphi_t]$ be the homotopy class of a symplectic isotopy connecting $\varphi_0 = \mathrm{id}$ to $\varphi_1 = \varphi$. For each $\epsilon > 0$, define*

$$\hbar([\varphi_t], \epsilon) := \limsup_{k \to \infty} \frac{\log^+ b_\epsilon([\varphi_t^k])}{k}.$$

Then, the barcode entropy of $[\varphi_t]$ *is defined to be*

$$\hbar([\varphi_t]) := \lim_{\epsilon \searrow 0} \hbar([\varphi_t], \epsilon).$$

Note that if the sequence $\{b_\epsilon[\varphi_t^k]\}_{k\in\mathbb{N}}$ eventually stabilizes, for instance if it is constant, then $\hbar([\varphi_t]) = 0$. In particular, $\hbar([\mathrm{id}]) = 0$.

Moreover, we recall that if φ is a Hamiltonian symplectomorphism, then for every $k \in \mathbb{N}$, the barcode $\mathcal{B}([\varphi_t^k])$ coincides with the barcode $\mathcal{B}(\varphi^k)$ as constructed in [ÇGG21].

The following proposition lists some properties of barcode entropy that are analogous to those of topological entropy.

Proposition 5.1.3. *Let $\varphi \in \mathrm{Symp}_0(M, \omega)$ and $\{\varphi_t\}$ be a symplectic isotopy connecting $\varphi_0 = \mathrm{id}_M$ to $\varphi_1 = \varphi$.*

(i) For every $m \in \mathbb{N}$, $\hbar([\varphi_t^m]) \leq m\hbar([\varphi_t])$.

(ii) If (M', ω') is a closed connected symplectic manifold, $\psi \in \mathrm{Symp}_0(M', \omega')$, and $\{\psi_t\}$ is a symplectic isotopy connecting $\psi_0 = \mathrm{id}_{M'}$ to $\psi_1 = \psi$, then

$$\hbar([\varphi_t \times \psi_t]) \leq \hbar([\varphi_t]) + \hbar([\psi_t]).$$

(iii) For the symplectic isotopy $\{\varphi_t^{-1}\}$ connecting id_M to φ^{-1}, we have

$$\hbar([\varphi_t^{-1}]) = \hbar([\varphi_t]).$$

(iv) For any symplectomorphism $\psi : M \to M$,

$$\hbar([\psi \circ \varphi_t \circ \psi^{-1}]) = \hbar([\varphi_t]).$$

That is, barcode entropy is invariant under symplectic conjugacy.

Proof. Let $\epsilon > 0$.

(i) For any $m \in \mathbb{N}$,

$$\begin{aligned} \hbar([\varphi_t^m], \epsilon) &= \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\varphi_t^{mk}])}{k} \\ &= m \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\varphi_t^{mk}])}{mk} \\ &\leq m \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\varphi_t^{k}])}{k} \\ &= m\hbar(\{\varphi_t\}, \epsilon) \end{aligned}$$

The result then follows by passing to the limit as $\epsilon \searrow 0$.

(ii) First of all, note that naturally,

$$CF([\varphi_t \times \psi_t]) \cong CF([\varphi_t]) \oplus CF([\psi_t]),$$

and hence by the Normal Theorem for Persistence Modules, $\mathcal{B}([\varphi_t \times \psi_t])$ will just be the multiset union of $\mathcal{B}([\varphi_t])$ and $\mathcal{B}([\psi_t])$. Thus,

$$b_\epsilon([\varphi_t \times \psi_t]) = b_\epsilon([\varphi_t]) + b_\epsilon([\psi_t]),$$

and hence

$$\begin{aligned} \hbar([\varphi_t \times \psi_t], \epsilon) &= \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\varphi_t^k \times \psi_t^k])}{k} \\ &= \limsup_{k\to\infty} \frac{\log^+ \left(b_\epsilon([\varphi_t^k]) + b_\epsilon([\psi_t^k])\right)}{k} \\ &\leq \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\varphi_t^k]) + \log^+ b_\epsilon([\psi_t^k])}{k} \\ &\leq \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\varphi_t^k])}{k} + \limsup_{k\to\infty} \frac{\log^+ b_\epsilon([\psi_t^k])}{k} \\ &= \hbar([\varphi_t], \epsilon) + \hbar([\psi_t], \epsilon) \end{aligned}$$

The result then follows by passing to the limit as $\epsilon \searrow 0$.

(iii) We first observe that the fixed points of φ^{-1} are precisely the fixed points of φ. Moreover, since Flux is a group homomorphism,

$$\mathrm{Flux}[\varphi_t^{-1}] = -\,\mathrm{Flux}[\varphi_t].$$

Indeed, if X_t is a vector field generating $\{\varphi_t\}$, then $-X_t$ generates $\{\varphi^{-1}\varphi_{1-t}\}$, which is homotopic to $\{\varphi_t^{-1}\}$. Hence, we get the usual Poincaré duality: $\mathcal{A}_{[\varphi_t^{-1}]}$ is just the negative of $\mathcal{A}_{[\varphi_t]}$ (up to a constant) and the usual reversal of flowlines for $\mathcal{A}_{[\varphi_t]}$ yields the connecting trajectories for $\mathcal{A}_{[\varphi_t^{-1}]}$. Thus, the bars in the barcodes for $[\varphi_t]$ and $[\varphi_t^{-1}]$ are exactly the same, except that the γ_i and η_i are switched. Trivially then,

$$b_\epsilon([\varphi_t^{-1}]) = b_\epsilon([\varphi_t]),$$

and hence the result immediately follows.

(iv) Note that if X_t is a vector field generating the isotopy $\{\varphi_t\}$, it will also generate the isotopy $\{\psi \circ \varphi_t \circ \psi^{-1}\}$. Moreover, if G_t is a Hamiltonian corresponding to $[\psi_t]$ via the Deformation Lemma, then $G_t \natural H_t \natural G_t^{-1}$ is a Hamiltonian for $[\psi \circ \varphi_t \circ \psi^{-1}]$. Finally, note that $x \in \mathrm{Fix}(\psi \circ \varphi \circ \psi^{-1})$ if and only if $\psi^{-1}(x) \in \mathrm{Fix}(\varphi)$. Thus, $\psi(\mathrm{Fix}(\varphi)) = \mathrm{Fix}(\psi \circ \varphi \circ \psi^{-1})$.

Thus $\mathcal{B}([\varphi_t]) = \mathcal{B}([\psi\varphi_t\psi^{-1}])$, and so the result immediately follows.

□

5.2 Barcode Entropy as a Lower Bound for Topological Entropy

Analogous to Theorem A in [ÇGG21] for Hamiltonian diffeomorphisms, the barcode entropy of a homotopy class of a symplectic isotopy connecting the identity to a symplectomorphism also provides a lower bound for the topological entropy of the symplectomorphism.

Theorem 5.2.1 (Restatement of Theorem A)**.** *Let $\varphi \in \mathrm{Symp}_0(M, \omega)$ and $\{\varphi_t\}$ be a symplectic isotopy connecting $\varphi_0 = \mathrm{id}_M$ to $\varphi_1 = \varphi$. Then*

$$\hbar([\varphi_t]) \leq h_{\mathrm{top}}(\varphi).$$

Recall that in order to prove Theorem A, Çineli, Ginzburg, and Gürel constructed in [ÇGG21] a Lagrangian tomograph and then used it to prove a Crofton-type inequality. We follow a similar program. We begin by recalling the notion of Lagrangian tomographs. We follow the definition and results as presented in [ÇGG21] and [ÇGG22a]. Recall that the *Hofer distance* $d_H(L, L')$ between two Lagrangian submanifolds L, L' of a symplectic manifold (M', ω') is given by

$$d_H(L, L') := \inf\{\|\psi\| : \psi \in \mathrm{Ham}(N, \omega_N), \psi(L) = L'\}.$$

Definition 5.2.2. *If L is a Lagrangian submanifold of a symplectic manifold M', a* Lagrangian tomograph *is a map $\Psi : B \times L \to M'$, where B is some closed ball in some $\mathbb{R}^d$, satisfying the following:*

(i) Ψ is a submersion onto its image,

(ii) for each $s \in B$, the map $\Psi_s := \Psi|_{\{s\}\times L}$ is a smooth embedding,

(iii) $\Psi_0 = \mathrm{id}$, and

(iv) for each $s \in B$, the image $L_s := \Psi(\{s\} \times L)$ is a Lagrangian submanifold of N that is Hamiltonian isotopic to L, with

$$d_H(L_0, L_s) \leq O(\|s\|).$$

The number d will be called the dimension *of the Lagrangian tomograph.*

The following lemma provides us a setting in which Lagrangian tomographs exist:

Lemma 5.2.3 (Lemma 5.6 in [ÇGG21])**.** *A Lagrangian tomograph with dimension d exists if and only if the Lagrangian submanifold L admits an immersion into $\mathbb{R}^d$.*

See the proof in [ÇGG21]. Moreover, Çineli, Ginzburg, and Gürel proved in the same paper a variant of Crofton's inequality:

Lemma 5.2.4 (Crofton's Inequality)**.** *Let L be a Lagrangian submanifold of a symplectic manifold (X, ω_X) and let $\Psi : B \times L \to X$ be a Lagrangian tomograph, with $L_s := \Psi(\{s\} \times L)$. Then, there exists a constant $C > 0$ that for any Lagrangian submanifold L' of X, the function $N : B \to \mathbb{R}$ defined by*

$$N(s) := |L_s \cap L'|.$$

is integrable and satisfies

$$\int_B N(s) \mathrm{d}s \leq C \operatorname{vol}(L').$$

This inequality is called a version of Crofton's inequality as it resembles the original Crofton formula that relates, which in turn inspired Crofton densities in the context of double fibrations. See for instance the works of Álvarez-Paiva and Fernandes ([APF98] and [APF07]) and Gelfand and Smirnov ([GS]).

We now use this lemma to prove a result that relates the numbers of fixed points of Hamiltonian perturbations of a symplectomorphism φ isotopic to the identity and the volume of the graph of φ.

Lemma 5.2.5 (Crofton's Inequality for Symplectomorphisms)**.** *Let (M, ω) be a closed symplectic manifold. Then, for each $\epsilon > 0$, there exist sufficiently large $d > 0$, a closed ball $B \subseteq \mathbb{R}^d$, and a family*

$$\{\psi_s : s \in B\} \subseteq \mathrm{Ham}(M, \omega)$$

of Hamiltonian diffeomorphisms parametrized by B satisfying the following:

(i) $\psi_0 = \mathrm{id}_M$*;*

(ii) $d_H(\psi_s, \mathrm{id}_M) < \epsilon/2$ *for every $s \in B$; and*

(iii) there exists $C > 0$ such that for any $\varphi \in \mathrm{Symp}(M, \omega)$, the function $N : B \to \mathbb{R}$ defined by

$$N(s) = |\,\mathrm{Fix}(\psi_s \circ \varphi)| \tag{5.1}$$

is integrable and satisfies

$$\int_B N(s)\mathrm{d}s \leq C \,\mathrm{vol}\,\mathrm{Gr}(\varphi).$$

Proof of Lemma 5.2.5. Let $L = \Delta = \{(x, x) : x \in M\}$, the diagonal in M. Clearly L is a Lagrangian submanifold of $X := M \times M$ that admits an immersion into $\mathbb{R}^d$ for some $d \geq 4 \dim M$. Thus, by Lemma 5.2.3, there exists a Lagrangian tomograph $\Psi : B \times L \to X$, for some ball $B \subset \mathbb{R}^d$. Note that since the Hofer distance between Δ and $L_s := \Psi(\{s\} \times \Delta)$ is $O(\|s\|)$, given $\epsilon > 0$, we may choose the ball B small enough so that

$$d_H(\Delta, L_s) < \epsilon/2.$$

Moreover, for each $s \in B$, L_s is Hamiltonian isotopic to Δ, and hence there exists $\psi_s \in \mathrm{Ham}(M, \omega)$ such that

$$\mathrm{Gr}(\psi_s)^{-1} = L_s.$$

Note that $\psi_0 = \mathrm{id}_M$ and, for each $s \in B$, $d_H(\psi_s, \mathrm{id}) = d_H(L_s, \Delta)$. Now, since graphs of symplectomorphisms are Lagrangian submanifolds of the product, taking $L' = \mathrm{Gr}(\varphi)$ in Lemma 5.2.4, we conclude that there exists $C > 0$ such that for any $\varphi \in \mathrm{Symp}_0(M, \omega)$, the function $N_1 : B \to \mathbb{R}$ defined by

$$N_1(s) := |L_s \cap L'|$$

satisfies Crofton's inequality.

Finally, observe that $(x, y) \in L_s \cap L' = \mathrm{Gr}(\psi_s)^{-1} \cap \mathrm{Gr}(\varphi)$ if and only if $(\psi_s)^{-1}(x) = \varphi(x)$, i.e. $x = \psi_s(\varphi(x))$. Thus, there is a one-to-one correspondence between $L_s \cap L'$ and $\mathrm{Fix}(\psi_s \circ \varphi)$. Thus, N_1 coincides with the function N defined by (5.1), and hence

$$\int_B N(s)\mathrm{d}s \leq C \,\mathrm{vol}(L') = C \,\mathrm{vol}(\mathrm{Gr}\,\varphi),$$

as desired. □

We are now in a position to prove Theorem 5.2.1.

Proof of Theorem A. Without loss of generality, we may assume that $\hbar([\varphi_t]) > 0$, since otherwise the inequality is trivial. Fix $\epsilon > 0$. Let $\alpha := \hbar([\varphi_t], 2\epsilon)$ and $\delta > 0$. Then, by definition, there exists a subsequence $\{k_i\}_{i=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ such that

$$\alpha - \delta < \lim_{i\to\infty} \frac{\log b_{2\epsilon}([\varphi_t^{k_i}])}{k_i} \leq \alpha.$$

Hence

$$2^{(\alpha-\delta)k_i} \leq b_{2\epsilon}([\varphi_t^{k_i}]). \tag{5.2}$$

Now, by Lemma 5.2.5, there exist a ball $B \subseteq \mathbb{R}^d$ for some large enough d, a family of Hamiltonian diffeomorphisms $\{\psi_s : s \in B\}$ parametrized by B, and a constant C depending on ϵ, B and $\{\psi_s\}$ such that for each $s \in B$, $d_H(\psi_s, \mathrm{id}_M) < \epsilon/2$ and for each $k \in \mathbb{N}$, the function $N_k : B \to \mathbb{R}$ defined by

$$N_k(s) = \left|\mathrm{Fix}(\psi_s \circ \varphi^k)\right|$$

satisfies

$$\int_B N_k(s)\mathrm{d}s \leq C \operatorname{vol}\operatorname{Gr}(\varphi^k).$$

Thus, with Lemma 5.1.1, we have

$$b_{2\epsilon}([\varphi_t^{k_i}]) \leq b_\epsilon([(\psi_s \circ \varphi^{k_i})_t]) \leq N_{k_i}(s)$$

and hence, with (5.2),

$$2^{(\alpha-\delta)k_i} \operatorname{vol}(B) \leq \int_B N_{k_i}(s)\mathrm{d}s \leq C \operatorname{vol}\operatorname{Gr}(\varphi^{k_i}).$$

Note that C and B are independent of i. Thus, with Yomdin's Theorem, we obtain

$$\alpha - \delta \le \limsup_{k\to\infty} \frac{\log \operatorname{vol} \operatorname{Gr}(\varphi^k)}{k} \le h_{\mathrm{top}}(\varphi)$$

Since $\delta > 0$ was arbitrary, we have $\alpha \le h_{\mathrm{top}}(\varphi)$, i.e.

$$\hbar([\varphi_t], 2\epsilon) \le h_{\mathrm{top}}(\varphi).$$

Passing to the limit as $\epsilon \to 0$, we obtain the desired inequality. □

5.3 Topological Entropy as a Lower Bound for Barcode Entropy

A lower bound for the barcode entropy that involves topological entropy can also be obtained in the context of hyperbolic sets such as horseshoes, as is the case with Theorem B in [ÇGG21].

We are now ready to state our version of Theorem B in [ÇGG21] for symplectomorphisms isotopic to the identity.

Theorem 5.3.1 (Restatement of Theorem B)**.** *Let $\varphi \in \mathrm{Symp}_0(M, \omega_M)$ and $\{\varphi_t\}$ be a symplectic isotopy connecting $\varphi_0 = \mathrm{id}_M$ to $\varphi_1 = \varphi$. If $K \subseteq M$ is a locally maximal hyperbolic subset, then*

$$\hbar([\varphi_t]) \ge h_{\mathrm{top}}\left(\varphi|_K\right).$$

We also recall the Crossing Energy Theorem, which asserts that the energy required for a Floer trajectory to approach a hyperbolic orbit and cross a fixed neighborhood of it is bounded below by a positive constant independent of the iteration. It Let $S^1_k := \mathbb{R}/k\mathbb{Z}$. Fix a 1-periodic family of almost complex structures $\{J_t\} \subset \mathcal{J}(M,\omega)$. Let $\Sigma \subset \mathbb{R} \times S^1_k$ be a closed domain, i.e. a closed subset with non-empty interior. It will be convenient to work on the extended phase spaces $\tilde{M}_k := S^1_k \times M$. The time-dependent flow φ^t lifts to the genuine flow $\tilde{\varphi}^t$ on $\tilde{M}_k$ defined by

$$\tilde{\varphi}^t(\theta, p) = \left(\theta + t, \varphi^t(p)\right),$$

generated by the vector field $\frac{\partial}{\partial\theta} + X_{\eta_t}$, where t in the first coordinate is treated as an element of S^1_k. For every map $u : \mathbb{R} \times S^1_k \to U \subset M$, we will denote its lift by $\tilde{u} : \mathbb{R} \times S^1_k \to \tilde{U} \subset \tilde{M}$.

Suppose x is a k-periodic orbit of φ. A solution $u : \Sigma \to M$ of the Floer equation (2.8) is said to be *asymptotic to x at ∞* if there exists $s_0 \in \mathbb{R}$ such that $[s_0, \infty) \times S^1_k \subset \Sigma$ and

$$\lim_{s\to\infty} \|u(s,\cdot) - x\|_M = 0,$$

where $\|\cdot\|_M$ denotes the C^0-norm on M. Moreover, we say that u is *asymptotic to K at ∞* if for any neighborhood $\tilde{U}$ of $\tilde{K}$, there exists $s_{\tilde{U}} \in \mathbb{R}$ such that $[s_{\tilde{U}}, \infty) \times S^1_k \subset \Sigma$ and

$$\tilde{u}([s_{\tilde{U}}, \infty)) \subset \tilde{U}.$$

In particular, if u is asymptotic at ∞ to x and $x(0) \in K$, then u is asymptotic to K at ∞. Note that it is sufficient that $x(0) \in K$; it is not necessary that the entire orbit x is contained in K.

We similarly define the notion of a solution *asymptotic to K at $-\infty$*.

The *energy* of a solution $u : \Sigma \to M$ of the Floer equation (2.8) is given by

$$E(u) := \int_\Sigma \left|\frac{\partial u}{\partial s}\right|_g^2 \mathrm{d}s\mathrm{d}t.$$

Recall that when $\Sigma = \mathbb{R} \times S^1_k$ and u is asymptotic to the k-periodic orbit x at $-\infty$ and to the k-periodic orbit y at ∞, we have

$$E(u) = \mathcal{A}_{[(\varphi^k)^t]}(x) - \mathcal{A}_{[(\varphi^k)^t]}(y).$$

In the following, $\overline{U}$ denotes the closure of U.

Theorem 5.3.2 (Crossing Energy Theorem)**.** *Let U be a sufficiently small open neighborhood of x with smooth boundary $\partial U := \overline{U} \setminus U$. Then, there exists a constant $c_\infty > 0$, independent of k and Σ, such that for all k-periodic almost complex structures sufficiently C^∞-close to J uniformly on $\mathbb{R} \times U$ and for any solution u of the Floer equation (2.8) for $\{J'_t\}$ that is asymptotic to K as $s \to \infty$ or $s \to -\infty$ and satisfying either*

(i) $\partial\Sigma \neq \emptyset$ and $u(\partial\Sigma) \subset \partial U$, or

(ii) $\Sigma = \mathbb{R} \times S^1_k$ and $u(\Sigma) \not\subset U$,

we have $E(u) > c_\infty$.

A key result that will be useful in the proof of Theorem 5.3.1 is the following lemma that gives a lower bound to action differences of periodic orbits.

Lemma 5.3.3. *Let K be a locally maximal hyperbolic invariant set for φ. There exists a constant $\epsilon_K > 0$ such that for any $k \in \mathbb{N}$, there exists $\delta > 0$ such that for any k-periodic family $\{J'_t\}$ of almost complex structures that is C^∞-close to $\{J_t\}$ by δ, any k-periodic orbits x and y of φ^t with $x(0) \in K$, and any Floer trajectory u asymptotic to x and y with positive energy,*

$$E(u) = \left|\mathcal{A}_{[\varphi^k_t]}(x) - \mathcal{A}_{[\varphi^k_t]}(y)\right| > \epsilon_K.$$

In order to prove this lemma, we need the following result:

Lemma 5.3.4. *There exists sufficiently small $\delta > 0$ such that for any solution $u : \mathbb{R} \times S^1_k \to M$ of the Floer equation (2.8) with $E(u) < \delta$ and for any $s \in \mathbb{R}$, the set*

$$\hat{z} := \{z_i = u(s, i) : i \in \mathbb{Z}/k\mathbb{Z}\}$$

is an ϵ-pseudo-orbit of φ, where $\epsilon \to 0$ as $E(u) \to 0$.

Proof. First, observe that whenever $\dfrac{\partial u}{\partial t}$ is uniformly C^∞-close to X_{η_t} by ϵ, $\hat{z}$ is an η-pseudo-orbit of φ. Indeed, if $\Psi : M \hookrightarrow \mathbb{R}^N$ is some embedding of M into some Euclidean space of sufficiently high dimension, then for some $0 \leq t' \leq 1$,

$$\begin{aligned}
d(\varphi(z_i), z_{i+1}) &= |\Psi(\varphi(z_i)) - \Psi(z_{i+1})| \\
&= |\Psi(\varphi(z_i)) - \Psi(z_i) + \Psi(z_i) - \Psi(z_{i+1})| \\
&= |\Psi(\varphi_1(z_i)) - \Psi(\varphi_0(z_i)) + \Psi(u(s,i)) - \Psi(u(s,i+1))| \\
&= \left| \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t'} \{\Psi(\varphi_t(z_i)) - \Psi(u(s,i+t))\} \right| \\
&= \left| \left.\left(X_{\eta_t} - \frac{\partial u}{\partial t} \right)\right|_{t=t'} \right| \\
&\leq \left\| \frac{\partial u}{\partial t} - X_{\eta_t} \right\|.
\end{aligned}$$

Now, for any solution u of the Floer equation (2.8),

$$\left\| \frac{\partial u}{\partial t} - X_{\eta_t} \right\| = \left\| J \frac{\partial u}{\partial s} \right\| = \left\| \frac{\partial u}{\partial s} \right\|.$$

Finally, by Fish's target-local compactness result (see [Fi]),

$$\left\| \frac{\partial u}{\partial s} \right\| \to 0 \quad \text{as} \quad E(u) \to 0.$$

Thus, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$E(u) < \delta \quad \Rightarrow \quad \left\| \frac{\partial u}{\partial t} - X_{\eta_t} \right\| < \epsilon.$$

□

We now give a proof of the previous lemma.

Proof of Lemma 5.3.3. Let U be an isolating neighborhood for K. Suppose x and y are k-periodic orbits of φ^t with $x(0) \in K$, so that the entire orbit $\{x(t) : t \in \mathbb{R}\}$ is also contained in K. Let c_∞ be the lower bound in the Crossing Energy Theorem for U. We consider two cases. First, suppose $y(0) \notin K$. Take $\epsilon_K = c_\infty$. Note that there exists $t_0 \in S^1_k$ such that $y(t_0) = \varphi^{t_0}(y(0)) \notin U$. Now then there exists sufficiently large $s_\infty > 0$ such that $u(s_\infty, t_0) \notin U$. Thus $u(\Sigma) \not\subset U$, and by the Crossing Energy Theorem, $E(u) > c_\infty = \epsilon_K$.

Now suppose $y(0) \in K$, so that both orbits $\{x(t) : t \in \mathbb{R}\}$ and $\{y(t) : t \in \mathbb{R}\}$ are entirely contained in K. We proceed by contradiction: Suppose that for any $\epsilon > 0$ there exists a Floer trajectory u asymptotic to x and y such that $0 < E(u_\epsilon) \leq \epsilon$. Without loss of generality, we may assume that U is sufficiently small so that both the Crossing Energy Theorem and Anosov Closing Lemma apply. Moreover, we may assume without loss of generality that u is asymptotic to x at $-\infty$. Let c_∞ be as in the conclusion of the Crossing Energy Theorem, ϵ_0 be as in the Anosov Closing Lemma, and δ and η be as in Lemma 5.3.4. By choosing $\epsilon < c_\infty$ and the Crossing Energy Theorem, we have $\tilde{u}_\epsilon(\Sigma) \subset \tilde{U}$. By choosing $\epsilon < \delta$, we have by Lemma 5.3.4 that for any $s \in \mathbb{R}$,

$$\hat{z}^s := \{z^s_i = u_\epsilon(s, i) : i \in \mathbb{Z}/k\mathbb{Z}\}$$

is a periodic η-orbit of φ. Since $\eta \to 0$ as $E(u) \to 0$, we can choose ϵ small enough so that

$$E(u_\epsilon) < \epsilon \quad \Rightarrow \quad \eta < \epsilon_0,$$

and so by the Anosov Closing Lemma, for every $s \in \mathbb{R}$, there exists a true periodic orbit $\hat{w}^s$ in K shadowing $\hat{z}$, i.e. for some constant $C > 0$ that depends only on U and φ, we have

$$d(z_i^s, w_i^s) < C\eta$$

for all $i \in \mathbb{Z}/k\mathbb{Z}$. Now, since K is a hyperbolic set, $\varphi|_K$ is expansive. Thus, for some constant $\delta' > 0$ that depends only on φ and K, any two distinct orbits of φ in K are separated by a distance of at least δ'. Thus, by choosing ϵ sufficiently small so that $C\eta < \delta'/2$ as well, we have for each $s \in \mathbb{R}$ that $\hat{z}^s = \hat{w}^s$. Clearly

$$\lim_{s \to -\infty} \hat{z}^s = x$$

uniformly with respect to the C^0-norm, with each $\hat{z}^s$ a periodic orbit of φ. By nondegeneracy of x, x must be isolated, and therefore $\hat{z}^s = x$ for all s sufficiently large negative. By continuity of $\hat{z}^s$ in s, we then have $\hat{z}^s = x$ for all $s \in \mathbb{R}$. That is, $u(s,t) = x(t)$ for all $s \in \mathbb{R}$ and so $x = y$ and $E(u) = 0$. Contradiction. $\square$

Finally, it would be convenient to recall the notion of a Floer graph, which was also used in [ÇGG21] to prove the analogous result for Hamiltonian diffeomorphisms and introduced for the first time in [ÇGG22b]. The *Floer graph* for a Floer package $(\mathcal{C}, \{x_i\}, \partial, \mathcal{A})$ is the directed graph whose vertices are the generators x_i of $\mathcal{C}$ and edges are (x_i, x_j), where x_j appears in ∂x_i. That is, if

$$\partial x_i = \sum_j f_{ij} T^{a_{ij}} x_j.$$

we have an edge (x_i, x_j) in the Floer graph if and only if $f_{ij} = 1$. We label the edge (x_i, x_j) by a_{ij}. We let the length of the arrow representing this edge be the action difference

$$\mathcal{A}(x_i) - \mathcal{A}(T^{a_{ij}} x_j) = \mathcal{A}(x_i) - \mathcal{A}(x_j) + a_{ij}.$$

A vertex x_i is said to be *ϵ-isolated* if every possible edge (x_i, x_j) or (x_j, x_i) has length strictly greater than ϵ. We recall Proposition 3.8 in [ÇGG21], which allows us to extract from the Floer graph a lower bound for $b_\epsilon(\mathcal{C})$:

Lemma 5.3.5. *Let $\epsilon > 0$. Suppose that the Floer graph of $(\mathcal{C}, \{x_i\}, \partial, \mathcal{A})$ has p ϵ-isolated vertices. Then*

$$b_\epsilon(\mathcal{C}) \geq \frac{p}{2}.$$

See [ÇGG21] for a proof.

Proof of Theorem B. First, by Lemma 4.3.3, we can assume without loss of generality that K is a locally maximal hyperbolic set. Thus, by Theorem 4.3.2,

$$h_{\mathrm{top}}(\varphi|_K) = \limsup_{k\to\infty} \frac{\log^+ P_k(\varphi|_K)}{k}.$$

Thus, it suffices to show that

$$b_\epsilon([\varphi_t^k]) \geq \frac{P_k(\varphi|_K)}{2}.$$

Indeed, taking the logarithm, dividing through by k, passing to the upper limit as $k \to \infty$, and passing to the limit as $\epsilon \searrow 0$ yields $\hbar([\varphi_t]) \geq h_{\mathrm{top}}(\varphi|_K)$.

Let $\epsilon_K > 0$ be the lower bound as in Lemma 5.3.3. That is, for any positive-energy Floer trajectory u asymptotic to k-periodic orbits x and y with $x(0) \in K$,

$$E(u) = \left|\mathcal{A}_{[\varphi_t^k]}(x) - \mathcal{A}_{[\varphi_t^k]}(y)\right| > \epsilon_K.$$

Note that there are exactly $P_k(\varphi|_K)$ k-periodic points of φ in K. Thus, there are at least this many generators for the Floer complex for $[\varphi_t^k]$. By Lemma 5.3.3, for any $0 < \epsilon < \epsilon_K$, each of these generators corresponds to an ϵ-isolated vertex of the Floer graph for the Floer package associated to $[\varphi_t^k]$. Therefore, by Lemma 5.3.5, for any $0 < \epsilon < \epsilon_K$,

$$b_\epsilon([\varphi_t^k]) \geq \frac{P_k(\varphi|_K)}{2},$$

as desired. □

As a consequence, in the case of a surface, we achieve equality between barcode entropy and topological entropy.

Corollary 5.3.6 (Restatement of Theorem C)**.** *If (M, ω) is a closed symplectic surface and $\varphi \in \mathrm{Symp}_0(M, \omega)$ with symplectic isotopy $\{\varphi_t\}$, then*

$$\hbar([\varphi_t]) = h_{\mathrm{top}}(\varphi).$$

Proof. We already generally have $\hbar([\varphi_t]) \leq h_{\mathrm{top}}(\varphi)$ by Theorem 5.2.1. Thus, it only remains to show that $\hbar([\varphi_t]) \geq h_{\mathrm{top}}(\varphi)$ for the case when $\dim M = 2$.

To this end, recall from [BP] that if $\dim M = 2$, then

$$h_{\mathrm{top}}(\varphi) = \sup\{h_{\mathrm{top}}(\varphi|_K) : K \text{ is a hyperbolic horseshoe}\},$$

where in this context, K is said to be a *horseshoe* if K is a closed invariant set such that $\varphi|_K$ is topologically conjugate to a subshift of finite type. Now, observe that in this case, then for any two points $p, q \in K$ sufficiently near each other, the local stable manifold $W_s(p)$ of p and the local unstable manifold $W_u(q)$ intersect transversely at a unique point. This is equivalent to saying that K is locally maximal, for instance by Theorem 5.4 of [AY].

Thus, every hyperbolic horseshoe is a hyperbolic locally maximal subset, and so

$$h_{\text{top}}(\varphi) \leq \sup\{h_{\text{top}}(\varphi|_K) : K \text{ is hyperbolic and locally maximal}\}.$$

Now, by Theorem 5.3.1, $\hbar([\varphi_t]) \geq h_{\text{top}}(\varphi|_K)$ for any locally maximal hyperbolic subset K. Thus, $\hbar([\varphi_t]) \geq h_{\text{top}}(\varphi)$, as desired. $\square$

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