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Authors
Hu, Chao
Hu, Hanxiong
Zhang, Xiaowei
et al.

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Chao Hu · Hanxiong Hu · Xiaowei Zhang · Fai Ma

Refined theory for vibration of thick plates with the lateral and tangential loads

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Abstract Based on the three-dimensional elastodynamics, without using the classical assumption, an operator method is established to refine the dynamic theory of an infinite homogenous isotropic plate by using the spectral decomposition of operators. By this method, the governing equations of the bending and stretching vibrations of plates with the lateral and tangential loads on the surface are derived from the Boussinesq–Galerkin solution of the three-dimensional elasticity, respectively. To effectively deduce the governing equations, a complex differential operator is introduced. Dispersion relations based on the refined equations and the three-dimensional elastodynamics are compared to verify the refined theory of plates. It is shown that the dispersion relation of the refined theory of plates agrees more with the result based on the three-dimensional elastodynamics than Mindlin's theory. Therefore, the refined equations are accurate that can be used to solve the vibration of thick plates and determine high-order vibration modes of plates. The applicable conditions of the refined plate theory are analyzed and discussed.

Keywords Refined dynamic theory of plates · Spectral decomposition of operators · Typical low-dimensional structure · Dynamic equations of plate bending and stretching · Lateral and tangential loads

1 Introduction

Potential theory of elasticity proposed in the late nineteenth century is still being used in engineering, and an overview of the theory was presented [1]. The complex-valued holomorphic potential is one of the most useful methods to solve elasticity. Recently, the algebra of real quaternion is used in solving three-dimensional elasticity [2]. The plate theory in elasticity is still a problem worthy of attention in engineering, especially, the non-classical modeling dynamics of the low-dimensional structure [3–7]. The mini-symposium on the topic of the refined theory of plates and shells is held in the 8th European solid mechanics conference [8].

Numerical methods can be used to solve various complex mechanical problems including nonlinear problems. Nevertheless, isotropic linear elasticity is a frequent problem in engineering. Numerical methods are extensively used at present. However, analytic methods have their own merits. For instance, when solving boundary-value problems, we can use boundary integral equation to decrease dimensions. Another way is to find the stationary value of a properly functional defined in the domain.

C. Hu (✉) · X. Zhang
Yangzhou, China
E-mail: chaohu1961@163.com

H. Hu
University Park, PA, USA

F. Ma
Berkeley, CA, USA

Plates and beams, as typical low-dimensional structures, are not only used in aerospace and civil engineering, but also applied in electronic and micro–nanomechanical devices [9,10]. With the development of modern science and technology, this kind of the low-dimensional structure frequently operates at high frequency. Therefore, it is very important to investigate the dynamical modeling of plates, which are related to the prediction accuracy of dynamic properties, elastic wave propagation and localized vibration in the low-dimensional structure [11–13].

Much work has been done in the area of plate theories by using some classical assumptions and operator methods. In the middle of twentieth century, Reissner utilized the generalized variational principle of complementary energy to derive the static theory of thick plates including the effect of lateral shear deformation [14]. The refined static equation of plate bending was proposed, which yields from the 3-dimensional elasticity by using the operator method [3]. Based on the general solution of elasticity, the refined static equation is extended to transversely isotropic plates [7].

A fundamental contribution to the development of the Timoshenko theory for plate vibration was made by Mindlin, who considered the effect of transverse shear deformation and rotational inertia in frequency domain [15]. Based on the general solution of elastodynamics, using the spectral decomposition of operators, the refined dynamic equation of plate bending was developed, which introduced the imaginary differential operator to successfully get the governing equation of plate vibration [4]. The dynamics of the elastic plate of finite dimensions was investigated by Stoyan, who considered the case of discrete and continuous sets of the initial and boundary conditions that are satisfied by the means of square criterion [16]. It can be seen that these papers usually use some assumptions to construct the governing equation of low-dimensional structures.

In this paper, based on the spectral decomposition of operators, we investigate the operator method to construct the refined dynamical equations of thick plates from three-dimensional elastodynamics without using the well-known straight normal assumption.

2 Refined dynamic equation of plates

The governing equations of linear elastodynamics in the absence of body forces are given by the monographs [17,18].

$$\mu_M \nabla_0^2 \boldsymbol{u} + (\lambda_M + \mu_M)\, \nabla_0 \left(\nabla_0 \cdot \boldsymbol{u}\right) = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{1}$$

where $\lambda_M = \frac{vE_M}{(1+v)(1-2v)}$ and $\mu_M = \frac{E_M}{2(1+v)}$ are Lame's constants of materials, E_M and v are elastic modules and Poisson ratio of materials, respectively, ρ is the mass density, t is time, $\nabla_0 = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y} + \mathbf{e}_3 \frac{\partial}{\partial z}$ denotes the Hamiltonian operator in Cartesian coordinate system $oxyz$, $\mathbf{e}_j$ $(j = 1, 2, 3)$ are unit basis vectors, $\nabla_0^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is corresponding Laplacian operator in 3D space, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the corresponding Laplacian operator in 2D space, and $\boldsymbol{u}$ is the displacement vector.

The general solution of Eq. (1) is presented by Boussinesq–Galerkin as follows [17].

$$\boldsymbol{u} = 2\,(1-\nu)\left(\nabla_0^2 - T_1^2\right)\boldsymbol{G} - \nabla_0\left(\nabla_0 \cdot \boldsymbol{G}\right), \tag{2}$$

where $T_j^2 = \frac{1}{c_j^2} \frac{\partial^2}{\partial t^2}$ $(j = 1, 2)$ are the time differential operators, $c_1 = \sqrt{(\lambda_M + 2\mu_M)/\rho}$ and $c_2 = \sqrt{\mu_M/\rho}$ are the velocity of longitudinal and the transverse waves, $\boldsymbol{G} = G_1\mathbf{e}_1 + G_2\mathbf{e}_2 + G_3\mathbf{e}_3$ is the Somigliana's vector potential, which satisfies the following equation

$$\prod_{j=1}^{2} \left(\nabla_0^2 - T_j^2\right) \boldsymbol{G} = 0. \tag{3}$$

Employing the Taylor series expansion about $z = 0$, the displacements in plates can be described in terms of exponential functions of the differential operator:

$$u_j\,(x, y, z, t) = \exp\left(z \frac{\partial}{\partial z}\right) u_j\,(x, y, 0, t) \quad (j = 1, 2, 3), \tag{4}$$

where u_j $(j = 1, 2, 3)$ denote u_x, u_y and u_z, respectively.

The motion in the plate may be decomposed into two parts, in which the symmetric and antisymmetric with reference to middle plane are the longitudinal and transverse displacements in the plate. It can be seen that u_j $(j = 1, 2)$ and u_3 are odd and even functions of the z-coordinate, respectively. Thus, the displacements can be written:

$$u_j(x, y, z, t) = \sinh\left(z\frac{\partial}{\partial z}\right) u(x, y, 0, t) \quad (j = 1, 2) \tag{5a}$$

$$u_3(x, y, z, t) = \cosh\left(z\frac{\partial}{\partial z}\right) u_3(x, y, 0, t), \tag{5b}$$

where $\sinh(\cdot)$ and $\cosh(\cdot)$ are the hyperbolic sine and cosine functions.

Consider Eqs. (2) and (4), the following expression can be obtained:

$$\begin{aligned} G_k(x, y, z, t) &= \exp\left(z\frac{\partial}{\partial z}\right) G_k(x, y, 0, t) \\ &= \exp\left(z\frac{\partial}{\partial z}\right) \sum_{j=1}^{2} G_k^j(x, y, 0, t) \\ &= 2\,\mathrm{Re} \sum_{j=1}^{2} \left[\exp\left(\mathrm{i}z\Box_j\right) g_k^{j1}\right], \end{aligned} \tag{6}$$

where $\mathrm{Re}(\cdot)$ means to take the real part of complex variables, $\Box_j^2 = \nabla^2 - T_j^2$ $(j = 1, 2)$ is Lorentz operator. $\mathrm{i}\Box_j$ is the introduced imaginary differential operator, $\boldsymbol{G} = \sum_{j=1}^{2} \boldsymbol{G}^j = \sum_{j=1}^{2} \sum_{k=1}^{2} \boldsymbol{G}^{jk}$, $\left(\Box_j^2 + \frac{\partial^2}{\partial z^2}\right) \boldsymbol{G}^j = 0$, $\left(\frac{\partial}{\partial z} - \mathrm{i}\Box_j\right) \boldsymbol{G}^{j1} = 0$ and $\left(\frac{\partial}{\partial z} + \mathrm{i}\Box_j\right) \boldsymbol{G}^{j2} = 0$, $j = 1, 2$.

To eliminate the non-uniqueness of the unknown function in potential, the following gauge conditions are introduced [17,19]

$$\frac{\partial}{\partial x} g_1^j + \frac{\partial}{\partial y} g_2^j = 0 \quad (j = 1, 2), \tag{7}$$

Thus, we are arrive at

$$\begin{aligned} \nabla_0 (\nabla_0 \cdot \boldsymbol{G}) = &-2\mathrm{Im} \sum_{j=1}^{2} \left[\exp\left(\mathrm{i}z\Box_j\right) \Box_j \frac{\partial}{\partial x} g_3^{j1}\right] \mathbf{e}_1 \\ &-2\mathrm{Im} \sum_{j=1}^{2} \left[\exp\left(\mathrm{i}z\Box_j\right) \Box_j \frac{\partial}{\partial y} g_3^{j1}\right] \mathbf{e}_2 \\ &-2\,\mathrm{Re} \sum_{j=1}^{2} \left[\exp\left(\mathrm{i}z\Box_j\right) \Box_j^2 g_3^{j1}\right] \mathbf{e}_3, \end{aligned} \tag{8}$$

where $\mathrm{Im}(\cdot)$ means to take the imaginary part of complex variables.

Substituting Eqs. (6) and (8) into Eq. (2), we can obtain:

$$u_k = 2\,\mathrm{Re}\left[\exp\left(\mathrm{i}z\Box_2\right) T_2^2 g_k^{21}\right] + 2\mathrm{Im} \sum_{j=1}^{2} \left[\exp\left(\mathrm{i}z\Box_j\right) \Box_j \frac{\partial}{\partial x_k} g_3^{j1}\right] \quad (k = 1, 2), \tag{9a}$$

$$u_3 = 2\,\mathrm{Re}\left[\exp\left(\mathrm{i}z\Box_2\right) T_2^2 g_3^{21}\right] + 2\,\mathrm{Re} \sum_{j=1}^{2} \left[\exp\left(\mathrm{i}z\Box_j\right) \Box_j^2 g_3^{j1}\right]. \tag{9b}$$

Therefore, the displacements at the middle plane, rotational angles of normal to the middle plane, and the lateral strain at the middle plane of plates can be described as:

$$U_k = u_k|_{z=0} = 2\,\mathrm{Re}\left(T_2^2 g_k^{21}\right) + 2\mathrm{Im}\sum_{j=1}^{2}\left(\Box_j \frac{\partial}{\partial x_k} g_3^{j1}\right) \quad (k=1,2)\,, \tag{10a}$$

$$W = u_z|_{z=0} = 2\,\mathrm{Re}\left(T_2^2 g_3^{21}\right) + 2\,\mathrm{Re}\sum_{j=1}^{2}\left(\Box_j^2 g_3^{j1}\right), \tag{10b}$$

$$\psi_k = -\frac{\partial u_k}{\partial z}\bigg|_{z=0} = \mathrm{Im}\left(\Box_2 T_2^2 g_k^{21}\right) \quad - 2\,\mathrm{Re}\sum_{j=1}^{2}\left(\Box_j^2 \frac{\partial}{\partial x_k} g_3^{j1}\right) \quad (k=1,2), \tag{10c}$$

$$E = \frac{\partial u_z}{\partial z}\bigg|_{z=0} = -2\mathrm{Im}\left(\Box_2 T_2^2 g_3^{21}\right) - 2\mathrm{Im}\sum_{j=1}^{2}\left(\Box_j^3 g_3^{j1}\right). \tag{10d}$$

The functions in Eqs. (10a) and (10c) may be changed into the following expressions by the decomposition method of generalized displacements [20]

$$\begin{aligned}\psi_1 &= \frac{\partial}{\partial x}F^{(1)} + \frac{\partial}{\partial y}f^{(1)}, \psi_2 = \frac{\partial}{\partial y}F^{(1)} - \frac{\partial}{\partial x}f^{(1)},\\ U_1 &= \frac{\partial}{\partial x}F^{(2)} + \frac{\partial}{\partial y}f^{(2)}, U_2 = \frac{\partial}{\partial y}F^{(2)} - \frac{\partial}{\partial x}f^{(2)}.\end{aligned} \tag{11}$$

Substituting Eq. (11) into Eq. (10), the following expressions are obtained

$$\begin{aligned}&\mathrm{Im}\left(g_1^{21}\right) = \frac{1}{2}\Box_2^{-1}T_2^{-2}\frac{\partial}{\partial y}f^{(1)}, \mathrm{Im}\left(g_2^{21}\right) = -\frac{1}{2}\Box_2^{-1}T_2^{-2}\frac{\partial}{\partial x}f^{(1)},\\ &\mathrm{Re}\left(g_1^{21}\right) = \frac{1}{2}T_2^{-2}\frac{\partial}{\partial y}f^{(2)},\ \mathrm{Re}\left(g_2^{21}\right) = -\frac{1}{2}T_2^{-2}\frac{\partial}{\partial x}f^{(2)},\\ &\mathrm{Re}\left(g_3^{11}\right) = -\frac{1}{2}\Box_1^{-2}T_2^{-2}\left(\Box_2^2 W + \nabla^2 F^{(1)}\right),\ \mathrm{Re}\left(g_3^{21}\right) = \frac{1}{2}T_2^{-2}\left(W + F^{(1)}\right),\\ &\mathrm{Im}\left(g_3^{11}\right) = \frac{1}{2}\Box_1^{-1}T_1^{-2}\left(E + \nabla^2 F^{(2)}\right), \mathrm{Im}\left(g_3^{21}\right) = -\frac{1}{2}T_1^{-2}\Box_2^{-1}\left(E + \Box_1^2 F^{(2)}\right).\end{aligned} \tag{12}$$

Here, the negative exponent in the differential operator means the inverse operator, which can be represented by the integration of Green's function. Thus, the displacements in the bending and stretching vibration of plates are

$$\begin{aligned}u_k &= 2T_2^2\left[\cos\left(z\Box_2\right)\ \mathrm{Re}\left(g_k^{21}\right) - \sin\left(z\Box_2\right)\ \mathrm{Im}\left(g_k^{21}\right)\right]\\ &\quad +2\sum_{j=1}^{2}\Box_j\frac{\partial}{\partial x_k}\left[\cos\left(z\Box_j\right)\ \mathrm{Im}\left(g_3^{j1}\right) + \sin\left(z\Box_j\right)\ \mathrm{Re}\left(g_3^{j1}\right)\right] \quad (k=1,2)\,,\\ u_3 &= 2T_2^2\left[\cos\left(z\Box_2\right)\ \mathrm{Re}\left(g_3^{21}\right) - \sin\left(z\Box_2\right)\ \mathrm{Im}\left(g_3^{21}\right)\right]\\ &\quad +2\sum_{j=1}^{2}\Box_j^2\left[\cos\left(z\Box_j\right)\ \ \mathrm{Re}\left(g_3^{j1}\right) - \sin\left(z\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right)\right].\end{aligned} \tag{13}$$

Here, $\frac{\sin(z\Box_j)}{\Box_j}$ and $\cos\left(z\Box_j\right)$ $(j=1,2)$ are differential operators whose values can be obtained by expanding the functions $\sin\left(z\Box_j\right)$ and $\cos\left(z\Box_j\right)$ into the series in power of $z\Box_j$ and returning the operator value to the operator $\Box_j^2$.

According to Hooke's law, the stress components in plates can be described as

$$\begin{aligned}
\tau_{zx} =& \mu_M \Big\{ -2\Box_2 T_2^2 \left[\sin(z\Box_2)\ \mathrm{Re}\left(g_1^{21}\right) + \cos(z\Box_2)\,\mathrm{Im}\left(g_1^{21}\right) \right] \\
& -4\sum_{j=1}^{2} \sin\left(z\Box_j\right) \Box_j^2 \frac{\partial}{\partial x} \mathrm{Im}\left(g_3^{j1}\right) + 4\sum_{j=1}^{2} \cos\left(z\Box_j\right) \Box_j^2 \frac{\partial}{\partial x}\ \mathrm{Re}\left(g_3^{j1}\right) \\
& +2T_2^2 \left[\cos(z\Box_2) \frac{\partial}{\partial x}\ \mathrm{Re}\left(g_3^{21}\right) - \sin(z\Box_2) \frac{\partial}{\partial x} \mathrm{Im}\left(g_3^{21}\right) \right] \Big\}, \\
\tau_{zy} =& \mu_M \Big\{ -2\Box_2 T_2^2 \left[\sin(z\Box_2)\ \mathrm{Re}\left(g_2^{21}\right) + \cos(z\Box_2)\,\mathrm{Im}\left(g_2^{21}\right) \right] \\
& -4\sum_{j=1}^{2} \sin\left(z\Box_j\right) \Box_j^2 \frac{\partial}{\partial y} \mathrm{Im}\left(g_3^{j1}\right) + 4\sum_{j=1}^{2} \cos\left(z\Box_j\right) \Box_j^2 \frac{\partial}{\partial y}\ \mathrm{Re}\left(g_3^{j1}\right) \\
& +2T_2^2 \left[\cos(z\Box_2) \frac{\partial}{\partial y}\ \mathrm{Re}\left(g_3^{21}\right) - \sin(z\Box_2) \frac{\partial}{\partial y} \mathrm{Im}\left(g_3^{21}\right) \right] \Big\}, \\
\sigma_z =& 2(\lambda_M + 2\mu_M) \Box_1 T_1^2 \left[\cos(z\Box_1)\mathrm{Im}\left(g_3^{11}\right) + \sin(z\Box_1)\ \mathrm{Re}\left(g_3^{11}\right) \right] \\
& -4\mu_M \sum_{j=1}^{2} \nabla^2 \Box_j \left[\cos\left(z\Box_j\right)\mathrm{Im}\left(g_3^{j1}\right) + \sin\left(z\Box_j\right)\ \mathrm{Re}\left(g_3^{j1}\right) \right].
\end{aligned} \tag{14}$$

The surfaces of plates are assumed to be loaded. Thus, the boundary conditions on the upper and lower surfaces of plates to be satisfied are

$$\tau_{zx}\big|_{z=\pm\frac{h}{2}} = \mu_M \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = \pm\frac{1}{2} q_x \quad \text{for antisymmetric condition} \tag{15a}$$

$$\tau_{zx}\big|_{z=\pm\frac{h}{2}} = \mu_M \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = \frac{1}{2} p_x \quad \text{for symmetric condition} \tag{15b}$$

$$\tau_{zy}\big|_{z=\pm\frac{h}{2}} = \mu_M \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) = \pm\frac{1}{2} q_y \quad \text{for antisymmetric condition} \tag{16a}$$

$$\tau_{zy}\big|_{z=\pm\frac{h}{2}} = \mu_M \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) = \pm\frac{1}{2} p_y \quad \text{for symmetric condition} \tag{16b}$$

$$\sigma_z\big|_{z=\pm\frac{h}{2}} = \lambda_M \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + 2\mu_M \frac{\partial u_z}{\partial z} = \pm\frac{1}{2} q_n \quad \text{for antisymmetric condition} \tag{17a}$$

$$\sigma_z\big|_{z=\pm\frac{h}{2}} = \lambda_M \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) + 2\mu_M \frac{\partial u_z}{\partial z} = \frac{1}{2} p_n \quad \text{for symmetric condition} \tag{17b}$$

Here h is the thickness of the plate $q_x = q_x^+ - q_x^-$, $p_x = q_x^+ + q_x^-$, $q_y = q_y^+ - q_y^-$, $q_n = q_n^+ - q_n^-$, $p_n = q_n^+ + q_n^-$.

Divide the load on the upper and lower surface of plates ($z = \pm h/2$) into the symmetric and antisymmetric parts with respect to the middle plane, satisfying the boundary conditions of the upper and lower surface of plates. We can get the following equations from Eqs. (14), (15), (16) and (17).

$$\begin{aligned}
& \frac{\partial}{\partial x} \left[2\cos\left(\frac{h}{2}\Box_1\right) F^{(1)} + \cos\left(\frac{h}{2}\Box_2\right)\left(W + F^{(1)}\right) - 2\Box_2^2 T_2^{-2} \sum_{j=1}^{2} (-1)^{j-1} \cos\left(\frac{h}{2}\Box_j\right)\left(W + F^{(1)}\right) \right] \\
& \pm \frac{\partial}{\partial x} \left[2\Box_1 \sin\left(\frac{h}{2}\Box_1\right) F^{(2)} - \frac{\sin\left(\frac{h}{2}\Box_2\right)}{\kappa\Box_2} \left(E + \Box_1^2 F^{(2)}\right) + 2T_1^{-2} \sum_{j=1}^{2} (-1)^{j-1} \Box_j \sin\left(\frac{h}{2}\Box_j\right)\left(E + \Box_1^2 F^{(2)}\right) \right] \\
& = \frac{\partial}{\partial y} \left[\cos\left(\frac{h}{2}\Box_2\right) f^{(1)} \pm \Box_2 \sin\left(\frac{h}{2}\Box_2\right) f^{(2)} \right] \pm \frac{1}{2\mu_M} q_x + \frac{1}{2\mu_M} p_x,
\end{aligned} \tag{18}$$

$$\frac{\partial}{\partial y}\left[2\cos\left(\frac{h}{2}\square_1\right)F^{(1)}+\cos\left(\frac{h}{2}\square_2\right)\left(W+F^{(1)}\right)-2\square_2^2T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\square_j\right)\left(W+F^{(1)}\right)\right]$$
$$\pm\frac{\partial}{\partial y}\left[2\square_1^2\frac{\sin\left(\frac{h}{2}\square_1\right)}{\square_1}F^{(2)}-\frac{\sin\left(\frac{h}{2}\square_2\right)}{\kappa\square_2}\left(E+\square_1^2F^{(2)}\right)+2T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\square_j\sin\left(\frac{h}{2}\square_j\right)\left(E+\square_1^2F^{(2)}\right)\right]$$
$$=-\frac{\partial}{\partial x}\left[\cos\left(\frac{h}{2}\square_2\right)f^{(1)}\pm\square_2\sin\left(\frac{h}{2}\square_2\right)f^{(2)}\right]\pm\frac{1}{2\mu_M}q_y+\frac{1}{2\mu_M}p_y, \tag{19}$$

$$\left[\cos\left(\frac{h}{2}\square_1\right)\left(E+\nabla^2F^{(2)}\right)-2\kappa\nabla^2\cos\left(\frac{h}{2}\square_1\right)F^{(2)}-2\kappa\nabla^2T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\square_j\right)\left(E+\square_1^2F^{(2)}\right)\right]$$
$$\pm\left[2\kappa\square_2^2\nabla^2T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\frac{\sin\left(\frac{h}{2}\square_j\right)}{\square_j}\left(W+F^{(1)}\right)+2\kappa\nabla^2\frac{\sin\left(\frac{h}{2}\square_1\right)}{\square_1}F^{(1)}-\kappa\frac{\sin\left(\frac{h}{2}\square_1\right)}{\square_1}\left(\square_2^2W+\nabla^2F^{(1)}\right)\right]$$
$$=\pm\frac{\kappa}{2\mu_M}q_n+\frac{\kappa}{2\mu_M}p_n. \tag{20}$$

where $\kappa=\frac{1-2v}{2(1-v)}$.

Solving the simultaneous equations of Eqs. (18) and (19), the following two equations can be obtained

$$2\cos\left(\frac{h}{2}\square_1\right)F^{(1)}-\cos\left(\frac{h}{2}\square_2\right)\left(W+F^{(1)}\right)+2\square_2^2T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\square_j\right)\left(W+F^{(1)}\right)$$
$$\pm\left[2\square_1\sin\left(\frac{h}{2}\square_1\right)F^{(2)}-\frac{1}{\kappa}\frac{\sin\left(\frac{h}{2}\square_2\right)}{\square_2}\left(E+\square_1F^{(2)}\right)+2T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\sin\left(\frac{h}{2}\square_j\right)\left(E+\square_1^2F^{(2)}\right)\right]$$
$$=\frac{1}{2\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}p_x+\frac{\partial}{\partial y}p_y\right)\pm\frac{1}{2\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}q_x+\frac{\partial}{\partial y}q_y\right), \tag{21}$$

$$\left[\cos\left(\frac{h}{2}\square_2\right)f^{(1)}+\frac{1}{2\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial y}p_x-\frac{\partial}{\partial x}p_y\right)\right]\pm\left[\square_2^2\frac{\sin\left(\frac{h}{2}\square_2\right)}{\square_2}f^{(2)}+\frac{1}{2\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial y}q_x-\frac{\partial}{\partial x}q_y\right)\right]=0. \tag{22}$$

The plate vibration can be decomposed into symmetric and antisymmetric motion, which represents the bending vibration and stretching vibration of plates, respectively. Eqs. (21) and (22) can be changed into the following equations

$$\cos\left(\frac{h}{2}\square_1\right)F^{(1)}-\frac{1}{2}\cos\left(\frac{h}{2}\square_2\right)\left(W+F^{(1)}\right)+\square_2^2T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\square_j\right)\left(W+F^{(1)}\right)$$
$$=\frac{1}{4\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}p_x+\frac{\partial}{\partial y}p_y\right), \tag{23a}$$

$$\square_1^2\frac{\sin\left(\frac{h}{2}\square_1\right)}{\square_1}F^{(2)}-\frac{1}{2\kappa}\square_2^2\frac{\sin\left(\frac{h}{2}\square_2\right)}{\square_2}\left(E+\square_1^2F^{(2)}\right)+T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\square_j^2\frac{\sin\left(\frac{h}{2}\square_j\right)}{\square_j}\left(E+\square_1^2F^{(2)}\right)$$
$$=\frac{1}{4\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}q_x+\frac{\partial}{\partial y}q_y\right), \tag{23b}$$

$$\cos\left(\frac{h}{2}\square_2\right)f^{(1)}=\frac{1}{2\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}p_y-\frac{\partial}{\partial y}p_x\right), \tag{24a}$$

$$\square_2^2\frac{\sin\left(\frac{h}{2}\square_2\right)}{\square_2}f^{(2)}=\frac{1}{2\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}q_y-\frac{\partial}{\partial y}q_x\right). \tag{24b}$$

According to the holomorphism function theory, the cosine and sine function of operators in Eq. (24) can be expanded into the following form

$$\cos\left(\frac{h}{2}\Box_2\right) f^{(1)} = \prod_{m=1}^{\infty}\left[1-\frac{h^2\Box_2^2}{(2m-1)^2\pi^2}\right] f^{(1)}, \tag{25a}$$

$$\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2} f^{(2)} = \prod_{m=1}^{\infty}\left[1-\frac{h^2\Box_2^2}{4m^2\pi^2}\right] f^{(2)}. \tag{25b}$$

Substituting Eq. (25) into Eq. (24), truncating the infinite product series, the two-order wave equations arrive at

$$\nabla^2 f^{(1)} - \left(\frac{\pi^2}{h^2}+T_2^2\right) f^{(1)} = \frac{\pi^2}{2\mu_M h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y;\xi,\eta)\left[\frac{\partial p_x(\xi,\eta)}{\partial \eta}-\frac{\partial p_y(\xi,\eta)}{\partial \xi}\right]\mathrm{d}\xi\mathrm{d}\eta, \tag{26a}$$

$$\nabla^2 f^{(2)} - T_2^2 f^{(2)} = \frac{\pi^2}{2\mu_M h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y;\xi,\eta)\left[\frac{\partial q_x(\xi,\eta)}{\partial \eta}-\frac{\partial q_y(\xi,\eta)}{\partial \xi}\right]\mathrm{d}\xi\mathrm{d}\eta. \tag{26b}$$

where $G(x,y;\xi,\eta)$ is Green's function and $G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}$.

The simultaneous equation can be composed by Eqs. (20) and (23a) to get the governing equation of plate bending.

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix} W \\ F^{(1)}\end{bmatrix} = \begin{bmatrix} \frac{1}{4\mu_M}\nabla^{-2}\left(\frac{\partial}{\partial x}p_x+\frac{\partial}{\partial y}p_y\right) \\ \frac{1}{4\mu_M}q_n\end{bmatrix}, \tag{27}$$

where the expressions of these operators in Eq. (27) are

$$L_{11} = T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\Box_j\right)\nabla^2 - \cos\left(\frac{h}{2}\Box_1\right) + \frac{1}{2}\cos\left(\frac{h}{2}\Box_2\right),$$

$$L_{12} = T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\Box_j\right)\nabla^2 + \frac{1}{2}\cos\left(\frac{h}{2}\Box_2\right),$$

$$L_{21} = T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\frac{\sin\left(\frac{h}{2}\Box_j\right)}{\Box_j}\nabla^2\nabla^2 - \left[\frac{3}{2}\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1} - \frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\right]\nabla^2 + \frac{1}{2}\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1}T_2^2,$$

$$L_{22} = T_2^{-2}\sum_{j=1}^{2}(-1)^{j-1}\frac{\sin\left(\frac{h}{2}\Box_j\right)}{\Box_j}\nabla^2\nabla^2 - \left[\frac{1}{2}\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1} - \frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\right]\nabla^2.$$

On the base of determinant of the operator matrix in Eq. (27), the lateral displacement function of plates would satisfy the following equation:

$$T_2^{-2}\left[\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1}\cos\left(\frac{h}{2}\Box_2\right) - \frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\cos\left(\frac{h}{2}\Box_1\right)\right]\Box_2^2\nabla^2 W + \frac{1}{4}T_2^2\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1}\cos\left(\frac{h}{2}\Box_2\right)W$$
$$= \frac{1}{4\mu_M}L_{12}q_n - \frac{1}{4\mu_M}\nabla^{-2}L_{22}\left(\frac{\partial p_x}{\partial x}+\frac{\partial p_y}{\partial y}\right), \tag{28}$$

The corresponding generalized displacement function $F^{(1)}$ can be expressed as:

$$L_{22}F^{(1)} = \frac{1}{4\mu_M}q_n - L_{21}W. \tag{29}$$

Truncating the infinite operator series in Eq. (28), and then we have the governing equation of bending vibration of plates

$$
\begin{aligned}
&D\nabla^2\nabla^2 W-(2-\upsilon)\,DT_2^2\nabla^2 W+CT_2^2 W+\frac{7-8\upsilon}{8}DT_2^4 W\\
&=q_n-\frac{3\,(2-\upsilon)\,D}{4C}\left(\nabla^2-\frac{1-\upsilon}{2-\upsilon}T_2^2\right)q_n-\frac{h}{2}\left\{1-\frac{(2-\upsilon)\,D}{4C}\left[\nabla^2-\frac{3-2\upsilon}{2\,(2-\upsilon)}T_2^2\right]\right\}\left(\frac{\partial p_x}{\partial x}+\frac{\partial p_y}{\partial y}\right),
\end{aligned}
\tag{30}
$$

where C and D are the shear and bending stiffness of plates, $C=\frac{E_M h}{2(1+\upsilon)}$ and $D=\frac{E_M h^3}{12(1-\upsilon^2)}$.

The simultaneous equations can be composed by Eqs. (20) and (23b) to get the governing equation of the stretching plate. We can arrive at

$$
\begin{aligned}
&\Box_1\sin\left(\frac{h}{2}\Box_1\right)F^{(2)}-\frac{1}{2\kappa}\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\left(E+\Box_1^2F^{(2)}\right)\\
&+T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\Box_j\sin\left(\frac{h}{2}\Box_j\right)\left(E+\Box_1^2F^{(2)}\right)=\frac{1}{4\mu_M}\nabla^{-2}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right),
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\nabla^2T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\Box_j\right)\left(E+\Box_1^2F^{(2)}\right)\\
&+\frac{1}{2\kappa}\cos\left(\frac{h}{2}\Box_1\right)\left(E+\nabla^2F^{(2)}\right)+\cos\left(\frac{h}{2}\Box_1\right)\nabla^2F^{(2)}=\frac{1}{4\mu_M}p_n.
\end{aligned}
\tag{32}
$$

Based on Eqs. (31) and (32), the governing equation of the stretching plate can be written

$$
\begin{bmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\begin{bmatrix} E\\ F^{(2)}\end{bmatrix}=\begin{bmatrix}\frac{1}{4\mu_M}\nabla^{-2}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right)\\ \frac{1}{4\mu_M}p_n\end{bmatrix}.
\tag{33}
$$

Here, the expressions of these operators are

$$
\begin{aligned}
L_{11}&=T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\frac{\sin\left(\frac{h}{2}\Box_j\right)}{\Box_j}\nabla^2-\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1}+\frac{1}{2\kappa}\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2},\\
L_{12}&=T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\frac{\sin\left(\frac{h}{2}\Box_j\right)}{\Box_j}\nabla^2\Box_1^2+\frac{1}{2\kappa}\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\Box_1^2,\\
L_{21}&=\frac{1}{2\kappa}\cos\left(\frac{h}{2}\Box_1\right)-T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\Box_j\right)\nabla^2,\\
L_{22}&=\left[\frac{1}{2\kappa}\cos\left(\frac{h}{2}\Box_1\right)-\cos\left(\frac{h}{2}\Box_2\right)\right]\nabla^2-T_1^{-2}\sum_{j=1}^{2}(-1)^{j-1}\cos\left(\frac{h}{2}\Box_j\right)\nabla^2\nabla^2.
\end{aligned}
$$

Based on the operator matrix determinant of Eq. (33), the lateral strain function E at the middle plane would satisfy the following equation

$$
\begin{aligned}
&T_1^{-2}\left[\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1}\cos\left(\frac{h}{2}\Box_2\right)-\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\cos\left(\frac{h}{2}\Box_1\right)\right]\nabla^2\nabla^2E\\
&-\left[\frac{\sin\left(\frac{h}{2}\Box_1\right)}{\Box_1}\cos\left(\frac{h}{2}\Box_2\right)-\frac{1}{\kappa}\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\cos\left(\frac{h}{2}\Box_1\right)\right]\nabla^2E-\frac{1}{4\kappa}T_2^2\frac{\sin\left(\frac{h}{2}\Box_2\right)}{\Box_2}\cos\left(\frac{h}{2}\Box_1\right)E\\
&=\frac{1}{4\mu_M}L_{12}p_n-\frac{1}{4\mu_M}L_{22}\nabla^{-2}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right).
\end{aligned}
\tag{34}
$$

Table 1 Comparison of all kinds of governing equations of bending vibration of plates

A kind of plate theories	Refined theory for plate vibration
Static governing equation	$D\nabla^2\nabla^2 W = \left[1 - \frac{3(2-v)D}{4C}\nabla^2\right] q_n - \frac{h}{2}\left[1 - \frac{(2-v)D}{4C}\nabla^2\right]\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right)$ $\nabla^2 f^{(1)} - \frac{\pi^2}{h^2} f^{(1)} = \frac{\pi^2}{2\mu_M h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y; \xi, \eta)\left[\frac{\partial p_x(\xi,\eta)}{\partial \eta} - \frac{\partial p_y(\xi,\eta)}{\partial \xi}\right] \mathrm{d}\xi\,\mathrm{d}\eta$
Dynamic governing equation	$D\nabla^2\nabla^2 W - (2-v) DT_2^2\nabla^2 W + CT_2^2 W + \frac{7-8v}{8} DT_2^4 W$ $= \left[1 - \frac{3(2-v)D}{4C}\left(\nabla^2 - \frac{1-v}{2-v}T_2^2\right)\right] q_n - \frac{h}{2}\left\{1 - \frac{(2-v)D}{4C}\left[\nabla^2 - \frac{3-2v}{2(2-v)}T_2^2\right]\right\}\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right)$ $\nabla^2 f^{(1)} - \left(\frac{\pi^2}{h^2} + T_2^2\right) f^{(1)} = \frac{\pi^2}{2\mu_M h^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y; \xi, \eta)\left[\frac{\partial p_x(\xi,\eta)}{\partial \eta} - \frac{\partial p_y(\xi,\eta)}{\partial \xi}\right] \mathrm{d}\xi\,\mathrm{d}\eta$
Kind of plate theories	Lagrange–Germain's plate theory
Static governing equation	$D\nabla^2\nabla^2 W = q_n$
Dynamic governing equation	$D\nabla^2\nabla^2 W + CT_2^2 W = q_n$
A kind of plate theories	Reissner's plate theory
Static governing equation	$D\nabla^2\nabla^2 W = q_n - \frac{2-v}{10(1-v)} h^2\nabla^2 q_n$ $\nabla^2 f^{(1)} - \frac{10}{h^2} f^{(1)} = 0$
Dynamic governing equation	N/A
A kind of plate theories	Mindlin's plate theory
Static governing equation	No corresponding equation
Dynamic governing equation	Mindlin's plate theory is given in frequency domain
A kind of plate theories	Hencky's plate theory
Static governing equation	$D\nabla^2\nabla^2 W = q_n - \frac{1}{6(1-v)} h^2\nabla^2 q_n$
Dynamic governing equation	N/A
A kind of plate theories	Panc's plate theory
Static governing equation	$D\nabla^2\nabla^2 W = q_n - \frac{1}{5(1-v)} h^2\nabla^2 q_n$
Dynamic governing equation	N/A

Truncating the infinite product series in Eq. (34), a fourth-order wave equation of stretching vibration of plates can be obtained.

$$\nabla^2\nabla^2 E - 12\left[\frac{1}{h^2} + \frac{2-\kappa^2}{24(1-\kappa)}T_2^2\right]\nabla^2 E + \frac{3}{1-\kappa}\left[\frac{1}{h^2} + \frac{1+3\kappa}{24}T_2^2\right]T_2^2 E$$
$$= \frac{3}{2(1-\kappa)\mu_M h^2}\Box_1^2 p_n - \frac{3}{8(1-\kappa)\mu_M h}\left(\Box_1^2 - \frac{8(1-2\kappa)}{h^2}\right)\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right). \tag{35}$$

Here, the corresponding generalized displacement function $F^{(2)}$can be expressed in terms of the function E as follows:

$$L_{12}F^{(2)} = -L_{11}E. \tag{36}$$

The static governing equation corresponding to Eq. (35) can be written as follows

$$\nabla^2\nabla^2 E - \frac{12}{h^2}\nabla^2 E = \frac{3}{2(1-\kappa)\mu_M h^2}\nabla^2 p_n - \frac{3}{8(1-\kappa)\mu_M h}\left(\nabla^2 - \frac{8(1-2\kappa)}{h^2}\right)\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right). \tag{37}$$

From Table 1, it can be seen that the static equation of plate bending proposed by this paper and the governing equations of the classical theories of plate bending are very close in the form. When the some high-order terms of time derivatives in the refined theory of plates are ignored, the corresponding reduced equation is close the one of all kinds of governing equations of the classical theory of plate bending.

Without loss of generality, consider the steady-state solution of bending vibration of plates, let

$$W = \operatorname{Re}\left(\tilde{W}e^{-i\omega t}\right),\ F^{(1)} = \operatorname{Re}\left(\tilde{F}^{(1)}e^{-i\omega t}\right),\ f^{(1)} = \operatorname{Re}\left(\tilde{f}^{(1)}e^{-i\omega t}\right). \tag{38}$$

where ω is the angular frequency.

In the following formula, omit the time factor and symbol $\sim$ on the unknown generalized displacement potential for convenience.

Substituting Eqs. (38) into (30), the dispersion relation in terms of the general displacement function W of plate bending is obtained by the following equation

$$\alpha^4 W - (2-\upsilon)\,k_2^2\alpha^2 W - \frac{6\,(1-\nu)}{h^2}k_2^2 W + \frac{7-8\nu}{8}k_2^4 W = 0, \tag{39}$$

$$\left(\nabla^2+\alpha_1^2\right)\left(\nabla^2+\alpha_2^2\right)W = 0, \tag{40}$$

where α_i $(i=1,2)$ are the traveling wavenumber of plates in a bending state, which satisfy

$$\alpha^4 - (2-\nu)\,k_2^2\alpha^2 - 6\,(1-\nu)\,k_2^4\left[\frac{1}{k_2^2h^2} - \frac{7-8\nu}{48\,(1-\nu)}\right] = 0. \tag{41}$$

Here k_2 is the wavenumber of the transverse wave, and $k_2^2 = \omega^2/c_2^2$.

According to the Vieta theorem, we can see that at least a propagational wave exists along the x direction of the plate in a bending state because $\alpha_1^2\alpha_2^2 = -6\,(1-\nu)\,k_2^4\left(\frac{1}{k_2^2h^2} - \frac{1}{48}\frac{7-8\nu}{(1-\nu)}\right)$ from Eq. (41). That is, without losing the generality, the wavenumber α_1 always represents the propagating wave; thus, the wavenumber α_2 represents the propagating wave or attenuating wave.

In the same way, consider the steady solution of stretching vibration of plates, let

$$E = \operatorname{Re}\left(\tilde{E}\mathrm{e}^{-\mathrm{i}\omega t}\right),\; F^{(2)} = \operatorname{Re}\left(\tilde{F}^{(2)}\mathrm{e}^{-\mathrm{i}\omega t}\right),\; f^{(2)} = \operatorname{Re}\left(\tilde{f}^{(2)}\mathrm{e}^{-\mathrm{i}\omega t}\right). \tag{42}$$

Substituting Eq. (42) into Eq. (35), the dispersion relation in terms of the generalized displacement function E of plate stretching is obtained by the following equation

$$\nabla^2\nabla^2 E - 12\left[\frac{1}{h^2} - \frac{2-\kappa^2}{24\,(1-\kappa)}k_2^2\right]\nabla^2 E - \frac{3}{1-\kappa}\left[\frac{1}{h^2} - \frac{1+3\kappa}{24}k_2^2\right]k_2^2 E = 0, \tag{43}$$

$$\left(\nabla^2+\alpha_1^2\right)\left(\nabla^2+\alpha_2^2\right)E = 0, \tag{44}$$

where α_i $(i=1,2)$ is the traveling wavenumber of plate in a stretching state, which would satisfy the following dispersion relation:

$$\alpha^4 + 12\left[\frac{1}{h^2} - \frac{2-\kappa^2}{24\,(1-\kappa)}k_2^2\right]\alpha^2 - \frac{3}{1-\kappa}\left[\frac{1}{h^2} - \frac{1+3\kappa}{24}k_2^2\right]k_2^2 = 0, \tag{45}$$

According to the Vieta theorem, we can see that at least a propagational wave exists along the x direction of plates in a stretching state because $\alpha_1^2\alpha_2^2 = -\frac{3}{1-\kappa}\left(\frac{1}{h^2} - \frac{1+3\kappa}{24}k_2^2\right)k_2^2$ from Eq. (45). That is, without losing the generality, the wavenumber α_1 always represents the propagating wave, thus the wavenumber α_2 represents the propagating wave or attenuating wave.

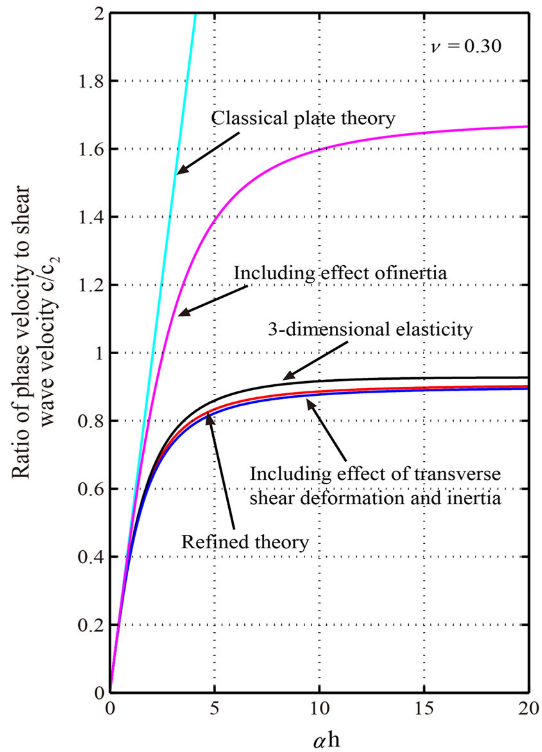

Fig. 1 Dispersion relation by the various plate theories ($\nu = 0.30$)

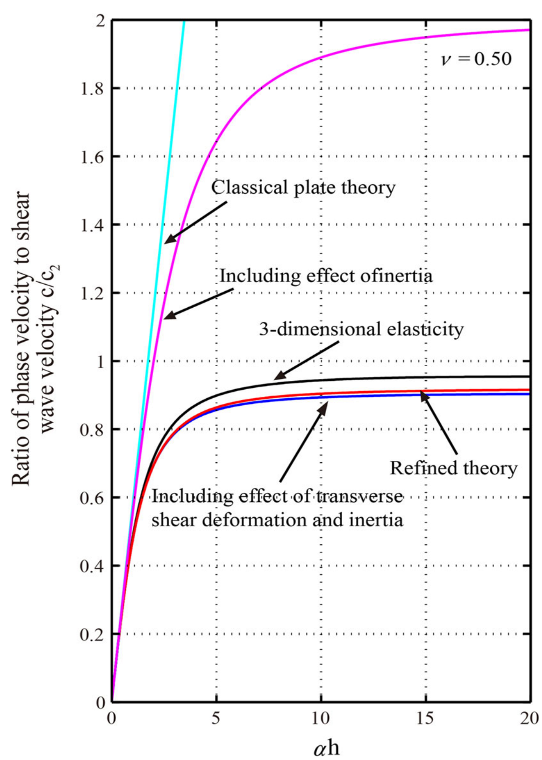

Fig. 2 Dispersion relation by the various plate theories ($\nu = 0.50$)

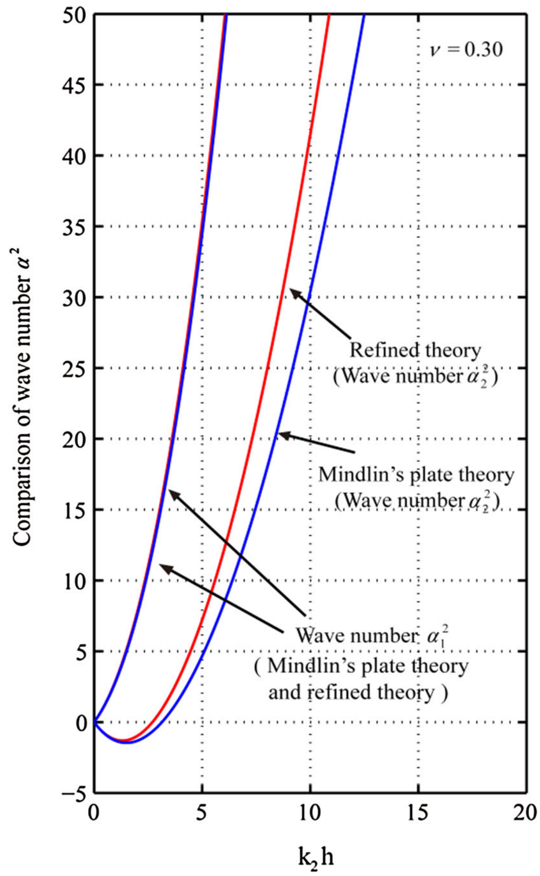

Fig. 3 Wavenumber by the various theories ($\nu = 0.30$)

3 Validation of Refined Plate Theory

3.1 Refined Equation of Plate Bending

The dispersion relations, which are from the given refined theory, 3-dimensional elasticity and Mindlin's plate theory, are graphically presented and compared with each other. And the dispersion equations are as follows

$$\left(\frac{c}{c_2}\right)^2 = \frac{2\pi^2}{3(1-\nu)}\left(\frac{h}{\lambda}\right)^2 \quad \text{based on the classical theory of thin plates} \tag{46}$$

$$\left(\frac{c}{c_2}\right)^2 = \frac{2\pi^2}{3(1-\nu)}\left[1+\frac{\pi^2}{3}\left(\frac{h}{\lambda}\right)^2\right]^{-1}\left(\frac{h}{\lambda}\right)^2 \quad \text{from the theory including the effect of inertia} \tag{47}$$

$$\frac{\pi^2}{3}\left(\frac{h}{\lambda}\right)^2\left[1-\frac{1}{K^2}\left(\frac{c}{c_2}\right)^2\right]\left(\frac{c_p^2}{c_2^2}-1\right) = 1 \quad \text{based on the Mindlin's theory of plates} \tag{48}$$

$$4\frac{\left[1-\kappa\left(\frac{c}{c_2}\right)^2\right]^{1/2}\left[1-\left(\frac{c}{c_2}\right)^2\right]^{1/2}}{\left[2-\left(\frac{c}{c_2}\right)^2\right]^2} = \frac{\tanh\left\{\frac{h}{\lambda}\pi\left[1-\kappa\left(\frac{c}{c_2}\right)^2\right]^{1/2}\right\}}{\tanh\left\{\frac{h}{\lambda}\pi\left[1-\left(\frac{c}{c_2}\right)^2\right]^{1/2}\right\}} \quad \text{based on the 3-dimensional elasticity} \tag{49}$$

where $c_p^2 = \frac{E_M}{\rho(1-\nu^2)}$, $\frac{\lambda}{h}$ and $\frac{\lambda_2}{h}$ are the traveling wavelength and the shear wavelength ratio to the thickness of plates, respectively; K^2 is the correction factor of the shear strain in plates, $K^2 = \frac{\pi^2}{12}$. The dispersion relation of elastic waves in plates proposed by this paper is Eq. (30).

In Figs. 1 and 2, the Poisson's ratio is 0.50 and 0.30, respectively. In Fig. 1, it can be seen that the difference between dispersion curves from Mindlin's plate theory and the one from the 3-dimensional elasticity

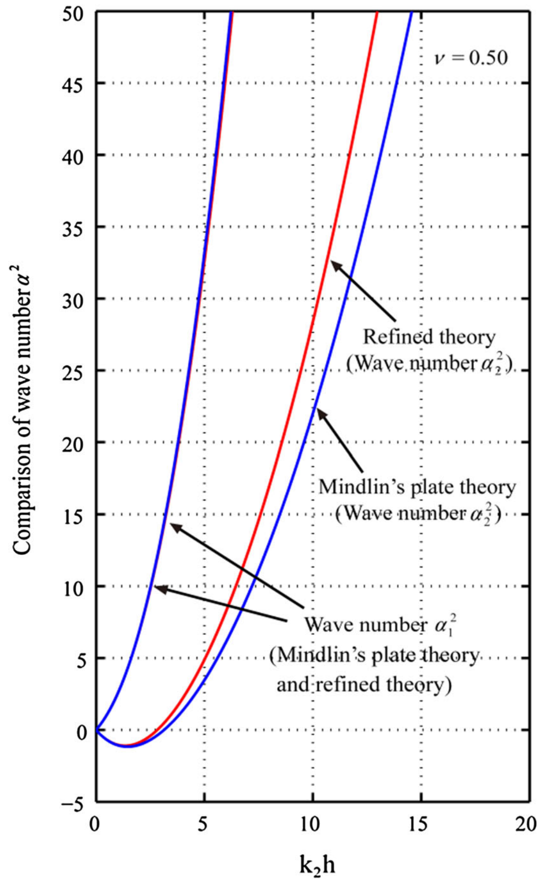

Fig. 4 Wavenumber by the various theories ($\nu = 0.50$)

is larger than the difference between dispersion curves given by this paper and the one from the 3-dimensional elasticity.

The comparison of wavenumber curves of traveling waves is made by Eq. (41) and Mindlin's plate theory on the different Poisson's ratio in Figs. 3 and 4, respectively. It can be seen that the wavenumber α_1^2 of traveling waves given by the refined theory of plate bending is very close to the one from the Mindlin's theory, but the wavenumber α_2^2 of traveling waves given by the refined theory is not close to the one from the Mindlin's plate theory at the higher frequency. It is demonstrated that the Mindlin's plate theory has limitations because the refined theory of plate bending is accurate without the assumptions.

In addition to, the comparison of the mode coefficients from the various theories of plate bending are made on the different Poisson's ratio in Figs. 5 and 6, respectively.

3.2 Refined equation of plate stretching

The dispersion relations, which are from the refined theory, 3D elasticity and the classical theory of plate stretching, respectively, are graphically presented and are compared with each other. And the dispersion equations are as follows:

$$\left(\frac{c}{c_2}\right)^2 = \frac{2}{1-\nu} \text{ from the classical theory of plate stretching} \tag{50}$$

$$\frac{\left[2-\left(\frac{c}{c_2}\right)^2\right]^2}{4\left[1-\kappa\left(\frac{c}{c_2}\right)^2\right]^{1/2}\left[1-\left(\frac{c}{c_2}\right)^2\right]^{1/2}} = \frac{\tanh\left\{\frac{h}{\lambda}\pi\left[1-\kappa\left(\frac{c}{c_2}\right)^2\right]^{1/2}\right\}}{\tanh\left\{\frac{h}{\lambda}\pi\left[1-\left(\frac{c}{c_2}\right)^2\right]^{1/2}\right\}} \text{ from the 3D elastodynamics}$$

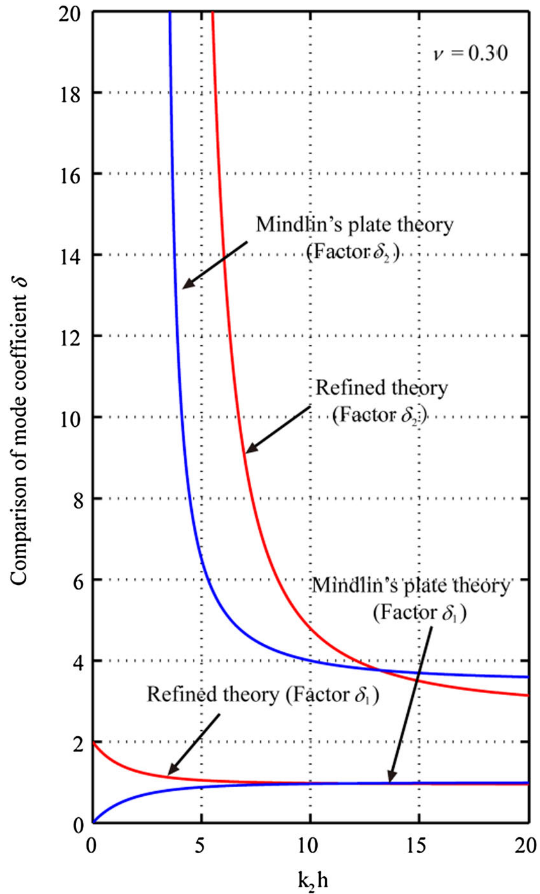

Fig. 5 Mode coefficients by the various theories ($\nu = 0.30$)

(51)

In Figs. 7 and 8, the Poisson's ratio is 0.30 and 0.50, respectively. From Fig. 7, it can be seen that as the Poisson's ratio is smaller, the dispersion curve of the refined theory has little difference with that of the 3D elastodynamics at the lower frequency; From Fig. 8, it can be seen that as the Poisson's ratio is larger, the dispersion curve of the refined theory has larger difference with that of the 3D elastodynamics at the lower frequency.

Besides, from Figs. 7 and 8, we can see that when the elastic wavelength is larger or the vibration frequency is lower, the dispersion relation from the classical theory of plate stretching is the same as one from the 3-dimensional elastodynamics, and the difference goes larger when the elastic wavelength is smaller or the vibration frequency is larger.

4 Conclusion

In this paper, based on 3-dimensional elastodynamics without the classical assumption in plates, the refined dynamic theory of plates, which involves the governing equation of the bending and stretching vibration of plates, is established by the spectral decomposition of operators and the gauge theory, instead of the geometrical mechanics method, which use the force and moment equilibrium equations.

The classical modeling in the plate theory, which is from the force and moment equilibrium, may loss the some influences of the dynamical boundary condition at upper and lower surfaces. Unlike the classical plate theory, the refined equation presented by this paper is directly based on the operator method from the general

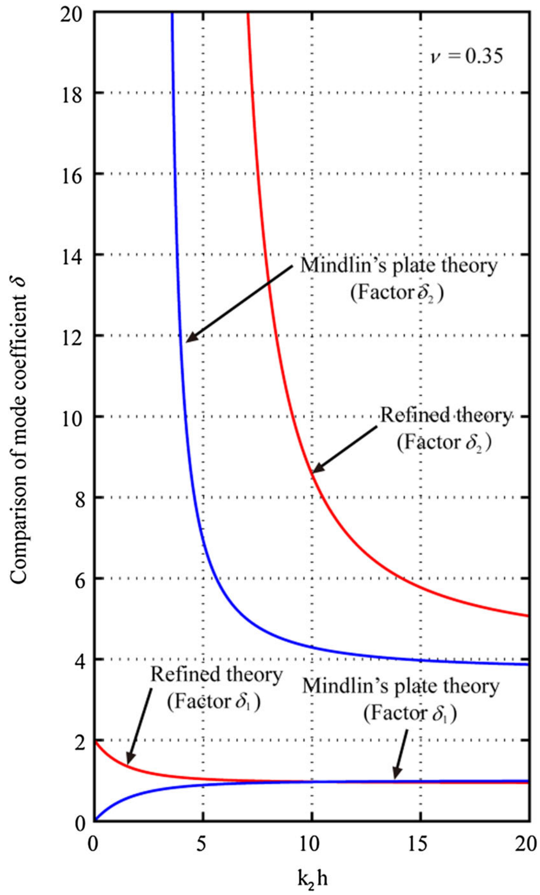

Fig. 6 Mode coefficients by the various theories ($\nu = 0.35$)

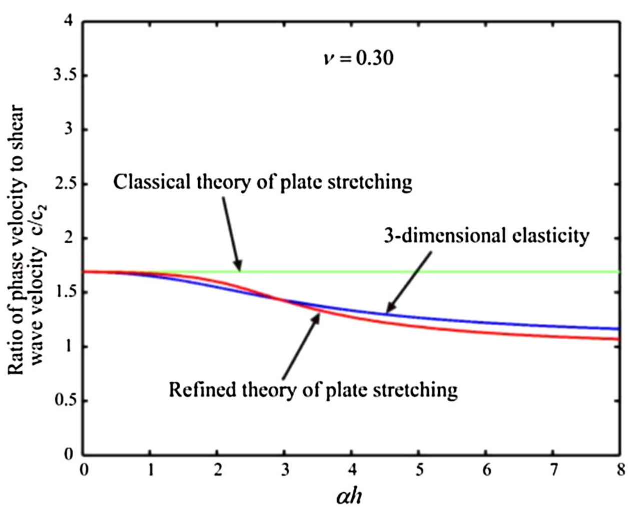

Fig. 7 Dispersion relation by the various plate theories ($\nu = 0.30$)

solution in 3-dimensional elastodynamics. The modeling for the refined theory of plates is one of analytic and algebraic method as opposed to the classical geometric mechanics.

When the deflection of plates is a small, the stress state in plates would be decomposed into the antisymmetric and symmetric vibration with respect to the middle plane without action coupling, in which the bending vibration is governed by a four-order wave equation of lateral displacement and a second-order wave equation

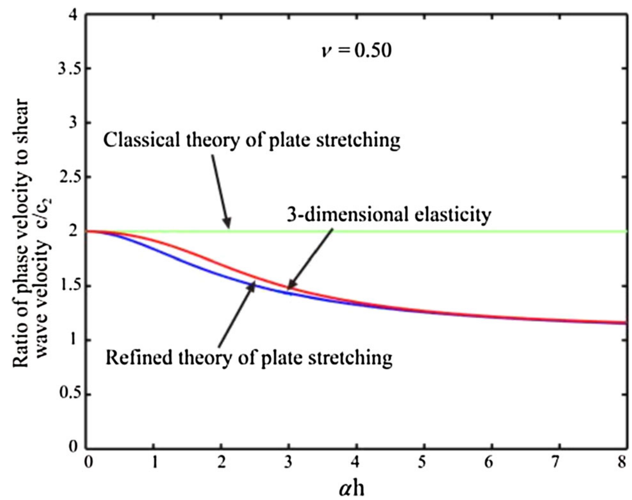

Fig. 8 Dispersion relation by the various plate theories ($\nu = 0.50$)

of the corresponding shear deformation field, and the stretching vibration is governed by a fourth-order wave equation of the lateral strain and a second-order wave equation of the other shear deformation field.

In this paper, we present a general method for constructing the refined dynamical theory of plates, which would be treated as the uniform method to establish the governing equation of the low-dimensional structure from the 3-dimensional elasticity. The comparison of the dispersion relations from the various theories is made to verify the proposed method for establish the refined theory of plates. The feature of the refined theory of plates on the physical meaning is that not only the effect of transverse shear and rotational inertia but also the lateral and tangential loads are included. Unlike the classical plate theory, we can see that the refined equation contains the fourth-order time derivative, which means the initial condition would involve initial displacement, velocity, acceleration and the jerk in plates. The refined dynamical equations of plate bending and stretching in the time domain would be used to analyze and calculate the vibration of thick plates and to determine the high-order vibration modes of plates, because the refined equations of plates are directly from three-dimensional elasticity without using the well-known straight normal assumption, so the refined theory is more accurate than the classical plate theory.

References

1. Barber, J.: Elasticity Solid Mechanics and Its Applications, vol. 107. Springer, Berlin (2003)
2. Weisz-Patrault, D., Bock, S., Gurlebeck, K.: Three-dimensional elasticity based on quaternion-valued potentials. Int. J. Solids Struct. **51**(9), 3422–3430 (2014). doi:10.1016/j.ijsolstr.2014.06.002
3. Cheng, S.: Elasticity theory of plates and a refined theory. J. Appl. Mech. **46**(3), 644–650 (1979). doi:10.1115/1.3424620
4. Clough, R.W., Penzien, J.: Dynamics of Structures. McGraw-Hill Companies, New York (1975)
5. Eringen, A.C., Suhubi, E.S.: Elastodynamics Linear Theory, vol. 2. Academic Press, New York (1975)
6. Hu, C., Ma, F., Ma, X.R., Huang, W.H.: Refined dynamic equations of the plate bending without any assumptions (in Chinese). Scientia Sinica Physica, Mechanica & Astronomica **41**(6), 781–790 (2011). doi:10.1360/132010-788
7. Hu, C., Fang, X.Q., Long, G., Huang, W.H.: Hamiltonian systems of propagation of elastic waves and localized vibrations in the strip plate. Int. J. Solids Struct. **43**(21), 6568–6573 (2006). doi:10.1016/j.ijsolstr.2006.01.011
8. Hu, H.C.: Variational Principles in Elasticity and Its Applications (**in Chinese**). Science Press, Beijing (1981)
9. Karnovsky, I.A., Lebed, O.I.: Non-classical Vibrations of Aches and Beams. McGraw-Hill Companies, New York (2003)
10. Lasiecka, I., Triggiani, R.: Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Cambridge University Press, Cambridge (2000)
11. Li, F.M., Hu, C., Huang, W.H.: One-dimensional localization of elastic waves in rib-stiffened plates (in Chinese). Chin. J. Aeronaut. **15**(4), 208–212 (2002). doi:10.1016/S1000-9361(11)60154-4
12. Liu, D.K., Hu, C.: Scattering of flexural wave and dynamic stress concentration in Mindlin thick plates. Acta. Mech. Sin. **12**(2), 169–185 (1996). doi:10.1007/s10483-014-1883-6
13. Mindlin, R.D.: Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. J. Appl. Mech. **18**, 31–36 (1951). doi:10.1007/978-1-4613-8865-4
14. Pao, Y.H., Mow, C.C.: Diffraction of Elastic Waves and Dynamic Stress Concentrations. Grane Russak, New York (1971)
15. Reissner, E.: The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. **12**(3), 69–77 (1945)
16. Song, W.M.: Dyadic Green's Function and Operator Theory of Electromagnetic Waves (in Chinese). Hefei University of Science and Technology of China Press, Hefei (1991)

17. Stoyan, V.A., Dvirnychuk, K.V.: Mathematical modeling of three-dimensional fields of transverse dynamic displacements of thick elastic plates. Cybern. Syst. Anal. **49**(6), 852–864 (2013). doi:10.1007/s10559-013-9575-3
18. Vasil'ev, V.V., Lur'e, S.A.: On refined theories, plates, and shells. J. Compos. Mater. **26**(4), 546–557 (1992). doi:10.1016/0010-4361(92)90019-Q
19. Victor, A.E., Wojciech, P.: Editorial: Refined theories of plates and shells. Zeitschrift fur Angewandte Mathematik und Mechanik 94: 1-2 and 5-6. doi:10.1002/zamm.201300148 (2014)
20. Wang, M.Z., Zhao, B.S.: The decomposed form of the three-dimensional elastic plate. Chin. J. Theoret. Appl. Mech. **166**, 207–216 (2003). doi:10.1007/s00707-003-0029-2