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Dear Editors,

This paper uses extremum seeking control (ESC) method to study the resource allocation problem with unknown functions in the constraints. In contrast to the literature, the novelty of this paper is summarized as below.

A distributed ESC-based primal-dual algorithm is developed to search for the optimal solution of the optimization problem with unknown functions in the constraints, and the semi-globally practically asymptotically (SPA) stability is proved.

The ESC-based primal-dual algorithm is applied to generate the demand response strategies in electricity markets when the relationship between the operation strategy and the power consumption of the appliance is unknown.

To the best of our knowledge, this is the first work to use ESC in the research of the primal-dual algorithm and the first application of ESC to demand response.

We deeply appreciate your consideration of our manuscript and look forward to receiving comments from the reviewers.

Best regards,
The authors

Resource Allocation with Unknown Constraints: An Extremum Seeking Control Approach and Applications to Demand Response*

Kai Ma[1], Guoqiang Hu[1], and Costas J. Spanos[2]

Abstract— **This paper studies a resource allocation problem with unknown functions in the constraints. The resource allocation problem is formulated as a nonlinear optimization problem. A sufficient condition is established to guarantee a unique global optimal solution in the optimization problem, and an extremum seeking control (ESC)-based primal-dual algorithm is developed to generate the optimal solution. To implement extremum seeking, the gradients of the unknown functions are estimated by adding dither signals to the measurable inputs and outputs. We prove the semi-globally practically asymptotically (SPA) stability of the ESC-based primal-dual algorithm. The results are further applied to the demand response program with distributed heating ventilation air conditioning (HVAC) systems with unknown relationship between the temperature settings and the power consumption. Simulation results demonstrate that the ESC-based primal-dual algorithm converges to a neighborhood of the optimal solution and achieves the balance between supply and demand in electricity markets.**

I. INTRODUCTION

A. Motivation

Resource allocation has been an active topic in the research of networked system, such as power systems [1], [2], communication networks [3]–[5], and building systems [6], [7]. Generally, the objective of resource allocation is to optimize the social welfare under the resource limitation. Both the efficiency and the robustness of the system can be improved by optimizing the distribution of resource. To make it applicable to large-scale systems, distributed algorithms are required to search for the optimal resource allocation strategies. For example, a primal-dual algorithm was developed to generate the optimal solution for a convex optimization problem [8], a better response strategy was proposed to search for the Nash equilibrium of a noncooperative game [9], and a local replicator dynamics was given for learning in evolutionary game theory [10]. The existing literature on resource allocation usually assume that the constraints are exactly known, which is reasonable when physical operations are not included in the resource allocation model. In fact, the physical operations can not be ignored in many applications, such as the wind turbine speed regulation for the wind energy conversion system, the transmission power allocation for the wireless transmitter, and the temperature setting adjustment for the heating ventilation air conditioning (HVAC) system. Typically, the relationship between the physical operations and resource consumption is unknown. In that case, we have the unknown constraints in the resource allocation model. However, few papers are devoted to the resource allocation problem with unknown constraints. In this study, we use the extremum seeking control (ESC) method to solve this problem and apply it to the demand response program. Specifically, we add dither signals to the input and output of the equipment and measure the operation states and the resource consumption to estimate the gradients of the unknown functions. The novelty of this work is two-fold. First, we develop a distributed ESC-based primal-dual algorithm to generate the optimal solution of an optimization problem with unknown functions in the constraints. Second, we apply the ESC-based primal-dual algorithm to the demand response program with unmodeled appliances. To the best of our knowledge, this is the first work to use ESC in the research of primal-dual algorithm and also the first application of ESC to demand response.

B. Related Works

ESC is an adaptive learning method to search for the optimal solution of a model-free optimization problem [11] and has been applied to the wind energy conversion system [12], axial-flow compressor [13], photovoltaic system [14], and so on. Recently, some works applied ESC to the noncooperative game [15] and the constrained convex optimization with unknown utility functions [16]–[18]. However, It has not been used for solving the optimization problem with unknown functions in the constraints.

Distributed optimization has been used for modeling demand response in smart grid. For example, noncooperative game theory was utilized to study the cost minimization of interactive consumers [19], [20] and the charging control of plug-in electric vehicles (PEV) [21]–[23]. Stackelberg game theory was employed to model the interactions between the consumers and the utility companies [24]. Convex optimization was used for minimizing the total costs to all consumers, which can be decomposed into the minimization of the individual cost to each consumer. Then, the power consumption and the price were adjusted based on a distributed primal-dual algorithm [25]. The volatility of electricity markets under real-time price (RTP) was studied

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based on the primal-dual algorithm [26]. Furthermore, the operation constraints of appliances were introduced to the optimization models [27], [28]. The common assumption in the above works is that the relationship between the operation strategy and the resource consumption of the appliance is exactly known to the consumer. In fact, the relationship is unclear because of many uncertainties in the operations, such as external environments, operation conditions, and human factors. Thus, the operation-to-consumption function that relates the operation strategies and the power consumption is unknown. In that case, the primal-dual algorithm given in the above works can not be used for generating the optimal strategies because of the dependence on the gradients of the unknown functions.

C. Organization

The rest of the paper is organized as follows. Preliminaries are given in Section II. In Section III, the resource allocation problem is formulated as an optimization problem, which is equivalent to a convex optimization problem given the established conditions. Then, the ESC-based primal-dual algorithm is developed, and the semi-globally practically asymptotically (SPA) stability is proved. Section IV applies the ESC-based primal-dual algorithm to the demand response program with distributed HVAC systems. Numerical results are shown in Section V, and conclusions are summarized in Section VI.

II. PRELIMINARIES

In this section, we introduce some notations and definitions that will be used in the paper. Given a vector x, we define $\|x\|$ denotes the Euclidean norm and $x \in L_\infty$ denotes $\|x\|_{L_\infty} = \text{ess sup}_{t\geq 0}\|x(t)\| < \infty$.

Definition 1: [29] A continuous function $\beta : R_+ \times R_+ \to R_+$ is of class $\mathscr{K}\mathscr{L}$ if it is nondecreasing in its first argument and converging to zero in its second argument.

Definition 2: [30] A vector function $f(x,\varepsilon) \in R^n$ is said to be $O(\varepsilon)$ if for any compact set $\mathscr{D}$ if there exist positive constants k and ε such that $\|f(x,\varepsilon)\| \leq k\varepsilon$, for $\varepsilon \in (0,\varepsilon^*]$, $x \in \mathscr{D}$.

Definition 3: [30] Given a parameterized family of systems:

$$\dot{x} = f(t,x,\varepsilon), \tag{1}$$

where $x \in R^n$, $t \in R_+$, and $\varepsilon \in R_+^l$ are the state vector, time variable, and parameter vector, respectively. The system (1) is said to be SPA stable, uniformly in $(\varepsilon_1,\ldots,\varepsilon_j)$, $j \in \{1,\ldots,l\}$, if there exists $\beta \in \mathscr{K}\mathscr{L}$ and constructed parameters $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_l)$ such that

$$\|x\| \leq \beta(\|x(0)\|, (\varepsilon_1 \cdot \varepsilon_2 \cdot \cdots \cdot \varepsilon_l)(t-t_0)) + \nu,$$

for all $t \geq t_0$, where the constructed parameters ε and the initial state vector $x_0 = x(t_0)$ satisfy: For each pair of strictly positive real numbers (Δ,ν), the initial state $\|x(0)\| \leq \Delta$ and there exist real numbers $\varepsilon_k^* = \varepsilon_k^*(\Delta,\nu) > 0, k = 1,\ldots,j$ and for each fixed $\varepsilon_k \in (0,\varepsilon_k^*), k = 1,\ldots,j$ there exist $\varepsilon_i = \varepsilon_i(\varepsilon_1,\ldots,\varepsilon_{i-1},\Delta,\nu)$, with $i = j+1,\ldots,l$.

Definition 4: [31] The Taguchi loss function is a statistical method that captures the cost to society due to the manufacture of imperfect products. The loss function is given as

$$V = \gamma(y-\hat{y})^2,$$

where y is the value of quality characteristic, $\hat{y}$ is the target value of y, V is the loss in dollars, and γ is a constant coefficient. The quadratic representation of the loss function is minimum at $y = \hat{y}$, increases as y deviates from $\hat{y}$. The Taguchi loss function defines the relationship between the economic loss and the deviation of the quality characteristic from the target value. For a product with target value $\hat{y}$, $\hat{y} \pm \Delta_0$ represents the deviation at which functional failure of the product occurs. When a product is manufactured with the quality characteristic at the extremes, $\hat{y}+\Delta_0$ or $\hat{y}-\Delta_0$, some countermeasure must be undertaken by the customers. Assuming the cost of countermeasure is A_0 at $\hat{y}+\Delta_0$ or $\hat{y}-\Delta_0$, we define the constant γ as

$$\gamma = \frac{A_0}{\Delta_0^2}.$$

Lemma 1: [29] Suppose that $W : [0,\infty) \to R$ satisfies

$$D^\dagger W(t) \leq -\alpha W(t) + \gamma(t),$$

where $D^\dagger$ denotes the upper Dini derivative, α is a positive constant, and $\gamma(t) \in L_\infty$. Then,

$$\|W(t)\| \leq e^{-\alpha t}\|W(0)\| + \alpha^{-1}\|\gamma(t)\|_{L_\infty}.$$

III. MAIN RESULTS

A. Problem Formulation

We consider a networked system consisting of N consumers that are served by a single resource coordinator. Suppose that consumer i ($i \in \mathbb{N} = \{1,\ldots,N\}$) has K_i equipments, the set of operation strategies for the equipments are denoted as $x_i = (x_{i,1},\ldots,x_{i,j},\ldots,x_{i,K_i})^{\mathrm{T}}$, where $x_{i,j}$ is operation strategy of consumer i on equipment j ($j \in \mathbb{K}_i = \{1,\ldots K_i\}$). For each equipment, the consumer has a cost function $c_{i,j}(x_{i,j})$ that denotes the discomfort caused by changing the normal operation strategy $x_{i,j}^n$ to the actual operation strategy $x_{i,j}$ and an operation-to-consumption function $f_{i,j}(x_{i,j})$ that defines the relationship between the operation strategy and the resource consumption of the equipment. We denote $x_{i,j}^{\min}$ and $x_{i,j}^{\max}$ as the lower limit and the upper limit of the operation strategy of consumer i on equipment j, respectively. The cost to the coordinator is represented by a function $w(q)$, where q is the resource supply.

The objective of resource allocation is to minimize the total costs under the balancing constraints and operation restrictions and can be formulated as the following optimization problem:

$$\begin{aligned}
\text{(P1)}\quad \text{maximize}\quad & -\tau\sum_{i=1}^{N}\sum_{j=1}^{K_i} c_{i,j}(x_{i,j}) - (1-\tau)w(q)\\
\text{subject to}\quad & \sum_{i=1}^{N}\sum_{j=1}^{K_i} f_{i,j}(x_{i,j}) = q\\
& x_{i,j}^{\min} \leq x_{i,j} \leq x_{i,j}^{\max}, \quad i \in \mathbb{N}, j \in \mathbb{K}_i,
\end{aligned}$$

where $\tau \in [0,1)$ is the parameter to achieve the desired tradeoff between the costs to the consumers and the resource coordinator. Without loss of generality, we assume that each consumer has only one equipment. Then, (P1) is reduced to

$$\text{(P2)}\quad \text{maximize}\quad -\tau\sum_{i=1}^{N} c_i(x_i) - (1-\tau)w(q)$$
$$\text{subject to}\quad \sum_{i=1}^{N} f_i(x_i) = q$$
$$x_i^{\min} \le x_i \le x_i^{\max},\quad i \in \mathbb{N},$$

Before proceeding further, we give an assumption used in this paper:

Assumption 1: $c_i(x_i)$ and $f_i(x_i)$ are convex on $[x_i^{\min}, x_i^{\max}]$ and $w(q)$ is convex on $[0,\infty)$.

Generally, (P2) is nonconvex when $f_i(x_i)$ is not an affine function in the balancing constraints [32]. Next, we will give the conditions to guarantee a unique global optimal solution in (P2).

Theorem 1: The optimization problem (P2) has a unique global optimal solution if $w(q)$ is strictly increasing, and the global optimal solution of (P2) is the same as the global optimal solution of the following optimization problem:

$$\text{(P3)}\quad \text{maximize}\quad -\tau\sum_{i=1}^{N} c_i(x_i) - (1-\tau)w(q)$$
$$\text{subject to}\quad \sum_{i=1}^{N} f_i(x_i) \le q$$
$$x_i^{\min} \le x_i \le x_i^{\max},\quad i \in \mathbb{N}.$$

Proof: We first prove that (P2) has a unique global optimal solution. (P2) is equivalent to

$$\text{(P4)}\quad \text{maximize}\quad -\tau\sum_{i=1}^{N} c_i(x_i) - (1-\tau)w\Big(\sum_{i=1}^{N} f_i(x_i)\Big)$$
$$\text{subject to}\quad x_i^{\min} \le x_i \le x_i^{\max},\quad i \in \mathbb{N}.$$

Taking the second derivative of $w(\sum_{i=1}^{N} f_i(x_i))$ with respect to x_i, we obtain

$$\frac{\mathrm{d}^2 w(\sum_{i=1}^{N} f_i(x_i))}{\mathrm{d}x_i^2} = \frac{\mathrm{d}^2 w(\sum_{i=1}^{N} f_i(x_i))}{\mathrm{d}(\sum_{i=1}^{N} f_i(x_i))^2}\cdot\Big(\sum_{i=1}^{N}\frac{\mathrm{d}f_i(x_i)}{\mathrm{d}x_i}\Big)^2 + \frac{\mathrm{d}w(\sum_{i=1}^{N} f_i(x_i))}{\mathrm{d}(\sum_{i=1}^{N} f_i(x_i))}\cdot\Big(\sum_{i=1}^{N}\frac{\mathrm{d}^2 f_i(x_i)}{\mathrm{d}x_i^2}\Big).$$

Since $w(\cdot)$ is strictly increasing and convex and $f_i(x_i)$ is convex, we prove that $\mathrm{d}^2 w(\sum_{i=1}^{N} f_i(x_i))/\mathrm{d}x_i^2 > 0$, i.e., $w(\sum_{i=1}^{N} f_i(x_i))$ is convex. Combining with the convexity of $c_i(x_i)$, we prove that (P4) is a convex optimization problem. Assuming x^* is the unique global optimal solution of (P4), we obtain a feasible solution of (P2) as (x^*, q^*), where $q^* = \sum_{i=1}^{N} f_i(x_i^*)$. Next, we use the mathematical induction to prove that (x^*, q^*) is also the unique global optimal solution of (P2).

Let $-\tau\sum_{i=1}^{N} c_i(x_i') - (1-\tau)w(q') \ge -\tau\sum_{i=1}^{N} c_i(x_i^*) - (1-\tau)w(q^*)$ for some $(x_i', q_i') \in \{x_i | \sum_{i=1}^{N} f_i(x_i) = q, x_i^{\min} \le x_i \le x_i^{\max}, i \in \mathbb{N}\}$. Combining with $q' = \sum_{i=1}^{N} f_i(x_i')$, we obtain $-\tau\sum_{i=1}^{N} c_i(x_i') - (1-\tau)w(\sum_{i=1}^{N} f_i(x_i')) \ge -\tau\sum_{i=1}^{N} c_i(x_i^*) - (1-\tau)w(\sum_{i=1}^{N} f_i(x_i^*))$ for some $x_i' \in \{x_i^{\min} \le x_i \le x_i^{\max}, i \in \mathbb{N}\}$, which is contradictory to the unique global optimality of x_i^*. Then, we conclude that (x^*, q^*) is the unique global optimal solution of (P2). Similarly, if (x^*, q^*) is the unique global optimal solution of (P2) and $q^* = \sum_{i=1}^{N} f_i(x_i^*)$, we can also prove that x^* is the unique global optimal solution of (P4).

Since $c_i(x_i)$, $f_i(x_i)$, and $w(q)$ are all convex, (P3) is a convex optimization problem that generates a unique global optimal solution (x^*, q^*). Next, we will prove the global optimal solution of (P3) is also the global optimal solution of (P2). The Karush-Kuhn-Tucker (KKT) condition [32] of (P3) is denoted as

$$\begin{cases} -\tau\frac{\mathrm{d}c_i(x_i)}{\mathrm{d}x_i}|_{x_i^*} - \lambda^*\frac{\mathrm{d}f_i(x_i)}{\mathrm{d}x_i}|_{x_i^*} + \mu_i^* - \nu_i^* = 0 \\ \lambda^* - (1-\tau)\frac{\mathrm{d}w(q)}{\mathrm{d}q}|_{q^*} = 0 \\ \lambda^*(\sum_{i=1}^{N} f_i(x_i^*) - q^*) = 0 \\ \mu_i^*(x_i^{\min} - x_i^*) = 0 \\ \nu_i^*(x_i^* - x_i^{\max}) = 0 \\ \lambda^* \ge 0 \\ \mu_i^* \ge 0 \\ \nu_i^* \ge 0 \end{cases} \tag{2}$$

where λ^*, μ_i^*, and ν_i^* are the optimal Lagrangian multipliers. Recalling that $w(q)$ is strictly increasing and $\tau \in [0,1)$, we obtain $\lambda^* > 0$ and $\sum_{i=1}^{N} f_i(x_i^*) - q^* = 0$. The KKT condition (2) can be reduced to

$$\begin{cases} -\tau\frac{\mathrm{d}c_i(x_i)}{\mathrm{d}x_i}|_{x_i^*} - (1-\tau)\frac{\mathrm{d}w(\sum_{i=1}^{N} f_i(x_i))}{\mathrm{d}(\sum_{i=1}^{N} f_i(x_i))}|_{x_i^*}\cdot\frac{\mathrm{d}f_i(x_i)}{\mathrm{d}x_i}|_{x_i^*} + \mu_i^* - \nu_i^* = 0 \\ \mu_i^*(x_i^{\min} - x_i^*) = 0 \\ \nu_i^*(x_i^* - x_i^{\max}) = 0 \\ \mu_i^* \ge 0 \\ \nu_i^* \ge 0 \end{cases}$$

which is the KKT condition of (P4). Since (P4) is convex, the KKT condition is the sufficient condition for the optimality. Thus, x^* is the global optimal solution of (P4). Following the equivalence of (P2) and (P4), (x^*, q^*) is the unique global optimal solution of (P2). Similarly, we can also prove that the global optimal solution of (P2) is also the global optimal solution of (P3) given the conditions in Theorem 1. ■

Theorem 1 shows that the optimal solution of (P2) can be obtained by solving (P3) given the established conditions. Since (P3) is a convex optimization problem, we can solve it by the Lagrangian dual method [32]. The Lagrangian function of (P3) is defined as

$$L(x_i, q, \lambda, \mu_i, \nu_i) = \sum_{i=1}^{N}(-\tau c_i(x_i) - \mu_i(x_i^{\min} - x_i) - \nu_i(x_i - x_i^{\max})) - \lambda\Big(\sum_{i=1}^{N} f_i(x_i) - q\Big) - (1-\tau)w(q),$$

where λ, μ_i, and ν_i are the Lagrangian multipliers. Specifically, λ can be seen as the unit price for resource consumption, μ_i and ν_i are the ancillary variables, which ensure x_i to be within the interval $[x_i^{\min}, x_i^{\max}]$. Since the Lagrangian function is concave in x_i and q and convex in λ, μ_i, and

ν_i, the saddle point of the Lagrangian function is the global optimal solution of (P3). Then, we can transform (P3) into the following individual optimization problems:

$$x_i^* = \arg\max -\tau c_i(x_i) - \lambda f_i(x_i) + \mu_i x_i - \nu_i x_i, \qquad (3)$$

and

$$q^* = \arg\max \lambda q - (1-\tau)w(q). \qquad (4)$$

The corresponding dual problems are defined as

$$\lambda^* = \arg\min_{\lambda \geq 0} D(\lambda, \mu_i, \nu_i), \qquad (5)$$

$$\mu_i^* = \arg\min_{\mu_i \geq 0} D(\lambda, \mu_i, \nu_i), \qquad (6)$$

and

$$\nu_i^* = \arg\min_{\nu_i \geq 0} D(\lambda, \mu_i, \nu_i), \qquad (7)$$

where $D(\lambda, \mu_i, \nu_i) = L(x_i^*, q^*, \lambda, \mu_i, \nu_i)$ is the dual function. Then, the primal-dual algorithm corresponding to (3)–(7) are given as

$$\dot{x}_i = k_i^x\left(-\tau \frac{\mathrm{d}c_i(x_i)}{\mathrm{d}x_i} - \lambda \frac{\mathrm{d}f_i(x_i)}{\mathrm{d}x_i} + \mu_i - \nu_i\right), \qquad (8)$$

$$\dot{q} = k^q\left(\lambda - (1-\tau)\frac{\mathrm{d}w(q)}{\mathrm{d}q}\right), \qquad (9)$$

$$\dot{\lambda} = g^\lambda\left[\sum_{i=1}^{N} f_i(x_i) - q\right]_\lambda^+, \qquad (10)$$

$$\dot{\mu}_i = g_i^\mu [x_i^{\min} - x_i]_{\mu_i}^+, \qquad (11)$$

$$\dot{\nu}_i = g_i^\nu [x_i - x_i^{\max}]_{\nu_i}^+, \qquad (12)$$

where k_i^x, k^q, g^λ, g_i^μ, and g_i^ν are the adaptive gains. $[\phi]_\psi^+ = \phi$ if $\phi > 0$ or $\psi > 0$, and $[\phi]_\psi^+ = 0$ otherwise. In practice, (8) models the dynamics of the operation strategy, (9) models the dynamics of the resource supply, and (10) models the dynamics of the resource price.

Remark 1: If τ is equal to 1 in (P2), the objective of resource allocation is to minimize the costs to consumers without considering the cost to the resource coordinator. The resource allocation problem can be reformulated as

$$\text{(P5)} \quad \text{maximize} \quad -\sum_{i=1}^{N} c_i(x_i)$$
$$\text{subject to} \quad x_i^{\min} \leq x_i \leq x_i^{\max}, \quad i \in \mathbb{N},$$

which is also a convex optimization problem and can be solved by the lagrangian dual method.

Remark 2: In practice, the operation-to-consumption function may be a combination of multiple step functions. In that case, we can employ the continuous convex function to approximate it and obtain the optimal operation strategy for the equipment. Then, the sub-optimal strategy can be obtained by approximating the optimal operation strategy to the step value.

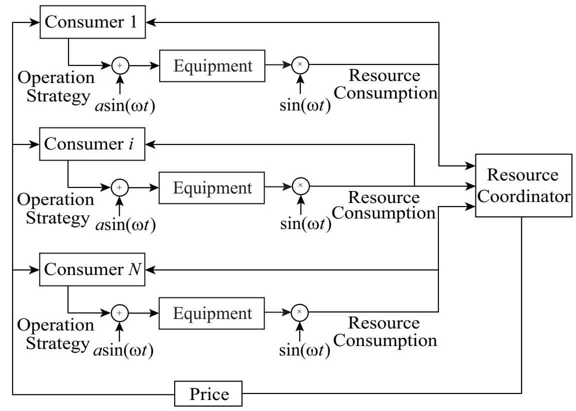

Fig. 1. ESC-based resource allocation scheme.

B. ESC-Based Primal-Dual Algorithm

The primal-dual algorithm needs the gradient information of $f_i(x_i)$, which is easy to obtain when $f_i(x_i)$ is accurately known to the consumers. In fact, the accurate formulations are infeasible because of many unpredictable and time-varying factors in the operations of the equipment. Next, we utilize ESC to estimate the gradients of the unknown operation-to-consumption functions. As shown in Fig. 1, the core idea is to estimate the gradients by adding dither signals to the inputs and outputs of the equipments. Then, the ESC-based primal-dual algorithm is denoted as

$$\dot{\hat{x}}_i = k_i^x\left(-\tau \frac{\mathrm{d}c_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} - \hat{\lambda}\xi_i + \hat{\mu}_i - \hat{\nu}_i\right), \qquad (13)$$

$$\dot{\hat{q}} = k^q\left(\hat{\lambda} - (1-\tau)\frac{\mathrm{d}w(\hat{q})}{\mathrm{d}\hat{q}}\right), \qquad (14)$$

$$\dot{\hat{\lambda}} = g^\lambda\left[\sum_{i=1}^{N} f_i(\hat{x}_i + a\sin(\omega t)) - \hat{q}\right]_{\hat{\lambda}}^+, \qquad (15)$$

$$\dot{\hat{\mu}}_i = g_i^\mu [x_i^{\min} - \hat{x}_i]_{\hat{\mu}_i}^+, \qquad (16)$$

$$\dot{\hat{\nu}}_i = g_i^\nu [\hat{x}_i - x_i^{\max}]_{\hat{\nu}_i}^+, \qquad (17)$$

$$\dot{\xi}_i = -\hat{\omega}_i^c\left(\xi_i - \frac{2}{a}\hat{f}_i(\hat{x}_i + a\sin(\omega t))\sin(\omega t)\right), \qquad (18)$$

where $\hat{\omega}_i^c$ is the adaptive gain, $\sin(\omega t)$ is the dither signal, a is the signal amplitude, and ξ_i is a filtered signal that represents the gradient estimation of $f_i(x_i)$. To separate the dynamics of ξ_i from the other variables, we assume that $\hat{\omega}_i^c$ is much larger than k_i^x, k^q, g^λ, g_i^μ, and g_i^ν. We denote that $\hat{\omega}_i^c = \omega_L \omega_i^c$, $k_i^x = \delta\omega_L\omega_i^x$, $k^q = \delta\omega_L\omega^q$, $g^\lambda = \delta\omega_L\omega^\lambda$, $g_i^\mu = \delta\omega_L\omega_i^\mu$, and $g_i^\nu = \delta\omega_L\omega_i^\nu$, where δ is a small scalar, ω_L is a positive, real number, and ω_i^c, ω_i^x, ω^q, ω^λ, ω_i^μ, and ω_i^ν are positive and rational numbers. Different from the traditional primal-dual algorithm, the ESC-based primal-dual algorithm only requires to measure the input and output values of $f_i(x_i)$ without regard to the mathematical formulations. Next, we will prove that the ESC-based primal-dual algorithm converges to a neighborhood of the global optimal solution of (P3).

Theorem 2: The ESC-based primal-dual algorithm (13)–(18) is SPA stable at the global optimal solution of (P3) with respect to a, δ, and ω_L if $\mathrm{d}^2 c_i(x_i)/\mathrm{d}x_i^2 \geq \eta_1$ for $i \in \mathbb{N}$ and $\mathrm{d}^2 w(q)/\mathrm{d}q^2 \geq \eta_2$, where η_1 and η_2 are positive scalars.

Proof: Let $\kappa = \omega_L t$, we obtain the ESC-based primal-dual algorithm in the new time-scale κ:

$$\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}\kappa} = \delta\omega_i^x(-\tau\frac{\mathrm{d}c_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} - \hat{\lambda}\xi_i + \hat{\mu}_i - \hat{\nu}_i), \quad (19)$$

$$\frac{\mathrm{d}\hat{q}}{\mathrm{d}\kappa} = \delta\omega^q(\hat{\lambda} - (1-\tau)\frac{\mathrm{d}w(\hat{q})}{\mathrm{d}\hat{q}}), \quad (20)$$

$$\frac{\mathrm{d}\hat{\lambda}}{\mathrm{d}\kappa} = \delta\omega^\lambda[\sum_{i=1}^{N} f_i(\hat{x}_i + a\sin(\omega t)) - \hat{q}]^+_{\hat{\lambda}}, \quad (21)$$

$$\frac{\mathrm{d}\hat{\mu}_i}{\mathrm{d}\kappa} = \delta\omega_i^\mu[x_i^{\min} - \hat{x}_i]^+_{\hat{\mu}_i}, \quad (22)$$

$$\frac{\mathrm{d}\hat{\nu}_i}{\mathrm{d}\kappa} = \delta\omega_i^\nu[\hat{x}_i - x_i^{\max}]^+_{\hat{\nu}_i}, \quad (23)$$

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}\kappa} = -\omega_i^c(\xi_i - \frac{2}{a}f_i(\hat{x}_i + a\sin(\omega t))\sin(\omega t)). \quad (24)$$

According to the averaging theory [29], the dynamic system with periodic disturbance can be approximated by its average system:

$$\frac{\mathrm{d}\hat{x}_i^A}{\mathrm{d}\kappa} = \delta\omega_i^x(-\tau\frac{\mathrm{d}c_i(\hat{x}_i^A)}{\mathrm{d}\hat{x}_i^A} - \hat{\lambda}^A\xi_i^A + \hat{\mu}_i^A - \hat{\nu}_i^A), \quad (25)$$

$$\frac{\mathrm{d}\hat{q}^A}{\mathrm{d}\kappa} = \delta\omega^q(\hat{\lambda}^A - (1-\tau)\frac{\mathrm{d}w(\hat{q}^A)}{\mathrm{d}\hat{q}^A}), \quad (26)$$

$$\frac{\mathrm{d}\hat{\lambda}^A}{\mathrm{d}\kappa} = \delta\omega^\lambda[\sum_{i=1}^{N} h_i^A - \hat{q}^A]^+_{\hat{\lambda}^A}, \quad (27)$$

$$\frac{\mathrm{d}\hat{\mu}_i^A}{\mathrm{d}\kappa} = \delta\omega_i^\mu[x_i^{\min} - \hat{x}_i^A]^+_{\hat{\mu}_i^A}, \quad (28)$$

$$\frac{\mathrm{d}\hat{\nu}_i^A}{\mathrm{d}\kappa} = \delta\omega_i^\nu[\hat{x}_i^A - x_i^{\max}]^+_{\hat{\nu}_i^A}, \quad (29)$$

$$\frac{\mathrm{d}\xi_i^A}{\mathrm{d}\kappa} = -\omega_i^c(\xi_i^A - \frac{2}{a}f_i^A), \quad (30)$$

where f_i^A and h_i^A are defined as

$$f_i^A = \frac{1}{2\pi}\int_0^{2\pi} f_i(\hat{x}_i + a\sin(\omega t))\sin(\omega t)\mathrm{d}t,$$

and

$$h_i^A = \frac{1}{2\pi}\int_0^{2\pi} f_i(\hat{x}_i + a\sin(\omega t))\mathrm{d}t.$$

Approximating $f_i(\hat{x}_i + a\sin(\omega t))$ with the Taylor series, we have

$$\begin{aligned}\frac{2}{a}f_i^A &= \frac{1}{a\pi}\int_0^{2\pi}(f_i(\hat{x}_i) + a\sin(\omega t)\frac{\mathrm{d}f_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} \\ &\quad + \sum_{n=2}^{\infty}\frac{(a\sin(\omega t))^n}{n!}\frac{\mathrm{d}^n f_i(\hat{x}_i)}{\mathrm{d}(\hat{x}_i)^n})\sin(\omega t)\mathrm{d}t \\ &= \frac{\mathrm{d}f_i(\hat{x}_i^A)}{\mathrm{d}\hat{x}_i^A} + O_i^f(a^2), \quad (31)\end{aligned}$$

and

$$\begin{aligned}h_i^A &= \frac{1}{2\pi}\int_0^{2\pi}(f_i(\hat{x}_i) + a\sin(\omega t)\frac{\mathrm{d}f_i(\hat{x}_i)}{\mathrm{d}\hat{x}_i} \\ &\quad + \sum_{n=2}^{\infty}\frac{(a\sin(\omega t))^n}{n!}\frac{\mathrm{d}^n h_i(\hat{x}_i)}{\mathrm{d}(\hat{x}_i)^n})\mathrm{d}t \\ &= f_i(\hat{x}_i^A) + O_i^h(a^2). \quad (32)\end{aligned}$$

Let $\alpha = \delta\kappa$ and substitute (31) and (32) into the average system (25)–(30), we have the dynamic system in time-scale α:

$$\frac{\mathrm{d}\hat{x}_i^A}{\mathrm{d}\alpha} = \omega_i^x(-\tau\frac{\mathrm{d}c_i(\hat{x}_i^A)}{\mathrm{d}\hat{x}_i^A} - \hat{\lambda}^A\xi_i^A + \hat{\mu}_i^A - \hat{\nu}_i^A), \quad (33)$$

$$\frac{\mathrm{d}\hat{q}^A}{\mathrm{d}\alpha} = \omega^q(\hat{\lambda}^A - (1-\tau)\frac{\mathrm{d}w(\hat{q}^A)}{\mathrm{d}\hat{q}^A}), \quad (34)$$

$$\frac{\mathrm{d}\hat{\lambda}^A}{\mathrm{d}\alpha} = \omega^\lambda[\sum_{i=1}^{N}(f_i(\hat{x}_i^A) + O_i^h(a^2)) - \hat{q}^A]^+_{\hat{\lambda}^A}, \quad (35)$$

$$\frac{\mathrm{d}\hat{\mu}_i^A}{\mathrm{d}\alpha} = \omega_i^\mu[x_i^{\min} - \hat{x}_i^A]^+_{\hat{\mu}_i^A}, \quad (36)$$

$$\frac{\mathrm{d}\hat{\nu}_i^A}{\mathrm{d}\alpha} = \omega_i^\nu[\hat{x}_i^A - x_i^{\max}]^+_{\hat{\nu}_i^A}, \quad (37)$$

$$\delta\frac{\mathrm{d}\xi_i^A}{\mathrm{d}\alpha} = -\omega_i^c(\xi_i^A - \frac{\mathrm{d}f_i(\hat{x}_i^A)}{\mathrm{d}\hat{x}_i^A} - O_i^f(a^2)). \quad (38)$$

The system (33)–(38) is the standard singular perturbation form with fast dynamics ξ_i^A when δ is small. "Freezing" the dynamics (38) at the equilibrium $\xi_i^{A*} = \mathrm{d}f_i(\hat{x}_i^A)/\mathrm{d}\hat{x}_i^A + O_i^f(a^2)$, we obtain the reduced system:

$$\begin{aligned}\frac{\mathrm{d}\hat{x}_i^r}{\mathrm{d}\alpha} &= \omega_i^x(-\tau\frac{\mathrm{d}c_i(\hat{x}_i^r)}{\mathrm{d}\hat{x}_i^r} - \hat{\lambda}^r(\frac{\mathrm{d}f_i(\hat{x}_i^r)}{\mathrm{d}\hat{x}_i^r} + O_i^f(a^2)) \\ &\quad + \hat{\mu}_i^r - \hat{\nu}_i^r), \quad (39)\end{aligned}$$

$$\frac{\mathrm{d}\hat{q}^r}{\mathrm{d}\alpha} = \omega^q(\hat{\lambda}^r - (1-\tau)\frac{\mathrm{d}w(\hat{q}^r)}{\mathrm{d}\hat{q}^r}), \quad (40)$$

$$\frac{\mathrm{d}\hat{\lambda}^r}{\mathrm{d}\alpha} = \omega^\lambda[\sum_{i=1}^{N}(f_i(\hat{x}_i^r) + O_i^h(a^2)) - \hat{q}^r]^+_{\hat{\lambda}^r}. \quad (41)$$

$$\frac{\mathrm{d}\hat{\mu}_i^r}{\mathrm{d}\alpha} = \omega_i^\mu[x_i^{\min} - \hat{x}_i^r]^+_{\hat{\mu}_i^r}, \quad (42)$$

$$\frac{\mathrm{d}\hat{\nu}_i^r}{\mathrm{d}\alpha} = \omega_i^\nu[\hat{x}_i^r - x_i^{\max}]^+_{\hat{\nu}_i^r}, \quad (43)$$

Next, we will prove the stability of the reduced system. We assume that the optimal solution of (P3) is denoted as $(x_i^*, q^*, \lambda^*, \mu_i^*, \nu_i^*)$ and define $\hat{x}^r = (\hat{x}_1^r, \ldots, \hat{x}_i^r, \ldots, \hat{x}_N^r)^\mathrm{T}$, $\hat{x}^{r*} = (\hat{x}_1^{r*}, \ldots, \hat{x}_i^{r*}, \ldots, \hat{x}_N^{r*})^\mathrm{T}$, $\hat{x}^{\min} = (\hat{x}_1^{\min}, \ldots, \hat{x}_i^{\min}, \ldots, \hat{x}_N^{\min})^\mathrm{T}$, $\hat{x}^{\max} = (\hat{x}_1^{\max}, \ldots, \hat{x}_i^{\max}, \ldots, \hat{x}_N^{\max})^\mathrm{T}$, $\hat{\mu}^r = (\hat{\mu}_1^r, \ldots, \hat{\mu}_i^r, \ldots, \hat{\mu}_N^r)^\mathrm{T}$, $\hat{\mu}^{r*} = (\hat{\mu}_1^{r*}, \ldots, \hat{\mu}_i^{r*}, \ldots, \hat{\mu}_N^{r*})^\mathrm{T}$, $\hat{\nu}^r = (\hat{\nu}_1^r, \ldots, \hat{\nu}_i^r, \ldots, \hat{\nu}_N^r)^\mathrm{T}$, $\hat{\nu}^{r*} = (\hat{\nu}_1^{r*}, \ldots, \hat{\nu}_i^{r*}, \ldots, \hat{\nu}_N^{r*})^\mathrm{T}$, $\tilde{x}^r = \hat{x}^r - \hat{x}^{r*}$, $\tilde{q}^r = \hat{q}^r - \hat{q}^{r*}$, $\tilde{\mu}^r = \hat{\mu}^r - \hat{\mu}^{r*}$, $\tilde{\nu}^r = \hat{\nu}^r - \hat{\nu}^{r*}$, and $\tilde{\lambda}^r = \hat{\lambda}^r - \hat{\lambda}^{r*}$, where "–" denotes element-wise operation. We choose the following candidate Lyapunov function:

$$\begin{aligned}V &= V_1 + V_2 + V_3 + V_4 + V_5 \\ &= \frac{1}{2}\tilde{x}^{r\mathrm{T}}\Phi^{-1}\tilde{x}^r + \frac{1}{2}\tilde{\mu}^{r\mathrm{T}}\Psi^{-1}\tilde{\mu}^r + \frac{1}{2}\tilde{\nu}^{r\mathrm{T}}\Theta^{-1}\tilde{\nu}^r \\ &\quad + \frac{1}{2\omega^q}(\tilde{q}^r)^2 + \frac{1}{2\omega^\lambda}(\tilde{\lambda}^r)^2,\end{aligned}$$

where $\Phi = \mathrm{diag}\{\omega_i^x\}$, $\Psi = \mathrm{diag}\{\omega_i^\mu\}$, and $\Theta = \mathrm{diag}\{\omega_i^\nu\}$ are diagonal matrices. Defining $O^f = (O_1^f, \ldots, O_i^f, \ldots, O_N^f)^\mathrm{T}$, $O^h = (O_1^h, \ldots, O_i^h, \ldots, O_N^h)^\mathrm{T}$, and $R_N = (1, \ldots, 1)$ with $|R_N| = N$. Then, the derivative of the Lyapunov function along the reduced system (39)–(43) is denoted as

$$\begin{aligned}\dot V \;=\;& \tilde x^{r\mathrm T}(-\tau\hat c'(\hat x^r)-\hat\lambda^r(f'(\hat x^r)+O^f(a^2))+\hat\mu^r-\hat\nu^r)\\ &+\tilde\mu^{r\mathrm T}[x^{\min}-\hat x^r]^+_{\hat\mu^r}+\tilde\nu^{r\mathrm T}[\hat x^r-x^{\max}]^+_{\hat\nu^r}+\tilde q^r(\hat\lambda^r-\\ &(1-\tau)w'(\hat q^r))+\tilde\lambda^r[R_Nf(\hat x^r)+R_NO^h(a^2)-\hat q^r]^+_{\hat\lambda^r},\end{aligned}$$

where $c'(\hat x^r)=(\mathrm dc_1(\hat x^r_1)/\mathrm d\hat x^r_1,\ldots,\mathrm dc_i(\hat x^r_i)/\mathrm d\hat x^r_i,\ldots,\mathrm dc_N(\hat x^r_N)/\mathrm d\hat x^r_N)^{\mathrm T}$, $f'(\hat x^r)=(\mathrm df_1(\hat x^r_1)/\mathrm d\hat x^r_1,\ldots,\mathrm df_i(\hat x^r_i)/\mathrm d\hat x^r_i,\ldots,\mathrm df_N(\hat x^r_N)/\mathrm d\hat x^r_N)^{\mathrm T}$, $f(\hat x^r)=(f_1(\hat x^r_1),\ldots,f_i(\hat x^r_i),\ldots,f_N(\hat x^r_N))^{\mathrm T}$, and $w'(\hat q^r)=\mathrm dw(\hat q^r)/\mathrm d\hat q^r$. Combining with $\tilde\mu^{r\mathrm T}[x^{\min}-\hat x^r]^+_{\hat\mu^r}\le\tilde\mu^{r\mathrm T}(x^{\min}-\hat x^r)$, $\tilde\nu^{r\mathrm T}[\hat x^r-x^{\max}]^+_{\hat\nu^r}\le\tilde\nu^{r\mathrm T}(\hat x^r-x^{\max})$, and $\tilde\lambda^r[R_Nf(\hat x^r)+R_NO^h(a^2)-\hat q^r]^+_{\hat\lambda^r}\le\tilde\lambda^r(R_Nf(\hat x^r)+R_NO^h(a^2)-\hat q^r)$, we obtain

$$\begin{aligned}\dot V \;\le\;& \tilde x^{r\mathrm T}(-\tau\hat c'(\hat x^r)-\hat\lambda^r(f'(\hat x^r)+O^f(a^2))+\hat\mu^r-\hat\nu^r)\\ &+\tilde\mu^{r\mathrm T}(x^{\min}-\hat x^r)+\tilde\nu^{r\mathrm T}(\hat x^r-x^{\max})+\tilde q^r(\hat\lambda^r-(1-\tau)w'(\hat q^r))\\ &+\tilde\lambda^r(R_Nf(\hat x^r)+R_NO^h(a^2)-\hat q^r).\end{aligned}$$

At the global optimal solution of (P3), we have $\hat q^{r*}=R_Nf(\hat x^{r*})$, $\tau R_Nc'(\hat x^{r*})=-\hat\lambda^{r*}R_Nf'(\hat x^{r*})+R_N\hat\mu^{r*}-R_N\hat\nu^{r*}$, and $(1-\tau)w'(\hat q^{r*})=\hat\lambda^{r*}$. Then, the derivative of the Lyapunov function can be further bounded by

$$\begin{aligned}\dot V \;\le\;& -\tau\tilde x^{r\mathrm T}c'(\hat x^r)+\tau\tilde x^{r\mathrm T}c'(\hat x^{r*})-\tilde x^{r\mathrm T}\hat\lambda^r(f'(\hat x^r)+O^f(a^2))\\ &+\tilde x^{r\mathrm T}(\hat\mu^r-\hat\nu^r)-\hat x^{r\mathrm T}(\tilde\mu^r-\tilde\nu^r)+(x^{\min})^{\mathrm T}\tilde\mu^r\\ &-(x^{\max})^{\mathrm T}\tilde\nu^r-\tilde x^{r\mathrm T}(\hat\mu^{r*}-\hat\nu^{r*})+\tilde q^r\hat\lambda^r-(1-\tau)\tilde q^rw'(\hat q^r)\\ &+(1-\tau)\tilde q^rw'(\hat q^{r*})-\tilde q^r\hat\lambda^{r*}+\hat\lambda^{r*}\tilde x^{r\mathrm T}f'(\hat x^{r*})\\ &-\hat\lambda^{r*}(R_Nf(\hat x^r)+R_NO^h(a^2)-\hat q^r)\\ &+\hat\lambda^r(R_Nf(\hat x^r)+R_NO^h(a^2)-\hat q^r)\\ =\;& -\tau\tilde x^{r\mathrm T}c'(\hat x^r)+\tau\tilde x^{r\mathrm T}c'(\hat x^{r*})\\ &-(1-\tau)\tilde q^rw'(\hat q^r)+(1-\tau)\tilde q^rw'(\hat q^{r*})\\ &-\tilde x^{r\mathrm T}\hat\lambda^r(f'(\hat x^r)+O^f(a^2))+\hat\lambda^{r*}\tilde x^{r\mathrm T}f'(\hat x^{r*})\\ &+\tilde\lambda^r(R_Nf(\hat x^r)-R_Nf(\hat x^{r*}))+\tilde\lambda^rR_NO^h(a^2)\\ &-\hat x^{r*\mathrm T}\tilde\mu^r+\hat x^{r*\mathrm T}\tilde\nu^r+(x^{\min})^{\mathrm T}\tilde\mu^r-(x^{\max})^{\mathrm T}\tilde\nu^r.\end{aligned}\tag{44}$$

Using the Mean Value Theorem [33], we have

$$R_Nf(\hat x^r)-R_Nf(\hat x^{r*})=\tilde x^{r\mathrm T}f'(\hat x^{m1}),\tag{45}$$

where $\hat x^{m1}=(\hat x^{m1}_1,\ldots,\hat x^{m1}_i,\ldots,\hat x^{m1}_N)^{\mathrm T}$ such that $\hat x^{m1}_i\in[\hat x^r_i,\hat x^{r*}_i]$ or $\hat x^{m1}_i\in[\hat x^{r*}_i,\hat x^r_i]$. Substituting (45) into (44), we obtain

$$\begin{aligned}\dot V \;\le\;& -\tau\tilde x^{r\mathrm T}c'(\hat x^r)+\tau\tilde x^{r\mathrm T}c'(\hat x^{r*})\\ &-(1-\tau)\tilde q^rw'(\hat q^r)+(1-\tau)\tilde q^rw'(\hat q^{r*})\\ &-\hat x^{r*\mathrm T}\tilde\mu^r+\hat x^{r*\mathrm T}\tilde\nu^r+(x^{\min})^{\mathrm T}\tilde\mu^r-(x^{\max})^{\mathrm T}\tilde\nu^r\\ &+\hat\lambda^r\tilde x^{r\mathrm T}(f'(\hat x^{m1})-f'(\hat x^r))+\hat\lambda^{r*}\tilde x^{r\mathrm T}(f'(\hat x^{r*})-f'(\hat x^{m1}))\\ &+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2).\end{aligned}\tag{46}$$

Following the Mean Value Theorem, we obtain

$$R_Nf'(\hat x^{r*})-R_Nf'(\hat x^{m1})=(\hat x^{r*}-\hat x^{m1})A_Nf''(\hat x^{m2}),\tag{47}$$

and

$$R_Nf'(\hat x^{m1})-R_Nf'(\hat x^r)=(\hat x^{m1}-\hat x^r)A_Nf''(\hat x^{m3}),\tag{48}$$

where $A_N=\mathrm{diag}\{1,\ldots,1\}$ with $|A_N|=N$, $\hat x^{m2}=(\hat x^{m2}_1,\ldots,\hat x^{m2}_i,\ldots,\hat x^{m2}_N)^{\mathrm T}$ such that $\hat x^{m2}_i\in[\hat x^{m1}_i,\hat x^{r*}_i]$ or $\hat x^{m2}_i\in[\hat x^{r*}_i,\hat x^{m1}_i]$, $\hat x^{m3}=(\hat x^{m3}_1,\ldots,\hat x^{m3}_i,\ldots,\hat x^{m3}_N)^{\mathrm T}$ such that $\hat x^{m3}_i\in[\hat x^{m1}_i,\hat x^r_i]$ or $\hat x^{m3}_i\in[\hat x^r_i,\hat x^{m1}_i]$, and $f''(\hat x^r)=(\mathrm df'_1(\hat x^r_1)/\mathrm d\hat x^r_1,\ldots,\mathrm df'_i(\hat x^r_i)/\mathrm d\hat x^r_i,\ldots,\mathrm df'_N(\hat x^r_N)/\mathrm d\hat x^r_N)^{\mathrm T}$. Substituting (47) and (48) into (46), we have

$$\begin{aligned}\dot V \;\le\;& -\tau\tilde x^{r\mathrm T}c'(\hat x^r)+\tau\tilde x^{r\mathrm T}c'(\hat x^{r*})\\ &-(1-\tau)\tilde q^rw'(\hat q^r)+(1-\tau)\tilde q^rw'(q^{\hat r*})\\ &-\hat x^{r*\mathrm T}\tilde\mu^r+\hat x^{r*\mathrm T}\tilde\nu^r+(x^{\min})^{\mathrm T}\tilde\mu^r-(x^{\max})^{\mathrm T}\tilde\nu^r\\ &+\hat\lambda^{r*}(\hat x^r-\hat x^{r*})^{\mathrm T}(\hat x^{r*}-\hat x^{m1})R_Nf''(\hat x^{m2})\\ &+\hat\lambda^r(\hat x^r-\hat x^{r*})^{\mathrm T}(\hat x^{m1}-\hat x^r)R_Nf''(\hat x^{m3})\\ &+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2).\end{aligned}\tag{49}$$

Combining with the convexity of $f_i(x_i)$ and the positivity of $\hat\lambda^r$ and $\hat\lambda^{r*}$, we obtain $\hat\lambda^{r*}(\hat x^r-\hat x^{r*})^{\mathrm T}(\hat x^{r*}-\hat x^{m1})R_Nf''(\hat x^{m2})\le0$ and $\hat\lambda^r(\hat x^r-\hat x^{r*})^{\mathrm T}(\hat x^{m1}-\hat x^r)R_Nf''(\hat x^{m3})\le0$. Then, the derivative of Lyapunov function can be further bounded by

$$\begin{aligned}\dot V \;\le\;& -\tau\tilde x^{r\mathrm T}c'(\hat x^r)+\tau\tilde x^{r\mathrm T}c'(\hat x^{r*})\\ &-(1-\tau)\tilde q^rw'(\hat q^r)+(1-\tau)\tilde q^rw'(q^{\hat r*})\\ &-\hat x^{r*\mathrm T}\tilde\mu^r+\hat x^{r*\mathrm T}\tilde\nu^r+(x^{\min})^{\mathrm T}\tilde\mu^r-(x^{\max})^{\mathrm T}\tilde\nu^r\\ &+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2).\end{aligned}\tag{50}$$

Using the Mean Value Theorem, we have

$$R_Nc'(\hat x^r)-R_Nc'(\hat x^{r*})=c''(\hat x^{m4})^{\mathrm T}\tilde x^r,\tag{51}$$

and

$$w'(\hat q^r)-w'(\hat q^{r*})=w''(\hat q^m)\tilde q^r,\tag{52}$$

where $\hat x^{m4}=(\hat x^{m4}_1,\ldots,\hat x^{m4}_i,\ldots,\hat x^{m4}_N)^{\mathrm T}$ such that $\hat x^{m4}_i\in[\hat x^r_i,\hat x^{r*}_i]$ or $\hat x^{m4}_i\in[\hat x^{r*}_i,\hat x^r_i]$, $\hat q^m\in[\hat q^r,\hat q^{r*}]$ or $\hat q^m\in[\hat q^{r*},\hat q^r]$, $c''(\hat x^r)=(\mathrm dc'_1(\hat x^r_1)/\mathrm d\hat x^r_1,\ldots,\mathrm dc'_i(\hat x^r_i)/\mathrm d\hat x^r_i,\ldots,\mathrm dc'_N(\hat x^r_N)/\mathrm d\hat x^r_N)^{\mathrm T}$, and $w''(\hat q^r)=\mathrm dw'(\hat q^r)/\mathrm d\hat q^r$. Substituting (51) and (52) into (50) and combining with the convexity of $c_i(\hat x^r_i)$ and $w(\hat q^r)$, we obtain

$$\begin{aligned}\dot V \;\le\;& -\tau\eta_1\|\tilde x^r\|^2-(1-\tau)\eta_2(\tilde q^r)^2+(x^{\min}-\hat x^{r*})^{\mathrm T}\tilde\mu^r\\ &+(\hat x^{r*}-x^{\max})^{\mathrm T}\tilde\nu^r+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2),\\ \le\;& -2\tau\eta_1\omega^x_{\min}V_1-2(1-\tau)\eta_2\omega^qV_4+(x^{\min}-\hat x^{r*})^{\mathrm T}\tilde\mu^r\\ &+(\hat x^{r*}-x^{\max})^{\mathrm T}\tilde\nu^r+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2),\end{aligned}$$

where $\omega^x_{\min}=\min\{\omega^x_1,\ldots,\omega^x_N\}$, η_1 and η_2 are the lower bounds of $c''_i(\hat x^r_i)$ and $w''(\hat q^r)$. There exists a positive scalar η^* such that

$$\eta^*V=2\tau\eta_1\omega^x_{\min}V_1+2(1-\tau)\eta_2\omega^qV_4.$$

When $\eta\in[0,\eta^*]$, we have

$$\begin{aligned}\dot V \;\le\;& -\eta V+(x^{\min}-\hat x^{r*})^{\mathrm T}\tilde\mu^r+(\hat x^{r*}-x^{\max})^{\mathrm T}\tilde\nu^r\\ &+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2).\end{aligned}$$

For sufficiently small a, there exists θ such that

$$\begin{aligned}\theta\sqrt V \;\ge\;& (x^{\min}-\hat x^{r*})^{\mathrm T}\tilde\mu^r+(\hat x^{r*}-x^{\max})^{\mathrm T}\tilde\nu^r\\ &+\tilde\lambda^rR_NO^h(a^2)-\tilde x^{r\mathrm T}\hat\lambda^rO^f(a^2),\end{aligned}$$

and

$$\dot V\le-\eta V+\theta\sqrt V.$$

Following Lemma 1, we have

$$\|V\| \leq e^{-\frac{\eta}{2}\alpha}\|W(0)\| + \frac{2}{\eta}\theta.$$

Let $\tilde{z}^r = (\tilde{x}_1^r, \ldots, \tilde{x}_N^r, \tilde{\mu}_1^r, \ldots, \tilde{\mu}_N^r, \tilde{\nu}_1^r, \ldots, \tilde{\nu}_N^r, \tilde{q}^r, \tilde{\lambda}^r)^{\mathrm{T}}$, we obtain

$$\begin{aligned} \|\tilde{z}^r\| &\leq \sqrt{2\omega_{\max}}\|W\| \\ &\leq \sqrt{2\omega_{\max}}(e^{-\frac{\eta}{2}\alpha}\|W(0)\| + \frac{2}{\eta}\theta), \end{aligned}$$

where $\omega_{\max} = \max\{\omega_1^x, \ldots, \omega_N^x, \omega_1^\mu, \ldots, \omega_N^\mu, \omega_1^\nu, \ldots, \omega_N^\nu, \omega^q, \omega^\lambda\}$. Thus, the reduced system (39)–(43) is SPA stable with respect to a.

Defining the boundary system as $e_i^x = \xi_i^A - \frac{2}{a}f_i^A$ for $i = 1, \ldots, N$. According to (38), the boundary system is globally asymptotically stable. Combining with the SPA stability of the reduced system and Lemma 2 in [34], the average system (25)–(30) is SPA stable with respect to a and δ in the κ-time scale. Following the Lemma 1 in [34], the original system (13)–(18) is SPA stable with respect to a, δ, and ω_L. ■

IV. Applications to Demand Response with Distributed HVAC Systems

In this section, we apply the results to the demand response program with inverter-based HVAC systems. The discomfort costs to consumers can be denoted by the following Taguchi loss function:

$$c_i(T_i) = \gamma_i(T_i - T_i^N)^2,$$

where γ_i is a constant coefficient, T_i and T_i^N are the actual and normal temperature settings, respectively. Each consumer has a temperature requirement denoted by $T_i^{\min} \leq T_i \leq T_i^{\max}$, where $T_i^{\min}$ and $T_i^{\max}$ are the minimal and maximal temperature requirements of consumer i. In the demand response program, we need to balance the total power consumption and the supply in the sense of

$$\sum_{i=1}^{N} l_i = q,$$

where l_i is the power consumption of HVAC i, which is determined by the actual temperature setting T_i. We assume that the relationship between the actual temperature setting and the power consumption can be defined by a convex function $l_i = f_i(T_i)$. Then, the balancing constraints can be denoted as

$$\sum_{i=1}^{N} f_i(T_i) = q.$$

According to [35], the cost to the utility company is denoted as

$$w(q) = \rho_1 q^2 + \rho_2 q + \rho_3,$$

where ρ_1, ρ_2, and ρ_3 are positive cost coefficients.

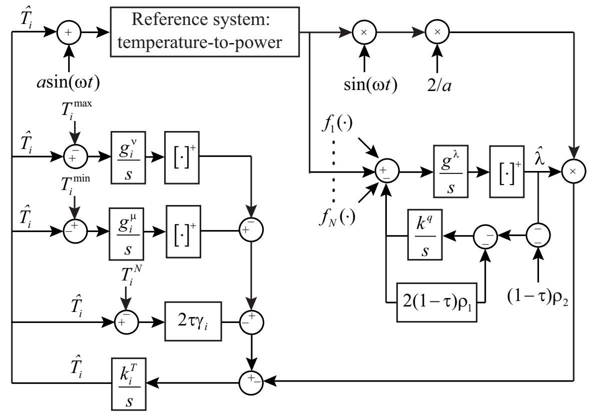

Fig. 2. ESC-based primal-dual algorithm for demand response with distributed HVAC systems.

Then, the demand response program with distributed HVAC systems can be formulated as the following optimization problem:

$$\begin{aligned} \text{(P6)} \quad \text{maximize} \quad & -\tau\sum_{i=1}^{N} c_i(T_i) - (1-\tau)w(q) \\ \text{subject to} \quad & \sum_{i=1}^{N} f_i(T_i) = q \\ & T_i^{\min} \leq T_i \leq T_i^{\max}, \end{aligned}$$

where τ is a parameter to achieve the desirable tradeoff between the costs to the consumers and the utility company. Following Theorem 1, (P6) has a unique optimal solution.

In general, the relationship between l_i and T_i is complex because of the non-linear relationship among the characteristic parameters, such as the power consumption, the compressor frequency, the mass flow rate of refrigerant, the cooling/heating capacity, and the temperature settings. The analytical models for the power consumption at different temperature settings are hardly to obtain, i.e., $f_i(T_i)$ is unknown. The traditional gradient-based methods can not be used for generating the optimal response strategies. Recently, some calculation methods have been proposed to obtain the power consumption of the HVAC at different temperature settings [36], [37]. The calculation methods provide a reference system for the unmodeled HVAC system and make it feasible to measure the temperature settings and the power consumption. Thus, we can use the following ESC-based algorithm to generate the optimal response strategies:

$$\begin{aligned} \dot{\hat{T}}_i &= k_i^T(-2\tau\gamma_i(\hat{T}_i - T_i^N) - \hat{\lambda}\xi_i + \hat{\mu}_i - \hat{\nu}_i), && (53) \\ \dot{\hat{q}} &= k^q(\hat{\lambda} - 2(1-\tau)\rho_1\hat{q} - (1-\tau)\rho_2), && (54) \\ \dot{\hat{\lambda}} &= g^\lambda[\sum_{i=1}^{N} f_i(\hat{T}_i) - \hat{q}]_{\hat{\lambda}}^+. && (55) \\ \dot{\hat{\mu}}_i &= g_i^\mu[T_i^{\min} - \hat{T}_i]_{\hat{\mu}_i}^+, && (56) \\ \dot{\hat{\nu}}_i &= g_i^\nu[\hat{T}_i - T_i^{\max}]_{\hat{\nu}_i}^+, && (57) \\ \dot{\xi}_i &= -\hat{\omega}_i^c(\xi_i - \frac{2}{a}\hat{f}_i(\hat{T}_i + a\sin(\omega t))\sin(\omega t)), && (58) \end{aligned}$$

Following Theorem 2, the ESC-based algorithm (53)–(58) is SPA stable with respect to a, δ, and ω_L. We give the schematic diagram of the ESC-based algorithm for demand response in Fig. 2, where the ESC is used for estimating the gradient of the temperature-to-consumption function.

V. Numerical Results

We consider a retail electricity market consisting of a single utility company and 10 consumers with HVACs. Without loss of generality, we assume that the HVAC is a cooler. The normal temperature settings for all the consumers are assumed to be 24°C, and the cost functions of the consumers are denoted as $c_i(T_i) = 10(T_i - 24)^2$. The temperature-to-consumption functions are assumed to be $l_i = \varphi_i(T_i - 30)^2$, where φ_i varies with different consumers and the outdoor temperature is 30°C. The cost function of the utility company is assumed to be $w(q) = 0.1q^2 + 0.5q + 2$. The maximal and minimal temperature settings are assumed to be 26°C and 24°C, respectively. The adaptive gains of the primal-dual algorithm are defined as $k_i^x = 0.01$, $k^q = 0.05$, $g^\lambda = 0.1$, $g_i^\mu = 0.1$, and $g_i^\nu = 0.1$. We set the parameters of the ESC as $a = 0.1$, $\hat{\omega}_i^c = 2$, and $\omega = 20$. To evaluate the balance between supply and demand, we define the matching errors:

$$\mathrm{E} = \frac{\sum_{i=1}^N \hat{x}_i - q}{\sum_{i=1}^N \hat{x}_i} \times 100\%.$$

The SPA stability is demonstrated in figures 3–5. It is shown that the ESC-based primal-dual algorithm converges to a neighborhood of the optimal temperature settings, the optimal electricity supply, and the optimal price, respectively. The matching errors versus the iterations of the ESC-based primal-dual algorithm are shown in Fig. 6. It is shown that the matching errors converge to a neighborhood of 0 and the fluctuations are bounded within [-6%, +6%].

The costs to the consumers and the utility company versus the tradeoff parameter are shown in Fig. 7. The costs to the consumers decrease with τ, and the cost to the utility company increases with τ. Therefore, different tradeoffs can be achieved between the costs to the consumers and the utility company by tuning the tradeoff parameter. As shown in figures 8–10, we can obtain different temperature settings, electricity supplies, and retail prices by changing τ. It means that various market equilibriums can be achieved by tuning the tradeoff parameter.

VI. Conclusions

In this study, we utilize ESC to study the resource allocation problem with unknown functions in the constraints. A distributed ESC-based primal-dual algorithm is developed to generate the optimal resource allocation strategies. It is shown that the ESC-based algorithm can converge to a small neighborhood of the optimal solution. The algorithm is further applied to the demand response with distributed HVACs. It is also shown that the balance between supply and demand is achieved and various market equilibriums can be obtained by tuning the value of the tradeoff parameter.

As shown in the numerical results, the dither signals will cause fluctuations to the system. Therefore, it is necessary to study the impact of the frequencies of the dither signals on the fluctuations. Furthermore, the ESC-based primal-dual algorithm will be executed in a discrete form in practice. It is meaningful to find the critical sampling rate to stabilize the discrete-time primal-dual algorithm. For the applications to demand response, the ESC-based primal-dual algorithm is applied to the reference system of the HVAC. The objective is to obtain the optimal power consumption and price in an offline mode, which can be used by the consumers and the utility company to set the electricity usage and pricing plans, respectively. The actual electricity usage may not coincide with the plan. In that case, the direct load control method can be used as a supplement for load balancing. Another interesting issue is to consider more complicated market models with the integration of renewable power, which gives a stochastic optimization problem.

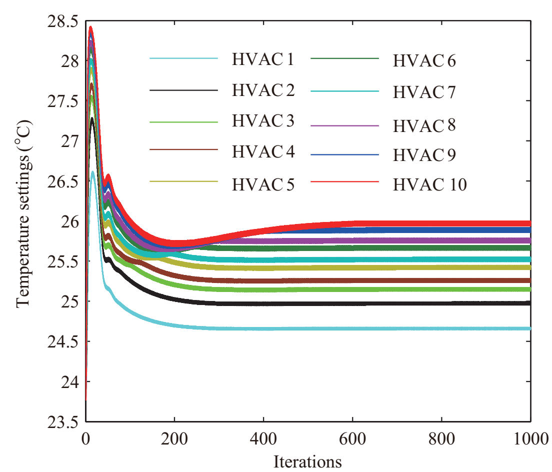

Fig. 3. Convergence of the temperature settings.

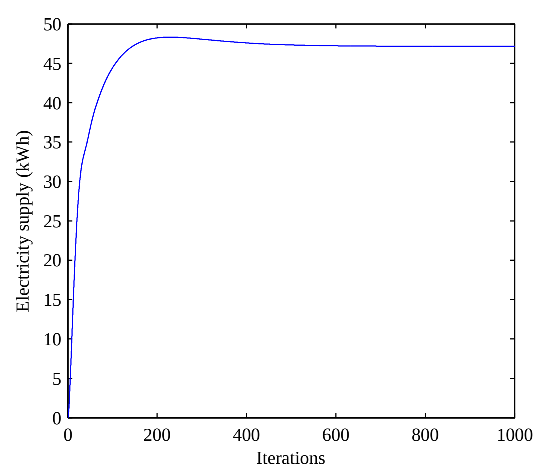

Fig. 4. Convergence of the electricity supply.

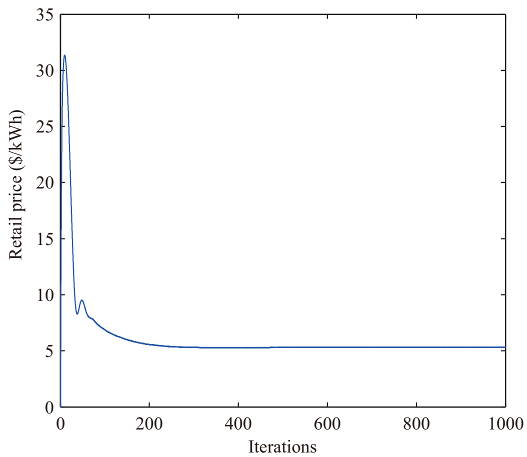

Fig. 5. Convergence of the retail price.

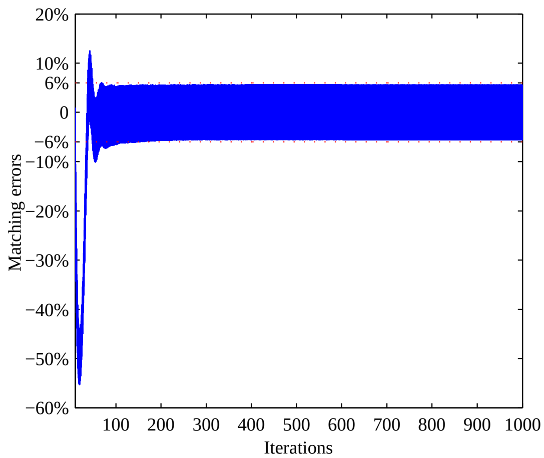

Fig. 6. Matching errors versus the iterations of ESC-based primal-dual algorithm.

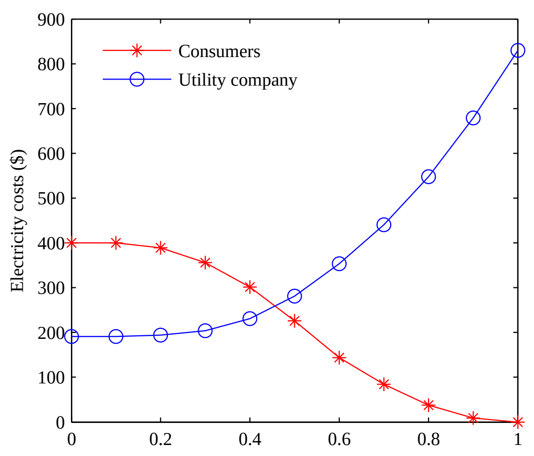

Fig. 7. Electricity costs to the utility company and consumers versus τ.

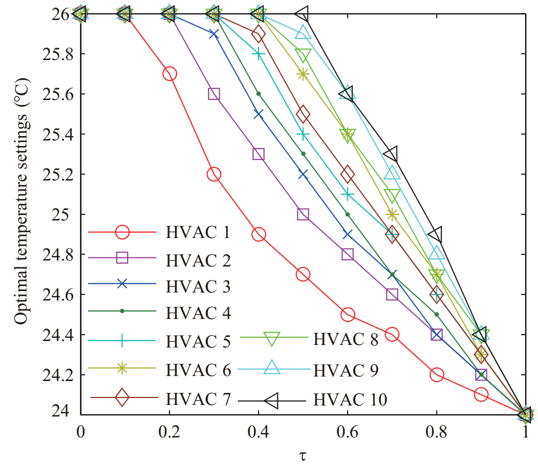

Fig. 8. The optimal temperature settings versus τ.

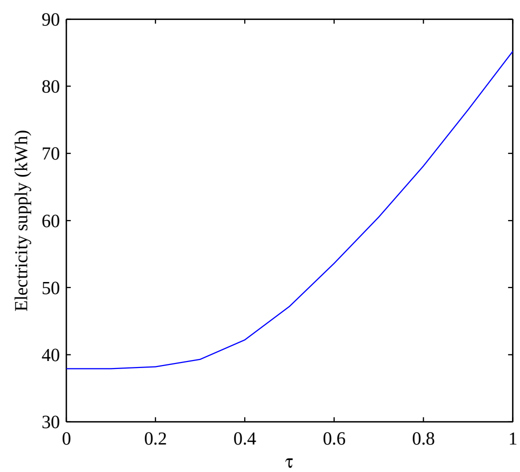

Fig. 9. The optimal electricity supply versus τ.

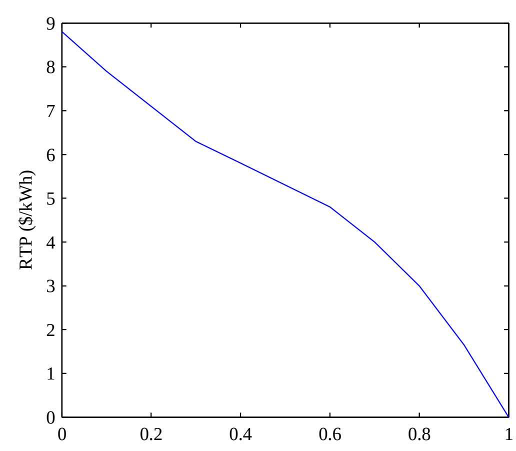

Fig. 10. The optimal retail price versus τ.

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