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ABSTRACT

An investigation of the passage of very-high-energy (up to 200 BeV) nucleons through matter is carried out with phenomenological nucleon-nucleon cross sections recently obtained by George Trilling. A model for nucleon-nucleus interactions is assumed in which the nucleus is represented as a constant-density sphere of nucleons. The intra-nuclear and internuclear cascades are then treated analytically to obtain the secondary nucleon spectrum as a function of secondary energy and depth in the matter. In addition, another phenomenological formula developed by Trilling for pion production is used to obtain the secondary pion spectrum with the same nuclear model. The effect of pion production by pions has not been included. Numerical results for a variety of nuclei are given as examples.

I. INTRODUCTION

In the past, a number of calculations for intranuclear cascades have been carried out with existing experimental data used for the fundamental nucleon interactions. These calculations have principally been done by the Monte Carlo technique, because of the complexity of dealing with the complicated energy and angular dependence of the various cross sections involved.¹ Other types of calculations have made necessary extensive simplifying assumptions.²

The present investigation deals with cascades associated with nucleons and pions at very high energies; i.e., from 5 to 200 BeV. In this "asymptotic" energy region, a number of simplifications occur. In the first place, the total cross sections for nucleon-nucleon and pion-nucleon interactions are essentially constant. Recently Trilling³ has made an empirical fit to the experimental data for nucleon-nucleon and pion-nucleon scattering around 20 BeV, and has achieved a reasonable fit to the nucleon-nucleon data with a functional form which becomes quite simple at very high energies. Extrapolation to energies greater than can be obtained in the laboratory was achieved by use of empirical evidence from cosmic ray experiments. A further simplification at these energies is that such considerations as the nuclear potentials and the exclusion principle, which are significant at lower energies, may be ignored. Further, at very high energies secondary particles are predominantly forward, so that the angular spread of the secondaries can be neglected. In the analysis to be described the nucleus is treated as a sphere of nuclear matter having a constant density, although

this simplification is not at all basic, and a more complicated density distribution could be used.

In Section IIA, an analytic treatment of the energy spectrum of nucleons in matter on the basis of the high-energy limit of the Trilling formula is carried out for cascades in hydrogen. The model is extended to other materials in Section IIB. These results should be valid for energies above about 10 BeV. At lower energies, the Trilling model is somewhat complicated, and the analytic treatment must be supplemented by a perturbation calculation and a numerical integration. This calculation is outlined in Section IIC. Finally, pion spectra are obtained by use of another formula due to Trilling for pion production by nucleons, in Section III. In the concluding section, representative numerical results are presented for nucleon and pion spectra at depths up to 10 interaction mean free paths in various materials with a variety of incident energies.

II. NUCLEON ENERGY SPECTRA IN MATTER

In the calculations here detailed we ignore the difference between protons and neutrons, and simply treat them together as nucleons.

It is expected that charged pion production is such that after a number of collisions the charge distribution of the secondaries is determined only by the distribution of protons and neutrons in the medium.⁴ The distribution of nucleons can then be obtained from the one-dimensional cascade equation

$$\frac{d\rho_N(E,L)}{dL} = -\frac{1}{\lambda_N} \rho_N(E,L) + \frac{1}{\lambda_N} \int_E^{E_0} \rho_N(E',L) R_N(E,E') dE', \quad (1)$$

where $\rho_N(E,L)dE$ is the number of nucleons between E and $E + dE$ at the depth L in the matter. The first term on the right represents the removal of particles from the spectrum at energy E by collisions, and the second, the addition to the spectrum at E by collisions of particles with higher energy E' . The function $R_N(E,E')$ is the probability distribution for secondary nucleons of energy E , as generated by collisions of primary nucleons of energy E' . The mean free path for nucleon-nucleon collisions is designated by λ_N . The distribution is normalized so that

$$\int_0^{E'} R_N(E,E') dE = 2,$$

since two secondary nucleons generally emerge from a high-energy collision. The formula of Trilling for $R_N(E,E')$ characterizes only the inelastic part of the secondary distribution and could be supplemented by the elastic scattering, but at very high energies the elastic scattering is very nearly in the forward direction and the corresponding $R_N(E,E')$ becomes almost proportional to $\delta(E - E')$. A δ function in R_N will not contribute to changes in the distribution and may be removed. Thus we choose the interaction mean free path to include only the inelastic cross section

$$\lambda_N = [n \sigma_{inel}^N]^{-1},$$

where n is the nuclear density in the matter considered, and we neglect elastic terms in Eq. (1). At high energies σ_{inel}^N is well

approximated by a constant. As will be seen, this feature leads to separation between the dependence on energy and on depth of the secondary spectra for a given number of collisions. As was mentioned in the introduction, the angular distribution of secondaries at very high energies is generally quite narrow: The average transverse momentum for the Trilling formula is about 500 MeV/c, so that for 10-BeV/c secondary nucleons the angular spread would be of the order of 0.05 radian. Thus in the present model we assume that the cascades associated with a given primary develop collinearly with that particle, both in nuclear and in ordinary matter. In equations such as Eq. (1) we will therefore use cross sections and production probabilities which have been integrated over the angular distribution.

A. Cascades Within Nuclei

For the passage of nucleons through hydrogen, Eq. (1) can be used directly, with $R_N(E,E')$ corresponding to the nucleon-nucleon cross sections. For other kinds of matter, however, we first obtain the secondary distribution from nucleon-nucleus collisions, using a model in which intranuclear cascades are produced by collisions with independent nucleons in the nucleus. At lower energies previous Monte Carlo calculations indicate that this model represents the data quite well, and there seems to be no evidence up to 9 BeV and beyond that cooperative multiparticle effects are important.⁵ The determination of $R(E,E')$ for a nucleus is made by use of Eq. (1), but with λ_N calculated with the nucleonic density in nuclear matter, and with an average obtained over all impact parameters for the nucleus.

Since the mean free path is assumed to be constant, we can readily obtain the solution of the differential equation in L as

$$\rho(E, \ell) = \rho(E, 0)e^{-\ell} + \int_0^{\ell} d\ell' e^{-(\ell-\ell')} \int_E^{E_0} dE' \rho(E', \ell') R(E, E') \quad (2)$$

where ℓ is the depth measured in mean free paths, $\ell \equiv L/\lambda_N$. For the incident beam ($\ell = 0$) we choose $\rho(E, 0) = \delta(E - E_0)$. Equation (2) can be solved by iteration to obtain

$$\rho_N(E, \ell) = e^{-\ell} \left\{ \delta(E - E_0) + \sum_{m=1}^{\infty} \frac{\ell^m}{m!} F_m(E, E_0) \right\} \quad (3)$$

where $F_m(E, E_0)$ is the energy distribution of nucleons after m collisions:

$$\begin{aligned} F_m(E, E_0) &\equiv \int_E^{E_0} dE_1 \int_E^{E_1} dE_2 \cdots \int_E^{E_{m-2}} dE_{m-1} R_N(E, E_{m-1}) \cdots R_N(E_2, E_1) R_N(E_1, E_0) \\ &= \int_E^{E_0} dE_{m-1} \int_{E_{m-1}}^{E_0} dE_{m-2} \cdots \int_{E_2}^{E_0} dE_1 R_N(E, E_{m-1}) \cdots R_N(E_2, E_1) R_N(E_1, E_0) \end{aligned} \quad (4)$$

Hence the energy and depth dependence of the distribution in $\rho_N(E, \ell)$ are factorizable for a given number of nucleonic interactions.

To continue, we must now introduce the formula for the differential inelastic cross section which Trilling has obtained. In the laboratory system his result is

$$\frac{d^2\sigma}{dp d\Omega} = \frac{p^2}{\gamma(\beta + \beta_\ell^*)} (a + bp_\ell^*) e^{-3.0p_t^{*2}}$$

where γ, β are the usual quantities needed for the Lorentz transformation from laboratory to center-of-mass variables; and β_ℓ^* is the longitudinal velocity, p_ℓ^* the longitudinal momentum, and p_t^* the transverse momentum of the secondary nucleon, all in the center-of-mass system. The momentum in the laboratory system is p , and the corresponding element of solid angle is $d\Omega$. Experimental data at 18.8 and 23.1 BeV/c were used to obtain the numerical coefficients in the formula, and the best fit was found for $a = 6.0 \text{ mb}/(\text{BeV}/c)^3$ and $b = 1.57 \text{ mb}/(\text{BeV}/c)^4$.⁶ To maintain a constant total cross section, he assumed that for other energies

$$a(p_0) = a(18.8 \text{ BeV}/c) \frac{p_{\max}^*(18.8)}{p_{\max}^*(p_0)}$$

and

$$b(p_0) = b(18.8 \text{ BeV}/c) \left[\frac{p_{\max}^*(18.8)}{p_{\max}^*(p_0)} \right]^2$$

where p_0 is the incident momentum and p_{\max}^* is the maximum secondary momentum in the center-of-mass-system. If one integrates over all solid angles, one finds approximately⁷

$$\frac{d\sigma}{dp} \approx \frac{\pi(a + bp^*)}{3\gamma(\beta + \beta_l^*)}$$

In the appendix it is shown that for very high energies,

$$\left(\frac{d\sigma}{dE}\right) \approx \frac{1}{E_0} \left(A + \frac{BE}{E_0} \right) \equiv \left(\frac{d\sigma}{dE}\right)^{(0)} \quad (5)$$

while for lesser energies (down to 5 BeV) this result should be corrected by

$$\left(\frac{d\sigma}{dE}\right)^{(1)} = \frac{1}{E_0} \left\{ A \left(\frac{M}{2E_0} + \frac{ME_0}{2E^2} \right) + \frac{BE}{E_0} \left(\frac{M}{E_0} - \frac{M^2 E_0^2}{4E^4} \right) \right\} \quad (6)$$

The parameters in Eq. (5) are found to be $A = 0.72$ and $B = 0.56$. In the range of energies from 5 to 200 BeV, the most significant corrections are those having a parameter of smallness $\alpha = ME_0/E^2$. These affect the distribution most importantly for cases in which a low-energy secondary emerges from a very-high-energy primary. For example, if $E_0 = 200$ BeV, and $E = 5$ BeV, $\alpha = 8$. The large corrections to the distribution are associated with secondary particles which are emitted backwards in the center-of-mass system. Since half

of the secondaries do go backwards and thereby have a low laboratory-system energy, the secondary spectrum shows a large peak for small energies. Nevertheless, the effect of $(d\sigma/dE)^{(1)}$ on the calculated secondary nucleon spectra is not generally large because of the requirement of a high-energy primary and a low-energy secondary, which can take place only once in a cascade chain, and so this correction may be treated as a perturbation.

The contribution from $d\sigma^{(0)}/dE$ to $F_m(E, E_0)$ can be treated analytically. For this contribution,

$$R_N^{(0)}(E, E_0) \equiv 1/E_0 f(E/E_0) .$$

Let us now introduce⁸ $x \equiv \ln(E_0/E)$, and define

$$f(E/E_0) \equiv g(x) .$$

Then we obtain, from Eq. (4),

$$\begin{aligned} F_m^{(0)}(E, E_0) &= \int_E^{E_0} dE_{m-1} \int_{E_{m-1}}^{E_0} dE_{m-2} \cdots \int_{E_2}^{E_0} dE_1 \\ &\times \frac{g(x_1)}{E_0} \frac{g(x_2 - x_1)}{E_1} \cdots \frac{g(x - x_{m-1})}{E_{m-1}} \\ &= \frac{1}{E_0} \int_0^x dx_{m-1} \int_0^{x_{m-1}} dx_{m-2} \cdots \int_0^{x_2} dx_1 \\ &\times g(x_1) g(x_2 - x_1) \cdots g(x - x_{m-1}) . \end{aligned}$$

If we now define

$$g^{(m)}(x) \equiv E_0 F_m^{(0)}(E, E_0) \quad .$$

we see that

$$g^{(m)}(x) = \int_0^x dx_{m-1} g(x - x_{m-1}) g^{(m-1)}(x_{m-1}) \quad .$$

where $g^{(1)}(x) \equiv g(x)$. Thus $g^{(m)}(x)$ is given as a repeated "faltung" integration. From the theory of Laplace transforms, the Laplace transform of $g^{(m)}$, $\mathcal{L}\{g^{(m)}\}$ is given by

$$\mathcal{L}\{g^{(m)}\} = [\mathcal{L}\{g\}]^m \quad .$$

Since

$$g(x) = A + B e^{-x} \quad ,$$

we find

$$\mathcal{L}\{g\} = \frac{A}{s} + \frac{B}{s+1} \quad ,$$

where s is the independent variable in the transform space. Finally, we can write the solution as

$$g^{(m)}(x) = \frac{1}{2\pi i} \int_0 e^{zx} \left(\frac{A}{z} + \frac{B}{z+1} \right)^m dz \quad , \quad (7)$$

where the contour is taken counterclockwise around the poles at $z = 0, -1$.

This integral may be evaluated by using the series expansions for the exponential and the polynomial near the poles, to obtain the result

$$g^{(m)}(x) = g_0^{(m)}(x) + g_1^{(m)}(x)$$

where

$$g_0^{(m)}(x) = \sum_{k=1}^{m-1} \binom{m}{k} A^k B^{m-k} \sum_{n=0}^{k-1} \frac{(-1)^{k-n-1}}{n!} \binom{m-n-2}{k-n-1} x^n + \frac{x^{m-1} A^m}{(m-1)!}$$

and

$$g_1^{(m)}(x) = e^{-x} \left\{ \sum_{k=1}^{m-1} (-1)^k \binom{m}{k} A^k B^{m-k} \sum_{n=0}^{(m-k-1)} \frac{x^n}{n!} \binom{m-n-2}{m-k-n-1} + \frac{B^m x^{m-1}}{(m-1)!} \right\}$$

Although this result is exact and finite, it is not completely satisfactory for numerical calculations for large m . As m increases, the individual terms in the series rapidly increase because of the binomial coefficients, but as a result of cancellations between terms the total sum decreases with m , as will now be seen. Thus in the numerical

calculations round-off errors were found to dominate the results even for moderate values of m . An alternative form without this defect will now be obtained.

Let us consider the contour for Eq. (7) to be a circle about $z = z_0 - B/(A + B)$, in which the radius is larger than both $|z_0|$ and $|1 + z_0|$. If we now set $u = z - z_0$, then

$$\begin{aligned} \frac{A}{z} + \frac{B}{z+1} &= \frac{A}{u+z_0} + \frac{B}{u+(1+z_0)} \\ &= \frac{A}{u} \left[1 - \frac{z_0}{u} + \left(\frac{z_0}{u} \right)^2 - \dots \right] + \frac{B}{u} \left[1 - \frac{(1+z_0)}{u} + \dots \right] \end{aligned}$$

The two series are uniformly and absolutely convergent on the contour.

Inserting the value of z_0 , we find

$$\frac{A}{z} + \frac{B}{z+1} = \frac{A+B}{u} \sum_{n=0}^{\infty} \frac{a_n}{u^n}$$

where

$$a_n = \left(\frac{B^{n-1} - (-A)^{n-1}}{(A+B)^{n-1}} \right) \frac{AB}{(A+B)^2}$$

Note that $a_0 = 1$, and $a_1 = 0$. To obtain the m th power, we use

$$\left(\sum_{n=0}^{\infty} \frac{a_n}{u^n} \right)^m = \sum_{k=0}^{\infty} u^{-k} \sum_{n_1} \binom{m}{n_1, n_2, \dots} \prod_{i=0}^{\infty} a_i^{n_i}$$

$$= \sum_{k=0}^{\infty} u^{-k} S_{mk}$$

where

$$S_{m,k} = \sum_{n_1} \binom{m}{n_1, n_2, \dots} \prod_{i=0}^{\infty} a_i^{n_i}$$

and the \sum is carried over all n_i so that $\sum_{i=0}^{\infty} n_i = m$, and $\sum_{i=1}^{\infty} i n_i = k$. After inserting this series into Eq. (7), we find

$$g^{(m)}(x) = e^{-\frac{B}{A+B}x} (A+B)^m \sum_{k=0}^{\infty} \frac{S_{m,k} x^{m+k-1}}{(m+k-1)!} \quad (8)$$

For numerical calculations Eq. (8) is very convenient, since $S_{m,k}$ need be calculated only once for given A, B and the series can then be evaluated as needed. In practice it converges quite well.⁹ The asymptotic behavior of $g^{(m)}(x)$ for large x can be readily obtained as

$$g^{(m)}(x) \underset{m \rightarrow \infty}{\sim} e^{\frac{xB}{A+B}} \frac{(A+B)^m x^{m-1}}{(m-1)!} \left\{ 1 + \frac{a_2 x^2}{m+1} + \frac{a_3 x^3}{(m+1)(m+2)} + \dots \right\}$$

B. Cascades in Matter for Which $A_n > 1$

To treat cascades in matter for which the atomic weight (A_n) > 1 , we must introduce a model for the nucleus. For simplicity we assume a spherical nucleus of radius R , and constant density d , though a model including a continuous distribution could easily be treated. Evidently $(4/3)\pi R^3 d = A_n$. The secondary spectrum arising from a nucleon-nucleus collision comes about from sequences of single nucleon-nucleon events within the nucleus, and the probability of n collisions in a cascade will be determined by the path length in the nucleus. It is convenient to characterize the collisions by the impact parameter, b , which are distributed according to

$$P(b)db = \frac{2b db}{R^2}$$

where $P(b)db$ is the probability of an impact parameter between b and $b + db$ in a collision. The probability, P_n , of n interparticle collisions on a particular trajectory through the nucleus (assuming straight-line propagation of the cascade) satisfies a Poisson distribution,

$$P_n(l) = \frac{(\alpha L)^n}{n!} e^{-\alpha L}$$

where $\alpha = (\rho\sigma_{\text{inel}})$ is the inverse of the collision mean free path in nuclear matter. Using $L^2 = 4(R^2 - b^2)$, we obtain the probability for n collisions in traversing a nucleus:

$$\bar{P}_n = \frac{1}{2n!(\alpha R)^2} \int_0^{2\alpha R} x^{n+1} e^{-x} dx \quad (9)$$

These probabilities may be computed by using the recursion relation obtained by integration by parts:

$$\bar{P}_n = \left(\frac{n+1}{n} \right) \bar{P}_{n-1} - \frac{2(2\alpha D)^{n-1}}{n!} e^{-\alpha D}$$

where $D = 2R$. This formula can be used for the smaller n 's together with $\bar{P}_0 = \frac{1}{2} (\alpha R)^{-2} [1 - (1 + \alpha D) \exp(-\alpha D)]$, but for large n cancellations occur which reduce the accuracy of the result. Alternatively, the recursion relation may be solved to obtain

$$\bar{P}_n = \frac{2(n+1)e^{-\alpha D}}{(\alpha D)^2} \sum_{i=n+2}^{\infty} \frac{(\alpha D)^i}{i!}$$

With this result, we can now give the average spectrum to be expected from a nucleon-nucleus collision:

$$R_A^{(0)}(E, E_0) dE = \bar{P}_0 \delta(E - E_0) + \frac{1}{E_0} \sum_n \bar{P}_n g^{(n)}(x) dE \quad (10)$$

The δ function arises from the probability of a nucleon's passing through a nucleus without interaction. In the model, the sum over n was cut off at $n = A_n$, although one would expect that a smaller cutoff should perhaps be made. The precise cutoff is not too important, since the likelihood of high- n collisions is quite small. With this cutoff, the probabilities $\bar{\rho}_n$ were renormalized so that the total overall probability for $n \leq A_n$ was 1.

The term in $\bar{\rho}_0$ can be removed from consideration by a redefinition of the mean free path in bulk matter. Up to now, the mean free path that would enter the calculations is $\lambda_N = [\pi R^2 n]^{-1}$. The presence of the $\bar{\rho}_0$ in R_A leads to a correction such that

$$\lambda_N^{-1} = (1 - \bar{\rho}_0) \pi R^2 n \quad (11)$$

If the distance in matter is measured in units of this new λ , we find that Eq. (10) now becomes

$$R_A^{(0)'}(E, E_0) = \frac{1}{E_0} \sum_{n=1}^{A_n} \bar{\rho}_n' g^{(n)}(x) \quad (12)$$

where

$$\bar{\rho}_n' = \bar{\rho}_n [1 - \bar{\rho}_0 - \sum_{n=A_n+1}^{\infty} \bar{\rho}_n]^{-1}$$

The sum over n is introduced to take account of the cutoff in n . Clearly $\bar{\phi}_n'$ is the probability that if any interaction takes place then n interparticle collisions occur. We see that $\sum_{n=1}^{A_n} \bar{\phi}_n' = 1$.

It is clear that if the fundamental secondary distribution has a shape independent of E_0 ($R(E, E_0)$ is a function of x times E_0^{-1}), then R_A' will also be of this form in the independent collision cascade model. Thus we can introduce the R_A' given by Eq. (12) into Eqs. (3) and (4) to obtain the complete spectrum in matter. We find

$$\rho_N^{(0)}(E, L) dE = e^{-L} \{ \delta(E - E_0) + \frac{1}{E_0} \sum_{M=1}^{\infty} \frac{L^M}{M!} \sum_{K=1}^{\infty} g^{(K)}(x) U_{MK} \} dE, \quad (13)$$

where

$$U_{M,K} = \sum_{m_1, m_2, \dots} \binom{M}{m_1, m_2, \dots} (\bar{\phi}_1')^{m_1} (\bar{\phi}_2')^{m_2} \dots,$$

and the \sum indicates the sum over all m_i with the restriction that $\sum_{i=1}^{\infty} m_i = M$ and $\sum_{i=1}^{\infty} i m_i = K$. Evidently M is the total number of collisions with nuclei that a nucleon has undergone in the distance L , and K is the number of nucleon-nucleon interactions it has had. In terms of the energy spectrum at a certain depth it is seen that the energy distribution, $g^{(K)}(x)$, after K collisions is needed, but it is not necessary to take account of the number of collisions that took place in each of the M nuclear encounters.

One might wonder whether the results in a medium of atomic weight A_n could be directly related to those for hydrogen without going through Eq. (13) and the probabilities $\bar{\rho}_m'$. This is not generally the case, as can be seen easily: In hydrogen, the ratio between the probability for m collisions in a distance L and that for one is given by the Poisson distribution. Thus if we consider $\lambda \ll 1$, we find $\bar{\rho}_2/\bar{\rho}_1 \ll 1$. On the other hand, for nuclei with $A_n > 1$ the probability of m collisions depends on λ and on A_n . In this case for $\lambda \ll 1$, the probability of m collisions depends primarily on A_n and in any case is not to be given by a Poisson distribution, but rather is given by Eq. (9).

C. Corrections to the Proton Distribution

It has already been mentioned that the preceding analysis deals only with the form-invariant part of the elementary cross section as given in Eq. (5). There is an additional term, $d\sigma^{(1)}/dE$ in Eq. (6), which is of importance particularly in collisions in which high-energy primary nucleons produce low-energy secondaries. These contributions may be treated as perturbations to the distribution. To obtain the correction terms we begin with Eq. (4), and express R_N as $R_N^{(0)} + R_N^{(1)}$. The first correction to $F_m(E, E_0)$ can then be written as

$$F_m^{(1)}(E, E_0) = \sum_{n=0}^{m-1} \int_E^{E_0} dE_1 F_n^{(0)}(E_1, E_0) \int_E^{E_1} F^{(1)}(E_2, E_1) F_{k-n-1}^{(0)}(E, E_2) dE_2$$

In this expression, $F_n^{(0)}(E_1, E_0)$ is form-invariant and has been developed in Section IIA, but $F_n^{(1)}(E_2, E_1)$ is not. For these contributions to F_m , the integrations were carried out numerically as follows.

Using Eq. (6), one finds

$$F_1(E, E') = A_1 \left[\left(\frac{E_0}{E'} \right)^2 + \left(\frac{E_0}{E} \right)^2 \right] + B_1 \left(\frac{E}{E_0} \right) \left(\frac{E_0}{E'} \right)^3 - B_2 \left(\frac{E_0}{E} \right)^3,$$

where $A_1 = \frac{1}{2} AM/E_0^2$, $B_1 = BM/E_0^2$, and $B_2 = \frac{1}{4} BM^2/E_0^3$. The first integration can then be expressed as

$$\begin{aligned} I_n(E, E_0) de &= \int_E^{E_0} F_n^{(0)}(x') \frac{dE'}{E_0} F_n^{(1)}(E, E') dE \\ &= dE \{ A_1 G_n^{(-1)}(E) + h(E) G_n^{(1)}(E) + B_1 G_n^{(-2)}(E) \}, \end{aligned}$$

where

$$G_n^{(k)}(E) = \int_E^{E_0} F_n^{(0)}(x') \frac{dE'}{E'} \left(\frac{E'}{E_0} \right)^k = \int_0^x F_n^{(0)}(k') e^{-kx'} dx',$$

and

$$h(E) = A_1 (E_0/E)^2 - B_2 (E_0/E)^3.$$

It is necessary to carry out the integrals $G_n^{(k)}(E)$ only once over the desired range in x in order to obtain $I_n(E, E_0)$ for all needed points,

because the $G_n^{(k)}$'s are functions only of x and can be calculated sequentially. The I_n does not have this property, however, and to perform the last integration needed for $F_m^{(1)}$ a full numerical integration over x' is necessary for each value of E, E_0 .

III. PION SPECTRA

In treating pion spectra in matter, experimental data for pion production by high-energy protons are sufficiently extensive to provide a phenomenological extrapolation to high energies. However, information about pion production by pions does not exist in enough detail to allow a similar extrapolation for this process, so in the present calculations this source of secondary pions has been ignored. Thus the results of the calculation would represent a lower limit to the total pion spectra.

The pion cascade equation is analogous to Eq. (1):

$$\frac{d\rho_\pi(E, L)}{dL} = -\frac{1}{\lambda_\pi} \rho_\pi(E, L) + \frac{1}{\lambda_N} \int_E^{E_0} R_\pi(E, E') \rho_N(E', L) dE' \quad (14)$$

In this equation $R_\pi(E, E')$ is the distribution of charged pions¹⁰ of energy E contributed by collisions from nucleons of energy E' . The inelastic cross section for pions, σ_{inel}^π , at high energies is approximately constant, so we assume that λ_π is constant. Trilling has recently developed a formula for $R_\pi(E, E_0)$, which is

$$R_\pi(E, E_0) = \frac{1.66}{E_0} e^{-10.4E^2/E_0^2} + 4.0 \frac{E}{E_0^{1/2}} e^{-4.8E/(E_0)^{1/2}} \quad (15)$$

The first term contributes a high-energy pion distribution of constant multiplicity, while the second represents a cloud of lower-energy pions of multiplicity proportional to $E_0^{1/2}$. This formula fits known experimental data fairly well. It is somewhat similar to the formula developed by Cocconi et al.,¹¹ but differs by its separation into a high- and a low-energy part, and also in the behavior of the pion multiplicity with energy. This formula is meant to apply only to mesons produced in the forward direction in the center-of-mass system. There would also be a group of very-low-energy pions in the laboratory system which are produced in the backward direction.

For hydrogen or nuclear matter cascades, Eq. (3) can be used directly with Eq. (14) to obtain ρ_π . After solving the resulting differential equation, we find

$$\rho_\pi(E, L) = \sum_{m=0}^{\infty} d_m(L) G_m(E, E_0) \quad , \quad (16)$$

where

$$d_m(L) = e^{-\frac{L}{\lambda_\pi}} \int_0^L \frac{dL'}{\lambda_N} e^{\left(\frac{1}{\lambda_\pi} - \frac{1}{\lambda_N}\right)L'} \frac{(L'/\lambda_N)^m}{m!} \quad (17)$$

and

$$G_m(E, E_0) = \int_E^{E_0} dE' R_\pi(E, E') F_m(E', E_0) \quad . \quad (18)$$

If $\lambda_\pi = \lambda_N$, the expression for d_m could be integrated to give a Poisson distribution, but since the two inelastic cross sections are quite different, we resort to the recursion relation for d_m :

$$d_m(L) = \frac{\lambda_\pi}{\lambda_\pi - \lambda_N} \left[d_{m-1}(L) - \frac{(L/\lambda_N)^m}{m!} e^{-\frac{L}{\lambda_N}} \right]$$

and:

$$d_0(L) = \frac{\lambda_\pi}{\lambda_\pi - \lambda_N} \left[e^{-\frac{L}{\lambda_\pi}} - e^{-\frac{L}{\lambda_N}} \right]$$

These equations can be solved to give

$$d_m(L) = \left(\frac{\lambda_\pi}{\lambda_\pi - \lambda_N} \right)^{m+1} e^{-L/\lambda_N} \sum_{k=m+1}^{\infty} \frac{1}{k!} \left[\left(\frac{1}{\lambda_N} - \frac{1}{\lambda_\pi} \right) L \right]^k \quad (19)$$

To extend the model to matter for $A_n > 1$, we proceed, as for protons, by first treating the cascades in a nucleus and secondly by going to a medium consisting of a distribution of nuclei. For the first step we are again led to an averaging over impact parameters. If δ_m is the probability that a nucleon makes m interactions with nucleons on passing through the nucleus and then in the $(m+1)$ th interaction produces a pion which escapes from the nucleus, we find, from Eq. (17),

$$\delta_m = \frac{2}{D^2} \int_0^D L dL \left\{ e^{-\frac{L}{\lambda_\pi}} \int_0^L \frac{dL'}{\lambda_N} e^{(\frac{1}{\lambda_\pi} - \frac{1}{\lambda_N})L'} \frac{(L'/\lambda_N)^m}{m!} \right\}$$

Again an integration by parts gives a recursion relation,

$$\delta_m = \left(\frac{\lambda_\pi}{\lambda_\pi - \lambda_N} \right) [\delta_{m-1} - \bar{\rho}_m] ,$$

and we also find

$$\delta_0 = \left(\frac{\lambda_\pi}{\lambda_\pi - \lambda_N} \right) \left\{ 2 \left(\frac{\lambda_\pi}{D} \right)^2 [1 - (1 + D/\lambda_\pi) e^{-D/\lambda_\pi}] - \bar{\rho}_0 \right\} .$$

Alternatively, we can use Eq. (19) directly, and on averaging over L we find

$$\delta_m = \left(\frac{\lambda_\pi}{\lambda_\pi - \lambda_N} \right)^{m+1} \sum_{k=m+1}^{\infty} \left[\left(\frac{\lambda_\pi - \lambda_N}{\lambda_\pi} \right)^k \bar{\rho}_k \right] ,$$

where $\bar{\rho}_k$ is given by Eq. (9) .

With these results, we can now give the average distribution of secondary pions of energy E to be expected from a collision between a nucleon of energy E' and a nucleus. From Eq. (16) and the above averaging over impact parameters we obtain

$$R_{\pi, A_N}(E, E') = \sum_{m=0}^{\infty} \delta_m G_m(E, E') .$$

To obtain the pion spectrum for the passage of nucleons through matter consisting of such nuclei, we use Eq. (14) modified for nuclear cascades as above:

$$\frac{d\rho_{\pi}(E, L)}{dL} = -\epsilon\rho_{\pi}(E, L) + \frac{1}{\lambda_N} \int_E^{E_0} R_{\pi, A}(E, E')\rho_N(E', L)dE' . \quad (20)$$

The first term corresponds to the loss of pions on passage through nuclei. For the present calculations, this term has been chosen as

$$\epsilon = \pi R^2 v (1 - \xi) ,$$

where ξ is the probability that the pion not make an inelastic collision in passing through the nucleus and v is the density of nuclei in the medium. This choice overestimates ϵ , since an inelastic collision of a high-energy pion with a nucleus may lead to a distribution of less energetic pions that still are high-energy pions. Unfortunately, the information about such secondaries is too meager to provide a significant model and so the above choice has been made. As better information becomes available this assumption should be reconsidered. The probability ξ for pions is analogous to $\bar{\rho}_0$, and has the same form except that the inelastic cross section for pions must replace that for protons in $\bar{\rho}_0$.

Before giving the solution to Eq. (20), we note that to express the results for pions, using the same depth variable as for the nucleons, it is convenient to express L in units of λ_N as given by Eq. (11). The pion distribution is then

$$\rho_{\pi}(E, \lambda) = e^{-\frac{(1-\xi)\lambda}{(1-\bar{\rho}_0)}} \int_0^{\lambda} e^{\frac{(1-\xi)\lambda'}{(1-\bar{\rho}_0)}} \frac{d\lambda'}{(1-\bar{\rho}_0)} \int_E^{E_0} dE' e^{-\lambda'}$$

$$\left\{ \delta(E' - E_0) + \sum_{m=1}^{\infty} \frac{(\lambda')^m}{m!} \sum_{k=1}^{\infty} F_k(E', E_0) U_{mk} \right\} \\ \times \sum_{n=0}^{\infty} \delta_n G_n(E, E')$$

Using the relation

$$\int_E^{E_0} dE' F_k(E', E_0) G_n(E, E') = G_{n+k}(E, E_0)$$

we obtain

$$\rho_{\pi}(E, \lambda) = \sum_{m=0}^{\infty} \frac{I_m(\lambda)}{m!} \sum_{k=0}^{\infty} G_k(E, E_0) D_{m,k}$$

where

$$I_m(\lambda) = e^{-\frac{(1-\xi)\lambda}{(1-\bar{\rho}_0)}} \int_0^{\lambda} e^{\frac{(\bar{\rho}_0 - \xi)\lambda'}{(1-\bar{\rho}_0)}} (\lambda')^m \frac{d\lambda'}{(1-\bar{\rho}_0)}$$

and

$$D_{m,k} = \sum_{n=0}^k U_{m,k-n} \delta_n \quad .$$

Again we obtain a recursion relation for $I_m(\lambda)$ of the form

$$I_m(\lambda) = \frac{\lambda^m e^{-\lambda}}{(\bar{\rho}_0 - \xi)} - m \frac{(1 - \bar{\rho}_0)}{(\bar{\rho}_0 - \xi)} I_{m-1}(\lambda) \quad .$$

which can be solved by use of

$$I_0(\lambda) = \frac{1}{(\bar{\rho}_0 - \xi)} \left[e^{-\lambda} - e^{-\left(\frac{1-\xi}{1-\bar{\rho}_0}\right)\lambda} \right]$$

to give

$$I_m(\lambda) = \frac{\lambda^{m+1} e^{-\lambda}}{(1 - \bar{\rho}_0)} \sum_{k=0}^{\infty} \left(\frac{\xi - \bar{\rho}_0}{1 - \bar{\rho}_0} \lambda \right)^k \frac{m!}{(m + k + 1)!} \quad .$$

IV. NUMERICAL CALCULATIONS AND RESULTS

A number of calculations using the above formulae were carried out for various representative values of A_n and E . The numerical integrations needed for the first-order corrections to the proton spectra as discussed in Section IIC, and for the pion spectra, were evaluated with an integration formula that retains second and fourth differences of the integrand:¹²

$$\int_x^{x+h} f(x') dx' \approx h \left\{ f\left(x + \frac{h}{2}\right) - \frac{1}{12} \Delta^{II}\left(x + \frac{h}{2}\right) + \frac{11}{720} \Delta^{IV}\left(x + \frac{h}{2}\right) \right\}, \quad (22)$$

where each function on the right is evaluated as the average of its value at x and at $x + h$. The quantities $\Delta^{II}(x)$ and $\Delta^{IV}(x)$ are the second and fourth differences of $f(x)$ at x .

The accuracy of this formula was compared with that for one retaining only second differences in iterating the "faltung" integration of Eq. (4). An energy range of $E/E_0 = 40$ was covered in 10, 20, and 40 steps. When R_N was set equal to $R_N^{(0)}$, we found that Eq. (22) gave agreement with the exact result for $F_m^{(0)}$ to within a few percent for most values of x and m using 20 points. The agreement using the second-difference form was much less satisfactory, even for 40 points. Because the final spectrum for pions involved three numerical integrations, the number of points was very important in determining the running time of the program, and so the more complicated fourth-difference formula was selected, together with 20 integration steps plus a few more which were needed to determine differences at the end points of the integration range. Ten points did not provide the desired accuracy in the end results. Since the numerical integrations involved $F_m^{(0)}(x)$ times rather smooth functions, the preceding analysis seemed to be a good test of the accuracy of the integration.

The formula selected will give an exact result for polynomials of 5th degree or less. Near $x = 0$, the functions $F_m^{(0)}(x)$ vary as

x^{m-1} , however, so that for $m > 6$ one expects errors to appear. In this region of x , $F_m^{(0)}(x)$ is very small, and for $x > 0$ is positive definite. The numerical calculations were checked at each step to determine that the increment to the integral was positive. In a few cases the increment was negative for x near zero, and since it was expected that the integral should be very small there in any case, those increments were set equal to zero. This occurred for a small fraction of the steps one unit from $x = 0$, and only very rarely for the second step away. Values of the integrals were needed for $x < 0$ to obtain differences of the integrands at some end points. It is easily seen that for m even, F_m is an odd function of x , while for m odd, F_m is positive definite. This dependence was guaranteed for $x < 0$ in a manner analogous to that described above for $x > 0$.

Finally, a few second-order perturbation terms were evaluated to compare them with the zero- and first-order contributions to $F_m(E, E_0)$. As has been stated, the first-order terms are largest for large E_0 and small E . For m values from 2 to 6 and $E_0 = 200$ BeV, it was found that the second-order correction was about 10% of the first order at $E = 5$ BeV. As the secondary energy is increased the ratio rapidly becomes much smaller. Thus, since the basic cross-section formulas are not expected to be as accurate as this, corrections to the spectra from corrections beyond first order have not been included.

Certain other parameters are needed for numerical calculations. For hydrogen cascades the depth is measured in mean free paths, so for

the nucleon spectra nothing more is needed. For nuclei with $A_n > 1$, however, the \bar{P}_n 's depend on R , and this quantity must be specified. We have chosen $R = 1.25 \times 10^{-3} \cdot A^{1/3}$ cm, and $\sigma_{inel}^N = 32$ mb. This leads to $R = 0.50 \cdot A^{1/3}$. For the pion calculations it is necessary to know λ_π/λ_N . For this we have chosen $\sigma_{inel}^\pi = 20$ mb, which leads to $\lambda_\pi/\lambda_N = 1.6$. The results of the numerical calculations for secondary proton spectra are given in Figs. 1 through 11, and those for pions are given in Figs. 12 through 18. It is seen that for $E_0 = 200$ BeV and $A_n = 1$, the low-energy correction to the elementary interaction is very important and contributes strongly to the low-energy secondaries, especially at small depths. For complex nuclei or larger depths, however, the correction becomes small. One also sees that for complex nuclei the logarithms of the secondary distributions are well approximated by straight lines when plotted as a function of the logarithm of the energy, with a slope which depends on the depth. It might also be noted that although the incident beam decreases exponentially with depth, the secondary protons produced may also have significant energies for high-energy pion production. Thus it is seen that over a wide range of energies the pion distribution is approximately constant in depth over several mean free paths. In our calculation the pions are assumed to be absorbed whenever they undergo an interaction with a nucleon. This is not a realistic assumption, as they are much more likely to reappear at a somewhat lesser energy. Thus the pions produced are assumed to disappear too rapidly. A more realistic treatment of the pions would probably show that the pion distribution increases with depth for

several mean free paths, so that to maximize the pion beam of a given moderate secondary energy from a target, for example, the target would be much more than a mean free path long.

Unfortunately the calculations are not readily compared with experimental data from accelerators or cosmic radiation at these energies, because such data are primarily obtained as the number of stars produced in photographic emulsions. To extend the analysis to include star production would probably require Monte Carlo calculations such as have already been reported in the literature, and would introduce a new dimension of complexity in the calculations. Thus no comparison of the results with experiments has been attempted.

ACKNOWLEDGMENT

The author would like to thank Dr. Denis Keefe, Dr. George Trilling, Dr. Ralph Thomas, and Dr. Joseph Lepore for numerous valuable discussions during the course of this work.

Appendix ASYMPTOTIC FORM OF THE TRILLING NUCLEON SPECTRUM

The secondary nucleon spectrum as developed by Trilling is of the form

$$\frac{d\sigma}{dp} = \frac{\pi(a(E_0) + b(E_0)p^*)}{3 \gamma(\beta + \beta^*)}$$

as given in Section IIA. The transformation between the laboratory and center-of-mass systems is accomplished with $\beta = ((E_0 - M)/(E_0 + M))^{1/2}$ and $\gamma = ((E_0 + M)/2M)^{1/2}$. We neglect transverse motions, and then find $p^* = \gamma p(1 - \beta/\beta_p)$, where $\beta_p = p/E$. [p^* is the c.m. momentum and p the laboratory-system momentum of the secondary nucleon.] One can then deduce that

$$\begin{aligned} \gamma(\beta + \beta^*) &= (1 - \beta^2)^{-1/2} \left[\beta + \frac{\beta_p(1 - \beta/\beta_p)}{1 - \beta\beta_p} \right] \\ &= \beta_p [\gamma(1 - \beta\beta_p)]^{-1} \end{aligned}$$

The dependence of $a(E_0)$, and $b(E_0)$ is then obtained using $p_{\max}^* = \beta\gamma M = ((E_0 - M)M/2)^{1/2}$ so that

$$a = a'/\beta\gamma \quad b = b'/\beta^2\gamma^2$$

where a' , b' are constants.

On converting from momentum to energy as independent variable, we find

$$\frac{d\sigma}{dE} = \frac{\pi}{3} \frac{E^2}{p^2} \left\{ \left(\frac{1}{\beta} - \beta_p \right) a' + \left[\beta_p \left(\frac{1}{\beta} - 1 \right)^2 - \frac{1}{\beta} (1 - \beta_p)^2 \right] b'E \right\} \quad (A1)$$

Now $E^2/p^2 = (1 - m^2/E^2)^{-1} \approx 1 + M^2/E^2$, and since we are interested only in $E \approx 5$ BeV, the term in M^2/E^2 may be neglected as a small correction to the distribution. Further,

$$\frac{1}{\beta} - 1 \approx \frac{M}{E_0} + \frac{M^2}{2E_0^2} + O \left[\left(\frac{M}{E_0} \right)^3 \right] \quad (A2)$$

and

$$1 - \beta_p \approx \frac{M^2}{2E^2} + O \left[\left(\frac{M}{E} \right)^4 \right]$$

The second term in Eq. (A2) can be as large as 10% of the leading term, while the higher terms will be very small and are neglected. If these results are introduced in Eq. (A1), we then find that coefficient of a' becomes

$$\frac{1}{\beta} - \beta_p \approx M/E_0 + \frac{1}{2} (M/E_0)^2 + \frac{1}{2} (M/E)^2$$

The first term is the leading one, and the second can give a correction of up to 10%. The third term can be large, however, if $E/E_0 \ll 1$. As has been discussed, this situation corresponds to the spectrum generated by secondaries going backwards in the center-of-mass system.

For $E_0 = 200$ BeV and $E = 5$ BeV, this correction is four times the leading term, thus drastically affecting the low-energy spectrum for a single collision. On the other hand, for multiple collision processes the dominant contributions come from smaller energy changes, with the result that this correction term becomes less significant.

For the coefficient of b'

$$\beta_p \left(\frac{1}{\beta} - 1 \right)^2 - \frac{1}{\beta} (1 - \beta_p)^2 \approx \left(\frac{M}{E} \right)^2 + \left(\frac{M}{E_0} \right)^3 + \frac{M^4}{4E_0^4} - \frac{M^4}{2E_0^2 E_0^2} - \frac{M^4}{4E_0^4}$$

the first term is the leading one, and the second may produce a correction of 20%. The third and fourth terms will be neglected as small. The fifth term is the more important correction, and can be large compared with the first if $E/E_0 \ll 1$, as has been discussed for the corresponding term in the a' coefficient. Thus we are led to the final form,

$$R_N(E, E_0) = \frac{1}{E_0} \left\{ A \left(1 + \frac{M}{2E_0} + \frac{ME_0}{4E^4} \right) + \frac{BE}{E_0} \left(1 + \frac{M}{E_0} - \frac{M^2 E_0^2}{4E^4} \right) \right\}$$

where A and B are constants.

The coefficients A and B have been deduced from Trilling's values for a, b and by assuming conservation of nucleons in the collisions, thus neglecting such effects as hyperon and antinucleon production. For very high energies, we require

$$\int_0^{E_0} \left(A + \frac{BE}{E_0} \right) \frac{dE}{E_0} = A + \frac{B}{2} = 1$$

Use of only the large-E portion of the spectrum corresponds to inclusion of only one of the secondary nucleons in a collision, and hence should integrate to 1 rather than 2. Further, at $E_0 = 18.8$ BeV, Trilling finds

$$\frac{d\sigma}{dE} \propto a + bp^*$$

$$\approx a + b \left(\frac{E_0 M}{2} \right)^{1/2} \frac{E}{E_0}$$

Inserting his values for a and b, we find

$$\frac{d\sigma}{dE} \propto 6.0 + 1.57 \frac{E}{E_0}$$

Since we also have

$$\frac{d\sigma}{dE} \propto A + BE/E_0$$

we finally can solve for A and B to obtain

$$A = 0.72$$

$$B = 0.56$$

FOOTNOTES AND REFERENCES

- * This work was performed under the auspices of the United States Atomic Energy Commission.
1. There are numerous Monte Carlo calculations in the literature; for example: Hugo W. Bertini, Phys. Rev. 131, 1801 (1963); Metropolis, Bivins, Storm, Miller, Friedlander, and Turkevich, Phys. Rev. 110, 185 and 204 (1958); E. Gradsztajn, Phys. Rev. Letters 13, 240 (1964); and J. Ranft, Monte Carlo Calculation of Nucleon-Meson Cascade in Shielding Materials Initiated by Incoming Proton Beams Between 10 and 1000 GeV, CERN 64-47, 18 November 1964.
 2. R. G. Alsmiller, Jr., F. S. Alsmiller, and J. E. Murphy, ORNL-3289, 1962; ORNL-3365, 1963; ORNL-3412, 1963; R. G. Alsmiller, Jr., and J. E. Murphy, ORNL-3367, 1963; R. G. Alsmiller, Jr., ORNL-3570, 1963.
 3. George Trilling (Lawrence Radiation Laboratory) private communication.
 4. One would expect that the nucleon charge distribution in the secondary spectrum would be determined eventually by the medium, but because of neutron-proton differences one would not generally expect equal numbers of pions of opposite charges. It is observed experimentally, however, that there is a marked tendency for the high-energy nucleon emerging from an interaction to retain the charge of the high-energy incident particle, so that the charge may "persist" through a number of collisions.
 5. V. S. Barashnikov, V. M. Maltsev, and E. K. Mihul, Nucl. Phys. 24, 642 (1961); V. S. Barashnikov and S. M. Eliseev, Particle Interaction

with Matter at Superhigh Energies, Preprint, Joint Institute for Nuclear Research, Dubna, 1964.

6. These coefficients give the secondary proton spectra in proton-proton collisions. To obtain the total nucleon spectrum these coefficients must be increased by the ratio of the total inelastic cross section (32 mb) to the partial cross section for high-energy proton secondaries (26 mb).
7. The approximation consists of neglecting the difference between p^* and p_l^* . These differences enter quadratically in the form $(p_t^*/p_l^*)^2$, and since $p_{\max}^* = (T_0 M/2)^{1/2}$ and $p_t^* \lesssim 0.5 \text{ BeV/c}$, for $E > 5 \text{ BeV}$ such effects will be small.
8. The quantity x is sometimes called the "lethargy."
9. The convergence of the series is easily demonstrated. If we set $T_{m,k} \equiv \sum_{n_i}^{m} (n_0 n_1 n_2 \dots) > S_{m,k}$, we find that $T_{m,k} = \binom{m+k-1}{k}$. The latter step follows easily through the use of a generating function. From this result the convergence follows directly.
10. Neutral pions in hydrogen decay almost immediately to gamma rays, and so do not produce any cascade effect. In a nucleus, however, such pions could interact before decaying, but for lack of information these effects are ignored in this work.
11. G. Cocconi, L. J. Koester, and D. H. Perkins, Particle Fluxes from Proton Synchrotrons, UCRL-10022, April 1963, p. 167.
12. D. R. Hartree, Numerical Analysis (Oxford University Press, London, 1952), Eq. (6.2).

FIGURE CAPTIONS

Figs. 1-11. Secondary proton distribution for various initial proton energies and atomic weights.

Figs. 12-18. Secondary pion distributions for various initial proton energies and atomic weights.

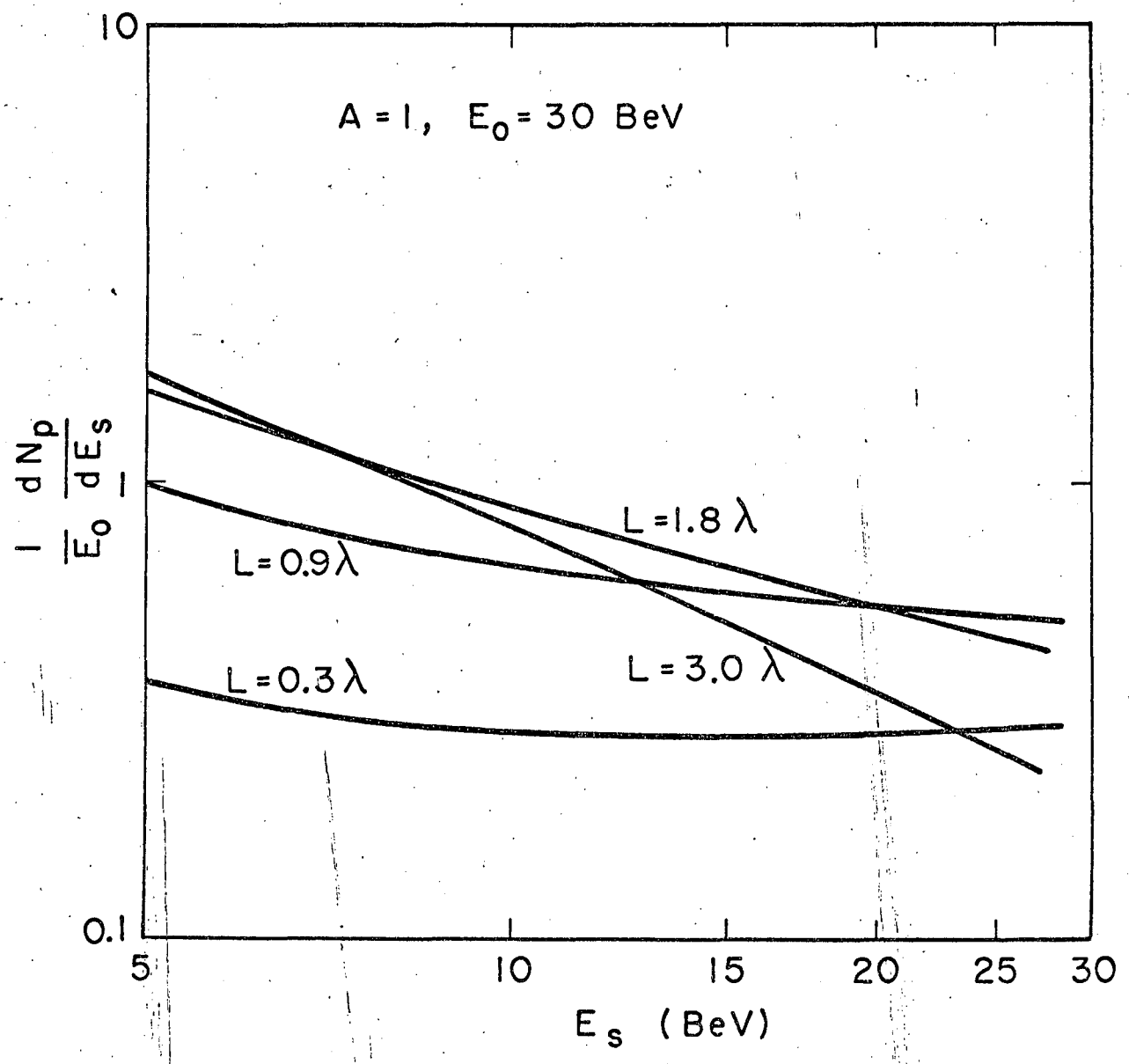


Fig. 1

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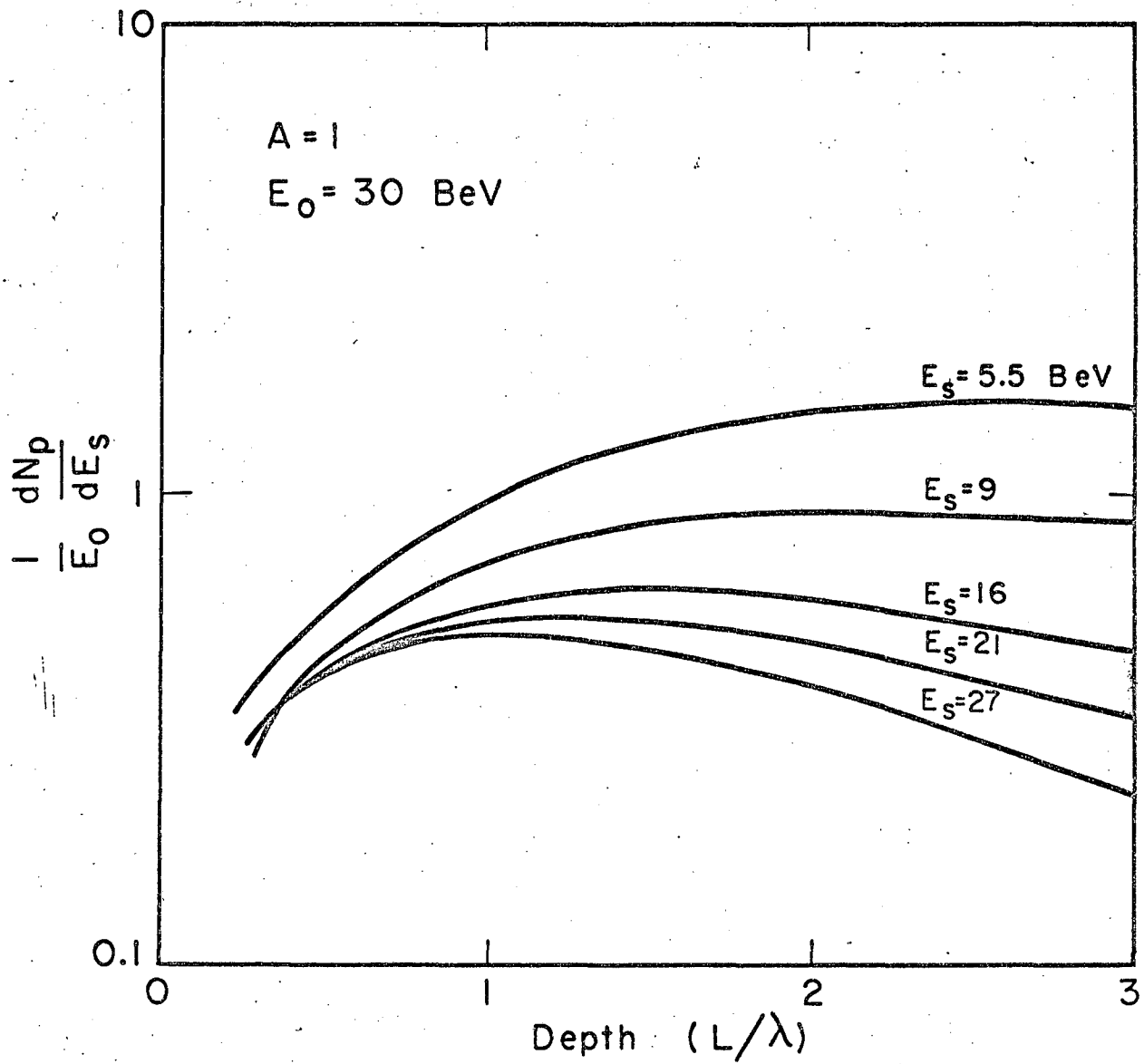


Fig. 2

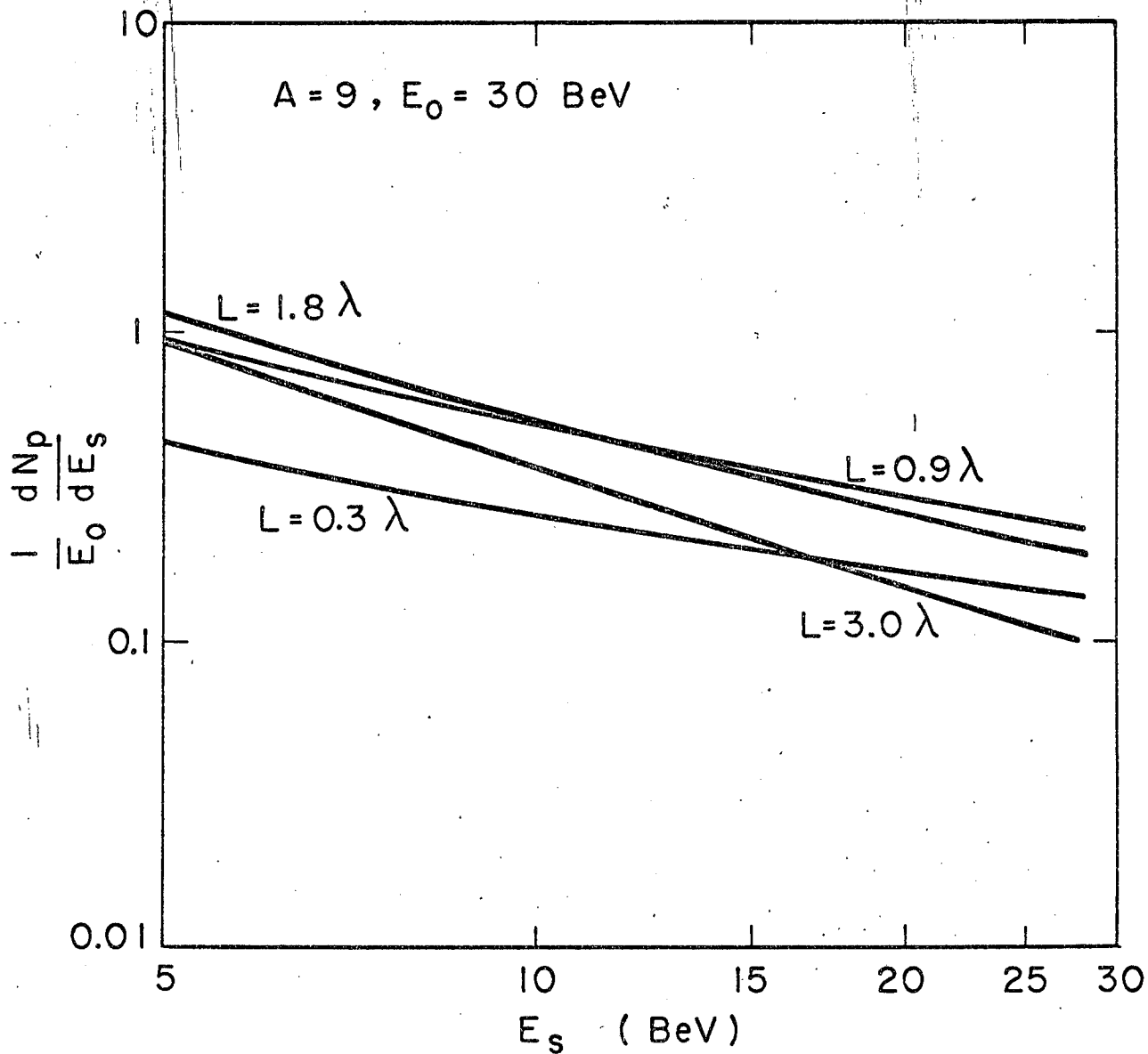
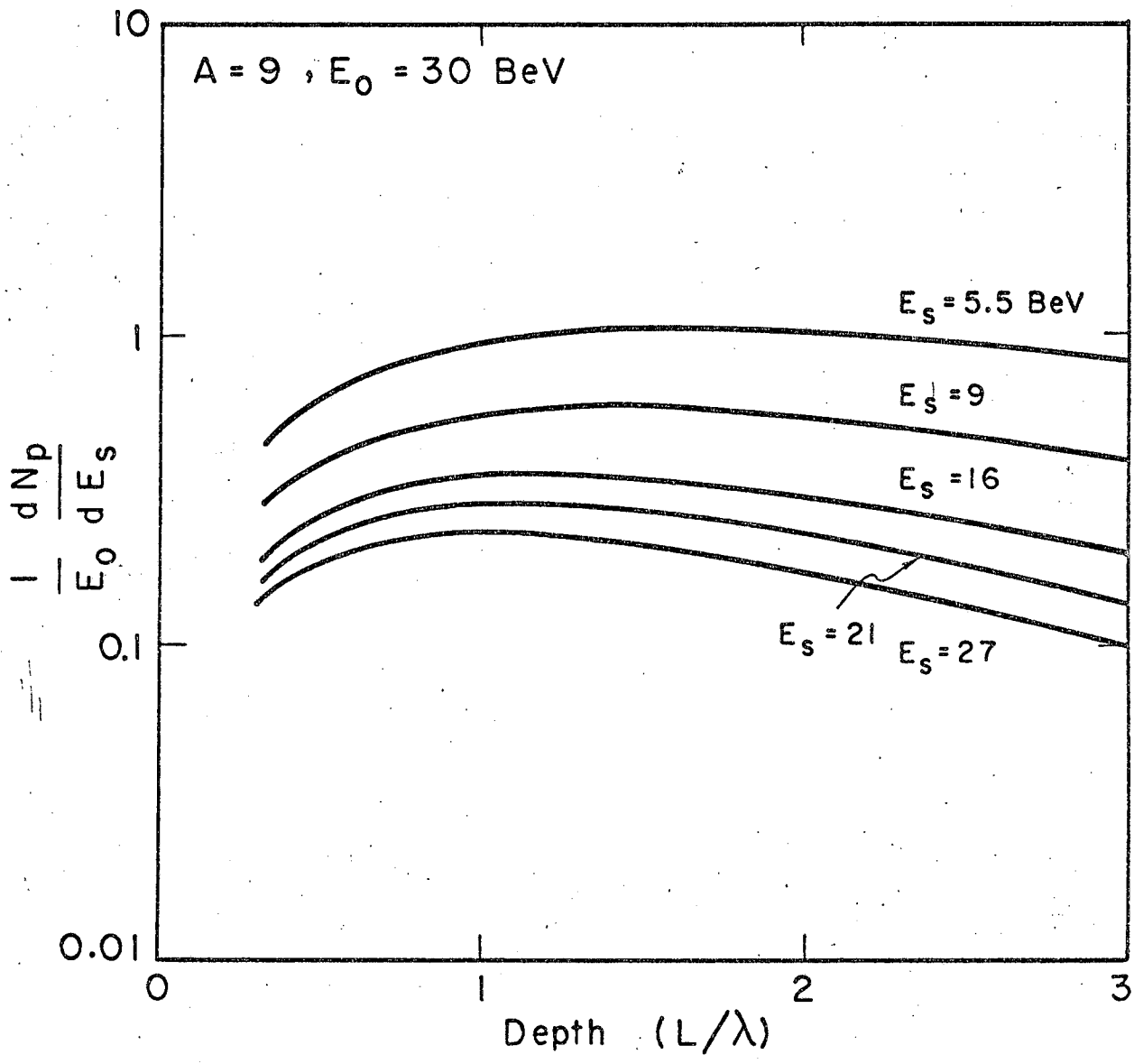


Fig. 3



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Fig. 4

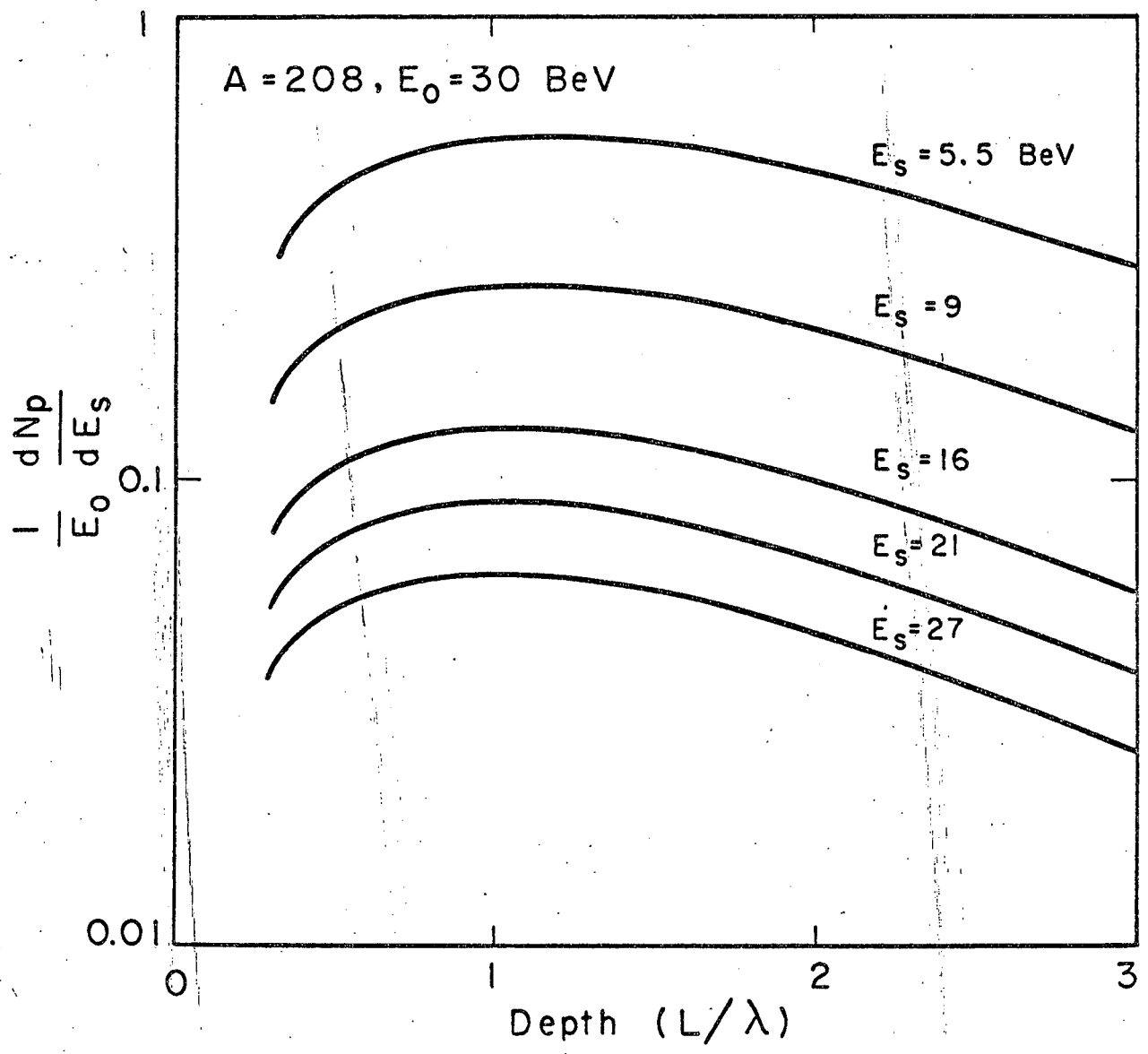


Fig. 5

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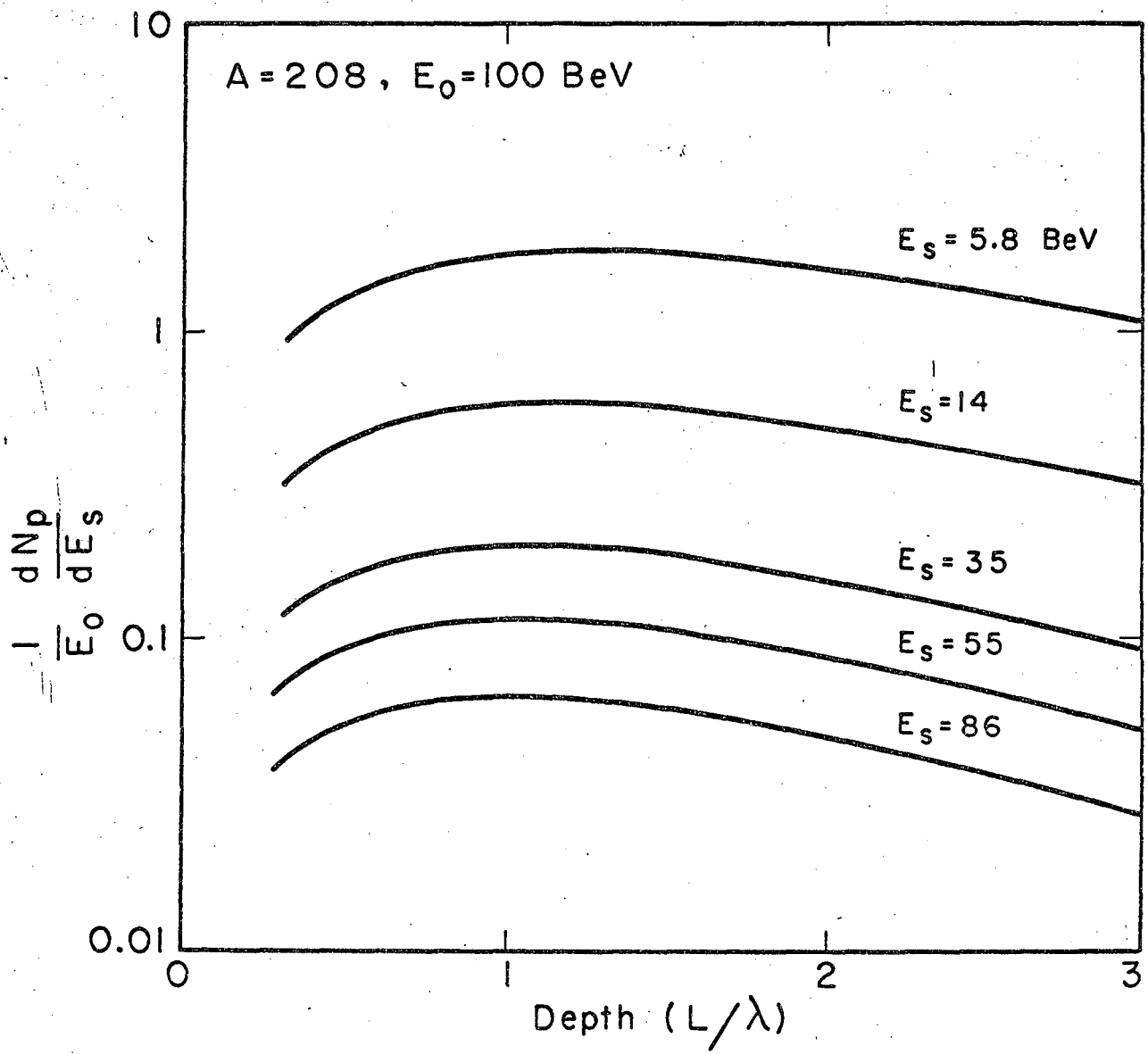


Fig. 6

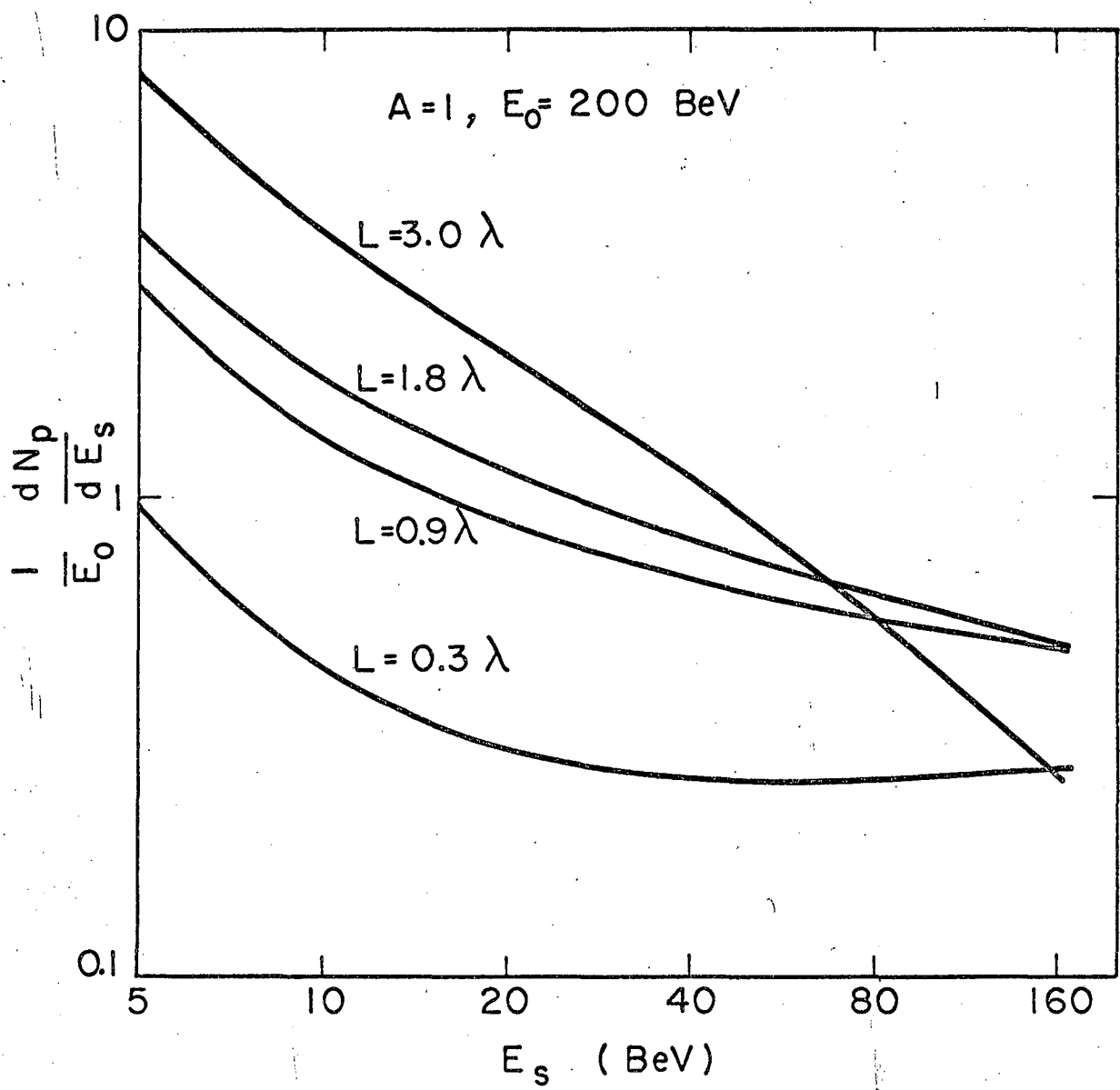


Fig. 7

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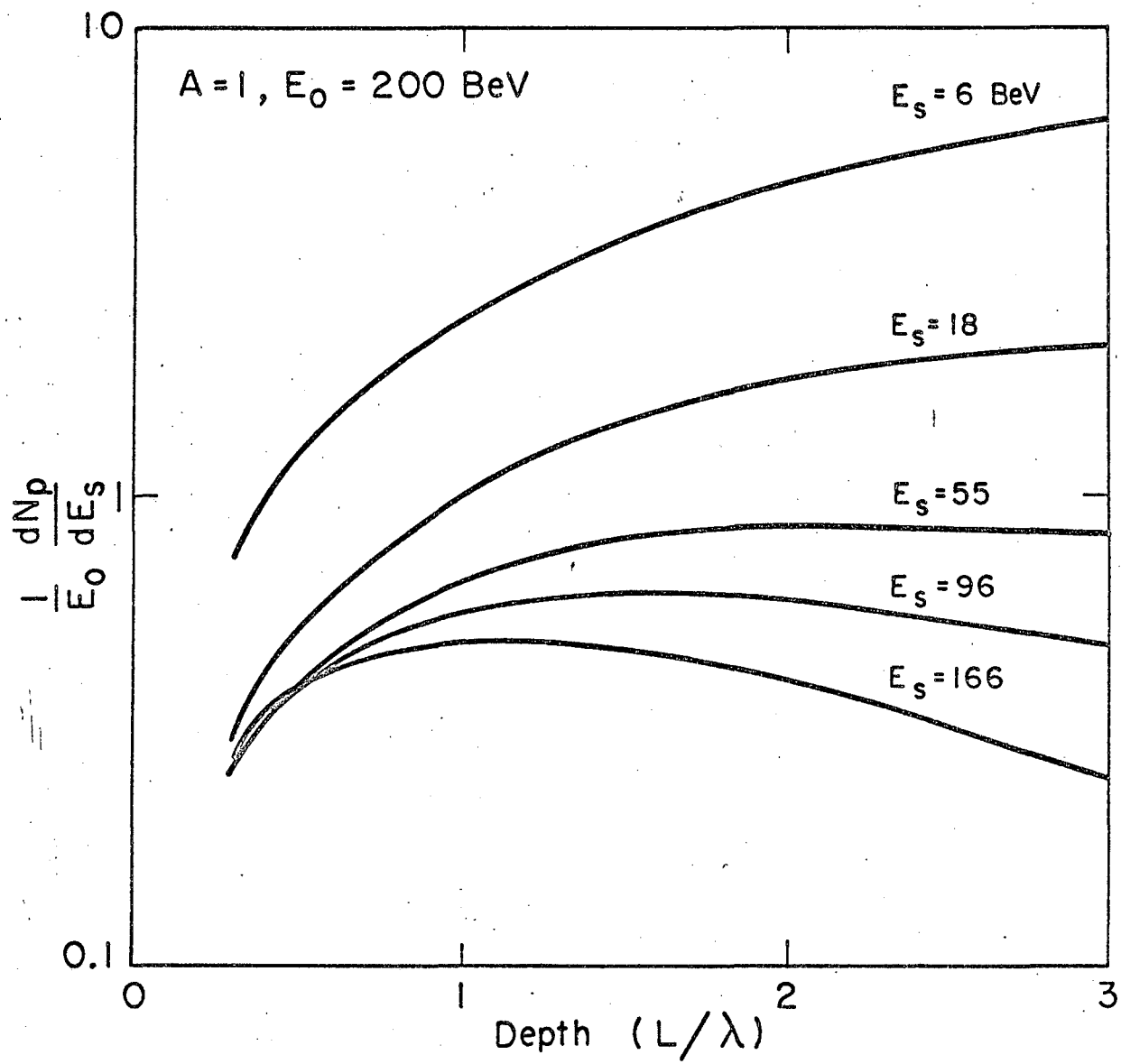


Fig. 8

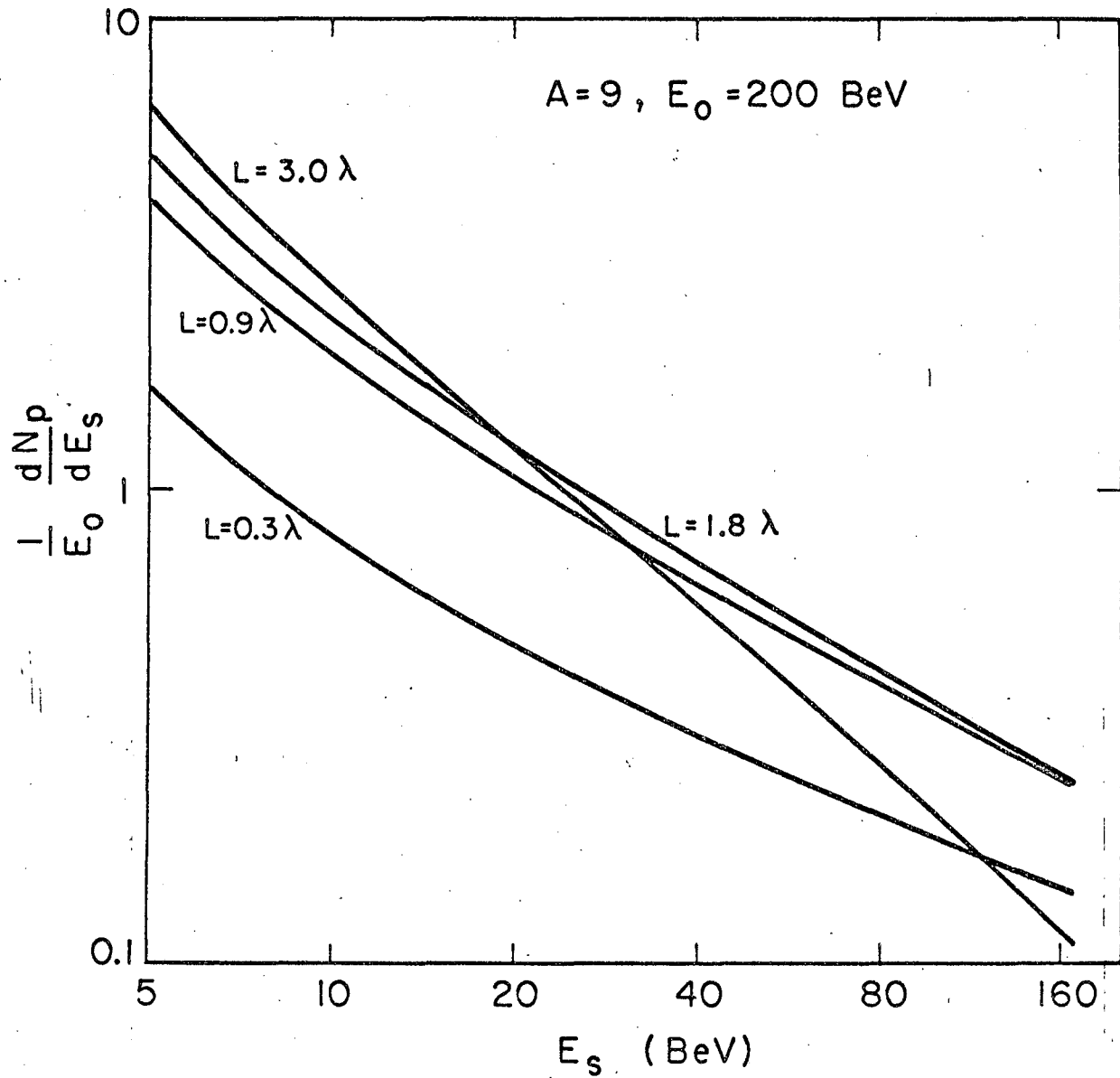


Fig. 9

MUB-7442

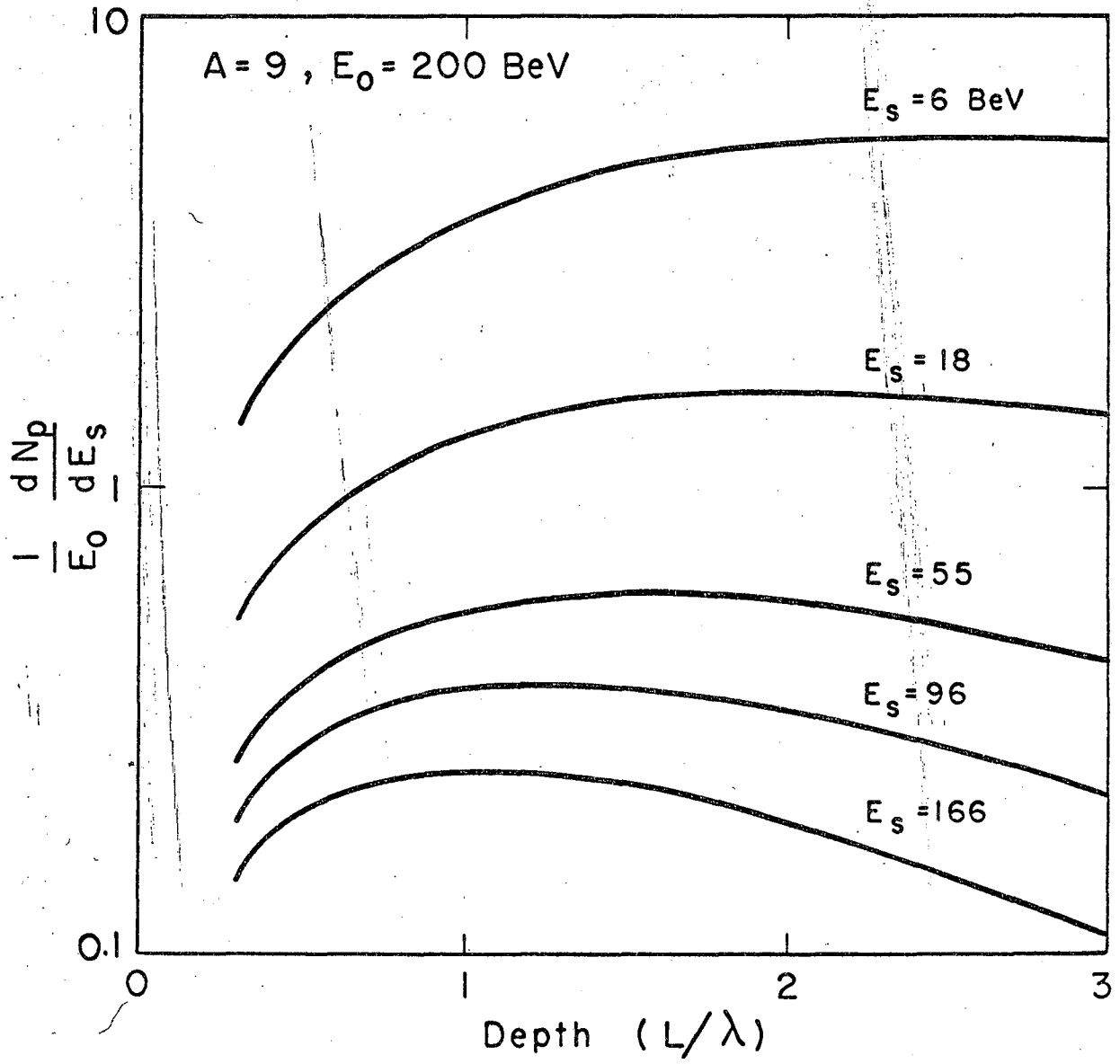


Fig. 10

MUB-7443

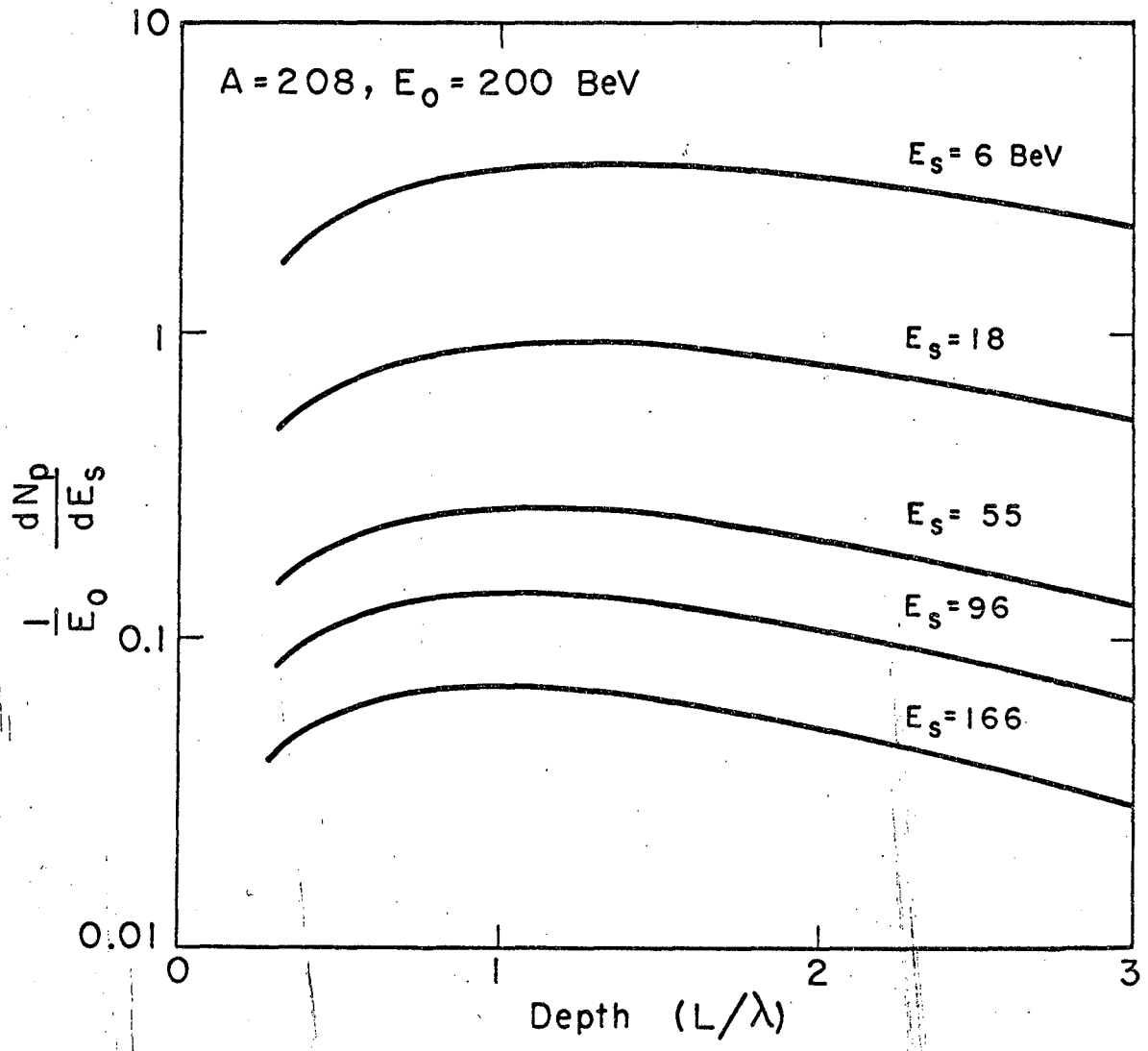


Fig. 11

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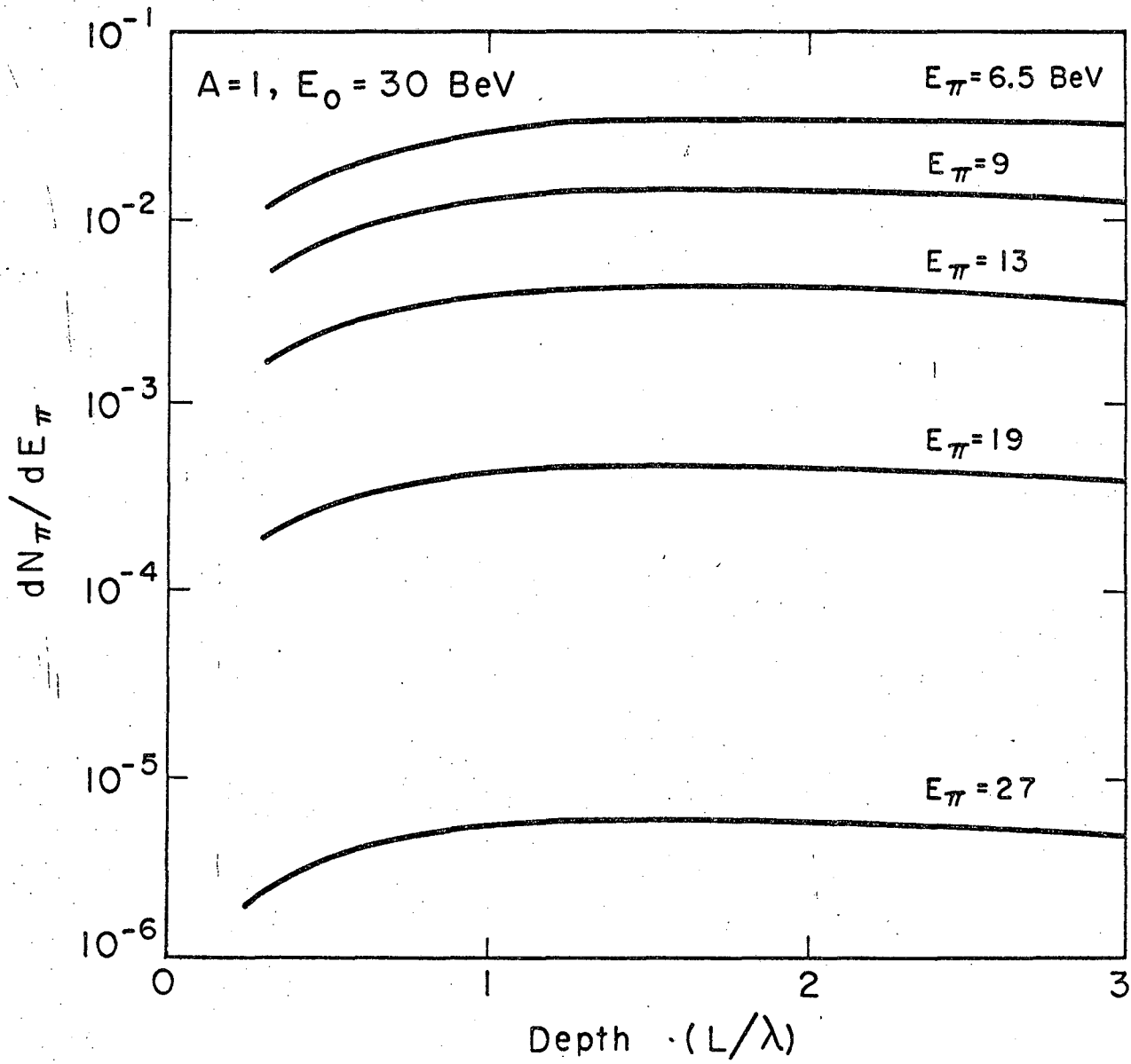


Fig. 12

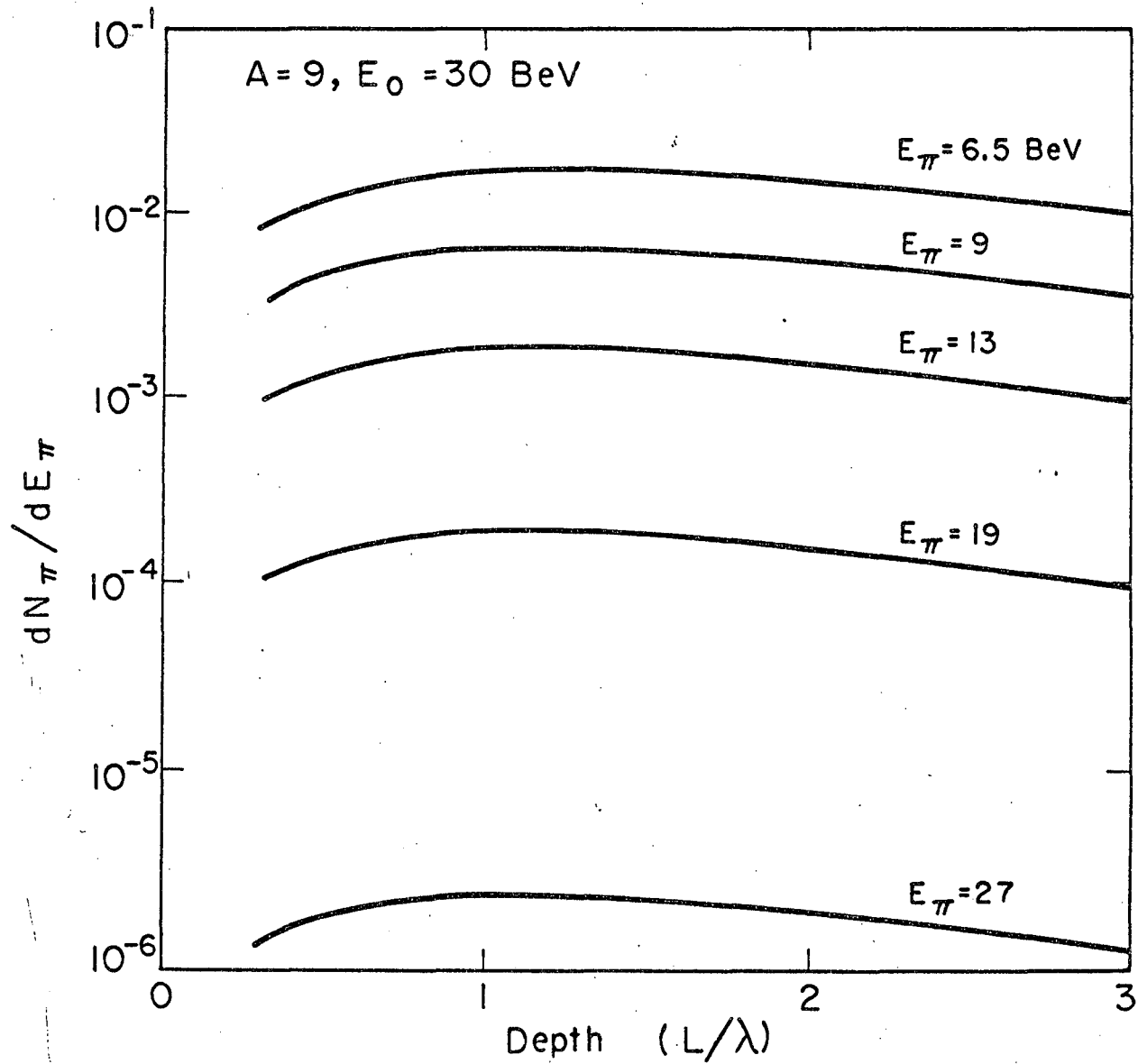


Fig. 13

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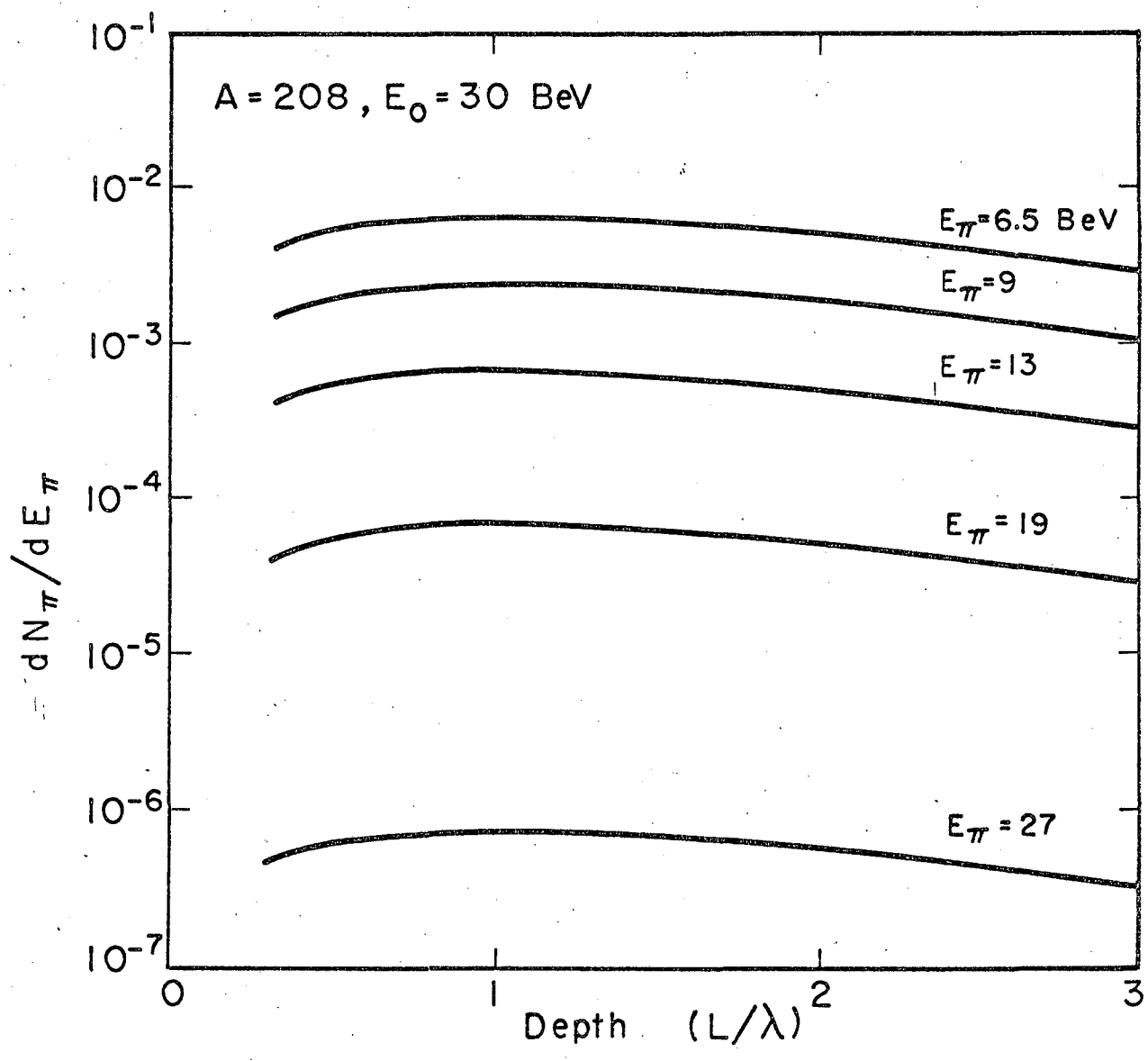


Fig. 14

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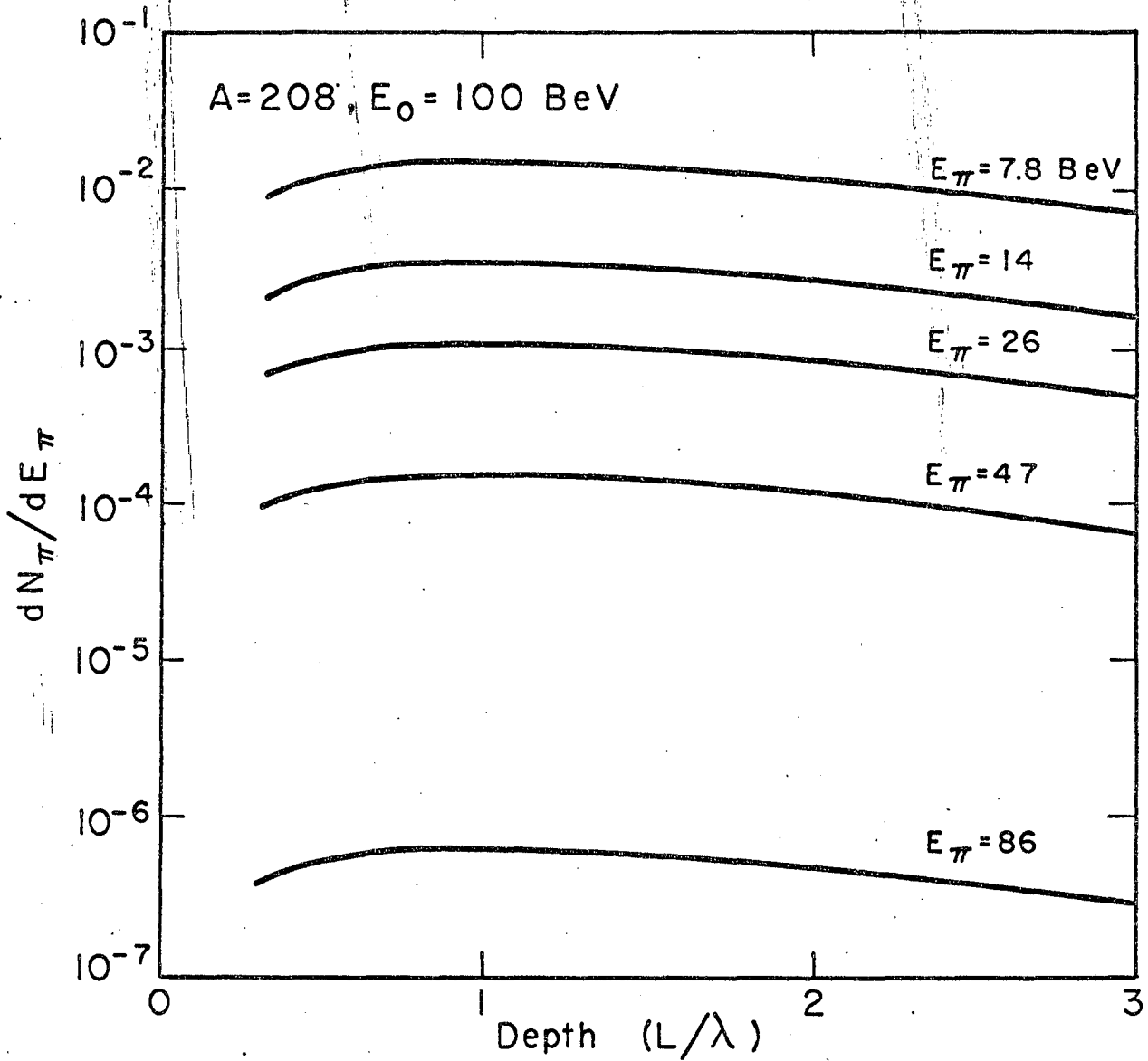


Fig. 15

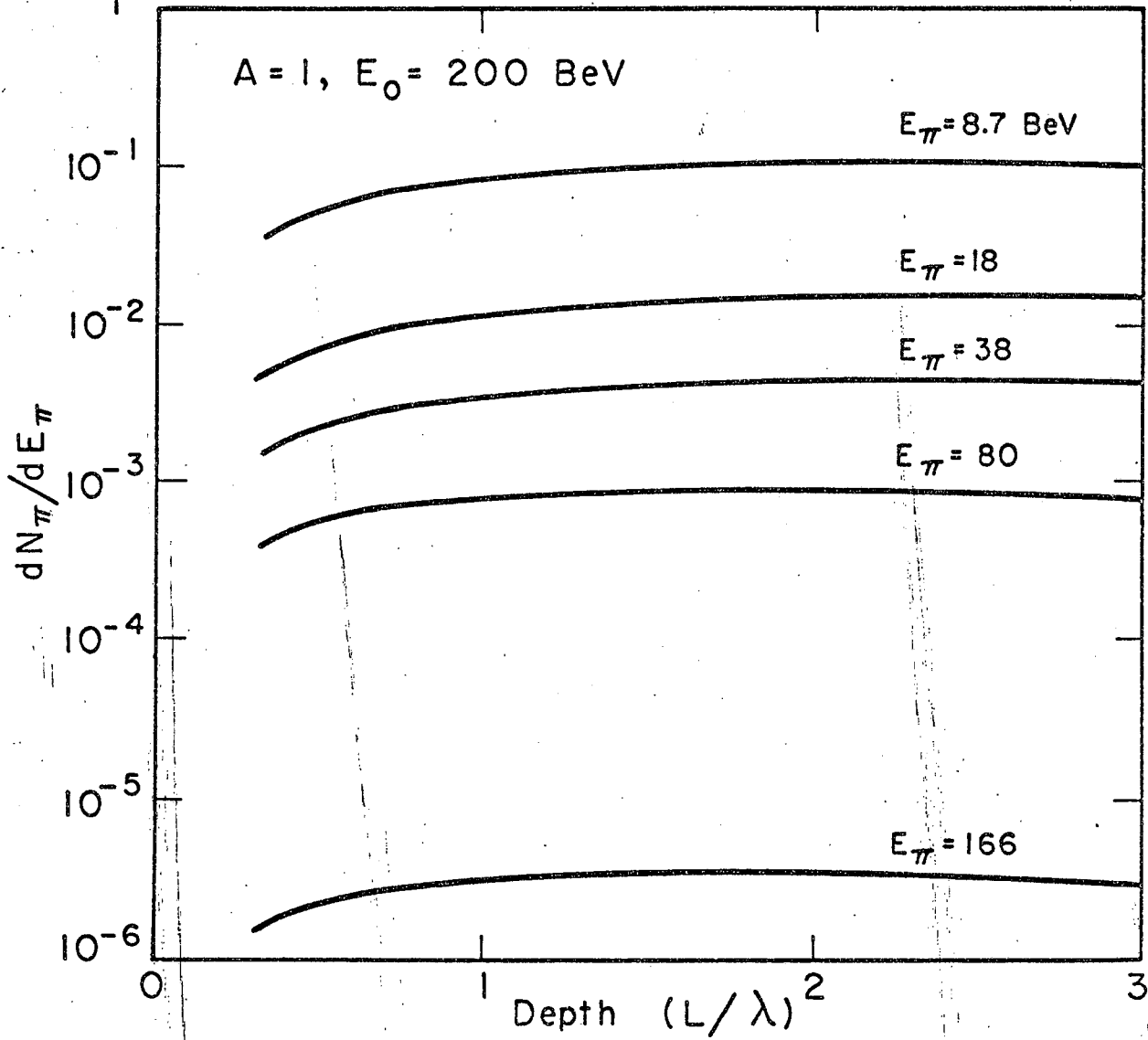


Fig. 16

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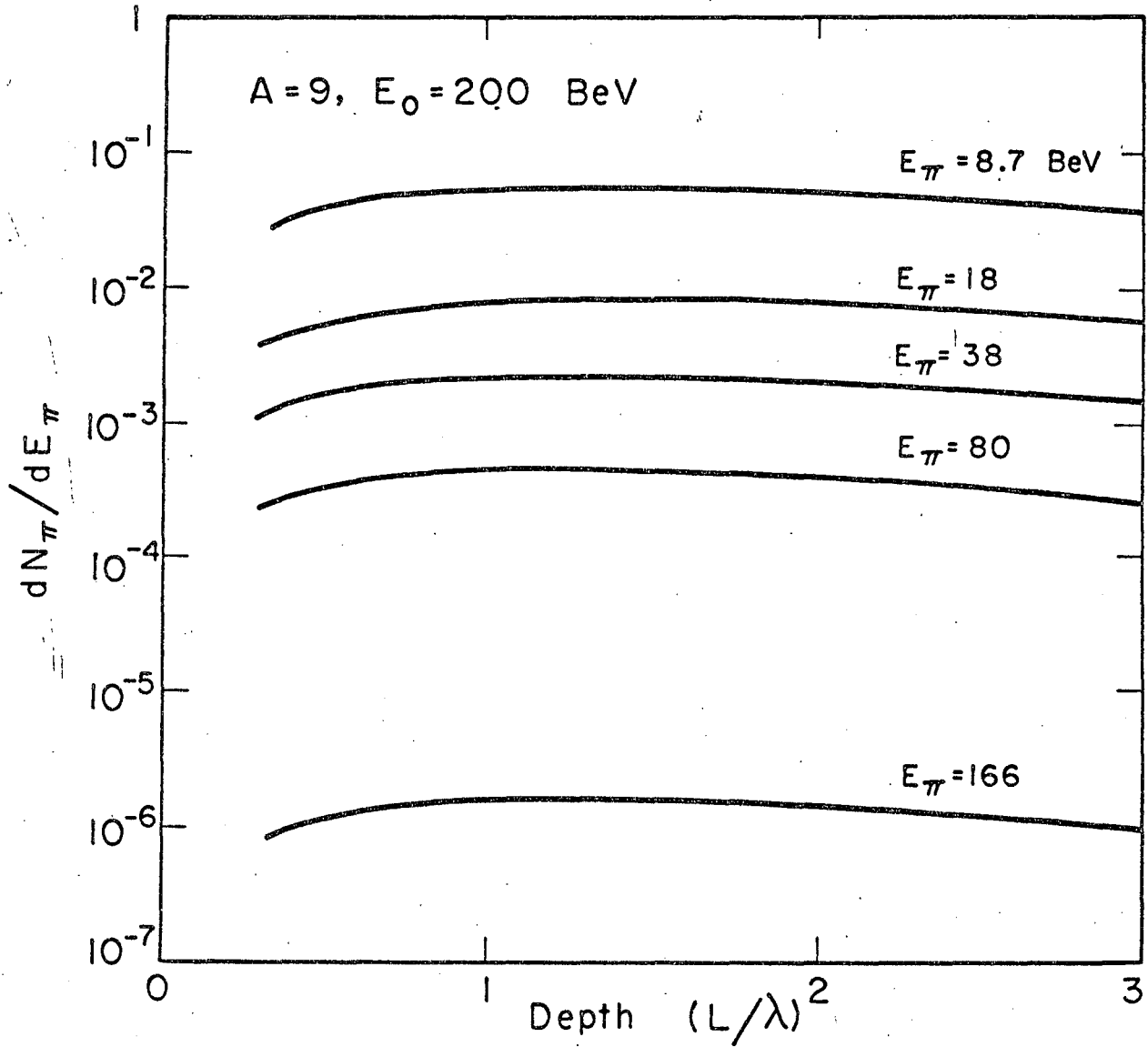


Fig. 17

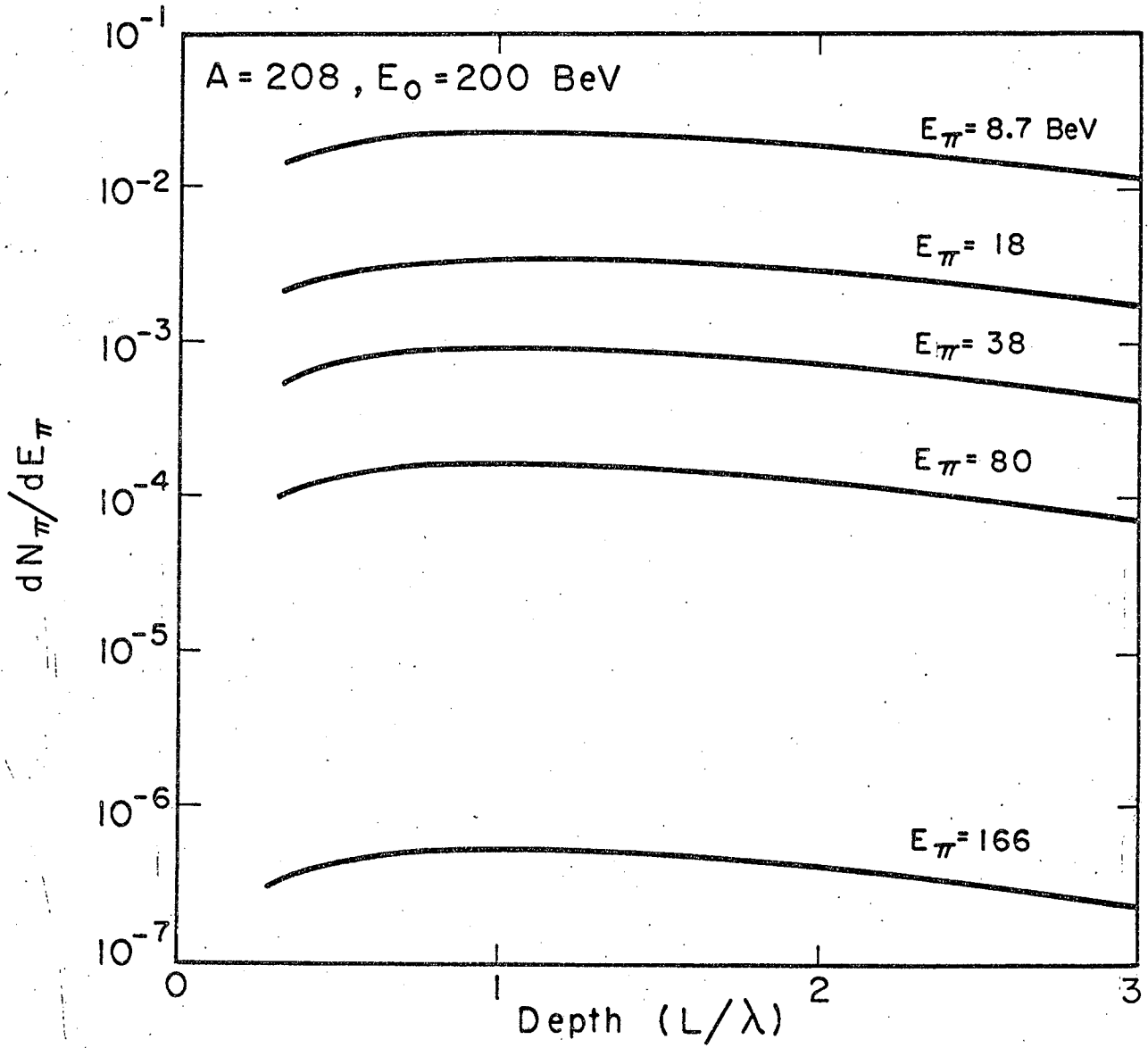


Fig. 18

MUB-7451

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