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by

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Abstract

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Given a singular scheme X over a field k, we consider the problem of resolving the singularities of X by an algebraic stack. When X is a toroidal embedding or is étale locally the quotient of a smooth scheme by a linearly reductive group scheme, we show that such "stacky resolutions" exist. Moroever, these resolutions are canonical and easily understandable in terms of the singularities of X.

We give three applications of our stacky resolution theorems: various generalizations of the Chevalley-Shephard-Todd Theorem, a Hodge decomposition in characteristic p, and a theory of toric Artin stacks extending the work of Borisov-Chen-Smith. While these applications are seemingly different, they are all related by the common theme of using stacky resolutions to study singular schemes.

Contents

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Chapter 1

Introduction

In [Hi], Hironaka proved his celebrated theorem that every scheme X in characteristic 0 has a resolution of singularities. A resolution of singularities of X, however, is usually not canonical, and oftentimes hard to control. In this thesis, we show that in certain cases one can introduce a canonical smooth stack which well-approximates X and serves as a replacement for the resolution of singularities. The first known instance of this is if X has quotient singularities: it is a folklore theorem (see for example [Vi, 2.9]), that if X is a scheme over a field k with quotient singularities prime to the characteristic of k (i.e. X is étale locally a quotient of a smooth scheme by a group whose order is prime to the characteristic of k), then there is a canonical smooth Deligne-Mumford stack $\mathfrak{X}$ with coarse space X such that the stacky structure of $\mathfrak{X}$ is supported on the singular locus of X. We refer to this folklore theorem as the stacky resolution theorem for quotient singularities.

We generalize this above stacky resolution theorem in two different directions. The first of these generalizations is most interesting in positive characteristic. We say that a scheme X has *linearly reductive singularities* if it is étale locally the quotient of a smooth scheme by a finite linearly reductive group scheme. In characteristic 0, this simply recovers the notion of quotient singularities. The focus of Chapter 2 is to prove the following stacky resolution theorem for linearly reductive singularities:

Theorem 2.1.10. *If k is a perfect field of characteristic p, and X is a k-scheme with linearly reductive singularities, then there exists a smooth tame Artin stack $\mathfrak{X}$ (as defined in* [AOV, Def 3.1]*) and a morphism $f : \mathfrak{X} \to X$ realizing X as the coarse space of $\mathfrak{X}$. Moreover, the base change of f to X^{sm} is an isomorphism.*

In Chapter 3 we present the second of our stacky resolution theorems, which holds for toroidal embeddings (which are not necessarily strict). We recall that a pair (U, X) consisting of a k-scheme X and an open subset $U \subset X$ is called a *toroidal embedding* if for every point of $x \in X$ there is an étale neighborhood X' of x, a toric variety Y_x, and an isomorphism $X' \to Y_x$ sending $U \times_X X'$ to the torus of Y_x. A toroidal embedding (U, X) is called *strict* if every irreducible component of $X - U$ is normal.

It follows from [Ka1, Thm 4.8] that toroidal embeddings are precisely fs log smooth log schemes over k endowed with the trivial log structure. Under this equivalence, the log scheme $(X, \mathcal{M}_X)$ corresponds to the (not necessarily strict) toroidal embedding (X^{triv}, X). We can now state the stacky resolution theorem for toroidal embeddings:

Theorem 3.3.2. *Let k be field and X be a toroidal embedding over k. Then there exists a smooth Artin stack $\mathfrak{X}$ over k and a morphism $f : \mathfrak{X} \to X$ over k which realizes X as the good moduli space of $\mathfrak{X}$ (in the sense of* [Al, Def 1.4]*). Moreover, the base change of f to the smooth locus of X is an isomorphism, and $\mathfrak{X}$ carries a natural log structure in the sense of* [Ka2, 1.2] *which makes it log smooth.*

The technique of studying a singular scheme through a canonical smooth stack sitting over it seems to be quite broadly applicable. We apply the two resolution theorems above to Invariant theory, Hodge theory, and toric geometry. These applications are the focuses of Chapters 4, 5, and 6 respectively.

1.1 Generalizations of the Chevalley-Shephard-Todd Theorem

We begin by recalling the Chevalley-Shephard-Todd theorem ([Bo, §5 Thm 4]). Suppose G is a finite group with order prime to the characteristic of a field k and suppose G acts faithfully on a finite-dimensional k-vector space V. We say an element g of G is a *pseudo-reflection* if the fixed subspace V^g is a hyperplane. The Chevalley-Shephard-Todd theorem then states

Theorem 1.1.1 ([Bo, §5 Thm 4])**.** *If $G \to \mathrm{Aut}_k(V)$ is a faithful representation of a finite group and the order of G is not divisible by the characteristic of k, then $k[V]^G$ is polynomial if and only if G is generated by pseudo-reflections.*

There is a strong connection between the Chevalley-Shephard-Todd theorem and the stacky resolution theorem for quotient singularities. When constructing the canonical smooth Deligne-Mumford stack, the local problem one is faced with is precisely the Chevalley-Shephard-Todd theorem. Therefore, the stacky resolution theorem for quotient singularities gives a global reformulation of the Chevalley-Shephard-Todd theorem.

In the same vein, the local manifestations of the two stacky resolution theorems we prove give two generalizations of the Chevalley-Shephard-Todd theorem. Given a faithful action of a finite linearly reductive group scheme G on a finite-dimensional k-vector space V, we define a subgroup scheme N to be a *pseudo-reflection* if V^N is a hyperplane. The local version of Theorem 2.1.10 is then:

Theorem 2.1.3. *If G is a finite linearly reductive group scheme with a faithful action on a finite-dimensional k-vector space V and k is algebraically closed, then $k[V]^G$ is polynomial if and only if G is generated by pseudo-reflections.*

We also prove a more technical verison of this theorem for fields which are not algebraically closed (see Theorem 2.1.6). Similarly, we show that the local version of Theorem 3.3.2 gives a generalization of the Chevalley-Shephard-Todd theorem to the case of arbitrary diagonalizable group schemes (see Theorem 4.1.2). We remark that this generalization to the case of diagonalizable group schemes recovers [We, Thm 5.6], where the result is shown for tori.

Although the following is not the main point of the thesis, it should nonetheless be mentioned that as a subtheme we hope to convince the reader that Chevalley-Shephard-Todd type theorems (at least within the setting of reductive group schemes) are roughly equivalent to proving stacky resolution theorems. The proof of the stacky resolution theorem for quotient singularities (and for linearly reductive singularities) rests on first proving the Chevalley-Shephard-Todd theorem to obtain a local version of the stacky resolution theorem. These local solutions are then "glued" (using canonicity of the stack) to obtain a global solution. However, the generalization of the Chevalley-Shephard-Todd theorem we obtain for diagonalizable group schemes is proved by *first* proving a stacky resolution theorem for toroidal embeddings and then noticing that the local manifestation of this theorem is a generalization of the Chevalley-Shephard-Todd theorem to the case of diagonalizable group schemes.

1.2 De Rham theory for schemes with linearly reductive singularities

In Chapter 5, we use the stacky resolution theorem for linearly reductive singularities to obtain a positive characteristic analogue of a theorem due to Steenbrink. In [St, Thm 1.12], Steenbrink shows that if k is a field of characteristic 0 and X is a proper k-scheme with isolated quotient singularities and smooth locus $j : X^{sm} \to X$, then the hypercohomology spectral sequence

$$E_1^{st} = \mathrm{H}^t(j_*\Omega^s_{X^{sm}}) \Rightarrow \mathrm{H}^n(j_*\Omega^\bullet_{X^{sm}})$$

of the Steenbrink complex $j_*\Omega^\bullet_{X^{sm}}$ degenerates. (Here $j_*\Omega^\bullet_{X^{sm}}$ denotes the complex obtained by applying the functor j_* term-by-term to $\Omega^\bullet_{X^{sm}}$, as opposed to the derived push-forward.) Moreover, Steenbrink shows that if $k = \mathbb{C}$, then $\mathrm{H}^n(j_*\Omega^\bullet_{X^{sm}})$ agrees with the singular cohomology $\mathrm{H}^n(X^{an}, \mathbb{C})$. Our characteristic p analogue of this theorem is:

Theorem 5.4.7. *Let k be a field of characteristic p and X be a k-scheme with isolated linearly reductive singularities. If X lifts mod p^2 and the dimension of X is at least 4, then the hypercohomology spectral sequence*

$$E_1^{st} = \mathrm{H}^t(j_*\Omega^s_{X^{sm}}) \Rightarrow \mathrm{H}^n(j_*\Omega^\bullet_{X^{sm}})$$

degenerates for $s + t < p$, where $j : X^{sm} \to X$ is the inclusion of the smooth locus.

The idea of the proof is as follows. We show that the method of Deligne-Illusie [DI] extends to tame Artin stacks; that is, if $\mathfrak{X}$ is a smooth proper tame Artin stack which lifts mod p^2, then the "Hodge-de Rham spectral sequence" for $\mathfrak{X}$ degenerates. Here, "Hodge-de Rham spectral sequence" is in quotes because it is not *a priori* clear what the differentials on an Artin stack should be. In Section 3 of Chapter 5, a definition is chosen and it is shown that this yields a resonable theory. We then make use of Theorem 2.1.10 to show that degeneracy of the Hodge-de Rham spectral sequence for a smooth proper tame Artin stack $\mathfrak{X}$ implies degeneracy of the hypercohomology spectral sequence of $j_*\Omega^\bullet_{X^{sm}}$, where X is the coarse space of $\mathfrak{X}$. We remark that Theorem 5.4.7 is a purely scheme-theoretic statement, but we are able to prove it by using Theorem 2.1.10 which allows us to pass through smooth stacks.

1.3 Toric Artin Stacks.

In [BCS], Borisov, Chen, and Smith develop a theory of toric Deligne-Mumford stacks by extending techniques developed in [Cox]. They introduce a notion of a stacky fan $\mathbf{\Sigma}$ which is essentially a fan Σ together with marked points along the rays of the fan. When the underlying fan of $\mathbf{\Sigma}$ is simplicial, they associate to $\mathbf{\Sigma}$ a smooth Deligne-Mumford stack $\mathfrak{X}(\mathbf{\Sigma})$ whose coarse space is $X(\Sigma)$. Moreover, $\mathfrak{X}(\mathbf{\Sigma})$ has a dense open stacky torus whose action on itself extends to an action on the stack. Toric Deligne-Mumford stacks are therefore the natural analogue of simplicial toric varieties within the class of smooth Deligne-Mumford stacks.

Smith asked whether the theory of toric Deligne-Mumford stacks can be extended to a theory of toric Artin stacks. In other words, what is the analogue of *arbitrary* toric varieties within the class of smooth algebraic stacks? There have so far been three main approaches to developing a theory of toric Deligne-Mumford stacks: one is a log geometric approach due to Iwanari [Iw]; another is the aforementioned theory of Borisov, Chen, and Smith using stacky fans; and the last is an approach taken by Fantechi, Mann, and Nironi [FMN] giving an intrinsic geometric description of toric Deligne-Mumford stacks. In Chapter 6, we generalize and unite the first two of these three approaches to produce a theory of toric Artin stacks.

Our generalization of Iwanari's approach rests on a variant of Theorem 3.3.2. We show in Chapter 6 that if X is a toric variety rather than just a toroidal embedding, then there are many log smooth log Artin stacks $\mathfrak{X}$ other than the canonical stack given by Theorem 3.3.2. All of these stacks have X as a good moduli space and they all have moduli interpretations in terms of log geometry (however, only the canonical stack is isomorphic to X over X^{sm}). Each of these Artin stacks has a dense open torus whose action on itself extends to an action on the stack. Thus, these stacks should all naturally be thought of as toric Artin stacks.

We also give a stacky fan "explanation" of this theory which both generalizes the approach to toric Deligne-Mumford stacks taken in [BCS] and unites it with Iwanari's approach. That is, we define a notion of generalized stacky fan and recast the above construction of toric Artin stacks in terms of these stacky fans. The key difference between the generalized stacky fans we introduce and the stacky fans of [BCS, p.193] is that we allow marked points which do not lie on the rays of the fan. It is this extra bit of leverage that allows one to capture big stabilizers which occur in the setting of non-simplicial toric varieties.

Chapter 2

Stacky Resolutions of Schemes with Linearly Reductive Singularities

2.1 Introduction

Given a field k and an action of a finite (abstract) group G on a k-vector space V, we obtain a linear action of G on the polynomial ring $k[V]$. A central theme in Invariant Theory is determining when certain nice properties of a ring with G-action are inherited by its invariants. In particular, it is natural to ask when $k[V]^G$ is polynomial. If G acts faithfully on V, we say $g \in G$ is a *pseudo-reflection* (with respect to the action of G on V) if V^g is a hyperplane. The classical Chevalley-Shephard-Todd Theorem states

Theorem 2.1.1 ([Bo, §5 Thm 4])**.** *If $G \to \mathrm{Aut}_k(V)$ is a faithful representation of a finite group and the order of G is not divisible by the characteristic of k, then $k[V]^G$ is polynomial if and only if G is generated by pseudo-reflections.*

In this chapter we generalize this theorem to the case of finite linearly reductive group schemes. To do so, we first need a notion of pseudo-reflection in this setting.

Definition 2.1.2. Let k be a field and V a finite-dimensional k-vector space with a faithful action of a finite linearly reductive group scheme G over $\operatorname{Spec} k$. We say that a subgroup scheme N of G is a *pseudo-reflection* if V^N has codimension 1 in V. We define the *subgroup scheme generated by pseudo-reflections* to be the intersection of the subgroup schemes which contain all of the pseudo-reflections of G. We say G is *generated by pseudo-reflections* if G is the subgroup scheme generated by pseudo-reflections.

Over algebraically closed fields, Theorem 2.1.1 generalizes to

Theorem 2.1.3. *Let k be an algebraically closed field and V a finite-dimensional k-vector space with a faithful action of a finite linearly reductive group scheme G over* $\operatorname{Spec} k$*. Then G is generated by pseudo-reflections if and only if $k[V]^G$ is polynomial.*

A more technical version of this theorem holds over fields which are not algebraically closed; however, the "only if" direction does not hold for finite linearly reductive group schemes in general (see Example 2.2.3). We instead prove the "only if" direction for the smaller class of stable group schemes, which we now define (see Proposition 2.2.1 for examples). Over an algebraically closed field, the class of stable group schemes coincides with that of finite linearly reductive group schemes. Recall from [AOV, Def 2.9] that G is called *well-split* if it is isomorphic to a semi-direct product $\Delta \rtimes Q$, where Δ is a finite diagonalizable group scheme and Q is a finite constant tame group scheme; here, tame means that the degree is prime to the characteristic.

Definition 2.1.4. A group scheme G over a field k is called *stable* if the following two conditions hold:

(a) for all finite field extensions K/k, every subgroup scheme of G_K descends to a subgroup scheme of G

(b) there exists a finite Galois extension K/k such that G_K is well-split.

Remark 2.1.5. If G is a finite linearly reductive group scheme over a perfect field k, then [AOV, Lemma 2.11] shows that condition (b) above is automatically satisfied.

Theorem 2.1.3 is then a special case of the following generalization of the Chevalley-Shephard-Todd theorem. This is the first main result of this chapter.

Theorem 2.1.6. *Let k be a field and V a finite-dimensional k-vector space with a faithful action of a finite linearly reductive group scheme G over* $\operatorname{Spec} k$. *If G is generated by pseudo-reflections, then $k[V]^G$ is polynomial. The converse holds if G is stable.*

We also prove a version of this theorem for an action of a finite linearly reductive group scheme on a smooth scheme.

Definition 2.1.7. Let U be a smooth affine scheme over $\operatorname{Spec} k$ with a faithful action of a finite linearly reductive group scheme G which fixes a field-valued point $x \in U(K)$. The cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ at x is therefore a G_K-representation. We say a subgroup scheme N of G is a *pseudo-reflection at x* if N_K is a pseudo-reflection with respect to this representation. We define what it means for G to be generated by pseudo-reflections at x in the same manner as in Definition 2.1.2.

Theorem 2.1.6 then has the following corollary whose proof is given in Section 2.3.

Corollary 2.1.8. *Let k be a field, let U be a smooth affine k-scheme with a faithful action by a finite linearly reductive group scheme G over* $\operatorname{Spec} k$. *Let $x \in U(K)$, where K/k is a finite separable field extension, and suppose x is fixed by G. If G is generated by pseudo-reflections at x, then U/G is smooth at the image of x. The converse holds if G is stable.*

The second main result of this chapter is

Theorem 2.1.9. *Let k be a field and let U be a smooth affine k-scheme with a faithful action by a stable group scheme G over* $\operatorname{Spec} k$. *Suppose K/k is a finite separable field extension and G fixes a point $x \in U(K)$. Let $M = U/G$, let M^0 be the smooth locus of M, and let $U^0 = U \times_M M^0$. If G has no pseudo-reflections at x, then after possibly shrinking M to a smaller Zariski neighborhood of the image of x, we have that U^0 is a G-torsor over M^0.*

We remark that in the classical case, Theorem 2.1.9 follows directly from Corollary 2.1.8 and the purity of the branch locus theorem [SGA1, X.3.1]. For us, however, a little more work is needed since G is not necessarily étale.

As an application of Theorem 2.1.9, we generalize the well-known result (see for example [Vi, 2.9] or [FMN, Rmk 4.9]) that schemes with quotient singularities prime to the characteristic are coarse spaces of smooth Deligne-Mumford stacks. We say a scheme has *linearly reductive singularities* if it is étale locally the quotient of a smooth scheme by a finite linearly reductive group scheme. We show that every such scheme M is the coarse space of a smooth tame Artin stack (in the sense of [AOV, Def 3.1]) whose stacky structure is supported at the singular locus of M:

Theorem 2.1.10. *Let k be a perfect field and M a k-scheme with linearly reductive singularities. Then it is the coarse space of a smooth tame stack $\mathfrak{X}$ over k such that f^0 in the diagram*

$$\begin{array}{ccc} \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \\ {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle f} \\ M^0 & \xrightarrow[j]{} & M \end{array}$$

is an isomorphism, where j is the inclusion of the smooth locus of M and $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$.

This chapter is organized as follows. In Section 2.2, we prove the "if" direction of Theorem 2.1.6 and reduce the proof of the "only if" direction to the special case of Theorem 2.1.9 in which $U = \mathbb{V}^\vee(V)$ for some k-vector space V with G-action (see the Notation section below). This special case is proved in Section 2.3. The key input for the proof is a result of Iwanari [Iw, Thm 3.3] which we reinterpret in the language of pseudo-reflections. We finish the section by proving Corollary 2.1.8. In Section 2.4, we use Corollary 2.1.8 to complete the proof of Theorem 2.1.9. In Section 2.5, we prove Theorem 2.1.10.

Notation. Throughout this chapter, k is a field and $S = \operatorname{Spec} k$. If V is a k-vector space with an action of a group scheme G, then we denote by $\mathbb{V}^\vee(V)$, or simply $\mathbb{V}^\vee$ if V is understood, the scheme $\operatorname{Spec} k[V]$ whose G-action is given by the dual representation on functor points. Said another way, if $G = \operatorname{Spec} A$ is affine and its action on V is given by the co-action map $\sigma : V \longrightarrow V \otimes_k A$, then the co-action map $k[V] \longrightarrow k[V] \otimes_k A$ defining the G-action on $\mathbb{V}^\vee$ is given by $\sum a_i v_i \mapsto \sum a_i \sigma(v_i)$.

All Artin stacks in this chapter are assumed to have finite diagonal so that, by Keel-Mori [KM], they have coarse spaces. Given a scheme U with an action of a finite flat group scheme G, we denote by U/G the coarse space of the stack $[U/G]$.

If R is a ring and $\mathcal{I}$ an ideal of R, then we denote by $V(\mathcal{I})$ the closed subscheme of $\operatorname{Spec} R$ defined by $\mathcal{I}$.

2.2 Linear Actions on Polynomial Rings

Our goal in this section is to prove the "if" direction of Theorem 2.1.6 and show how the "only if" direction follows from the special case of Theorem 2.1.9 in which $U = \mathbb{V}^\vee$. We begin with examples of stable group schemes and with some basic results about the subgroup scheme generated by pseudo-reflections.

Proposition 2.2.1. *Let G be a finite group scheme over S. Consider the following conditions:*

1. *G is diagonalizable.*
2. *G is a constant group scheme.*
3. *k is perfect, the identity component Δ of G is diagonalizable, and G/Δ is constant.*

If any of the above conditions hold, then G is stable.

Proof. It is clear that finite diagonalizable group schemes and finite constant group schemes are stable, so we consider the last case. Let $Q = G/\Delta$. Since k is perfect, the connected-étale sequence

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is functorially split (see [Ta, 3.7 (IV)]). Let K/k be a finite extension and let P be a subgroup scheme of G_K. Letting $\Delta' = P \cap \Delta_K$ and $Q' = P/\Delta'$, we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_K & \longrightarrow & G_K & \longrightarrow & Q_K & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle\varphi} & & \uparrow & & \uparrow{\scriptstyle\psi} & & \\ 1 & \longrightarrow & \Delta' & \longrightarrow & P & \longrightarrow & Q' & \longrightarrow & 1 \end{array}$$

with exact rows. Since Δ is connected and has a k-point, [EGA4, 4.5.14] shows that Δ is geometrically connected. In particular, Δ_K is the connected component of the identity of G_K, and so Δ' is the connected component of the identity of P. Therefore, the bottom row of the above diagram is the connected-étale sequence of P, and so

$$P = \Delta' \rtimes Q'$$

as k is perfect. Since Q' is a constant group scheme, it clearly descends to a subgroup scheme of Q over k. Similarly, since Δ' is diagonalizable, it is of the form $\operatorname{Spec} K[A]$ for some finite abelian group A, and so it descends to a subgroup scheme of Δ. To show that P descends, note that its underlying scheme is $\Delta' \times_K Q'$ and its group structure is given by an action of Q' on Δ'. This action of Q' is equivalent to an action of Q' on A. We can therefore define a group scheme structure on $\operatorname{Spec} k[A] \times_k Q'$ using the same action of Q' on A. This group scheme P_0 over k then pulls back to P over K.

Lastly, we must show that P_0 is a subgroup scheme of G. Let $*$ denote the action of Q_K (resp. Q')

on Δ_K (resp. Δ'). Since the splitting of the connected-étale sequence of a finite group scheme over a perfect field is functorial, we see that for all $q' \in Q'$ and local sections δ' of Δ',

$$\psi(q') * \varphi(\delta') = \varphi(q' * \delta').$$

We therefore obtain a morphism from P_0 to G whose pullback to K is the morphism from P to G_K. □

Lemma 2.2.2. *Let V be a finite-dimensional k-vector space with a faithful action of a stable group scheme G over S, and let H be the subgroup scheme generated by pseudo-reflections. If K/k is a finite field extension, then a subgroup scheme of G_K is a pseudo-reflection if and only if it descends to a pseudo-reflection over k. Furthermore, H_K is the subgroup scheme of G_K generated by pseudo-reflections.*

Proof. Note that if N is a pseudo-reflection of G, then N_K is a pseudo-reflection of G_K, as

$$(V_K)^{N_K} = (V^N)_K.$$

Since G is stable, this proves the first claim. The second claim follows from the fact that if P' and P'' are subgroup schemes of G, then P'_K contains P''_K if and only if P' contains P''. □

We remark that even in characteristic zero, Lemma 2.2.2 is false for general finite linearly reductive group schemes G, as the following example shows. Note that this example also shows that the "only if" direction of Theorem 2.1.6 and of Corollary 2.1.8 is false for general finite linearly reductive group schemes.

Example 2.2.3. Let k be a field contained in $\mathbf{R}$ or let $k = \mathbf{F}_p$ for p congruent to 3 mod 4. Let $K = k(i)$, where $i^2 = -1$, and let G be the locally constant group scheme over $\operatorname{Spec} k$ whose pullback to $\operatorname{Spec} K$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$ with the Galois action that switches the two $\mathbb{Z}/2$ factors. Let g_1 and g_2 be the generators of the two $\mathbb{Z}/2$ factors and consider the action

$$\rho : G_K \longrightarrow \operatorname{Aut}_K(K^2)$$

on the K-vector space K^2 given by

$$\rho(g_1) : (a, b) \mapsto (-bi, ai)$$

$$\rho(g_2) : (a, b) \mapsto (bi, -ai).$$

Then ρ is Galois-equivariant and hence comes from an action of G on k^2. Note that $\mathbb{Z}/2 \times 1$ and $1 \times \mathbb{Z}/2$ are both pseudo-reflections of G_K, as the subspaces which they fix are $K \cdot (1, i)$ and $K \cdot (1, -i)$, respectively. Since G_K is not a pseudo-reflection, it follows that there are no Galois-invariant pseudo-reflections of G_K, and hence, the subgroup scheme generated by pseudo-reflections of G is trivial; the subgroup scheme generated by pseudo-reflections of G_K, however, is G_K.

Corollary 2.2.4. *If V is a finite-dimensional k-vector space with a faithful action of a stable group scheme G over S, then the subgroup scheme generated by pseudo-reflections is normal in G.*

Proof. We denote by H the subgroup scheme generated by pseudo-reflections. Let T be an S-scheme and let $g \in G(T)$. We must show the subgroup schemes H_T and gH_Tg^{-1} of G are equal. To do so, it suffices to check this on stalks and so we can assume $T = \operatorname{Spec} R$, where R is strictly Henselian. By [AOV, Lemma 2.17], we need only show that these two group schemes are equal over the closed fiber of T, so we can further assume that $R = K$ is a field. Since G is finite over S, the residue fields of G are finite extensions of k. We can therefore assume that K/k is a finite field extension.

By Lemma 2.2.2, we know that H_K is the subgroup scheme of G_K generated by pseudo-reflections. Note that if N' is a pseudo-reflection of G_K, then $gN'g^{-1}$ is as well since

$$V_K^{gN'g^{-1}} = g(V_K^{N'}).$$

As a result, $gH_Kg^{-1} = H_K$, which completes the proof. □

Lemma 2.2.5. *Given a finite-dimensional k-vector space V with a faithful action of a finite linearly reductive group scheme G over S, let $\{N_i\}$ denote the set of pseudo-reflections of G and let H be the subgroup scheme generated by pseudo-reflections. Then*

$$k[V]^H = \bigcap_i k[V]^{N_i}.$$

Proof. Let $R = \bigcap_i k[V]^{N_i}$. Consider the functor

$$F : (k\text{-}alg) \longrightarrow (Groups)$$

$$A \longmapsto \{g \in G(A) \mid g(m) = m \text{ for all } m \in R \otimes_k A\}.$$

Since each $k[V]^{N_i}$ is finitely-generated, we see that R is as well. Let $r_i \in k[V]$ be generators for R. We see then that F is the intersection of the stabilizers G_{r_j}, and so is represented by a closed subgroup scheme of G. Since F contains every pseudo-reflection, we see $H \subset F$. We therefore have the containments

$$R \subset k[V]^F \subset k[V]^H \subset \bigcap_i k[V]^{N_i}$$

from which the lemma follows. □

If N is any subgroup scheme of G, it is linearly reductive by [AOV, Prop 2.7]. It follows that

$$V \simeq V^N \oplus V/V^N \tag{2.1}$$

as N-representations. If N is a pseudo-reflection, then $\dim_k V/V^N = 1$. Let v be a generator of the 1-dimensional subspace V/V^N and let $\sigma : V \to V \otimes_k B$ be the coaction map, where $N = \operatorname{Spec} B$. Then via the isomorphism (2.1), σ is given by

$$V^N \oplus V/V^N \longrightarrow (V^N \otimes_k B) \oplus (V/V^N \otimes_k B)$$

$$(w, w') \longmapsto (w \otimes 1, w' \otimes b)$$

for some $b \in B$. It follows that there is a k-linear map $h : V \to B$ such that for all $w \in V$,

$$\sigma(w) - (w \otimes 1) = v \otimes h(w).$$

If we continue to denote by σ the induced coaction map $k[V] \longrightarrow k[V] \otimes_k B$, we see that h extends to a $k[V]^N$-module homomorphism $k[V] \longrightarrow k[V] \otimes_k B$, which we continue to denote by h, such that for all $f \in k[V]$,

$$\sigma(f) - (f \otimes 1) = (v \otimes 1) \cdot h(f).$$

We are now ready to prove the "if" direction of Theorem 2.1.6. Our proof is only a slight variant of the proof of the classical Chevalley-Shephard-Todd Theorem presented in [Sm].

Proof of "if" direction of Theorem 2.1.6. By Lemma 2.2.5, we know that the intersection R of the $k[V]^N$ is $k[V]^G$, where N runs through the pseudo-reflections of G. By the proposition on page 225 of [Sm], to show R is polynomial, we need only show that $k[V]$ is a free R-module. By graded Nakayama, the projective dimension of $k[V]$ is the smallest integer i such that $\operatorname{Tor}^R_{i+1}(k, k[V]) = 0$, where k is viewed as an R-module via the augmentation map

$$\epsilon : k[V]^G \to k[V] \to k$$

sending all positively graded elements to 0. We must therefore show $\operatorname{Tor}^R_1(k, k[V]) = 0$.

Tensoring the short exact sequence defined by ϵ with $k[V]$, we obtain a long exact sequence

$$0 \longrightarrow \operatorname{Tor}^R_1(k, k[V]) \longrightarrow \ker\epsilon \otimes_R k[V] \xrightarrow{\phi} R \otimes_R k[V] \xrightarrow{\epsilon\otimes 1} k \otimes_R k[V] \longrightarrow 0.$$

To show $\operatorname{Tor}^R_1(k, k[V]) = 0$, we must prove that ϕ is injective. We in fact show

$$\phi \otimes 1 : \ker\epsilon \otimes_R k[V] \otimes_k C \longrightarrow k[V] \otimes_k C$$

is injective for all finite-dimensional k-algebras C. If this is not the case, then the set

$$\{\xi \mid C \text{ is a finite-dimensional } k\text{-algebra}, 0 \neq \xi \in \ker\epsilon \otimes_R k[V] \otimes_k C, (\phi \otimes 1)(\xi) = 0\}$$

is non-empty and we can choose an element ξ of minimal degree, where $\ker\epsilon$ is given its natural grading as a submodule of $k[V]$ and the elements of C are defined to be of degree 0. We begin by showing $\xi \in \ker\epsilon \otimes_R R \otimes_k C$. That is, we show ξ is fixed by all pseudo-reflections.

Let $N = \operatorname{Spec} B$ be a pseudo-reflection. Let $\sigma : k[V] \longrightarrow k[V] \otimes B$ be the coaction map. As explained above, we get a $k[V]^N$-module homomorphism $h : k[V] \longrightarrow k[V] \otimes B$. Note that this morphism has degree -1. Since

$$(1 \otimes \sigma \otimes 1)(\xi) - \xi \otimes 1 = (1 \otimes h \otimes 1)(\xi) \cdot (1 \otimes v \otimes 1 \otimes 1),$$

the commutativity of

$$\begin{array}{ccc}
\ker\epsilon \otimes k[V] \otimes B \otimes C & \xrightarrow{\phi\otimes 1\otimes 1} & k[V] \otimes B \otimes C \\
\Big\uparrow{\scriptstyle 1\otimes\sigma\otimes 1} & & {\scriptstyle \sigma\otimes 1}\Big\uparrow \\
\ker\epsilon \otimes k[V] \otimes C & \xrightarrow{\phi\otimes 1} & k[V] \otimes C
\end{array}$$

implies

$$(\phi \otimes 1 \otimes 1)(1 \otimes h \otimes 1)(\xi) \cdot (v \otimes 1 \otimes 1) = 0.$$

It follows that $(1 \otimes h \otimes 1)(\xi)$ is killed by $\phi \otimes 1 \otimes 1$. Since h has degree -1, our assumption on ξ shows that $(1 \otimes h \otimes 1)(\xi) = 0$. We therefore have $(1 \otimes \sigma \otimes 1)(\xi) = \xi \otimes 1$, which proves that ξ is N-invariant.

Since G is linearly reductive, we have a section of the inclusion $k[V]^G \hookrightarrow k[V]$. We therefore, also obtain a section s of the inclusion $j : R \hookrightarrow k[V]$. Let $\psi : \ker \epsilon \otimes_R R \longrightarrow R$ be the canonical map, and consider the diagram

$$\begin{array}{ccc} \ker \epsilon \otimes k[V] \otimes C & \xrightarrow{\phi \otimes 1} & k[V] \otimes C \\ {\scriptstyle 1 \otimes j \otimes 1} \uparrow \downarrow {\scriptstyle 1 \otimes s \otimes 1} & & {\scriptstyle j \otimes 1} \uparrow \downarrow {\scriptstyle s \otimes 1} \\ \ker \epsilon \otimes R \otimes C & \xrightarrow{\psi \otimes 1} & R \otimes C \end{array}$$

We see that

$$(j \otimes 1)(\psi \otimes 1)(1 \otimes s \otimes 1)(\xi) = (\phi \otimes 1)(1 \otimes j \otimes 1)(1 \otimes s \otimes 1)(\xi) = (\phi \otimes 1)(\xi) = 0.$$

But $j \otimes 1$ and $\psi \otimes 1$ are injective, so $(1 \otimes s \otimes 1)(\xi) = 0$. Since $\xi \in \ker \epsilon \otimes_R R \otimes_k C$, it follows that $\xi = 0$, which is a contradiction. □

Now that we have proved the "if" direction of Theorem 2.1.6, we work toward reducing the "only if" direction to the special case of Theorem 2.1.9 where $U = \mathbb{V}^\vee$. The main step in this reduction is showing that if G acts faithfully on V, and H denotes the subgroup scheme generated by pseudo-reflections, then the action of G/H on $\mathbb{V}^\vee/H$ has no pseudo-reflections at the origin. In the classical case, the proof of this statement relies on the fact that G has no pseudo-reflections if and only if $\mathbb{V}^\vee \to \mathbb{V}^\vee/G$ is étale in codimension one. In our case, however, this relation between pseudo-reflections and ramification no longer holds. For example, if k has characteristic 2 and $G = \mu_2$ acts on $V = kx \oplus ky$ by sending x to ζx and y to ζy, then $\mathbb{V}^\vee \to \mathbb{V}^\vee/G$ is ramified at every height 1 prime, but G has no pseudo-reflections.

Nonetheless, we introduce the following functor which, for our purposes, should be thought of as an analogue of the inertia group. If $v \in V$ and $\mathfrak{P}$ is the ideal of $k[V]$ generated by v, then let

$$I_{\mathfrak{P}} : (k\text{-}alg) \longrightarrow (Groups)$$

$$\begin{array}{rcl} R & \longmapsto & \{g \in G(R) \mid \text{for all } R\text{-algebras } R' \text{ and all } f \in (V \otimes R')^\vee \\ & & \text{such that } f(v \otimes 1) = 0, \text{ we have } (g(f))(v \otimes 1) = 0\}. \end{array}$$

Note that in the above example, $I_{\mathfrak{P}} = 1$ for all homomgeneous height one primes $\mathfrak{P}$. So our "inertia groups" do not capture information about ramification, but they are related to pseudo-reflections, as the following lemma shows.

Lemma 2.2.6. *If G acts faithfully on V and stabilizes the closed subscheme $V(\mathfrak{P})$ of $\mathbb{V}^\vee$ defined by $\mathfrak{P} = (v)$ for some $v \in V$, then $I_{\mathfrak{P}} = 1$ if and only if G has no pseudo-reflections acting trivially on $\mathfrak{P}$.*

Proof. Since G stabilizes $V(\mathfrak{P})$, we have a morphism $G \to \mathcal{A}ut(V(\mathfrak{P}))$ of sheaves of groups. We see then that $I_{\mathfrak{P}}$ is a closed subgroup scheme of G, as it is the kernel of the above morphism. If N is any pseudo-reflection of G which acts trivially on $V(\mathfrak{P})$, then it is clearly contained in $I_{\mathfrak{P}}$. Conversely, since $I_{\mathfrak{P}}$ is a closed subgroup scheme of G which acts trivially on $V(\mathfrak{P})$, it is a pseudo-reflection. This completes the proof. □

We now prove a general result concerning faithful actions by group schemes.

Lemma 2.2.7. *Let G be a finite group scheme which acts faithfully on an affine scheme U. If H is a normal subgroup scheme of G, then the action of G/H on U/H is faithful.*

Proof. Let $\mathfrak{X} = [U/H]$ and let $\pi : U \to U/H$ be the natural map. We must show that if G' is a subgroup scheme of G such that G'/H acts trivially on U/H, then $G' = H$. Replacing G by G', we can assume $G' = G$.

Since G acts faithfully on U, there is a non-empty open substack of $\mathfrak{X}$ which is isomorphic to its coarse space. That is, we have a non-empty open subscheme V of U/H over which π is an H-torsor. Let $P = V \times_{U/H} U$. Since G acts on P over V, we obtain a morphism

$$s : G \longrightarrow \mathcal{A}ut(P) = H.$$

Note that s is a section of the closed immersion $H \to G$, so $H = G$. □

With the above results in place, we are ready to prove that after quotienting by the subgroup scheme generated by pseudo-reflections, there are no pseudo-reflections in the resulting action. We then use this result to prove the "only if" direction of Theorem 2.1.6, assuming the special case of Theorem 2.1.9 in which $U = \mathbb{V}^\vee$.

Proposition 2.2.8. *Let G be a finite linearly reductive group scheme over S with a faithful action on a finite-dimensional k-vector space V. Let $U = \mathbb{V}^\vee$ and H be the subgroup scheme of G generated by pseudo-reflections. Then the induced action of G/H on $U/H \simeq \mathbb{A}^n_k$ has no pseudo-reflections at the origin.*

Proof. By the "if" direction of Theorem 2.1.6, we have $k[V]^H = k[W]$ for some subvector space W of $k[V]$. The proof of [Ne, Prop 6.19] shows that the degrees of the homogeneous generators of $k[V]^H$ are determined. As a result, the action of G/H on $k[W]$ is linear. Lemma 2.2.7 further tells us that this action is faithful.

Suppose N/H is a pseudo-reflection at the origin of the G/H-action on U/H. Then there is some $w \in W$ such that N/H acts trivially on the closed subscheme $V(\mathfrak{p})$ of U/H defined by $\mathfrak{p} = (w)$.

Let $H = \operatorname{Spec} A$, $N = \operatorname{Spec} B$, and $N/H = \operatorname{Spec} C$. We have a commutative diagram

$$\begin{array}{ccc}
 & & k[V] \otimes_k A \\
 & \overset{\sigma_A}{\nearrow} & \uparrow \\
k[V] & \xrightarrow[\sigma_B]{} & k[V] \otimes_k B \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\
k[V]^H & \xrightarrow[\sigma_C]{} & k[V]^H \otimes_k C
\end{array}$$

where the σ denote the corresponding coaction maps. Since H is linearly reductive, there is a section s of i. Since N/H is a pseudo-reflection, we see that $w \otimes 1$ divides $\sigma_C(f) - f \otimes 1$ for all $f \in k[V]^H$. In particular, this holds when $f = s(v)$ for any $v \in V$. There is some $v \in V$ such that $\sigma_B(v) - v \otimes 1$ is non-zero, as the action of G on V is faithful. Since $w \otimes 1$ divides $\sigma_C(s(v)) - s(v) \otimes 1$, applying j shows that $w \otimes 1$ divides $\sigma_B(v) - v \otimes 1$. Since $\sigma_B(v) - v \otimes 1$ lies in $V \otimes B$, it follows that w has no higher degree terms; that is, $w \in V$.

Let $\mathfrak{P}$ be the height one prime of $k[V]$ generated by w. Since $j\sigma_C(w) = \sigma_B(i(w))$, we see that N stabilizes the closed subscheme $V(\mathfrak{P})$ of U. Note that the diagram

$$\begin{array}{ccc}
V(\mathfrak{P}) & \longrightarrow & \mathbb{V}^\vee \\
\downarrow & & \downarrow \\
V(\mathfrak{p}) & \longrightarrow & \mathbb{V}^\vee/H
\end{array}$$

is cartesian, as $\mathfrak{P} = i(\mathfrak{p})k[V]$. Since H is linearly reductive, it follows that

$$V(\mathfrak{P})/H = V(\mathfrak{p}).$$

Let $I_{\mathfrak{P}}$ denote the "inertia group" of the N-action on $\mathbb{V}^\vee$. Then, by Lemma 2.2.6, to complete the proof we must show that $I_{\mathfrak{P}}$ is non-trivial. Since $I_{\mathfrak{P}}$ is the kernel of

$$N \to \mathcal{A}ut(V(\mathfrak{P})),$$

if $I_{\mathfrak{P}} = 1$, then N acts faithfully on $V(\mathfrak{P})$. Since the action of N/H on $V(\mathfrak{p})$ is trivial, Lemma 2.2.7 shows that $N = H$, and so N/H is not a pseudo-reflection. □

Proof of "only if" direction of Theorem 2.1.6. Let H be the subgroup scheme generated by pseudo-reflections. By the "if" direction, $k[V]^H$ is polynomial and as explained in the proof of Proposition 2.2.8, the G/H-action on $k[V]^H$ is linear. Since G/H acts on U/H without pseudo-reflections at the origin by Proposition 2.2.8, and since $M = U/G$ is smooth by assumption, Theorem 2.1.9 implies that U/H is a G/H-torsor over U/G after potentially shrinking U/G. Since the origin of U/H is a fixed point, we conclude that $G = H$. □

2.3 Theorem 2.1.9 for Linear Actions on Polynomial Rings

In Section 2.2, we reduced the proof of the "only if" direction of Theorem 2.1.6 to

Proposition 2.3.1. *Let G be a stable group scheme over S which acts faithfully on a finite-dimensional k-vector space V. Then Theorem 2.1.9 holds when $U = \mathbb{V}^\vee$ and x is the origin.*

The proof of this proposition is given in two steps. We handle the case when G is diagonalizable in Subsection 2.3.1 and then handle the general case in Subsection 2.3.2 by making use of the diagonalizable case.

2.3.1 Reinterpreting a Result of Iwanari

The key to proving Proposition 2.3.1 for diagonalizable G is provided by Theorem 3.3 and Proposition 3.4 of [Iw] after we reinterpret them in the language of pseudo-reflections. We refer the reader to [Iw, p.4-6] for the basic definitions concerning monoids. We recall the following definition given in [Iw, Def 2.5].

Definition 2.3.2. An injective morphism $i : P \to F$ from a simplicially toric sharp monoid to a free monoid is called a *minimal free resolution* if i is close and if for all injective close morphisms $i' : P \to F'$ to a free monoid F' of the same rank as F, there is a unique morphism $j : F \to F'$ such that $i' = ji$.

Given a faithful action of a finite diagonalizable group scheme Δ over S on a k-vector space V of dimension n, we can decompose V as a direct sum of 1-dimensional Δ-representations. Therefore, after choosing an appropriate basis, we have an identification of $k[V]$ with $k[\mathbb{N}^n]$ and can assume that the Δ-action on $U = \mathbb{V}^\vee$ is induced from a morphism of monoids

$$\pi : F = \mathbb{N}^n \longrightarrow A,$$

where A is the finite abelian group such that Δ is the Cartier dual $D(A)$ of A. We see then that

$$U/\Delta = \operatorname{Spec} k[P],$$

where P is the submonoid $\{p \mid \pi(p) = 0\}$ of F. Note that P is simplicially toric sharp, that $i : P \to F$ is close, and that $A = F^{gp}/i(P^{gp})$.

We now give the relationship between minimal free resolutions and pseudo-reflections.

Proposition 2.3.3. *With notation as above, $i : P \to F$ is a minimal free resolution if and only if the action of Δ on V has no pseudo-reflections.*

Proof. If i is not a minimal free resolution, then without loss of generality, $i = ji'$, where $i' : P \to F$ is close and injective, and $j : F \to F$ is given by

$$j(a_1, a_2, \ldots, a_n) = (ma_1, a_2, \ldots, a_n)$$

with $m \neq 1$. We have then a short exact sequence

$$0 \longrightarrow F^{gp}/i'(P^{gp}) \longrightarrow F^{gp}/i(P^{gp}) \longrightarrow F^{gp}/(m,1,\ldots,1)(F^{gp}) \longrightarrow 0.$$

Let N be the Cartier dual of $F^{gp}/(m,1,\ldots,1)(F^{gp})$, which is a subgroup scheme of Δ. Letting $\{x_i\}$ be the standard basis of F, we see that

$$k[F]^N = k[x_1^m, x_2, \ldots, x_n],$$

and so V^N, which is the degree 1 part of $k[F]^N$, has codimension 1 in V. Therefore, N is a pseudo-reflection.

Conversely, suppose N is a pseudo-reflection. Since N is a subgroup scheme of Δ, it is diagonalizable as well. Let $N = \operatorname{Spec} k[B]$, where B is a finite abelian group and let $\psi : A \to B$ be the induced map. We see that

$$V^N = \bigoplus_{i \neq j} kx_i$$

for some j. Without loss of generality, $j = 1$. It follows then that

$$\{f \in F \mid \psi\pi(f) = 0\} = (m,1,\ldots,1)F$$

for some m dividing $|B|$. Since the Δ action on V is assumed to be faithful, we see, in fact, that $m = |B|$. Therefore, i factors through $\cdot(m,1,\ldots,1) : F \longrightarrow F$, which shows that i is not a minimal free resolution. □

Having reinterpreted minimal free resolutions, the proof of Proposition 2.3.1 for diagonalizable group schemes G follows easily from Iwanari's work.

Proposition 2.3.4. *Let $G = \Delta$ be a finite diagonalizable group scheme over S which acts faithfully on a finite-dimensional k-vector space V. Then Theorem 2.1.9 holds when $U = \mathbb{V}^\vee$ and x is the origin. In this case it is not necessary to shrink M to a smaller Zariski neighborhood of the image of x.*

Proof. Let F and P be as above, and let $\mathfrak{X} = [U/\Delta]$. By Proposition 2.3.3, the morphism $i : P \to F$ is a minimal free resolution. Theorem 3.3 (1) and Proposition 3.4 of [Iw] then show that the natural morphism $\mathfrak{X} \times_M M^0 \to M^0$ is an isomorphism. Since $\mathfrak{X} \times_M M^0 = [U^0/\Delta]$, we see U^0 is a Δ-torsor over M^0. □

2.3.2 Finishing the Proof

The goal of this subsection is to prove Proposition 2.3.1. The main result used in the proof of this proposition, as well as in the proof of Theorem 2.1.9, is the following.

Proposition 2.3.5. *Let notation and hypotheses be as in Theorem 2.1.9. Let $X = U/\Delta$ and $G = \Delta \rtimes Q$, where Δ is diagonalizable and Q is constant and tame. If in addition to assuming that G acts without pseudo-reflections at x, we assume that Δ is local and that the base change of U to X^{sm} is a Δ-torsor over X^{sm}, then after possibly shrinking M to a smaller Zariski neighborhood of the image of x, the quotient map $f : X \to M$ is unramified in codimension 1.*

Proof. Let g be the quotient map $U \to X$. For every $q \in Q$, consider the cartesian diagram

$$\begin{array}{ccc} Z_q & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ U & \xrightarrow{\Gamma_q} & U \times U \end{array}$$

where $\Gamma_q(u) = (u, qu)$. We see that Z_q is a closed subscheme of U and that $Z_q(T)$ is the set of $u \in U(T)$ which are fixed by q. Let Z be the closed subset of U which is the union of the Z_q for $q \neq 1$. Since the action of G on U is faithful, Z is not all of U. Let Z' be the union of the codimension 1 components of Z. Since fg is finite, we see that $fg(Z')$ is a closed subset of M. Moreover, $fg(Z')$ does not contain the image of x, as G is assumed to act without pseudo-reflections at x. By shrinking M to $M - fg(Z')$, we can assume that no non-trivial $q \in Q$ acts trivially on a divisor of U.

Let $U = \operatorname{Spec} R$. The morphism f is unramified in codimension 1 if and only if the (traditional) inertia groups of all height 1 primes $\mathfrak{p}$ of R^{Δ} are trivial. So, we must show that if $q \in Q$ acts trivially on $V(\mathfrak{p})$, then $q = 1$. Since g is finite, and hence integral, the going up theorem shows that

$$\mathfrak{p}R = \mathfrak{P}_1^{e_1} + \cdots + \mathfrak{P}_n^{e_n},$$

where the $\mathfrak{P}_i$ are height 1 primes and the e_i are positive integers. Note that X is normal and so the complement of X^{sm} in X has codimension at least 2. As a result,

$$h : U \times_X \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}} \longrightarrow \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}$$

is a Δ-torsor. Since Δ is local, h is a homeomorphism of topological spaces, so there is exactly one prime $\mathfrak{P}$ lying over $\mathfrak{p}$. We see then that $U \times_X V(\mathfrak{p}) = V(\mathfrak{P}^e)$ for some e.

Let $V(\mathfrak{p})^0$ be the intersection of $V(\mathfrak{p})$ with X^{sm}, and let $Z^0 = U \times_X V(\mathfrak{p})^0$. Then Z^0 is a Δ-torsor over $V(\mathfrak{p})^0$. Since q acts trivially on $V(\mathfrak{p})$, we obtain an action of q on Z^0 over $V(\mathfrak{p})^0$, and hence a group scheme homomorphism

$$\varphi : Q'_{V(\mathfrak{p})^0} \longrightarrow \mathcal{A}ut(Z^0/V(\mathfrak{p})^0) = \Delta_{V(\mathfrak{p})^0},$$

where Q' denotes the subgroup of Q generated by q. Since $V(\mathfrak{p})^0$ is reduced, we see that φ factors through the reduction of $\Delta_{V(\mathfrak{p})^0}$, which is the trivial group scheme. Therefore, q acts trivially on Z^0.

Since the complement of X^{sm} in X has codimension at least 2, and since g factors as a flat map $U \to [U/\Delta]$ followed by a coarse space map $[U/\Delta] \to X$, both of which are codimesion-preserving (see Definition 4.2 and Remark 4.3 of [FMN]), we see that the complement of Z^0 in $V(\mathfrak{P}^e)$ has codimension at least 2. Note that if Y is a normal scheme and W is an open subscheme of Y whose complement has codimension at least 2, then any morphism from W to an affine scheme Z extends uniquely to a morphism from Y to Z. Since the action of q on $V(\mathfrak{P}^e)$ restricts to a trivial action on Z^0, the action of q on $V(\mathfrak{P}^e)$ is trivial. Therefore, q acts trivially on a divisor of U, and so $q = 1$. □

Proof of Proposition 2.3.1. Let k'/k be a finite Galois extension such that $G_{k'} \simeq \Delta \rtimes Q$, where Δ is diagonalizable and Q is constant and tame. Let $S' = \operatorname{Spec} k'$ and consider the diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where the squares are cartesian. We denote by x' the induced k'-rational point of U'. Since Δ is the product of a local diagonalizable group scheme and a locally constant diagonalizable group scheme, replacing k' by a further extension if necessary, we can assume that Δ is local.

Since G is stable, $G_{k'}$ has no pseudo-reflections at x'. It follows then from Proposition 2.3.5 that there exists an open neighborhood W' of x' such that $U' \times_{M'} W' \longrightarrow W'$ is unramified in codimension 1. Since k'/k is a finite Galois extension, replacing W' by the intersection of the $\tau(W')$ as τ ranges over the elements of $\mathrm{Gal}(k'/k)$, we can assume W' is Galois-invariant. Hence, $W' = W \times_M M'$ for some open subset W of M. We shrink M to W.

To check that U^0 is a G-torsor over M^0, we can look étale locally. We can therefore assume $S = S'$. Let $X = U/\Delta$, and let $g : U \to X$ and $f : X \to M$ be the quotient maps. We denote by X^0 the fiber product $X \times_M M^0$ and by f^0 the induced morphism $X^0 \to M^0$.

By Proposition 2.3.4, we know that the base change of U to X^{sm} is a Δ-torsor over X^{sm}. Since f is unramified in codimension 1, we see that f^0 is as well. Since M^0 is smooth and X^0 is normal, the purity of the branch locus theorem [SGA1, X.3.1] implies that f^0 is étale, and hence a Q-torsor. Since X^0 is étale over M^0, it is smooth. As a result, U^0 is a Δ-torsor over X^0 from which it follows that U^0 is a G-torsor over M^0. □

This finishes the proof of Proposition 2.3.1, and hence also of Theorem 2.1.6. We conclude this section by proving Corollary 2.1.8.

Proof of Corollary 2.1.8. Let $U = \operatorname{Spec} R$ and $M = U/G$. We denote by y the image of x. Since G being generated by pseudo-reflections at x implies that G_K is generated by pseudo-reflections at x for arbitrary finite linearly reductive group schemes G, and since smoothness of M at y can be checked étale locally, we can assume that x is k-rational. Let $V = \mathfrak{m}_x/\mathfrak{m}_x^2$ be the cotangent space of x. As G is linearly reductive, there is a G-equivariant section of $\mathfrak{m}_x \to V$. This yields a G-equivariant map $\mathrm{Sym}^\bullet(V) \to R$, which induces an isomorphism $k[[V]] \longrightarrow \hat{\mathcal{O}}_{U,x}$ of G-representations. That is, complete locally, we have linearized the G-action. Since $\hat{\mathcal{O}}_{M,y} = k[[V]]^G$, the corollary follows from Theorem 2.1.6, as M is smooth at y if and only if $\hat{\mathcal{O}}_{M,y}$ is a formal power series ring over k. □

2.4 Actions on Smooth Schemes

Having proved Theorem 2.1.9 for polynomial rings with linear actions, we now turn to the general case. We begin with two preliminary lemmas and a technical proposition.

Lemma 2.4.1. *Let U be a smooth affine scheme over S with an action of a finite diagonalizable group scheme Δ. Then there is a closed subscheme Z of U on which Δ acts trivially, and with the property that every closed subscheme Y on which Δ acts trivially factors through Z. Furthermore, the construction of Z commutes with flat base change on U/Δ.*

Proof. Let $U = \operatorname{Spec} R$ and $\Delta = \operatorname{Spec} k[A]$, where A is a finite abelian group written additively. The Δ-action on U yields an A-grading

$$R = \bigoplus_{a \in A} R_a.$$

We see that if $\mathcal{J}$ is an ideal of R, then Δ acts trivially on $Y = \operatorname{Spec} R/\mathcal{J}$ if and only if $\mathcal{J}$ contains the R_a for $a \neq 0$. Letting $\mathcal{I}$ be the ideal generated by the R_a for $a \neq 0$, we see that $\operatorname{Spec} R/\mathcal{I}$ is our desired Z.

We now show that the formation of Z commutes with flat base change. Note that

$$U/\Delta = \operatorname{Spec} R_0.$$

Let R_0' be a flat R_0-algebra and let $R' = R_0' \otimes_{R_0} R$. The induced Δ-action on $\operatorname{Spec} R'$ corresponds to the A-grading

$$R' = \bigoplus_{a \in A} (R_0' \otimes_{R_0} R_a).$$

Since R_0' is flat over R_0, we see that $\mathcal{I} \otimes_{R_0} R_0'$ is an ideal of R', and one easily shows that it is the ideal generated by the $R_0' \otimes_{R_0} R_a$ for $a \neq 0$. □

Recall that if G is a group scheme over a base scheme B which acts on a B-scheme U, and if $y : T \to U$ is a morphism of B-schemes, then the stabilizer group scheme G_y is defined by the cartesian diagram

$$\begin{array}{ccc} G_y & \longrightarrow & G \times_B U \\ \downarrow & & \downarrow \varphi \\ T & \xrightarrow{y \times y} & U \times_B U \end{array}$$

where $\varphi(g, u) = (gu, u)$. If U is separated over B, then G_y is a closed subgroup scheme of G_T.

Lemma 2.4.2. *Let B be a scheme and G a finite flat group scheme over B. If G acts on a B-scheme U, then $U \to U/G$ is a G-torsor if and only if the stabilizer group schemes G_y are trivial for all closed points y of U.*

Proof. The "only if" direction is clear. To prove the "if" direction, it suffices to show that the stabilizer group schemes G_y are trivial for all scheme valued points $y : T \to U$. This is equivalent to showing that the universal stabilizer G_u is trivial, where $u : U \to U$ is the identity map. Since G_u is a finite group scheme over U, it is given by a coherent sheaf $\mathcal{F}$ on U. The support of $\mathcal{F}$ is a closed subset, and so to prove G_u is trivial, it suffices to check this on stalks of closed points. Nakayama's Lemma then shows that we need only check the triviality of G_u on closed fibers. That is, we need only check that the G_y are trivial for closed points y of U. □

Proposition 2.4.3. *Let U be a smooth affine scheme over S with a faithful action of a stable group scheme G fixing a k-rational point x. If N has a pseudo-reflection at x, then there is an étale neighborhood $T \longrightarrow U/G$ of x and a divisor D of U_T defined by a principal ideal on which N_T acts trivially.*

Proof. Let $M = U/G$ and let y be the image of x in M. As in the proof of Corollary 2.1.8, we have an isomorphism $k[[V]] \longrightarrow \hat{\mathcal{O}}_{U,x}$ of G-representations, where $V = \mathfrak{m}_x/\mathfrak{m}_x^2$. If N is a pseudo-reflection at x, then there is some $v \in V$ such that N acts trivially on the closed subscheme of $\operatorname{Spec} k[[V]]$ defined by the prime ideal generated by v.

Consider the contravariant functor F which sends an M-scheme T to the set of divisors of U_T defined by a principal ideal on which N_T acts trivially. As F is locally of finite presentation and $U \times_M \operatorname{Spec} \hat{\mathcal{O}}_{M,y} = \operatorname{Spec} \hat{\mathcal{O}}_{U,x}$, Artin's Approximation Theorem [Ar] finishes the proof. □

We are now ready to prove Theorem 2.1.9. Our method of proof is similar to that of Proposition 2.3.1; we first prove the theorem in the case that G is diagonalizable and then make use of this case to prove the theorem in general.

Proposition 2.4.4. *Theorem 2.1.9 holds when $G = \Delta$ is a finite diagonalizable group scheme.*

Proof. Let $g : U \to M$ be the quotient map. Since any subgroup scheme N of Δ is again finite diagonalizable, Lemma 2.4.1 shows that for every N, there exists a closed subscheme Z_N of U on which N acts trivially, and with the property that every closed subscheme Y on which N acts trivially factors through Z_N. Let Z be the union of the finitely many closed subsets Z_N for $N \neq 1$. Since the action of Δ on U is faithful, Z has codimension at least 1. Let Z' be the union of all irreducible components of Z which have codimension 1. Since Δ acts without pseudo-reflections at x, we see $x \notin Z'$. Note that $g(Z')$ is closed as g is proper. Since the construction of Z commutes with flat base change on M and since flat morphisms are codimension-preserving, replacing M with $M - g(Z')$, we can assume that there are no non-trivial subgroup schemes of Δ which fppf locally on M act trivially on a divisor of U.

By Lemma 2.4.2, to show U^0 is a Δ-torsor over M^0, it suffices to show that for every closed point y of U which maps to M^0, the stabilizer group scheme Δ_y is trivial. Fix such a closed point y and let $T = \operatorname{Spec} k(y)$. Since T is fppf over S, we see from Proposition 2.4.3 that the closed subgroup scheme Δ_y of Δ_T acts faithfully on U_T without pseudo-reflections at the $k(y)$-rational point y' of U_T induced by y. Since y maps to a smooth point of M, it follows that y' maps to a smooth point of M_T. Corollary 2.1.8 then shows that Δ_y is generated by pseudo-reflections. Since Δ_y has no pseudo-reflections, it is therefore trivial. □

Proof of Theorem 2.1.9. If $G = \Delta \rtimes Q$, where Δ is diagonalizable and Q is constant and tame, then letting Z' be as in Proposition 2.4.4 and letting U, X, f, and g be as in the proof of Proposition 2.3.1, the proof of Proposition 2.4.4 shows that after replacing M by $M - fg(Z')$, the base change of U to X^{sm} is a Δ-torsor over X^{sm}. As in the proof of Proposition 2.3.1, we can then reduce the general case to the case when $G = \Delta \rtimes Q$, where Δ is local diagonalizable and Q is constant tame. The last paragraph of the proof of Proposition 2.3.1 then shows that U^0 is a G-torsor over M^0. □

2.5 Schemes with Linearly Reductive Singularities

Let k be a perfect field of characteristic p.

Definition 2.5.1. We say a scheme M over S has *linearly reductive singularities* if there is an étale cover $\{U_i/G_i \to M\}$, where the U_i are smooth over S and the G_i are linearly reductive group schemes which are finite over S.

Note that if M has linearly reductive singularities, then it is automatically normal and in fact Cohen-Macaulay by [HR, p.115].

Our goal in this section is to prove Theorem 2.1.10, which generalizes the result that every scheme with quotient singularities prime to the characteristic is the coarse space of a smooth Deligne-Mumford stack. We remark that in the case of quotient singularities, the converse of the analogous theorem is true as well; that is, every scheme which is the coarse space of a smooth Deligne-Mumford stack has quotient singularities. It is not clear, however, that the converse of Theorem 2.1.10 should hold. We know from Theorem 3.2 of [AOV] that $\mathfrak{X}$ is étale locally $[V/G_0]$, where G_0 is a finite flat linearly reductive group scheme over V/G_0, but V need not be smooth and G_0 need not be the base change of a group scheme over S. On the other hand, Proposition 2.5.2 below shows that $\mathfrak{X}$ is étale locally $[U/G]$ where U is smooth and G is a group scheme over S, but here G is not finite.

Before proving Theorem 2.1.10, we begin with a technical proposition followed by a series of lemmas.

Proposition 2.5.2. *Let $\mathfrak{X}$ be a tame stack over S with coarse space M. Then there exists an étale cover $T \to M$ such that*

$$\mathfrak{X} \times_M T = [U/\mathbb{G}^r_{m,T} \rtimes H],$$

where H is a finite constant tame group scheme and U is affine over T. Furthermore, $\mathbb{G}^r_{m,T} \rtimes H$ is the base change to T of a group scheme $\mathbb{G}^r_{m,S} \rtimes H$ over S, so $\mathfrak{X} \times_M T = [U/\mathbb{G}^r_{m,S} \rtimes H]$.

Proof. Theorem 3.2 of [AOV] shows that there exists an étale cover $T \to M$ and a finite flat linearly reductive group scheme G_0 over T acting on a finite finitely presented scheme V over T such that

$$\mathfrak{X} \times_M T = [V/G_0].$$

By [AOV, Lemma 2.20], after replacing T by a finer étale cover if necessary, we can assume there is a short exact sequence

$$1 \to \Delta \to G_0 \to H \to 1,$$

where $\Delta = \operatorname{Spec} \mathcal{O}_T[A]$ is a finite diagonalizable group scheme and H is a finite constant tame group scheme. Since Δ is abelian, the conjugation action of G_0 on Δ passes to an action

$$H \to \operatorname{Aut}(\Delta) = \operatorname{Aut}(A).$$

Choosing a surjection $F \to A$ in the category of $\mathbb{Z}[H]$-modules from a free module F, yields an H-equivariant morphism $\Delta \hookrightarrow \mathbb{G}^r_{m,T}$. Using the H-action on $\mathbb{G}^r_{m,T}$, we define the group scheme $\mathbb{G}^r_{m,T} \rtimes G_0$ over T. Note that there is an embedding

$$\Delta \hookrightarrow \mathbb{G}^r_{m,T} \rtimes G_0$$

sending δ to (δ, δ^{-1}), which realizes Δ as a normal subgroup scheme of $\mathbb{G}^r_{m,T} \rtimes G_0$. We can therefore define

$$G := (\mathbb{G}^r_{m,T} \rtimes G_0)/\Delta.$$

One checks that there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & G_0 & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
1 & \longrightarrow & \mathbb{G}^r_{m,T} & \longrightarrow & G & \xrightarrow{\pi} & H & \longrightarrow & 1
\end{array}$$

where the rows are exact and the vertical arrows are injective.

We show that étale locally on T, there is a group scheme-theoretic section of π, so that $G = \mathbb{G}^r_{m,T} \rtimes H$. Let P be the sheaf on T such that for any T-scheme W, $P(W)$ is the set of group scheme-theoretic sections of $\pi_W : G_W \to H_W$. Note that the sheaf $\underline{\operatorname{Hom}}(H, G)$ parameterizing group scheme homomorphisms from H to G is representable since it is a closed subscheme of $G^{\times |H|}$ cut out by suitable equations. We see that P is the equalizer of the two maps

$$\underline{\operatorname{Hom}}(H, G) \underset{p_2}{\overset{p_1}{\rightrightarrows}} H^{\times |H|}$$

where $p_1(\phi) = (\pi\phi(h))_h$ and $p_2(\phi) = (h)_h$. That is, there is a cartesian diagram

$$\begin{array}{ccc}
P & \longrightarrow & \underline{\operatorname{Hom}}(H, G) \\
\downarrow & & \downarrow{\scriptstyle (p_1, p_2)} \\
H^{\times |H|} & \xrightarrow{\Delta} & H^{\times |H|} \times H^{\times |H|}
\end{array}$$

Since H is separated over T, we see that P is a closed subscheme of $\underline{\operatorname{Hom}}(H, G)$. In particular, it is representable and locally of finite presentation over T. Furthermore, $P \to T$ is surjective as

[AOV, Lemma 2.16] shows that it has a section fppf locally. To show P has a section étale locally, by [EGA4, 17.16.3], it suffices to prove P is smooth over T.

Given a commutative diagram

$$\begin{array}{ccc} X_0 = \operatorname{Spec} A/\mathcal{I} & \longrightarrow & P \\ \downarrow & \nearrow & \downarrow \\ X = \operatorname{Spec} A & \longrightarrow & T \end{array}$$

with $\mathcal{I}$ a square zero ideal, we want to find a dotted arrow making the diagram commute. That is, given a group scheme-theoretic section $s_0 : G_{W_0} \to H_{W_0}$ of π_{W_0}, we want to find a group scheme homomorphism $s : G_W \to H_W$ which pulls back to s_0 and such that $\pi_W \circ s$ is the identity. Note first that any group scheme homomorphism s which pulls back to s_0 is automatically a section of π_W since H is a finite constant group scheme and $\pi_W \circ s$ pulls back to the identity over W_0. By [SGA3, Exp. III 2.3], the obstruction to lifting s_0 to a group scheme homomorphism lies in

$$H^2(H, Lie(G) \otimes \mathcal{I}),$$

which vanishes as H is linearly reductive. This proves the smoothness of P.

To complete the proof of the lemma, let $U := V \times^{G_0} G$ and note that

$$\mathfrak{X} \times_M T = [V/G_0] = [U/G].$$

Since V is finite over T and G is affine over T, it follows that U is affine over T as well. Replacing T by a finer étale cover if necessary, we have

$$\mathfrak{X} \times_M T = [U/\mathbb{G}^r_{m,T} \rtimes H].$$

Lastly, the scheme underlying $\mathbb{G}^r_{m,T} \rtimes H$ is $\mathbb{G}^r_{m,T} \times_T H$ and its group scheme structure is determined by the action $H \to \operatorname{Aut}(\mathbb{G}^r_{m,T})$. Since $\operatorname{Aut}(\mathbb{G}^r_{m,T}) = \operatorname{Aut}(\mathbb{Z}^r)$, we can use this same action to define the semi-direct product $\mathbb{G}^r_{m,S} \rtimes H$ and it is clear that this group scheme base changes to $\mathbb{G}^r_{m,T} \rtimes H$. ☐

Lemma 2.5.3. *If V is a smooth S-scheme with an action of finite linearly reductive group scheme G_0 over S, then $[V/G_0]$ is smooth over S.*

Proof. Let $\mathfrak{X} = [V/G_0]$. To prove $\mathfrak{X}$ is smooth, it suffices to work étale locally on S, where, by [AOV, Lemma 2.20], we can assume G_0 fits into a short exact sequence

$$1 \to \Delta \to G_0 \to H \to 1,$$

where Δ is a finite diagonalizable group scheme and H is a finite constant tame group scheme. Let G be obtained from G_0 as in the proof of Proposition 2.5.2 and let $U = V \times^{G_0} G$. Since $\mathfrak{X} = [U/G]$, it suffices to show U is smooth over S. The action of G_0 on $V \times G$, given by $g_0 \cdot (v, g) = (vg_0, g_0 g)$, is free as the G_0-action on G is free. As a result, $U = [(V \times G)/G_0]$ and $G/G_0 = [G/G_0]$. Since the projection map $p : V \times G \to G$ is G_0-equivariant, we have a cartesian diagram

$$\begin{array}{ccc} V \times G & \xrightarrow{p} & G \\ \downarrow & & \downarrow \\ U & \xrightarrow{q} & G/G_0 \end{array}$$

Since p is smooth, q is as well. Since $G \to [G/G_0] = G/G_0$ is flat and G is smooth, [EGA4, 17.7.7] shows that G/G_0 is smooth, and so U is as well. □

Lemma 2.5.4. *Let X be a smooth S-scheme and $i : U \hookrightarrow X$ an open subscheme whose complement has codimension at least 2. Let P be a G-torsor on U, where $G = \mathbb{G}_m^r \rtimes H$ and H is a finite constant étale group scheme. Then P extends uniquely to a G-torsor on X.*

Proof. The structure map from P to U factors as $P \to P_0 \to U$, where P is a $\mathbb{G}_m^r$-torsor over P_0 and P_0 is an H-torsor over U. Since the complement of U in X has codimension at least 2, we have $\pi_1(U) = \pi_1(X)$ and so P_0 extends uniquely to an H-torsor Q_0 on X. Let $i_0 : P_0 \hookrightarrow Q_0$ be the inclusion map. Since Q_0 is smooth and the complement of P_0 in Q_0 has codimension at least 2, the natural map $\mathrm{Pic}(Q_0) \to \mathrm{Pic}(P_0)$ is an isomorphism. It follows that any line bundle over P_0 can be extended uniquely to a line bundle over Q_0. We can therefore inductively construct a unique lift of P over X. □

Our proof of the following lemma closely follows that of [FMN, Thm 4.6].

Lemma 2.5.5. *Let $f : \mathcal{Y} \to M$ be an S-morphism from a smooth tame stack $\mathcal{Y}$ to its coarse space which pulls back to an isomorphism over the smooth locus M^0 of M. If $h : \mathfrak{X} \to M$ is a dominant, codimension-preserving morphism (see* [FMN, Def 4.2]*) from a smooth tame stack, then there is a morphism $g : \mathfrak{X} \to \mathcal{Y}$, unique up to unique isomorphism, such that $fg = h$.*

Proof. We show that if such a morphism g exists, then it is unique. Suppose g_1 and g_2 are two such morphisms. We see then that $g_1|_{h^{-1}(M^0)} = g_2|_{h^{-1}(M^0)}$. Since h is dominant and codimension-preserving, $h^{-1}(M^0)$ is open and dense in $\mathfrak{X}$. Proposition 1.2 of [FMN] shows that if $\mathfrak{X}$ and $\mathcal{Y}$ are Deligne-Mumford with $\mathfrak{X}$ normal and $\mathcal{Y}$ separated, then g_1 and g_2 are uniquely isomorphic. The proof, however, applies equally well to tame stacks since the only key ingredient used about Deligne-Mumford stacks is that they are locally $[U/G]$ where G is a separated group scheme.

By uniqueness, to show the existence of g, we can assume by Proposition 2.5.2 that $\mathcal{Y} = [U/G]$, where U is smooth and affine, and $G = \mathbb{G}_m^r \rtimes H$, where H is a finite constant tame group scheme. Let $p : V \to \mathfrak{X}$ be a smooth cover by a smooth scheme. Since smooth morphisms are dominant and codimension-preserving, uniqueness implies that to show the existence of g, we need only show there is a morphism $g_1 : V \to \mathcal{Y}$ such that $fg_1 = hp$. So, we can assume $\mathfrak{X} = V$.

Given a stack $\mathcal{Z}$ over M, let $\mathcal{Z}^0 = M^0 \times_M \mathcal{Z}$. Given a morphism $\pi : \mathcal{Z}_1 \to \mathcal{Z}_2$ of M-stacks, let $\pi^0 : \mathcal{Z}_1^0 \to \mathcal{Z}_2^0$ denote the induced morphism. Since f^0 is an isomorphism, there is a morphism $g^0 : V^0 \to \mathcal{Y}^0$ such that $f^0 g^0 = h^0$. It follows that there is a G-torsor P^0 over V^0 and a G-equivariant map from P^0 to U^0 such that the diagram

$$\begin{array}{ccc} P^0 & \longrightarrow & U^0 \\ \downarrow & & \downarrow \\ V^0 & \longrightarrow & \mathcal{Y}^0 \\ \downarrow & \swarrow_{\simeq} & \\ M^0 & & \end{array}$$

commutes and the square is cartesian. By Lemma 2.5.4, P^0 extends to a G-torsor P over V.

Note that if X is a normal algebraic space and $i : W \hookrightarrow X$ is an open subalgebraic space whose complement has codimension at least 2, then any morphism from W to an affine scheme Y extends uniquely to a morphism $X \to Y$. As a result, the morphism from P^0 to U^0 extends to a morphism $q : P \to U$. Consider the diagram

$$\begin{array}{ccc} G \times P & \xrightarrow{id \times q} & G \times U \\ \downarrow & & \downarrow \\ P & \xrightarrow{q} & U \end{array}$$

where the vertical arrows are the action maps. Precomposing either of the two maps in the diagram from $G \times P$ to U by the inclusion $G \times P^0 \hookrightarrow G \times P$ yields the same morphism. That is, the two maps from $G \times P$ to U are both extensions of the same map from $G \times P^0$ to the affine scheme U, and hence are equal. This shows that q is G-equivariant, and therefore yields a map $g : V \to \mathcal{Y}$ such that $fg = h$. $\square$

Proof of Theorem 2.1.10. We begin with the following observation. Suppose U is smooth and affine over S with a faithful action of a finite linearly reductive group scheme G over S. Let y be a closed point of U mapping to $x \in U/G$. After making the étale base change $\operatorname{Spec} k(y) \to S$, we can assume y is a k-rational point. Let G_y be the stabilizer subgroup scheme of G fixing y. Since

$$U/G_y \longrightarrow U/G$$

is étale at y, replacing U/G by an étale cover, we can further assume that G fixes y. Then by Corollary 2.1.8, we can assume G has no pseudo-reflections at y, and hence, Theorem 2.1.9 shows that after shrinking U/G about x, we can assume that the base change of U to the smooth locus of U/G is a G-torsor.

We now turn to the proof. Since M has linearly reductive singularities, there is an étale cover $\{U_i/G_i \to M\}$, where U_i is smooth and affine over S and G_i is a finite linearly reductive group scheme over S which acts faithfully on U_i. By the above discussion, replacing this étale cover by a finer étale cover if necessary, we can assume that the base change of U_i to the smooth locus of

U_i/G_i is a G_i-torsor. Let $M_i = U_i/G_i$ and $\mathfrak{X}_i = [U_i/G_i]$. We see that the $\mathfrak{X}_i$ are locally the desired stacks, so we need only glue the $\mathfrak{X}_i$. Let $M_{ij} = M_i \times_M M_j$ and let $V_i \to \mathfrak{X}_i$ be a smooth cover. Since M_{ij} is the coarse space of both $\mathfrak{X}_i \times_{M_i} M_{ij}$ and $\mathfrak{X}_j \times_{M_j} M_{ij}$, and since coarse space maps are dominant and codimension-preserving, Lemma 2.5.5 shows that there is a unique isomorphism of $\mathfrak{X}_i \times_{M_i} M_{ij}$ and $\mathfrak{X}_j \times_{M_j} M_{ij}$. Identifying these two stacks via this isomorphism, let I_{ij} be the fiber product over the stack of $V_i \times_{M_i} M_{ij}$ and $V_j \times_{M_j} M_{ij}$. We see then that we have a morphism $I_{ij} \to U_i \times_M U_j$. This yields a groupoid

$$\coprod I_{ij} \to \coprod U_i \times_M U_j,$$

which defines our desired glued stack $\mathfrak{X}$. Note that $\mathfrak{X}$ is smooth and tame by [AOV, Thm 3.2]. □

Chapter 3

Stacky Resolutions of Toroidal Embeddings

3.1 Introduction

This chapter is concerned with proving the second of our stacky resolution theorems. As mentioned in Chapter 1, it is a well-known result (see for example [Vi, 2.9] or [FMN, Rmk 4.9]) that if k is a field and X is a k-scheme with quotient singularities prime to the characteristic of k, then there is a canonical smooth Deligne-Mumford stack $\mathfrak{X}$ with coarse space X such that the stacky structure of $\mathfrak{X}$ is supported on the singular locus of X. If the singularities of X are worse than quotient singularities, however, we can no longer hope to find a stacky resolution of X by using Deligne-Mumford stacks. Typically Artin stacks do not have coarse spaces and the appropriate notion that replaces coarse space is that of good moduli space (in the sense of J. Alper [Al, Def 1.4]).

Inspired by the work of Iwanari [Iw], we take a different approach toward the problem of finding a stacky resolution of X by a smooth algebraic stack. Namely, we restrict attention to a class of schemes which carry more structure in hopes of being able to both construct our desired stacky resolution and say more about that stack than we could for an arbitrary scheme. This richer class of schemes we look at is that of fs log smooth log schemes X over k, where $\operatorname{Spec} k$ is given the trivial log structure (or equivalently, the class of toroidal embeddings which are not necessarily strict). Our main stacky resolution theorem of this chapter is then:

Theorem 3.3.2. *Let k be field and X be an fs log scheme which is log smooth over $S = \operatorname{Spec} k$, where S is given the trivial log structure. Then there exists a smooth, log smooth log Artin stack $\mathfrak{X}$ over S and a morphism $f : \mathfrak{X} \to X$ over S which realizes X as the good moduli space of $\mathfrak{X}$. Moreover, the base change of f to the smooth locus of X is an isomorphism.*

This is a generalization of [Iw, Thm 3.3] where the result is proved for X all of whose charts are given by simplicial toric varieties. Our method of proof is a direct generalization of Iwanari's. In particular, our stack $\mathfrak{X}$ has a moduli interpretation (in terms of log geometry) and agrees with the stack Iwanari constructs when X is as in [Iw, Thm 3.3]. We also give a slight improvement of [Iw, Thm 3.3(2)] in Remark 3.3.3.

This chapter is organized as follows. In Section 3.2, we define minimal free resolutions and sliced resolutions. In Section 3.3, we construct the canonical stack of Theorem 3.3.2 as the moduli space of sliced resolutions. We also construct several other smooth log smooth log Artin stacks having our given log scheme as a good moduli space. These stacks are constructed as moduli spaces parameterizing what we call admissible sliced resolutions. This line of thought is continued further in Chapter 6 where we restrict to the case when X is a toric variety and construct several other "toric Artin stacks" having X as their good moduli space.

3.2 Minimal Free Resolutions

In this section, we define the objects which will be parameterized by the canonical stack $\mathfrak{X}$ of Theorem 3.3.2. A morphism $f : P \to Q$ of monoids is called *close* if for all $q \in Q$ there is a positive integer n such that nq is in the image of f. A fine sharp monoid P with P^{gp} free of rank r is called *simplicially toric* if there is a submonoid Q generated by r elements such that Q is close to P. We recall ([Iw, Def 2.5]) that an injective morphism $i : P \to F$ from a saturated simplicially toric sharp monoid to a free monoid is called a *minimal free resolution* if i is close and if for all injective close morphisms $i' : P \to F'$ to a free monoid F' of the same rank as F, there is a unique morphism $j : F \to F'$ such that $i' = ji$. The stack Iwanari constructs in [Iw, Thm 3.3] has a moduli interpretation in terms of minimal free resolutions, so our first step is to generalize this notion. The key is to replace his use of closeness in the above definition with that of exactness.

Definition 3.2.1. A morphism $f : P \to Q$ of integral monoids is *exact* if the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow & & \downarrow \\ P^{gp} & \xrightarrow{f^{gp}} & Q^{gp} \end{array}$$

is set-theoretically cartesian.

Note that if $f : P \to Q$ is sharp and exact, then it is automatically injective.

Let P be an fs sharp monoid and let $C(P)$ denote the rational cone of P in $P^{gp} \otimes \mathbb{Q}$. Let d be the number of rays of the dual cone $C(P)^\vee$. For $1 \leq i \leq d$, we denote by v_i the first lattice point on each of the rays of $C(P)^\vee$ and by $F(P)$ the free monoid on the v_i. We obtain a morphism $i : P \to F(P)$ defined by $p \mapsto (v_i(p))$.

Proposition 3.2.2. *The morphism $i : P \to F(P)$ is exact. Moreover, for any exact morphism $i' : P \to F$ to a free monoid F of the same rank as $F(P)$, there is a unique morphism $j : F(P) \to F$ such that $i' = ji$.*

Proof. The exactness of i follows easily from the discussion on the top of page 12 of [Fu]. Arbitrarily choosing an isomorphism of F with $F(P)$, we can assume $F(P) = F$. Let $i' = (\varphi_i)$. Exactness of i' shows that $p \in P^{gp}$ is in P if and only if $\varphi_i^{gp}(p) \geq 0$ for all i. Since p is in P if and only if $f(p) \geq 0$ for all $f \in C(P)^\vee$, we see

$$\mathrm{Cone}(\varphi_i) = C(P)^\vee = \mathrm{Cone}(v_i).$$

Since $C(P)^\vee$ has d rays, it follows that every φ_i lies on a ray. Composing i' by a uniquely determined permutation, we can assume that φ_i lies on ray generated by v_i. Since φ_i takes integer values on P, we see that φ_i is a lattice point of $C(P)^\vee$. Since v_i is defined to be the first lattice point on the ray defined by v_i, we have $\varphi_i = n_i v_i$ for uniquely determined $n_i \in \mathbb{N}$. Hence, multiplication by $(n_1, \ldots, n_d)$ is our desired j. ☐

In light of this proposition, we make the following definition.

Definition 3.2.3. Let P be an fs sharp monoid and let $F(P)$ have rank d. A morphism $i : P \to F$ to a free monoid of rank d is a *minimal free resolution* if it is exact and if for every exact morphism $i' : P \to F'$ to a free monoid F' of rank d, there is a unique morphism $j : F \to F'$ such that $i' = ji$.

If all of the charts of X are given by simplicial toric varieties (as in the case Iwanari considers), then in constructing the canonical stack $\mathfrak{X}$, one need only consider minimal free resolutions. As we see shortly, in the non-simplicial case, however, certain quotients of minimal free resolutions naturally arise:

Definition 3.2.4. Let P be an fs sharp monoid and let $F(P)$ have rank d. A morphism $i' : P \to F'$ to a free monoid is a *sliced resolution* if it is of the form

$$P \xrightarrow{i} F \xrightarrow{\pi} F/H,$$

where i is a minimal free resolution, H is a face of F such that $i(P) \cap H = 0$, and π is the natural projection.

Example 3.2.5. For example, let P be the submonoid of $\mathbb{N}^4$ generated by $x = (1, 0, 0, 1)$, $y = (0, 1, 1, 0)$, $z = (1, 0, 1, 0)$, and $w = (0, 1, 0, 1)$. Note that the only relation among the generators is $x + y = z + w$. This is an fs sharp monoid which is non-simplicial. Its minimal free resolution is given by the embedding into $F = \mathbb{N}^4$ that is used to define it. If H is any of the four faces of F which are generated by a single element, then $P_0 := H \cap P = 0$. So, $P = P/P_0 \to F/H \simeq \mathbb{N}^3$ cannot be a minimal free resolution since the rank of F/H is too small. Nonetheless, this morphism is a sliced resolution.

If P is a saturated simplicially toric sharp monoid, then the rank of $F(P)$ is equal to the rank of P^{gp}. If $i : P \to F$ is a morphism to a free monoid of rank equal to P^{gp}, then i is exact if and only if it is injective and close. Hence, i is a minimal free resolution in the sense of [Iw, Def 2.5] if and only if it is a minimal free resolution in the sense of Definition 3.2.3 if and only if it is a sliced resolution. As in [Iw, Def 2.11], we define a minimal free resolution morphism of log schemes.

Definition 3.2.6. A morphism $f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)$ of fs log schemes is a *minimal free resolution*, resp. is a *sliced resolution* if for all geometric points y of Y, the induced morphism

$$\bar{\mathcal{M}}_{X,f(\bar{y})} \to \bar{\mathcal{M}}_{Y,\bar{y}}$$

is a minimal free resolution, resp. is a sliced resolution.

Proposition 3.2.7. *Let P be an fs sharp monoid with minimal free resolution $i : P \to F$. If P_0 is a face of P and F_0 is the face of F generated by P_0, then the induced morphism $P/P_0 \to F/F_0$ is a minimal free resolution.*

Proof. One easily checks that P/P_0 is an fs sharp monoid. Let $\pi : P \to P/P_0$ be the natural morphism. Let $\{v_i\}_{i=1}^d$ be the extremal rays of $C(P)^\vee$. If $v_i(P_0) = 0$, then we obtain a well-defined morphism $w_i : P/P_0 \to \mathbb{N}$ given by $w_i(\bar{p}) = v_i(p)$. Note that the span of w_i is an extremal ray of $C(P)^\vee$ since if $\psi + \psi' = w_i$, then $\psi\pi + \psi'\pi = v_i$. It follows that $\psi\pi = av_i$ for some $a \in \mathbb{Q}_{\geq 0}$, and so $\psi = aw_i$.

We claim that the w_i generate $C(P/P_0)^\vee$. Let $\psi \in C(P/P_0)^\vee$. Then $\psi\pi = \sum_j a_j v_j$ for some $a_j \in \mathbb{Q}_{\geq 0}$. If there is some $p_0 \in P_0$ such that $v_k(p_0) \neq 0$, then since

$$0 = \psi\pi(p_0) = \sum_j a_j v_j(p_0),$$

we see $a_k v_k(p_0) = 0$ and so $a_k = 0$. Therefore, the w_i generate $C(P/P_0)^\vee$.

To complete the proof of the proposition, we need only show that $e_i \notin F_0$ if and only if $v_i(P_0) = 0$. If $v_i(p_0) \neq 0$ for some $p_0 \in P_0$, then

$$e_i + ((v_i(p_0) - 1)e_i + \sum_{j \neq i} v_j(p_0)e_j)$$

is in the image of P_0 and so $e_i \in F_0$. Conversely, if $e_i \in F_0$, then there exists some $p_0 \in P_0$ and $b_j \in \mathbb{N}$ such that

$$e_i + \sum_j b_j e_j = \sum_j v_j(p_0)e_j.$$

As a result, $v_i(p_0) \neq 0$. □

We remark that if H is any face of F and $P_0 = H \cap P$, then $P/P_0 \to F/H$ is not in general a minimal free resolution: in Example 3.2.5, the morphism $P \to F$ is a minimal free resolution, but $P = P/P_0 \to F/H \simeq \mathbb{N}^3$ is not; it is however a sliced resolution. This general phenomenon is the content of the following proposition which generalizes [Iw, Prop 2.12].

Proposition 3.2.8. *Let P be an fs sharp monoid and let $i : P \to F$ be its minimal free resolution. If R is a ring, then the induced morphism $f : \operatorname{Spec} R[F] \to \operatorname{Spec} R[P]$ on log schemes is a sliced resolution.*

Proof. Let $\bar{t}$ be a geometric point of $\operatorname{Spec} R[F]$ and let $\mathfrak{p}$ be the corresponding prime ideal of $R[F]$. Let H be the face of F consisting of elements which map to units under $F \to R[F] \to R[F]_{\mathfrak{p}}$. Then $\bar{\mathcal{M}}_{P,f(\bar{t})} \to \bar{\mathcal{M}}_{F,\bar{t}}$ is given by the natural map $\eta : P/P_0 \to F/H$, where $P_0 = H \cap P$. If we let F_0 be the face of F generated by P_0, we see $F_0 \subset H$ and so η factors as

$$P/P_0 \xrightarrow{\pi} F/F_0 \to F/H.$$

By Proposition 3.2.7, we see that π is a minimal free resolution. Since $H/F_0 \cap P/P_0 = 0$, we see then that η is a sliced resolution. □

We now prove an analogue of [Iw, Prop 2.17].

Proposition 3.2.9. *Let P be an fs sharp monoid and $i : P \to F$ an injective morphism to a free monoid F. Let R be a ring and let $(f,h) : (T, \mathcal{M}_T) \to \operatorname{Spec} R[P]$ be a morphism of fine log schemes. If we have a commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{i} & F \\ \downarrow & & \downarrow \alpha \\ f^{-1}\bar{\mathcal{M}}_{P,\bar{s}} & \xrightarrow{\bar{h}_{\bar{s}}} & \bar{\mathcal{M}}_{\bar{s}} \end{array}$$

and α étale locally lifts to a chart, then there is an fppf neighborhood of $\bar{s}$ and a chart $\epsilon : F \to \mathcal{M}$ making the diagram

$$\begin{array}{ccc} P & \xrightarrow{i} & F \\ \downarrow & & \downarrow \alpha \\ f^{*}\mathcal{M}_{P} & \xrightarrow{h} & \mathcal{M} \end{array}$$

commute.

Proof. The proof is the same as that of [Iw, Prop 2.17], except here the ranks of P^{gp} and F^{gp} are no longer the same. Letting the ranks be r and d, respectively, but otherwise keeping Iwanari's notation, we can choose isomorphisms $\phi : P^{gp} \to \mathbb{Z}^r$ and $\psi : F^{gp} \to \mathbb{Z}^d$ so that a is given by $e_i \mapsto \lambda_i e_i$ for $1 \le i \le r$. Note that the λ_i are positive integers since $P^{gp} \to F^{gp}$ is injective. Letting

$$\mathcal{O}' = \mathcal{O}_{T,\bar{s}}[T_1, \ldots, T_r]/(T_i^{\lambda_i} - u_i),$$

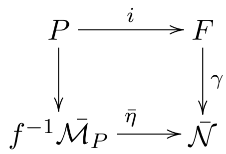
we define $\eta : F^{gp} \to \mathcal{M}_{\bar{t}}^{gp}$ as Iwanari does on the e_i for $1 \le i \le r$, and for $r < i \le d$, we send the e_i to 0. □

3.3 The Stacky Resolution Theorem

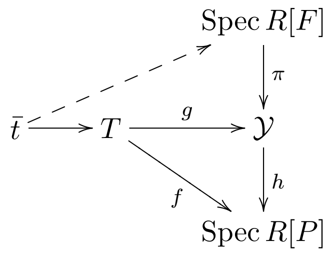
Throughout this section k is a field and $S = \operatorname{Spec} k$ has the trivial log structure. Given X a log scheme over S, we define a fibered category $\mathfrak{X}$ over X-schemes as follows. Objects are sliced resolutions $(T, \mathcal{N}) \to (X, \mathcal{M}_X)$, where $\mathcal{N}$ is a fine log structure on T, and morphisms are maps of $(X, \mathcal{M}_X)$-log schemes $h : (T, \mathcal{N}) \to (T', \mathcal{N}')$ with $h^*\mathcal{N}' \to \mathcal{N}$ an isomorphism. Then $\mathfrak{X}$ is a stack on the étale site of X, and in fact also on the fppf site by [Ol2, Thm A.1].

Proposition 3.3.1. *Let P be an fs sharp monoid with minimal free resolution $i : P \to F$. Let R be a ring and G be the group scheme* $\operatorname{Spec} R[F^{gp}/P^{gp}]$*. If $X = \operatorname{Spec} R[P]$, then $\mathfrak{X}$ is isomorphic to $\mathcal{Y} := [\operatorname{Spec} R[F]/G]$ over* $\operatorname{Spec} R[P]$.

Proof. Let $h : \mathcal{Y} \to \operatorname{Spec} R[P]$ and $\pi : \operatorname{Spec} R[F] \to \mathcal{Y}$ be the natural morphisms. By [Ol2, Prop 5.20], the stack $\mathcal{Y}$ has the following moduli interpretation. The fiber over $f : T \to \operatorname{Spec} R[P]$ is the groupoid of triples $(\mathcal{N}, \eta, \gamma)$, where $\mathcal{N}$ is a fine log structure on T, where $\gamma : F \to \bar{\mathcal{N}}$ is a morphism which étale locally lifts to a chart, and where $\eta : f^*\mathcal{M}_P \to \mathcal{N}$ is a morphism of log structures such that



commutes. We claim that η is a sliced resolution. Let $g : T \to \mathcal{Y}$ be the morphism representing $(\mathcal{N}, \eta, \gamma)$ and let $\bar{t}$ be a geometric point of T. Since smooth morphisms étale locally have sections by [EGA4, 17.16.3], we have a dotted arrow making the diagram



commute. Recall from the proof of [Ol2, Prop 5.20] that η is simply the pullback under g of the natural morphism $h^*\mathcal{M}_P \to \mathcal{M}_{\mathcal{Y}}$. Therefore, $\bar{\eta}_{\bar{t}}$ is the morphism

$$\bar{\mathcal{M}}_{P,h\bar{\pi(t)}} = (\pi^*h^*\bar{\mathcal{M}}_P)_{\bar{t}} \to (\pi^*\bar{\mathcal{M}}_{\mathcal{Y}})_{\bar{t}} = \bar{\mathcal{M}}_{F,\bar{t}},$$

which is a sliced resolution by Proposition 3.2.8.

We have, then, a morphism $\Phi : \mathcal{Y} \to \mathfrak{X}$ of stacks which forgets γ. To prove full faithfulness of Φ, we must show that if

$$(\eta_i : f^*\mathcal{M}_P \to \mathcal{N}_i, \gamma_i : F \to \bar{\mathcal{N}_i})$$

are objects of $\mathcal{Y}$ for $i = 1, 2$, then any isomorphism of log structures $\xi : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\xi\eta_1 = \eta_2$ automatically satisfies $\bar{\xi}\gamma_1 = \gamma_2$. The equality $\bar{\xi}\gamma_1 = \gamma_2$ can be checked on stalks. Let $t \in T$. Since the γ_i étale locally lift to charts $\epsilon_i : F \to \mathcal{N}_i$, we see that $\bar{\mathcal{N}}_{i,\bar{t}} \simeq \mathbb{N}^r$ for some r and that $(\epsilon_i)_{\bar{t}}$ is a projection followed by a permutation of coordinates. We have therefore reduced to proving that if

$$(\phi_j), (\psi_j) : F \to \mathbb{N}^r$$

are morphisms given by projecting and permuting coordinates, and if

$$(\phi_j(p)) = (\psi_j(p))$$

for all $p \in P$, then $\phi_j = \psi_j$ for all j. Post-composing (ϕ_j) and (ψ_j) by the projection to the j^{th} factor for some fixed j, we may assume that $r = 1$. That is, we have reduced to the statement that if $j \neq j'$, then there is some $p \in P$ such that $v_j(p) \neq v_{j'}(p)$, which is clearly true.

We now prove essential surjectivity of Φ. Let $f : (T, \mathcal{N}) \to \operatorname{Spec} R[P]$ be a sliced resolution. By full faithfullness, we need only show that f is fppf locally in the image of Φ. Let $\bar{t}$ be a geometric point of T. Then $\bar{\mathcal{M}}_{P,f(\bar{t})} = P/P_0$ for some face P_0 of P. If F_0 is the face of F generated by P_0, then by Proposition 3.2.7, the natural morphism $P/P_0 \to F/F_0$ is a minimal free resolution. Since $\bar{f}_{\bar{t}} : \bar{\mathcal{M}}_{P,f(\bar{t})} \to \bar{\mathcal{N}}_{\bar{t}}$ is a sliced resolution by assumption, it has the form

$$P/P_0 \to F/F_0 \to (F/F_0)/F_1$$

for some face F_1 of F/F_0 such that $F_1 \cap P/P_0 = 0$. Letting H be the face of F such that $H/F_0 = F_1$, we see that

$$P \to f^{-1}\bar{\mathcal{M}}_{P,\bar{t}} \to \bar{\mathcal{N}}_{\bar{t}}$$

factors as

$$P \to F \to F/H.$$

Proposition 3.2.9 then shows that fppf locally, f is in the image of Φ. □

Theorem 3.3.2. *Let X be an fs log scheme which is log smooth over S. Then there exists a smooth, log smooth log Artin stack $\mathfrak{X}$ over S and a morphism $f : \mathfrak{X} \to X$ over S which realizes X as the good moduli space of $\mathfrak{X}$. Moreover, the base change of f to the smooth locus of X is an isomorphism.*

Proof. By [Ka1, Thm 4.8], since X is log smooth over S, we have an étale cover $h : Y \to X$ and a smooth strict morphism $g : Y \to Z$, where $Z = \operatorname{Spec} k[P]$ with P an fs sharp monoid. Let $i : P \to F$ be the minimal free resolution. Then Proposition 3.3.1 shows that $\mathfrak{X}$ is étale locally $[U/G]$, where

$$U = Y \times_Z \operatorname{Spec} k[F]$$

and G is $\operatorname{Spec} k[F^{gp}/P^{gp}]$. Hence, $\mathfrak{X}$ is a smooth Artin stack. Moreover, the log structure on $[\operatorname{Spec} k[F]/G]$ induces a log structure on $\mathfrak{X}$ which makes it log smooth over S. We see that $\mathfrak{X} \to X$ is a good moduli space by [Al, Ex 8.3]. Lastly, the base change of this map to X^{sm} is an isomorphism since $\bar{\mathcal{M}}_{X,\bar{x}}$ is free if and only if $x \in X^{sm}$ by [Iw, Lemma 3.5], which shows that if $(f, h) : (T, \mathcal{N}) \to (X^{sm}, \mathcal{M}_X|_{X^{sm}})$ is a sliced resolution, then h is an isomorphism. □

Remark 3.3.3. We can improve slightly on [Iw, Thm 3.3(2)]. If X is a good toroidal embedding ([Iw, 1.2]), then $\mathfrak{X}$ is a tame Artin stack in the sense of [AOV, Def 3.1]. It follows from Lemma 2.5.5 that [Iw, Thm 3.3(2)] still holds for such X. That is, we do not need to assume that X is a tame toroidal embedding ([Iw, 1.2]).

We end this section by showing that Iwanarai's stack of admissible free resolutions can also be generalized to the case when the charts of X are given by toric monoids which are not necessarily simplicial. Throughout the rest of this section, k is a field, $S = \operatorname{Spec} k$ has the trivial log structure, and X is a log scheme which is log smooth over S.

Definition 3.3.4. If P is an fs sharp monoid, $i : P \to F$ is its minimal free resolution, and b_i are positive integers for every irreducible element v_i of F, then $P \to F'$ is an *admissible free resolution of type* (b_i) if it is isomorphic to

$$P \xrightarrow{i} F \xrightarrow{\cdot(b_i)} F.$$

We say that $P \to F'$ is an *admissible sliced resolution of type* (b_i) if it is isomorphic to

$$P \xrightarrow{i'} F \to F/H,$$

where i' is an admissible free resolution of type (b_i), and where H is a face of F such that $H \cap i'(P) = 0$.

Note that if P is simplicial, then $i' : P \to F'$ is an admissible free resolution of type (b_i) in the sense of [Iw, Def 2.5] if and only if it is in the sense of Definition 3.3.4 if and only if it is an admissible sliced resolution of type (b_i).

To define the corresponding notions for morphisms of log schemes, we first generalize [Iw, Prop 3.1].

Proposition 3.3.5. *For every geometric point $\bar{x}$ of X, there is a canonical bijection between the irreducible elements of the minimal free resolution of $\bar{\mathcal{M}}_{X,\bar{x}}$ and the irreducible components of $X - X^{triv}$ on which $\bar{x}$ lies.*

Proof. As the proof of [Iw, Prop 3.1] shows, we need only address the case when $X = \operatorname{Spec} k[P]$, where P is an fs sharp monoid. We can further assume that $\bar{x}$ maps to the torus-invariant point, so that $\bar{\mathcal{M}}_{X,\bar{x}} = P$. Then the irreducible components of $X - X^{triv}$ are the torus-invariant divisors of X, and we see that $\bar{x}$ lies on all of them. The torus-invariant divisors are in canonical bijection with the extremal rays of $C(P)^\vee$, which are precisely the irreducible elements of the minimal free resolution of P. □

Definition 3.3.6. Let b_i be a positive integer for every irreducible component D_i of $X - X^{triv}$. For every geometric point $\bar{x}$ of X, let $I(\bar{x})$ be the set of irreducible components of $X - X^{triv}$ on which $\bar{x}$ lies. Then a morphism $f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)$ from a fine log scheme is an *admissible free resolution of type* (b_i), resp. is an *admissible sliced resolution of type* (b_i) if for all geometric points $\bar{y}$ of Y, the induced morphism

$$\bar{\mathcal{M}}_{X,f(\bar{y})} \to \bar{\mathcal{M}}_{Y,\bar{y}}$$

is an admissible free resolution of type $(b_i)_{i \in I(f(\bar{y}))}$, resp. is an admissible sliced resolution of type $(b_i)_{i \in I(f(\bar{y}))}$.

With this definition in place, for any choice (b_i) of positive integers indexed by the irreducible components of $X - X^{triv}$, let $\mathfrak{X}_{(b_i)}$ be the fibered category over X-schemes whose objects are morphisms $(T, \mathcal{N}) \to (X, \mathcal{M}_X)$ which are admissible sliced resolutions of type (b_i), and whose morphisms are maps of $(X, \mathcal{M}_X)$-log schemes $h : (T, \mathcal{N}) \to (T', \mathcal{N}')$ with $h^*\mathcal{N}' \to \mathcal{N}$ an isomorphism. As before, this fibered category is a stack on the fppf site by [Ol2, Thm A.1].

The proofs of Propositions 3.2.8 and 3.3.1 apply word for word after replacing "minimal free resolution" by "admissible free resolution of type (b_i)", and "sliced resolution" by "admissible sliced resolution of type (b_i)" to show the following two propositions:

Proposition 3.3.7. *Let P be an fs sharp monoid and $i : P \to F$ its minimal free resolution. Let b_i be a positive integer for every irreducible element v_i of F. If $i' : P \to F'$ is an admissible free resolution of type (b_i) and $X = \operatorname{Spec} k[P]$, then the induced morphism*

$$f : X \to \operatorname{Spec} k[F']$$

of log schemes is an admissible sliced resolution of type (b_i); here we are using Proposition 3.3.5 to identify the irreducible components of $X - X^{triv}$ and the irreducible elements of F.

Proposition 3.3.8. *Let P be an fs sharp monoid and $i : P \to F$ its minimal free resolution. Let b_i be a positive integer for every irreducible element v_i of F. If $i' : P \to F'$ is an admissible free resolution of type (b_i) and $X = \operatorname{Spec} k[P]$, then*

$$\mathfrak{X}_{(b_i)} \simeq [\operatorname{Spec} k[F']/D(F'^{gp}/i'(P^{gp}))]$$

over X.

Using Proposition 3.3.8, we prove an analogue of Theorem 3.3.2. Note that if some $b_i > 1$, then a morphism $F \to F'$ which is an admissible sliced resolution of type (b_i) has automorphisms. As a result, the stacks $\mathfrak{X}_{(b_i)}$ are not isomorphic to X over X^{sm}; they are, however, isomorphic to X over X^{triv}:

Theorem 3.3.9. *Let b_i be a positive integer for every irreducible component D_i of $X - X^{triv}$. Then $\mathfrak{X}_{(b_i)}$ is a smooth, log smooth Artin stack over S. The natural morphism $\mathfrak{X}_{(b_i)} \to X$ is a good moduli space and the base change of this morphism to X^{triv} is an isomorphism.*

Proof. The proof is the same as that of Theorem 3.3.2. We address only the last assertion. If $(f, h) : (T, \mathcal{N}) \to (X^{triv}, \mathcal{M}_X|_{X^{triv}})$ is an admissible sliced resolution of type (b_i), then $\bar{h} = 0$ is an isomorphism, and so h is strict. □

The $\mathfrak{X}_{(b_i)}$ are all root stacks over the stack $\mathfrak{X} = \mathfrak{X}_{(1)}$ in Theorem 3.3.2. As we will see in Chapter 6, if we restrict X to being a toric variety, rather than an arbitrary log smooth log scheme, then we can construct many other smooth log smooth stacks having X as a good moduli space.

Chapter 4

The Chevalley-Shephard-Todd Theorem for Diagonalizable Group Schemes

4.1 Polynomial Invariants of Diagonalizable Group Schemes

Throughout this chapter, k is a field and $D(B)$ denotes the diagonalizable group scheme over k associated to a finitely-generated abelian group B. Throughout this chapter, we fix a finitely-generated abelian group A, we let $G = D(A)$ and we fix a faithful action of G on a finite-dimensional k-vector space V. Our goal is to give necessary and sufficient conditions for when the invariants $k[V]^G$ is a polynomial algebra over k. In the process of working toward this goal, we give necessary and sufficient conditions for when $\mathrm{Spec}(k[V]^G)$ is a simplicial toric variety (see Theorem 4.1.5).

Since G is diagonalizable, there is a free monoid F' such that $k[V] = k[F']$ and such that the action of G on $k[V]$ is induced from a morphism of monoids $\pi : F' \to A$. Then there is an exact morphism $i' : P \to F'$ such that $A = F'^{gp}/i'(P^{gp})$ and the induced morphism $k[P] \to k[F']^G$ is an isomorphism.

We state the main theorem of this chapter after first giving a definition.

Definition 4.1.1. Given a torus $T = \mathrm{Spec}\, k[M]$ over k and a faithful action of T on a finite-dimensional k-vector space W, we say the action *is orderly* if there are weights $m_1, \ldots, m_k \in M$ such that the $m_i \otimes 1$ are a basis for $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and all other weights are non-positive linear combinations of the m_i. We say that $m_1, \ldots, m_k$ is an *orderly basis*.

Theorem 4.1.2. *If the induced action of the torus of G on V is not orderly, then $k[V]^G$ is not polynomial. If the action is orderly, then fix an orderly basis $m_1, \ldots, m_k$ and let F'_0 be the face generated by the elements of F' which map to the positive orthant of $m_1, \ldots, m_k$. If B denotes the image of $F_0'^{gp}$ in A under π^{gp}, then A/B is a finite abelian group and the induced action of*

$D(A/B)$ on $k[F'/F'_0]$ has the property that

$$k[V]^G = k[F'/F'_0]^{D(A/B)};$$

therefore $k[V]^G$ is polynomial if and only if the action of $D(A/B)$ on $k[F'/F'_0]$ is generated by pseudo-reflections, as defined in Definition 2.1.2.

We begin with some observations. Let $i : P \to F$ be the minimal free resolution (as in Section 3.2) and let the ranks of F and F' be d and d', respectively. Choosing an isomorphism $F' \simeq \mathbb{N}^{d'}$, we see that $i'(p) = (w_j(p))_j$, where the w_j are morphisms from P to $\mathbb{N}$. As in the proof of Proposition 3.2.2, exactness of i' shows that $\mathrm{Cone}(w_j) = C(P)^\vee$ and so rechoosing our isomorphism of F' with $\mathbb{N}^{d'}$ if necessary, we can assume $w_j = c_j v_j$ for $1 \le j \le d$ and positive integers c_j. We see then that $d \le d'$. For $d < j \le d'$, we have

$$w_j = \sum_{i=1}^{d} c_{ij} v_i$$

for $c_{ij} \in \mathbb{Q}_{\geq 0}$. We can therefore find a positive integer n and isomorphisms $F \simeq \mathbb{N}^d$ and $F' \simeq \mathbb{N}^{d'}$ such that the diagram

$$\begin{array}{ccccc} F' & \xrightarrow{\cdot n} & F' & \xrightarrow{\simeq} & \mathbb{N}^{d'} \\ {\scriptstyle i'}\uparrow & & \uparrow & & \uparrow{\scriptstyle \psi} \\ P & \xrightarrow{i} & F & \xrightarrow{\simeq} & \mathbb{N}^d \end{array}$$

commutes, where $b_i = c_i n$ and $b_{ij} = c_{ij} n$, and ψ is given by the $d' \times d$ matrix whose top $d \times d$ block is $\mathrm{diag}(b_i)$ and whose bottom block has entries b_{ij}. For the rest of this section, we identify F with $\mathbb{N}^d$ and F' with $\mathbb{N}^{d'}$ via the above isomorphisms.

We now show the necessity of the condition that the action be orderly in Theorem 4.1.2.

Proposition 4.1.3. *If $k[V]^G$ is a polynomial algebra, then the action of the torus of G on V is orderly.*

Proof. If $k[V]^G$ is polynomial, then P is free. So, i is an isomorphism and we can take $n = 1$. If C denotes the torsion-free part of A, then the torus T of G is $D(C)$ and the action of T on $k[F']$ is induced from the morphism $\sigma : F' \xrightarrow{\pi} A \to C$. From the explicit description of ψ, we see $\sigma(e_i) \otimes 1$ for $i > d$ form a basis for $C \otimes_{\mathbb{Z}} \mathbb{Q}$ and the $\sigma(e_j)$ for $j \le d$ are all non-positive linear combinations of the $\sigma(e_i)$ for $i > d$. Hence, the action of T is orderly. □

Lemma 4.1.4. *The action of the torus of G on V is orderly if and only if there is a subset S of $\{1, 2, \ldots, d'\}$ and a basis $\{p_j\}_{j \notin S}$ of $P^{gp} \otimes \mathbb{Q}$ with each $p_j \in P$ satisfying $w_j(p_j) > 0$ and $w_k(p_j) = 0$ for $j \ne k \notin S$. There is a canonical bijection*

$$S \mapsto \{\bar{e}_i \otimes 1 \mid i \in S\}$$

between such subsets and the set of orderly bases, where $\bar{e}_i$ denotes the image in A of the standard generators $e_i \in \mathbb{N}^{d'} \simeq F'$.

Proof. To prove the "if" direction we show that the $\bar{e}_i \otimes 1$ with $i \in S$ form an orderly basis. Let $a_j = w_j(p_j)$ and let $a_{ij} = w_i(p_j) \geq 0$ for $i \in S$. Since the p_j form a basis for $P^{gp} \otimes \mathbb{Q}$, we see that

$$|S| = d' - \operatorname{rank} P^{gp},$$

which is the rank of the torus of G. To prove that the $\bar{e}_i \otimes 1$ form a basis, we therefore need only show that they are linearly independent. If $\sum_{i \in S} c_i \bar{e}_i = 0$ for some $c_i \in \mathbb{Q}$, then

$$\sum_{i \in S} c_i e_i = i'(p)$$

in F'^{gp} for some $p \in P^{gp} \otimes \mathbb{Q}$. Writing $p = \sum_{j \notin S} c'_j p_j$ for $c'_j \in \mathbb{Q}$, and comparing the e_j coordinates above, we see $c'_j a_j = 0$, which shows that $p = 0$, and so the $c_i = 0$ as well.

To show that the $\bar{e}_i \otimes 1$ form an orderly basis, note that for $j \notin S$,

$$a_j e_j + \sum_{i \in S} a_{ij} e_i = \sum_{k=1}^{d'} w_k(p_j) e_k = i'(p_j).$$

Since the $a_{ij} \geq 0$, we see then that $\bar{e}_j \otimes 1$ is a non-positive linear combination of the $\bar{e}_i \otimes 1$ for $i \in S$.

We now prove the "only if" direction. If $j \notin S$, then $\bar{e}_j \otimes 1$ is a non-positive linear combination of the $\bar{e}_i \otimes 1$ for $i \in S$. We therefore have some positive integer a_j, non-negative integers a_{ij}, and some $p_j \in P^{gp}$ such that

$$a_j e_j = -\sum_{i \in S} a_{ij} e_i + \sum_{k=1}^{d'} w_k(p_j) e_k.$$

We see then that $w_j(p_j) = a_j$, $w_i(p_j) = a_{ij}$ if $i \in S$, and $w_k(p_j) = 0$ otherwise. Since $w_\ell(p_j)$ is non-negative for all ℓ, exactness of i' shows that $p_j \in P$.

We prove that the p_j form a basis for $P^{gp} \otimes \mathbb{Q}$. Since the rank of P^{gp} is $d' - |S|$, we need only show that the p_j are linearly independent. If $\sum_{j \notin S} c_j p_j = 0$ for $c_j \in \mathbb{Q}$, then applying w_{j_0} for $j_0 \notin S$ shows that $c_{j_0} a_{j_0} = 0$. Since a_{j_0} is positive, we have $c_{j_0} = 0$, as desired. □

The following theorem gives necessary and sufficient conditions for when $k[V]^G$ is simplicial (*c.f.* [We, Thm 4.1]).

Theorem 4.1.5. *The action of the torus of G on V is orderly if and only if P is simplicial. Furthermore, when these equivalent conditions are satisfied, if we let $S \subset \{1, 2, \ldots, d'\}$ correspond to our choice of an orderly basis as in Lemma 4.1.4, then the extremal rays of $C(P)^\vee$ are precisely the rays defined by the w_j for $j \notin S$, each lying on a distinct extremal ray.*

Proof. Suppose first that P is simplicial. Then the minimal free resolution $i : P \to F$ is a close morphism (see the first paragraph of Section 3.2). For all $j \leq d$, let $p_j \in P$ such that $i(p_j) = \lambda_j e_j$ for some $\lambda_j \in \mathbb{N}$. Since P is simplicial, we see that the p_j form a basis for $P^{gp} \otimes \mathbb{Q}$, that $v_j(p_j) = \lambda_j > 0$, and that $v_k(p_j) = 0$ for $j \neq k \leq d$. Since w_j is a multiple of v_j for $j \leq d$, we see that Lemma 4.1.4 finishes the proof of the "if" direction.

Assume now that the action of the torus of G on V is orderly, and let S be as in Lemma 4.1.4. We prove the "only if" direction in the process of proving the second assertion of the theorem. We begin by showing that the w_j for $j \notin S$ lie on extremal rays. Fix $j_0 \notin S$ and suppose that w_{j_0} does not lie on an extremal ray. Then we have

$$w_{j_0} = \sum_{k=1}^{r} c'_k v_{i_k}$$

for positive rational numbers c'_k, distinct i_k, and $r \geq 2$. If $j \notin S$ and $j \neq j_0$, then $w_{j_0}(p_j) = 0$. Since the $p_j \in P$, we see the $v_{i_k}(p_j) \geq 0$, and so $v_{i_k}(p_j) = 0$. Since the p_j for $j \notin S$ form a basis for $P^{gp} \otimes \mathbb{Q}$, we see that the v_{i_k} are determined by their values on p_{j_0}. They therefore all define the same extremal ray, which is a contradiction.

To prove that the w_j for $j \notin S$ lie on distinct extremal rays, assume that $w_k = cw_j$ for some positive rational number c. Then we see

$$0 = w_k(p_j) = cw_j(p_j) = ca_j > 0$$

which is a contradiction.

Lastly, we show that every extremal ray actually does occur as one of the w_j for $j \notin S$. Note that this implies that P is simplicial, for then

$$d = d' - |S| = \operatorname{rank} P^{gp}.$$

Since the w_j for $j \notin S$ lie on distinct extremal rays, they define a basis for the $\mathbb{Q}$-vector space $\operatorname{Hom}(P^{gp}, \mathbb{Z}) \otimes \mathbb{Q}$. Hence for $i \in S$, we have $c'_{ij} \in \mathbb{Q}$ such that

$$w_i = \sum_{j \notin S} c'_{ij} w_j.$$

Since $0 \leq w_i(p_j) = c'_{ij} a_j$, we see that the $c'_{ij} \geq 0$. Therefore, if w_i is an extremal ray, it must be one of the w_j for $j \notin S$. This finishes the proof, as every extremal ray of $C(P)^\vee$ must occur as some w_k. □

Proof of Theorem 4.1.2. The necessity of the orderly condition is shown in Theorem 4.1.3, so we assume now that the action of the torus of G on V is orderly. By Proposition 4.1.5, we can assume that the $\bar{e}_i \otimes 1$ for $i > d$ is an orderly basis. Then F'_0 is the face of F' generated by the e_i for $i > d$. Note that F'/F'_0 is free of rank d and that we have a commutative diagram

$$\begin{array}{ccccc}
F'/F'_0 & \xrightarrow{\cdot n} & F'/F'_0 & \xrightarrow{\simeq} & \mathbb{N}^d \\
{\scriptstyle \pi'}\uparrow & & {\scriptstyle \pi'}\uparrow & & \uparrow{\scriptstyle \psi'} \\
F' & \xrightarrow{\cdot n} & F' & \xrightarrow{\simeq} & \mathbb{N}^{d'} \\
{\scriptstyle i'}\uparrow & & \uparrow & & \uparrow{\scriptstyle \psi} \\
P & \xrightarrow{i} & F & \xrightarrow{\simeq} & \mathbb{N}^d
\end{array}$$

where $\psi'(a_1, \ldots, a_{d'}) = (a_1, \ldots, a_d)$. We see then that $\psi'\psi = \mathrm{diag}(b_i)$ and is therefore exact. Since the multiplication by n map from F'/F'_0 to itself is exact, [Og, Prop I.4.1.3(2)] shows that $\pi' i'$ is exact as well. Hence,

$$k[V]^G = k[P] = k[F'/F'_0]^{D((F'/F'_0)^{gp}/\pi' i'(P^{gp}))}.$$

Furthermore, since P is simplicial by Theorem 4.1.5, and since F'/F'_0 is free of rank d, we see that $(F'/F'_0)^{gp}/\pi' i'(P^{gp})$ is a finite group. Note that by the definition of B, there exists a morphism φ making the diagram

$$\begin{array}{ccc}
F'^{gp} & \xrightarrow{\pi'} & F'^{gp}/F_0'^{gp} \\
\downarrow & & \downarrow{\scriptstyle \varphi} \\
A & \longrightarrow & A/B
\end{array}$$

commute. One easily checks that the kernel of φ is $\pi' i'(P^{gp})$, and so A/B is isomorphic to $(F'/F'_0)^{gp}/\pi' i'(P^{gp})$. This proves that A/B is a finite group and that

$$k[V]^G = k[F'/F'_0]^{D(A/B)}$$

as desired. Lastly, since $D(A/B)$ is a finite diagonalizable group scheme, we see from Theorem 2.1.6 and Proposition 2.2.1 that $k[V]^G$ is polynomial if and only if the action of $D(A/B)$ on $k[F'/F'_0]$ is generated by pseudo-reflections. $\square$

Chapter 5

de Rham Theory for Schemes with Linearly Reductive Singularities

Given a scheme X smooth and proper over a field k, the cohomology of the algebraic de Rham complex $\Omega^\bullet_{X/k}$ is an important invariant of X, which, when $k = \mathbb{C}$, recovers the singular cohomology of $X(\mathbb{C})$. When the Hodge-de Rham spectral sequence

$$E_1^{st} = H^t(\Omega^s_{X/k}) \Rightarrow H^n(\Omega^\bullet_{X/k})$$

degenerates, the invariants $\dim_k H^n(\Omega^\bullet_{X/k})$ break up into sums of the finer invariants $\dim_k H^t(\Omega^s_{X/k})$. The degeneracy of this spectral sequence for smooth proper schemes in characteristic 0 was first proved via analytic methods. It was not until much later that Faltings [Fa] gave a purely algebraic proof by means of p-adic Hodge Theory. Soon afterwards, Deligne and Illusie [DI] gave a substantially simpler algebraic proof by showing that the degeneracy of the Hodge-de Rham spectral sequence in characteristic 0 is implied by its degeneracy for smooth proper schemes in characteristic p that lift mod p^2. Their method therefore extends de Rham Theory to the class of smooth proper schemes in positive characteristic which lift. A form of de Rham Theory also exists for certain singular schemes. Steenbrink showed [St, Thm 1.12] that if k is a field of characteristic 0, M a proper k-scheme with quotient singularities, and $j : M^0 \hookrightarrow M$ its smooth locus, then the hypercohomology spectral sequence

$$E_1^{st} = H^t(j_*\Omega^s_{M^0/k}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/k})$$

degenerates and $H^n(j_*\Omega^\bullet_{M^0/k})$ agrees with $H^n(M(\mathbb{C}), \mathbb{C})$ when $k = \mathbb{C}$. As we explain in this chapter, a version of this theorem is true in positive characteristic as well: if k has characteristic p and M is proper with quotient singularities by groups whose orders are prime to p, then the above spectral sequence degenerates for $s + t < p$ provided a certain liftability criterion is satisfied (see Theorem 5.1.14 for precise hypotheses).

As a warm-up for the rest of the chapter, we begin by showing how Steenbrink's result can be reproved using the theory of stacks. The idea is as follows. Every scheme M as above is the coarse space of a smooth Deligne-Mumford stack $\mathfrak{X}$ whose stacky structure is supported at the singular locus of M. We show that the de Rham cohomology $H^n(\Omega^\bullet_{\mathfrak{X}/k})$ of the stack agrees with $H^n(j_*\Omega^\bullet_{M^0/k})$. After checking that the method of Deligne-Illusie extends to Deligne-Mumford stacks, we recover Steenbrink's result as a consequence of the degeneracy of the Hodge-de Rham spectral sequence for $\mathfrak{X}$.

The above extends de Rham Theory to the class of schemes with quotient singularities by groups whose orders are prime to the characteristic, but in positive characteristic this class of schemes contains certain "gaps" and it is natural to ask if de Rham Theory can be extended further. For example, in all characteristics except for 2, the affine quadric cone $\operatorname{Spec} k[x,y,z]/(xy-z^2)$ can be realized as the quotient of $\mathbb{A}^2$ by $\mathbb{Z}/2\mathbb{Z}$ under the action $x \mapsto -x$, $y \mapsto -y$. In characteristic 2, however, this action is trivial. If we allow quotients not just by finite groups, but rather finite group schemes, then we can realize the cone as $\mathbb{A}^2/\mu_2$ where $\zeta \in \mu_2(T)$ acts as $x \mapsto \zeta x$, $y \mapsto \zeta y$. This is an example of what we call a scheme with *linearly reductive singularities*; that is, a scheme which is étale locally the quotient of a smooth scheme by a finite flat linearly reductive group scheme.

One of the main results of this chapter is that de Rham Theory can be extended to the class of schemes with isolated linearly reductive singularities. As with Steenbrink's result, we prove this by passing through stacks. In light of our stacky resolution theorem for schemes with linearly reductive singularities (Theorem 2.1.10), we begin by showing the degeneracy of a type of Hodge-de Rham spectral sequence for tame stacks. We should emphasize that, unlike in the case of Deligne-Mumford stacks, there are technical barriers to extending the method of Deligne-Illusie to Artin stacks or even tame stacks, first and foremost being that relative Frobenius does not behave well under smooth base change. It should also be noted that it is *a priori* not clear what the definition of the de Rham complex of a tame stack $\mathfrak{X}$ should be. One can use the cotangent complex $L_{\mathfrak{X}}$ of the stack (see [LMB, §15] and [Ol3, §8]) to define the derived de Rham complex $\bigwedge^\bullet L_{\mathfrak{X}}$; alternatively, one can use a more naive sheaf of differentials $\varpi^1_{\mathfrak{X}}$ on the lisse-étale site of $\mathfrak{X}$ whose restriction to each U_{et} is Ω^1_U, for every U smooth over $\mathfrak{X}$. The latter has the advantage that it is simpler, but it is *not coherent*; the cotangent complex, on the other hand, has coherent cohomology sheaves. We take the naive de Rham complex as our definition, but it is by comparing this complex with the derived de Rham complex that we prove our main result for tame stacks:

Theorem 5.3.7. *Let $\mathfrak{X}$ be a smooth proper tame stack over a perfect field k of characteristic p. If $\mathfrak{X}$ lifts mod p^2, then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\varpi^s_{\mathfrak{X}/k}) \Rightarrow H^n(\varpi^\bullet_{\mathfrak{X}/k})$$

degenerates for $s+t<p$ (see the Notation section below).

From Theorem 5.3.7 and Theorem 2.1.10, we are able to deduce

Theorem 5.4.6. *Let M be a proper k-scheme with isolated linearly reductive singularities, where k is a perfect field of characteristic p. Let $j : M^0 \hookrightarrow M$ be the smooth locus of M and let $\mathfrak{X}$ be as in Theorem 2.1.10. If $\mathfrak{X}$ lifts mod p^2, then the hypercohomology spectral sequence*

$$E_1^{st} = H^t(j_*\Omega^s_{M^0/k}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/k})$$

degenerates for $s+t<p$.

We should mention that unlike in the case of quotient singularities, the cohomology groups $H^n(\varpi^\bullet_{\mathfrak{X}/k})$ and $H^n(j_*\Omega^\bullet_{M^0/k})$ no longer agree, so some care is needed in showing how Theorem 5.4.6 follows from the degeneracy of the Hodge-de Rham spectral sequence of the stack.

It is desirable, of course, to remove the stack from the statement Theorem 5.4.6. We show in Theorem 5.4.7 that if the dimension of M is at least 4, the liftability of M implies the liftability of $\mathfrak{X}$. In this case, we therefore have a purely scheme-theoretic statement of Theorem 5.4.6. We end the chapter by proving a type of Kodaira vanishing theorem within this setting.

This chapter is organized as follows. In Section 5.1, we begin by reviewing some background material and giving an outline of [DI, Thm 2.1] as some of the technical details will be used later. We then consider de Rham Theory for Deligne-Mumford stacks and show how stacks can be used to recast Steenbrink's result. The purpose of Section 5.2 is to find a way around the problem that the method of Deligne and Illusie does not carry over directly to the lisse-étale site of Artin stacks. Since relative Frobenius *does* behave well under étale base change, our solution is to prove a Deligne-Illusie result on the étale site of $X_\bullet$, where $X\to\mathfrak{X}$ is a smooth cover of a smooth tame stack by a scheme, and $X_\bullet$ is the simplicial scheme obtained by taking fiber products over $\mathfrak{X}$. The key technical point here is showing that étale locally on the coarse space of $\mathfrak{X}$, the relative Frobenius for $\mathfrak{X}$ lifts mod p^2. In Section 5.3, we prove that the naive de Rham complex and the derived de Rham complex above compute the same cohomology, and show how this result implies the degeneracy of the Hodge-de Rham spectral sequence for smooth proper tame stacks which lift mod p^2. In Section 5.4, we prove Theorem 5.4.6.

Notation. Unless otherwise stated, all Artin stacks in this chapter are assumed to have finite diagonal. If $\mathfrak{X}$ is an Artin stack over a scheme S, we let $\mathfrak{X}'$ denote the pullback of $\mathfrak{X}$ by the absolute Frobenius F_S. We usually drop the subscript on the relative Frobenius $F_{\mathfrak{X}/S}$, denoting it by F. Given a morphism $g:\mathfrak{X}_1\to\mathfrak{X}_2$ of S-stacks, we denote by $g':\mathfrak{X}_1'\to\mathfrak{X}_2'$ the induced morphism.

Given a morphism $g:\mathfrak{X}_1\to\mathfrak{X}_2$ of Artin stacks and complex of sheaves $\mathcal{F}^\bullet$ on $\mathfrak{X}_1$, we do not use the shorthand $g_*\mathcal{F}^\bullet$ when we mean $Rg_*\mathcal{F}^\bullet$. For us, $g_*\mathcal{F}^\bullet$ always denotes the complex obtained by applying the functor g_* to the complex $\mathcal{F}^\bullet$.

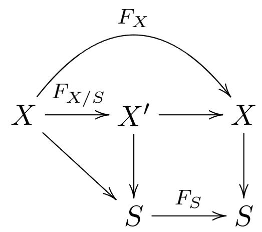
Lastly, we say a first quadrant spectral sequence E_{r_0} "degenerates for $s+t<N$" if for all $r\geq r_0$ and all s and t satisfying $s+t<N$, all of the differentials to and from the E_r^{st} are zero.

5.1 Steenbrink's Result via Stacks

5.1.1 Review of Deligne-Illusie

We briefly review the proof of [DI, Thm 2.1] and explain how it generalizes to Deligne-Mumford stacks. Having an outline of this proof will be useful for us in Section 5.2.

Let $S = \operatorname{Spec} k$ be a perfect field of characteristic p. For any S-scheme X, let $F_X : X \to X$ be the absolute Frobenius, which acts as the identity on topological spaces and sends a local section $s \in \mathcal{O}_X(U)$ to s^p. We have the following commutative diagram, where $F_{X/S}$ is the relative Frobenius and the square is cartesian



We drop the subscript on the relative Frobenius $F_{X/S}$, denoting it by F. If X is locally of finite type over S, so that it is locally $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, where $f_j = \sum a_{j,I} x^I$, then X' is locally $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1^{(p)}, \ldots, f_m^{(p)})$, where $f_j^{(p)} := \sum a_{j,I}^p x^I$. The relative Frobenius morphism is then given by sending x_i to x_i^p and $a \in k$ to a.

Our primary object of study is the de Rham complex $\Omega^\bullet_{X/S}$. The maps $d : \Omega^k_{X/S} \to \Omega^{k+1}_{X/S}$ in the complex are defined as the composite

$$\Omega^k_{X/S} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \xrightarrow{d \otimes id} \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^k_{X/S} \longrightarrow \Omega^{k+1}_{X/S},$$

where the last map is given by $\eta \otimes \omega \mapsto \eta \wedge \omega$. Note that the maps in the complex $\Omega^\bullet_{X/S}$ are not $\mathcal{O}_X$-linear. To correct this "problem" we instead consider $F_*\Omega^\bullet_{X/S}$, whose maps *are* $\mathcal{O}_{X'}$-linear. It is now reasonable to ask how the cohomology of this new complex compares with the cohomology of the de Rham complex on X'. An answer is given by:

Theorem 5.1.1 (Cartier isomorphism)**.** *If X is smooth over S, then there is a unique isomorphism of $\mathcal{O}_{X'}$-graded algebras*

$$C^{-1} : \bigoplus \Omega^i_{X'/S} \longrightarrow \bigoplus \mathcal{H}^i(F_*\Omega^\bullet_{X/S})$$

such that $C^{-1}d(x \otimes 1)$ is the class of $x^{p-1}dx$ for all local sections x of $\mathcal{O}_{X'}$.

Note that once C^{-1} is shown to exist, uniqueness is automatic. For a proof of this theorem, see [Ka, Thm 7.2].

We are now ready to discuss [DI, Thm 2.1].

Theorem 5.1.2. *Let $W_2(k)$ be the ring of truncated Witt vectors and let $\tilde{S} = \operatorname{Spec} W_2(k)$. If X is smooth over S, then to every smooth lift $\tilde{X}$ of X to $\tilde{S}$, there is an associated isomorphism*

$$\varphi : \bigoplus_{i<p} \Omega^i_{X'/S}[-i] \longrightarrow \tau_{<p} F_*\Omega^\bullet_{X/S}$$

in the derived category of $\mathcal{O}_{X'}$-modules such that $\mathcal{H}^i(\varphi) = C^{-1}$ for all $i < p$.

We give a sketch of the argument. To define φ, we need only define $\varphi^i : \Omega^i_{X'/S}[-i] \to \tau_{<p} F_* \Omega^\bullet_{X/S}$ such that $\mathcal{H}^i(\varphi) = C^{-1}$ for each $i < p$. We take φ^0 to be the composite

$$\mathcal{O}_{X'} \xrightarrow{C^{-1}} \mathcal{H}^0 F_* \Omega^\bullet_{X/S} \longrightarrow F_* \Omega^\bullet_{X/S}.$$

Suppose for the moment that φ^1 has already been defined. For $i > 1$, we can then define φ^i to be the composite

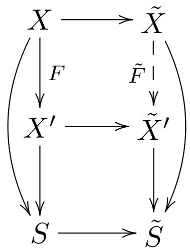
$$\Omega^i_{X/S}[-i] \xrightarrow{a[-i]} (\Omega^1_{X/S})^{\otimes i}[-i] \xrightarrow{(\varphi^1)^{\otimes i}} (F_* \Omega^\bullet_{X/S})^{\otimes i} \xrightarrow{b} F_* \Omega^\bullet_{X/S},$$

where

$$a(\omega_1 \wedge \cdots \wedge \omega_i) = \frac{1}{i!} \sum_{\sigma \in S_i} (\operatorname{sign} \sigma)\, \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}$$

and $b(\omega_1 \otimes \cdots \otimes \omega_i) = \omega_1 \wedge \cdots \wedge \omega_i$.

Thus, we are reduced to defining φ^1. Suppose first that Frobenius lifts; that is, there exists $\tilde{F}$ filling in the diagram



where $\tilde{X}' = \tilde{X} \times_{\tilde{S},\sigma} \tilde{S}$ and σ is the Witt vector Frobenius automorphism. Let $\mathbf{p} : \mathcal{O}_X \xrightarrow{\simeq} p\mathcal{O}_{\tilde{X}}$ be the morphism sending x_0 to px for any local section x of $\mathcal{O}_{\tilde{X}}$ reducing mod p to x_0. Note that if $x \otimes 1$ is a local section of $\mathcal{O}_{\tilde{X}} \otimes_{W_2(k),\sigma} W_2(k) = \mathcal{O}_{\tilde{X}'}$, then $\tilde{F}^*(x \otimes 1) = x^p + \mathbf{p}(u(x))$ for a unique local section $u(x)$ of $\mathcal{O}_X$. We define a morphism $f : \Omega^1_{X'/S} \to F_* \Omega^1_{X/S}$ by

$$f(dx_0 \otimes 1) = x_0^{p-1} dx_0 + du(x).$$

Deligne and Illusie show that φ^1 can be taken to be f. Given two different choices $\tilde{F}_1$ and $\tilde{F}_2$ of F, we obtain a homotopy h_{12} relating f_1 and f_2, defined by $h_{12}(dx_0 \otimes 1) = u_2(x) - u_1(x)$.

Note that F lifts locally since the obstruction to lifting it lies in

$$\operatorname{Ext}^1(F^* \Omega^1_{X'/S}, \mathcal{O}_X) = H^1(X, F^* T_{X'/S}).$$

So, to define φ^1 in general, we need only patch together the local choices. This is done as follows. Let $\mathcal{U} = \{U_i\}$ be a cover on which Frobenius lifts and let $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ denote the sheafified version of the Čech complex of a sheaf $\mathcal{F}$. We define φ^1 to be the morphism in the derived category

$$\Omega^1_{X'/S}[-1] \xrightarrow{\Phi} \operatorname{Tot}(F_* \check{\mathcal{C}}^\bullet(\mathcal{U}, \Omega^\bullet_{X/S})) \xleftarrow{\simeq} F_* \Omega^\bullet_{X/S},$$

where

$$\Phi = (\Phi_1, \Phi_2) : \Omega^1_{X'/S} \to F_*\check{\mathcal{C}}^1(\mathcal{U}, \mathcal{O}_X) \oplus F_*\check{\mathcal{C}}^0(\mathcal{U}, \Omega^\bullet_{X/S})$$

is given by $(\Phi_1(\omega))_{ij} = h_{ij}(\omega|U'_{ij})$ and $(\Phi_2(\omega))_i = f_i(\omega|U'_i)$. Deligne and Illusie further show that this is independent of the choice of covering. This completes the proof.

Remark 5.1.3. In the local case where Frobenius lifts, φ is a morphism of complexes. It is only when patching together the local choices that we need to pass to the derived category.

Remark 5.1.4. Using the fact that for any étale morphism $g : Y \to Z$ of S-schemes, the pullback of $F_{Z/S} : Z \to Z'$ by g is $F_{Y/S}$, one can check that the proof of Theorem 5.1.2 works when X is a Deligne-Mumford stack. Alternatively, this will follow from the proof of Theorem 5.2.5 below.

Given any abelian category $\mathcal{A}$ with enough injectives, a left exact functor $G : \mathcal{A} \to \mathcal{B}$ to another abelian category, and a bounded below complex of objects $A^\bullet$ of $\mathcal{A}$, we obtain a hypercohomology spectral sequence

$$E_1^{st} = R^tG(A^s) \Rightarrow R^nG(A^\bullet).$$

If $\mathfrak{X}$ is a Deligne-Mumford stack over a scheme Y, the hypercohomology spectral sequence $E_1^{st} = H^t(\Omega^s_{\mathfrak{X}/Y}) \Rightarrow H^n(\Omega^\bullet_{\mathfrak{X}/Y})$ obtained in this way is called the Hodge-de Rham spectral sequence.

As Deligne and Illusie show, Theorem 5.1.2 implies the degeneracy of the Hodge-de Rham spectral sequence for smooth proper schemes. We reproduce their proof, which requires no modification to handle the case of Deligne-Mumford stacks, after first isolating the following useful fact from homological algebra.

Lemma 5.1.5. *Let K be a field and r_0 a positive integer. Let $E_{r_0}^{st} \Rightarrow E^{s+t}$ be a first quadrant spectral sequence whose terms are finite-dimensional K-vector spaces and whose morphisms are K-linear. If n is a non-negative integer and*

$$\sum_{s+t=n} \dim_K E_{r_0}^{st} = \dim_K E^n,$$

then for all $r \geq r_0$ the differentials to and from the $E_r^{s,n-s}$ are zero. Hence, if the above equality holds for all $n < N$, then the spectral sequence degenerates for $s + t < N$.

Proof. Note that for all $r \geq r_0$

$$\sum_{s+t=n} \dim_K E_{r+1}^{st} \leq \sum_{s+t=n} \dim_K E_r^{st}$$

with equality if and only if all of the differentials to and from the $E_r^{s,n-s}$ are zero. Hence

$$\sum_{s+t=n} \dim_K E_\infty^{st} \leq \sum_{s+t=n} \dim_K E_{r_0}^{st}$$

with equality if and only if the differentials to and from the $E_r^{s,n-s}$ are zero for all $r \geq r_0$. Since the E_∞ terms are K-vector spaces, the extension problem is trivial, and so

$$\dim_K E^n = \sum_{s+t=n} \dim_K E_\infty^{st} \leq \sum_{s+t=n} \dim_K E_{r_0}^{st} = \dim_K E^n,$$

which completes the proof. □

Corollary 5.1.6 ([DI, Cor 2.5]). *If $\mathfrak{X}$ is a Deligne-Mumford stack over S, which is smooth, proper, and lifts mod p^2, then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\Omega^s_{\mathfrak{X}/S}) \Rightarrow H^n(\Omega^\bullet_{\mathfrak{X}/S})$$

degenerates for $s + t < p$.

Proof. By Theorem 5.1.2 and Remark 5.1.4, we have an isomorphism

$$\bigoplus_{s<p} \Omega^s_{\mathfrak{X}'/S}[-s] \longrightarrow \tau_{<p} F_* \Omega^\bullet_{\mathfrak{X}/S}$$

in the derived category of $\mathcal{O}_{\mathfrak{X}'}$-modules. It follows that for all $n < p$,

$$\bigoplus_{s+t=n} H^t(\Omega^s_{\mathfrak{X}'/S}) = H^n(\Omega^\bullet_{\mathfrak{X}/S}).$$

Using the fact that $H^t(\Omega^s_{\mathfrak{X}'/S}) = H^t(\Omega^s_{\mathfrak{X}/S}) \otimes_{k,F_k} k$, we see

$$\sum_{s+t=n} \dim_k H^t(\Omega^s_{\mathfrak{X}/S}) = \sum_{s+t=n} \dim_k H^t(\Omega^s_{\mathfrak{X}'/S}) = \dim_k H^n(\Omega^\bullet_{\mathfrak{X}/S}),$$

which, by Lemma 5.1.5, proves the degeneracy of the spectral sequence. □

Deligne and Illusie further show that the degeneracy of the Hodge-de Rham spectral sequence in positive characteristic implies the degeneracy in characteristic 0. While its degeneration in characteristic 0 had previously been known by analytic means, this provided a purely algebraic proof.

Corollary 5.1.7. *Let $\mathfrak{X}$ be a Deligne-Mumford stack which is smooth and proper over a field K of characteristic 0. Then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\Omega^s_{\mathfrak{X}/K}) \Rightarrow H^n(\Omega^\bullet_{\mathfrak{X}/K})$$

degenerates.

The proof given in [DI, Cor 2.7] for schemes requires only a minor modification. It uses that if X is a smooth proper scheme over a field K of characteristic 0, then there is an integral domain A of finite type over $\mathbb{Z}$, a morphism $A \to K$, and a smooth proper scheme Y over $\operatorname{Spec} A$ which pulls back over $\operatorname{Spec} K$ to X. Since this statement remains true when we allow X and Y to be Deligne-Mumford stacks ([MO, p.2]), the proof given in [DI, Cor 2.7] implies Corollary 5.1.7 above.

5.1.2 De Rham Theory for Schemes with Quotient Singularities

Let k be a field of characteristic 0 and let $S = \operatorname{Spec} k$. Our goal in this subsection is to use stacks to reprove [St, Thm 1.12] which states:

Theorem 5.1.8. *Let M be a proper S-scheme with quotient singularities, and let $j : M^0 \to M$ be its smooth locus. Then the hypercohomology spectral sequence*

$$E_1^{st} = H^t(j_*\Omega^s_{M^0/S}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$$

of the complex $j_*\Omega^\bullet_{M^0/S}$ degenerates. Furthermore, if $k = \mathbb{C}$, then $H^n(j_*\Omega^\bullet_{M^0/S})$ agrees with the Betti cohomology $H^n(M(\mathbb{C}), \mathbb{C})$ of M.

The following proposition gives the relationship between Deligne-Mumford stacks and schemes with quotient singularities.

Proposition 5.1.9. *Let M be an S-scheme and let $j : M^0 \to M$ be its smooth locus. Then M has quotient singularities if and only if it is the coarse space of a smooth Deligne-Mumford stack $\mathfrak{X}$ such that f^0 in the diagram*

$$\begin{array}{ccc} \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \\ {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle f} \\ M^0 & \xrightarrow[j]{} & M \end{array}$$

is an isomorphism, where $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$.

For a proof, see [FMN, Rmk 4.9] or [Vi, Prop 2.8]. Vistoli's proposition is slightly more general than the proposition above.

We give the proof of Theorem 5.1.8 after first proving a lemma which compares $j_*\Omega^\bullet_{M^0/S}$ to the de Rham complex of a Deligne-Mumford stack.

Lemma 5.1.10. *If M is an S-scheme with quotient singularities and $\mathfrak{X}$ is as in Proposition 5.1.9, then*

$$j_*\Omega^\bullet_{M^0/S} = f_*\Omega^\bullet_{\mathfrak{X}/S}.$$

Proof. To prove this equality, we need only show $j^0_*\Omega^\bullet_{\mathfrak{X}^0/S} = \Omega^\bullet_{\mathfrak{X}/S}$. That is, given an étale morphism $U \to \mathfrak{X}$, we want to show $i_*\Omega^\bullet_{U^0/S} = \Omega^\bullet_{U/S}$, where $U^0 := M^0 \times_M U$ and i is the projection to U. Since $\Omega^k_{U/S}$ is locally free, hence reflexive, the following lemma completes the proof. □

Lemma 5.1.11. *Let X be a normal scheme and $i : U \hookrightarrow X$ an open subscheme whose complement has codimension at least 2. If $\mathcal{F}$ is a reflexive sheaf on X, then the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ is an isomorphism.*

Proof. Since $\mathcal{F}$ is reflexive, $\mathcal{F} = \mathcal{H}om(\mathcal{G}, \mathcal{O}_X)$, where $\mathcal{G} = \mathcal{F}^\vee$. Therefore,

$$i_*i^*\mathcal{F} = i_*\mathcal{H}om(i^*\mathcal{G}, \mathcal{O}_U) = \mathcal{H}om(\mathcal{G}, i_*\mathcal{O}_U)$$

and since X is normal, $i_*\mathcal{O}_U = \mathcal{O}_X$. □

Proof of Theorem 5.1.8. Let $\mathfrak{X}$ be as in Proposition 5.1.9. From Lemma 5.1.10, we see that $j_*\Omega^\bullet_{M^0/S} = f_*\Omega^\bullet_{\mathfrak{X}/S}$ and $j_*\Omega^s_{M^0/S} = f_*\Omega^s_{\mathfrak{X}/S}$ for all s. Since the $\Omega^s_{\mathfrak{X}/S}$ are coherent, it follows from [AV, Lemma 2.3.4] that

$$j_*\Omega^\bullet_{M^0/S} = Rf_*\Omega^\bullet_{\mathfrak{X}/S} \quad \text{and} \quad j_*\Omega^s_{M^0/S} = Rf_*\Omega^s_{\mathfrak{X}/S}.$$

We see then that

$$H^n(j_*\Omega^\bullet_{M^0/S}) = H^n(\Omega^\bullet_{\mathfrak{X}/S}) \quad \text{and} \quad H^t(j_*\Omega^s_{M^0/S}) = H^t(\Omega^s_{\mathfrak{X}/S}).$$

Keel-Mori [KM] shows that f is proper, and so the Hodge-de Rham spectral sequence for $\mathfrak{X}$ degenerates by Corollary 5.1.7. It follows that

$$\sum_{s+t=n} \dim_k H^t(j_*\Omega^s_{M^0/S}) = \sum_{s+t=n} \dim_k H^t(\Omega^s_{\mathfrak{X}/S}) = \dim_k H^n(\Omega^\bullet_{\mathfrak{X}/S}) = \dim_k H^n(j_*\Omega^\bullet_{M^0/S}),$$

which, by Lemma 5.1.5, proves the degeneracy of the hypercohomology spectral sequence for $j_*\Omega^\bullet_{M^0/S}$.

We now show that if $k = \mathbb{C}$, then $H^n(j_*\Omega^\bullet_{M^0/S}) = H^n(M(\mathbb{C}), \mathbb{C})$. We have shown $H^n(j_*\Omega^\bullet_{M^0/S}) = H^n(\Omega^\bullet_{\mathfrak{X}})$, and GAGA for Deligne-Mumford stacks ([To, Thm 5.10]) shows

$$H^n(\Omega^\bullet_{\mathfrak{X}}) = H^n(\Omega^\bullet_{\mathfrak{X}^{an}}),$$

where $\mathfrak{X}^{an}$ is defined in [To, Def 5.6]. Note that $\mathbb{C} \to \Omega^\bullet_{\mathfrak{X}^{an}}$ is a quasi-isomorphism since this can be checked étale locally. It follows that

$$H^n(\Omega^\bullet_{\mathfrak{X}^{an}}) = H^n(\mathfrak{X}^{an}, \mathbb{C}).$$

Lastly, the singular cohomology of $\mathfrak{X}^{an}$ and that of its coarse space, $M(\mathbb{C})$, are the same. This is shown in [Be, Prop 36] for topological Deligne-Mumford stacks with $\mathbb{Q}$-coefficients, but the proof works equally well in our situation once it is combined with [To, Prop 5.7], which states $[U^{an}/G] = [U/G]^{an}$. □

We end this section with some remarks about the situation in positive characteristic. Suppose k is a perfect field of characteristic p and let $S = \operatorname{Spec} k$.

Definition 5.1.12. We say an S-scheme M (necessarily normal) has *good quotient singularities* if it has an étale cover $\{U_i/G_i \to M\}$, where the U_i are smooth over S and the G_i are finite groups of order prime to p.

Both the proof in [FMN] and in [Vi] (along with Vistoli's Remark 2.9) cited above work in positive characteristic. So, we have the following generalization of Proposition 5.1.9.

Proposition 5.1.13. *Let M be an S-scheme, and let $j : M^0 \to M$ be its smooth locus. Then M has good quotient singularities if and only if it is the coarse space of a smooth tame Deligne-Mumford stack $\mathfrak{X}$ ([AV, Def 2.3.1]) such that f^0 in the diagram*

$$\begin{array}{ccc} \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \\ {\scriptstyle f^0}\downarrow & & \downarrow{\scriptstyle f} \\ M^0 & \xrightarrow[j]{} & M \end{array}$$

is an isomorphism, where $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$.

If $\mathfrak{X}$ is a smooth proper tame Deligne-Mumford stack, then the Hodge-de Rham spectral sequence for $\mathfrak{X}$ degenerates by Corollary 5.1.6, and $f_*\mathcal{F} = Rf_*\mathcal{F}$ for any quasi-coherent sheaf on $\mathfrak{X}$ by [AV, Lemma 2.3.4]. The proof of Theorem 5.1.8 therefore gives the following result as well.

Theorem 5.1.14. *Let M be a proper S-scheme with good quotient singularities, and let $j : M^0 \to M$ be its smooth locus. If $\mathfrak{X}$, as in Proposition 5.1.13, lifts mod p^2, then*

$$E_1^{st} = H^t(j_*\Omega^s_{M^0/S}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$$

degenerates for $s + t < p$.

As will follow from Theorem 5.4.7 below, if M has dimension at least 4, lifts mod p^2, and has isolated singularities, then $\mathfrak{X}$ automatically lifts mod p^2.

5.2 Deligne-Illusie for Simplicial Schemes

Let k be a perfect field of characteristic p and let $S = \operatorname{Spec} k$. In this section, we prove a Deligne-Illusie result at the simplicial level. To do so, we must first make sense of the Cartier isomorphism for simplicial schemes.

Lemma 5.2.1. *Let X and Y be smooth schemes over S and let $\rho : X \to Y$ be a morphism of S-schemes. If C^{-1} denotes the Cartier isomorphism, then the following diagram commutes*

$$\begin{array}{ccc} \rho'^*\Omega^i_{X'/S} & \xrightarrow{\rho'^*C^{-1}} & \rho'^*\mathcal{H}^i(F_*\Omega^\bullet_{X/S}) \\ \downarrow & & \downarrow \\ \Omega^i_{Y'/S} & \xrightarrow{C^{-1}} & \mathcal{H}^i(F_*\Omega^\bullet_{Y/S}) \end{array}$$

Proof. Using the canonical morphism $\rho'^*\mathcal{H}^i(F_*\Omega^\bullet_{X/S}) \to \mathcal{H}^i(\rho'^*F_*\Omega^\bullet_{X/S})$ and the multiplicativity property of the Cartier isomorphism, we need only check that the diagram commutes for $i = 0, 1$. For $i = 0$, the Cartier isomorphism is simply the kernel map, so the $i = 0$ case follows from the commutativity of

$$\begin{array}{ccc} \rho'^*F_*\mathcal{O}_X & \xrightarrow{\rho'^*d} & \rho'^*F_*\Omega^1_{Y/S} \\ \downarrow & & \downarrow \\ F_*\mathcal{O}_Y & \xrightarrow{d} & F_*\Omega^1_{X/S} \end{array}$$

To handle the $i = 1$ case, let f be a local section of $\mathcal{O}_X$ and note that

$$\begin{array}{ccc} df & \longmapsto & f^{p-1}df \\ \downarrow & & \downarrow \\ d(\rho(f)) & \longmapsto & \rho(f)^{p-1}d(\rho(f)) \end{array}$$

□

Corollary 5.2.2. *Let $\mathfrak{X}$ be a smooth Artin stack over S and let $X_0 \to \mathfrak{X}$ be a smooth cover by a scheme. If $X_\bullet$ is the simplicial scheme obtained by taking fiber products of X_0 over $\mathfrak{X}$, and $X'_\bullet$ is its pullback by F_S, then there exists a unique isomorphism*

$$C^{-1} : \Omega^i_{X'_\bullet/S} \to \mathcal{H}^i(F_*\Omega^\bullet_{X_\bullet/S})$$

such that $C^{-1}(1) = 1$, $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$, and $C^{-1}(df)$ is the class of $f^{p-1}df$.

Proof. If such a C^{-1} exists, then its restriction to the n^{th} level of the simplicial scheme is the Cartier isomorphism for X_n. Therefore, we need only show existence, which follows from Lemma 5.2.1. □

We have now proved the Cartier isomorphism for simplicial schemes. The other main ingredient in extending Deligne-Illusie to simplicial schemes $X_\bullet$, is showing that relative Frobenius for $X_\bullet$ lifts locally. We note that there are, in fact, simplicial schemes for which relative Frobenius does not lift.

Example 5.2.3. Let $X_\bullet$ be obtained by taking fiber products of S over $B\mathbb{G}_a$. Lifting Frobenius for $X_\bullet$ is then equivalent to lifting Frobenius F of $\mathbb{G}_a$ to a morphism $\tilde{F}$ of group schemes

$$\operatorname{Spec} W_2(k)[x] = \mathbb{G}_{a,\tilde{S}} \to \mathbb{G}_{a,\tilde{S}} = \operatorname{Spec} W_2(k)[x].$$

Since $\tilde{F}$ reduces to F, we must have $\tilde{F}(x) = x^p + pf(x)$ for some $f(x) \in W_2(k)[x]$. The condition that $\tilde{F}$ be a group scheme homomorphism implies

$$(x + y)^p + pf(x + y) = x^p + y^p + p(f(x) + f(y)),$$

and an easy check shows that this is not possible.

Although the above example shows that relative Frobenius need not lift locally for an arbitrary simplicial scheme, we show that relative Frobenius does lift locally for those simplicial schemes which come from smooth tame stacks. This is the key technical point of this section.

Proposition 5.2.4. *Let $\mathfrak{X}$ be a smooth tame stack over S with coarse space M. Then étale locally on M, both $\mathfrak{X}$ and the relative Frobenius $F_{\mathfrak{X}/S}$ lift mod p^2.*

Proof. Since the statement of the proposition is étale local, by Proposition 2.5.2, we can assume that M is affine and $\mathfrak{X} = [U/G]$, where $G = \mathbb{G}^r_{m,S} \rtimes H$ and H is a finite étale constant group scheme. Note that U is affine and that the smoothness of G and $\mathfrak{X}$ imply that U is smooth over S.

As a first step in showing that $\mathfrak{X}$ and $F_{\mathfrak{X}/S}$ lift mod p^2, we begin by showing that BG and its relative Frobenius lift. Since the underlying scheme of G is $\mathbb{G}^r_{m,S} \times_S H$ and its group structure is determined by the action

$$H \to \mathrm{Aut}(\mathbb{G}^r_m) = \mathrm{Aut}(\mathbb{Z}^r),$$

we can use this same action to define a group scheme $\tilde{G} = \mathbb{G}^r_{m,\tilde{S}} \rtimes H$ which lifts G. It follows that $B\tilde{G}$ is a lift of BG. Lifting the relative Frobenius of BG is the same as lifting the relative Frobenius $F_{G/S} : \mathbb{G}^r_{m,S} \rtimes H \longrightarrow \mathbb{G}^r_{m,S} \rtimes H$ to a group scheme homomorphism. Note that $F_{G/S}$ is given by the identity on H and component-wise multiplication by p on $\mathbb{G}^r_{m,S}$. It therefore has a natural lift mod p^2 to the group scheme homomorphism given by the identity on H and component-wise multiplication by p on $\mathbb{G}^r_{m,\tilde{S}}$.

We now prove that $\mathfrak{X}$ and $F_{\mathfrak{X}/S}$ lift. There is a natural map $\pi : \mathfrak{X} \to BG$ which makes

$$\begin{array}{ccc} U & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{\pi} & BG \end{array}$$

a cartesian diagram. To lift $\mathfrak{X}$ mod p^2, it suffices to show that there a stack $\tilde{\mathfrak{X}}$ and a cartesian diagram

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \tilde{\mathfrak{X}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ BG & \longrightarrow & B\tilde{G} \end{array}$$

The obstruction to the existence of such a diagram lies in $\mathrm{Ext}^2(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}})$; here $L_{\mathfrak{X}/BG}$ denotes the cotangent complex. Since π is representable and smooth, $L_{\mathfrak{X}/BG}$ is a locally free sheaf. It follows that

$$R\mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}) = \mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}),$$

which is a quasi-coherent sheaf. Since π is affine and G is linearly reductive, for any quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$, we have

$$R\Gamma(\mathfrak{X}, \mathcal{F}) = R\Gamma(BG, R\pi_*\mathcal{F}) = \Gamma(BG, \pi_*\mathcal{F}).$$

In particular,

$$R\,\mathrm{Hom}(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}) = \Gamma(\mathfrak{X}, \mathcal{H}om(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}))$$

and so $\mathrm{Ext}^2(L_{\mathfrak{X}/BG}, \mathcal{O}_{\mathfrak{X}}) = 0$.

To show that $F_{\mathfrak{X}/S}$ lifts mod p^2, it suffices to show that it lifts over our choice $\tilde{F}_{BG/S}$. That is, it suffices to show that there exists a dotted arrow making the diagram

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \tilde{\mathfrak{X}} \\ {\scriptstyle F_{\mathfrak{X}/S}}\downarrow & & \downarrow \\ \mathfrak{X}' & \longrightarrow & \tilde{\mathfrak{X}}' \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}'} \\ BG' & \longrightarrow & B\tilde{G}' \end{array} \qquad (\tilde{\mathfrak{X}} \xrightarrow{\tilde{F}_{BG/S}\circ\tilde{\pi}} B\tilde{G}')$$

commute. The obstruction to finding such a dotted arrow lies in $\mathrm{Ext}^1(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}})$. As before, we have

$$R\mathcal{H}om(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}}) = \mathcal{H}om(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}_{\mathfrak{X}}),$$

which is again a quasi-coherent sheaf. An argument similar to the one above then shows $\mathrm{Ext}^1(L_{\mathfrak{X}'/BG'}, (F_{\mathfrak{X}/S})_*\mathcal{O}$ 0, thereby completing the proof. □

We now prove Deligne-Illusie for simplicial schemes coming from smooth tame stacks.

Theorem 5.2.5. *Let $\mathfrak{X}$ be a smooth tame stack over S. Let $X_0 \to \mathfrak{X}$ be a smooth cover by a scheme and let $X_\bullet$ be the simplicial scheme obtained by taking fiber products of X_0 over $\mathfrak{X}$. Then, to every lift $\tilde{X}_0 \to \tilde{\mathfrak{X}}$ of $X_0 \to \mathfrak{X}$, there is a canonically associated isomorphism*

$$\varphi : \bigoplus_{i<p} \Omega^i_{X'_\bullet/S}[-i] \to \tau_{<p} F_* \Omega^\bullet_{X_\bullet/S}$$

in the derived category of $\mathcal{O}_{X'_\bullet}$-modules such that $\mathcal{H}^i(\varphi) = C^{-1}$ for $i < p$.

Proof. To prove this theorem we simply check that all of the morphisms in the proof of Deligne-Illusie extend to morphisms on the simplicial level (see Section 5.1.1 for an outline of Deligne-Illusie and relevant notation).

Let $\rho : X_n \to X_m$ be a face or a degeneracy map of $X_\bullet$. To ease notation, we denote X_n by Y and X_m by X. In addition, we use F to denote all relative Frobenii.

To show that φ^0 extends to a morphism $\mathcal{O}_{X'_\bullet} \to F_*\Omega^\bullet_{X_\bullet/S}$, we show

$$\begin{array}{ccccc} \rho'^*\mathcal{O}_{X'} & \xrightarrow{\rho'^*C^{-1}} & \rho'^*\mathcal{H}^0 F_*\Omega^\bullet_{X/S} & \longrightarrow & \rho'^* F_*\Omega^\bullet_{X/S} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{Y'} & \xrightarrow{C^{-1}} & \mathcal{H}^0 F_*\Omega^\bullet_{Y/S} & \longrightarrow & F_*\Omega^\bullet_{Y/S} \end{array}$$

commutes. The left square commutes by Lemma 5.2.1. The right square commutes since for any morphism $A^\bullet \to B^\bullet$ of complexes concentrated in non-negative degrees, the following diagram commutes

$$\begin{array}{ccc} \ker d_A^0 & \longrightarrow & A^0 \\ \downarrow & & \downarrow \\ \ker d_B^0 & \longrightarrow & B^0 \end{array}$$

To show that φ^i extends to a morphism on the simplicial level for $i > 0$, we must check

$$\begin{array}{ccccccc} \rho'^*\Omega^i_{X'/S}[-i] & \xrightarrow{\rho'^* a[-i]} & (\Omega^1_{X'/S})^{\otimes i}[-i] & \xrightarrow{\rho'^*(\varphi^1)^{\otimes i}} & (F_*\Omega^\bullet_{X/S})^{\otimes i} & \xrightarrow{\rho'^* b} & F_*\Omega^\bullet_{X/S} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^i_{Y'/S}[-i] & \xrightarrow{a[-i]} & (\Omega^1_{Y'/S})^{\otimes i}[-i] & \xrightarrow{(\varphi^1)^{\otimes i}} & (F_*\Omega^\bullet_{Y/S})^{\otimes i} & \xrightarrow{b} & F_*\Omega^\bullet_{Y/S} \end{array}$$

commutes. It is clear that the outermost squares commute, and so we are reduced to checking the commutativity of

$$\begin{array}{ccc} \rho'^*\Omega^1_{X'/S}[-1] & \xrightarrow{\rho'^*\varphi^1} & \rho'^* F_*\Omega^\bullet_{X/S} \\ \downarrow & & \downarrow \\ \Omega^1_{Y'/S}[-1] & \xrightarrow{\varphi^1} & F_*\Omega^\bullet_{Y/S} \end{array}$$

Suppose now that Frobenius (for the simplicial scheme) lifts. So, we have a commutative square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{F}} & \tilde{Y}' \\ {\scriptstyle \tilde{\rho}}\downarrow & & \downarrow{\scriptstyle \tilde{\rho}'} \\ \tilde{X} & \xrightarrow{\tilde{F}} & \tilde{X}' \end{array}$$

of $\tilde{S}$-schemes which pulls back to

$$\begin{array}{ccc} Y & \xrightarrow{F} & Y' \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho'} \\ X & \xrightarrow{F} & X' \end{array}$$

over S. In this case $\varphi^1 = f$, and to check that it defines a morphism $\Omega^1_{X'_\bullet/S}[-1] \to F_*\Omega^1_{X_\bullet/S}$, we need to check that

$$\begin{array}{ccc} \rho'^*\Omega^1_{X'/S}[-1] & \xrightarrow{\rho'^* f} & \rho'^* F_*\Omega^1_{X/S} \\ \downarrow & & \downarrow \\ \Omega^1_{Y'/S}[-1] & \xrightarrow{f} & F_*\Omega^1_{Y/S} \end{array}$$

commutes. Under these morphisms,

$$\begin{array}{ccc} dx_0 \otimes 1 & \longmapsto & x_0^{p-1} dx_0 + du(x) \\ \downarrow & & \downarrow \\ d\rho(x_0) \otimes 1 & & \rho(x_0)^{p-1} d\rho(x_0) + d\rho(u(x)) \end{array}$$

We see $d\rho(x_0) \otimes 1$ is sent to $\rho(x_0)^{p-1} d\rho(x_0) + d\rho(u(x))$ since

$$\tilde{F}^*(\tilde{\rho}(x) \otimes 1) = \tilde{F}\tilde{\rho}'(x \otimes 1) = \tilde{\rho}\tilde{F}^*(x \otimes 1) = \tilde{\rho}(x^p + \mathbf{p}(u(x))) = \tilde{\rho}(x)^p + \mathbf{p}(u(\tilde{\rho}(x))).$$

Given two different choices $\tilde{F}_1$ and $\tilde{F}_2$ of F, we obtain a homotopy h_{12} relating f_1 and f_2. It is clear that h_{12} extends to a morphism on the simplicial level since $h_{12}(dx_0 \otimes 1) = u_2(x) - u_1(x)$ and $\tilde{\rho}(\mathbf{p}(u_i(x))) = \mathbf{p}(u_i(\tilde{\rho}(x)))$.

We now need to handle the general case when Frobenius does not lift. We begin by proving that Frobenius lifts étale locally on $X_\bullet$. To do so, we can, by Proposition 5.2.4, assume that there is a lift $\tilde{F}$ of $F_{\mathfrak{X}/S}$. Let $\mathcal{U}_0 = \{U_i\}$ be a Zariski cover of X_0 where Frobenius lifts and let $\tilde{F}_i$ be a lift of $F_{U_i/S}$. Then $\mathcal{U}_n := \{U_{i_1} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} U_{i_n}\}$ is a Zariski cover of X_n and $\tilde{F}_{i_1} \times_{\tilde{F}} \cdots \times_{\tilde{F}} \tilde{F}_{i_n}$ is a lift of Frobenius on $U_{i_1} \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} U_{i_n}$. Moreover, these lifts of Frobenius are compatible so we see that Frobenius for the simplicial scheme does lift étale locally.

To finish the proof of the theorem, we need only prove the commutativity of

$$\begin{array}{ccccc} \rho'^* F_* \Omega^\bullet_{X/S} & \xrightarrow{\simeq} & \mathrm{Tot}(\rho'^* F_* \check{\mathcal{C}}^\bullet(\mathcal{U}_m, \Omega^\bullet_{X/S})) & \xleftarrow{\rho'^*\Phi} & \rho'^* \Omega^1_{X'/S}[-1] \\ \downarrow & & \downarrow & & \downarrow \\ F_* \Omega^\bullet_{Y/S} & \xrightarrow{\simeq} & \mathrm{Tot}(F_* \check{\mathcal{C}}^\bullet(\mathcal{U}_n, \Omega^\bullet_{Y/S})) & \xleftarrow{\Phi} & \Omega^1_{Y'/S}[-1] \end{array}$$

The right square commutes because the Φ are defined in terms of the f's and h's. The middle vertical map is induced by the morphism of the respective double complexes given by

$$(\omega_{1,s} \wedge \cdots \wedge \omega_{a,s})_{s \in S_m} \mapsto (\rho_s^{st}(\omega_{1,s} \wedge \cdots \wedge \rho_s^{st}(\omega_{a,s}))_{s \in S_m, t \in S_n}$$

where $\rho_s^{st} : U_s \times_{\mathfrak{X}} U_t \to U_s$ and S_k is the symmetric group. So, under the morphisms in the left square,

$$\begin{array}{ccc} \omega_1 \wedge \ldots \omega_a & \longmapsto & (\omega_1|U_s \wedge \cdots \wedge \omega_a|U_s)_s \\ \downarrow & & \\ \rho(\omega_1) \wedge \ldots \rho(\omega_a) & \longmapsto & (\rho(\omega_1)|U_s \times U_t \wedge \cdots \wedge \rho(\omega_a)|U_s \times U_t)_{s,t} \end{array}$$

Under the middle vertical map, $(\omega_1|U_s \wedge \cdots \wedge \omega_a|U_s)_s$ is sent to $(\rho_s^{st}(\omega_1|U_s) \wedge \cdots \wedge \rho_s^{st}(\omega_a|U_s))_{s,t}$. But

$$\begin{array}{ccc} U_s \times_{\mathfrak{X}} U_t & \longrightarrow & Y \\ \downarrow{\scriptstyle \rho_s^{st}} & & \downarrow{\scriptstyle \rho} \\ U_s & \longrightarrow & X \end{array}$$

commutes, so this completes the proof. □

Remark 5.2.6. If $\mathfrak{X}$ is a smooth Artin stack which lifts mod p^2, then there automatically exists a smooth cover $X \to \mathfrak{X}$ by a smooth scheme such that the cover lifts mod p^2. This can be seen as follows. Let $Y \to \mathfrak{X}$ be any smooth cover by a smooth scheme Y and let $\bigcup U_i = Y$ be a Zariski cover of Y by open affine subschemes. We can then take $X = \coprod U_i$.

5.3 De Rham Theory for Tame Stacks

Let S be a scheme and $\mathfrak{X} \to \mathcal{Y}$ a morphism of Artin stacks over S. We denote by $\varpi^1_{\mathfrak{X}/\mathcal{Y}}$ the sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules on the lisse-étale site of $\mathfrak{X}$ such that $\varpi^1_{\mathfrak{X}/\mathcal{Y}}|U_{et} = \Omega^1_{U/\mathcal{Y}}$ for all U smooth over $\mathfrak{X}$. We define $\varpi^s_{\mathfrak{X}/\mathcal{Y}}$ to be $\bigwedge^s \varpi^1_{\mathfrak{X}/\mathcal{Y}}$. Given a morphism $f : V \to U$ of smooth $\mathfrak{X}$-schemes, note that the transition function

$$f^*\Omega^1_{U/\mathcal{Y}} \longrightarrow \Omega^1_{V/\mathcal{Y}}$$

need not be an isomorphism, and so the $\varpi^s_{\mathfrak{X}/\mathcal{Y}}$ are never coherent. Note also that $\varpi^1_{\mathfrak{X}/\mathfrak{X}}$ is not the zero sheaf.

As mentioned in the introduction, the sheaf $\varpi^1_{\mathfrak{X}/S}$ gives us a naive de Rham complex $\varpi^\bullet_{\mathfrak{X}/S}$. In this section we prove that when S is spectrum of a perfect field of characteristic p, the hypercohomology spectral sequence

$$E_1^{st} = H^t(\varpi^s_{\mathfrak{X}/S}) \Rightarrow H^n(\varpi^\bullet_{\mathfrak{X}/S})$$

degenerates for smooth proper tame stacks $\mathfrak{X}$ that lift mod p^2. The reason the proof of Corollary 5.1.6 and the Deligne-Illusie result proved in the last section do not immediately imply the degeneracy of this spectral sequence is that, as mentioned above, the $\varpi^s_{\mathfrak{X}/S}$ are not coherent, and so we do not yet know that the $H^t(\varpi^s_{\mathfrak{X}/S})$ and $H^n(\varpi^\bullet_{\mathfrak{X}/S})$ are finite-dimensional k-vector spaces. The main goal of this section, which implies the degeneracy of the above spectral sequence, is to prove that they are by comparing them with the cohomology of the cotangent complex.

We begin by proving three general lemmas and a corollary which require no assumptions on the base scheme S. The first two lemmas are concerned with relative cohomological descent. For background material on cohomological descent, we refer the reader to [Ol3, §2] and [Co, §6].

In what follows, given a smooth hypercover $a : X_\bullet \to \mathfrak{X}$ of an Artin stack by a simplicial algebraic space, $\mathfrak{X}_{lis-et}|X_s$ denotes the topos of sheaves over the representable sheaf defined by X_s and $\mathfrak{X}_{lis-et}|X_\bullet$ denotes the associated simplicial topos.

Lemma 5.3.1. *Let $\mathfrak{X}$ be an Artin stack over S and let $a : X_\bullet \to \mathfrak{X}$ be a smooth hypercover by a simplicial algebraic space. If $f : \mathfrak{X} \to M$ is a morphism to a scheme, then for any $\mathcal{F}_\bullet \in Ab(\mathfrak{X}_{lis-et}|X_\bullet)$, there is a spectral sequence*

$$E_1^{st} = R^t(fa_s)_*(\mathcal{F}_s|X_{s,et}) \Rightarrow \epsilon_* R^n(f_*a_*)\mathcal{F}_\bullet,$$

where $\epsilon : M_{lis-et} \to M_{et}$ *is the canonical morphism of topoi. If* $\mathcal{F}_\bullet = a^*\mathcal{F}$ *for some* $\mathcal{F} \in Ab(\mathfrak{X}_{lis-et})$*, then* $\epsilon_* R^n(f_* a_*)\mathcal{F}_\bullet = \epsilon_* R^n f_* \mathcal{F}$*.*

Proof. Let $\eta_s : \mathfrak{X}_{lis-et}|X_s \to X_{s,et}$ be the canonical morphism of topoi and note that

$$\begin{array}{ccccccc} Ab(X_{s,et}) & & & \xrightarrow{\quad (fa_s)_* \quad} & & & Ab(M_{et}) \\ \uparrow{\scriptstyle \eta_{s*}} & & & & & & \\ Ab(\mathfrak{X}_{lis-et}|X_s) & \xrightarrow{a_{s*}} & Ab(\mathfrak{X}_{lis-et}) & \xrightarrow{f_*} & Ab(M_{lis-et}) & \xrightarrow{\epsilon_*} & Ab(M_{et}) \end{array}$$

commutes. By general principles (see proof of [Co, Thm 6.11]), there is a spectral sequence

$$E_1^{st} = R^t(\epsilon_* f_* a_{s*})(\mathcal{F}_s) \Rightarrow R^n(\epsilon_* f_* a_*)\mathcal{F}_\bullet.$$

As ϵ_* is exact, $R^n(\epsilon_* f_* a_*)\mathcal{F}_\bullet = \epsilon_* R^n(f_* a_*)\mathcal{F}_\bullet$. Since η_{s*} is exact and takes injectives to injectives, the commutativity of the above diagram implies that $E_1^{st} \simeq R^t(fa_s)_*(\mathcal{F}_s|X_{s,et})$, which shows the existence of our desired spectral sequence. Lastly, since

$$a^* : Ab(\mathfrak{X}_{lis-et}) \to Ab(\mathfrak{X}_{lis-et}|X_\bullet)$$

is fully faithful, it follows ([Co, Lemma 6.8]) that $Ra_*a^* = id$. As a result, $\epsilon_* R(f_* a_*)a^*\mathcal{F} = \epsilon_* Rf_*\mathcal{F}$. □

Lemma 5.3.2. *With notation and hypotheses as in Lemma 5.3.1, we have*

$$R^n(fa)_*(\eta_*\mathcal{F}_\bullet) = \epsilon_* R^n(f_* a_*)\mathcal{F}_\bullet,$$

where $\eta : \mathfrak{X}_{lis-et}|X_\bullet \to X_{\bullet,et}$ *is the canonical morphism of topoi.*

Proof. We see that the diagram

$$\begin{array}{ccccccc} Ab(X_{\bullet,et}) & & & \xrightarrow{\quad (fa)_* \quad} & & & Ab(M_{et}) \\ \uparrow{\scriptstyle \eta_*} & & & & & & \\ Ab(\mathfrak{X}_{lis-et}|X_\bullet) & \xrightarrow{a_*} & Ab(\mathfrak{X}_{lis-et}) & \xrightarrow{f_*} & Ab(M_{lis-et}) & \xrightarrow{\epsilon_*} & Ab(M_{et}) \end{array}$$

commutes. It follows that

$$R(fa)_*(\eta_*\mathcal{F}_\bullet) = R(fa)_*(R\eta_*\mathcal{F}_\bullet) = \epsilon_* R(f_* a_*)\mathcal{F}_\bullet,$$

as ϵ_* and η_* are exact and take injectives to injectives. □

Using Lemma 5.3.1, we prove a base change result for sheaves on an Artin stack which are not necessarily quasi-coherent, but are level-by-level quasi-coherent on a smooth hypercover of the stack.

Corollary 5.3.3. *Let $f : \mathfrak{X} \to M$ be a morphism from an Artin stack to a scheme and let $a : X_\bullet \to \mathfrak{X}$ be a smooth hypercover by a simplicial algebraic space. Let $h : T \to M$ be an étale morphism and consider the diagram*

$$\begin{array}{ccc} Y_\bullet & \xrightarrow{j} & X_\bullet \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ \mathcal{Y} & \xrightarrow{i} & \mathfrak{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{h} & M \end{array}$$

where all squares are cartesian. If $\mathcal{F}$ is an $\mathcal{O}_{\mathfrak{X}}$-module such that each $\mathcal{F}|X_{s,et}$ is quasi-coherent, then the canonical map

$$h^*\epsilon_* R^n f_* \mathcal{F} \longrightarrow \alpha_* R^n g_* i^* \mathcal{F}$$

is an isomorphism, where ϵ and α denote the canonical morphisms of topoi $M_{lis-et} \to M_{et}$ and $T_{lis-et} \to T_{et}$, respectively.

Proof. By Lemma 5.3.1, we have a spectral sequence

$$E_1^{st} = R^t(fa_s)_*(\mathcal{F}|X_{s,et}) \Rightarrow \epsilon_* R^n f_* \mathcal{F}.$$

Applying h^*, we obtain another spectral sequence

$$`E_1^{st} = h^* R^t(fa_s)_*(\mathcal{F}|X_{s,et}) \Rightarrow h^*\epsilon_* R^n f_* \mathcal{F}.$$

Flat base change shows

$$h^* R^t(fa_s)_*(\mathcal{F}|X_{s,et}) = R^t(gb_s)_* j_s^*(\mathcal{F}|X_{s,et}) = R^t(gb_s)_*(i^*\mathcal{F})|Y_{s,et}.$$

Another application of Lemma 5.3.1 then shows that $`E$ in fact abuts to $\alpha_* R^n g_* i^* \mathcal{F}$. □

Before stating the next lemma, we introduce the following definitions. Let Z be an S-scheme equipped with an action $\rho : G \times_S Z \to Z$ of a smooth reductive group scheme G over S and let $p : G \times Z \to Z$ be the projection. We denote by (G-*lin* $\mathcal{O}_{Z_{et}}$-*mod*) the category of G-linearized $\mathcal{O}_{Z_{et}}$-modules. That is, the category of quasi-coherent $\mathcal{O}_{Z_{et}}$-modules $\mathcal{F}$ together with an isomorphism $\phi : p^*\mathcal{F} \to \rho^*\mathcal{F}$ satisfying a cocycle condition. From such a ϕ we can define a "coaction map"

$$\sigma : \mathcal{F} \longrightarrow p_* p^* \mathcal{F} \xrightarrow{p_*(\phi)} p_* \rho^* \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_Z, \rho} \mathcal{O}_{Z \times_S G}$$

which satisfies an associativity relation as in [GIT, p.31]; here $\mathcal{F} \to p_* p^* \mathcal{F}$ is the canonical map. Letting $f : Z \to Z/G$ be the natural map, we define the G-invariants $\mathcal{F}^G$ of $\mathcal{F}$ to be the equalizer of

$$f_*\sigma : f_*\mathcal{F} \longrightarrow f_* p_* \rho^* \mathcal{F} = f_* p_* p^* \mathcal{F}$$

and f_* of the canonical map $s \mapsto s \otimes 1$.

If Y is also an S-scheme equipped with a G-action and $h : Z \to Y$ is a G-equivariant map over S, then for every G-linearized $\mathcal{O}_{Z_{et}}$-module $\mathcal{F}$, there is a natural G-linearization on $h_*\mathcal{F}$. So, we have a commutative diagram of categories

$$\begin{array}{ccc}
(G\text{-}lin\ \mathcal{O}_{Z_{et}}\text{-}mod) & \longrightarrow & (\mathcal{O}_{Z_{et}}\text{-}mod) \\
\downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_*} \\
(G\text{-}lin\ \mathcal{O}_{Y_{et}}\text{-}mod) & \longrightarrow & (\mathcal{O}_{Y_{et}}\text{-}mod)
\end{array}$$

where the horizontal arrows are the obvious forgetful functors. If $g : Z/G \to Y/G$ denotes the map induced by h, then it is not hard to see that $(h_*\mathcal{F})^G = g_*\mathcal{F}^G$. In particular, $\mathcal{F}^G = (f_*\mathcal{F})^G$ where G acts trivially on Z/G. Note that for any sheaf $\mathcal{G}$ of $\mathcal{O}_{Z/G}$-modules, $f^*\mathcal{G}$ comes equipped with a canonical G-linearization. If the G-action on Z is free, so that f is a G-torsor, then $(f^*\mathcal{G})^G = (f_*f^*\mathcal{G})^G = \mathcal{G}$.

By descent theory, $(G\text{-}lin\ \mathcal{O}_{Z_{et}}\text{-}mod)$ is equivalent to the category of quasi-coherent sheaves on $[Z/G]$. Under this equivalence, taking G-invariants in the above sense corresponds to pushing forward to the coarse space Z/G.

When the action of G on Z is trivial, we denote $(G\text{-}lin\ \mathcal{O}_{Z_{et}}\text{-}mod)$ by $(G\text{-}\mathcal{O}_{Z_{et}}\text{-}mod)$. We can similarly define the categories $(G\text{-}lin\ \mathcal{O}_{Z_{\bullet,et}}\text{-}mod)$ and $(G\text{-}\mathcal{O}_{Z_{\bullet,et}}\text{-}mod)$ for simplicial schemes $Z_\bullet$.

Lemma 5.3.4. *Let U be a smooth S-scheme with an action of a smooth affine linearly reductive group scheme G over S. Let $\mathfrak{X} = [U/G]$ and $a : X_\bullet \to \mathfrak{X}$ be the hypercover obtained by taking fiber products of U over $\mathfrak{X}$. Consider the diagram*

$$\begin{array}{ccc}
Y_\bullet & \xrightarrow{\pi} & X_\bullet \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} \\
U & \xrightarrow{a_0} & \mathfrak{X} \\
& {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\
& & M
\end{array}$$

where the square is cartesian and M is a scheme. Then

$$R^n(fa)_*\mathcal{F}_\bullet = (R^n(gb)_*\pi^*\mathcal{F}_\bullet)^G$$

for all $\mathcal{O}_{X_{\bullet,et}}$-modules $\mathcal{F}_\bullet$ such that the $\mathcal{F}_s$ are quasi-coherent.

Proof. Note that the following diagram

$$\begin{array}{ccccc}
(G\text{-}lin\ \mathcal{O}_{Y_{\bullet,et}}\text{-}mod) & & \xrightarrow{\quad (gb)_* \quad} & & (G\text{-}\mathcal{O}_{M_{et}}\text{-}mod) \\
\downarrow{\scriptstyle \pi_*} & & & & \downarrow{\scriptstyle (-)^G} \\
(G\text{-}\mathcal{O}_{X_{\bullet,et}}\text{-}mod) & \xrightarrow{(-)^G} & (\mathcal{O}_{X_{\bullet,et}}\text{-}mod) & \xrightarrow{(fa)_*} & (\mathcal{O}_{M_{et}}\text{-}mod)
\end{array}$$

of categories commutes. As a result,

$$R(fa)_*R(-)^G(R\pi_*\pi^*\mathcal{F}_\bullet) = R(-)^G(R(gb)_*\pi^*\mathcal{F}_\bullet) = (R(gb)_*\pi^*\mathcal{F}_\bullet)^G,$$

where the second equality holds because $R(gb)_*\pi^*\mathcal{F}_\bullet$ has quasi-coherent cohomology. It suffices then to prove

$$\mathcal{F}_\bullet = R(-)^G(R\pi_*\pi^*\mathcal{F}_\bullet).$$

We begin by showing $R\pi_*\pi^*\mathcal{F}_\bullet = \pi_*\pi^*\mathcal{F}_\bullet$. Let

$$0 \to \pi^*\mathcal{F}_\bullet \to \mathcal{I}^0_\bullet \to \mathcal{I}^1_\bullet \to \dots$$

be an injective resolution of $\mathcal{O}_{X_{\bullet,et}}$-modules. To show $R^n\pi_*\pi^*\mathcal{F}_\bullet = 0$ for $n > 0$, we need only do so after restricting to each level X_s. Since the restriction functor $res_s : Ab(X_{\bullet,et}) \to Ab(X_{s,et})$ is exact, we see

$$res_s R^n\pi_*\pi^*\mathcal{F}_\bullet = res_s\mathcal{H}^n(\pi_*\mathcal{I}^\bullet_\bullet) = \mathcal{H}^n(\pi_*\mathcal{I}^\bullet_s) = R^n\pi_*\pi^*\mathcal{F}_s = 0,$$

where the last equality holds because π is affine and $\mathcal{F}_s$ is quasi-coherent.

A similar argument shows $R(-)^G(\pi_*\pi^*\mathcal{F}_\bullet) = (\pi_*\pi^*\mathcal{F}_\bullet)^G$ as every $\pi_*\pi^*\mathcal{F}_s$ is quasi-coherent. The lemma then follows from the fact that π is a G-torsor, and so $(\pi_*\pi^*\mathcal{F}_\bullet)^G = \mathcal{F}_\bullet$. □

For the rest of the section, we let $S = \operatorname{Spec} k$, where k is a perfect field of characteristic p. We remind the reader that if $\mathfrak{X}$ is a smooth Artin stack and $X_\bullet \to \mathfrak{X}$ is a hypercover, then the cotangent complex $L_{\mathfrak{X}/S}$ of the stack ([Ol3, §8]) is the bounded complex of $\mathcal{O}_{\mathfrak{X}}$-modules with quasi-coherent cohomology such that

$$L_{\mathfrak{X}/S}|X_{\bullet,et} = \Omega^1_{X_\bullet/S} \to \Omega^1_{X_\bullet/\mathfrak{X}}$$

with $\Omega^1_{X_\bullet/S}$ in degree 0; that is,

$$L_{\mathfrak{X}/S} = \varpi^1_{\mathfrak{X}/S} \to \varpi^1_{\mathfrak{X}/\mathfrak{X}}.$$

In Theorem 5.3.5 below, we compare $\varpi^s_{\mathfrak{X}/S}$ with $\bigwedge^s L_{\mathfrak{X}/S}$, the s^{th} derived exterior power of $L_{\mathfrak{X}/S}$. Given an abelian category $\mathcal{A}$, the derived exterior powers $L\bigwedge^s$, as well as the derived symmetric powers LS^s, of a complex $E \in D^-(\mathcal{A})$ are defined in [Il, I.4.2.2.6]. Since $L_{\mathfrak{X}/S}$ is not concentrated in negative degrees, we cannot directly define $\bigwedge^s L_{\mathfrak{X}/S}$; however, it is shown in [Il, I.4.3.2.1] that for $E \in D^-(\mathcal{A})$,

$$LS^s(E[1]) = (L\bigwedge^s E)[s]$$

so we may define $\bigwedge^s L_{\mathfrak{X}/S}$ as $LS^s(L_{\mathfrak{X}/S}[1])[-s]$. It follows, then, from [Il, I.4.3.1.7] that

$$\bigwedge^s L_{\mathfrak{X}/S} = \varpi^s_{\mathfrak{X}/S} \to \varpi^{s-1}_{\mathfrak{X}/S} \otimes \varpi^1_{\mathfrak{X}/\mathfrak{X}} \to \dots \to \varpi^1_{\mathfrak{X}/S} \otimes S^{s-1}\varpi^1_{\mathfrak{X}/\mathfrak{X}} \to S^s\varpi^1_{\mathfrak{X}/\mathfrak{X}}$$

with $\varpi^s_{\mathfrak{X}/S}$ in degree 0. Note that we have a canonical map from $\bigwedge^s L_{\mathfrak{X}/S}$ to $\varpi^s_{\mathfrak{X}/S}$.

We remark that $\bigwedge^s L_{\mathfrak{X}/S} \in D^b_{coh}(\mathfrak{X})$ for all s. This can be seen as follows. We have an exact triangle

$$a_0^* L_{\mathfrak{X}/S} \longrightarrow L_{X_0/S} \longrightarrow \Omega^1_{X/\mathfrak{X}}.$$

By [Il, II.2.3.7], $L_{X_0/S}$ has coherent cohomology. Since $\Omega^1_{X/\mathfrak{X}}$ is coherent and coherence can be checked smooth locally, we see $L_{X/S}$ and hence all $\bigwedge^s L_{\mathfrak{X}/S}$ are in $D^b_{coh}(\mathfrak{X})$.

We are now ready to prove the comparison theorem.

Theorem 5.3.5. *If $\mathfrak{X}$ is smooth and tame over S and $f : \mathfrak{X} \to M$ is its coarse space, then the canonical map*

$$\epsilon_* R^t f_* (\bigwedge^s L_{\mathfrak{X}/S}) \longrightarrow \epsilon_* R^t f_* \varpi^s_{\mathfrak{X}/S}$$

is an isomorphism.

Proof. By Lemma 2.5.2, there exists an étale cover $h : T \to M$ and a cartesian diagram

$$\begin{array}{ccc} [U/G] & \longrightarrow & \mathfrak{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{h} & M \end{array}$$

where G is linearly reductive, affine, and smooth over S. Since $\mathfrak{X}$ and G are smooth, we see that U is as well. Let $\mathcal{Y} = [U/G]$ and let $\varphi : \epsilon_* R^t f_* (\bigwedge^s L_{\mathfrak{X}/S}) \to \epsilon_* R^t f_* \varpi^s_{\mathfrak{X}/S}$ be the canonical map. By Corollary 5.3.3, we see that $h^*\varphi$ is the canonical map

$$\epsilon_* R^t g_* (\bigwedge^s L_{\mathcal{Y}/S}) \longrightarrow \epsilon_* R^t g_* \varpi^s_{\mathcal{Y}/S}.$$

To show that φ is an isomorphism, we can therefore assume $\mathfrak{X} = [U/G]$ and $M = T$.

To prove the theorem, it suffices to show $\epsilon_* R^t f_* (\varpi^{s-k}_{\mathfrak{X}/S} \otimes S^k \varpi^1_{\mathfrak{X}/\mathfrak{X}}) = 0$ for all $k > 0$ and all t. With notation as in Lemma 5.3.4, we see

$$\epsilon_* R^t f_* (\varpi^{s-k}_{\mathfrak{X}/S} \otimes S^k \varpi^1_{\mathfrak{X}/\mathfrak{X}}) = \epsilon_* R^t (f_* a_*) a^* (\varpi^{s-k}_{\mathfrak{X}/S} \otimes S^k \varpi^1_{\mathfrak{X}/\mathfrak{X}}) = R^t (fa)_* (\Omega^{s-k}_{X_\bullet/S} \otimes S^k \Omega^1_{X_\bullet/\mathfrak{X}}),$$

where the first equality is by Lemma 5.3.1 and the second is by Lemma 5.3.2. It now follows from Lemma 5.3.4 that

$$R^t (fa)_* (\Omega^{s-k}_{X_\bullet/S} \otimes S^k \Omega^1_{X_\bullet/\mathfrak{X}}) = (R^t (gb)_* (\pi^* \Omega^{s-k}_{X_\bullet/S} \otimes S^k \Omega^1_{Y_\bullet/U}))^G.$$

Fix t and $k > 0$. It suffices then to prove by (strong) induction on s that for every flat $\mathcal{O}_{X_\bullet}$-module $\mathcal{G}$ which is restriction to $Y_{\bullet,et}$ of some $\mathcal{O}$-module $\mathcal{F}$ on the lisse-étale site of U,

$$R^n (gb)_* (\pi^* \Omega^s_{X_\bullet/S} \otimes \mathcal{G} \otimes S^k \Omega^1_{Y_\bullet/U}) = 0.$$

We begin with the case $s = 0$, which is handled separately. An application of Lemmas 5.3.1 and 5.3.2 shows

$$R^n(gb)_*(\mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U}) = \epsilon_* R^n g_*(\mathcal{F} \otimes S^k\varpi^1_{U/U}).$$

If we let $\alpha : U_{lis-et} \to U_{et}$ be the canonical morphism of topoi, we see then that

$$\epsilon_* R^n g_*(\mathcal{F} \otimes S^k\varpi^1_{U/U}) = R^n g_*(\alpha_*\mathcal{F} \otimes S^k\Omega^1_{U/U}) = 0,$$

where the last equality holds since $k > 0$.

Assume now that $s > 0$. Since π is smooth, we have a short exact sequence

$$0 \longrightarrow \pi^*\Omega^1_{X_\bullet/S} \longrightarrow \Omega^1_{Y_\bullet/S} \longrightarrow \Omega^1_{Y_\bullet/X_\bullet} \longrightarrow 0.$$

As a result, we have a filtration $\Omega^s_{Y_\bullet/S} \supset \mathcal{K}^1 \supset \cdots \supset \mathcal{K}^s \supset 0$ with $\mathcal{K}^s = \pi^*\Omega^s_{X_\bullet/S}$ and short exact sequences

$$0 \longrightarrow \mathcal{K}^1 \longrightarrow \Omega^s_{Y_\bullet/S} \longrightarrow \Omega^s_{Y_\bullet/X_\bullet} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K}^2 \longrightarrow \mathcal{K}^1 \longrightarrow \pi^*\Omega^1_{X_\bullet/S} \otimes \Omega^{s-1}_{Y_\bullet/X_\bullet} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \pi^*\Omega^s_{X_\bullet/S} \longrightarrow \mathcal{K}^{s-1} \longrightarrow \pi^*\Omega^{s-1}_{X_\bullet/S} \otimes \Omega^1_{Y_\bullet/X_\bullet} \longrightarrow 0.$$

Since $\mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U}$ is flat, tensoring each of the above short exact sequences by it results in a new list of short exact sequences. Since

$$\Omega^1_{Y_\bullet/S} \otimes \mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U} = (\varpi^1_{U/S} \otimes \mathcal{F} \otimes S^k\varpi^1_{U/U})|Y_{\bullet,et}$$

and

$$\Omega^1_{Y_\bullet/X_\bullet} \otimes \mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U} = (L_{U/\mathfrak{X}} \otimes \mathcal{F} \otimes S^k\varpi^1_{U/U})|Y_{\bullet,et},$$

the $s = 0$ case shows

$$R^n(gb)_*(\Omega^1_{Y_\bullet/S} \otimes \mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U}) = R^n(gb)_*(\Omega^1_{Y_\bullet/X_\bullet} \otimes \mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U}) = 0.$$

As a result, $R^n(gb)_*(\mathcal{K}^1 \otimes \mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U}) = 0$. Using the inductive hypothesis, we conclude

$$R^n(gb)_*(\mathcal{K}^i \otimes \mathcal{G} \otimes S^k\Omega^1_{Y_\bullet/U}) = 0$$

for all i, in particular for $i = s$. □

Corollary 5.3.6. *If $\mathfrak{X}$ is a smooth proper tame stack over S, then $H^t(\varpi^s_{\mathfrak{X}/S})$ and $H^n(\varpi^\bullet_{\mathfrak{X}/S})$ are finite-dimensional k-vector spaces for all s, t, and n.*

Proof. Let $f : \mathfrak{X} \to M$ be the coarse space of $\mathfrak{X}$. For each s, there is a Leray spectral sequence

$$E_2^{ij} = H^i(\epsilon_* R^j f_* \varpi^s_{\mathfrak{X}/S}) \Rightarrow H^t(\varpi^s_{\mathfrak{X}/S}).$$

By Theorem 5.3.5, the canonical map

$$\epsilon_* R^j f_*(\bigwedge^s L_{\mathfrak{X}/S}) \longrightarrow \epsilon_* R^j f_* \varpi^s_{\mathfrak{X}/S}$$

is an isomorphism. As we remarked above, $\bigwedge^s L_{\mathfrak{X}/S} \in D^b_{coh}(\mathfrak{X})$. Since f is proper by Keel-Mori [KM], and M is proper by [Ol1, Prop 2.10], we see the E_2^{ij} are finite-dimensional k-vector spaces. It follows that $H^t(\varpi^s_{\mathfrak{X}/S})$ is a finite-dimensional k-vector space for every s and t.

Since the morphisms in the complex $\varpi^\bullet_{\mathfrak{X}/S}$ are k-linear, the hypercohomology spectral sequence

$$E_1^{st} = H^t(\varpi^s_{\mathfrak{X}/S}) \Rightarrow H^n(\varpi^\bullet_{\mathfrak{X}/S})$$

consists of finite-dimensional k-vector spaces with k-linear maps. As a result, $H^n(\varpi^\bullet_{\mathfrak{X}/S})$ is a finite-dimensional k-vector space as well. □

Theorem 5.3.7. *Let $\mathfrak{X}$ be a smooth proper tame stack over S that lifts mod p^2. Then the Hodge-de Rham spectral sequence*

$$E_1^{st} = H^t(\varpi^s_{\mathfrak{X}/S}) \Rightarrow H^n(\varpi^\bullet_{\mathfrak{X}/S})$$

degenerates for $s + t < p$.

Proof. By Remark 5.2.6, there exists a smooth cover $X \to \mathfrak{X}$ by a smooth scheme such that the cover lifts mod p^2. Theorem 5.2.5 now shows

$$\bigoplus_{s<p} \Omega^s_{X'_\bullet/S}[-s] \simeq \tau_{<p} F_* \Omega^\bullet_{X_\bullet/S},$$

where $X_\bullet$ is obtained from X by taking fiber products over $\mathfrak{X}$. Since $H^t(\varpi^s_{\mathfrak{X}/S}) = H^t(\Omega^s_{X_\bullet/S})$ and $H^n(\varpi^\bullet_{\mathfrak{X}/S}) = H^n(\Omega^\bullet_{X_\bullet/S})$, we see that for $n < p$,

$$\dim_k H^n(\varpi^\bullet_{\mathfrak{X}/S}) = \sum_{s+t=n} \dim_k H^t(\Omega^s_{X'_\bullet/S}) = \sum_{s+t=n} \dim_k H^t(\Omega^s_{X_\bullet/S}) = \sum_{s+t=n} \dim_k H^t(\varpi^s_{\mathfrak{X}/S}),$$

which proves the degeneracy of the spectral sequence by Lemma 5.1.5. □

5.4 De Rham Theory for Schemes with Isolated Linearly Reductive Singularities

Let k be a perfect field of characteristic p and let $S = \operatorname{Spec} k$. Recall that a scheme M over S has *linearly reductive singularities* if there is an étale cover $\{U_i/G_i \to M\}$, where the U_i are smooth over S and the G_i are linearly reductive group schemes which are finite over S. Note that if

M has linearly reductive singularities, then it is automatically normal and in fact Cohen-Macaulay by [HR, p.115].

Our goal in this section is to prove that if M is proper over S, and $j : M^0 \to M$ is its smooth locus, then under suitable liftability conditions, the hypercohomology spectral sequence $E_1^{st} = H^t(j_*\Omega^s_{M^0/S}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$ degenerates.

5.4.1 Relationship with Tame Stacks, and the Cartier Isomorphism

Let M and $\mathfrak{X}$ be as in Theorem 2.1.10. Then the proof of Lemma 5.1.10 goes through word for word (after replacing "an étale morphism $U \to \mathfrak{X}$" by "a smooth morphism $U \to \mathfrak{X}$") to show

$$j_*\Omega^\bullet_{M^0/S} = \epsilon_* f_* \varpi^\bullet_{\mathfrak{X}/S},$$

where $\epsilon : M_{lis-et} \to M_{et}$ is the canonical morphism of topoi.

Remark 5.4.1. Since $\epsilon_* f_* \varpi^s_{\mathfrak{X}/S} = \epsilon_* f_* \mathcal{H}^0(\bigwedge^s L_{\mathfrak{X}/S})$, the above equality shows that $j_*\Omega^s_{M^0/S}$ is coherent, which is not *a priori* obvious.

To simplify notation, throughout the rest of this subsection we suppress ϵ.

Proposition 5.4.2 (Cartier isomorphism)**.** *Let $\mathfrak{X}$ be a smooth tame stack over S which lifts mod p^2, and let $f : \mathfrak{X} \to M$ be its coarse space. Then there is a canonical isomorphism*

$$\mathcal{H}^t(f'_* F_* \varpi^\bullet_{\mathfrak{X}/S}) \xrightarrow{\simeq} f'_* \varpi^t_{\mathfrak{X}'/S}.$$

If we further assume that $\mathfrak{X}$ and M are as in Theorem 2.1.10, then

$$\mathcal{H}^t(F_* j_* \Omega^\bullet_{M^0/S}) \xrightarrow{\simeq} j'_* \Omega^t_{M^0/S}.$$

Proof. For any left exact functor $G : \mathcal{A} \to \mathcal{B}$ of abelian categories and any complex $A^\bullet$ of objects of $\mathcal{A}$, there is a canonical morphism $\mathcal{H}^t(GA^\bullet) \to G\mathcal{H}^t(A^\bullet)$: the map $GA^\bullet \to RGA^\bullet$ induces a morphism from $\mathcal{H}^t(GA^\bullet)$ to the E_2^{0t}-term of the spectral sequence $E_2^{st} = R^s G\mathcal{H}^t(A^\bullet) \Rightarrow R^n G(A^\bullet)$.

For us this yields the (global) map $\phi : \mathcal{H}^t(f'_* F_* \varpi^\bullet_{\mathfrak{X}/S}) \to f'_* \mathcal{H}^t(F_* \varpi^\bullet_{\mathfrak{X}/S}) = f'_* \varpi^t_{\mathfrak{X}'/S}$. To prove this is an isomorphism, we need only do so locally. So, by Lemma 2.5.2 and Proposition 5.2.4, we are reduced to the case $\mathfrak{X} = [U/G]$, where U is smooth and affine, $G = \mathbb{G}^r_{m,S} \rtimes H$ for some finite étale constant group scheme H, and both $\mathfrak{X}$ and the relative Frobenius lift mod p^2. Let $X_\bullet$ be the simplical scheme obtained by taking fiber products of U over $\mathfrak{X}$, and let $a : X_\bullet \to \mathfrak{X}$ be the augmentation map. Since $U \to \mathfrak{X}$ lifts mod p^2, Theorem 5.2.5 yields a quasi-isomorphism

$$\varphi : \bigoplus_{t<p} \Omega^t_{X'_\bullet/S}[-t] \xrightarrow{\simeq} \tau_{<p} F_* \Omega^\bullet_{X_\bullet/S}.$$

In this local setting, φ is a *morphism of complexes* by Remark 5.1.3. We can therefore apply $(f'a)_*$. Subsequently taking cohomology, we have a morphism $f'_* \varpi^t_{\mathfrak{X}'/S} \xrightarrow{f'_* \varphi^t} \mathcal{H}^t(f'_* F_* \varpi^\bullet_{\mathfrak{X}/S})$. We show that

$$\psi : f'_* \mathcal{H}^t(F_* \varpi^\bullet_{\mathfrak{X}/S}) \xrightarrow{(f'_* C^{-1})^{-1}} f'_* \varpi^t_{\mathfrak{X}'/S} \xrightarrow{f'_* \varphi^t} \mathcal{H}^t(f'_* F_* \varpi^\bullet_{\mathfrak{X}/S})$$

and ϕ are inverses. Note that in this local setting f'_* is simply "take G-invariants", and that $\phi : \mathcal{H}^t((F_*\Omega^\bullet_U)^G) \to \mathcal{H}^t(F_*\Omega^\bullet_U)^G$ is $[\alpha] \mapsto (\alpha)$, where we use square, resp. round brackets to denote classes in $\mathcal{H}^t((F_*\Omega^\bullet_U)^G)$, resp. $\mathcal{H}^t(F_*\Omega^\bullet_U)^G$.

In general, one does not expect the map $(\alpha) \mapsto [\alpha]$ to be well-defined, but we show here that this is precisely what ψ is. Let $(\omega) \in \mathcal{H}^t(F_*\Omega^\bullet_U)^G$. Via the Cartier isomorphism $\Omega^t_{U'} \stackrel{(C^{-1})^G}{\longrightarrow} \mathcal{H}^t(F_*\Omega^\bullet_U)^G$, we know that (ω) is of the form

$$(\sum f_{i_1,\ldots,i_t} x_{i_1}^{p-1} \ldots x_{i_t}^{p-1} dx_{i_1} \wedge \cdots \wedge dx_{i_t}),$$

where

$$\sum f_{i_1,\ldots,i_t}(dx_{i_1} \otimes 1) \wedge \cdots \wedge (dx_{i_t} \otimes 1) \in (\Omega^t_{U'})^G.$$

The Deligne-Illusie map φ^q sends this G-invariant form to

$$\eta = \sum f_{i_1,\ldots,i_t}(x_{i_1}^{p-1} dx_{i_1} + du(x_{i_1})) \wedge \cdots \wedge (x_{i_t}^{p-1} dx_{i_t} + du(x_{i_t})),$$

where $u(x)$ is the reduction mod p of any y satisfying $\tilde{F}^*(d\tilde{x} \otimes 1) = \tilde{x}^p d\tilde{x} + py$. So, ψ sends (ω) to (η). But since $(du(x)) = 0$, we see that ψ is the map sending (α) to $[\alpha]$. □

5.4.2 Degeneracy of Various Spectral Sequences and a Vanishing Theorem

Let $\mathfrak{X}$ and M be as in Theorem 2.1.10. Our immediate goal is to show the degeneracy of the hypercohomology spectral sequence for $j_*\Omega^\bullet_{M^0/S}$ when $\mathfrak{X}$ is proper and lifts mod p^2. If $\varpi^1_{\mathfrak{X}/S}$ were coherent, then since $\mathfrak{X}$ is tame, we would have $j_*\Omega^\bullet_{M^0/S} = \epsilon_* f_* \varpi^\bullet_{\mathfrak{X}/S} = \epsilon_* R f_* \varpi^\bullet_{\mathfrak{X}/S}$. The proof of Theorem 5.1.8 would then apply directly to show the degeneracy of $E_1^{st} = H^t(j_*\Omega^s_{M^0/S}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$. Since $\varpi^1_{\mathfrak{X}/S}$ is not coherent, we must take a different approach. As we explain below, the Cartier isomorphism for $j_*\Omega^\bullet_{M^0/S}$ proved in the last subsection implies that the degeneracy of the above hypercohomology spectral sequence is equivalent to the degeneracy of the conjugate spectral sequence $E_2^{st} = H^s(\mathcal{H}^t(j_*\Omega^\bullet_{M^0/S})) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$. We show that this latter spectral sequence degenerates by comparing it to the spectral sequence $E_2^{st} = H^s(R^t f_* \varpi^\bullet_{\mathfrak{X}/S}) \Rightarrow H^n(\varpi^\bullet_{\mathfrak{X}/S})$ over which we have more control due to the Deligne-Illusie result of Section 5.2.

As in the last subsection, we suppress $\epsilon : M_{et} \to M_{lis-et}$. The following is the key technical lemma we use to prove the degeneracy of the hypercohomology spectral sequence for $j_*\Omega^\bullet_{M^0/S}$.

Lemma 5.4.3. *Let E and $`E$ be two first quadrant $E_2$ spectral sequences. Suppose that for $s \neq 0$, every differential $`E_r^{st} \to `E_r^{s+r,t-(r-1)}$ is zero. Suppose further that we are given a morphism $E \to `E$ of spectral sequences such that the induced morphism $E_r^{st} \to `E_r^{s+r,t-(r-1)}$ is zero for all $r$, $s$, and $t$, and such that $E_2^{st} \to `E_2^{st}$ is an injection for all s and t. Then E degenerates.*

Proof. We claim that the morphism $E_r^{st} \to {}^{\backprime}E_r^{st}$ is an injection for $s \geq r$. Note that this is enough to prove the lemma since for all s, the square

$$\begin{array}{ccc} E_r^{st} & \longrightarrow & {}^{\backprime}E_r^{st} \\ {\scriptstyle d_r^{st}}\downarrow & & \downarrow \\ E_r^{s+r,t-(r-1)} & \longrightarrow & {}^{\backprime}E_r^{s+r,t-(r-1)} \end{array}$$

commutes, the composite is zero, and $E_r^{s+r,t-(r-1)} \to {}^{\backprime}E_r^{s+r,t-(r-1)}$ is an injection; this shows that all differentials d_r^{st} are zero.

We now prove the claim by induction. It is true for $r = 2$, so we may assume $r > 2$. Let $s \geq r$ and consider the commutative diagram

$$\begin{array}{ccccc} E_{r-1}^{s-(r-1),t+(r-2)} & \longrightarrow & E_{r-1}^{st} & \longrightarrow & E_{r-1}^{s+r-1,t-(r-2)} \\ \downarrow & & \downarrow & & \downarrow \\ {}^{\backprime}E_{r-1}^{s-(r-1),t+(r-2)} & \longrightarrow & {}^{\backprime}E_{r-1}^{st} & \longrightarrow & {}^{\backprime}E_{r-1}^{s+r-1,t-(r-2)} \end{array}$$

By the inductive hypothesis, all vertical arrows are injective and all arrows on the bottom row are zero. It follows that all arrows on the top arrow are zero, and so $E_r^{st} \to {}^{\backprime}E_r^{st}$ is injective. $\square$

Theorem 5.4.4. *Let $\mathfrak{X}$ and M be as in Theorem 2.1.10. If M has isolated singularities, and $\mathfrak{X}$ is proper and lifts mod p^2, then the conjugate spectral sequence*

$$E_2^{st} = H^s(\mathcal{H}^t(j_*\Omega^\bullet_{M^0/S})) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$$

degenerates for $s + t < p$.

Proof. Let $X_\bullet$ be as in Remark 5.2.6 and let $a : X_\bullet \to \mathfrak{X}$ be the augmentation map. By Theorem 5.2.5, we have an isomorphism $\bigoplus_{i<p} \Omega^i_{X'_\bullet/S}[-i] \xrightarrow{\simeq} \tau_{<p} F_*\Omega^\bullet_{X_\bullet/S}$ in the derived category, and therefore, also an isomorphism

$$\bigoplus_{i<p} R(f'_* a_*)\Omega^i_{X'_\bullet/S}[-i] \xrightarrow{\simeq} \tau_{<p} R(f'_* a_*) F_*\Omega^\bullet_{X_\bullet/S}.$$

The first of these isomorphisms implies that the Leray spectral sequence

$${}^{\backprime\backprime}E_2^{st} = R^s f'_* \mathcal{H}^t(F_*\varpi^\bullet_{\mathfrak{X}/S}) \Rightarrow R^n f'_* F_*\varpi^\bullet_{\mathfrak{X}/S}$$

degenerates and that the extension problem is trivial. The second of the two isomorphisms shows that the spectral sequence

$${}^{\backprime}E_2^{st} = H^s(R^t f'_* F_*\varpi^\bullet_{\mathfrak{X}/S}) \Rightarrow H^n(\varpi^\bullet_{\mathfrak{X}/S})$$

decomposes as the direct sum $\bigoplus {}^{\backprime}_{i}E$ of Leray spectral sequences, where

$$ {}^{\backprime}_{i}E_2^{st} = H^s(R^{t-i} f'_{*} \varpi^{i}_{\mathfrak{X}'/S}) \Rightarrow H^{n-i}(\varpi^{i}_{\mathfrak{X}'/S}). $$

Note that the morphism $f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S} \to Rf'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}$ induces a morphism of spectral sequences $E \to {}^{\backprime}E$, where

$$ E_2^{st} = H^s(\mathcal{H}^t(f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S})) \Rightarrow H^n(f_* \varpi^{\bullet}_{\mathfrak{X}/S}). $$

By the degeneracy of ${}^{\backprime\backprime}E$, the morphism $\varphi : \mathcal{H}^t(f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}) \to R^t f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}$ factors as

$$ \mathcal{H}^t(f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}) \to f'_* F_* \varpi^{t}_{\mathfrak{X}/S} \hookrightarrow R^t f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}. $$

But this first morphism is precisely how the Cartier isomorphism of Proposition [?] was defined. From this and the fact that the extension problem for ${}^{\backprime\backprime}E$ is trivial, we have a split short exact sequence

$$ 0 \longrightarrow \mathcal{H}^t(f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}) \xrightarrow{\varphi} R^t f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S} \longrightarrow \bigoplus_{\substack{i+j=t \\ j>0}} R^j f'_* \varpi^{i}_{\mathfrak{X}'/S} \longrightarrow 0. $$

It follows that E_2^{st} is mapped isomorphically to the direct summand ${}^{\backprime}_{t}E_2^{st}$ of ${}^{\backprime}E_2^{st}$. This implies that for all r, s, and t, the induced morphism $E_r^{st} \to {}^{\backprime}E_r^{s+r,t-(r-1)}$ is zero.

Note that

$$ j'^* R^t f'_* \varpi^{i}_{\mathfrak{X}'/S} = j'^* R^t f'_* \bigwedge^{i} L_{\mathfrak{X}'/S} = (f^0)'_* \mathcal{H}^t(\bigwedge^{i} L_{(\mathfrak{X}^0)'/S}) = 0 $$

It follows that $R^t f'_* \varpi^{i}_{\mathfrak{X}'/S}$ is supported at the singular locus of M', and since M is assumed to have isolated singularities, $H^s(R^t f'_* \varpi^{s}_{\mathfrak{X}/S}) = 0$ for s and t positive. We see then that ${}^{\backprime}_{i}E_2^{st}$ is zero if $t > i$ and $s > 0$, or if $t < i$. Therefore, the differential ${}^{\backprime}E_r^{st} \to {}^{\backprime}E_r^{s+r,t-(r-1)}$ is zero if $s \neq 0$. From Lemma 5.4.3, it follows that E degenerates. $\square$

Remark 5.4.5. *Let $\mathcal{E}$ be a locally free sheaf on M'. Tensoring the isomorphism*

$$ \bigoplus_{i<p} R(f'_* a_*)\Omega^{i}_{X'_{\bullet}/S}[-i] \xrightarrow{\simeq} \tau_{<p} R(f'_* a_*) F_* \Omega^{\bullet}_{X_{\bullet}/S} $$

with $\mathcal{E}$, we see that the Leray spectral sequence

$$ {}^{\backprime}E_2^{st} = H^s(R^t f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S} \otimes \mathcal{E}) \Rightarrow H^n(Rf'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S} \otimes \mathcal{E}) $$

decomposes as the direct sum of spectral sequences. The proof of Theorem 5.4.4 then shows that the spectral sequence

$$ E_2^{st} = H^s(\mathcal{H}^t(f'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S}) \otimes \mathcal{E}) \Rightarrow H^n(Rf'_* F_* \varpi^{\bullet}_{\mathfrak{X}/S} \otimes \mathcal{E}). $$

degenerates for $s + t < p$.

Theorem 5.4.6. *Let $\mathfrak{X}$ and M be as in Theorem 2.1.10. If M has isolated singularities, and $\mathfrak{X}$ is proper and lifts mod p^2, then the hypercohomology spectral sequence*

$$E_1^{st} = H^t(j_*\Omega^s_{M^0/S}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$$

degenerates for $s+t<p$.

Proof. By the Cartier isomorphism

$$H^s(\mathcal{H}^t(j_*\Omega^\bullet_{M^0/S})) = H^s(\mathcal{H}^t(f'_*F_*\varpi^\bullet_{\mathfrak{X}/S})) = H^s(f'_*\varpi^t_{\mathfrak{X}'/S}).$$

But $H^s(f'_*\varpi^t_{\mathfrak{X}'/S}) = H^s(f_*\varpi^t_{\mathfrak{X}/S}) \otimes_{k,F_k} k$; in particular,

$$\dim_k H^s(\mathcal{H}^t(j_*\Omega^\bullet_{M^0/S})) = \dim_k H^s(f_*\varpi^t_{\mathfrak{X}/S}).$$

By Corollary 5.3.7, the above cohomology groups are finite-dimensional k-vector spaces. The degeneracy of the conjugate spectral sequence shows

$$H^n(j_*\Omega^\bullet_{M^0/S}) \simeq \bigoplus_{s+t=n} H^s(\mathcal{H}^t(j_*\Omega^\bullet_{M^0/S})),$$

and so

$$\dim_k H^n(j_*\Omega^\bullet_{M^0/S}) = \sum_{s+t=n} \dim_k H^s(f_*\varpi^t_{\mathfrak{X}/S}),$$

which implies the degeneracy of the hypercohomology spectral sequence by Lemma 5.1.5. □

Although our proof of Theorem 5.1.8 goes through stacks, the statement of the theorem is purely scheme-theoretic. We would similarly like to remove the stack from the statement of Theorem 5.4.6. We can do so when M has large enough dimension.

Theorem 5.4.7. *Let M be a proper S-scheme with isolated linearly reductive singularities. If* $\dim M \geq 4$ *and M lifts mod p^2, then*

$$E_1^{st} = H^t(j_*\Omega^s_{M^0/S}) \Rightarrow H^n(j_*\Omega^\bullet_{M^0/S})$$

degenerates for $s+t<p$.

Proof. Let $m = \dim M$ and let $\mathfrak{X}$ be as in Theorem 2.1.10. If we can prove $\mathfrak{X}$ lifts mod p^2, then we are done. The exact triangle

$$Lf^*L_{M/S} \longrightarrow L_{\mathfrak{X}/S} \longrightarrow L_{\mathfrak{X}/M}$$

gives rise to the long exact sequence

$$\ldots \longrightarrow \mathrm{Ext}^2(L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow \mathrm{Ext}^2(L_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow \mathrm{Ext}^2(Lf^*L_{M/S}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow \mathrm{Ext}^3(L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow \ldots.$$

Note that

$$RHom(Lf^*L_{M/S}, \mathcal{O}_{\mathfrak{X}}) = RHom(L_{M/S}, Rf_*\mathcal{O}_{\mathfrak{X}}) = RHom(L_{M/S}, \mathcal{O}_M)$$

since $Rf_*\mathcal{O}_{\mathfrak{X}} = f_*\mathcal{O}_{\mathfrak{X}}$ by tameness and $f_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_M$ by Keel-Mori [KM]. Since the obstruction to lifting $\mathfrak{X}$ lies in $\mathrm{Ext}^2(L_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}})$, we need only show $\mathrm{Ext}^2(L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}}) = 0$. We in fact prove $\mathcal{E}xt^s(L_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}}) = 0$ for $s \leq m-2$.

Since $(j^0)^* L_{\mathfrak{X}/M} = L_{\mathfrak{X}^0/M^0} = 0$, we see

$$0 = Rj^0_* R\mathcal{H}om((j^0)^* L_{\mathfrak{X}/M}, \mathcal{O}_{\mathfrak{X}^0}) = R\mathcal{H}om(L_{\mathfrak{X}/M}, Rj^0_*\mathcal{O}_{\mathfrak{X}^0}).$$

A local cohomology argument given below will show $R^t j^0_* \mathcal{O}_{\mathfrak{X}^0} \neq 0$ if and only if $t = 0, m-1$. Assuming this for the moment, let us complete the proof. We have a spectral sequence

$$E_2^{st} = R^s\mathcal{H}om(L_{\mathfrak{X}/M}, R^t j^0_*\mathcal{O}_{\mathfrak{X}^0}) \Rightarrow R^n\mathcal{H}om(L_{\mathfrak{X}/M}, Rj^0_*\mathcal{O}_{\mathfrak{X}^0}) = 0.$$

The only page with non-zero differentials, then, is the m^{th}. Since $L_{\mathfrak{X}/M}$ is concentrated in degrees at most 1, $R^s\mathcal{H}om(L_{\mathfrak{X}/M}, R^t j^0_*\mathcal{O}_{\mathfrak{X}^0}) = 0$ for $s < -1$. It follows that

$$R^s\mathcal{H}om(L_{\mathfrak{X}/M}, j^0_*\mathcal{O}_{\mathfrak{X}^0}) = 0$$

for $s \leq m-2$, which proves the theorem since $j^0_*\mathcal{O}_{\mathfrak{X}^0} = \mathcal{O}_{\mathfrak{X}}$.

We now turn to the local cohomology argument. To prove $R^t j^0_*\mathcal{O}_{\mathfrak{X}^0} \neq 0$ if and only if $t = 0, m-1$, we can make an étale base change. We can therefore assume $\mathfrak{X} = [U/G]$, where U is smooth and affine, and G is finite linearly reductive. Since M has isolated singularities, we can further assume $U^0 = U \backslash \{x\}$, where U^0 is the pullback

$$\begin{array}{ccc} U^0 & \xrightarrow{i} & U \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ \mathfrak{X}^0 & \xrightarrow{j^0} & \mathfrak{X} \end{array}$$

The following lemma, then, completes the proof. □

Lemma 5.4.8. *Let U be a normal affine scheme of dimension m and let $x \in U$ be Cohen-Macaulay. If $U^0 = U\backslash\{x\}$ and $i : U^0 \hookrightarrow U$ is the inclusion, then $R^t i_*\mathcal{O}_{U^0} \neq 0$ if and only if $t = 0, m-1$.*

Proof. Note that $R^t i_*\mathcal{O}_{U^0}$ is the skyscraper sheaf $H^t(\mathcal{O}_{U^0})$ at x. By normality, $H^0(\mathcal{O}_{U^0}) = H^0(\mathcal{O}_U)$. Since U is affine, the long exact sequence

$$\ldots \longrightarrow H^n_x(\mathcal{O}_U) \longrightarrow H^n(\mathcal{O}_U) \longrightarrow H^n_x(\mathcal{O}_{U^0}) \longrightarrow H^n_x(\mathcal{O}_U) \longrightarrow \ldots$$

shows $H^t(\mathcal{O}_{U^0}) = H^{t+1}_x(\mathcal{O}_U)$ for $t > 0$. Since x is Cohen-Macaulay, $H^{t+1}_x(\mathcal{O}_U) \neq 0$ if and only if $t + 1 = m$. □

We now prove an analogue of [DI, Lemma 2.9] which Deligne and Illusie use to deduce Kodaira Vanishing.

Lemma 5.4.9. *Let $\mathfrak{X}$ and M be as in Theorem 2.1.10. Suppose M has isolated singularities, and $\mathfrak{X}$ is proper and lifts mod p^2. Let d be the dimension of M and let N be an integer such that $N \leq \inf(d,p)$. If $\mathcal{M}$ is an invertible sheaf on M such that*

$$H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{M}^p) = 0$$

for all $s + t < N$, then

$$H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{M}) = 0$$

for all $s + t < N$.

Proof. Let $\mathcal{M}'$ be the pullback of $\mathcal{M}$ to M'. Since $F^*\mathcal{M}' = \mathcal{M}^p$, the projection formula shows

$$H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{M}^p) = H^t(f'_*F_*\varpi^s_{\mathfrak{X}/S} \otimes \mathcal{M}').$$

From the hypercohomology spectral sequence

$$E_2^{st} = H^t(f'_*F_*\varpi^s_{\mathfrak{X}/S} \otimes \mathcal{M}') \Rightarrow H^n(f'_*F_*\varpi^\bullet_{\mathfrak{X}/S} \otimes \mathcal{M}'),$$

we see that $H^n(f'_*F_*\varpi^\bullet_{\mathfrak{X}/S} \otimes \mathcal{M}') = 0$ for all $n < N$. Remark 5.4.5 shows that the Leray spectral sequence

$$E_2^{st} = H^s(f'_*\varpi^t_{\mathfrak{X}'/S} \otimes \mathcal{M}') \Rightarrow H^n(f'_*F_*\varpi^\bullet_{\mathfrak{X}/S} \otimes \mathcal{M}')$$

degenerates, and so $H^s(f'_*\varpi^t_{\mathfrak{X}'/S} \otimes \mathcal{M}') = 0$ for all $s + t < N$. Since

$$\dim_k H^s(f'_*\varpi^t_{\mathfrak{X}'/S} \otimes \mathcal{M}') = \dim_k H^s(j_*\Omega^t_{M^0/S} \otimes \mathcal{M}),$$

the lemma follows. □

Unfortunately, we cannot quite deduce from Lemma 5.4.9 a general Kodaira Vanishing result. Following Deligne and Illusie, we would like to show that if M is a projective scheme of dimension d with isolated linearly reductive singularities and $\mathcal{L}$ is an ample line bundle on M, then $H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{L}^{-p^m}) = 0$ for m sufficiently large. Lemma 5.4.9 would then imply that m can be taken to be 1. The issue is that the vanishing of these cohomology groups for m large enough is not clear. Under certain hypothesis, however, we obtain a vanishing theorem.

Proposition 5.4.10. *Let M be a projective scheme of dimension d with isolated linearly reductive singularities. Let $\mathcal{L}$ be an ample line bundle on M. If the $j_*\Omega^s_{M^0/S}$ are Cohen-Macaulay for all s, then*

$$H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{L}^{-1}) = 0$$

for all $s + t < \inf(d,p)$.

Proof. By Lemma 5.4.9, we need only prove that $H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{L}^{-p^m}) = 0$ for m sufficiently large. Grothendieck Duality shows

$$H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{L}^{-p^m})^\vee = \operatorname{Ext}^{d-t}(j_*\Omega^s_{M^0/S} \otimes \mathcal{L}^{-p^m}, \omega^0_M).$$

Since the $\mathcal{E}xt^{d-t}(j_*\Omega^s_{M^0/S}, \omega^0_M)$ are coherent, the local-global Ext spectral sequence shows that for m sufficiently large,

$$H^t(j_*\Omega^s_{M^0/S} \otimes \mathcal{L}^{-p^m})^\vee = \Gamma(\mathcal{E}xt^{d-t}(j_*\Omega^s_{M^0/S}, \omega^0_M) \otimes \mathcal{L}^{p^m}).$$

For all $x \in M$,

$$\mathcal{E}xt^{d-t}(j_*\Omega^s_{M^0/S}, \omega^0_M)_x = \mathrm{Ext}^{d-t}_{\mathcal{O}_x}((j_*\Omega^s_{M^0/S})_x, \omega^0_{\mathcal{O}_x}).$$

Since M and the $j_*\Omega^s_{M^0/S}$ are Cohen-Macaulay, local duality then shows that for $t < d$,

$$\mathcal{E}xt^{d-t}(j_*\Omega^s_{M^0/S}, \omega^0_M) = 0,$$

thereby completing the proof. □

We conclude by showing that the hypercohomology spectral sequence

$$E_1^{st} = R^t f'_* F_* \varpi^s_{\mathfrak{X}/S} \Rightarrow R^n f'_* F_* \varpi^\bullet_{\mathfrak{X}/S}$$

degenerates at E_2 and that the only potentially non-zero differentials on the first page are those on the zero-th row.

Lemma 5.4.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose that $\mathcal{A}$ has enough injectives. If $A^\bullet$ is a complex of objects in $\mathcal{A}$ and $C^\bullet$ denotes the cone in the derived category $D(\mathcal{A})$ of the canonical morphism $FA^\bullet \to RFA^\bullet$, then there is a spectral sequence*

$${}^{\backprime}E_1^{st} = \begin{cases} R^tFA^s & t > 0 \\ 0 & t = 0 \end{cases} \Rightarrow \mathcal{H}^n(C^\bullet).$$

If in the hypercohomology spectral sequence $E_1^{pq} = R^qFA^p \Rightarrow R^nFA^\bullet$, the differentials $E_r^{s-r,r-1} \to E_r^{s,0}$ are zero for all $r \geq 2$, then for every n,

$$0 \to \mathcal{H}^n(FA^\bullet) \to R^nFA^\bullet \to \mathcal{H}^n(C^\bullet) \to 0$$

is a short exact sequence.

Proof. The existence of the spectral sequence ${}^{\backprime}E$ is shown as follows. Let $A^s \to I^{s,\bullet}$ be an injective resolution of A^s. The cone $C^\bullet$ is then quasi-isomorphic to the total complex of

$$\begin{array}{ccc} \vdots & \vdots & \\ FI^{01} & FI^{11} & \dots \\ FI^{00} & FI^{10} & \dots \\ FA^0 & FA^1 & \dots \end{array}$$

where FA^0 has bidegree $(-1, 0)$. The spectral sequence associated to this double complex in which we begin by taking cohomology vertically is our desired ${}^{\backprime}E$.

Note that there is a morphism of spectral sequences $E \to {}^{\backprime}E$. If the differentials $E_r^{s-r,r-1} \to E_r^{s,0}$ are zero for all $r \geq 2$, then the morphism of spectral sequences induces an isomorphism $E_\infty^{st} \xrightarrow{\simeq} {}^{\backprime}E_\infty^{st}$ for $t \neq 0$. It follows that $\mathcal{H}^n(C^\bullet)$ is equal to $R^nFA^\bullet$ modulo the bottom part of its filtration, namely $E_\infty^{n0} = \mathcal{H}^n(FA^\bullet)$. □

Proposition 5.4.12. *Let $\mathfrak{X}$ and M be as in Theorem 2.1.10. If M has isolated singularities, and $\mathfrak{X}$ is proper and lifts mod p^2, then the hypercohomology spectral sequence*

$$E_1^{st} = R^t f'_* F_* \varpi^s_{\mathfrak{X}/S} \Rightarrow R^n f'_* F_* \varpi^\bullet_{\mathfrak{X}/S}$$

degenerates at E_2, and for $t \neq 0$, the differentials $E_1^{st} \to E_1^{s+1,t}$ are zero.

Proof. Let $C^\bullet$ be the cone of the canonical morphism $f'_* F_* \varpi^\bullet_{\mathfrak{X}/S} \to Rf'_* F_* \varpi^\bullet_{\mathfrak{X}/S}$. Note that for $t > 0$, we have

$$j'^* R^t f'_* F_* \varpi^s_{\mathfrak{X}/S} = F_* j^* R^t f_* \bigwedge^s L_{\mathfrak{X}/S} = (f^0)'_* \mathcal{H}^t (\bigwedge^s L_{(\mathfrak{X}^0)'/S}) = 0,$$

and so $R^t f'_* F_* \varpi^s_{\mathfrak{X}/S}$ is supported at the singular locus of M'; in particular, the $R^t f'_* F_* \varpi^s_{\mathfrak{X}/S}$ are torsion. On the other hand, $R^0 f'_* F_* \varpi^s_{\mathfrak{X}/S} = F_* j_* \Omega^s_{M^0/S}$, which is reflexive, and hence torsion-free. As a result, for $r \geq 2$ every differential $E_r^{s-r,r-1} \to E_r^{s,0}$ is zero, and E_r^{st} is supported at the singular locus of M' for all $t \neq 0$ and all s and r. So, to prove the proposition, we need only show that the spectral sequence

$$`E_1^{st} = \begin{cases} R^t f'_* F_* \varpi^s_{\mathfrak{X}/S} & t > 0 \\ 0 & t = 0 \end{cases} \Rightarrow \mathcal{H}^n(C^\bullet)$$

of Lemma 5.4.11 degenerates.

Since M is assumed to have isolated singularities, for any short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$$

with $\mathcal{F}$ supported at the singular locus,

$$0 \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{G}) \to \Gamma(\mathcal{Q}) \to 0$$

is short exact as well. Furthermore, $\Gamma(\mathcal{F}) = \bigoplus_{x \in M} \mathcal{F}_x$, so $\mathcal{F}$ is zero if and only if $\Gamma(\mathcal{F})$ is zero. It follows that we have a spectral sequence

$$``E_1^{st} = \begin{cases} \Gamma(R^t f'_* F_* \varpi^s_{\mathfrak{X}/S}) & t > 0 \\ 0 & t = 0 \end{cases} \Rightarrow \Gamma(\mathcal{H}^n(C^\bullet))$$

whose degeneracy is equivalent to that of $`E$. By Lemma 5.4.11, there is a short exact sequence

$$0 \longrightarrow \mathcal{H}^n(f'_* F_* \varpi^\bullet_{\mathfrak{X}/S}) \longrightarrow R^n f'_* F_* \varpi^\bullet_{\mathfrak{X}/S} \longrightarrow \mathcal{H}^n(C^\bullet) \longrightarrow 0.$$

Comparing this with the short exact sequence

$$0 \longrightarrow \mathcal{H}^t(f'_* F_* \varpi^\bullet_{\mathfrak{X}/S}) \xrightarrow{\varphi} R^t f'_* F_* \varpi^\bullet_{\mathfrak{X}/S} \longrightarrow \bigoplus_{\substack{i+j=t \\ j>0}} R^j f'_* \varpi^i_{\mathfrak{X}'/S} \longrightarrow 0$$

proved in Theorem 5.4.4, we see

$$\mathcal{H}^n(C^\bullet) = \bigoplus_{\substack{s+t=n \\ t>0}} R^t f'_* \varpi^s_{\mathfrak{X}'/S}.$$

It follows that

$$\sum_{s+t=n} \dim_k {}''E_1^{st} = \dim_k \Gamma(\mathcal{H}^n(C^\bullet))$$

which shows the degeneracy of $''E$ by Lemma 5.1.5. □

Chapter 6

Toric Artin Stacks

In this chapter, we develop of a theory of toric Artin stacks which generalizes and unites the approach to toric Deligne-Mumford stacks taken by Iwanari in [Iw] and Borisov-Chen-Smith in [BCS]. In Chapter 3 we proved the stacky resolution theorem for toroidal embeddings X and through a slight variant of the proof (see Theorem 3.3.9), also constructed several other smooth log smooth Artin stacks each of which has X as its good moduli space and is isomorphic to X over the trivial locus X^{triv}.

When X is a toric variety, rather than just a toroidal embedding, we can construct even more such smooth log smooth Artin stacks (see Theorem 6.2.6). Each of these Artin stacks has a dense open torus whose action on itself extends to an action on the stack. Thus, these stacks should all be thought of as toric Artin stacks. Along the lines of [BCS], we develop a theory of generalized stacky fans and recast the construction of the above toric Artin stacks in terms of these stacky fans (see Theorem 6.2.8). We imagine that along the lines of [FMN], there is a more intrinsic description of the toric Artin stacks we introduce in this chapter, but we do not attempt to prove such a result here.

The key difference between our stacky fans and those of [BCS, p.193], is that we allow marked points which do not lie on an extremal ray. More precisely, we define a generalized stacky fan $\mathbf{\Sigma}$ to be the choice of a finitely-generated abelian group N, a rational fan $\Sigma \subset N \otimes \mathbb{Q}$, a choice of $r \in \mathbb{N}$, and a morphism $\beta : \mathbb{Z}^{\Sigma(1)} \times \mathbb{Z}^r \to N$. We require that if $\rho_i \in \Sigma(1)$, then $\beta(\rho_i) \otimes 1$ lie on the the ray ρ_i, and we require that if $e_j \in \mathbb{Z}^r$, then $\beta(e_j) \otimes 1$ lie in some cone of the fan. In particular, we allow β to be the zero map. We remark that the toric Artin stack associated to $(N = 0, \Sigma = 0, \beta : \mathbb{Z}^n \to N)$ is Lafforgue's toric Artin stack $[\mathbb{A}^n/\mathbb{G}_m^n]$. Therefore, the theory of toric Artin stacks we develop here helps to unite that of [BCS] with that of [La].

Throughout this chapter, let k be a field and $S = \operatorname{Spec} k$ have trivial log structure. In Section 6.1, we generalize the definition of stacky fan given in [BCS, p.193] and associate to a generalized stacky fan $\mathbf{\Sigma}$ a smooth log smooth Artin stack $\mathfrak{X}(\mathbf{\Sigma})$ having $X(\Sigma)$ as a good moduli space. In Section 6.2, we show that if N is torsion-free, then $\mathfrak{X}(\mathbf{\Sigma})$ has a natural moduli intepretation in terms of log geometry.

6.1 Generalized Stacky Fans

In keeping with the notation of [BCS], for this section only, we let $A^* = \mathrm{Hom}(A, \mathbb{Z})$ for an abelian group A; recall that P^* usually denotes the units of a monoid P. Given a finitely-generated abelian group N and a rational fan $\Sigma \subset N \otimes \mathbb{Q}$, we let $\Sigma(1)$ be the set of rays of the fan. We denote by d the rank of N and n the order of $\Sigma(1)$.

We introduce the following notion of a generalized stacky fan. We frequently drop the word "generalized" when referring to it.

Definition 6.1.1. A *generalized stacky fan* $\mathbf{\Sigma}$ consists of a finitely-generated abelian group N, a rational fan $\Sigma \subset N \otimes \mathbb{Q}$, a choice of $r \in \mathbb{N}$, and a morphism $\beta : \mathbb{Z}^{\Sigma(1)} \times \mathbb{Z}^r \to N$. We require that if $\rho_i \in \Sigma(1)$, then $\beta(\rho_i) \otimes 1$ lie on the the ray ρ_i, and we require that if $e_j \in \mathbb{Z}^r$, then $\beta(e_j) \otimes 1$ lie in some cone σ_j of the fan. We often suppress r and write $\mathbf{\Sigma} = (N, \Sigma, \beta)$.

Throughout this section, we fix for every stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ an ordering on the rays of $\Sigma(1)$ so that β is a map from $\mathbb{Z}^{n+r}$. In the next section, however, canonicity will be more important.

Note that in the above definition we do not require that the rays ρ_i span $N \otimes \mathbb{Q}$ or that the $\beta(e_j)$ be distinct. Some of the $\beta(e_j)$ can even be zero, which as we will see, corresponds to the fact that the associated stack $\mathfrak{X}(\mathbf{\Sigma})$ contains a dense "Artin stacky torus".

We remark that in [Ji, §2], Jiang introduces a notion of extended stacky fans which is equivalent to our definition above, but the stacks he associates to them are all Deligne-Mumford. His goal is to obtain suitable presentations of toric Deligne-Mumford stacks rather than construct toric Artin stacks.

We show now how to associate to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ an Artin stack $\mathfrak{X}(\mathbf{\Sigma})$, which we refer to as a *toric Artin stack*. We follow the procedure in [BCS]. As in [BCS, p. 195], we obtain an exact sequence

$$N^* \xrightarrow{\beta^*} (\mathbb{Z}^{n+r})^* \to H^1(\mathrm{Cone}(\beta)^*) \to \mathrm{Ext}^1_{\mathbb{Z}}(N, \mathbb{Z}) \to 0.$$

Letting $DG(\beta) := H^1(\mathrm{Cone}(\beta)^*)$, we define $\beta^\vee : (\mathbb{Z}^{n+r})^* \to DG(\beta)$ to be the connecting homomorphism above. More concretely, let

$$0 \to \mathbb{Z}^\ell \xrightarrow{Q} \mathbb{Z}^{d+\ell} \to N \to 0$$

be a projective resolution of N. If $B : \mathbb{Z}^{n+r} \to \mathbb{Z}^{d+\ell}$ is a lift of β, then

$$DG(\beta) = \mathrm{coker}([BQ]^*)$$

and $\beta^\vee$ is the composite

$$(\mathbb{Z}^{n+r})^* \to (\mathbb{Z}^{d+r+\ell})^* \to DG(\beta).$$

The construction of $\mathfrak{X}(\mathbf{\Sigma})$ is then essentially the same as in [BCS, p.198]. Consider the ideal

$$J_{\mathbf{\Sigma}} = \langle \prod_{\beta(e_i)\otimes 1 \notin \sigma} x_i \mid \sigma \in \Sigma \rangle$$

of $k[x_1, \ldots, x_{d+r}]$. Letting $G_{\mathbf{\Sigma}}$ be the diagonalizable group scheme associated to $DG(\beta)$, then via $\beta^\vee$ we obtain a morphism $G_{\mathbf{\Sigma}} \to \mathbb{G}_m^{n+r}$ and hence an action of $G_{\mathbf{\Sigma}}$ on $\mathbb{A}^{d+r} = \operatorname{Spec} k[x_1, \ldots, x_{d+r}]$ via the action of $\mathbb{G}_m^{n+r}$. Since $V(J_{\mathbf{\Sigma}})$ is a union of coordinate subspaces, we see that $Z_{\mathbf{\Sigma}} := \mathbb{A}^{d+r} - V(J_{\mathbf{\Sigma}})$ is $G_{\mathbf{\Sigma}}$-invariant. We obtain a log structure $\mathcal{M}_{Z_{\mathbf{\Sigma}}}$ on $Z_{\mathbf{\Sigma}}$ by pulling back the canonical log structure on $\mathbb{A}^{d+r}$. Note that the $G_{\mathbf{\Sigma}}$-action on $Z_{\mathbf{\Sigma}}$ extends to the log scheme $(Z_{\mathbf{\Sigma}}, \mathcal{M}_{Z_{\mathbf{\Sigma}}})$. We define

$$\mathfrak{X}(\mathbf{\Sigma}) = [Z_{\mathbf{\Sigma}}/G_{\mathbf{\Sigma}}]$$

and obtain a log structure on $\mathfrak{X}(\mathbf{\Sigma})$ by descent theory. We see then that $\mathfrak{X}(\mathbf{\Sigma})$ is smooth and log smooth.

Example 6.1.2. If $\mathbf{\Sigma} = (N, 0, \beta : \mathbb{Z}^d \to 0)$ with N torsion-free of rank d, then $\mathfrak{X}(\mathbf{\Sigma}) = [\mathbb{A}^d/\mathbb{G}_m^d]$, which is a smooth toric Artin stack in the sense of Lafforgue [La, IV.1.a].

Remark 6.1.3. Given N and a rational fan Σ, we can define a canonical stacky fan $\mathbf{\Sigma}^{can} = (N, \Sigma, \beta^{can} : \mathbb{Z}^n \to N)$ by letting $\beta^{can}(e_i)$ be the first lattice point on the i^{th} ray. However, unlike in the theory of toric Deligne-Mumford stacks, given a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, we do not necessarily have a morphism from $\mathfrak{X}(\mathbf{\Sigma})$ to $\mathfrak{X}(\mathbf{\Sigma}^{can})$ as in [FMN, Thm I]. It is sometimes necessary to take a root construction of $\mathfrak{X}(\mathbf{\Sigma})$ in order to get a map to $\mathfrak{X}(\mathbf{\Sigma}^{can})$.

Note that given a stacky fan $\mathbf{\Sigma}$, we can always write $N = N' \times N''$, where N'' is a free abelian group, the rays of Σ span $N' \otimes \mathbb{Q}$, and the span of the rays of Σ does not intersect N''. We then obtain another stacky fan $\mathbf{\Sigma}' = (N', \Sigma, \beta)$ in the evident way, and the above construction shows that $\mathfrak{X}(\mathbf{\Sigma}) = \mathbb{G}_m^{d-n} \times \mathfrak{X}(\mathbf{\Sigma}')$.

We work now toward showing that $X(\Sigma)$ is the good moduli space of $\mathfrak{X}(\mathbf{\Sigma})$. Given a stacky fan $\mathbf{\Sigma}$ and $\sigma \in \Sigma$, along the lines of [Cox, §1], we let

$$x^\sigma = \prod_{\beta(e_i)\otimes 1 \notin \sigma} x_i$$

and let $U_\sigma = \mathbb{A}^{d+r} - V(x^\sigma)$. Note that U_σ is $G_{\mathbf{\Sigma}}$-invariant and that $Z_{\mathbf{\Sigma}}$ is the union of the U_σ.

Proposition 6.1.4. *Let $\mathbf{\Sigma}$ be a stacky fan and $\sigma \in \Sigma$. If $X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$, then there is a natural map $[U_\sigma/G_{\mathbf{\Sigma}}] \to X_\sigma$ which is a good moduli space.*

Proof. We may assume $n = d$. Let $P_\sigma = \sigma^\vee \cap M$ and note that $U_\sigma = \operatorname{Spec} k[F_\sigma]$, where

$$F_\sigma = \mathbb{N}^{\{i \mid \beta(e_i)\otimes 1 \in \sigma\}} \times \mathbb{Z}^{\{i \mid \beta(e_i)\otimes 1 \notin \sigma\}}.$$

Let $i_\sigma : P_\sigma \to F_\sigma$ be defined by $i_\sigma(p) = ((\beta(e_i) \otimes 1)(p))$. If N is torsion-free, we can choose $\ell = 0$ and $B = \beta$ in the above construction of $\mathfrak{X}(\mathbf{\Sigma})$ so that the diagram

$$\begin{array}{ccc} P^{gp} & \xrightarrow{\ i_\sigma\ } & F_\sigma^{gp} \\ {\scriptstyle id}\downarrow & & \simeq\downarrow{\scriptstyle \varphi} \\ N^* & \xrightarrow{\ B^*\ } & (\mathbb{Z}^{n+r})^* \end{array}$$

commutes; here φ sends e_i to the dual basis vector e_i^*. This shows that

$$k[P_\sigma] = k[F_\sigma]^{G_{\mathbf{\Sigma}}}$$

and so the morphism $[U_\sigma/G_{\mathbf{\Sigma}}] \to X_\sigma$ induced by i_σ is a good moduli space.

We now handle the case when N is not torsion-free. Let $N_{tors} \simeq \prod \mathbb{Z}/m_i\mathbb{Z}$, where the $m_i > 1$. Let $N' = N/N_{tors}$ and $\mathbf{\Sigma}' = (N', \Sigma, \beta')$ where β' is the composite of β and the projection of N to N'. We have then a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{Z}^\ell & & \\ & & \downarrow{\scriptstyle Q=0\times\mathrm{diag}(m_i)} & & \\ & & N' \times \mathbb{Z}^\ell & \xrightarrow{\ \pi\ } & N' \\ & {\scriptstyle B}\nearrow & \downarrow & & \downarrow{\scriptstyle id} \\ \mathbb{Z}^{n+r} & \xrightarrow[\beta]{} & N & \longrightarrow & N' \end{array}$$

where the columns are projective resolutions. Let $B' = \pi B$. We then obtain a commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}^n)^* & \xrightarrow{(B')^*} & (\mathbb{Z}^{n+r})^* & \xrightarrow{(\beta')^\vee} & DG(\beta') \\ \downarrow & & \downarrow & \searrow^{\beta^\vee} & \downarrow{\scriptstyle \eta} \\ (N' \times \mathbb{Z}^\ell)^* & \xrightarrow[(BQ)^*]{} & (\mathbb{Z}^{n+r+\ell})^* & \longrightarrow & DG(\beta) \end{array}$$

The left and middle vertical arrows are injective. One easily checks that the left square is cartesian, and so η is injective. This shows that the induced map

$$k[F_\sigma]^{G_{\mathbf{\Sigma}'}} \to k[F_\sigma]^{G_{\mathbf{\Sigma}}}$$

is an isomorphism, and hence, the composite

$$[U_\sigma/G_{\mathbf{\Sigma}}] \to [U_\sigma/G_{\mathbf{\Sigma}'}] \to X_\sigma$$

is a good moduli space as well. $\square$

Theorem 6.1.5. *If $\mathbf{\Sigma}$ is a stacky fan, then there is a natural map $\mathfrak{X}(\mathbf{\Sigma}) \to X(\Sigma)$ which is a good moduli space such that for all $\sigma \in \Sigma$,*

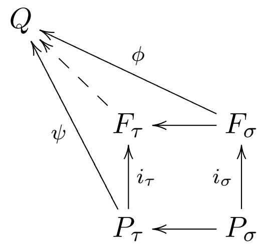
$$\begin{array}{ccc} [U_\sigma/G_{\mathbf{\Sigma}}] & \longrightarrow & \mathfrak{X}(\mathbf{\Sigma}) \\ \downarrow & & \downarrow \\ X_\sigma & \longrightarrow & X(\Sigma) \end{array}$$

is cartesian; here $X_\sigma = \operatorname{Spec} k[\sigma^\vee \cap M]$ and $[U_\sigma/G_{\mathbf{\Sigma}}] \to X_\sigma$ is the morphism constructed in Proposition 6.1.4.

Proof. We may assume $n = d$. For every $\sigma \in \Sigma$, let P_σ, F_σ, and i_σ be as in the proof of Proposition 6.1.4. We denote also by i_σ the induced morphism $U_\sigma \to X_\sigma$. Note that if τ is face of σ, then the diagram

$$\begin{array}{ccc} U_\tau & \longrightarrow & U_\sigma \\ {\scriptstyle i_\tau}\downarrow & & \downarrow{\scriptstyle i_\sigma} \\ X_\tau & \longrightarrow & X_\sigma \end{array}$$

commutes. We claim that it is, in fact, cartesian. To prove this, we show that if we have a commutative diagram of monoids



then there is a unique dotted arrow making the diagram commute. This is equivalent to showing that if $\beta(e_i) \otimes 1$ is in σ but not in τ, then $\phi(e_i)$ is a unit. By [Fu, §1.2 Prop 2], there is some $p \in P_\sigma$ such that $\tau = \sigma \cap p^\perp$ and $P_\tau = P_\sigma + \mathbb{N} \cdot (-p)$. Note then that $\psi(p)$ is a unit and that

$$\psi(p) = \phi\, i_\sigma(p) = \sum_i ((\beta(e_i) \otimes 1)(p))\phi(e_i).$$

Let i be such that $\beta(e_i) \otimes 1$ is in σ but not in τ. Since it is in σ, we see that $(\beta(e_i) \otimes 1)(p) \geq 0$. Since $\beta(e_i) \otimes 1$ is not in τ and since $\tau = \sigma \cap p^\perp$, we must have $(\beta(e_i) \otimes 1)(p) > 0$, and so $\phi(e_i)$ is a unit, as desired.

We see then that

$$\begin{array}{ccc} [U_\tau/G_{\mathbf{\Sigma}}] & \longrightarrow & [U_\sigma/G_{\mathbf{\Sigma}}] \\ \downarrow & & \downarrow \\ X_\tau & \longrightarrow & X_\sigma \end{array}$$

is cartesian. By Lemma 6.3 and Proposition 7.9 of [Al], it follows that there is a natural map $\mathfrak{X}(\mathbf{\Sigma}) \to X(\Sigma)$ which is a good moduli space whose base change to X_σ is as claimed. $\square$

6.2 Admissible Sliced Resolutions and a Moduli Interpretation of $\mathfrak{X}(\mathbf{\Sigma})$

We begin this subsection by associating to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ with N torsion-free, a smooth log smooth Artin stack $\mathfrak{X}_{\mathbf{\Sigma}}$ having $X(\Sigma)$ as a good moduli space. The stack $\mathfrak{X}_{\mathbf{\Sigma}}$ is constructed as a moduli space along the same lines of the constructions in Theorems 3.3.2 and 3.3.9. We then show that $\mathfrak{X}(\mathbf{\Sigma})$ is isomorphic to $\mathfrak{X}_{\mathbf{\Sigma}}$ as log stacks over $X(\Sigma)$, thereby giving a moduli interpretation to $\mathfrak{X}(\mathbf{\Sigma})$.

Note that if $\mathbf{\Sigma} = (N, \Sigma, \beta)$ is a stacky fan with N torsion-free, then giving the map β is equivalent to choosing a positive integer b_i for every $\rho_i \in \Sigma(1)$ and choosing for every $j \in \{1, 2, \ldots, r\}$ an element $w_j \in N$ which lies in some cone $\sigma_j \in \Sigma$. Given this equivalence, throughout this subsection, we denote stacky fans by $\mathbf{\Sigma} = (N, \Sigma; b_i; w_j)$.

Let us work toward defining the morphisms which $\mathfrak{X}_{\mathbf{\Sigma}}$ will parameterize.

Definition 6.2.1. If P is an fs sharp monoid and $i : P \to F$ its minimal free resolution, then we say a *datum D for P* is a choice of $r \in \mathbb{N}$, a positive integer b_i for every irreducible element v_i of F, and morphisms $w_j : P \to \mathbb{N}$ for $j \in \{1, 2, \ldots, r\}$. We frequently suppress r and write $D = (b_i; w_j)$.

If $\mathbf{\Sigma} = (N, \Sigma, b_i; w_j)$ is a stacky fan with N torsion-free, then given a geometric point $\bar{x}$ of $X = X(\Sigma)$, let $I(\bar{x})$ be the set of irreducible components of $X - X^{triv}$ on which $\bar{x}$ lies and let

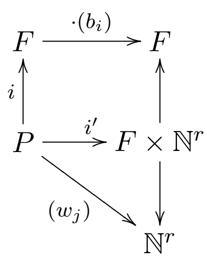
$$F_{\bar{x}} = \alpha_{\bar{x}}^{-1}(\mathcal{O}_{X,\bar{x}}^{*}),$$

where $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ is the structure morphism of the log structure $\mathcal{M}_X$. If $w_j(F_{\bar{x}}) = 0$, then we obtain a morphism $\bar{w}_j : \bar{\mathcal{M}}_{X,\bar{x}} \to \mathbb{N}$. We can therefore define a datum for $\bar{\mathcal{M}}_{X,\bar{x}}$ by

$$D_{\mathbf{\Sigma},\bar{x}} = (b_i \text{ s.t. } i \in I(\bar{x}); \bar{w}_j \text{ s.t. } w_j(F_{\bar{x}}) = 0);$$

here we are using Proposition 3.3.5 to identify $I(\bar{x})$ with the set of irreducible elements of the minimal free resolution of $\bar{\mathcal{M}}_{X,\bar{x}}$.

Definition 6.2.2. If P is an fs sharp monoid, $i : P \to F$ its minimal free resolution, and $D = (b_i; w_j)$ a datum for P, then we say a morphism $P \to F'$ is an *admissible free resolution of type D* if it is isomorphic to $i' : P \to F \times \mathbb{N}^r$, where i' is such that the diagram



commutes; the two arrows out of $F \times \mathbb{N}^r$ are the natural projections. We say that $P \to F'$ is an *admissible sliced resolution of type* D if it is isomorphic to

$$P \xrightarrow{i'} F \times \mathbb{N}^r \to (F \times \mathbb{N}^r)/H,$$

where i' is an admissible free resolution of type D and $H \cap i'(P) = 0$.

Definition 6.2.3. If $\mathbf{\Sigma}$ is a stacky fan with N torsion-free and if $X = X(\Sigma)$, then a morphism $f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)$ from a fine log scheme is an *admissible sliced resolution of type* $\mathbf{\Sigma}$ if for all geometric points $\bar{y}$ of Y, the induced morphism $\bar{\mathcal{M}}_{X,f(\bar{y})} \to \bar{\mathcal{M}}_{Y,\bar{y}}$ is an admissible sliced resolution of type $D_{\mathbf{\Sigma},f(\bar{y})}$.

Note that if D is a datum for P in which $r = 0$, then $P \to F'$ is an admissible free resolution of type D, resp. an admissible sliced resolution of type D in the sense of Definition 6.2.2 if and only if it is in the sense of Definition 3.3.4. Similarly, if $\mathbf{\Sigma}$ is a stacky fan for which $r = 0$ (*i.e.* a stacky fan in the sense of [BCS, p.193]) and for which N is torsion-free, then a morphism $f : (Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)$ from a fine log scheme is an admissible sliced resolution of type $\mathbf{\Sigma}$ if and only if it is an admissible sliced resolution of type (b_i) in the sense of Definition 3.3.6.

Given a stacky fan $\mathbf{\Sigma}$, let $X = X(\Sigma)$. We define $\mathfrak{X}_{\mathbf{\Sigma}}$ as the fibered category over X-schemes whose objects are morphisms $(T, \mathcal{N}) \to (X, \mathcal{M}_X)$ which are admissible sliced resolutions of type $\mathbf{\Sigma}$, and whose morphisms are maps of $(X, \mathcal{M}_X)$-log schemes $h : (T, \mathcal{N}) \to (T', \mathcal{N}')$ with $h^*\mathcal{N}' \to \mathcal{N}$ an isomorphism. As before, this fibered category is a stack on the fppf site by [Ol2, Thm A.1].

The proof that these stacks are algebraic and have the properties mentioned earlier is similar to the proofs of Theorems 3.3.2 and 3.3.9, so we indicate only where changes are necessary.

Proposition 6.2.4. *Let* $\mathbf{\Sigma}$ *be a stacky fan such that* $X := X(\Sigma) = \operatorname{Spec} k[P]$ *for some fs sharp monoid* P*. Let* $D = D_{\mathbf{\Sigma},\bar{0}}$*, where* $0 \in X$ *is the point such that* $\bar{\mathcal{M}}_{X,\bar{0}} = P$*. If* $i' : P \to F'$ *is an admissible free resolution of type* D*, then the induced morphism* $f : X \to \operatorname{Spec} k[F']$ *on log schemes is an admissible sliced resolution of type* $\mathbf{\Sigma}$*.*

Proof. Choosing an appropriate isomorphism, we can assume that $F' = F \times \mathbb{N}^r$ and that i' is as in Definition 6.2.2. Let $H'' = H \times H'$ be a face of F' and let $P_0 = i'(P) \cap H''$. Let $\bar{\imath}'$ be the resulting morphism which makes the diagram

$$\begin{array}{ccc} P & \xrightarrow{i'} & F' \\ \downarrow & & \downarrow \\ P/P_0 & \xrightarrow{\bar{\imath}'} & F/H \times \mathbb{N}^r/H' \end{array}$$

commute. We must show that $\bar{\imath}'$ is an admissible sliced resolution of the appropriate type. Note first that if H'_0 denotes the face of $\mathbb{N}^r$ generated by the e_j with $w_j(P_0) \neq 0$, then commutativity of the above diagram shows that $H' \supset H'_0$. Similarly, we see that $H \supset F_0$, where F_0 denotes the face of F generated by $i(P_0)$. As a result, we have a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{i'} & F' & \searrow & \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \\ P/P_0 & \xrightarrow{i''} & F/F_0 \times \mathbb{N}^r/H'_0 & \xrightarrow{\pi'} & F/H \times \mathbb{N}^r/H' \end{array}$$

where the bottom row composes to $\overline{i}'$, and π and π' are the natural projections. Recall that by Proposition 3.2.7, the natural morphism from P/P_0 to F/F_0 is a minimal free resolution. Note now that i'' is given by $i''(\bar{p}) = (b_i v_i(p); w_j(p))$ for i such that $v_i(P_0) = 0$ and j such that $w_j(P_0) = 0$; that is, i'' is an admissible free resolution of the correct type.

To complete the proof, we must therefore show $(H/F_0 \times H'/H'_0) \cap i''(P/P_0) = 0$. This amounts to showing that if $b_i v_i(p) v_i \in H$ for all i such that $v_i(P_0) = 0$ and if $w_j(p) e_j \in H'$ for all $w_j(P_0) = 0$, then $b_i v_i(p) v_i \in H$ for all i and $w_j(p) e_j \in H'$ for all j. If i is such that $v_i(P_0) \neq 0$, then $v_i \in F_0 \subset H$ and so $b_i v_i(p) v_i \in H$. Similarly, if j is such that $w_j(P_0) \neq 0$, then $e_j \in H'_0 \subset H'$ and so $w_j(p) e_j \in H'$. This shows that the above intersection is trivial. □

The proof of Proposition 3.3.1 then yields:

Proposition 6.2.5. *Let $\mathbf{\Sigma}$ be a stacky fan such that $X := X(\Sigma) = \operatorname{Spec} k[P]$ for some fs sharp monoid P. Let $D = D_{\mathbf{\Sigma},\bar{0}}$, where $0 \in X$ is the point such that $\bar{\mathcal{M}}_{X,\bar{0}} = P$. If $i' : P \to F'$ is an admissible free resolution of type D and $G = D(F'^{gp}/i'(P^{gp}))$, then $\mathfrak{X}_{\mathbf{\Sigma}}$ is isomorphic to $[\operatorname{Spec} k[F']/G]$ over X.*

The first main theorem of this section is then:

Theorem 6.2.6. *If $\mathbf{\Sigma} = (N, \Sigma; b_i; w_j)$ is a stacky fan and $X = X(\Sigma)$, then $\mathfrak{X}_{\mathbf{\Sigma}}$ is a smooth log smooth Artin stack over $\operatorname{Spec} k$ having X as a good moduli space. Moreover, we have a cartesian diagram*

$$\begin{array}{ccc} T \times [\mathbb{A}^z/\mathbb{G}_m^z] & \longrightarrow & \mathfrak{X}_{\mathbf{\Sigma}} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

where T is the dense open torus of X, π is the natural projection, and z is the number of w_j which are zero.

Proof. Zariski locally, X is of the form $Y \times T'$, where T' is a torus and $Y = \operatorname{Spec} k[P]$ for some fs sharp monoid P. The proof of Theorem 3.3.2 then shows that $\mathfrak{X}_{\mathbf{\Sigma}}$ is a smooth log smooth Artin stack and that the natural map $\mathfrak{X}_{\mathbf{\Sigma}} \to X$ is a good moduli space. Now note that we have a cartesian diagram

$$\begin{array}{ccc} \mathfrak{X}_{\mathbf{\Sigma}'} & \longrightarrow & \mathfrak{X}_{\mathbf{\Sigma}} \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \end{array}$$

where $\mathbf{\Sigma}' = (N, 0; \emptyset; w_j \text{ s.t. } w_j = 0)$. Then by the first part of this proof, we see that $\mathfrak{X}_{\mathbf{\Sigma}'} = T \times \mathfrak{X}_{\mathbf{\Sigma}''}$, where $\mathbf{\Sigma}'' = (0, 0; \emptyset; w_j \text{ s.t. } w_j = 0)$. Proposition 6.2.5 then shows that

$$\mathfrak{X}_{\mathbf{\Sigma}''} \simeq [\mathbb{A}^z / \mathbb{G}_m^z],$$

thereby completing the proof. □

We work now toward comparing the stacks $\mathfrak{X}_{\mathbf{\Sigma}}$ and $\mathfrak{X}(\mathbf{\Sigma})$. If $\sigma \in \Sigma$, then let P_σ, F_σ, i_σ, U_σ, and X_σ be as in the proof of Proposition 6.1.4.

Proposition 6.2.7. *Let $\mathbf{\Sigma} = (N, \Sigma; b_i; w_j)$ be a stacky fan in which N is torsion-free. If $\sigma \in \Sigma$ is a maximal cone, then*

$$[U_\sigma / G_{\mathbf{\Sigma}}] \simeq \mathfrak{X}_{\mathbf{\Sigma}} \times_{X(\Sigma)} X_\sigma$$

as log stacks over X_σ.

Proof. We may assume that $n = d$, so that P_σ is sharp. Let $D_\sigma = D_{\mathbf{\Sigma}, \bar{0}}$, where $0 \in X_\sigma$ is a point such that $\bar{\mathcal{M}}_{X_\sigma, \bar{0}} = P_\sigma$. Let $i'_\sigma : P_\sigma \to F'_\sigma$ be an admissible free resolution of type D_σ as in Definition 6.2.2. Let $A = F_\sigma^{gp} / i_\sigma(P_\sigma^{gp})$, $A' = F_\sigma'^{gp} / i'_\sigma(P_\sigma^{gp})$, and $G_\sigma = D(A')$. From the proof of Theorem 6.2.6, we have a cartesian diagram

$$\begin{array}{ccc}
[\operatorname{Spec} k[F'_\sigma]/G_\sigma] & \longrightarrow & \mathfrak{X}_{\mathbf{\Sigma}} \\
{\scriptstyle \epsilon}\downarrow & & \downarrow \\
X_\sigma & \longrightarrow & X(\Sigma)
\end{array}$$

where ϵ is induced from i'_σ. Let I be the set of rays in σ union the set of $w_j \in \sigma$. Let J be the set of rays not in σ union the set of $w_j \notin \sigma$. Then, we see that F'_σ is a direct sum of copies of $\mathbb{N}$ indexed by I. We have a commutative diagram

$$\begin{array}{ccc}
F_\sigma & \xrightarrow{\pi} & F'_\sigma \\
{\scriptstyle i_\sigma}\uparrow & \nearrow{\scriptstyle i'_\sigma} & \\
P_\sigma & &
\end{array}$$

where π is the natural projection.

To prove the proposition, we show that U_σ and the pushout $G_{\mathbf{\Sigma}} \times^{G_\sigma} \operatorname{Spec} k[F'_\sigma]$ are isomorphic as schemes with $G_{\mathbf{\Sigma}}$-action. By definition,

$$G_{\mathbf{\Sigma}} \times^{G_\sigma} \operatorname{Spec} k[F'_\sigma] = \operatorname{Spec} k[A \times \mathbb{N}^I]^{A'}.$$

Since i_σ and i'_σ are injective, we see that the induced morphism

$$\mathbb{Z}^J = \ker(F_\sigma^{gp} \to F_\sigma'^{gp}) \to \ker(A \to A')$$

is an isomorphism. It follows that

$$k[A \times \mathbb{N}^I]^{A'} \simeq k[\mathbb{Z}^J] \otimes_k k[A' \times \mathbb{N}^I]^{A'} \simeq k[\mathbb{Z}^J] \otimes_k k[Q],$$

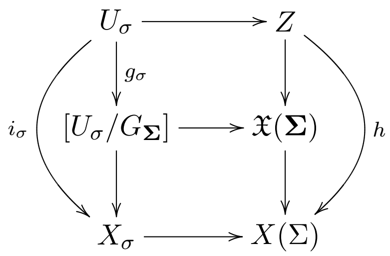
where $Q \subset A' \times \mathbb{N}^I$ is the submonoid of elements of the form $(-\bar{q}, q)$. Since the projection of $A' \times \mathbb{N}^I$ to $\mathbb{N}^I$ induces an isomorphism of Q with $\mathbb{N}^I$, we have an induced isomorphism

$$k[A \times \mathbb{N}^I]^{A'} \simeq k[\mathbb{Z}^J] \otimes_k k[\mathbb{N}^I].$$

Via the isomorphism $\varphi : \mathbb{Z}^J \times \mathbb{N}^I \to F_\sigma$ sending (a, b) to $(-a, b)$, and the automorphism of $G_{\mathbf{\Sigma}}$ sending g to its inverse, we obtain an isomorphism of $G_{\mathbf{\Sigma}} \times^{G_\sigma} \operatorname{Spec} k[F'_\sigma]$ and U_σ respecting the $G_{\mathbf{\Sigma}}$-actions. We see that this isomorphism respects the log structures as well. □

Theorem 6.2.8. *If $\mathbf{\Sigma} = (N, \Sigma; b_i; w_j)$ is a stacky fan in which N is torsion-free, then $\mathfrak{X}(\mathbf{\Sigma})$ and $\mathfrak{X}_{\mathbf{\Sigma}}$ are isomorphic as log stacks over $X(\Sigma)$.*

Proof. We show first that the composite $Z_{\mathbf{\Sigma}} \to \mathfrak{X}(\mathbf{\Sigma}) \to X(\Sigma)$ is an admissible sliced resolution of type $\mathbf{\Sigma}$. Note that this can be checked Zariski locally on $X(\Sigma)$. By Theorem 6.1.5, we have a cartesian diagram



for any cone $\sigma \in \Sigma$. By Proposition 6.2.7, we see that if σ is maximal, then the morphism i_σ is an admissible sliced resolution of type $\mathbf{\Sigma}$. Since the X_σ for σ maximal form a Zariski cover of $X(\Sigma)$, we see that h is an admissible sliced resolution of type $\mathbf{\Sigma}$.

We therefore have a strict morphism $f : Z \to \mathfrak{X}_{\mathbf{\Sigma}}$ over $X(\Sigma)$. We have a $G_{\mathbf{\Sigma}}$-action on Z over $X(\Sigma)$. Since this action respects the log structure of Z, we see that $G_{\mathbf{\Sigma}}$ acts on Z over $\mathfrak{X}_{\mathbf{\Sigma}}$.

We claim that f is a $G_{\mathbf{\Sigma}}$-torsor. This can be checked Zariski locally on $X(\Sigma)$. Since the above diagram is cartesian, we obtain a morphism

$$g'_\sigma : U_\sigma \to \mathfrak{X}_{\mathbf{\Sigma}} \times_{X(\Sigma)} X_\sigma$$

over X_σ. By the construction of f, we see that for σ a maximal cone, $\varphi_\sigma g'_\sigma = g_\sigma$, where φ_σ is the isomorphism from Proposition 6.2.7. It follows that g'_σ is a $G_{\mathbf{\Sigma}}$-torsor, and since the X_σ for σ maximal form a Zariski cover of $X(\Sigma)$, we see that f is as well. Hence,

$$\mathfrak{X}_{\mathbf{\Sigma}} \simeq [Z_{\mathbf{\Sigma}}/G_{\mathbf{\Sigma}}] = \mathfrak{X}(\mathbf{\Sigma}).$$

This is, moreover, an isomorphism of log stacks as the morphisms from $Z_{\mathbf{\Sigma}}$ to $\mathfrak{X}_{\mathbf{\Sigma}}$ and to $\mathfrak{X}(\mathbf{\Sigma})$ are strict. □

Bibliography

[Al] J. Alper, *Good moduli spaces for Artin stacks*, arxiv:0804.2242, 2008.

[Ar] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–58.

[AV] D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc **15** (2002), 27–75.

[AOV] D. Abramovich, M. Olsson, and A. Vistoli, *Tame stacks in positive characteristic*, Anneles de l'Institut Fourier 58 (2008), 1057-1091.

[Be] K. Behrend, *Cohomology of Stacks*, Intersection theory and moduli, 249–294 (electronic), ICTP Lect. Notes, XIX, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[BCS] L. Borisov, L. Chen, and G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc. 18 (2005), no. 1, 193–215.

[Bo] N. Bourbaki, *Groupes et algèbres de Lie*, Ch. V. Hermann, Paris, 1968.

[Co] B. Conrad, *Cohomological descent*, unpublished notes, available at http://math.stanford.edu/∼conrad/papers/cohdescent.pdf

[Cox] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebr. Geom. **4** (1995), 17–50.

[DI] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), no. 2, 247–270.

[EGA4] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas IV*. Inst. Hautes Études Sci. Publ. Math. **32** 1967.

[Fa] G. Faltings, *p-adic Hodge Theory*, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299.

[Fu] W. Fulton, *Introduction to toric varieties*, Princeton University Press, Princeton, NJ, 1993.

[FMN] B. Fantechi, E. Mann, and F. Nironi, *Smooth Toric DM Stacks*, arXiv:0708.1254v1

[GIT] D. Mumford, *Geometric invariant theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34 Springer-Verlag, Berlin-New York 1965.

[Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) 79 1964 205–326.

[HR] M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974), 115–175.

[Il] L. Illusie, *Complexe cotangent et déformations I.* Lecture Notes in Mathematics **239**, Springer-Verlag, Berlin, 1971.

[Iw] I. Iwanari, *Logarithmic geometry, minimal free resolutions and toric algebraic stacks*, Publ. Res. Inst. Math. Sci. 45 (2009), no. 4, 1095–1140.

[Ji] Y. Jiang, *The Orbifold Cohomology Ring of Simplicial Toric Stack Bundles*, Illinois J. Math. 52 (2009), no. 2, 493–514.

[Ka1] F. Kato, *Log smooth deformation theory.* Tohoku Math. J. (2) 48 (1996), no. 3, 317–354.

[Ka2] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in: Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Ka] N. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Inst. Hautes Études Sci. Publ. Math. No. 39 (1970), 175–232.

[KM] S. Keel and S. Mori, *Quotients by groupoids.* Ann. of Math. (2) **145** (1997), no. 1, 193–213.

[La] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002) 1–241.

[LMB] G. Laumon and L. Moret-Bailly, *Champs Algébriques.* Ergebnisse der Mathematik und ihrer Grenzgebiete 39, Springer-Verlag, 2000.

[MO] K. Matsuki and M. Olsson, *Kawamata-Viehweg vanishing as Kodaira vanishing for stacks*, Math. Res. Lett. 12 (2005), no. 2-3, 207–217.

[Ne] M. Neusel, *Invariant Theory*, Student Mathematical Library, 36, American Mathematical Society, Providence, RI, 2007.

[Og] A. Ogus, *Lectures on Logarithmic Algebraic Geometry*, book in preparation, available at http://math.berkeley.edu/∼ogus/preprints/log_book/logbook.pdf

[Ol1] M. Olsson, *Hom-stacks and restriction of scalars*, Duke Math. J. **134** (2006), 139–164.

[Ol2] M. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. École Norm. Sup. (4) 36 (2003), no. 5, 747–791.

[Ol3] M. Olsson, *Sheaves on Artin Stacks*, J. Reine Angew. Math. (Crelle's Journal) **603** (2007), 55–112.

[SGA1] A. Grothendieck, Revêtements étales et groupe fondamental, Séminaire de Géométrie Algébrique du Bois Marie 1960–61 (SGA 1), Springer-Verlag, Berlin, 1971.

[SGA3] M. Demazure and A. Grothendieck, Schémas en groupes. I: Propriétés générales des schémas en groupes, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151, Springer-Verlag, Berlin, 1970.

[Sm] L. Smith, *On the invariant theory of finite pseudoreflection groups*, Arch. Math. (Basel) **44** (1985), no. 3, 225–228.

[St] J. Steenbrink, "Mixed Hodge structure on the vanishing cohomology" P. Holm (ed.), Real and Complex Singularities (Oslo, 1976). Proc. Nordic Summer School, Sijthoff & Noordhoff (1977), 525–563

[Ta] J. Tate, *Finite flat group schemes* in Cornell, Silverman, Stevens: "Modular forms and Fermat's Last Theorem", Springer-Verlag, New York, 1997, p. 121-154.

[To] B. Toen, *K-théorie et cohomologie des champs algébriques: Théorèmes de Riemann-Roch, D-modules et théorèmes GAGA*, Ph.D Thesis, L' Université Paul Sabatier de Toulouse, 1999. arXiv:math.AG/9908097.

[Vi] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670.

[We] D. Wehlau, *When is a ring of torus invariants a polynomial ring?*, Manuscripta Mathematica **82** (1994), 161–170.