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Geometric constructions of mapping cones in the Fukaya Category

Author
Fang, Kuan-Ying

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Geometric Constructions of Mapping Cones in the Fukaya Category

by

Kuan-Ying Fang

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Committee in charge:

Professor Denis Auroux, Chair
Professor Katrin Wehrheim
Professor Vern Paxson

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Abstract

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Kuan-Ying Fang

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Professor Denis Auroux, Chair

We present two geometric constructions of Lagrangian surgeries between two Lagrangian submanifolds intersecting cleanly along a 1-dimensional submanifold. We show in a concrete example in 4-dimensions that the two constructions are isomorphic in the Fukaya Category and represent a mapping cone.

To Amano

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Chapter 1

Introduction

Exact triangles and mapping cones in the Fukaya Category of a symplectic manifold have been useful tools to help us understand the Fukaya Category as a whole, and seeking a geometric understanding of such mapping cones has been helpful in furthering our understanding of them. Geometric interpretations of mapping cones have been understood for a few cases. One such example was presented by Seidel. Seidel showed that in an exact symplectic manifold, the Dehn twist of a Lagrangian submanifold L about S, a Lagrangian sphere, is an example of a mapping cone of a certain evaluation map [5].

Another important way to understand mapping cones in the Fukaya Category geometrically is through Lagrangian surgery. The Lagrangian surgery between two Lagrangian submanifolds of a symplectic manifold intersecting transversely in a single point was proposed by Polterovich [4]. Fukaya-Oh-Ohta-Ono proposed that the Lagrangian surgery between two Lagrangian submanifolds intersecting transversely in a single point p is quasi-isomorphic to $Cone(p)$ [3].

In this paper, we study the case when two Lagrangian submanifolds of a symplectic manifold, instead of intersecting in a single point, intersect cleanly along a 1-dimensional submanifold. We construct two different surgeries between such Lagrangian submanifolds: the Morse surgery and the Morse-Bott surgery, and demonstrate in a 4-dimensional example that they are isomorphic and both represent a mapping cone.

The structure of this paper is as follows: In Section 2, we introduce the setup — the symplectic manifolds and Lagrangian submanifolds considered, the moduli spaces that will be counted, and the higher products needed. In Section 3, we define the two different Lagrangian surgeries. In Section 4, we present the main result, which is a detailed computation in 4-dimensions.

Chapter 2

Setup

Consider (M, ω, σ, J), where M is a manifold, $\omega = d\sigma$ a symplectic form, J an ω-compatible almost complex structure. Let $i : L \to M$ be a Lagrangian immersion with only transverse double points. Assume there exists $f : L \to \mathbb{R}$ such that $df = i^*\sigma$. Let $R = \{(p, q) \in L \times L : i(p) = i(q), p \neq q\}$ be the self intersections. Let g be a Morse function on L.

Definition 1. *The cochain complex is*

$$CF(L, L) = CM(g) \oplus \bar{R}$$

where $CM(g) = \bigoplus \mathbb{Z}/2 \cdot q$ generated by q, the critical points of g just like in Morse chain complex, $\bar{R} = \bigoplus(\mathbb{Z}/2 \cdot \gamma_- \oplus \mathbb{Z}/2 \cdot \gamma_+)$ where γ_-, γ_+ are two generators corresponding to γ at each double point.

The immersed Lagrangian may possibly bound holomorphic disks. We first force a condition on the behavior of such disks.

Given a point $x \in CF(L, L)$, consider maps $u : (D, \partial D) \to (M, L)$ satisfying

1. $u : (D, \partial D) \to (M, L)$ non-constant holomorphic disk. D the closed unit disk in $\mathbb{C}$ with one marked point on the boundary $-1 \in \partial D$
2. If $x \in CM(g)$, there exists a $-\nabla g$ flow line from x to $u(-1)$
3. If $x \in \bar{R}$, then $u(-1) \to x$

The moduli space of all such u's representing a given class A will be denoted by $M(x, A)$. The moduli space $M(x, A)$ need not be regular, but when it is, it is a smooth manifold of dimension $Ind(A) - 2$ when $x \in \bar{R}$, where Ind denotes the Fredholm index of the holomorphic disk, and dimension $\mu(A) + |x| - 2$ when $x \in CM(g)$, where μ is the Maslov index and $|x|$ is the Morse index of x.

Condition 2. *For any $x \in \bar{R}$ (resp. $x \in CM(g)$ with $|x| = 0$) and $A \in H_2$ with positive symplectic area we have either*

1. *Ind* ≥ 3 *(resp.* $\mu(A) \geq 3$*), or*

2. *Ind* $= 2$ *(resp.* $\mu(A) = 2$*) but* $M(x, A)$ *is regular for all such* A*. We require that the count of index 2 disks with marked points mapping to* x *sum to* 0 *(counting mod 2). Namely*

$$\sum_A |M(x, A)| = 0$$

Given $x, y \in CF(L, L)$, $A \in H_2(M, L)$, consider $(u_1, ..., u_l)$:

1. $u_i : (D, \partial D) \to (M, L)$ non-constant holomorphic disk. D the closed unit disk in $\mathbb{C}$
2. If $x, y \in CM(g)$, there exists $-\nabla g$ flow lines from x to $u_1(-1)$, from $u_i(1)$ to $u_{i+1}(-1)$,and from $u_l(1)$ to y
3. If $x \in \bar{R}$, then $u_1(-1) \to x$. If $y \in \bar{R}$, then $u_l(1) \to y$
4. $[u_1] + ... + [u_l] = A$

The moduli space of all such sequences will be denoted by $M(x, y, A)$.

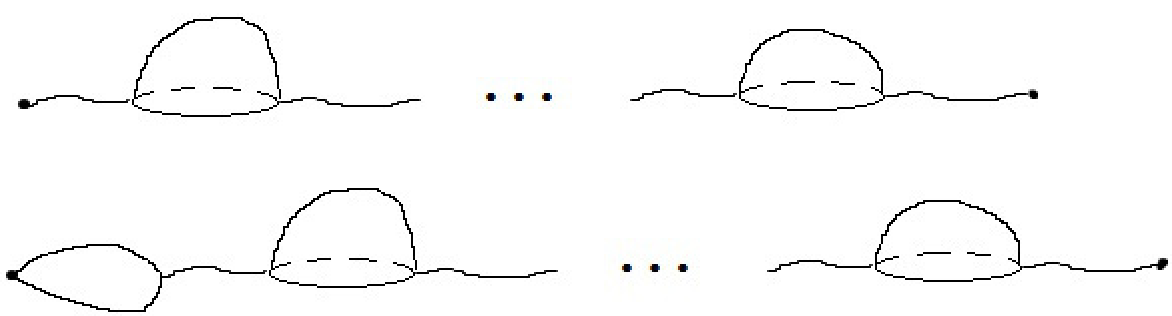

Figure 2.1: The pearly configurations in $M(x, y, A)$

In the special case when L is an embedded monotone Lagrangian with minimal Maslov number $\mu \geq 2$, Biran and Cornea showed in [2] the following:

Statement 3. *Let* $g : L \to \mathbb{R}$ *be a Morse function and* ρ *a Riemannian metric on* L *such that* (g, ρ) *is Morse-Smale. Then there exists an almost complex structure* J_{reg} *such that for every* $x, y \in Crit(g)$ *with* $\mu(A) + |x| - |y| - 1 \leq 1$*:*

1. *All the elements* $(u_1, ..., u_l)$ *are simple and absolutely distinct. The moduli space* $M(x, y, A)$ *is either empty or a smooth manifold of dimension* $\mu(A) + |x| - |y| - 1$*. In particular, if* $\mu(A) + |x| - |y| - 1 < 0$*, then the moduli space is empty.*

2. *If* $\mu(A) + |x| - |y| - 1 = 0$*, then* $M(x, y, A)$ *is a compact* 0*-dimensional manifold and hence consists of finite number of points.*

However, the L in our setting is an immersed Lagrangian submanifold so at best the theorem of Biran-Cornea considers the cases $x, y \in CM(g)$ but doesn't apply to the cases when x or $y \in \bar{R}$. As it turns out the moduli space when we include the configurations where x or $y \in \bar{R}$ need not be regular in general. When all the elements $(u_1, ..., u_l)$ are simple and absolutely distinct there will not be a problem. However, $M(x, y, A)$ can contain configurations where u_1 and u_l are the same exact holomorphic disk passing through x and $y \in \bar{R}$ the two generators associated to the same double point. In practice, we will address this issue by choosing our Morse function g so no such configurations can exist. If regularity holds then $M(x, y, A)$ is a smooth manifold, and its dimension will be $\mu(A) + |x| - |y| - 1$ for $x, y \in CM(g)$, $\mu(A) + Ind(u_1) - |y| - 1$ for $x \in \bar{R}$ and $y \in CM(g)$, $\mu(A) + |x| + Ind(u_l) - n - 1$ for $x \in CM(g)$ and $y \in \bar{R}$, and $\mu(A) + Ind(u_1) + Ind(u_l) - n - 1$ for $x, y \in \bar{R}$ and $l > 1$ (where $\mu(A)$ is the total Maslov index of the disk components without striplike ends).

We will state the definition of the differential δ assuming transversality (which we will address in our example later). The differential is given by

$$\delta : CF(L, L) \to CF(L, L) : a \mapsto \sum |M(b, a, A)| \cdot b$$

summing through all $M(b, a, A)$ that have dimension 0 and counting with mod 2 coefficients.

Statement 4. $\delta^2 = 0$*, assuming* L *satisfies condition 2 and transversality is achieved.*

A proof of the above statement for our specific example will be given explicitly in statement 12. In our case, L is exact, so all disks with boundary on L must involve double points

The Floer cohomology is

$$HF(L, L) = H(CF(L, L), \delta)$$

We will also need to define some higher products μ^2 for our computations later. Assume L_1 is another embedded monotone Lagrangian submanifold. In order to define $\mu^2 : CF(L, L_1) \otimes CF(L_1, L) \to CF(L, L)$ we consider the moduli space $M(L, L_1, L, a, b, c)$ where $a \in CF(L_1, L), b \in CF(L, L_1)$, and $c \in CF(L, L)$ consisting of

1. Holomorphic triangles with the vertices mapping to a, b, c and the edges mapping to L_1, L, L.

2. A holomorphic bigon with the boundaries mapping to L_1, L and the vertices mapping to a, b. On the edge of the holomorphic bigon that maps to L is a point y' where a pearly trajectory in $M(c, y', A)$ is attached.

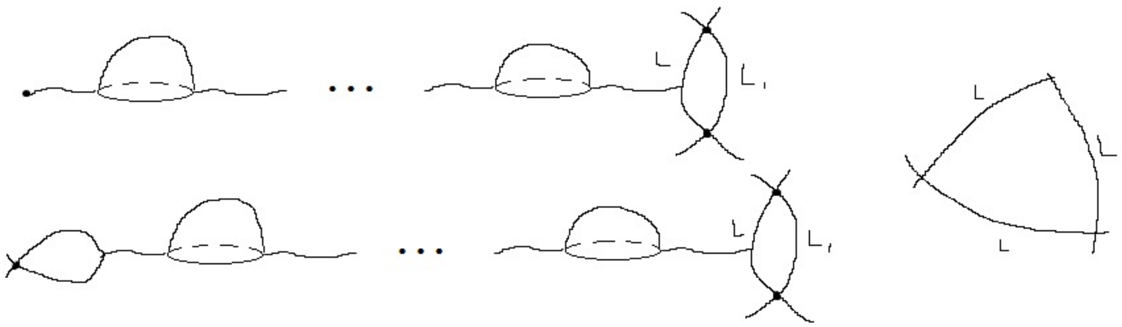

Figure 2.2: The pearly configurations in $M(L, L_1, L, a, b, c)$

The map $\mu^2 : CF(L, L_1) \otimes CF(L_1, L) \to CF(L, L)$ is defined by

$$\mu^2(b, a) = \sum |M(L, L_1, L, a, b, c)| \cdot c$$

summing through all the $M(L, L_1, L, a, b, c)$ of dimension 0.

The map $\mu^3 : CF(L, L_1) \otimes CF(L_1, L_2) \otimes CF(L_2, L) \to CF(L, L)$ is defined in the same manner.

Chapter 3

Surgery Construction

There are two types of Lagrangian surgery which we will perform in this section. The first one we refer to as Morse gluing or just regular Lagrangian surgery. The second one we refer to as Morse-Bott gluing. We will give an overview of the two constructions in this section.

Consider (M, ω) where M is a manifold and ω is a symplectic form on M. L_1, L_2 are two Lagrangians in M.

Suppose first that L_1 and L_2 intersect transversely at some point p. We will describe a local model for Lagrangian surgery at p, as first introduced by Polterovich [4].

There is always a Darboux chart in a neighborhood U of p and $i : U \to V \subset \mathbb{C}^n$ where

$$i(L_1 \cap U) = \mathbb{R}^n \cap V, i(L_2 \cap U) = \sqrt{-1}\mathbb{R}^n \cap V$$

Let ϵ be a sufficiently small real number, and let $f_\epsilon : \mathbb{R}^n - \{0\} \to \mathbb{R}$ be

$$f_\epsilon = \epsilon \log |x|$$

The graph of $df_\epsilon(x)$, which we will call H_ϵ in coordinates $z_j = x_j + \sqrt{-1}y_j$ can be described as

$$H_\epsilon = \left\{ (z_1, ..., z_n) : y_j = \frac{\epsilon x_j}{|x|^2}, j = 1, ..., n \right\}$$

H_ϵ is a Lagrangian submanifold of $\mathbb{C}^n$ which is asymptotic to $\mathbb{R}^n$ as $|x| \to \infty$, and approaches $\sqrt{-1}\mathbb{R}^n$ as $|x| \to 0$.

We will modify the above description a little bit to make sure that the Lagrangian we construct doesn't just approach L_1, L_2 asymptotically. Let $\tau : \mathbb{C}^n \to \mathbb{C}^n$ be the map that reflects along the diagonal $\Delta = \{(z_1, ..., z_n) : x_i = y_i\}$, sending $x_i + \sqrt{-1}y_i \mapsto y_i + \sqrt{-1}x_i$. Note that $\tau(H_\epsilon) = H_\epsilon$. τ is an anti-symplectomorphism: $\tau^*\omega_0 = -\omega_0$, which maps Lagrangians to Lagrangians. Instead of log, we consider a slightly different map $\rho : \mathbb{R}^+ \to \mathbb{R}$ by

$$\rho(r) = \begin{cases} \log r - |\epsilon| & \text{if } r \leq \sqrt{|\epsilon|}S_0 \\ \log \sqrt{|\epsilon|}S_0 & \text{if } r \geq 2\sqrt{|\epsilon|}S_0 \end{cases}$$

$$\rho'(r) \geq 0, \rho''(r) \leq 0$$

here S_0 is a fixed sufficiently large number and ϵ satisfies $\sqrt{|\epsilon|}S_0$ is sufficiently small. The modified function $\bar{f}_\epsilon : \mathbb{R}^n - \{0\} \to \mathbb{R}$ is defined by

$$\bar{f}_\epsilon = \epsilon\rho(|x|)$$

Construct a new Lagrangian manifold $\bar{H}_\epsilon$ of $\mathbb{C}^n$ that satisfies $\tau(\bar{H}_\epsilon) = \bar{H}_\epsilon$ and

$$\{(z_1, ..., z_n : x_i \geq y_i \ \forall i)\} \cap \bar{H}_\epsilon = \{(z_1, ..., z_n : x_i \geq y_i \ \forall i)\} \cap \text{graph } d\bar{f}_\epsilon$$

Since the graph of $d\bar{f}_\epsilon$ coincides with that of df_ϵ for $|x| \in (\sqrt{|\epsilon|}, \sqrt{|\epsilon|}S_0)$, $\bar{H}_\epsilon$ defined in this manner is smooth and coincides with H_ϵ inside the ball of radius $\sqrt{|\epsilon|}S_0$ around 0. Outside of the ball $B^{2n}(2\sqrt{|\epsilon|}S_0)$ around 0, $\bar{H}_\epsilon = \mathbb{R}^n \cup \sqrt{-1}\mathbb{R}^n$.

Thus for the given L_1 and L_2 intersecting transversely at some point p and the Darboux chart U in a neighborhood of p, we construct Lagrangian submanifold $L_\epsilon \subset M$ by replacing U with the local model $\bar{H}_\epsilon$ we constructed, i.e.

$$L_\epsilon - U = L_1 \cup L_2 - U, \ i(L_\epsilon \cap U) = \bar{H}_\epsilon \cap V$$

Definition 5. *Given L_1, L_2, the Lagrangian surgery of L_1 and L_2 at $p \in L_1 \cap L_2$ is the Lagrangian manifold L_ϵ, denoted as $L_\epsilon = L_1 \#_\epsilon L_2$.*

Now suppose the Lagrangian submanifolds L_1, L_2, instead of intersecting transversely at a point, now intersect cleanly along a 1 dimensional isotropic submanifold $K \subset L_1 \cap L_2$. We will also require L_1, L_2 to be orientable submanifolds.

Because $K = L_1 \cap L_2$ is a one dimensional submanifold, and since L_1 is orientable, the normal bundle K of inside L_1 is trivial. Thus there is a neighborhood of K in L_1 that is diffeomorphic to an open subset of $K \times \mathbb{R}^{n-1}$. Applying the Weinstein neighborhood theorem on L_1, we see that a neighborhood of L_1 inside M is symplectomorphic to T^*L_1. Thus locally we can view a neighborhood of K as $K \times \mathbb{R} \times \mathbb{C}^{n-1}$ with the standard symplectic form, where L_1 corresponds to $K \times 0 \times \mathbb{R}^{n-1}$. Denote the coordinates of $K \times \mathbb{R}$ by (s, t), and the coordinates of $\mathbb{C}^{n-1}$ by (x_i, y_i) for $i = 1, ..., n-1$. K is the s-axis, L_1 is the $(s, x_1, ..., x_{n-1})-$plane, and the symplectic form is $ds \wedge dt + \sum dx_i \wedge dy_i$

Since L_2 intersects L_1 cleanly along K, locally L_2 is a graph in the sense that $(t, x_1, ..., x_{n-1})$ are functions of $(s, y_1, ..., y_{n-1})$ where the functions vanish on K. Since L_2 is a Lagrangian submanifold and a graph, it is a graph of a closed 1-form over the s, y axes. Since $K \subset L_2$, the integral of this 1-form over $K \times 0$ is 0, the 1-form is exact. Thus there is some function $h(s, y)$ such that L_2 is a graph of dh, given by $x_i = -\frac{dh}{dy_i}$ and $t = \frac{dh}{ds}$. Moreover, h is constant along $K \times 0$ and its derivative vanishes on $K \times 0$. So substracting a constant we can assume that $h = O(|y|^2)$.

Consider the Hamiltonian $H = h(y, s)$. Its time 1 flow maps

$$(s, t, x, y) \mapsto \left(s, t - \frac{dh}{ds}, x_i + \frac{dh}{dy_i}, y_i\right)$$

which is identity on L_1 and maps the graph of dh to the (s, y) coordinate plane. Using this modification, we have arranged a neighborhood of K to be $K \times \mathbb{R} \times \mathbb{C}^{n-1}$ where L_1 is $K \times 0 \times \mathbb{R}^{n-1}$ (the s, x axes) and L_2 is $K \times 0 \times \sqrt{-1}\mathbb{R}^{n-1}$ (the s, y axes).

Under this arrangement of K, L_1, L_2, we perform regular Lagrangian surgery on the x, y coordinates. Given a small fixed ϵ, for every value of the s-coordinate we perform Lagrangian surgery on $s \times 0 \times \mathbb{R}^{n-1}$ and $s \times 0 \times \sqrt{-1}\mathbb{R}^{n-1}$. Since ϵ is a fixed small constant, the resulting submanifold is Lagrangian. This Lagrangian submanifold is called the Morse-Bott surgery or the Morse-Bott gluing in the following sections.

Remark 6. *Even though in our definition we only construct the Morse-Bott surgery for two Lagrangian submanifolds L_1, L_2 intersecting cleanly along a one dimensional submanifold K, we only used the fact that K is one dimensional for the argument that the normal bundle of K inside L_1 is topologically trivial. Thus the construction can be applied whenever the normal bundle to K inside L_1 is trivial, and dropping the assumption that $dim(K) = 1$. In the case where the normal bundle of K in L_1 is not trivial, one can still define Morse-Bott surgery, but it will not be described here as it is not needed in the following sections.*

Chapter 4

$M_1 \times M_2$

Let M_1 be a 2-dimensional cylinder and M_2 a punctured 2-torus, both with the standard symplectic form. Here we construct and compute examples of surgeries in $M = M_1 \times M_2$ with product symplectic form. Denote the S^1 in M_1 by N, and the longitude and the meridian of M_2 by C_1, C_2. Let $*$ be the intersection of C_1 and C_2. Let T_1 be the 2-torus $N \times C_1$, and T_2 the 2-torus $N \times C_2$. These are two Lagrangian tori intersecting along $N \times \{*\}$.

The Morse gluing of T_1 and T_2 is constructed as follows: Think of M_1 like T^*S^1 where N is the zero section. First choose a Morse function f on N with a max y and a min x. Let N_1 and N_2 be graph(df) and graph$(-df)$ respectively. $\bar{T_1} = N_1 \times C_1$ and $\bar{T_2} = N_2 \times C_2$ now intersect at the two points $(y, *), (x, *)$. Doing a regular Lagrangian surgery on $(y, *)$ gives the Morse gluing.

The Morse-Bott gluing, on the other hand, does a regular Lagrangian surgery on the intersection of C_1 and C_2 in M_2, and products $C' = C_1 \# C_2$ with N.

We denote $L_1(\epsilon)$ to be the Morse gluing (ϵ is the gluing parameter) and L_2 the Morse-Bott gluing. L_2 is an embedded Lagrangian torus which is N in the first factor and $C_1 \# C_2$ in the second factor, a product of Lagrangian S^1's. $L_1(\epsilon)$ is an immersed Lagrangian which topologically is a genus two surface with one self intersection.

Statement 7. *L_2 bounds no disc in $M_1 \times M_2$.*

Proof. L_2 is a product Lagrangian. Each factor bounds no disk. □

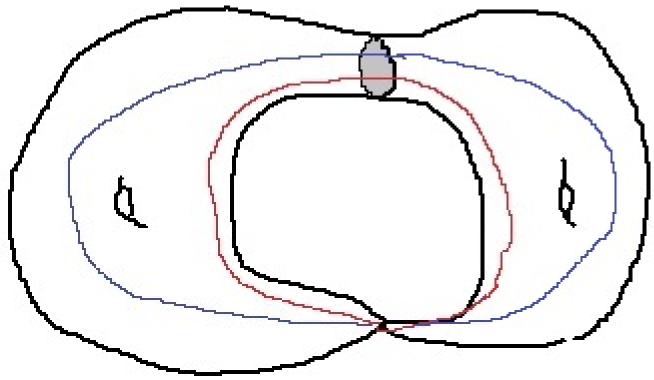

Figure 4.1: $L_1(\epsilon)$, a genus two surface with one transverse self intersection. The grey circle depicts the neck of the Morse surgery, while the blue and red lines shows the boundary of the two "teardrop".

Upon Lagrangian surgery, the two strips bounded by N_1 and N_2 in M_1 give rise to two teardrop-shaped regions with boundary on $L_1(\epsilon)$. The boundary loops of the two teardrops, shown in red and blue on Figure 4.1, run once through the double point and once through the neck of the surgery.

Statement 8. *Consider $L_1(\epsilon) \subset M$, the immersed genus two surface with one self intersection (as opposed to the abstract genus 2 surface S that is the domain of this immersion). $H_2(M, L_1(\epsilon))$ is generated by the two teardrops, denoted by A_+, A_-, and a third generator a, the class of the small disc bounded by the neck of the surgery at $(y, *)$.*

Proof. Consider the following section in the long exact sequence

$$... \to H_2(L_1(\epsilon)) \to H_2(M) \to H_2(M, L_1(\epsilon)) \to H_1(L_1(\epsilon)) \to H_1(M) \to ...$$

In the first map $H_2(L_1(\epsilon)) \to H_2(M)$, we have $H_2(L_1(\epsilon)) = \mathbb{Z}$, $H_2(M) = H_2(M_1 \times M_2) = H_1(M_1) \otimes H_1(M_2) = \mathbb{Z}^2$. The map $H_2(L_1(\epsilon)) \to H_2(M)$ is injective with cokernel $\mathbb{Z}$. $H_1(L_1(\epsilon)) = \mathbb{Z}^5$, generated by the 4 generators of $H_1(S)$, where S is the abstract genus two surface, and a loop γ_1 that starts and end on the immersed double point (say the red loop in Figure 4.1). We specify one of the four generators of $H_1(S)$ to be the sum of loop red and blue, denoted by γ_2. $H_1(M) = H_1(M_1 \times M_2) = (H_1(M_1) \otimes H_0(M_2)) \oplus (H_0(M_1) \otimes H_1(M_2)) = \mathbb{Z}^3$. The last map $H_1(L_1(\epsilon)) \to H_1(M)$ is onto, and its kernel is generated by γ_1 and γ_2. We conclude that $H_2(M, L_1(\epsilon)) = \mathbb{Z}^3$. Two of the generators of $H_2(M, L_1(\epsilon))$, denoted by A_+, A_-, are disks with boundaries on γ_1 and $\gamma_2 - \gamma_1$ (the red loop and the blue loop respectively). The last generator of $H_2(M, L_1(\epsilon))$ is a disk bounded by the neck of the surgery at $(y, *)$, depicted as the grey disk in Figure 4.1. □

We will show that only the classes A_+, A_- (and their multiples) can be represented by holomorphic disks.

Statement 9. *$L_1(\epsilon)$ bounds two holomorphic teardrops, and no other somewhere injective holomorphic disk.*

Proof. Let $L_1 = \bar{T}_1 \cup \bar{T}_2$ be $L_1(\epsilon)$ before gluing. L_1 is the union of $N_1 \times C_1$ and $N_2 \times C_2$. In the M_1 factor, the Riemann mapping theorem tells us that $N_1 \cup N_2$ bounds two homotopy classes of simple holomorphic maps, each with a unique holomorphic disk (mod reparametrization) and multiplicity 1. In the M_2 factor, $C_1 \cup C_2$ bounds no nonconstant holomorphic disk. If $u : (D, \partial D) \to (M, L_1)$ is a holomorphic disk that L_1 bounds, then $\pi_1 \circ u$ and $\pi_2 \circ u$ will be holomorphic maps in the first and second factor, and so any holomorphic disk that L_1 bounds must be constant in the second factor, and one of the two holomorphic disks in the first factor.

Now we move from disks in L_1 to disks with boundary in $L_1(\epsilon)$. Away from an arbitrarily small ϵ neighborhood of the self intersection y where we perform the gluing, the Lagrangian stays the same. Consider the local model for the Lagrangian gluing. In coordinates $z_i = x_i + \sqrt{-1}y_i$, the local gluing model can be written as

$$\left\{ (z_1, z_2) : y_i = \frac{\epsilon x_i}{x_1^2 + x_2^2}, i = 1, 2 \right\}$$

When projected to either of the two factors, it looks like $\left(x_i, \frac{\epsilon x_i}{x_1^2 + x_2^2} \right)$, covering all areas between $\left(x_i, \frac{\epsilon}{x_i} \right)$ and the axes.

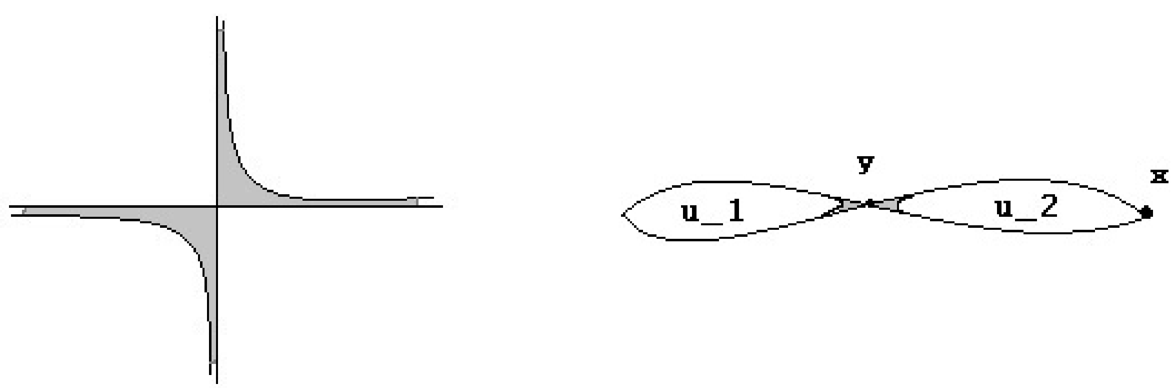

Figure 4.2: On the left: the local model for Morse surgery. On the right: What it looks like in the M_1 factor. The holomorphic "teardrops" are the regions marked by u_1, u_2. x is the self intersection, and y is where Morse surgery is performed.

In the M_1 factor of $L_1(\epsilon)$, there are two holomorphic $w_1, w_2 : (D, \partial D) \to (M_1, L_1(\epsilon))$. They pass through the "outermost" part of the projected gluing neck (the $\left(x_1, \frac{\epsilon}{x_1}\right)$ part in the local model). Let $u_i : (D, \partial D) \to (M, L_1(\epsilon))$ be the holomorphic disks that coincide with w_i in M_1 and constant in M_2. They are of homotopy class A_+, A_- respectively. Let $u : (D, \partial D) \to (M, L_1(\epsilon))$ be a holomorphic disk with boundary on $L_1(\epsilon)$ with homotopy class A_+. The image of w_1 is a subset of the image of $\pi_1 \circ u$. Since holomorphic disks are area-minimizing in their homotopy class, u_1 and u have to be the same holomorphic disk. The same argument can be applied to show that a holomorphic $(D, \partial D) \to (M, L_1(\epsilon))$ with homotopy class A_- will have to be u_2.

There are no other simple holomorphic disks with boundary on $L_1(\epsilon)$ with other homotopy classes. To show this, let u be a holomorphic disk in M with boundary on $L_1(\epsilon)$. Its image in the M_1 factor, $\pi_1 \circ u$, has boundary on $\pi_1(L_1(\epsilon))$. Open mapping principle tells us that $\pi_1 \circ u$ either covers w_1 or w_2 a certain number of times but not both, or stays in a small region. Thus the homology class of u in $H_2(M, L_1(\epsilon))$ must be either $k \cdot A_+ + l \cdot a$ or $k \cdot A_- + l \cdot a$ for $k \geq 0$ and l integers. Suppose the homology class is $k \cdot A_+ + l \cdot a$ (the case for $k \cdot A_- + l \cdot a$ works the same way). The symplectic area of the disk u is equal to k times the area of u_1. Since the image of $\pi_1 \circ u$ covers k times the entire image of w_1, we see that the disc is a k-fold cover of u_1 in the first factor and constant in the second factor. Thus u represents the class $k \cdot A_+$, and $l = 0$. $\square$

Statement 10. *The two teardrops are regular with index 2:*

Proof. In the previous statement we showed that the holomorphic disks with boundaries on $L_1(\epsilon)$ are "teardrops" in the first M_1 factor and constant in the M_2 factor. The tangent planes to the Lagrangian $L_1(\epsilon)$ along the boundary of the teardrop are products of lines in M_1 and M_2. Away from the point where gluing happens, $L_1(\epsilon)$ coincides with $L_1 = \bar{T}_1 \cup \bar{T}_2$, which is a union of two product Lagrangians $\bar{T}_1 = N_1 \times C_1$ and $\bar{T}_2 = N_2 \times C_2$, and thus locally the tangent planes split as a product of lines. On the part where we glue, we will take a look at the local model again. On the local model, when passing though the gluing, the boundary of the teardrop lives in

$$\left\{(z_1, z_2) : y_i = \frac{\epsilon x_i}{x_1^2 + x_2^2}, i = 1, 2, z_2 = 0\right\}$$

On $z_2 = 0$ the tangent space is spanned by $(1, \frac{-\epsilon}{x_1^2}, 0, 0), (0, 0, 1, \frac{\epsilon}{x_1^2})$. Thus the tangent planes to the $L_1(\epsilon)$ along the teardrops are indeed product of lines in each M_1, M_2 factor.

The linearized $\bar{\partial}$ operator splits into the direct sum of two $\bar{\partial}$ operators on $\mathbb{C}$-valued functions with boundary conditions given by a family of real lines: the real lines of the tangent direction to L.

The Fredholm index of u_1 will be computed as follows: We will consider u_1 as a holomorphic disk with one mark output point on the self intersection x. On the M_1 factor, after closing the family of boundary conditions to a closed loop in the Lagrangian Grassmannian by adding a short counterclockwise path at the double point, the Lagrangian tangent line rotates a full 2π. On the M_2 factor, again closing the family of boundary conditions to a closed loop in the Lagrangian Grassmannian in the same way, the Lagrangian tangent line ends up not rotating. The Fredholm index of u_2 can be computed the same way.

As the computation shows, the Fredholm index of a teardrop for each factor is non-negative. In the M_1 factor, the nonconstant teardrop has Fredholm index 2, while in the M_2 factor, the constant map has index 0. Since the linearized $\bar{\partial}$ operator splits into the direct sum of two $\bar{\partial}$ operators on $\mathbb{C}$-valued functions with boundary conditions given by a family of real lines, we can apply Lemma 11.5 in Seidel's book [6] for $\bar{\partial}$ operators on line bundles. The lemma states that in dimension 1, if the index is less than 0, then $\bar{\partial}$ is injective. Considering the adjoint operator which is also a $\bar{\partial}$ operator (with a different boundary condition), the lemma then states that if the index is ≥ 0, then the operator is surjective. □

Statement 11. *$L_1(\epsilon)$ satisfies the teardrop cancelling condition (Condition 2):*

Proof. Given that there is only one self intersection x, and the previous computation shows that there are only two index 2 holomorphic teardrops, one in $M(x_+, A_+)$ and the other in $M(x_+, A_-)$. Here x_+ is one of the two generators of $CF(L_1(\epsilon), L_1(\epsilon))$ that will be assigned to the self intersection x. Thus counting in $\mathbb{Z}/2$, $\sum_A |M(x, A)| = 0$ □

Define the chain complex. Choose a Morse function g on the genus 2 surface S which is the domain of the Lagrangian immersion $i : S \to L_1(\epsilon)$. $CF(L_1(\epsilon), L_1(\epsilon)) = CM(g) \oplus \bar{R}$ where $CM(g)$ has the usual 6 generators, and $\bar{R}$ has 2 generators: x_+ and x_-. The degree of the Morse critical points will be the Morse index, while the degrees of x_+ and x_- are 2 and 0 respectively. Here we choose our Morse function g so that the boundary of the two teardrops all live in a level set of the Morse function g (note that the boundaries of the two teardrops taken together form a homologically nontrivial embedded simple closed curve on S). We want to ensure that there are no Morse trajectories connecting the boundary of a holomorphic teardrop to itself so the moduli space considered in the proof that $\delta \circ \delta = 0$ is regular. Consider the 0-dimensional moduli space $M(p, q, A)$, where $deg(p) - deg(q) = 1$, consisting of configurations of the form:

1. A teardrop passing through self intersection p with homotopy class A, followed by a gradient flow line of $-\nabla g$ that goes from the boundary of the teardrop to a Morse critical point q.

2. A gradient flow line of $-\nabla g$ that goes from the Morse critical point p to the boundary of a teardrop that passes through a self intersection q.

3. A gradient flow line of $-\nabla g$ that goes from p to q.

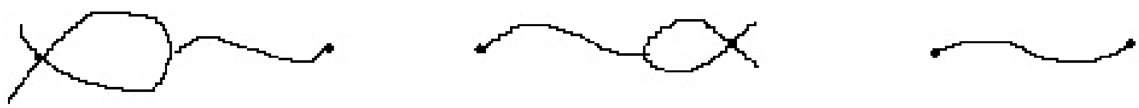

Figure 4.3: The configurations in a 0-dimensional $M(p, q, A)$

Transversality:

We showed that the two teardrops themselves are both regular. Our choice of g, which makes the boundary of teardrops live in a level set, ensures that the gradient flow lines will intersect the boundaries of teardrops transversely.

The differential is given by

$$\delta : CF(L_1(\epsilon), L_1(\epsilon)) \to CF(L_1(\epsilon), L_1(\epsilon)) : q \mapsto \sum_{dim(M(p,q,A))=0} |M(p, q, A)| \cdot p$$

Statement 12. $\delta \circ \delta = 0$

Proof. Consider a 1-dimensional moduli space $M(p, q, A)$. The boundary of $M(p, q, A)$ consists of the following configurations:

1. A teardrop passing through self intersection p with homotopy class A, followed by a broken gradient flow line of $-\nabla g$ that goes from the boundary of the teardrop to another Morse critical point q', and finally to the Morse critical point q.

2. Same as above, except starting with a broken gradient flow line from p to p', and then to the boundary of the teardrop passing through self intersection q with homotopy class A.

3. Broken Morse trajectory.

4. A teardrop passing through self intersection p with homotopy class A, with a Morse trajectory from one of the two preimages in S of the double point p to a Morse critical point q.

Of these four types of configurations, only the first three contribute to $\delta \circ \delta$. Moreover, $\delta \circ \delta$ also counts two other types of broken configurations:

(5) A Morse trajectory from p to the boundary of a teardrop passing through double point x, followed by another teardrop passing through x, followed by a Morse trajectory that goes from the boundary of the teardrop to a Morse critical point q.

(6) A teardrop passing through self intersection p, followed by a Morse trajectory that goes from the boundary of the teardrop to a Morse critical point r, followed by another Morse trajectory from r to the boundary of a teardrop that passes through self intersection q.

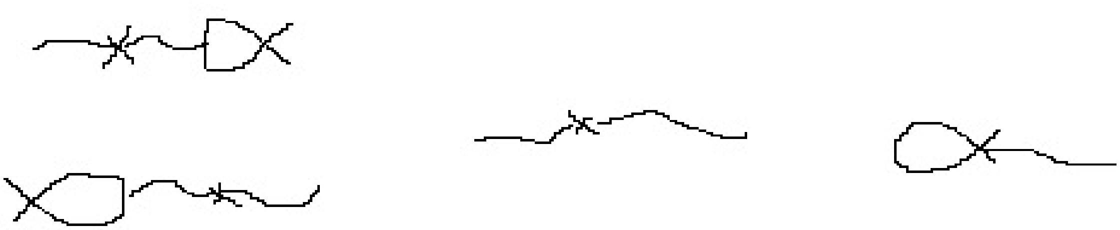

Figure 4.4: The boundary of a 1-dimensional $M(p, q, A)$

The configurations in (4) contribute 0 to the count because the condition that forces teardrops count to sum to 0 will also force this count to 0. This is because for each preimage of the double point, the number of such configurations in (4) has the same count as teardrops that pass through the double point p, and hence Condition 2 forces that count to be zero.

Configurations (5), (6) do not occur in our case. Configuration (5) does not occur because the output point of a teardrop is x_+, but the input point of a teardrop is x_-. On the other hand, configuration (6) also does not occur because we chose our Morse function g so that the boundary of the teardrops live in a level set of g, so there cannot be any configurations in (6) as no such critical point r can exist.

Since configurations (5) and (6) do not occur, the remaining configurations (those in (1), (2), (3)) count the coefficient of p that comes from $\delta \circ \delta(q)$. Because the signed count of the boundary of a compact smooth 1-dimensional manifold with boundary is 0 and the configurations in (4) cancel out, we see that $\delta \circ \delta = 0$. $\square$

Let $HF(L_1(\epsilon), L_1(\epsilon)) = H(CF(L_1(\epsilon), L_1(\epsilon)), \delta)$ be the cohomology of this chain complex.

Statement 13. *$HF(L_1(\epsilon), L_1(\epsilon))$ has the cohomology of a torus.*

Proof. Denote the index 2 and index 0 Morse critical points as p, q respectively, the index 1 critical points as $\gamma_1, ..., \gamma_4$, and the two generators from the self intersection as x_+, x_-. The boundary of the two teardrops together form a simple closed curve η on the genus two surface S (recall S is the domain of the immersion. $i(S) = L_1(\epsilon)$). Moreover, that simple closed curve lives in a level of our Morse function g. For the purpose of this proof, let's arrange the Morse function g as depicted in Figure 4.5. The boundary of the two teardrops intersects the ascending manifolds of $\gamma_1, \gamma_2, \gamma_3$ each at one point, while it intersects the descending manifolds of another γ, say γ_4, at another point. So $\delta(\gamma_1) = x_+$ (and also $\delta(\gamma_2) = x_+, \delta(\gamma_3) = x_+$), $\delta(x_-) = \gamma_4$, while all other δ are 0.

The differential $\delta : CF^0(L_1(\epsilon), L_1(\epsilon)) \to CF^1(L_1(\epsilon), L_1(\epsilon))$ thus has rank 1, and similarly $\delta : CF^1(L_1(\epsilon), L_1(\epsilon)) \to CF^2(L_1(\epsilon), L_1(\epsilon))$ also has rank 1. CF^0 has rank 2, CF^1 has rank 4, and CF^2 has rank 2. Thus in cohomology, $HF(L_1(\epsilon), L_1(\epsilon))$ now has HF^0 rank 1, HF^1 rank 2, and HF^2 rank 1. So cohomologically, $HF(L_1(\epsilon), L_1(\epsilon))$ looks like that of a 2-torus.

There is another way to compute the differential δ without arranging the Morse function g in a specific way like we did above. The simple closed curve η, formed by the boundaries of the two teardrops, is not 0 in homology $H_1(S)$. Since the ascending manifolds of the γ_i's form a basis for $H_1(S)$, at least one will have nonzero intersection number with η, say γ_1. Then $\delta(\gamma_1) = x_+$. On the other hand, the descending manifolds of the γ_i's also form a basis for $H_1(S)$, thus at least one of them will have nonzero intersection number with η. Therefore, $\delta(x_-)$ will be a nonzero linear combination of the γ_i's. An analogous argument to the previous paragraph shows that HF is that of a 2-torus. ☐

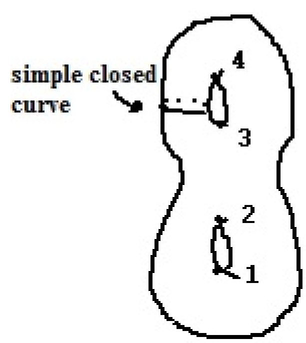

Figure 4.5: The genus two surface. The simple closed curve is the union of the boundaries of the two teardrops. $\gamma_1, ..., \gamma_4$ are the critical points of the Morse function.

The following computations build towards establishing the relation between $L_1(\epsilon)$ and L_2, the Morse and Morse-Bott gluing. To accomplish that, we will first work with L_1, the union of $N_1 \times C_1$ and $N_2 \times C_2$, which is $L_1(\epsilon)$ before gluing.

Let $p \in (N_2 \times C_2) \cap L_2$, $q \in L_2 \cap (N_1 \times C_1)$, $y \in (N_1 \times C_1) \cap (N_2 \times C_2)$ be the self intersection in L_1 that we eventually perform the Lagrangian surgery (Morse surgery), and $x \in (N_1 \times C_1) \cap (N_2 \times C_2)$ the immersed double point. q, p, x, and y are depicted in Figure 4.6 below. We start by counting holomorphic triangles with boundaries on $(N_2 \times C_2), L_2, (N_1 \times C_1)$ with the vertices going to (q, p, y) or (q, p, x).

In the M_2 factor, there are two simple triangles (depicted on the right of Figure 4.6). On the other hand, in the M_1 factor, a few cases can happen:

1. Concave triangles with vertices going to (q, p, y), each in a one-parameter family. There are three possible such triangles, depicted as the shaded grey area on the left of Figure 4.6. The one-parameter family comes from forming slits along the "red" and "blue" direction, with the parameter governing how big the slit is.

2. Convex simple triangles with vertices going to (q,p,x). There is one isolated such triangle in the M_1 factor.

Denote $\Delta(q, p, y)$ and $\Delta(q, p, x)$ to be the set of all such triangles with vertices going to (q, p, y) and (q, p, x) respectively.

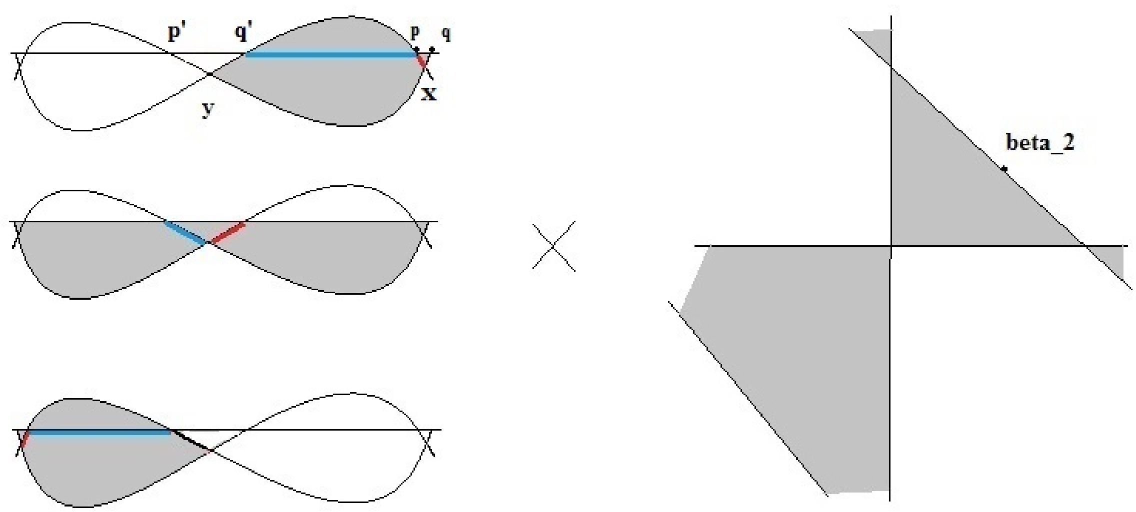

Figure 4.6: The holomorphic triangles in $\Delta(q, p, y)$

We will use the above counts of triangles to show:

Statement 14. *Given any point β on L_2, it lies on the boundary of some triangle in $\Delta(q, p, y)$. Moreover, for generic β the count of such triangles is equal to 1.*

Proof. Given any point β on L_2, since L_2 is a product Lagrangian, we write $\beta = (\beta_1, \beta_2)$. We want to find a holomorphic triangle T in $\Delta(q,p,y)$ with an extra marked point ξ on the L_2 boundary edge such that the holomorphic triangle T maps ξ to β. Asking that the extra marked point ξ on T maps, in the M_2 factor, to β_2 fixes the position of ξ on the boundary of T as the holomorphic triangles in the M_2 factor are rigid. In the M_1 factor, it will hit a point ξ_1, which we claim that we can make it equal to β_1.

In the M_1 factor, as we have listed above, the triangles $\Delta(q,p,y)$ come in three one-parameter families. As we let the parameter vary, the extra marked point ξ_1 will trace out a 1 dimensional path in N. Consider the top family of concave triangles. On the one end, as the red slit gets closer towards $(N_2 \times C_2)$, the concave holomorphic triangle breaks into a convex holomorphic triangle with vertices on (q,p,x) (a holomorphic triangle as described in (2) previously) and a holomorphic bigon with vertices to (x,y). Denote the position of ξ_1 at this configuration to be $p_0 \in N$. On the other end, as the blue slit gets closer towards $(N_1 \times C_1)$, the concave holomorphic triangle breaks into a convex holomorphic triangle with vertices on (q,q',y) and a holomorphic bigon with vertices to (q',p). Denote the position of ξ_1 at this configuration to be $p_1 \in N$. The point ξ_1 sweeps all the points in N from point p_0 to p_1.

In the middle family of concave triangles, on the one end as the red slit gets closer towards N, the concave holomorphic triangle breaks into a convex holomorphic triangle with vertices on (q,q',y), and a holomorphic bigon with vertices on (q',p). The convex holomorphic triangle with vertices on (q,q',y) coincides exactly with the convex holomorphic triangle described in the previous paragraph, and hence ξ_1 is again at $p_1 \in N$. On the other end of this family, as the blue slit approaches N, the concave triangle splits into a convex triangle with vertices on (p',p,y), and a holomorphic bigon with vertices on (p',q). Denote the position of ξ_1 on N to be p_2. In the second family of concave holomorphic triangles, ξ_1 traces all points in N from p_1 to p_2.

Applying a similar argument as before, following the third family of concave holomorphic disks ξ_1 now goes from p_2 back to p_0, concluding the proof of the statement that any point on L_2 lies on the boundary of some triangle in $\Delta(q,p,y)$, and that the count of such triangles is 1 (mod 2). $\square$

Repeating the proof of the previous statement with an extra fourth marked point ξ on the $N_1 \times C_1$ boundary edge (or $N_2 \times C_2$) shows the following statements:

Statement 15. *Given any point β on $N_1 \times C_1$ (or $N_2 \times C_2$), it lies on the boundary of some triangle in $\Delta(q,p,y)$. For generic β the count of such triangles is equal to 1.*

The μ^2 that we will be using below are between three distinct embedded Lagrangians, which will count the usual rigid (index 0) holomorphic triangles with edges going to the three distinct Lagrangians.

Statement 16. $\mu^2(y,q) = \mu^2(p,y) = \mu^2(q,p) = 0$

Proof. Let's compute $\mu^2(y, q)$. There is a single convex triangle with vertices on (q', y, q) in the M_1 factor. In the M_2 factor there are two simple triangles. Counting mod 2 the corresponding triangles in $M_1 \times M_2$ contribute 0 to the coefficient in front of $q' \in CF(L_2, N_2 \times C_2)$. A similar argument applies to $\mu^2(p, y)$ and $\mu^2(q, p)$ as well. □

The μ^3 considered below will be a specific case of μ^3 described in section 2 where the first and the last Lagrangian are identical. Since $L_2, N_1 \times C_1$, and $N_2 \times C_2$ are embedded, the only contributions are from holomorphic triangles with an extra marked point attached to a Morse flow line (left side of Figure 4.7).

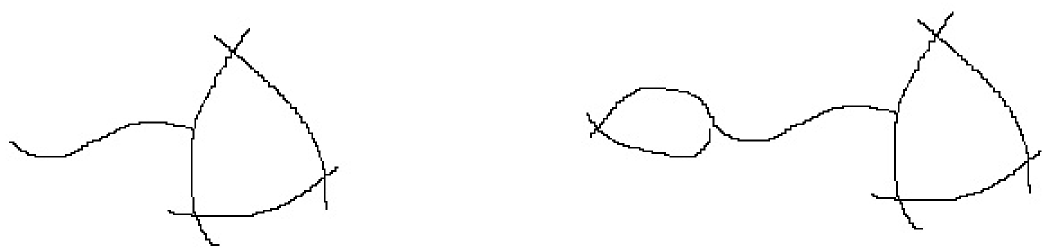

Figure 4.7: Configurations considered in μ^3

Statement 17. $\mu^3(q, p, y) = min(N_2 \times C_2), \mu^3(p, y, q) = min(L_2), \mu^3(y, q, p) = min(N_1 \times C_1)$ *where min is the generator corresponding to the minimum of the Morse function chosen to define the Morse complex.*

Proof. Let's compute $\mu^3(p, y, q) = min(L_2)$. Statement 14 shows that any point β on L_2 lies on the boundary of some triangle in $\Delta(p, y, q)$. In particular, $min(L_2)$ lies on the boundary of some triangle in $\Delta(p, y, q)$ and the count of such triangles is equal to one. This contributes a count of one to the coefficient of $min(L_2)$ when computing $\mu^3(p, y, q)$, and for dimension reasons there are no other contributions to $\mu^3(p, y, q)$. A similar argument applies to $\mu^3(q, p, y)$ and $\mu^3(y, q, p)$ as well. □

Recalling that the minima correspond to the identity endomorphisms, using the characterization of exact triangles ([6] Section 3) the last two statements imply:

Statement 18.

$$N_2 \times C_2 \xrightarrow{q} L_2 \xrightarrow{p} N_1 \times C_1 \xrightarrow{y} N_2 \times C_2$$

forms an exact triangle.

To move from L_1 to $L_1(\epsilon)$, we appeal to a result from Fukaya, Oh, Ohta, Ono ([3] Theorem 55.7):

Theorem 19 (FOOO)**.** *Let K be a compact subset of $M(L_0, L_1, L_2, u_{01}, u_{12}, u_{20})$ and U be a relatively compact open neighborhood of K inside $M(L_0, L_1, L_2, u_{01}, u_{12}, u_{20})$. Let $M(L_\epsilon, L_0, u_{01}, u_{20}, K, \epsilon_2)$ be the set of elements in $M(L_\epsilon, L_0, u_{01}, u_{20})$ represented by a J-holomorphic map w satisfying*

$$max_{z\in D^2} dist(w(z), w_{tri}(x)) \leq \epsilon_2$$

for some $w_{tri} \in K$.

Assume moreover that every $w_{tri} \in U$ has multiplicity one at u_{12} and is Fredholm regular. For each sufficiently small ϵ_2 and ϵ_1, there exists an open neighborhood $M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+$ of $M(L_\epsilon, L_0, u_{01}, u_{20}, K, \epsilon_2)$ inside $M(L_\epsilon, L_0, u_{01}, u_{20})$ and a map

$$\pi : M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+ \to U$$

such that:

1. *Every element of $M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+$ is Fredholm regular.*

2. *If $[w] \in M(L_{\epsilon_1}, L_0, u_{01}, u_{20}, K, \epsilon_2)^+$ and $\pi([w]) = [w_{tri}]$ then we have*

$$dist(w(z), w_{tri}(z)) \leq C\epsilon_2$$

   by re-choosing the representative w in the class $[w]$ if necessary.

3. *If $\epsilon_1 < 0$ then the restriction $\pi^{-1}(K) \to K$ of π is a diffeomorphism.*

When we apply this theorem, L_0 will be L_2, L_1 and L_2 will be $N_1 \times C_1$ and $N_2 \times C_2$ respectively.

The previous list of holomorphic triangles involving the generators q and p, after Lagrangian surgery, becomes:

1. One-parameter families of concave holomorphic bigons, roughly looking like the concave triangles before, but with one corner rounded.

2. The smaller triangle in M_1 times one of the triangles in M_2. (Unchanged as it does not pass trough the self intersection y to begin with.)

To define

$$\mu^2 : CF(L_2, L_1(\epsilon)) \otimes CF(L_1(\epsilon), L_2) \to CF(L_1(\epsilon), L_1(\epsilon))$$

we look at $M((L_1(\epsilon), L_2, L_1(\epsilon)), a, b, c)$, where $a \in CF(L_1(\epsilon), L_2), b \in CF(L_2, L_1(\epsilon))$, $c \in CF(L_1(\epsilon), L_1(\epsilon))$, consisting of:

1. If $c \in CM(g)$, holomorphic bigons with boundary on $L_1(\epsilon), L_2$ and vertices mapping to a, b, followed by a Morse flow trajectory of g on $L_1(\epsilon)$ from the $L_1(\epsilon)$ boundary of the bigon to c. (The left picture in Figure 4.8).

2. If $c \in \bar{R}$, there are two cases: a holomorphic triangle with boundary in $L_1(\epsilon), L_2, L_1(\epsilon)$ and vertices mapping to a, b, c; or a holomorphic bigon with boundary on $L_1(\epsilon), L_2$ and vertices to a, b, followed by a Morse flow trajectory from the $L_1(\epsilon)$ boundary of the bigon to the boundary of a teardrop that passes through the self-intersection c. (The other two pictures in Figure 4.8).

$$\mu^2(b, a) = \sum |M((L_1(\epsilon), L_2, L_1(\epsilon)), a, b, c)| \cdot c$$

Similarly, we can define $\mu^2 : CF(L_1(\epsilon), L_2) \otimes CF(L_2, L_1(\epsilon)) \to CF(L_2, L_2)$ by counting points in the 0-dimensional moduli space $M((L_2, L_1(\epsilon), L_2), a, b, c)$ consisting of holomorphic bigons with boundary on $L_2, L_1(\epsilon)$ and vertices at a, b, followed by a Morse flow trajectory of h on L_2 from the L_2 boundary of the bigon to c.

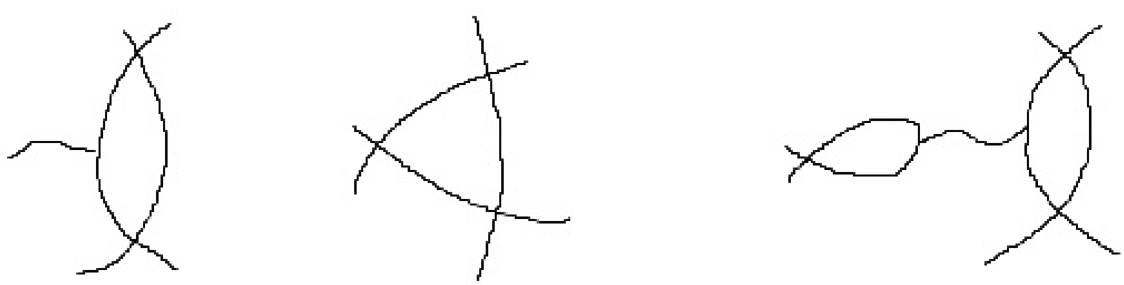

Figure 4.8: Configurations considered in $\mu^2(b, a)$

Statement 20. *$L_1(\epsilon)$ and L_2 are isomorphic*

Proof. We will proceed by showing that $\mu^2(q, p) \in CF(L_2, L_2)$ is the minimum of h in $CM(h)$ and $\mu^2(p, q) \in CF(L_1(\epsilon), L_1(\epsilon))$ is the minimum of g in $CM(g)$.

Since L_2 has no self intersections, $\mu^2(q, p)$ will only involve holomorphic bigon with boundary on $L_2, L_1(\epsilon)$ and vertices to p, q and Morse gradient flow lines in L_2. Consider $K \subset M(L_2, N_1 \times C_1, N_2 \times C_2, p, y, q)$ consisting of configurations considered in statement 14 where the triangle is not too close to a degenerate one and the extra marked point ξ is not too close to the vertices of the holomorphic triangles.

Let U be a relatively compact open neighborhood of K in $M(L_2, N_1 \times C_1, N_2 \times C_2, p, y, q)$. The theorem of FOOO tells us that there exists $M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+$ and a map π :

$M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+ \to U$ that is a diffeomorphism on $\pi^{-1}(K) \to K$. This means that we can find a one dimensional family of holomorphic bigons in $M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+$ that is diffeomorphic to K. Furthermore, each holomorphic bigon is ϵ_2 close to the corresponding holomorphic triangle in K. Note also that the limit as $\epsilon \to 0$ of any bigon with boundary on $L_1(\epsilon)$ and L_2 is a triangle with boundary on L_1 and L_2, so for ϵ small enough, all bigons of interest are covered.

We've shown in statements 14 and 17 that any point (in particular the minimum of h on L_2) lies on the boundary of a unique (mod 2) triangle in $M(L_2, N_1 \times C_1, N_2 \times C_2, p, y, q)$. Restricting to the subset K only restricts us to be outside of an arbitrarily small neighborhood of the vertices p, y, q and of the boundaries of the configuration when the domain degenerates. Thus the minimum of h can always be arranged to lie on the boundary of some holomorphic triangle in K and of no triangle outside of K. Applying the FOOO theorem now tells us that the minimum of h also lies on the boundary of some holomorphic bigon in $M(L_1(\epsilon), L_2, p, q, K, \epsilon_2)^+$ and no other bigon. Thus in the count of $\mu^2(q, p)$, the coefficient of $min(h)$ will be 1. Thus $\mu^2(q, p) = min(h)$. An identical argument shows that the coefficient of $min(g)$ in $\mu^2(p, q)$ is 1. On the other hand, the two convex triangles with vertices mapping to (q, p, x) contribute 0 mod 2 to the coefficient of x_- in $\mu^2(p, q)$, as shown in Statement 16. Hence $\mu^2(p, q) = min(g)$. Again since the minima correspond to the identity endomorphisms, we see that $\mu^2(q, p) = Id$ and $\mu^2(p, q) = Id$, proving that q, p are the desired isomorphisms between $L_1(\epsilon)$ and L_2.

□

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