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Ultraproducts of O-Minimal Structures

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by

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Abstract

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There are three main parts to this thesis, all centred around ultraproducts of o-minimal structures.

In the first part we investigate (for a fixed first-order language $\mathcal{L}$) what we call the *$\mathcal{L}$-theory of o-minimality*. It is the theory $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$ consisting of those $\mathcal{L}$-sentences true in *all* o-minimal $\mathcal{L}$-structures. We find that when $\mathcal{L}$ expands $\mathcal{L}_{RCF}$ by at least one new function or relation symbol, $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$ is not recursively axiomatizable. In particular, for any recursive list of axioms Λ which is consistent with $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$, we find that there are locally o-minimal, definably complete structures satisfying Λ which are not elementarily equivalent to an ultraproduct of o-minimal structures. We call the latter sort of structures *pseudo-o-minimal*.

In the second part we investigate uniform finiteness and cell decomposition in the pseudo-o-minimal setting. To do this, we introduce the notion of a pseudo-o-minimal structure *tallying* a discrete definable set. Investigating this notion, we answer some questions of uniqueness and existence. Finally, we show that under certain assumptions about the discrete definable sets that a given pseudo-o-minimal structure can tally, we have a version of uniform finiteness, at least in the planar case. This is the first step towards a cell decomposition theorem in this setting.

In the final section, we look into two classes of examples of ultraproducts of o-minimal structures. For the first class, we note the o-minimality of a certain subset of these structures, and show the non-o-minimality of another. In particular, we derive the o-minimality of a new structure related to $\mathbb{R}_{\exp}$. The second class is relatively intractable, but we discuss its relation to an important open problem in o-minimality.

For Goose.

Contents

Introduction

This dissertation is in the area of *o-minimality*, a branch of model theory and mathematical logic with ties to real analysis and real algebraic geometry. The research contained herein is focused on certain weakenings of o-minimality; namely first-order properties which can be said to hold in *all* o-minimal structures, such as *local o-minimality* and *definable completeness* (see below, or Definitions 1.5.1 and 1.5.3).

O-minimality can be thought of as a generalization of the study of the geometry of the semialgebraic sets.[1] In model theory generally and especially in o-minimality, the objects of primary consideration are the *definable sets* in a *structure*.[2] From this point of view, the semialgebraic sets are just the definable sets in one of the simplest o-minimal structures: the real field. A structure R is *o-minimal* just if every definable set in R^1 is a finite union of points and intervals (as the semialgebraic sets in $\mathbb{R}^1$ are). Thus, the real field is an o-minimal structure since the definable sets in $\mathbb{R}^1$ are just the zero- and positivity sets of polynomials (i.e finite unions of points and intervals.)[3] One of the most fundamental consequences of o-minimality is that definable sets in *any* ambient dimension can be shown to have finitely many (definably) connected components.[4]

Lou van den Dries states in the preface to his now classic [vdD98-a] that his aim was "*...to show that o-minimal structures provide an excellent framework for developing tame topology, or topologie modérée, as outlined in Grothendieck's prophetic 'Esquisse d'un Programme' of 1984.*" Since o-minimal structures seem to provide such a good framework for tame topology, it is natural to wonder what o-minimal structures exist. Thus, one of the major research areas in o-minimality is to classify various structures as either o-minimal or something weaker, thereby producing further tame or somewhat tame geometries. For

[1] Recall that the *semialgebraic sets*, defined in analogy with the *algebraic sets* are the class of sets defined by zero- and positivity-sets of systems of polynomials, closed under projection, complement, closure, union and intersection.

[2] A *structure* is a set together with *language*, consisting of a family of operations (e.g. addition, multiplication etc.) and relations (e.g. ordering, sharing an edge in a graph, an equivalence relation). In a structure, the *definable sets* are those sets that can be picked out by statements built up from logical symbols (like '$\wedge$', '$\vee$', '$\exists$', and variables like '$x, y, z, ...$') and the functions and relations in the language.

[3] This is by Tarski's Theorem (elimination of quantifiers for the real field), which is also known as the Seidenberg-Tarski Theorem (that the class of semi-algebraic sets is closed under projection).

[4] Often in model theory, since the *definable* sets are our objects of investigation, a condition such as connectedness, needs to be weakened to involve only definable sets. In this instance, a set A is *definably connected* if there are no two *definable* disjoint open sets whose union contains A.

instance, Wilkie [Wi99] showed that the structure of the real field together with the real exponential function is o-minimal, and thus that the class of exponential varieties (those constructed from zero- and positivity-sets of *exponential polynomials*[5]), share the same sorts of tameness results which hold for semialgebraic sets. As an aside, it is interesting to note that the decidability[6] of this structure was shown in [MW95] to be equivalent to a (weakened) version of the deep and as-of-now intractible number-theoretic conjecture of Schanuel.[7]

In this thesis, our primary object of consideration will be the class $\mathbb{K}_{\mathcal{L}}^{\text{o-min}}$ of all o-minimal structures (in a fixed language $\mathcal{L}$). In particular, we will be concerned with the set of (first-order) statements that are true of all structures in $\mathbb{K}_{\mathcal{L}}^{\text{o-min}}$; this set is called the *theory of o-minimality* (for that language), and we denote it $T_{\mathcal{L}}^{\text{o-min}}$. There are models of $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$ which are *not* o-minimal, so the property of being a model of $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$ is indeed a weakening of o-minimality. Understanding the theory $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$ and what it says about its models will be our specific focus. In particular, we will investigate the axiomatizability of $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$.[8]

In [Ax68], Ax proved that the *theory of finite fields*, consisting of all those sentences in the language of fields which are true in all finite fields, is recursively axiomatizable. He showed first that the infinite fields which are models of this theory, called *pseudofinite fields*, are precisely those fields which are perfect, pseudoalgebraically closed and have an algebraic extension of each degree. And second, he showed that these properties are all first-order definable by recursive axiom schemas.

Another positive result along the same lines comes from *pseudofinite linear orderings*: the theory of finite orderings is axiomatized by the statements that the ordering is discrete and has a first and last element (see [Va01]). A linear ordering is elementarily equivalent to an ultraproduct of finite linear orderings if and only if it is finite or has order type $\omega + L \cdot \mathbb{Z} + \omega^*$ for some linear order L.

All three of these cases fall into the following general framework: fix a language $\mathcal{L}$, let $\mathbb{K}$ be a class of $\mathcal{L}$-structures and let $\mathrm{T}^{\mathbb{K}}$ be the theory consisting of those $\mathcal{L}$-sentences which are true in all models $\mathcal{M} \in \mathbb{K}$.

In recent work by Fornasiero [Fo10] and Schoutens [Sch12], they ask whether or not certain first-order axioms are enough to axiomatize the theory of o-minimality (again, in a fixed language). Any such axioms, necessarily, will have to weaken the statement of o-minimality (since they have to be true in all o-minimal structures). Many weakenings of o-minimality, such as *weak o-minimality* ([MMS00]), *quasi-o-minimality* ([BPW00]), *d-*

[5]An *exponential polynomial* is a function $P(X)$ of the form $p(X, e^X)$ for p an ordinary polynomial, and X a tuple of variables $X_1, ..., X_n$.

[6]A structure is *decidable*, if there is an algorithm which can determine for any statement in the language of that structure, whether it is true of that structure or not.

[7]Schanuel's Conjecture states that given any n complex numbers $z_1, ..., z_n$, which are linearly independent over $\mathbb{Q}$, $\mathbb{Q}(z_1, ..., z_n, e^{z_1}, ..., e^{z_n})$ has transcendence degree at least n over $\mathbb{Q}$.

[8]A theory is *axiomatized* by a set of statements if every true statement of the theory follows from one of the axioms. An axiomatization is generally only helpful if it can be listed in some recursive fashion. For example, the set of *all true statements in a theory*, i.e. the whole theory itself, is an axiomatization technically, but generally there would not be a recursive way to list them. Thus, we restrict our search to *recursive axiomatizations.*

minimality ([Mi05]), *o-minimal open core* ([DMS09]) etc., have been studied in the literature (for more, see [vdD98-b, TV09, Mi01, Fo10]). However, since we are trying to write down an *axiomatization*, we are only interested in the first-order weakenings, and those which are true in all o-minimal structures.

Two weakenings in particular are of this form and are thus of interest to us: *local o-minimality* (that for every definable subset of the line, and every point, there is a neighbourhood of that point where the definable set is a finite union of points and intervals)[9]) and *definable completeness* (that every bounded definable subset of the line has a supremum).

Structures satisfying both of these properties have particularly nice definable sets: every definable $A \subseteq R^1$ has a discrete boundary which first, has no accumulation points in the topology on R, and second, is either finite, or has order-type $\omega + L \cdot \mathbb{Z} + \omega^*$ for some linear order L. That LOM and DC are expressible by first-order axiom schemas is an easy lemma, and it is clear that they are true in all o-minimal structures. Because of these two facts, these properties are a starting point to look for a recursive axiomatization of o-minimality.

With the *successful* examples of axiomatizations in other areas of model theory, like the theories of finite orderings and finite fields, in our mind, it would not be unreasonable to think that LOM+DC or something extending LOM+DC slightly might be enough to axiomatize $\mathrm{T}^{\text{o-min}}$. The fact that *there are no LOM+DC structures in the literature which are not ultraproducts of o-minimal structures (or actually o-minimal)* also gives us good reason to believe or intuitions for believing that LOM+DC might be sufficient.

One of the main results of this dissertation is the discovery that the project of axiomatizing $\mathrm{T}^{\text{o-min}}$ is actually impossible, at least when the language extends the language of ordered fields by at least one new function or relation symbol. In fact, *any* reasonable (i.e. recursive) list of axioms in such a language will be insufficient to axiomatize the theory $T^{\text{o-min}}$ (see Theorem 2.4.1.)

A classic model theoretic theorem tells us that a structure $\mathcal{M}$ satisfies $T^{\text{o-min}}$ if and only if $\mathcal{M}$ is (elementarily equivalent[10] to) an ultraproduct of a set $(\mathcal{M}_i)_{i\in I}$ of o-minimal structures. An ultraproduct of an indexed set $(\mathcal{M}_i)_{i\in I}$ of structures is a new structure $\mathcal{M}$ which in some sense 'averages' the index structures: $\mathcal{M}$ has a (first-order) property P if and only if 'most' index models $\mathcal{M}_i$ have P. Thus, $\mathcal{M}$ will have only those properties that there is a 'consensus' among the $\mathcal{M}_i$ about. Using this model-theoretic theorem, this result implies that ultraproducts of o-minimal expansions of fields are *irreducibly* more complicated (from the first-order perspective) than locally o-minimal (LOM), definably complete (DC) expansions of fields. In particular, there are LOM+DC expansions of fields which are not

[9]If we strenthen this by allowing the point to possibly be $\pm\infty$, we get a property that Schoutens calls *type completeness* in [Sch12]. Local o-minimality, though prima facie weaker, is only weaker if the structure in question does not have a *multiplicative* group structure, which will not be the case for us, as we will always assume we have a field structure. Nonetheless, it seems to be worth including in the general definition to keep local o-minimality more in line with o-minimality, and since this stronger version *is* true in all o-minimal structures. Thus, we will define it in this stronger fashion later.

[10]A structure is *elementarily equivalent* to another if they both satisfy the exact same first-order statements.

ultraproducts, and for any recursive extension Λ of the axioms LOM and DC, there will be structures satisfying Λ, but which are also not ultraproducts.

This result points us in the direction of studying the models of $T^{\text{o-min}}$ *themselves*, instead of trying to reduce them down to certain simple properties they have and studying what is true in those structures.[11] In particular, after this, we move on to study *ultra-o-minimal* and *pseudo-o-minimal* structures (our names for ultraproducts of o-minimal structures, and general models of $T^{\text{o-min}}$, respectively).

In this vein, the next goal is to prove a weakened version of an o-minimal tameness property: *Uniform Finiteness*. Uniform finiteness is an important step along the way to proving further tameness results, and in particular, the higher dimensional generalization of the property of o-minimality, the *Cell Decomposition* Theorem. The latter gives us a way to decompose definable sets in higher dimensions - not into intervals and points, as in one ambient dimension, but to higher dimensional analogues, called cells. These two pieces of 'machinery' have been used in one guise or another to prove many of the tameness properties that o-minimal structres enjoy.

Uniform Finiteness states that a definable set in ambient dimension $m + 1$ which has finite fibres has a uniform bound N on the size of all of its fibres. Now, since definable discrete sets in pseudo-o-minimal structures can be infinite, there is certainly no hope that a set with discrete fibres will have a finite bound. Thus, any analogue we hope to prove will have to generalize 'finite' in some way. To do this, we introduce a notion of 'counting', or as we call it 'tallying'[12] for discrete (possibly infinite) definable sets. The idea is to use a naturally occuring object, called an integer part, which all real closed fields have, to provide a 'number system' to use to 'tally' the discrete definable sets. The intuition here is that what we are missing from the o-minimal context is a way to tell two discrete sets apart based on something like cardinality. After defining our notion of the tally of a discrete set, we investigate some of their general properties, and in particular, show some forms of uniqueness which they enjoy, and which make our notion more intuitive. Finally, under the assumption that every discrete definable set can be tallied, and a further assumption about tallies, we are able to conclude a discrete analogue of Uniform Finiteness, at least in 2 ambient dimensions: if a definable set in two ambient dimensions has discrete fibres, then there is a single 'number' (in an integer part) which bounds the tallies of all the fibres of the set. We note here that proving a higher dimensional version would go hand-in-hand with proving a full blown Cell Decomposition-like theorem, which we save for future work.

At this point in the thesis we will have first looked at LOM+DC structures, the weakest we will consider, and we showed that they were in fact quite weak - weaker than pseudo-o-minimal structures. Then we proved a weakened tameness result for (some) pseudo-o-minimal structures. Following this, we look at ultra-o-minimal structures.

Ultra-o-minimal structures have all their tameness on display, in some sense: anything that is true in o-minimal structures will have a fairly immediate *ultraproduct version*: for

[11] We are not suggesting that the study of LOM+DC structures is not interesting in itself, of course.

[12] This is done to avoid our terminology clashing with 'countable' and 'uncountable'.

instance, the notion of tallies for pseudo-o-minimal structures, itself an analogue of counting finite sets, has a canonical, simpler version in ultraproducts: there is a canonical integer part, the hypernatural numbers, consisting of ultraproducts of natural numbers, which we can use to count any discrete set: since any discrete definable set in an ultra-o-minimal structure is just an ultraproduct of finite sets, and since each such set has a finite, natural number cardinality, the ultraproduct of these cardinalities will be the cardinality of the given discrete set.[13] Thus, a uniform bound on the size of the discrete fibres in a family of definable discrete sets is just the ultraproduct of the uniform bounds obtained in the index o-minimal models. Similarly, we can say that a cell decomposition in an ultra-o-minimal structure is just an ultraproduct of individual cell decompositions in the index models.[14]

Thus, instead of spending time recording all of the properties that hold as an ultraproduct of properties that hold in the index models, we instead focus on investigating examples of ultra-o-minimal structures. Since ultra-o-minimal structures can either be o-minimal or fail to be o-minimal, the project we set out to contribute to is to determine which ultraproducts are o-minimal and which are not.

We consider two families of examples: one given by ultraproducts of polynomials, which we call *pseudopolynomials*, and the other given by ultraproducts of iterates of functions (given a hypernatural number N there is a natural way to define the 'N^{th}' iterate of a given function). In the second case, we run into roadblocks quite early, as these functions end up being closely related to algebraic dynamics; an area which is quite new and quite open at the moment. Thus, even relatively simple examples involving only polynomials or rational functions cause trouble.

An example in this latter family which has a (much) longer-term target on it is the case of an infinite iterate of the exponential function. The exponential function is a key function in the study of o-minimality; the real field extended by it is o-minimal (and hence there are many other such examples), and in particular, there is a stark dichotomy for the growth of functions definable in o-minimal structures: in a given structure, either they are all eventually bounded by some power function, or else the structure defines the exponential function (and hence also the logarithm, and many other functions.) Showing that an infinite iterate of the exponential function was o-minimal would answer a long-standing open question from the time that the above growth dichotomy was first proved: are there any o-minimal structures which define a function which eventually bounds any (finite!) iterate of the exponential function (sometimes referred to as a *superexponential function*)? We will note that this is a very difficult problem to approach.

Coming back to the first family of examples, ultraproducts of polynomials, we may think on first glace that they are very innocuous. The first intuition one might have is that these ought to all be o-minimal. However it will end up that there are some easy and quick non-o-minimal examples; and that these are not just side cases: if f is *any* function

[13] This observation that ultra-o-minimal structures have this property was the original motivation to try to generalize it to pseudo-o-minimal structures.

[14] There is actually some subtlety here, but we do not explore this in this dissertation.

which is non-o-minimal, and we take an ultraproduct of polynomials that approximates f in a certain way, then the resulting pseudopolynomial will not be o-minimal. In addition to proving this negative result, one of the most important positive examples we are able to derive is that of the ultraproduct of the Taylor polynomials for the exponential function. The resulting pseudopolynomial is (up to an infinitesimal) equal to exp on the non-infinite elements of our ultraproduct, but eventually the two split off from each other. This is an example that has not appeared in the literature. There is a certain dearth of examples of interesting ultra- and pseudo-o-minimal structures, as we mentioned earlier. This structure seems to indicate that there are a number of interesting examples that should be investigated.

In the first Chapter, we introduce the necessary model theory, including ultraproducts, o-minimality, as well as LOM, DC, pseudo-o-minimality, ultra-o-minimality, and some of the basic relationships that these notions share.

Most of Chapter 2 is spent setting up and proving the first main fact of this thesis:

Theorem 1 (Theorem 2.4.1)**:** *Given any language $\mathcal{L}$ extending the language RCF by at least one new function or relation symbol and any recursively enumerable list Λ of $\mathcal{L}$-sentences, either* $\mathrm{T}^{\text{o-min}} \cup \mathrm{RCF} \cup \Lambda$ *is inconsistent or there is some model $\mathcal{M} \vDash T^{\text{o-min}} \cup \mathrm{RCF} \cup \Lambda$ which is not pseudo-o-minimal.*

In Chapter 3, some time is spent discussing various properties that definable 'counting' functions might have. In particular, after two versions of this concept are discarded, a third, the notion of a *tally* is then defined. Some questions about tallies are answered, before proving a Uniform Pseudofiniteness result:

Theorem 2 (Theorem 3.5.1)**:** *Let $\mathcal{R}$ be pseudo-o-minimal, and $\mathcal{N}$ be a fixed nn-part for $\mathcal{R}$. If $X \subset \mathcal{R}^2$ is definable and has that for all $r \in \mathcal{R}$ the fibre X_r of X above r is discrete, then assuming **(Tally)** and **(Unbounded Tally)** (see Section 3.5) we can find $\alpha \in \mathcal{N}$ such that for every r, the fibre X_r has tally $\leq \alpha$, and some fibre has tally equal to α. Futhermore, we can determine, for any pair of elements $z, z' \in X$, whether they have the same tally according to the tally of their respective fibres of X.*

The final section is spent dealing with a problem that tallies have, and a new, weaker concept is introduced: *approximate tallies*. I use these to prove a slightly weaker Uniform Pseudofiniteness result than the previous one, which has the advantage of assuming less about our structure.

In the final chapter, two families of ultra-o-minimal structures are considered: what we call pseudopolynomials, and pseudodynamical functions. We are able to prove that some examples of the first kind, and in particular, all those which are Pfaffian (see Definition 4.2.3) over an ultra-o-minimal structure are in fact o-minimal. This includes the function $\widetilde{\exp}$ (defined below). Along the way, we are also able to rule out a whole family of examples from being o-minimal:

Theorem 3 (Lemma 4.2.2, Corollary 4.2.1 and Theorem 4.2.1)**:**

- *If $\mathcal{R}$ is an ultraproduct of o-minimal expansions of a real closed field, then its expansion by a family of Pfaffian functions is also o-minimal.*
- $(\mathcal{R}, \widetilde{\exp})$ *is o-minimal, where* $\widetilde{exp} = \prod_{n\in\mathbb{N}} T_n/\mathcal{U}$ *and* $T_n \sim \exp$ *is the* n^{th} *Taylor polynomial for* $\exp$.
- *If f is a function such that $(\overline{\mathbb{R}}, f)$ is not o-minimal, and if $\widetilde{f}$ is some pseudopolynomial approximation of f (see Definition 4.2.2), then $(\mathcal{R}, \widetilde{f})$ is not o-minimal.*

Afterwards we dicuss the problems we ran into while trying to study functions of the second kind, and point to directions for future research.

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Chapter 1

The Basics

1.1 Model Theoretic Preliminaries

Some standard references for this material, and model theory in general are [Ma02], [Hod98] and [Poi00].

Definition 1.1.1:

- *The **complete $\mathcal{L}$-theory of** $\mathcal{M}$ is the set* $\mathrm{Th}(\mathcal{M}) \coloneqq \{\varphi \mid \mathcal{M} \models \varphi\}$*, the set of sentences true in* $\mathcal{M}$*.*
- *If* $\mathbb{K}$ *is a class of* $\mathcal{L}$*-structures, then we define*

$$\mathrm{T}^{\mathbb{K}} = \bigcap_{\mathcal{M} \in \mathbb{K}} \mathrm{Th}(\mathcal{M})$$

  That is, $\mathrm{T}^{\mathbb{K}}$ *is the set consisting of the sentences true in all the structures in* $\mathbb{K}$*.*

- *If* $\mathcal{M}$ *is an* $\mathcal{L}$*-structure, and* $\varphi(x;b)$ *is an* $\mathcal{L}$ *formula, with* $b \in \mathcal{M}^n$*, then the set*

$$\varphi(\mathcal{M};b) = \{a \in \mathcal{M}^m \mid \mathcal{M} \models \varphi(a;b)\}$$

  *is the **solution set** of* φ *in* $\mathcal{M}$*, or the **set defined by*** φ*. We will often omit the (implicit) parameters and just write* $\varphi(\mathcal{M})$*.*

- *We say that a subset* $X \subseteq \mathcal{M}^m$ *is **definable with parameters*** $b \in \mathcal{M}$ *if there is some formula* φ *such that* $X = \varphi(\mathcal{M};b)$*. When* b *is empty, we will say that* X *is* $\varnothing$***-definable** or* 0***-definable**, and when ignoring the issue of parameters, we will simply say that* X *is **definable** in* $\mathcal{M}$*.*
- $\mathrm{Def}(\mathcal{M})$ *is the class of all definable sets in* $\mathcal{M}$ *(of any arity), and we let* $\mathrm{Def}_n(\mathcal{M})$ *be the class of definable sets in* $\mathcal{M}^n$*.*
  - *Later, we will see that there is a notion of dimension on the sets in* $\mathrm{Def}(\mathcal{M})$ *for the particular structures* $\mathcal{M}$ *that we will consider. We will be interested in the 0-dimensional sets definable in* $\mathcal{M}$ *specifically, and we will write* $\mathrm{Def}_0(\mathcal{M})$ *for the class of* 0*-dimensional definable sets in* $\mathcal{M}$ *(of any arity). This notation does not clash with the above since we do not need a notation for the definable subsets of* $\mathcal{M}^0$*.*
- *If* $\mathcal{M}$ *is an* $\mathcal{L}$*-structure and* $X \subseteq \mathcal{M}^n$ *is a non-empty subset, then we define the following language* $\mathcal{L}'$*, and the* $\mathcal{L}'$*-structure* X_{ind}*:* $\mathcal{L}'$ *consists of a single relation symbol* R_φ *of arity* k *for each formula* $\varphi(x)$ *in* k *variables, where* $k = m \cdot n$ *for some* m*; then, we let* X_{ind} *have universe* X*, together with the interpretations* $R_\varphi^{X_{\mathrm{ind}}} = X^m \cap \varphi(\mathcal{M})$*. We call* X_{ind} *the **induced structure** on the definable set* X*.*

1.2 Ultraproducts

Definition 1.2.1: *Let I be an infinite set. A **filter** $\mathcal{F}$ on I is a set of subsets of I, i.e $\mathcal{F} \subseteq \mathcal{P}(I)$, such that:*

- *if $X \in \mathcal{F}$ and $Y \supset X$, then $Y \in \mathcal{F}$.*
- *if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.*
- *$I \in \mathcal{F}, \varnothing \notin \mathcal{F}$.*

*$\mathcal{F}$ is an **ultrafilter** if in addition, $\mathcal{F}$ satisfies:*

- *for all $X \subseteq I$, exactly one of X or $I - X$ is in $\mathcal{F}$.*

A filter is in a rough sense, a notion of when a subset of I is "most" of I. Thought of in this slightly vague way, the axioms state about elements of a filter $\mathcal{F}$, that "if X is most of I, and Y contains X, then Y is also most of I", that "if X and Y are both most of I, then they overlap on a set that is also most of I", that "the empty set is not most of I", and that "I itself is most of I". An ultrafilter is an extension of this notion: it says furthermore that "if X is not most of I, then its complement is". (Note that the filter axiom for intersections implies the converse: "if X is most of I, then the complement is not".)

A filter is **principal** if it contains some finite set, or equivalently, if it contains a singleton. We will generally be assuming in what follows that our ultrafilters are non-principal.[1]

From now on, we will write ultrafilters using $\mathcal{U}, \mathcal{V}, \mathcal{W}, \ldots$.

Importantly, any filter $\mathcal{F}$ can be extended, by Zorn's Lemma, to an ultrafilter $\mathcal{U}$. In a sense, extending a filter to an ultrafilter is a matter of, for every X, deciding whether X or its complement should be "most of I, according to $\mathcal{U}$", in a way consistent with the choices already made. Since, if $\mathcal{F}$ is not already an ultrafilter, there will be many consistent ways of doing this, there will be many different ultrafilters extending a given filter $\mathcal{F}$.

One of the most important filters is the *Frechet filter* consisting of all the cofinite subsets of I. Every ultrafilter $\mathcal{U}$ on I extends this filter (that is, every ultrafilter will contain all of the cofinite subsets of I).

Now, given an ultrafilter $\mathcal{U}$ on a set I, we can define *ultraproducts*:

Definition 1.2.2: *Suppose that $(\mathcal{M}_i)_{i \in I}$ is an I-indexed family of $\mathcal{L}$-structures. Let $\mathcal{M}^I$ be the product $\prod_{i \in I} \mathcal{M}_i$ consisting of all the I-indexed sequences of elements $(m_i)_{i \in I}$ from the $\mathcal{M}_i$. And then define $\prod_{i \in I} \mathcal{M}_i / \mathcal{U} = \mathcal{M}^I / \mathcal{U}$, the **ultraproduct of the $\mathcal{M}_i$ modulo $\mathcal{U}$** to be*

[1] Later on, when we consider ultraproducts of o-minimal structures, we do not mean omit the principal ultraproducts - they will just be trivial examples, in that they will always just be some particular o-minimal structure.

the $\mathcal{L}$-structure with universe the quotient of $\mathcal{M}^I$ by the following equivalence relation, where $x, y \in \mathcal{M}^I$:

$$x \sim_{\mathcal{U}} y \Leftrightarrow \{i \in I \mid x_i = y_i\} \in \mathcal{U}$$

and where for each function symbol f, relation symbol R and constant symbol c in $\mathcal{L}$, we define

$$f^{\mathcal{M}^I/\mathcal{U}}([x]_{\mathcal{U}}) \text{ to be } [(f^{\mathcal{M}_i}(x_i))_{i\in I}]_{\mathcal{U}}$$

$$R^{\mathcal{M}^I/\mathcal{U}}([x]_{\mathcal{U}}) \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models R(x_i)\} \in \mathcal{U}$$

and

$$c^{\mathcal{M}^I/\mathcal{U}} = [(c^{\mathcal{M}_i})_{i\in I}]_{\mathcal{U}}$$

We say that an ultraproduct is an ***ultrapower****, and we write $\mathcal{M}^{\mathcal{U}}$, if $\mathcal{M}_i = \mathcal{N}$ for all $i \in I$, for some fixed structure $\mathcal{N}$.*

It is straightforward to check that these are well defined and so give us an $\mathcal{L}$-structure.

The most important theorem about ultraproducts for us (though there will be other important theorems to come later) is Los' Theorem. It says, essentially, that truth in an ultraproduct is determined by "consensus" among the index models.

Theorem 1.2.1 (Los' Theorem)**:** *If $(\mathcal{M}_i)_{i\in I}$ is an I-indexed family of $\mathcal{L}$-structures for I some infinite set, and $\mathcal{U}$ is an ultrafilter on I, then for any $\mathcal{L}$-sentence φ,*

$$\prod_{i\in I} \mathcal{M}_i/\mathcal{U} \models \varphi \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi\} \in \mathcal{U}$$

In particular, an ultrapower $\mathcal{M}^{\mathcal{U}}$ is elementarily equivalent to $\mathcal{M}$.

To fill out the earlier analogy, Los' Theorem tells us that when $\mathcal{U}$-most index models $\mathcal{M}_i$ agree on a sentence φ, then that forces the ultraproduct to believe that φ is true as well. When $\{i \in I \mid \mathcal{M}_i \models \varphi\} \in \mathcal{U}$, we will say "$\mathcal{U}$-most $\mathcal{M}_i$ model φ".

1.3 Some Topological Notation and a Lemma

Let $\mathcal{R}$ be an expansion of a dense linear ordering (without endpoints).

We denote $\mathcal{R} \cup \{\pm\infty\}$ by $\mathcal{R}_{\pm\infty}$, the positive elements of $\mathcal{R}$ by $\mathcal{R}_{>0}$, and the non-negative elements by $\mathcal{R}_{\geq 0}$.

For the rest of the section, let $X \subseteq \mathcal{R}^n = \mathcal{R}^m \times \mathcal{R}^k$, $Y \subseteq \mathcal{R}$ be definable, $x \in \mathcal{R}^n$, $y \in \mathcal{R}$, $z \in \mathcal{R}^k$ and $r \in \mathcal{R}_{\geq 0} \cup \{+\infty\}$.

Notation:

- *The symbol* (a,b) *means the open interval between* a *and* b*, while the symbols* $[a,b]$ *and* $\langle a,b\rangle$ *mean the closed interval between* a *and* b*, and the point with coordinates* a *and* b *in* $\mathcal{R}^2$*, respectively.*

- *In general, we denote the set of elements of* Y *less than* y *by:*

$$Y_{<y} = \{z \in Y \mid z < y\} = Y \cap (-\infty, y)$$

And similarly for $Y_{\leq y}$*,* $Y_{>y}$ *and* $Y_{\geq y}$*.*

- *We say that* $B_r(x)$ *is the* ***(open)*** *box of radius* r*, centred at* x*:*

$$B_r(x) = (x_1 - r, x_1 + r) \times \ldots \times (x_n - r, x_n + r) \subseteq \mathcal{R}^n$$

- *We write* X_x *to denote the fibre of* X *above* $x \in \mathcal{R}^m$*:*

$$X_x = \{y \in \mathcal{R}^m \mid \langle x, y\rangle \in X\}$$

And we write $X_{[z]}$ *for the set* $X \cap (\{z\} \times \mathcal{R}^k)$*.*[2]

- *The interior, closure and complement of* X *in* $\mathcal{R}$*, written* $\operatorname{int}(X)$*,* $\operatorname{cl}(X)$ *and* X^c *respectively, are the ordinary topological interior, closure and complement of* X*, and in particular can be defined by*

$$\operatorname{int}(X) = \{x \in \mathcal{R}^n \mid \exists r \in \mathcal{R}_{>0}\ B_r(x) \subseteq X\}$$

$$\operatorname{cl}(X) = \{x \in \mathcal{R}^n \mid \forall r \in \mathcal{R}_{>0}\ B_r(x) \cap X \neq \varnothing\}$$

$$\text{and} \quad X^c = \{x \in \mathcal{R}^n \mid x \notin X\}$$

- *The boundary of* X*,* $\operatorname{bd}_{\mathcal{R}}(X)$*, is the set* $\operatorname{cl}_{\mathcal{R}}(X) - \operatorname{int}_{\mathcal{R}}(X)$*, and the frontier of a set* X*,* $\partial_{\mathcal{R}} X$ *is* $\operatorname{cl}_{\mathcal{R}}(X) - X$*.*

- *We write* $\operatorname{acc}(X)$ *for the set of accumulation points of* X *in* $\mathcal{R}$*. That is,*

$$\operatorname{acc}(X) = \{x \in \mathcal{R}^n \mid \forall r \in \mathcal{R}_{>0}\ \exists y \neq x \in \mathcal{R}^n\ y \in B_r(x) \cap X\}$$

Thus, we say that a point is an accumulation point of X *if* $x \in \operatorname{acc}(X)$*.*

Note first that since X and Y are definable, all of the above sets are definable as well (possibly with a parameter r).

We say that a set X is *discrete* if for every $x \in X$, there is an $r \in \mathcal{R}_{>0}$ such that $B_r(x) \cap X = \{x\}$; we say that X is *bounded* if there is an $r \in \mathcal{R}_{>0}$ and $x \in \mathcal{R}^n$ such that $X \subseteq B_r(x)$; and finally, we say that X is *closed* if $X = \operatorname{cl}(X)$. Furthermore, if $x \in \operatorname{acc}(X)$,

[2] Sometimes in the literature, these two notations are reversed.

then for any open set U containing x, it must be the case that $U \cap X$ is infinite (otherwise, since $\mathcal{R}$ is a dense ordering, we could shrink U to avoid the finitely many points away from x.)

It is trivial to see that a closed set contains all of its accumulation points, and that the closure of a set is just the union of the set and the set's accumulation points. That is, $\mathrm{cl}(X) = X \cup \mathrm{acc}(X)$, for all X, and that a set X is closed just if $X \cup \mathrm{acc}(X) = X$.

First a lemma relating some of definitions from the previous section:

Lemma 1.3.1: *If X is discrete and closed, then* $\mathrm{acc}(X)$ *is empty.*

Proof. Suppose for contradiction that $x \in \mathrm{acc}(X)$. Then for any ball of radius $r > 0$ centered at x, that ball intersects $X - \{x\}$. This would contradict the discreteness of X if $x \in X$. But $x \in X$ since X is closed; contradiction. ∎

One should note that these definitions are really just specializations of the ordinary topological notions of *closure*, *complement*, *interior*, etc. to the particular case of the product topology of the interval topology on a (dense) linearly ordered topological space; namely that of $\mathcal{R}$.

In particular, we should note here that an *interval* is not just any convex set in $\mathcal{R}$: it is specifically a *non-empty* set of the form $\{x \in \mathcal{R} \mid a \lhd x \lhd b\}$ with '$\lhd$' standing for either '$<$' or '$\leq$', and a and b *elements of* $\mathcal{R}_{\pm\infty}$. In particular, a set like $\{x \in \mathbb{Q} \mid x^2 < 2 \wedge x > 0\}$ would not be an 'interval' in $\mathbb{Q}$ in this sense. Note that intervals are thus clearly definable sets. Furthermore, note that an interval could be all of $\mathcal{R}$, or even just single point $\{x\}$ (in the latter case, if $a = b \neq \pm\infty$).

Definition 1.3.1:

- *We say that X is* ***connected****, if there is no pair of open sets U and V such that $U \cap V = \varnothing$ and $X \subseteq U \cup V$. However, we will almost never use this ordinary topological notion; instead, we will use the following definable analogue:*
- *We say that X is* ***definably connected*** *if there is no pair of* definable *open sets U and V which disconnect X. We say that $X' \subseteq X$ is a definably connected component of X if it is a maximal subset of X which is definably connected.*
- *If we say that a set is connected, we implicitly mean* definably connected.

Finally, we will mention projections and definable functions:

Notation:

- *Let $f : \mathcal{R}^n \to \mathcal{R}^m$. Then we write $\Gamma(f)$ for the graph of f. That is, for the subset:*

$$\Gamma(f) = \{\langle x, y\rangle \in \mathcal{R}^n \times \mathcal{R}^m \mid f(x) = y\}.$$

- *We say that a function f is definable in $\mathcal{R}$ just when $\Gamma(f)$ is definable.*

- *We will use π_m^{m+k} or π_m^n when $n = m + k$ to denote the projection map $\mathcal{R}^{m+k} \to \mathcal{R}^m$ onto the first m coordinates. Usually we will drop the superscript and just write π_m or both the super- and subscripts and just write π if these are clear from context.*

The projection maps are all definable:

$$\Gamma(\pi_m^{m+k}) = \{\langle x, y, z\rangle \mid x, z \in \mathcal{R}^m, y \in \mathcal{R}^k \wedge x = z\}$$

This points out that there is a correspondence between projections and the existential quantifier, showing that the result of applying it to any definable set X will be definable:

$$\pi_m^{m+k}(X) = \{x \in \mathcal{R}^m \mid \exists y \in \mathcal{R}^k \ \langle x, y\rangle \in X\}$$

Though in general, if X is definable, and f is a definable map, then the image $f(X)$ is clearly definable already:

$$f(X) = \{y \in \mathcal{R}^m \mid \exists x \in \mathcal{R}^n \ f(x) = y\}$$

1.4 O-Minimality

We include in this section summaries of and defintions of various pieces from the general theory of o-minimality that will either be used, or touched on later in this thesis. The book [vdD98-a] is classic, and is one of many good references for the general theory of o-minimality; we will cite it frequently.

Definition 1.4.1: *$\mathcal{R}$ is **o-minimal** if every one-variable definable set is a finite union of intervals.*[3]

The definition of o-minimality is equivalent to the assertion that one-variable definable sets have finitely many definably connected components. And in fact, the Cell Decomposition Theorem for o-minimal structures (see below) implies that a definable set in *any* number of variables is just a union of finitely many simple definable sets called cells. The easy fact that cells are definably connected implies the stronger statement that *all* definable sets have finitely many definably connected components.

Definition 1.4.2 ([vdD98-a], Definitions 4.2.3,10)**:** *Let $\langle i_1, .., i_m\rangle \in 2^m$. An $\langle i_1, ..., i_m\rangle$**-cell** is a definable subset of $\mathcal{R}^m$ obtained by induction on m as follows:*

(i) A $\langle 0\rangle$-cell is a singleton $\{x\}$ (i.e a point), and a $(\langle 1\rangle$-cell is an open interval in $\mathcal{R}^1$.

(ii) Now suppose that the $\langle i_1, ..., i_{m-1}\rangle$-cells have already been defined.

- *Then an $\langle i_1, ..., i_{m-1}, 0\rangle$-cell is the graph of a continuous function on any $\langle i_1, ..., i_{m-1}\rangle$-cell X,*

[3]This compact statement assumes our provisio that singletons are closed intervals of radius 0. If the reader is unhappy with this, then the statement becomes "... is a finite union of points and open intervals."

- *and an $\langle i_1, ..., i_{m-1}, 1\rangle$-cell is an 'interval' between two continous functions f and g on an $\langle i_1, ..., i_{m-1}\rangle$-cell X, with $f < g$ on X, i.e. a set of the form*

$$(f, g) := \{\langle x, y\rangle \in \mathcal{R}^{(m-1)+1} \mid x \in X \ \wedge \ f(x) < y < g(x)\}.$$

We also define a ***decomposition*** *$\mathcal{D}$ of $\mathcal{R}^m$: it is a partition of $\mathcal{R}^m$ into finitely many cells satisfying a projection hypothesis. It is defined by induction on m as well:*

(i) a decomposition $\mathcal{D}$ of $\mathcal{R}^1$ is a partititon of $\mathcal{R}^1$ into points (singletons, really) and intervals of the following form (where $a_1 < ... < a_k$):

$$\{(-\infty, a_1), \{a_1\}, (a_1, a_2), \{a_2\}, ..., \{a_k\}, (a_k, \infty)\}$$

(ii) a decomposition of $\mathcal{R}^m$ is a finite partition of $\mathcal{R}^m$ into cells C such that the set of projections onto the first $m-1$ coordinates $\{\pi(C) \mid C \in \mathcal{D}\}$ is a decomposition of $\mathcal{R}^{m-1}$.

We also say that a decomposition $\mathcal{D}$ of $\mathcal{R}^m$ is ***compatible*** *with a definable set A if every element of $\mathcal{D}$ is either contained in A, or disjoint from A.*

We commented before this definition that cells were always definably connected. This is an easy fact; but what else is true is that every cell (in any given fixed ambient dimension m) is homeomorphic to an open cell in $\mathcal{R}^k$ for some $1 \le k \le m$.[4] In particular, every open cell is homeomorphic to an open box in its ambient space. See [vdD98-a] for more details.

Theorem 1.4.1 (Cell Decomposition ([vdD98-a], Theorem 3.2.11))**:**

$(I)_n$ Given any definable sets $A_1, ..., A_k \subseteq \mathcal{R}^m$, there is a decomposition of $\mathcal{R}^m$ compatible with each of $A_1, ..., A_k$.

$(II)_n$ For each definable function $f : A \to \mathcal{R}$, $A \subseteq \mathcal{R}^m$, there is a decomposition $\mathcal{D}$ of $\mathcal{R}^m$ such that the restriction $f \restriction B : B \to \mathcal{R}$ is continuous, for each (cell) $B \in \mathcal{D}$ with $B \subseteq A$.

The statement $(I)_1$ is just a restatement of o-minimality (justifying calling this theorem a generalization to higher dimensions of the definition of o-minimality), and the statement $(II)_1$ is a consequence of the following theorem:

Theorem 1.4.2 (Monotonicity Theorem)**:** *Let $f : I \subseteq \mathcal{R} \to \mathcal{R}$ be definable for some interval I. Then there is a decomposition of $\mathcal{R}$ compatible with I such that on each subinterval where f is defined, it is continous, and either constant or strictly monotone.*

Other important results that we will use or reference are collected in the following theorem. We note that many of them are direct or nearly direct consequences of the Cell Decomposition theorem.

[4] Naming our set X, and letting X be an $\langle i_1, ..., i_m\rangle$-cell, we just project onto the set of coordinates j such that $i_j = 1$.

Theorem 1.4.3:

1. *Uniform Finiteness ([vdD98-a], Theorem 3.2.13): Every definable set* $X \subseteq \mathcal{R}^{m+1}$ *whose fibres* X_x *over elements* $x \in \mathcal{R}^m$ *are finite, have a uniform finite bound on their cardinality. That is, there is a natural number* N *such that no fibre of* X *has cardinality greater than* N.

2. *"Strong" O-Minimality: If* $\mathcal{R}$ *is o-minimal, then every model of* $\mathrm{Th}(\mathcal{R})$ *is as well.*[5]

3. *Definability of Skolem Functions ([vdD98-a], Proposition 6.1.2.i): Assuming that* $\mathcal{R}$ *expands an ordered group, if* $S \subseteq \mathcal{R}^{m+n}$ *is definable and* π *is the projection onto* $\mathcal{R}^m$, *then there is a definable map* $f : \pi(S) \to \mathcal{R}^n$ *such that* $\Gamma(f) \subseteq S$.[6]

4. *Curve Selection ([vdD98-a], Corollary 6.1.5): Assuming that* $\mathcal{R}$ *expands an ordered group, we have that if* X *is definable, and* $a \in \mathrm{cl}(X) - X$, *then there is a definable continuous injective map,* $\gamma : (0, t) \to X$, *for some* $t \in \mathcal{R}_{>0}$ *such that* $\lim_{t \to 0} \gamma(t) = a$.

5. *Dimension ([vdD98-a], Chapter 4): We can define a dimension on the cells in a natural way: if* X *is an* $\langle i_1, \ldots, i_n \rangle$ *cell, then* $\dim(X) := \sum_{j=1}^{n} i_j$. *Using this we can then define the dimension of a definable set* X *to be the largest dimension of a cell contained in it.*[7]

6. *([vdD98-a], Propositions 1.4.2 and 1.4.6): If an o-minimal structure* $\mathcal{R}$ *expands an ordered group, then it expands an ordered divisble abelian group; if* $\mathcal{R}$ *expands an ordered ring, then it expands a real closed field.*

7. *Growth Dichotomy ([Mi94]): If* $\mathcal{R}$ *expands a real closed field, then either every* $\mathcal{R}$*-definable one-variable function* f *is eventually bounded by some power function* $x \mapsto x^N$, *or* $\mathcal{R}$ *defines the exponential function. Structures of the former type are called* ***power-bounded***.[8,9]

1.5 Introduction to LOM, DC, and Pseudo-O-Minimality

In what follows, $\mathcal{R}$ is still an expansion of a dense linear ordering (without endpoints).

[5]This is an almost direct consequence of uniform finiteness and definability of Skolem functions (below).

[6]This is just a geometrical equivalent of the ordinary statement of definable Skolem functions.

[7]This, as it is classically developed, relies on cell decomposition. In weakenings of o-minimal structures, there is often still a dimension available to us, though sometimes we have to just through more hoops to get it.

[8]If the universe of $\mathcal{R}$ is the reals, then a power function just means an ordinary polynomial. In general, if $\mathcal{R}$ has infinite elements, then we need to take into account infinite N such that x^N is definable. (The set of such N is called the field of exponents of $\mathcal{R}$ in the literature.)

[9]Also note that there is a version of this for expansions of groups: see [MS98].

There are many ways in which we could weaken o-minimality. However, a fairly natural weakening, and a starting point for us is to "localize" o-minimality as follows (note that this notion was first introduced, or at least given this name, by Toffalori and Vozoris in [TV09]):

Definition 1.5.1: *A linearly ordered structure $\mathcal{R} = (\mathcal{R}, <, ...)$ is* ***locally o-minimal (LOM)*** *if for every one-variable definable set, X, and every element $x \in \mathcal{R}$, there is some interval $(a, b) \ni x$ such that $X \cap (a, b)$ is a finite union of intervals.*

Local o-minimality on its own, unlike o-minimality, is not strong enough to rule out certain 'bad' behaviour:

Example 1.5.1: *Consider $(\mathbb{R}, +, <, \sin, 0)$. In [MMS00], it is shown that this is locally o-minimal. Note that this structure is not o-minimal, since it defines an infinite discrete set*

$$\{z \in \mathbb{R} \mid \sin(z) = 0\}$$

However, a locally o-minimal *field* cannot define an unbounded subset of the line, such as the above.

Lemma 1.5.1: *Suppose that $\mathcal{R}$ is an expansion of a locally o-minimal ordered field. Then no unbounded discrete subset of $\mathcal{R}$ is definable.*

Proof. Suppose that Z is a non-empty, definable, unbounded discrete set. Then for any $c \in \mathcal{R}$, we have that $Z_{>c}$ has infinitely many connected components (each given by a point of Z). Then because we have the field structure, we can define the new set $1/Z = \{1/z \mid z \in Z_{\neq 0}\}$. Since Z was unbounded, we have that for each interval (a, b) containing $0, (a, b) \cap 1/Z$ is not a finite union of points and intervals. But this contradicts local o-minimality. ∎

Following Schoutens, we will from now on take local o-minimality to be the stronger statement where we have "... $x \in \mathcal{R}_{\pm\infty}$ ...". This modification rules out definable discrete sets which are unbounded. As we just noted, having or not having this strengthening is inconsequential if we are working in an expansion of an ordered field, but we believe that it was an oversight not to include this strenthening in the original statement:[10] in particular, the stronger version, as we will see, is a first-order consequence of o-minimality in any language.

Now, the non-existence of unbounded definable discrete sets draws our attention also to the potential definability of definable discrete sets with no greatest element (this could either be the result of an accumulation point, or a cut in the ordering, not of course, of unboundedness). The following weakening of o-minimality potentially allows such sets:

Definition 1.5.2: *Let $\mathcal{R}$ be an expansion of a dense ordered structure with no endpoints. We say that $\mathcal{R}$ is* ***weakly o-minimal*** *if every definable set $X \subseteq \mathcal{R}$ is a finite union of points and convex sets.*[11]

[10] Perhaps the weaker version should have been given a different name.

[11] A set X is *convex*, just if for all $x, z \in X$, if $y \in \mathcal{R}$ satisfies $x < y < z$, then $y \in X$.

There are weakly o-minimal structures which are not o-minimal, and there are locally o-minimal fields which are weakly o-minimal, but not o-minimal. (See [TV09] for examples.) Such structures cannot have an unbounded discrete set, but they *can* have discrete sets that have no greatest element. Every weakly o-minimal structure satisfies LOM, by [TV09], Proposition 2.2:

Lemma 1.5.2: *Every weakly o-minimal structure is locally o-minimal.*

Thus, a second property that we might consider in order to rule out these cousins of unbounded discrete sets - discrete sets that are contained in convex definable sets which are not intervals - and hence rule out weakly o-minimal structures, is the following:

Definition 1.5.3: *$\mathcal{R}$ is* ***definably complete (DC)*** *if for every non-empty definable set $X \subseteq \mathcal{R}$, if X is bounded, then $\sup(X)$ and $\inf(X)$ exist in $\mathcal{R}$ (and are definable).*

Lemma 1.5.3: *If $\mathcal{R}$ is definably complete and weakly o-minimal, then $\mathcal{R}$ is o-minimal.*

Proof. By weak o-minimality, every one variable definable set is a finite union of points and convex sets, but every convex definable set in $\mathcal{R}$, by DC, has a supremum in $\mathcal{R}$. Thus every convex definable set in $\mathcal{R}$ is an interval. ∎

So, DC rules out certain sorts of sets, but like LOM, DC is not *alone* enough to rule out certain kinds of bad behaviour, even with the field structure:

Example 1.5.2: *Consider an expansion of a real closed field $\mathcal{R}$ which satisfies DC. Now add a predicate to $\mathcal{R}$ for a dense substructure $\mathcal{Q}$ of $\mathcal{R}$. Then $(\mathcal{R}, \mathcal{Q})$ satisfies DC, but of course, is not locally o-minimal.*[12]

As we mentioned in the previous section, pair of classic results on o-minimal structures is that if an o-minimal structure expands an ordered group, then it is an ordered divisible abelian group (i.e a $\mathbb{Q}$-vector space), and if it expands an ordered ring, then it is an expansion of a real closed field. Similarly, LOM+DC rings are real closed, and this will focus our attention on LOM+DC expansions of real closed fields:

Lemma 1.5.4: *If $\mathcal{R}$ is LOM+DC and expands an ordered ring, then it is real closed.*

Proof. This follows from Proposition 2.5 in [Mi01]. ∎

The combination of LOM, DC (LOM+DC from now on) and the field structure leads to many nice properties, many of which Fornasiero outlines in [Fo10] and Schoutens discusses in [Sch12]. This includes a version of the o-minimal Monotonicity Theorem, and a weakened, and slightly tentative version of Cell Decomposition. In fact, many analogues of theorems true about o-minimal structures carry over to LOM+DC fields. This includes

[12] See the original paper [vdD98-b] for the details of how the theory of $(\mathcal{R}, \mathcal{Q})$ models DC and, incidentally, uniform finiteness.

definable Skolem functions, definable choice, a notion of dimension, etc. In the last section of this chapter we repeat the details of some of these (see [Fo10] for further details).

However, as the reader surely expects by this point, and as we mentioned in the Introduction, LOM+DC fields do not have to be o-minimal (from now on, $\overline{\mathbb{R}}$ will denote the usual real closed field structure on $\mathbb{R}$).

Example 1.5.3: *For $n \in \mathbb{N}$, let $\mathcal{R}_n = (\overline{\mathbb{R}}, P_n)$, where $P_n = \{1, ..., n\} \subseteq \mathbb{N}$. Then letting $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, define $\mathcal{R} = \prod_{n\in\mathbb{N}} \mathcal{R}_n/\mathcal{U} = (\mathbb{R}^{\mathbb{N}}/\mathcal{U}, P)$. $\mathcal{R}$ is an ultraproduct of o-minimal structures which is not o-minimal.*

Proof. This is straightforward. Since in each index model, the predicate P_n picks out a discrete set of size $> n-1$, we know two things by Los' Theorem: the set P must be discrete, and contain $> n-1$ elements for every $n \in \mathbb{N}$. That is, P must be an infinite discrete (definable) set, which is impossible in an o-minimal structure. ■

The previous example is typical of LOM+DC structures that have been given in the literature, in that it is really an ultraproduct of o-minimal structures. In fact, there seem be to no examples of LOM+DC fields in the literature that are not ultraproducts of o-minimal structures.

Now, a key difference that o-minimality has from LOM and DC is that even though they are both elementary properties (see the following definition), o-minimality is not first-order (again, see below), while the others are. Even though the following are basic definitions from model theory that we might have taken for granted, we record them due to there being some confusion about the difference between uses of "first-order" and "elementary". The following are the defintions that we will use.

Definition 1.5.4:

- *A property P of $\mathcal{L}$-structures which are models of a theory T is* ***first-order*** *if there is a set of sentences, or schema of sentences, $\mathcal{S}$, such that an $\mathcal{L}$-structure $\mathcal{M}$ has property P if and only if $\mathcal{M} \models \sigma$ for all $\sigma \in \mathcal{S}$.*
- *P is* ***elementary*** *if the class of $\mathcal{L}$-structures with property P is closed under elementary equivalence. (i.e if $\mathcal{M} \equiv \mathcal{N}$ then $\mathcal{M}$ has P if and only if $\mathcal{N}$ does too.)*

As I just mentioned, the statement of o-minimality is *not* first-order. That is, there is no first-order axiom schema that holds if and only if a structure is o-minimal. If there were, Los' Theorem would imply that all ultraproducts of o-minimal structures would themselves be o-minimal. However o-minimality *is* elementary (see [KPS86]):

Theorem 1.5.1: *If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, for some language $\mathcal{L}$ extending $\{<\}$, $\mathcal{M}$ is o-minimal, and $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{N}$ is also o-minimal.*

Unlike o-minimality, LOM *is* first-order:

Lemma 1.5.5: *LOM is equivalent to following the axiom schema, where $\mathcal{R}$ is a model:*

$$\mathrm{LOM}_{\varphi\in\mathcal{L},} \ : \ \forall c\ \forall x\ \exists a < x < b \text{ such that } \varphi(\mathcal{R},c)\cap(a,b) \text{ is one of the following eight sets:}$$

$$\varnothing, \{x\}, (a,x), (x,b), (a,x], [x,b), (a,x)\cup(x,b),\ \textit{or}\ (a,b)$$

Definable completeness is also first-order:

Lemma 1.5.6: *Definable completeness is equivalent to the following schema:*

$$\mathrm{DC}_{\varphi\in\mathcal{L}} \ : \ \forall c\ \exists r > 0\ \forall x(x\in\varphi(\mathcal{R},c) \to -r < x < r)$$

$$\longrightarrow \exists! s\ \left[\forall x\ (x\in\varphi(\mathcal{R},c)\to x\le s)\wedge\forall y\ (\forall x(x\in\varphi(\mathcal{R},c)\to y > x)\to s\le y)\right]$$

Thus, what we have noted is that LOM+DC is first-order, and that all the examples so far constructed have been ultraproducts. This leads one to wonder whether all LOM+DC fields are (at least elmentarily equivalent to) ultraproducts of o-minimal structures. Thus, we will make the following definition:

Definition 1.5.5: *A structure $\mathcal{R}$ is* ***pseudo-o-minimal*** *if it is elementarily equivalent to an ultraproduct of o-minimal structures.*

The following straightforward consequence of o-minimality has a straightforward analogue in pseudo-o-minimal structures:

Lemma 1.5.7: *If $\mathcal{R}$ is an o-minimal structure, and $A \subsetneq \mathcal{R}^n$ is a discrete definable set, then A is finite.*

Lemma 1.5.8: *Let $\mathcal{R}$ be a pseudo-o-minimal expansion of a real closed field. If $X \subsetneq \mathcal{R}^n$ is definable and discrete, then the reduct of X_{ind} to $\{<\}$ is pseudofinite (that is, elementarily equivalent to an ultraproduct of finite orderings).*

Proof. We may assume that $\mathcal{R} \equiv \mathcal{S} = \prod_{i\in I}\mathcal{S}_i/\mathcal{U}$ and assume that $X \subseteq \mathcal{R}^n$ is definable by $\varphi(x)$, and is discrete. Then $\varphi(\mathcal{S})$ is discrete, and so by Los' Theorem, in $\mathcal{U}$-most index models, $\varphi(\mathcal{S}_i)$ is also discrete. But a discrete set in an o-minimal structure is finite by Lemma 1.5.7. Thus, in $\mathcal{U}$-most index models $\varphi(\mathcal{S}_i)$ is finite, so the reduct of X_{ind} to $\{<\}$ is elementarily equivalent to their ultraproduct (themselves considered only in the language $\{<\}$). ■

Finally, before moving on, we collect one more result:

Lemma 1.5.9: *If $\mathcal{R}$ is a pseudo-o-minimal ordered ring, then $\mathcal{R}$ is LOM and DC; hence it is also an expansion of a real closed field.*

Proof. The first statement follows by simply noting that LOM and DC are both first-order (by Lemmas 1.5.5 and 1.5.6, respectively), and thus are true in ultraproducts of structures where they hold, by Los' Theorem.

Next, since LOM+DC fields are real closed (by Lemma 1.5.4), $\mathcal{R}$ is real closed too.

Alternately we could derive $\mathcal{R}$ being real closed from Los' Theorem as well, since $\mathcal{R}$ is elementarily equivalent to an ultraproduct of real closed fields, and the axioms for real closed fields are recursively axiomatizable (in fact the class of fields is finitely axiomatizable). ∎

1.6 Definable Sets in LOM+DC and Pseudo-O-Minimal Fields

Unless otherwise specified, $\mathcal{R}$ is an expansion of a real closed field satisfying LOM+DC in this section.

First a topological lemma:

Lemma 1.6.1: *Discrete definable subsets of $\mathcal{R}$ have no accumulation points, and hence are closed.*

Proof. By Lemma 1.3.1, if we show that a discrete definable subset $X \subseteq \mathcal{R}$ has no accumulation points, then it must be closed.

So suppose that x is an accumulation point of X. Let (a, b) be an interval containing x. Then since x is an accumulation point, $X \cap (a, b)$ contains infinitely many points. But since X contains no intervals, this is impossible by local o-minimality. ∎

Using this, we show the discrete definable sets in $\mathcal{R}$ are closed and bounded:

Lemma 1.6.2: *Every discrete definable set $X \subseteq \mathcal{R}$ is closed and bounded.*

Proof. By Lemma 1.6.1, X is closed, and by Lemma 1.5.1, X cannot be unbounded. ∎

The characterization of Lemma 1.6.2 is equivalent to other possible characterizations:

Lemma 1.6.3: *The following are equivalent, for any non-empty definable $X \subseteq \mathcal{R}$:*

a. X is discrete.

b. X is discrete, and has a first and last element, and every element of X except the first and last has an immediate predecessor and successor.

c. The structure $(X, <^{\mathcal{R}})$ is a pseudofinite ordering.[13]

[13]The followin proof, with a couple of modifications will actually work in any linearly ordered LOM+DC structure.

Proof. (1. ⇒ 2.)

Since X is discrete and closed, X has no accumulation points, by Lemma 1.3.1.

Since X is bounded, the definable completeness of $\mathcal{R}$ implies that it has a supremum in $\mathcal{R}$. That is, there is some element $x = \sup(X) \in \mathcal{R}$, such that if X had no greatest element, then x would be an accumulation point, which is impossible. Thus, it has a greatest element. Similarly, it has a least element.

Now, the existence of successors and predecessors can be seen by noting that if $x \in \mathcal{R}$, $X_{>x}$ must have a first element (if it is non-empty), and $X_{<x}$ must have a last element (similarly, if it is non-empty). This is true since $X_{>x}$ and $X_{<x}$ are both definable discrete sets, and hence closed and bounded by 1.6.2; thus the observations of the previous paragraph hold for them as well. □

(2. ⇒ 1.)

This is trivial. □

(2. ⇔ 3.)

In [Va01], it is demonstrated that the orderings (elementarily equivalent to) ultraproducts of finite orderings are exactly the orderings which satisfy condition 2 above. It is shown that an ordering elementarily equivalent to an ultraproduct of finite orderings is completely determined (up to elementary equivalence) by its order type, and that the order type of any such structure is always $\omega + L \cdot \mathbb{Z} + \omega^*$, where ω^* is a reversed copy of ω and L is some linear ordering.

It is clear that a set X satisfying condition 2. has this order type for some set L, and vice versa. □ ■

Since such sets always have a least and greatest elment, and all but the first and last elements have an immediate successor and predecessor, we make the following definitions:

Definition 1.6.1: *Suppose that P is non-empty and discrete.*

- *Let λP and γP be the least and greatest elements of P, respectively.*
- *For $p \in P$, if $p \neq \gamma P$, let $s_P(p)$ be the next element of P, and if $p \neq \lambda P$, let $p_P(p)$ be the previous element of P.*

We note now that a converse of Lemma 1.6.2 is true as well:

Lemma 1.6.4: *If for every definable non-empty, proper subset $X \subsetneq \mathcal{R}$, we have that* $\mathrm{bd}(X)$ *is a non-empty dcb with a least and a greatest element (possibly identical), then $\mathcal{R}$ satisfies LOM+DC.*

Proof. Suppose that $X \subseteq \mathcal{R}$ is definable, and bounded above. In order for DC to hold, we need only show that X has a supremum in $\mathcal{R}$. But, by the definition of $\mathrm{bd}(X)$, its greatest element is certainly the supremum of X.

Before we show that LOM follows, note the following claim:

Claim: *For every non-empty, definable proper subset $X \subsetneq \mathcal{R}$, every element of* $\text{bd}(X)$ *has a unique successor and predecessor (except the least and greatest elements, respectively.)*

Let $x \in \text{bd}(X)$ be a non-least element, if such an element exists, and consider the set $\text{bd}(X)_{<x}$: this set is definable, and since it equals its own boundary, it is the boundary of a definable set; and so by assumption, it has a greatest element. This element is the unique predeccessor of x in $\text{bd}(X)$. Similarly, we can determine the unique successor of x (this time if x is not the greatest element of $\text{bd}(X)$).

Returning to the proof of the lemma, we continue by showing LOM. Let $X \subsetneq \mathcal{R}$ be non-empty and definable, let $r \in \mathcal{R}$, and consider $\text{bd}(X_{\leq r})$: if this set has a greatest element less than r, define a to be that element. If the greatest element is r, define a to be the predeccessor of r in $\text{bd}(X_{\leq r})$, if it exists. If no such element exists, then $\text{bd}(X_{\leq r}) \subseteq \{r\}$, and thus $(-\infty, r) \subseteq X$ or $X_{<r} = \varnothing$; and we can then define a to be $r-1$. Similarly, if $\text{bd}(X_{\geq r}) \subseteq \{r\}$, define b to be $r+1$, and otherwise define b to be the least element of $\text{bd}(X_{\geq r})_{>r}$. By choosing a and b in this way, we have ensured that $X \cap (a, b)$ has at most two components, and since $r \in \mathcal{R}$ was arbitrary, LOM follows. ■

Now, for higher dimensional sets, we can prove the analogue of Lemma 1.6.2:

Lemma 1.6.5: *Every discrete definable set $X \subseteq \mathcal{R}^n$ is closed and bounded.*

But before proving this, another lemma is necessary:

Lemma 1.6.6: *For $X \subseteq \mathcal{R}^n$ definable, if for every $y \in \mathcal{R}$, for every coordinate choice, the sets X_y, and all of the projections $\pi_i^n(X)$, for $1 \leq i \leq n$ are discrete, closed and bounded, then X discrete, closed and bounded.*

This lemma will itself follow from the following more general statement:

Lemma 1.6.7: *Let P be a definable discrete, closed and bounded subset of $\mathcal{R}$, and suppose that $(X(p))_{p \in P}$ is a definable family of discrete, closed and bounded subsets of $\mathcal{R}^n$. Then $X' = \bigcup_{p \in P}(X(p))$ is discrete, closed, and bounded as well.*

Proof. Let $X'(p) := \bigcup_{q \leq p; q \in P}(X(q))$, for $p \in P$. Since $(X(p))_{p \in P}$ is a definable family, $(X'(p))_{p \in P}$ is a definable family as well. Thus, we may define the set

$$Q := \{p \in P \mid X'(p) \text{ is bounded}\}.$$

Since it is a definable set in $\mathcal{R}$, and is a subset of P, it is discrete, closed and bounded. Also, since $X'(\lambda P)$ is discrete, closed and bounded, we know that Q is non-empty. Then γQ exists; if it is not equal to γP, then $s_P(\gamma Q)$ exists, and we have that $X'(\gamma Q)$ is bounded, but $X'(s_P(\gamma Q))$ is not. But then since $X'(s_P(\gamma Q)) = X'(\gamma Q) \cup X(s_P(\gamma Q))$, we would have that $X(s_P(\gamma Q))$ is unbounded. But no $X(p)$ is unbounded by assumption, so this is a contradiction. ■

Proof of Lemma 1.6.6. We have finitely many discrete sets $P_1 = \pi_1^n(X)$, ..., $P_n = \pi_n^n(X)$, the projections of X onto each coordinate. Let us assume that after translating each P_i appropriately, we have that $\gamma P_1 < \lambda P_2$, ... $\gamma P_{n-1} < \lambda P_n$. Then, letting $P = P_1 \cup ... \cup P_n$, we have another definable dcb set. Furthermore, if $x \in X$, then there is some coordinate, say the i^{th}, such that for some $y \in \mathcal{R}$, $x \in X_y$, and thus, $x \in \bigcup_{p \in P} X_p$. Apply Lemma 1.6.7. ∎

Proof of Lemma 1.6.5. Suppose $X \subseteq \mathcal{R}^n$ was a discrete definable set with either (a) a projection onto some coordinate which was not dcb, or (b) a fibre above an element in some coordinate which was not dcb. If we can eliminate these possibilities, then Lemma 1.6.6 would apply.

We can rule out case (a) immediately, since the images of definable maps (and in particular, projection maps) are definable: because of this, $\pi_i(X)$ is a definable subset of $\mathcal{R}$, so Lemma 1.6.2 implies that it is dcb.

Now, we proceed by induction. In the base case, where $n = 2$, if there were $y \in \mathcal{R}$ such that in either coordinate X_y were not dcb, then we would also have that the projection of the fibre onto the other coordinate, $X_{[y]} \subseteq \mathcal{R}$ was a definable subset of $\mathcal{R}$ that was not dcb, a contradiction, by Lemma 1.6.2 again. In the higher dimensional case, we note that $X_y \subseteq \mathcal{R}^{n-1}$ and appeal to the inductive assumption. ∎

1.7 Some Analogues From O-Minimality

In the papers [Fo10], and [Sch12], Fornasiero and Schoutens each study various weakenings of o-minimality, including LOM+DC structures. We record here a few analogues of o-minimal results from these two papers that we will use freely later. There are many other results, many of them fairly fundamental, that are collected in these two papers. The ones that follow below are ones that will be explicitly used, and were not already mentioned in the previous sections.

Fact 1.7.1:

- *(Monotonicity Theorem for LOM+DC Structures - [Fo10], Theorem 6.3): We have the same Monotonicity theorem here, except that instead of having a partition*

$$\{(-\infty, a_1), \{a_1\}, (a_1, a_2), ..., \{a_k\}, (a_k, \infty)\}$$

  for k finite such that f is continuous and constant or strictly monotone on each open interval, we instead have a discrete definable set D such that f is continous and constant or strictly monotone on each open interval $(d, s_D(d))$ for $d \in D$, and on $(-\infty, \lambda D)$ and $(\gamma D, \infty)$.[14]

- *(Definable Skolem Functions - [Fo10], Lemma 6.16) - LOM+DC structures have definable Skolem functions.*

[14]The proof is a straightforward adaptation of the o-minimal proof.

- *(Dimension - [DMS09]) - Assuming $\mathcal{R}$ expands an ordered group, we have a notion of dimension for definable subsets of $\mathcal{R}^m$ such that the discrete definable sets are dimension 0, and the open definable sets are exactly the m-dimensional sets: the dimension of $X \subseteq \mathcal{R}^m$ is the greatest element d of $\{-\infty, 0, 1, ..., m\}$ such that the projection onto some d-dimensional (in the ordinary sense) coordinate hyperplane of a rotation (by an element of $\mathcal{R}$) of X has non-empty interior.*[15]

- *(Growth Dichotomy - [Fo10], Theorem 6.20): the same statement as for o-minimal structures holds.*[16]

1.8 Everything We Need to Know About PA

In this final preliminary section, we review facts about the theory PA which we will need in Chapter 2.

Definition 1.8.1:

- *The theory of **Peano Arithmetic*** **(PA)**, *is the $\mathcal{L}_{\text{PA}} = \{0, 1, S, <, +, \cdot\}$-theory consisting of the following axioms, where $\mathcal{N}$ is a purported model of the axioms:*
  - $(\mathcal{N}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, <^{\mathcal{N}}, 0^{\mathcal{N}}, 1^{\mathcal{N}})$ *is a commutative linearly-ordered semiring with additive identity $0^{\mathcal{N}}$, multiplicative identity $1^{\mathcal{N}}$ and annihiliator $0^{\mathcal{N}}$ with $0^{\mathcal{N}} <^{\mathcal{N}} 1^{\mathcal{N}}$.*
  - $\forall x \; S(x) = x + 1 > x$[17]
  - $Ind_{\varphi(x,\overline{y})\in\mathcal{L}} \; : \; \forall \overline{y} \; (\varphi(0, \overline{y}) \wedge \forall x \; (\varphi(x, \overline{y}) \rightarrow \varphi(S(x), \overline{y}))) \rightarrow \forall x \; \varphi(x, \overline{y})$

- *We call the $\mathcal{L}$-theory $Th(\mathbb{N})$ the **full theory of arithmetic**. It is, by definition a complete theory, and has PA as a* proper *subtheory.*

The most important theorem for us about PA is Gödel's Second Incompleteness Theorem:

Theorem 1.8.1 (Gödel's Second Incompleteness Theorem)**:** *If* T *is a consistent, recursively axiomatizable $\mathcal{L}$-theory for a language $\mathcal{L} \supseteq \mathcal{L}_{PA}$, and if* T *extends PA, then* T *does not entail* Con(T)*.*

See the discussion below (i.e. after Corollary 1.8.1) for the definition of "Con(T)". In order to use a slight modification of this theorem, first we need a couple of definitions:

Definition 1.8.2:

[15] We will mainly be concerned with discrete sets in what follows, so getting into the details of the consequences of this will not be important here.

[16] And Fornaseiro notes that the proof works almost verbatim.

[17] Or, we could have said more generally that $S^{\mathcal{N}}$ is an injective map from $\mathcal{N}$ with image $\mathcal{N} - \{0\}$.

- *An $\mathcal{L}$-structure $\mathcal{N}$ is* ***interpretable*** *in an $\mathcal{L}$-structure $\mathcal{M}$ if there is a definable set $X \subseteq \mathcal{M}^n$ and a definable equivalence relation E such that $\mathcal{N}$ is isomorphic to X/E and all the structure on $\mathcal{N}$ is definable on X/E in $\mathcal{M}$.*[18]

- *Given an $\mathcal{L}$-theory T, and an $\mathcal{L}'$-theory S, we say that* ***T interprets S*** *if for every $\mathcal{M} \models \mathrm{T}$, there is $\mathcal{N} \models S$ such that $\mathcal{M}$ interprets $\mathcal{N}$.*

Now, Gödel's Incompleteness Theorem can be slightly modified by noting that if a model $\mathcal{M}$ of a theory T interprets a model $\mathcal{N}$ of PA, then one can run the proof of the Incompleteness Theorem in $\mathcal{M}$ inside the interpreted model $\mathcal{N}$. See [Be10] for a discussion of interpretability and the Incompleteness Theorems. This gives us:

Theorem 1.8.2: *If* T *is a consistent, recursively axiomatizable theory, such that* T *interprets* PA*, then* T *does not entail* Con(T)*.*

In particular, a direct model theoretic consequence of this, which we will employ, is the following:

Corollary 1.8.1: *For such a theory* T, $\mathrm{T} + \neg\mathrm{Con}(\mathrm{T})$ *is consistent, and thus has a model.*

In addition to using the Incompleteness Theorem, we will need to understand a little bit about how it is proved. In particular, one of the main components of the proof is *Gödel coding*. Gödel coding uses elements of $\mathbb{N}$ (or later, some other model of PA) to code sentences of $\mathcal{L}$. There are many different ways in which one could code the formulas of $\mathcal{L}$ using natural numbers and their arithmetic properties, but abstractly what the process of coding gives us is ([Ka91], p.37) an injection $\varphi \to \ulcorner\varphi\urcorner$ taking formulas (strings of symbols from the language) to natural numbers, with the following properties:

(a) it is computable;

(b) its image is a computable subset of $\mathbb{N}$;

(c) and its inverse is also computable (i.e. we can go from a code $\ulcorner\varphi\urcorner$ to the formula φ computably.)

In addition to this, we need a uniform way of coding finite length sequences of natural numbers (and thus formulas). The existence of a coding of finite sequences uniform in their length is called "Gödel's Lemma" ([Ka91, Lemma 3.2]) which states that there is a Δ_0 formula[19] which accomplishes this uniform coding.[20] That this can be accomplished with

[18]This is equivalent to other standard definitions of interpretability.

[19]A formula is Δ_0 if any quantifiers which appear in the formula are *bounded*. That is, instead of the bare quantification "$\forall x$ (...)" appearing in the formula, we would always have, for some $\mathcal{L}$-term t, "$\forall x < t$ (...)" etc. (Recall that a *term* is a well-formed string of variables, constants and function symbols from the language $\mathcal{L}$.)

[20]Precisely, it states that there is a Δ_0 formula $\theta(x,y,z)$ such that $\mathbb{N} \models \forall x, y\ \exists! z\ \theta(x,y,z)$ such that for all $k \in \mathbb{N}$, and all $z_0, ..., z_{k-1} \in \mathbb{N}$, there is $x \in \mathbb{N}$ such that for all $i < k, \mathbb{N} \models \theta(x,i,z_i)$.

a Δ_0 formula will be important for us later (see Lemma 1.8.1). With these two pieces in hand, we would then define a Σ_1 provability formula[21] $\text{prov}(\ulcorner\varphi\urcorner)$ which expresses that "the sentence φ is provable in T" in T. It need only satisfy the following three conditions:

(a) $\mathrm{T} \vdash \varphi \Rightarrow \mathrm{T} \vdash \text{prov}(\ulcorner\varphi\urcorner)$

(b) $\mathrm{T} \vdash \text{prov}(\ulcorner\varphi \to \psi\urcorner) \to (\text{prov}(\ulcorner\varphi\urcorner) \to \text{prov}(\ulcorner\psi\urcorner))$

(c) $\mathrm{T} \vdash \text{prov}(\ulcorner\varphi\urcorner) \to \text{prov}(\ulcorner\text{prov}(\ulcorner\varphi\urcorner)\urcorner)$

Using prov, we can write down the statements Con(T) and ¬Con(T) as "$\neg\text{prov}(0 = 1)$" and "$\text{prov}(0 = 1)$" respectively. It is a short exercise to show that in a theory T which has coding and a provability predicate (for which it is enough to extend PA), T does not entail Con(T).

Now, since all of the coding can be accomplished with relatively simple formulas, we actually have the extension of coding, Gödel coding, and a provability predicate to non-standard models of PA. In particular, the coding is Δ_0, and this leads us to one last fact that we need:

Lemma 1.8.1: [Ka91, Theorem 2.7] *Let $\mathcal{N}_0$ be an initial segment*[22] *of $\mathcal{N}$. Then $\mathcal{N}_0$ is a Δ_0-elementary substructure.*[23]

In particular, if T is consistent and extends PA, then if $\mathcal{N} \vDash \neg\text{Con}(\mathrm{T})$, or precisely, if $\mathcal{N} \vDash \text{prov}(\ulcorner 0 = 1\urcorner)$, $\alpha \in \mathcal{N}_0$ is a code for $\text{prov}(\ulcorner 0 = 1\urcorner)$, then $\mathcal{N}_0 \vDash \text{prov}(\ulcorner 0 = 1\urcorner)$, and in particular $\mathcal{N}_0 \vDash \neg\text{Con}(\mathrm{T})$. This implies that since PA + ¬Con(T) is consistent, any code for a proof of 0 = 1 must be a non-standard element. That is, such an α as above would have to be non-standard (i.e. satisfy $\alpha > n$ for all $n \in \mathbb{N}$). If it were not, then since $\mathbb{N}$ is an initial segment of every model of PA, and hence of T, ([Ka91], Theorem 2.2) we would have that $\mathbb{N} \vDash \neg\text{Con}(\mathrm{T})$ by the above Lemma, and could conclude that PA was inconsistent.

[21] A formula is Σ_1 if it consists of a Δ_0 formula with a block of existential quantifiers at the beginning.

[22] That is, for all $x \in \mathcal{N}_0$ and for all $y \in \mathcal{N}$, $(\mathcal{N} \vDash y < x) \Rightarrow (y \in \mathcal{N}_0)$

[23] That is, whenever $a \in \mathcal{N}_0$ and $\varphi(x) \in \Delta_0$, then $(\mathcal{N}_0 \vDash \varphi(a)) \Leftrightarrow (\mathcal{N} \vDash \varphi(a))$.

Chapter 2

The Non-Axiomatizability of O-Minimality

2.1 Introduction

In this short chapter, we will show that for languages extending the language of ordered fields by at least one new non-constant symbol, *o-minimality is* ***not*** *(recursively) axiomatizable* (see Definition 2.2.2 below). That is, that in each fixed language $\mathcal{L}$ properly extending the language of ordered fields, there is no recursive axiomatization of $\mathrm{T}^{\text{o-min}}$, the set of all $\mathcal{L}$-sentences true in all o-minimal $\mathcal{L}$-structures. This is the opposite of the situation for the class of finite fields and the class of finite orderings, as I discuss below. In order to prove this, I will construct a model of a purported axiomatization which *could not* be (elementarily) equivalent to an ultraproduct of o-minimal structures. (See the discuss after Theorem 2.2.1 for details on the connection between ultraproducts and axiomatizations.)

2.2 Axiomatizations

Recall the following definitions (these are basic in model theory, but we record them here for maximum clarity):

Definition 2.2.1:

a. *For an $\mathcal{L}$-theory* T*, a subset* $\mathrm{S} \subseteq \mathrm{T}$ *is said to be an* ***axiomatization*** *of* T *if* S *and* T *have exactly the same set of models.* T *is* ***finitely axiomatizable*** *if there is a single* $\mathcal{L}$*-sentence* φ *such that* $\mathcal{M} \vDash \mathrm{T}$ *if and only if* $\mathcal{M} \vDash \varphi$*.*[1]

b. T *is* ***recursively axiomatizable*** *if* $\mathcal{L}$ *is a recursive language, and there is a recursive axiomatization* S *of* T*.*

c. *T is* ***decidable*** *if there is an algorithm which decides for any* $\mathcal{L}$*-sentence* φ*, whether* $\mathrm{T} \vdash \varphi$ *or* $\mathrm{T} \nvdash \varphi$

Note that if T is recursively axiomatizable and complete, then T is decidable.

Example 2.2.1:

1. *The theory of groups is undecidable. [No55]*
2. *The theory of* $(\mathbb{Q}, +, \cdot)$ *is undecidable. [Ro49]*
3. *The theory PA, of Peano Arithmetic (see Definition 1.8.1 below), a proper subtheory of the complete theory of* $(\mathbb{N}, +, \cdot, <)$ *is undecidable, despite being recursively axiomatizable. Furthermore, every consistent extension of PA is undecidable.*
4. *The theories of Real Closed Fields, of Euclidean geometry*[2] *and many others are all decidable by work of Alfred Tarski. See [Ta51] for the first two mentioned.*
5. *Presburger Arithmetic, the theory of the structure* $(\mathbb{N}, +, <)$ *is decidable. [Pr29]*

A natural extension of these notions is the following, where we recall that for a class $\mathbb{K}$ of $\mathcal{L}$-structures, ***the first-order theory of*** $\mathbb{K}$, $\mathrm{T}^{\mathbb{K}}$, is the set of $\mathcal{L}$-sentences true in *all* elements of $\mathbb{K}$:

Definition 2.2.2:

[1] Of course, any theory T can be axiomatized trivially by just taking S = T. However, if T is not decidable (definition follows in the body text), then there will be no such recursive listing and hence no reasonable listing that we can get our hands on; this is why we care about recursive or finite axiomatizations.

[2] The theory of Euclidean Geometry is equivalent to the theory of Real Closed Fields as long as the axiom of (definable) completeness is assumed. Without completeness, i.e. the theory of ruler-and-compass constructions, or the theory of Pythagorean numbers, things are more complicated, and probably undecidable.

a. *A class $\mathbb{K}$ of $\mathcal{L}$-structures is* ***axiomatizable*** *if there is a set T of $\mathcal{L}$-sentences such that a structure $\mathcal{M}$ is in $\mathbb{K}$ if and only if $\mathcal{M} \models T$.*[3]

b. *$\mathbb{K}$ is* ***finitely axiomatizable*** *if it is axiomatizable by a single $\mathcal{L}$-sentence.*[4]

c. *Similarly, $\mathbb{K}$ is* ***recursive*** *if there is a recursive axiomatization of $\mathrm{T}^{\mathbb{K}}$, and $\mathbb{K}$ is* ***decidable*** *if $\mathrm{T}^{\mathbb{K}}$ is decidable.*

Now, consider the following classic model-theoretic theorems relating axiomatizability to ultraproducts, and ultraproducts to elementary equivalence:

Theorem 2.2.1: [Hod98, Corollary 8.5.13] *$\mathbb{K}$ is axiomatizable if and only if $\mathbb{K}$ is closed under ultraproducts, and ultraroots.*[5,6]

Theorem 2.2.2: [Sh71](Keisler-Shelah Theorem) *Two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if and only if there is an ultrafilter $\mathcal{U}$ on a set I such that $\mathcal{A}^{\mathcal{U}} \cong \mathcal{B}^{\mathcal{U}}$.*[7]

These two can be combined to give us the following corollary:

Corollary 2.2.1: *If $\mathbb{K}$ is a class of $\mathcal{L}$-structures, then $\mathcal{N} \models T^{\mathbb{K}}$ if and only if $\mathcal{N}$ is elementarily equivalent to an ultraproduct of elements of $\mathbb{K}$.*

Example 2.2.2 (Pseudofinite Fields)**:**

In [Ax68], Ax proves that the theory of finite fields is decidable, and in particular, recursively axiomatizable. More precisely, he shows that the first-order theory of finite fields - all those statements true in *any* finite field - can be axiomatized by the following list (where $\mathcal{F}$ is a proposed model of the axioms):

- $\mathcal{F}$ is a field.
- $\mathcal{F}$ is perfect.[8]
- $\mathcal{F}$ is either finite, or pseudoalgebraically closed.[9]

---

[3]This is equivalent to saying that $\mathbb{K}$ is an *elementary class.*

[4]This is equivalent to saying that $\mathbb{K}$ is a *basic elementary class.*

[5]$\mathbb{K}$ is *closed under ultraroots* if whenever there is an ultrafilter $\mathcal{U}$ on a set I and $\mathcal{A}^{\mathcal{U}} \in \mathbb{K}$, then $\mathcal{A}$ is in $\mathbb{K}$ as well.

[6]Note also that our classes $\mathbb{K}$ will be assumed to be closed under isomorphism.

[7]That is, there is an ultrafilter, so that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic ultrapowers with respect to $\mathcal{U}$.

[8]There are many equivalent characterizations. A couple definitions are: "Every algebraic extension of $\mathcal{F}$ is separable", and "The separable closure is algebraically closed". But an equivalent definition with an obvious first-order schema is that "the Frobenius map is always onto", or in other words, either an element is 0, or has a p^{th} root for every p. That is, $F_{p \in \mathbb{N}} \,:\, (p = 0) \to (\forall y\ \exists x\ x^p = y)$

[9]Pseudoalgebraically closed means that every (absolutely irreducible) variety defined in a model $\mathcal{F}$ has an $\mathcal{F}$-valued point. See [Ax68, Lemma 1, §7] for first-order statements $B_{D,M,N}$ for $D, M, N \in \mathbb{N}$ such that $\mathcal{F}$ is pseudoalgebraically closed if and only if $\mathcal{F} \models B_{D,M,N}$ for all D, M, N. Thus, this axiom schema would say that $|\mathcal{F}| \leq N \vee \mathcal{F} \models B_{D,M,N}$.

- $\mathcal{F}$ has exactly one algebraic extension of each degree.[10]

This shows that the the perfect, pseudo-algebraically closed fields with one extension of each degree, are precisely the infinite models of the theory of finite fields. In the language of Definition 2.2.2 and Corollary 2.2.1, letting $\mathbb{K} = \{\mathcal{F} \mid \mathcal{F} \text{ a finite field}\}$, we can see that the theory $\mathrm{T}^{\mathbb{K}}$ is, by Ax's result, axiomatized by the above list of axioms. Thus, by Corollary 2.2.1, a field $\mathcal{F}$ is elementarily equivalent to an ultraproduct of finite fields, or is a *pseudofinite field*, if and only if it is a model of the above theory.

This is a complete first-order characterization of what it means to be an ultraproduct of finite fields, and at least *prima facie* seems quite surprising. And it might lead us to hope for an axiomatization of the class of o-minimal structures. In the same vein, the following is another, though perhaps less surprising example of this same phenomenon:

Example 2.2.3 (Pseudofinite Orderings)**:**

Recall from the proof of Lemma 1.6.3 that the class of *pseudofinite orderings* are the orderings which are elementarily eqiuvalent to an ultraproduct of finite orderings. They can be characterized as exactly the models of the theory of linear orderings which are finite, or of the form $\omega + L \cdot \mathbb{Z} + \omega^*$.[11] In turn, this theory (see [Va01]) can be axiomatized exactly by the following axioms (where $(\mathcal{D}, <)$ is a proposed model of the axioms):

- $\mathcal{D}$ is a linear ordering.[12]
- $\mathcal{D}$ has a first and last element.[13]
- $\mathcal{D}$ is discrete.[14]

Thus, with $\mathbb{K}$ the class of finite linear orderings, we have that the theory $\mathrm{T}^{\mathbb{K}}$, the theory of finite orderings, is exactly axiomatized by a recursive list of axioms (in fact, it is finitely axiomatizable since we can take the conjunction of all the axioms listed).

These examples all lead us to the following question:

Question: *Which classes $\mathbb{K}$ have recursively axiomatizable theories, $\mathrm{T}^{\mathbb{K}}$? That is, for which classes $\mathbb{K}$, are models elementarily equivalent to an ultraproduct of models from $\mathbb{K}$ reducible to models of a recursive theory?*

[10] See [Ax67, Theorem 5, §4] for a statement C_n, for $n \in \mathbb{N}$ such that the field $\mathcal{F}$ has precisely one algebaic extension of degree n if and only if $\mathcal{F} \vDash C_n$.

[11] Where $\overline{\omega}$ is a reverse ordered copy of ω.

[12] That is, $\forall x, y, z\ (x \nless x \wedge (x < y \rightarrow y \nless x) \wedge [(x < y \wedge y < z) \rightarrow x < z])$.

[13] i.e. the statement $\exists! x\ \forall y\ (x < y \vee x = y)$, and the similar statement for a greatest element. We will refer to these unique elements as $\lambda\mathcal{D}$ and $\gamma\mathcal{D}$.

[14] i.e. $\forall x \neq \gamma\mathcal{D}\ \exists y > x\ \forall z > x\ (z > y \vee z = y)$, to say that each element except the last has an immediate successor, and $\forall x \neq \lambda\mathcal{D}\ \exists y < x\ \forall z < x\ (z < y \vee z = y)$ to stipulate immediate predecessors for every element except the first.

2.3 Purported Axiomatizations of O-Minimality

In [Fo10] and [Sch12], two proposals are made for possible axiomatizations of the first-order theory of o-minimality. That is, for axiomatizations that would reduce ultraproducts of o-minimal structures to models of a recursive theory.

For a fixed language L, we will write $T_{\mathcal{L}}^{\text{o-min}}$ for the theory

$$\bigcap_{\mathcal{M}\text{ o-minimal}} \mathrm{Th}_{\mathcal{L}}(\mathcal{M})$$

i.e. the set of $\mathcal{L}$-sentences true in *all* o-minimal $\mathcal{L}$-structures. We will drop '$\mathcal{L}$' in general, as it should be clear from context. In this chapter we will assume that $\mathcal{L}$ expands the language of real closed fields, and that the theories we discuss all expand RCF. This of course, will leave an open problem; we mention this in the final subsection.

A first proposal is that of the axioms for fields together with the schemas of local o-minimality (LOM) and definable completeness (DC).[15] Since LOM, DC, and the field axioms are all first-order statements, we know that since they are true in ultraproducts of o-minimal structures (see Lemma 1.5.9), these schemas must be in, or implied by any purported axiomatization.

Part of the intuition behind the suggestion that LOM+DC is enough to axiomatize $T^{\text{o-min}}$ its that there are no LOM+DC fields in the literature which are not ultraproducts of o-minimal structures (or actually o-minimal). For this reason, it has been difficult to see how these two notions could come apart.

Another proposal for an axiomatization considers the addition of a further axiom schema.

Definition 2.3.1: *The* ***Discrete Pigeonhole Principle*** *(dPHP) is the statement that for any (definable) discrete set in a* LOM+DC *structure, and any definable selfmap of that set, if it is injective, it is surjective as well.*

This of course, generalizes the ordinary finite Pigeonhole Principle, which is the same statement, but for finite sets. It can be written as an axiom schema as follows:

$$(\text{dPHP})_{\varphi,\psi\in\mathcal{L}} : [\varphi \text{ is discrete } \wedge\ \psi \text{ defines an injective selfmap on } \varphi] \rightarrow \psi \text{ is surjective}$$

[15]By slightly strengthening the axiom schema (*LOM*) to hold 'at infinity' (i.e. for all definable sets X, LOM holds, but also, there are infinite intervals at $+\infty$ and $-\infty$ which intersect X in a finite union of points and intervals), we can eliminate the assumption of a field (and instead only work with a group). That is, instead of appealing to the field structure to obtain the boundedness of discrete sets, we could have strengthened the axiom LOM. Schoutens make this slightly more general proposal. He refers to the extended definition of LOM as type completeness (see [Sch12]). All that being said, his proposal will certainly fail if my slightly less general proposal fails.

And this can be rewritten as first-order schema as:

$$(\text{dPHP})_{\varphi,\psi\in\mathcal{L}} \ : \ \forall c,d \Bigg([\forall x \ (\varphi(x,c) \rightarrow \exists r > 0 \ \exists! z \ (z \in B_r(x) \wedge \varphi(z))] \wedge$$

$$[\forall x \ (\varphi(x,c) \rightarrow \exists! y \ (\psi(x,y,d))] \ \wedge \ [\forall x \neq x' \ ([\exists y, y' \ (\psi(x,y,d) \wedge \psi(x',y',d))] \rightarrow y \neq y')] \wedge$$

$$[\forall x,y \ (\psi(x,y,d) \rightarrow (\varphi(x,c) \wedge \varphi(y,c)))] \ \wedge \ [(\forall x,y \ (\psi(x,y,d) \rightarrow (\varphi(x,c) \wedge \varphi(y,c)))] \Bigg)$$

$$\longrightarrow \quad [\forall y \ (\varphi(y,c) \rightarrow \exists x \ (\varphi(x,c) \wedge \psi(x,y,d))]$$

Importantly, dPHP (i.e. the schema $(\text{dPHP})_{\varphi,\psi}$ for all $\varphi, \psi \in \mathcal{L}$) is true in *pseudo-o-minimal structures* (i.e. not only is it first-order, but it is true in o-minimal structures):

Lemma 2.3.1: *If $\mathcal{R}$ is pseudo-o-minimal, then $\mathcal{R} \vDash dPHP$.*

Proof. By the above, as dPHP is elementary, it is sufficient to prove the statement for $\mathcal{R}$ an ultraproduct of o-minimal structures. So suppose that we have a definable, discrete, closed and bounded set A, and a definable self-map $f : A \rightarrow A$ in $\mathcal{R} = \prod_{i\in I} \mathcal{R}_i/\mathcal{U}$, with $\mathcal{R}_i$ o-minimal. By Lemma 1.5.8, we know that A is pseudofinite, so we may write it as $A = \prod_{i\in I} A_i/\mathcal{U}$, and thus, $\Gamma(f) = \prod_{i\in I} \Gamma(f_i)/\mathcal{U}$. Thus, in $\mathcal{U}$-most index models, A_i is a finite set, and f_i is an injective self-map of A_i. But then by the ordinary Pigeonhole Principle, $\mathcal{U}$-most f_i are surjective, so f is too. ■

It is still open whether or not dPHP follows from LOM+DC or not (that is, whether or not any LOM+DC field must also satisfy dPHP) though it is clear that any purported axiomatization of o-minimality must either contain or imply dPHP by the above Lemma. Schoutens [Sch12] remarks that adding the dPHP may be necessary, and that it would of course be necessary if it were not a consequence of LOM+DC.

We will show in Section 2.4, that not only do these proposals fail, but any recursive axiomatization will fail. That is, no (recursive) list of axioms could possibly capture all the consequences of being o-minimal.

Recall (from Theorem 1.5.1) that o-minimality is an 'elementary' property in the sense of being preserved under elementary equivalence. But notice that the first-order $\mathcal{L}$-theory of o-minimality being axiomatizable is a very different statement: it would say that the set of sentences which are true in *all* o-minimal $\mathcal{L}$-structures is (recursively) axiomatizable. Being preserved by elementary equivalence is similar but weaker: as Example 1.5.3 shows, there are certainly structures elementarily equivalent to an ultraproduct of o-minimal sructures which are not o-minimal. We reiterate this, as unfortunately, there is a bit of a terminological overlap in the literature: sometimes authors use 'first-order' synonymously with 'elementary'; here we have made sure to split them apart.

2.4 The Non-Axiomatizability of O-Minimality

In this subsection, we will demonstrate that there is no recursive axiomatization of o-minimality. For the entirety, let Λ be some recursive list of sentences in a language $\mathcal{L}$ extending $\mathcal{L}_{RCF}(N)$, the language of real closed fields extended by a new unary predicate, such that every axiom $\varphi \in \Lambda$ is true in all ultraproducts of o-minimal $\mathcal{L}$-structures, and that in Λ, N is asserted to be discrete. That is, Λ is a purported (recursive) axiomatization of o-minimality, in which, without loss of generality, N is interpreted as a unary discrete set.[16] Finally, before stating the theorem, we need a simple definition:

Definition 2.4.1: *Let $\varphi(z)$ be a one-variable formula in $\mathcal{L}(N, \alpha, \mu)$ (where N is a new unary predicate, and α and μ are new ternary predicates). We fix a new variable x not occuring in φ, and define $\varphi_{\leq x}(z)$ as follows:*

Whenever "$N(t)$" appears in φ, for some $\mathcal{L}$-term t, replace it with "$N(t) \wedge t \leq x$".[17]

Also, let PA be the *relational* theory of Peano Arithmetic. That is, we consider PA in a language with relation symbols (only) for addition, multiplication and ordering. Note that this is essentially equivalent to the usual theory PA in a language with function symbols for addition, multiplication, and successor, in that a model of relational PA, can be definitionally expanded to a model of ordinary PA, and vice versa. The main difference is that a substructure of a model of relational PA can be finite.

Theorem 2.4.1: *There are LOM+DC+Λ structures which are not pseudo-o-minimal. Hence LOM+DC+Λ is not an axiomatization of o-minimality.*

Corollary 2.4.1: *There are LOM+DC+Λ' structures which are not pseudo-o-minimal, for Λ' any recursive list of axioms in any language $\mathcal{L}'$ extending $\mathcal{L}_{RCF}$ by at least one new predicate or function symbol which is consistent with $\mathrm{T}^{\text{o-min}}_{\mathcal{L}'} \cup \mathrm{RCF}$.*

The corollary follows from the theorem by noting that if $\mathcal{L}'$ extends $\mathcal{L}_{RCF}$ by any new predicate or function symbol, then we can expand a model of one of the counterexamples constructed in the proof of the theorem to a counterexample for Λ'.

Proof. Extend $\mathcal{L}$ by two ternary predicate symbols α and μ, and let T be the $\mathcal{L}$-theory consisting of the following informally stated, but nonetheless first-order axiom schemas (where $(R, +, \times, <, 0, 1, N, \alpha, \mu, ...)$ is a model of the axioms):

(I) $\overline{R} := (R, +, \times, <, 0, 1) \models RCF$

(II) $(N, \alpha, \mu, < \restriction N, 0, 1) \models PA$

[16] Any non-o-minimal model of the $\mathcal{L}'$-theory of o-minimality for some $\mathcal{L}'$ extending the language of real closed fields will define some new unary discrete set, so models of $\mathcal{L}'$ can be definitionally expanded to a model of a language $\mathcal{L}'(N)$. Thus, our assumption involves no real loss of generality.

[17] Note that in particular, t is forced to be non-negative.

(III) $\alpha = + \restriction N$ and $\mu = \times \restriction N$

(IV) $(\mathbf{LOM+DC})_{\varphi(z)\in\mathcal{L}} : (\overline{R}, N, \alpha, \mu) \models \forall x > 0$ "if $\varphi_{\leq x}(\mathcal{R})$ is a proper, non-empty subset of $\mathcal{R}$, then $\mathrm{bd}(\varphi_{\leq x}(\mathcal{R}))$ is discrete, closed and bounded and has a least and a greatest element (possibly identical)"

(V) $(\mathbf{\Lambda})_{\psi\in\Lambda} : (\overline{R}, N, \alpha, \mu) \models \forall x\ \psi_{\leq x}$

Unpacking this a bit, (I) contains all of the standard axioms for real closed fields. (II) states that all of the axioms of PA hold where all the quantifiers are relativized to the predicate N (i.e. instead of "$\forall x\ (...)$", we have "$\forall x\ (N(x) \rightarrow ...)$" etc.), and where "+" is replaced by α, and "×" is replaced by μ. (III) asserts that α and μ are strictly subsets of N^3. (IV) ensures that when the model N of PA is restricted to any initial segment, the set defined by φ in R with this initial segment of N has a discrete, closed and bounded boundary with (if non-empty) least and greatest elements. And finally, (V) forces every axiom in Λ to hold in R with N again restricted to any fixed initial segment.

We will show that not only does T have a model, but we will then show that there is a model of T with a reduct that satisfies LOM+DC+RCF+Λ but which could not possibly be an ultraproduct of o-minimal structures. **Note: we are not claiming that any model of T is LOM+DC: we will obtain LOM+DC structures from certain reducts of models of T.**

In order to accomplish this, we first note that T is consistent. To see this, observe that the real field, $\overline{\mathbb{R}} = (\mathbb{R}, +, \times, <, 0, 1)$ with an added predicate for $\mathbb{N}$ together with its usual addition and multiplication is a model of T. for (I), (II) and (III), this is clear; now for (IV) and (V) we can see that for any $r \in \mathbb{R}$, $\mathbb{N}_{\leq r}$ is a *finite* initial segment of $\mathbb{N}$, so this subset was definable inside $\overline{\mathbb{R}}$ already; indeed, the partial ternary subsets corresponding to restricted multiplication and addition (i.e. the interpretations of μ and α respectively) on this initial segment could already be defined in $\overline{\mathbb{R}}$ by the graphs of the restricted multiplication and addition functions on this initial segment. Thus, $(\overline{\mathbb{R}}, \mathbb{N}_{\leq r})$ is just a *definitional expansion*[18] of $\overline{\mathbb{R}}$. Now, since $\overline{\mathbb{R}}$ is o-minimal, $(\overline{\mathbb{R}}, \mathbb{N}_{\leq r})$ satisfies all the first-order consquences of o-minimality; in particular, it satisfies LOM+DC+RCF+Λ. To derive this consequence, we note that in the side case where $\varphi(x) \in \mathcal{L}(x)$ does not have $N(t)$ occuring for any term t, then $\varphi_{\leq x}((\overline{\mathbb{R}}, N)) = \varphi((\overline{\mathbb{R}}, N)) = \varphi(\overline{\mathbb{R}})$, and since Λ is a purported axiomatization of o-minimality for $\mathcal{L}$, any such statement would be true in all o-minimal $\mathcal{L}$-structures. In particular, the latter must be a finite union of points and intervals in $\mathcal{R}$ by o-minimality, and so have finite boundary (and in particular, have discrete, closed, bounded boundary with least and greatest element). Thus, $(\overline{\mathbb{R}}, \mathbb{N}) \models$ T, so the theory T is consistent.

[18]If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$ and $\mathcal{L}'$ structures respectively, which have equal underlying sets, and any definable set in $\mathcal{M}$ is definable in $\mathcal{N}$ and vice versa, then we say that $\mathcal{M}$ and $\mathcal{N}$ are *definitionally equivalent*, or that one is a *definitional expansion* of the other.

However, since a model of T interprets a model of PA (in fact, it *defines* one), Corollary 1.8.1 applies; so we can conclude that $T + \neg\mathrm{Con}(T)$ is also consistent. That is, there is a model $(\mathcal{R}, \mathcal{N})$ of $T + \neg\mathrm{Con}(T)$. In particular, $\mathcal{N}$ has a code, say the element $\alpha \in \mathcal{N}$, for a proof of $\neg\mathrm{Con}(T)$.

But then, letting $\alpha < x \in \mathcal{R}$ be sufficiently large, we have that the code for the proof of $\neg\mathrm{Con}(T)$ and the codes for any symbols occuring in its proof are contained in $\mathcal{N}_{\leq x}$. Since $(\mathcal{R}, \mathcal{N})$ satisfies (IV), we have that in $(\mathcal{R}, \mathcal{N}_{\leq x})$, the boundary of every non-empty definable proper subset of the line is discrete closed and bounded, and (if non-empty) has a least and greatest element; thus by Lemma 1.6.4, $(\mathcal{R}, \mathcal{N}_{\leq x})$ satisfies LOM+DC. And since $(\mathcal{R}, \mathcal{N})$ satisfies (V), $(\mathcal{R}, \mathcal{N}_{\leq x})$ satisfies Λ. Finally, since $\mathcal{N}_{\leq x}$ is an initial segment of a model $\mathcal{N}$ of PA with bounded portions of addition and multiplication, it is a Δ_0-elementary substructure of $\mathcal{N}$ (by Lemma 1.8.1). Since $\alpha < x$ and x is sufficiently large in $\mathcal{N}$, and since α being a code for a proof of $0 = 1$ in T is a Δ_0-property of $\alpha \in \mathcal{N}_{\leq x}$, we have that $\mathcal{N}_{\leq x} \models \neg\mathrm{Con}(T)$.

But $(\mathcal{R}, \mathcal{N}_{\leq x})$ could not possibly be pseudo-o-minimal. Suppose for contradiction that it was elementarily equivalent to an ultraproduct of o-minimal structures:

$$(\mathcal{R}, \mathcal{N}_{\leq x}) \equiv (\mathcal{S}, \mathcal{M}) = \prod_{i \in I} (\mathcal{S}_i, \mathcal{M}_i)/\mathcal{U}$$

with $\mathcal{U}$ a non-principal ultrafilter on I, and $\mathcal{U}$-most $(\mathcal{S}_i, \mathcal{M}_i)$ o-minimal. Then we would have $(\mathcal{S}, \mathcal{M}) \models \neg\mathrm{Con}(T)$ by elementary equivalence. But since $\mathcal{M}$ is discrete, $\mathcal{U}$-most of the sets $\mathcal{M}_i$ must be discrete by Los' Theorem. And since $\mathcal{U}$-most index models $(\mathcal{S}_i, \mathcal{M}_i)$ are o-minimal, $\mathcal{U}$-most $\mathcal{M}_i$ must then be finite, being discrete and definable (see Lemma 1.5.7). Since $\mathcal{N}_{\leq x}$ is an initial segment of a model of PA, so is $\mathcal{M}$. Since $\mathcal{U}$-most of the $\mathcal{M}_i$ are finite, $\mathcal{U}$-most of them are finite initial segments; but being a finite initial segment is the same as being isomorphic to the structure N_n consisting, for some $n \in \mathbb{N}$, of the first n elements of $\mathbb{N}$, together with the graphs of addition, multiplication, and ordering restricted to this set. That is, $\mathcal{U}$-most $\mathcal{M}_i$ are isomorphic, for some n_i to the structure $N_{n_i} = (\{0, 1, ..., n_i\}, \alpha \upharpoonright N_{n_i}, \mu \upharpoonright N_{n_i}, < \upharpoonright N_{n_i})$.

Finally, $\mathcal{M} \models \neg\mathrm{Con}(T)$, so there is $\alpha \in \mathcal{M}$ such that α is a code for a proof of $0 = 1$ in T. But then for an index i such that $(\mathcal{S}_i, \mathcal{M}_i)$ is o-minimal, and $\mathcal{M}_i$ is isomorphic to some N_{n_i} as above, then the i-th coordinate of α, i.e. the element $\alpha_i \in \mathcal{M}_i$, must be a code for a proof of $0 = 1$ in T as well. But this implies that there is a *standard* code for the proof of $\neg\mathrm{Con}(T)$ (by the same observation we made after Lemma 1.8.1). From the existence of a standard code for a proof we could recover an actual proof of $\neg\mathrm{Con}(T)$. Hence, T would actually be inconsistent, a contradiction. ∎

2.4.1 An Open Problem

As we mentioned when we introduced $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$, we have left an open problem: if $\mathcal{L}$ does not expand the language of real closed fields, we do not know the answer to whether or not $\mathrm{T}_{\mathcal{L}}^{\text{o-min}}$ is recursively axiomatizable. It is possible that without the field structure being around to induce (an initial segment of) the ordinary semiring structure of a model of PA onto a definable subset, that LOM+DC(+dPHP) might be enough to axiomatize o-minimality.

Chapter 3

Tallies and Uniform Pseudofiniteness

3.1 Introduction

Cell decomposition in the *o-minimal* setting is a generalization of the definition of o-minimality to higher dimensions. Recall that it tells us that we can obtain, for any definable set(s), a decomposition, or partition of the ambient space which is *compatible* with the given set(s), and which consists of a finite disjoint union of cells.[1] A fundamentally important step along the way to proving the o-minimal cell decomposition theorem is proving *uniform finiteness* (see Part 1 of Theorem 1.4.3). Though as we mentioned, it goes hand-in-hand with proving Cell Decomposition, it has also been shown to imply o-minimality or o-minimal open core together with other tameness conditions, and can be characterized as an 'elimination of the $\exists^\infty x$ quantifier'. That is, it can be seen as a strong and important component of o-minimality itself.[2] See [DMS09] for more on this.

In this chapter we will be concerned with proving a version of uniform finiteness for LOM+DC+dPHP, *though non-o-minimal*, expansions of real closed fields. Some statements may only hold for pseudo-o-minimal structures or for ultra-o-minimal structures, and some results are digressions from the main track, but the goal will be to generalize uniform finiteness in a reasonable way to the pseudo-o-minimal setting.

We will only, in the end, prove the statement of what we will call *Uniform Pseudofiniteness* in the planar case. (Though we will have an easy general-dimension version in a slightly weaker setting.) We note here that it is likely the case that as in the o-minimal setting, a proof of the higher ambient dimensional case would require that we simultaneously proved a cell decomposition result.[3]

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[1]See Theorem 1.4.1 for the full statement.

[2]Digressing slightly, I believe (though I am not willing to formally conjecture it at this point) that a version of Uniform Finiteness in the LOM+DC+dPHP case is analogously quite strong in this same sense, and could thus form part of a *neccessarily non-recursive* axiomatization of o-minimality, in the sense of implying pseudo-o-minimality with some additional tameness results.

[3]We are willing to conjecture at this point, and should be able to work out soon, that there will be such a full-blown Uniform Pseudofiniteness and Cell Decomposition result in the context we will work in.

3.2 Definable Isomorphisms and Ordermorphisms

3.2.1 Definitions and Basic Facts

Fix an expansion of a real closed field $\mathcal{R}$ such that $\mathcal{R} \models$ LOM+DC+dPHP. In particular, everything that follows holds for pseudo-o-minimal structures.

For $X \subsetneq \mathcal{R}$ discrete, *infinite* and definable, we use the following notation:[4]

$$X_{\text{ini}} \coloneqq \{x \in X \mid x = s_X^n(\lambda X) \text{ for some } n \in \mathbb{N}\}$$

$$X_{\text{fin}} \coloneqq \{x \in X \mid x = p_X^n(\gamma X) \text{ for some } n \in \mathbb{N}\}$$

$$X_{\text{mid}} \coloneqq X - (X_{\text{ini}} \sqcup X_{\text{fin}})$$

Thus, $X_{\text{ini}} < X_{\text{mid}} < X_{\text{fin}}$ and $X = X_{\text{ini}} \sqcup X_{\text{mid}} \sqcup X_{\text{fin}}$. **Note that for *any* (infinite) definable $X \subsetneq \mathcal{R}$ these three sets are *not* definable in any LOM+DC expansion of $\mathcal{R}$.**

Definition 3.2.1: *Let $X, Y \subseteq \mathcal{R}$ be two definable sets.*

- *X and Y are **definably isomorphic**,[5] and we write $[X] = [Y]$ if there is a definable bijection between X and Y. The relation $[X] = [Y]$ is an equivalence relation on* Def($\mathcal{R}$).
- *If X and Y are discrete, we say that X and Y are **definably ordermorphic**, and write $[X]_< = [Y]_<$, if there is a definable (lexicographic) order-preserving bijection between X and Y. The relation $[X]_< = [Y]_<$ is an equivalence relation on* $\mathrm{Def}_0(\mathcal{R})$.[6]
- *We define an ordering on the $[\cdot]_<$-classes, by defining $[X]_< \leq [Y]_<$ iff there is a definable, order-preserving bijection from X onto a (not necessarily proper) initial segment of Y.*

Definable isomorphism and ordermorphism classes of infinite discrete sets can be seen as generalizations of the finite cardinalities of finite sets. (This theme is elaborated on in [Sch12] in the o-minimal setting, and in general in [KS00], and [Kr04].) In particular, it is easy to show the following basic facts and analogues of statements about finite sets:

Fact 3.2.1:

[4] Below, “ini”, “mid”, and “fin” stand for ‘initial’, ‘middle’, and ‘final’ respectively.

[5] This terminology is perhaps unfortunate in that a definable *isomorphism* might be expected to somehow preserve all the structure on the definable set. The terminology comes from [KS00], and we think it is at least preferrable to the terminology of *definable equivalence* in [vdD98-a].

[6] Recall that this is the class of zero-dimensional definable sets (i.e. discrete sets in any ambient dimension).

(a) $[\{x\}] = [\{x'\}]$ *and* $[\{x\}]_< = [\{x'\}]_<$ *for any elements* $x, x' \in \mathcal{R}$*, and similarly for any two finite sets of the same cardinality. In general, the* $[\cdot]$*-class of a singleton is exactly the set* $\{\{r\} \mid r \in \mathcal{R}\}$*.*[7]

(b) For any two finite sets X *and* Y*, if* $|X| < |Y|$*, then* $[X]_< < [Y]_<$*.*

(c) $[X]_< < [Y]_<$ *for any non-empty discrete sets* X *and* Y *with* X *finite and* Y *infinite.*

(d) $[X]_<, [Y]_< < [X \sqcup Y]_<$ *for any two disjoint, non-empty discrete sets* X *and* Y*.*

Part (b) is a result of the following: in first-order logic, having n or less elements is definable via the formula $\exists^{\leq n} x\varphi(x)$ for any given formula $\varphi(x)$; using that together with the ordering, we can then define the i^{th} element of a given finite set (according to this ordering), for any $1 \leq i \leq |X|$. This is one of the key properties of finite cardinalities that we will have in mind in later sections in this chapter.

We note that that there are always going to be more definable bijections than *order-preserving* definable bijections (i.e. ordermorphisms). The proposition that follows below clarifies this gap, in that it shows that a definable ordermorphism between two sets (or onto an initial segment) is *always* unique.[8] In some sense, this relationship is similar to the relationship between cardinals and ordinals: there are many, many ordinals of any infinite cardinality (there is a wealth of ordinals, for example, below the ordinal ϵ_0, which is itself just the beginning of the ordinals of size $\aleph_0$.) Because there are more definable isomorphisms, there are fewer $[\cdot]$-classes than $[\cdot]_<$-classes, so the ordinals are playing the role of the $[\cdot]$-classes in this analogy.[9] First, a lemma:

Lemma 3.2.1: *Let* X *and* Y *be infinite discrete definable subsets of* $\mathcal{R}$*. If* $f : X \to Y$ *is a definable ordermorphism, then the following are true:*

(a) f^{-1} *is also a (definable) ordermorphism.*

(b) $f(\lambda X) = \lambda Y, f(\gamma X) = \gamma Y$*.*

(c) $\forall x \in X, \quad x \neq \gamma X \Rightarrow f(s_X(x)) = s_Y(f(x)) \quad$ *and* $\quad x \neq \lambda X \Rightarrow f(p_X(x)) = p_Y(f(x))$*.*

Proof.

(a) is clear.

(b) If $f(\lambda X) > \lambda Y$, then $\lambda X > f^{-1}(\lambda Y)$. But $f^{-1}(\lambda Y) \in X$, so this is an element less than the least element of X, a contradiction. The preservation of greatest elements follows similarly.

[7] It is worth noting here that this fact holds without the use of parameters by using the ordering. It is not true in general structures, without parameters, that two definable finite sets of the same cardinality are in definable bijection.

[8] Schoutens shows a slightly different statement in [Sch12]: if f is a monotone definable self-map of a set X, then it is either the identity or an involution.

[9] We will not be keeping this analogy in mind in what follows, but we thought it worth mentioning.

(c) Let $x \in X - \{\gamma X\}$. Suppose that $f(s_X(x)) \neq s_Y(f(x))$, and in particular, without loss of generality, that $f(s_X(x)) > s_Y(f(x))$. Let $x^* = f^{-1}(s_Y(f(x)))$.

Then obviously $x^* \neq x$, and since $f(x^*) = s_Y(f(x))$, we have that $x^* \neq s_X(x)$. Now either $x^* < x$, or $x^* > s_X(x)$ (if such an element exists). In the first case, $f(x^*) = s_Y(f(x)) < f(x)$, a contradiction. And in the second case, $x^* > s_X(x) > x$, so $f(x^*) = s_Y(f(x)) > f(s_X(x))$, contradicting our assumption.

The second statement follows similarly. ■

Proposition 3.2.1: *Let $X, Y \subsetneq \mathcal{R}$ be infinite discrete definable sets.*

(a) The identity map on X is the unique *definable orderautomorphism of X.*

(b) If $[X]_< \leq [Y]_<$, then there is a unique *ordermorphism $f : X \to I \subseteq Y$, with I an initial segment of Y.*

(c) If $[X]_< = [Y]_<$, then there is a unique *ordermorphism $f : X \to Y$.*

Proof.

(a) Since id_X is a definable orderautomorphism, we will show that if $f : X \to X$ is another definable orderautomorphism, then $f = id_X$. So suppose that they differ, and let $\{x \in X \mid id_X(x) \neq f(x)\}$. This is a discrete definable subset of X, and hence has a least element, λ. Then by Lemma 3.2.1, part (a), we know that $\lambda \neq \lambda X$, so it has a predecessor. Thus, since λ is minimal, we have that $p_X(\lambda) = id_X(p_X(\lambda)) = p_X(\lambda) = f(p_X(\lambda))$. But by part (b) of the lemma, we then know that $f(s_X(p_X(\lambda))) = f(\lambda) = s_X(f(p_X(\lambda)) = s_X(p_X(\lambda)) = \lambda$. That is, $f(\lambda) = \lambda$, so $f = id_X$ at λ, contradicting the choice of λ as a place of disagreement between id_X and f. Thus, $f = id_X$.

(b) Suppose that f and g were both definable ordermorphisms from X onto intial segments, I and I' of Y, respectively. Since X is infinite, and f and g are bijections, I and I' are both infinite as well.

Now consider $g^{-1} \circ f : X \to I \to g^{-1}(I) \subseteq X$. It is a definable injective, order-preserving selfmap of X. By the dPHP, it is bijective, and thus, a definable ordermorphism. Part (a) implies then that it is the identity map, and thus that $g^{-1} \circ f = id_X$, which implies $f = g$.

Part (c) is similar. ■

3.2.2 Discrete Over- and Under-spill

One lesson to take away is that even in any expanded language (where $\mathcal{R}$ remains LOM +DC+dPHP), there is very little room to work with when defining *order-preserving* maps between discrete definable sets. And in the case of automorphisms, the proposition says that orderautomorphisms have to fix everything. This contrasts with the situation for definable automorphisms: given any infinite discrete definable set X, there are infinitely many definable automorphisms of X: for instance, one such infinite family is the family of shift maps $(\mathrm{Sh}^n_X)_{n\in\mathbb{N}}$ which are defined as the n-th iterate of the successor function on X, except on the last n elements, which they map to the initial n elements, in order. In general, there may be many such definable permutations of a discrete infinite set which are not order-preserving.

This contrasts with a potential naive intuition about these sets: that if we take X and define an automorphism which fixes X_{ini} and X_{fin}, then it must in fact, fix an initial segment and a final segment[10] of X_{mid}. That is, we have no room to patch together a map $id_{X_{\text{ini}}} \sqcup F \sqcup id_{X_{\text{fin}}}$ with $F \neq id_{X_{\text{mid}}}$. This phenomenon of properties holding on infinite initial or final segments of discrete sets spilling into the middle part of the set is an instance of what we call discrete overspill and underspill.

Fact 3.2.2 (Discrete Over- and Under-spill)**:** *Let $\varphi(x)$ be a one-variable formula, possibly with parameters, in the language of $\mathcal{R}$. Then if there is a definable subset of an infinite discrete definable subset X with the property that $\varphi(\mathcal{R})$ contains a subset Y which is cofinal in X_{mid}, then it contains a subset Z which is coinitial in X_{fin}. Furthermore, if Y is not only cofinal in X_{mid}, it contains a set $(X_{\text{mid}})_{>r}$ for some $r \in X_{\text{mid}}$, then Z is not only coinitial in X_{fin}, but contains a set $(X_{\text{fin}})_{<s}$ for some $s \in X_{\text{fin}}$.*

Proof. Consider the definable set $X^{\varphi} := \{x \mid x \in X \wedge \varphi(x)\}$. Since it is a definable subset of a discrete definable set, it is also discrete, and thus has a least and greatest element. In particular, since it contains a cofinal subset of X_{mid}, we have that $\gamma X^{\varphi} \in X_{\text{fin}}$. We can easily check that $\{p^n_X(\gamma X^{\varphi}) \mid n \in \mathbb{N}\} = (X^{\varphi})_{\text{fin}} \subsetneq X^{\varphi}$ is contained in X_{fin} and is a coinitial there.

The further strengthening is easy: if Y contains a set $(X_{\text{mid}})_{>r}$ and yet Z did not contain a set $(X_{\text{fin}})_{<s}$, then since $Y \cup Z \subseteq X^{\varphi}$, we have that $Y \cup Z$ would have no least element greater than r which was not in X^{φ}. That is, $[(X^{\varphi})^c \cap (Y \cup Z)]_{>r}$, a discrete definable set, would have no least element.

The situation for coinitiality is similar. ∎

In the next subsection, we present another application of this kind of reasoning to - at least partially - answer the question of whether dPHP follows from LOM+DC.

3.2.3 Order-preserving Discrete Pigeonhole Principle

In this subsection we show that a weakening of the dPHP follows from LOM+DC.

Definition 3.2.2: *The **Order-preserving Discrete Pigeonhole Principle*** (OPdPHP) *is the statement that for every discrete definable set $X \subseteq \mathcal{R}$, if f is a definable,* order-preserving *self-map of X, then it is a bijection.*[11]

Proposition 3.2.2: *If $\mathcal{R}$ is an LOM+DC expansion of a field, then $\mathcal{R}$ satisfies the OPdPHP.*

[10] An initial segment in the reverse order.

[11] Note that injectivity is implied by order-preservation, which puts the statement more obviously in line with other variations on the PHP.

Proof. Let f be as in the statement of the OPdPHP. By Proposition 3.2.1, if f were in fact bijective, then $f = id_X$. So it is enough to assume for contradiction that $f \neq id_X$. Thus, let $Z = \{x \in X \mid f(x) \neq x\}$.

Now, since f is order-preserving (i.e. $x \leq y$ iff $f(x) \leq f(y)$), we have in particular that $x \leq f(x)$ (take $x = y$ in the definition). Thus, $f(\gamma) = \gamma$, and $f(p_X(\gamma)) = p_X(\gamma)$, etc. That is, $f = id_X$ on X_{fin}, and thus by Discrete Underspill, $f = id_X$ on a final segment of X. But then let γ^* be the greatest element of Z. In particular, $f(\gamma^*) \neq \gamma^*$, so by order-preservation, $f(\gamma^*) > \gamma^*$, and thus, $f(\gamma^*) > \gamma$ (a contradiction) since for every $x > \gamma^*$, $f(x) = x$ (if $f(\gamma^*) = f(x) = x$ for some $x > \gamma^*$, this would contradict the injectivity of f).

Thus $f = id_X$, and in particular, f is bijective. ∎

3.2.4 Higher Dimensions

We can extend the notion of definable ordermorphism to definable discrete sets in higher dimensions: a definable discrete set $X \subsetneq \mathcal{R}^k$ can be decomposed into

$$X = \bigsqcup_{p \in \pi(X)} X_p$$

where π is the projection onto the first coordinate. So, we will define $[\cdot]_<$ inductively on the ambient k-dimensional sets in $\text{Def}_0(\mathcal{R})$ for $k \geq 2$. But first, we define the *collapse* of a set:

Definition 3.2.3: *Let $X \subsetneq \mathcal{R}^2$ be infinite, definable and discrete, and write $X = \bigsqcup_{q \in \pi(X)} X_q$. Define* ***the collapse of*** *X, written $\underline{\downarrow} X \subsetneq \mathcal{R}$, to be the set defined as follows:*

- *for each $q \in \pi(X)$, and $y \in X_q$, define $\mu(q, y) := (y - \lambda X_q) \in \mathcal{R}_{>0}$ if $|X_q| \neq 1$, and let $\mu(q) = 1$ otherwise. (μ is the height of y relative to the bottom of the fibre X_q.)*

- *If $|X_q| \neq 1$, let $\epsilon(q) := (\gamma X_q - \lambda X_q) \in \mathcal{R}_{>0}$. Otherwise, let $\epsilon(q) = 1$. (ϵ is the total height of the fibre X_q.)*

- *Define $\delta(q) := (s_{\pi(X)}(q) - q) \in \mathcal{R}_{>0}$ if $q \neq \gamma\pi(X)$. And let $\delta(\gamma\pi(X)) = 1$. (δ is the width between one fibre and the next.)*

- *Define $\underline{\downarrow} X_q$,* ***the collapse of the fibre*** *X_q, to be the set*

$$\underline{\downarrow} X_q := \left\{ x \in \mathcal{R} \mid \exists y \in X_q,\ x = q + \frac{\mu(q, y) \cdot \delta(q)}{2 \cdot \epsilon(q)} \right\}$$

Note that by definition, $\underline{\downarrow} X_q$ is a discrete definable set contained in the interval $[q, q + \frac{\delta(q)}{2}] = [q, q + \frac{s_{\pi(X)}(q)+q}{2}]$; that is, the left half of the interval $[q, s_{\pi(X)}(q)]$ (except if $q = \gamma\pi(X)$, in which case, it is contained in $[q, q + 1/2]$.)

Finally, define $\underline{\downarrow}X$ to be the set $\{x \in \mathcal{R} \mid \exists q \in \pi(X),\ x \in \underline{\downarrow}X_q\}$. The family $\underline{\downarrow}X_q$ is a definable family, so $\underline{\downarrow}X$ is a definable set.[12]

Now, suppose that X is an infinite discrete definable subset of $\mathcal{R}^3$, and define the collapse of X, $\underline{\downarrow}X$, by taking the collapse of the fibre X_q, for each $q \in \pi(X)$. This gives us a new family $(\underline{\downarrow}X_q)_{q\in\pi(X)} \subsetneq \mathcal{R}^2$. We then define $\underline{\downarrow}X$ to be the collapse of this set. Defining the collapse of a subset of higher dimension is done by repeatedly collapsing in the same way. Finally, we define $\underline{\downarrow}X = X$ for $X \subsetneq \mathcal{R}$ discrete and definable.

This gives us a way to compare ordermorphism classes of higher dimensional definable discrete sets: we define, for $X \subsetneq \mathcal{R}^k$ definable, $[X]_< := [\underline{\downarrow}X]_<$. In particular, it brings our defintions of $[\cdot]$ and $[\cdot]_<$ more in line: the former is an equivalence relation on all definable sets, and the latter is now equivalence relation on all discrete definable sets (instead of just those in one ambient dimension). Furthermore, it preserves the lexicographical ordering of X.

3.2.5 Potential Problems With Ordermorphism Classes

Definable ordermorphism (and definable isomorphism), are unfortunately, at least apparently too weak to have all of the properties we might expect from a notion of 'cardinality'. First we will present a couple of potential problems, then below we give examples that (could possibly) witness them:

Remark 3.2.1: *Let X,Y, and Y be infinite discrete, definable sets in $\mathcal{R}$ (which is LOM+DC+dPHP).*

(a) (Non-Linearity of Ordermorphism Classes) *Even though the ordering on $[\cdot]_<$-classes is a quasi-order, it may not be linear. That is, it is possible some $[\cdot]_<$-classes may simply be incomparable. This problem has a few restatements in terms of properties that an abstract 'cardinality' notion might be expected to respect:*

- *If $[X]_< = [Y]_<$ and $Y \subsetneq Z$, then it might not necessarily be the case that $[X]_< \leq [Z]_<$, or even $[Y]_< \leq [Z]_<$.*
- *In particular, if there is some definable injective map from X into a set Y, it is not necessarily the case that $[X]_< \leq [Y]_<$.*
- *And if $[X] = [Y]$ (i.e. there is a definable bijection between two sets), then it is not necessarily the case that $[X]_< = [Y]_<$.*

(b) (Potential Failure of Uniformization) *If X is an infinite discrete definable set in ambient dimension > 1, and thus of the form $(X_q)_{q\in\pi(X)}$, and if X has infinitely many infinite fibres, such that the fibres are pairwise ordermorphism-equivalent (i.e. $[X_p]_< = [X_q]_<$ for*

[12] Intuitively, the collapse of a fibre X_q shrinks the fibre down, and places it between q and the successor of q in $\pi(X)$. Then the collapse of a set X is just the union of the collapse of all its fibres.

all $p, q \in P$), then it is does not seem to be necessary that with $\lambda := \lambda\pi(X)$, we have $[X]_< = [\pi(X) \times X_\lambda]_<$. This is actually a problem with $[\cdot]$ as well: there may not necessarily be a way to uniformize from the existence of pairwise definable isomorphisms/ordermorphisms to the existence of a single map.

Below we record the main idea of an example to illustrate problem (a). Though it is inconclusive at this stage, it points to the kind of problem that *could* potentially occur, and which seems quite likely to occur. We will show in the next section that strengthening definable ordermorphisms to make them even more closely mimic the finite situation can solve issue (a) and guarantee the uniformization desired in (b). In fact, we will address (b) in this new context en route to the main theorem of this chapter.

Example 3.2.1: *Let $\mathcal{R}$ be a LOM+DC+dPHP structure, and suppose that $U \subsetneq \mathcal{R}^1$ is a discrete subset which is* not *already definable in $\mathcal{R}$, and in fact, has no $\mathcal{R}$-definable infinite subset. Furthermore, suppose that $(\mathcal{R}, U)$ is still LOM+DC+dPHP. Let*

$$V := \{v \in \mathcal{R} \mid \exists u \in U \quad v = (s_U(u) + u)/2 \ \vee \ v = \gamma U + 1\}.$$

Then V is clearly interdefinable with U in $\mathcal{R}$, and hence not definable in $\mathcal{R}$. Thus, $(\mathcal{R}, U, V)$ is a definitional extension of $(\mathcal{R}, U)$, and so also LOM+DC+dPHP. Finally, define $W = U \cup V = U \sqcup V$ in this structure.

Now suppose that $F : U \to W$ is an ordermorphism onto an initial segment I of W. Then clearly, the set I would have to be a proper subset of W, and in fact, we would have that $F^{-1} \restriction U \cap I$ mapped U onto a subset U_{odd} of U satisfying $U_{\text{ini}} \cap U_{odd} = \{s_U^{2n}(\lambda U) \mid n \in \mathbb{N}\}$. Similarly, $F^{-1} \restriction V \cap I$ would map V onto a subset U_{even} satisfying $U_{\text{ini}} \cap U_{even} = \{s_U^{2n+1}(\lambda U) \mid n \in \mathbb{N}\}$.[13] Clearly in this situation, some such U_{odd} or U_{even} must be definable if F is, and such a set could not already be definable in $\mathcal{R}$, by choice of U. It seems very likely, though out of the grasp of our current tools, that these sets cannot be definable in general.

It is certainly not always the case that a set U_{odd} (or U_{even}) will fail to be definable when U is definable. For example, if U is an end extension of the set $\mathbb{N}$ in $\mathcal{R}$, then we can define U_{even} by $\{u \in U \mid u/2 \in U\}$ (and similarly with U_{odd}, and many other subsets of U). However, this does not seem likely to be the situation in general.

3.3 Tallies

Now, we specialize the notion of definable ordermorphism that we introduced in the previous section.

Definition 3.3.1: *Let $\mathcal{K}$ be an ordered field.*

[13] We are thinking of λU as the 1^{st} element of U instead of as the 0^{th}.

- *If $\mathcal{Z}$ is a subring such that every element of $\mathcal{K}$ has difference in absolute value at most 1 from a unique element of $\mathcal{Z}$, then $\mathcal{Z}$ is called an* ***integer part*** *of $\mathcal{K}$. (Note in particular that 0 and 1 are in $\mathcal{Z}$, that $z \in \mathcal{Z}$ if and only if $z \pm 1 \in \mathcal{Z}$, and thus that $\mathbb{Z}$ is a subring of every integer part.)*

- *We call a subset of $\mathcal{K}$ of the form $\mathcal{N} = \mathcal{Z}_{>0}$, where $\mathcal{Z}$ is an integer part, a* ***(positive) natural numbers part for*** *$\mathcal{K}$ or* ***nn-part for*** *$\mathcal{K}$.*[14]

Morgues and Ressayre showed in [MR93] that every real closed field has an integer part, so in particular, every real closed field has an nn-part as well. Note also that $\mathcal{N}$ is an nn-part for a real closed field iff $\mathcal{N} \models I\Delta_0$ (see [Shep64] for more on this.)

Definition 3.3.2: *We say that I is an* ***interval initial segment*** *of an nn-part $\mathcal{N}$ if it is the form $\mathcal{N}_{\leq N}$ for $N \in \mathcal{N}$.*[15]

For the rest of this section, let $\mathcal{R}$ be a LOM+DC+dPHP expansion of a real closed field, as in previous sections.

Definition 3.3.3: *Let $X \subsetneq \mathcal{R}^k$ be a discrete definable subset.*

- *We say that $\mathcal{R}$* ***tallies*** *X if $[X]_< = [I]_<$ for some interval initial segment $I = \mathcal{N}_{\leq N}$ of some nn-part $\mathcal{N}$ of $\mathcal{R}$, where $N \in \mathcal{N}$. Note in particular, that we are asking that I be definable.*

- *We call N* ***the tally of*** *X.*[16]

- *For any $x \in X$, the image of x under a tallymorphism for X is called* ***the tally of*** *x* ***(in*** *X***).**

- *An ordermorphism $f : X \to Y$, for X and Y discrete definable sets, is a* ***tallymorphism*** *if f factors into $f = c_Y^{-1} \circ c_X$ where $c_X : X \to I$, and $c_Y : Y \to I$ are tallies of X and Y respectively, each onto the same intial segment of the same nn-part.*

- *We write $[X]_t$ for the tallymorphism-class of X, in analogy with $[\cdot]$ and $[\cdot]_<$, and we say that X and Y are* ***tallymorphic****, or* ***have the same tally*** *and write $[X]_t = [Y]_t$ if there is a tallymorphism between them. It is easy to see that this also defines an equivalence relation, but now on the class $\mathrm{Def}_{0,t}(\mathcal{R})$, and we can again define a partial order on tallymorphism classes by defining $[X]_t \leq [Y]_t$ if there is a tallymorphism between X and an initial segment of Y. If $[X]_t \leq [Y]_t$, we say that* ***the tally of*** *X* ***is less than or equal to the tally of*** *Y.*

[14] We use $\mathcal{Z}_{>0}$ since we are thinking of the least element of a discrete set as the 'first' element, not the 'zeroth'. (This choice is not important - we could as easily have done it the other way.)

[15] This is as opposed to convex initial segments with no greatest element. In general, for $\mathcal{R}$ as in previous sections, an **interval initial segment** of X is just a definable initial segment of X, which is always of the form $X_{\leq x}$ for some $x \in X$. Interval initial segments of nn-parts are naturally models of $I\Delta_0^{top}$.

[16] We will see later that 'the' instead of 'a' is appropriate here.

- *It is easy to see that the map witnessing $[X]_t = [Y]_t$ is a tallymorphism, and that if $[X]_t = [I]_t = [Y]_t$, then composing the first implied map with the inverse of the second, we have a tallymorphism explicitly (and we will see that every equality of $[\cdot]_t$-classes arises this way).*

- *If $\mathcal{R}$ tallies all the discrete definable subsets of any ambient dimension then we simply say that $\mathcal{R}$ **tallies (or can tally) its discrete definable sets**, and in general, we write $\mathrm{Def}_{0,t}(\mathcal{R})$ for the class of all discrete definable sets that are tallied in $\mathcal{R}$.*

Note that the definition of tallies is really only important when $\mathcal{R}$ is not o-minimal: if $\mathcal{R}$ is o-minimal, then every discrete definable set X is finite, and there is already a unique natural number n (its ordinary cardinality), and there is a unique (and definable) order-preserving bijective function taking the set to the initial segment $\mathbb{N}_{\leq n}$. Thus:

From now on, we will assume that $\mathcal{R}$ is not o-minimal.

The main intuition behind considering tallymorphism classes or ordermorphism classes over isomorphism classes is that each notion is successively stronger than the next in its analogy with the way we might consider 'counting' finite sets. The notion of definable isomorphism captures the bare cardinality of a set, while ordermorphisms and tallymorphisms are more specific to the (linearly) ordered context: finite sets (and now discrete sets in general) are given to us with an induced ordering, and each of our notions tries to capture how we might compare the size of two sets by comparing them one at a time, in order. Ordermorphisms compare the size of sets by matching them up one-by-one, starting with the first, while tallymorphisms strengthen ordermorphisms by insisting that we compare the elements of the two sets by 'counting' them with a set of 'numbers'. Since in o-minimal structures, we always have this stronger notion of counting our discrete sets, we know that some more general version of this must be available to us in LOM+DC+dPHP expansions of real closed fields. In future research we hope to show how we can see what we are doing more generally in the context of Euler Characteristics and Grothendieck semirings, as Schoutens does with definable isomorphisms in [Sch12].

3.3.1 Uniqueness

By Proposition 3.2.1, we can conclude the following almost immediately:

Proposition 3.3.1 (Uniqueness of Tallies, Part 1)**:** *Let $X \subsetneq \mathcal{R}^k$ be an infinite discrete definable set, and $\mathcal{N}$ an nn-part in $\mathcal{R}$. Let $f : X \to \mathcal{N}_{\leq N}$ and $g : X \to \mathcal{N}_{\leq M}$ be definable order-preserving bijections for some elements $N, M \in \mathcal{N}$. Then $f = g$, and in particular, $N = M$.*

Proof. Suppose without loss of generality that $\mathbb{N} < N \leq M$. Let i be the inclusion map from the initial segment $\mathcal{N}^{\leq N}$ into the initial segment $\mathcal{N}_{\leq M}$, and define $F : X \to X$ by $F = g^{-1} \circ i \circ f$.

Then F is a definable, injective selfmap of the definable discrete (infinite) set X. Thus, by dPHP, F is a bijection. So, i is too, and $N = M$. Finally, Proposition 3.2.1, Part 3 implies that F is the identity map. Thus, $id_X = g^{-1} \circ id_{\mathcal{N}_{\leq N}} \circ f = g^{-1} \circ f$, so $g = f$. ■

Addressing problem (a) in Remark 3.2.1, we can say that tallymorphism classes (i.e. classes of elements of $\mathrm{Def}_{0,t}(\mathcal{R})$) are linearly-ordered:[17]

Lemma 3.3.1 (Linear Ordering of Tallies)**:** *The $[\cdot]_t$-classes of elements of $\mathrm{Def}_{0,t}(\mathcal{R})$ are linearly ordered.*

Proof. Let $X, Y \in \mathrm{Def}_{0,t}(\mathcal{R})$, and let f, g be their tallymorphisms to initial segments $\mathcal{N}_{\leq N}$ and $\mathcal{M}_{\leq M}$ of nn-parts $\mathcal{N}$ and $\mathcal{M}$ respectively. Since $\mathcal{R}$ is linearly ordered, and $\mathcal{N}, \mathcal{M} \subsetneq \mathcal{R}$, $N \leq M$ or $M \leq N$ as elements of $\mathcal{R}$. But in particular, we also need that there is either a tallymorphism of X onto an initial segment of Y or vice versa.

To conclude this, we use Theorem 3.3.1 below: we have either

$$(a)\ \mathcal{N}_{\leq N} = \mathcal{M}_{\leq N} \text{ and } M > N \in \mathcal{M}$$

$$\text{or,}\quad (b)\ \mathcal{M}_{\leq M} = \mathcal{N}_{\leq M} \text{ and } N > M \in \mathcal{N}.$$

In particular, if we assume the former (without loss of generality), we have that f is a map from X onto $\mathcal{M}_{\leq N}$. But then the inclusion map $i : \mathcal{M}_{\leq N} \to \mathcal{M}_{\leq M}$ is a definable injection, and the composition $F = g^{-1} \circ i \circ f : X \to g^{-1}(\mathcal{M}_{\leq N}) \subseteq Y$ is a tallymorphism onto an initial segment of Y. ■

We record a quick fact, and make two definitions before proving one of the main 'uniqueness' results we need to know about tallies.

Fact 3.3.1: *If $\mathcal{N}_{\leq N}$ is a definable interval initial segment in $\mathcal{R}$, then every proper interval initial segment of $\mathcal{N}_{\leq N}$ is definable (with parameters) in $\mathcal{R}$: if $\mathcal{M} \in \mathcal{N}_{\leq N}$, then we just define $\mathcal{N}_{\leq M} = \{x \in \mathcal{N}_{\leq N} \mid x \leq M\}$.*

Definition 3.3.4:

- *Let X be a discrete definable subset of $\mathcal{R}^1$. We say that X is* ***equally-spaced*** *(with spacing r) if $r \in \mathcal{R}_{>0}$ is such that $s_X(x) - x = r$ for all $\gamma X \neq x \in X$.*
    - *We say that a discrete definable set $X \subsetneq \mathcal{R}^k$, for $k > 1$, is* ***equally-spaced*** *(with spacings $r_1, ..., r_k$) if it is 'equally-spaced in every direction' (with spacing r_i in the i-th coordinate direction). That is, if for all $1 \leq i \leq k$, there are $r_i \in \mathcal{R}_{>0}$ such that each fibre X_q is equally-spaced with spacing r_i, for all $q \in \pi_i^k(X)$, (where π_i^k is the projection onto the i-th coordinate).*

[17] Of course there may be sets which $\mathcal{R}$ does not tally (more on that later). The difference from the situation with ordermorphism classes, and the reason why this always works for sets which are tallied is that there is a natural restriction of $\mathrm{Def}_0(\mathcal{R})$ so that the $[\cdot]$-classes are linearly ordered.

- *Given an interval initial segment $I = \mathcal{N}_{\le K}$ of an nn-part $\mathcal{N} \subsetneq \mathcal{R}$, we define the* ***convex definable hull of*** *I, written $\langle I \rangle$, to be the union of all the definable sets (with parameters) which extend I, and which are interval initial segments of nn-parts. In particular, $\langle I \rangle$ will still have least element 1, and will extend the set obtained by the closure of I under addition and multiplication. It is clear from this definition and the fact that we just recorded above, that $\langle I \rangle$ is, in fact, an initial segment of some nn-part for $\mathcal{R}$.*[18]

Theorem 3.3.1 (Uniqueness of Tallies, Part 2)**:** *Suppose $\mathcal{R}$ tallies a definable discrete set X, say with tally $N \in \mathcal{N}$, for some nn-part $\mathcal{N}$. And suppose that $\mathcal{R}$ tallies another subset Y, say with tally $M \in \mathcal{M}$ for some other nn-part $\mathcal{M}$. Then with $K = \min(N, M)$, we have $\mathcal{N}_{\le K} = \mathcal{M}_{\le K}$. That is, the two nn-parts $\mathcal{N}$ and $\mathcal{M}$ agree below K. In fact, they agree up to the convex definable hull of $\mathcal{N}_{\le K}$ in $\mathcal{R}$.*

Proof. Let $\mathcal{N}$, $\mathcal{M}$, X, Y, N, M and K be as in the statement. Let $f : X \to \mathcal{N}_{\le N}$ and $g : Y \to \mathcal{M}_{\le M}$ be the implied tallies. Without loss of generality we may assume that $N \le M$ (as elements of $\mathcal{R}$), and thus that we must show that the initial segment $\mathcal{M}_{\le N}$ of $\mathcal{M}$ is an interval initial segment of $\mathcal{N}$ as well.

To this end, we first note that $\mathbb{N}$ is contained in both, by the definition of an nn-part. Thus, if they differ, it must be at some infinite element $\alpha \in \mathcal{N}$. Suppose, without loss of generality, that $\alpha \le N$ is in $\mathcal{N}$, but $\alpha \notin \mathcal{M}$. We may further suppose that there is a least such α, by letting α be the least element of the following definable discrete set:

$$\{n \in \mathcal{R} \mid n \in \mathcal{N}_{\le N} \wedge n \notin \mathcal{M}_{\le M}\}$$

(This set is definable since $\mathcal{N}_{\le N}$ and $\mathcal{M}_{\le M}$, being images of definable maps, are both definable.) But now, we have that $\alpha - 1 \in \mathcal{N}$, and $\mathcal{N}$ and $\mathcal{M}$ agree here by the choice of α. Thus, $\alpha - 1 \in \mathcal{M}$, and thus $\alpha = (\alpha - 1) + 1 \in \mathcal{M}$, since $\mathcal{M}$ is an nn-part, a contradiction.

Thus, $\mathcal{N}$ and $\mathcal{M}$ agree for all $r \le K = \min(N, M)$.

Now, we finish by showing that $\mathcal{N}$ and $\mathcal{M}$ agree below L for all $L \in \langle \mathcal{N}_{\le K} \rangle$. First, we note that $\langle \mathcal{N}_{\le K} \rangle \subseteq \mathcal{N}$: by the definition of $\langle \cdot \rangle$, for each $L \in \langle \mathcal{N}_{\le K} \rangle$, since $\mathcal{N}_{\le K} \subsetneq \mathcal{N}$, we have that $\mathcal{N}_{\le L}$ is definable by Fact 3.3.1. Similarly, since $\mathcal{N}_{\le K} \subsetneq \mathcal{M}$, $\mathcal{M}_{\le L}$ is also definable for each $L \in \langle \mathcal{N}_{\le K} \rangle$. By the previous part of the theorem, $\mathcal{N}$ and $\mathcal{M}$ then agree below L. ∎

The following is an easy consequence of the final observation in the proof of the previous theorem:

Corollary 3.3.1: *Let $I \in \mathrm{Def}_{0,t}(\mathcal{R})$ be infinite (assuming such an element exists). Then $\langle I \rangle$ is equal to the union of $\langle J \rangle$ for all $J \in \mathrm{Def}_{0,t}(\mathcal{R})$.*

In particular, the theorem means that there is no need for more than one nn-part for tallying sets in a given structure $\mathcal{R}$: letting $\mathcal{N}$ be any nn-part which extends all of the convex

[18] Since it necessarily has no greatest element, $\langle I \rangle$ is of course *not* definable in $\mathcal{R}$.

definable hulls of all the tallied subsets of $\mathcal{R}$, or equivalently, the convex definable hull of any infinite tallied set, then any definable discrete subset of $\mathcal{R}$ which is tallied is in fact tallied by a tallymorphism onto an interval initial segment of $\mathcal{N}$. By fixing any such $\mathcal{N}$, we can say that $\mathcal{R}$ is tallying sets 'only' using $\mathcal{N}$.

The proof also highlights why we use initial segments of nn-parts in the definition of tallys instead of other discrete subsets: any two given discerete definable subsets of the line could disagree even if they agree up to some point α. But this is not the case for nn-parts since they are closed under $x \mapsto x + 1$, and since there is nothing between the elements n and $n + 1$ in an nn-part. That is, we are using the fact that initial segments of nn-parts are equally-spaced with spacing 1. Of course, we could use other equally-spaced sets with a different spacing to accomplish the same goal. However, if $f : X \to I$ is a ordermorphism, and I is equally-spaced, with spacing r, then we can define the set

$$I^* := \frac{1}{r}(I - \lambda I) + 1 = \left\{ \frac{1}{r}(x - \lambda I) + 1 \;\middle|\; x \in I \right\}$$

Then I^* is equally-spaced with spacing 1, and has $\lambda I^* = 1$. That is, I^* is an interval initial segment of some nn-part for $\mathcal{R}$. Thus, if $\mathcal{R}$ can 'tally' a definable set X with respect to I, for some equally-spaced set I, then it can tally X in the ordinary sense. An analogue of Theorem 3.3.1 for equidistant sets can be proved immediately:

Corollary 3.3.2: *If X and Y are two equally-spaced discrete, definable sets, both with spacing $r \in \mathcal{R}^{>0}$, then letting $\lambda = \lambda X - \lambda Y$, we have that X and $Y + \lambda$ are equal for all $x < \min(\gamma X, \gamma(Y + \lambda))$.*

3.3.2 More on Problem (a)

We noted earlier in Lemma 3.3.1 that tallymorphism classes of tallied sets are linearly ordered, rectifying problem (a) with ordermorphism classes mentioned in Remark 3.2.1. However, in that remark, though the linearity of the equivalence classes was the main point, there were three specific statements that made up problem (a), and whose analogues we now mention. For the next Lemma and its two Corollaries, let X, Y, and Z be definable discrete subsets of $\mathcal{R}$ (possibly in higher ambient dimension).

Proposition 3.3.2: *If $f : X \to Y$ is a definable, injective map, and $\mathcal{R}$ tallies X and Y, then $[X]_t \leq [Y]_t$.*

Proof. Suppose we have maps $g : X \to \mathcal{N}_{\leq N}$ and $h : Y \to \mathcal{N}_{\leq M}$, for $N, M \in \mathcal{N}$, for some nn-part $\mathcal{N}$ of $\mathcal{R}$. We first show that necessarily, $N \leq M$. So suppose that $N > M$ for contradiction, and note that we then have the inclusion map $i : \mathcal{N}_{\leq M} \to \mathcal{N}_{\leq N}$. So let $H = g^{-1} \circ i \circ h : Y \to X$. Then H is an ordermorphism onto a proper initial segment $I \subsetneq X$ (that is, $[Y]_< \leq [X]_<$). Now, consider $F = f \circ H$. Since f is a definable order-preserving map, and it is injective, so it is still injective on the initial segment $H(Y) = I \subsetneq X$. Thus, F is an injective, definable self-map of Y to itself, and by dPHP, it must be bijective. But then if F is bijective, f must be bijective too, since H is. Thus, letting $x \in X - I$, we have that there is some $i \in I$ such that $f(x) = f(i)$, contradicting the injectivity of f.

Thus, $N \le M$. But then we have the inclusion map $i : \mathcal{N}_{\le N} \to \mathcal{N}_{\le M}$, and taking $h^{-1} \circ i \circ g$, we get an ordermorphism of X onto the initial segment $h^{-1}(\mathcal{N}_{\le N})$ of Y (and thus, $[X]_t \le [Y]_t$.) ∎

Corollary 3.3.3: *If there is a definable bijection $f : X \to Y$, and $\mathcal{R}$ tallies X and Y, then $[X]_t = [Y]_t$.*

Proof. This follows immediately from Proposition 3.3.2 and Lemma 3.3.1 by noting that the assumption gives us an injective definable map $f : X \to Y$ and another $f^{-1} : Y \to X$, and thus $[X]_t \le [Y]_t$ and $[Y]_t \le [X]_t$. ∎

Corollary 3.3.4: *If $[X]_t \le [Y]_t$, $Y \subsetneq Z$, and $\mathcal{R}$ tallies all three sets, then $[Y]_t \le [Z]_t$ (and thus, $[X]_t \le [Z]_t$ as well).*

Proof. The fact that $[Y]_t \le [Z]_t$ follows immediately from Proposition 3.3.2 since the inclusion map $Y \subseteq Z$ is injective. Now, we have $[X]_t \le [Y]_t$ and $[Y]_t \le [Z]_t$, so we can compose the implied maps to get a tallymorphism $X \to I \subseteq Z$ for I an initial segment. That is, $[X]_t \le [Z]_t$. ∎

3.4 When Can We Tally Sets?

3.4.1 Canonical Tallies in Ultra-O-Minimal Structures

First of all, the situation is completely clear and as nice as we could ask for if $\mathcal{R}$ is actually ultra-o-minimal: then if $\mathcal{R}$ can tally a set, it neccessarily tallies it in the nn-part $\mathbb{N}^*$. That is, we have a canonical tally for any given discrete, definable set in an ultraproduct (or an expansion):

Proposition 3.4.1: *Let $\mathcal{R}$ be ultra-o-minimal.*

(a) If $X \subsetneq \mathcal{R}^k$ and is discrete and definable in an ultra-o-minimal expansion of $\mathcal{R}$, then X can be tallied in an ultra-o-minimal expansion of $\mathcal{R}$.

(b) In any ultra-o-minimal expansion that can tally X, the tally of X is an element of $\mathbb{N}^*$.

Proof. If necessary, we expand the language by a new predicate so that X is definable in $\mathcal{R}$. We expand the language with a new predicate I and a new function C, and let $\mathcal{R}'$ be the structure $\mathcal{R}$ expanded to this new language by interpreting I as $\prod_{i\in I} I_i/\mathcal{U}$, and C by the function $\prod_{i\in I} C_i/\mathcal{U}$, where I_i is, in $\mathcal{U}$-most index models the initial segment $[1, |X_i|]$ of $\mathbb{N}$, and the set $\{0\}$ otherwise, and the map $C_i : X_i \to I_i$ is the map that sends the j-th element of X_i to $j \in \mathbb{N}$, in $\mathcal{U}$-most index models, and maps everything to 0 otherwise. Since X is finite in $\mathcal{U}$-most index models, these maps exist in $\mathcal{U}$-most index models, so this definition is well-formed. This gives us an ultra-o-minimal expansion of $\mathcal{R}$ which tallies X.

Now for (b), if X was tallied in an expansion $\mathcal{R}'$ of $\mathcal{R}$ using an integer part $\mathcal{N}$ of $\mathcal{R}'$, then we could expand $\mathcal{R}'$, by the above construction to a new structure $\mathcal{R}''$ where X was also tallied in $\mathbb{N}^*$. But then we would have two tallies, $X \mapsto (\mathbb{N}^*)_{\leq N}$ and $X \mapsto \mathcal{N}_{\leq M}$. But then by Proposition 3.3.1, we have that $\mathcal{N}_{\leq K} = (\mathbb{N}^*)_{\leq K}$ where $K = \min(N, M)$. ∎

3.4.2 Untalliable Sets

The previous proposition seems to point to a major difference between general pseudo-o-minimal structures and ultraproducts of o-minimal structures: a priori it seems to be possible to tally the elements of discrete definable sets in more than one way in the former structures, or possibly not at all, while in ultraproducts, tallying always happens using the hypernaturals.

In fact, the latter situation is a possibility. That is, there are pseudo-o-minimal structures with definable subsets that are not only not tallied, but cannot be tallied in any pseudo-o-minimal expansion.

Example 3.4.1: *Let $\mathcal{R}$ be an ultrapower of the real field, in the language of real closed fields, and let Q be a new unary predicate symbol. We define P to be set of powers of 2 in $\mathcal{R}$ (i.e. the set $\{x \in \mathcal{R} \mid x = 2^n \text{ for some } n \in \mathbb{N}^*\}$, where $\mathbb{N}^*$ is the set of hypernatural numbers in $\mathcal{R}$). We expand $\mathcal{R}$ to an $\mathcal{L}_{rcf}(Q)$-structure by letting Q be interpreted as $P \cap [1, M]$ for some infinite power of 2, $M \in \mathcal{R}$. Then this structure $(\overline{\mathcal{R}}, Q)$ is pseudo-o-minimal: write $\mathcal{R} = \prod_{i \in I} \mathcal{R}_i/\mathcal{U}$, and $M = \prod_{i \in I} M_i/\mathcal{U}$, and note that $Q = \prod_{i \in I} Q_i/\mathcal{U}$, where $Q_i = \{1, 2, 4, ..., 2^{m_i}\}$, and m_i is the greatest natural number so that $2^{m_i} \leq M_i$ (in particular, $M_i \in \mathbb{N}$).)*

Now, by the main results in [**?**], the structure $(\overline{\mathcal{R}}, P)$ defines (in the language $\mathcal{L}_{rcf} \cup \{P\}$) an ordered abelian group structure on P, and this is essentially it. That is, the theory of this structure is completely axiomatized by just the axioms for real closed fields together with the statements that P is an ordered multiplicative group satisfying $2 \in P \wedge \forall x(1 < x < 2 \rightarrow x \notin P)$ and $\forall x(x > 0 \rightarrow \exists y \in P(y \leq x < 2y))$. But then, in the prime model $\mathcal{R}'$ of $\mathrm{Th}(\overline{\mathcal{R}}, Q)$, there is no infinite element N of the underlying field which is logarithmic in M (that is, an (infinite) element such that $N^k < M$ for all natural numbers k). This is because the prime model of $\mathrm{Th}(\overline{\mathcal{R}}, Q, M)$ is just the real closure of the field generated by the (relative) divisible hull of M in P (since we have definable Skolem functions, and the theory has quantifier elimination), and this clearly does not contain an element N as above. Since $(\overline{\mathcal{R}}, Q)$ is a reduct of $(\overline{\mathcal{R}}, Q, M)$, its prime model would be contained in the prime model of the latter (and so itself fail to have an element logarithmic in M.)

We will show that there can be no tallymorphism of Q with an interval initial segment in an nn-part $\mathcal{N}$ in this structure. The intuition here is that since the set Q just consists of the set $\{2^1, 2^2, ..., 2^{N-1}, 2^N\}$, at least seen from an elementary extension, and so Q 'has' N elements (with '$N = \log_2(M)$'); and that this would force a tallymorphism to map onto an interval $[1, N]$.

Making this intuition rigorous, if there were a tallymorphism $l : Q \to \mathcal{N}^{\leq N}$ in some LOM+DC+dPHP expansion of $\mathcal{R}'$ for some some nn-part $\mathcal{N}$ of $\mathcal{R}'$, with $N \in \mathcal{N}$, then we would have that N was logarithmic in M in the above sense. Thus:

Proof. Write $e = l^{-1}$. Now note that for any $k \in \mathbb{N}$, we have that if $x \in \mathcal{N}_{\leq N}$ is large enough, then $x^k < e(x)$. If this were not the case, let $x \in \mathcal{N}^{\leq N}$ be least with this property; that is, take the least infinite element of the definable set $\{x \in \mathcal{R}' \mid x \in \mathcal{N}_{\leq N} \wedge \ x^k > e(x)\}$. We can take an infinite element, since for any $k \in \mathbb{N}$, there is a finite x' so that for all $x > x'$, $x^k < e(x)$. This is because $e \restriction \mathbb{N} = 2^x \restriction \mathbb{N}$ since e is order-preserving, and Q_{ini} just consists of the set $\{2^n \mid n \in \mathbb{N}\}$.

So then, we have that $(x-1)^k < e(x-1)$, yet $x^k \geq e(x)$ by choice of x. But letting $p(Y)$ be the unique polynomial of degree $k-1$ such that $Y^k = (Y-1)^k + p(Y)$, we have $p(x) \leq (x-1)^k$, since x is infinite and p is of degree $< k$, so then:

$$x^k = (x-1)^k + p(x) \leq (x-1)^k + (x-1)^k = 2(x-1)^k < 2e(x-1) = e(x)$$

The final equality follows since e must send $p_{\mathcal{N}}(x)$ to $p_Q(e(x))$. But $p_Q(e(x)) = e(x)/2$ by definition of Q.

Thus, we have a contradiction, and thus, all $x \in \mathcal{N}_{\leq N}$ satisfy $x^k < e(x)$ as long as they are large enough. But for each k, the least such x satisfying this is finite, and we just showed that there is no infinite counterexample. Thus, in particular, the greatest element of $\mathcal{N}_{\leq N}$, namely N, satisfies $N^k < e(N) = M$ for all natural numbers k. That is, it is logarithmic in M. But this contradicts the choice of $\mathcal{R}'$. Thus, $\mathcal{R}'$ cannot tally Q. ∎

This example lead us to consider relaxing the definition of tallies slightly in a way that ensures (or at least seems likely to ensure) that they always exist (in an LOM+DC+dPHP expansion). However, so as not to lose the main thread, we present that material - on 'Approximate Tallies' - later, in Section 3.6.

3.5 Uniform Pseudofiniteness (Ambient Dimension 2)

From now on, we will work with LOM+DC+dPHP expansions of fields $\mathcal{R}$ which can tally all of their discrete definable sets (in any ambient dimension). That is, for every discrete definable set, there is a definable tallymorphism to an initial segment of some fixed nn-part $\mathcal{N}$ in $\mathcal{R}$. We will call this assumption **(Tally)**. This alone, however, can do work for us, but not get us all the way. We also make the following assumption, and use it to derive the uniformity result that follows:

(Unbounded Tally): Every interval initial segment $\mathcal{N}^{\leq N}$ of $\mathcal{N}$ is definable in $\mathcal{R}$. That is, there is no upper bound on the tallies of definable sets in $\mathcal{R}$.[19]

[19]By Fact 3.3.1, we know that this is equivalent to tallying a sequence of sets whose tallies are cofinal in $\mathcal{N}$.

The main result of this section is the following:

Theorem 3.5.1 (Uniform Pseudofiniteness with Tallies (Dim 2))**:** *Let $X \subsetneq \mathcal{R}^2$ be definable and suppose that for all $r \in \mathcal{R}$ the fibre X_r of X above r is discrete. Then assuming **(Tally)** and **(Unbounded Tally)** we can find $\alpha \in \mathcal{N}$ such that for every r, the fibre X_r has tally $\leq \alpha$, and some fibre has tally equal to α. Futhermore, we can determine, for any pair of elements $z, z' \in X$, whether they have the same tally according to the tally of their respective fibres of X (this notion is called fiberwise tally in X, and is defined below).*

What we do not claim we can derive just from these assumptions is a decomposition of X into the graphs of α-many different **definable** functions $f_1 < f_2 < ... < f_\alpha$ such that they are all continuous, except on a common discrete set of (possible) discontinuities. Certainly, we can get this for the initial ω-many functions f_1, f_2, ... by simply defining $f_n(x) = n^{th}$ element of the fibre above x,[20] and assuming every fibre had the same tally, we could also define the final ω-many f_α, $f_{\alpha-1}$, ... by $f_{\alpha-n}(x) := (n+1)^{st}$-last element of the fibre above x. These functions have a discrete set of discontinuities, but there is, at this point, no way to get the definability of *all* of these functions, let alone what we might really want: a uniform definition collecting all of them.

Before proceeding we will begin by recording a proposition and a definition.

Proposition 3.5.1 (Automatic Uniformity of Tallies)**:** *If $D \subsetneq \mathcal{R}^2$ is discrete, infinite and definable, with infinitely many fibres, all of which are infinite,[21] then letting $F : D \to \mathcal{N}_{\leq N}$ and $f_q : D_q \to \mathcal{N}_{\leq N_q}$ be tallymorphisms for all $q \in \pi(X)$, we can define a function $t_D : D \to \mathcal{N}_{\leq \alpha}$ for some $\alpha \in \mathbb{N}$ such that $t_D \restriction D_q = f_q$. In particular, we can define the set* $\{q \in \pi(D) \mid D_q \text{ has tally } \leq \beta\}$ *for any $\beta \leq \alpha$.*

Proof. We have that $f_q(z) = \beta$ iff $\beta = F(z) - F(\lambda D_q) =: t_D(z)$, a uniform definition in $q \in \pi(D)$. The α in our statement is just the greatest element of the set

$$\{x \in \mathcal{R} \mid x = F(z) - F(\lambda D_q) \text{ for some } q \in D\}$$

And the final statement is obvious: just define $\{q \in \pi(X) \mid F(z) - F(\lambda D_q) \leq \beta\}$. ∎

Definition 3.5.1: *For a set X as in the statement of Theorem 3.5.1, which has discrete fibres, and for $z \in X$, we say that α is **the fiberwise tally of** z **in** X if it has image α under the tally of the fibre $X_{\pi(z)}$. In particular, if X is a discrete set D, as in the proposition above, and we let $z = \langle q, x\rangle \in X$, then we can define the fibrewise tally of z in X: it is just $t_X(z)$.*

The proposition allows us, as an aside, to answer question (d) from Remark 3.2.1:

Corollary 3.5.1: *With D as in the proposition, if we additionally had that every D_q had tally $\alpha \in \mathcal{N}$, then letting $\lambda = \lambda\pi(D)$, we have $[D]_t = [\pi(D) \times D_\lambda]_t$.*

[20] This works since we can just write "$y = f_n(x) \leftrightarrow (y \in X_x \ \wedge \ \exists^{=n-1} z \in X_x$ such that $z < y)$"

[21] An unneccessary, but simplifying assumption.

Proof. Let f_λ be the tally for D_λ, and $f_{\pi(D)}$ the tally for $\pi(D)$. The map F is a tallymorphism of D to $\mathcal{N}^{\leq N}$, and the map

$$G(z) = f_\lambda(\gamma D_\lambda) \cdot (f_{\pi(D)}(\pi(z)) - 1) + t_D(z)$$

sends $\pi(D) \times D_\lambda$ to $\mathcal{N}_{\leq N}$, and it is easy to check that it is order-preserving. Thus, since F is a tallymorphism from D to $\mathcal{N}_{\leq N}$, $G^{-1} \circ F$ is a tallymorphism from D to $\pi(D) \times D_\lambda$. ∎

We will start the proof of the theorem by making the following definition, as in [vdD98-a] (Chapter 3, 1.7):

Definition 3.5.2: *Given a subset $X \subseteq \mathcal{R}^2$, a point $\langle a, b\rangle \in \mathcal{R}^2$ is called **normal** if there is an $r \in \mathcal{R}^{>0}$ such that either $B_r(\langle a,b\rangle) \cap X = \varnothing$ (and hence $\langle a,b\rangle \notin X$), or $B_r(\langle a,b\rangle) \cap X = \Gamma(f)$ for some continuous (necessarily unique and definable) function $f : (a-r, a+r) \to \mathcal{R}$ (and hence $\langle a,b\rangle \in X$). We also define $\langle a, +\infty\rangle$ to be **normal** if there is $r \in \mathcal{R}^{>0}$ and $b \in \mathcal{R}$ such that $((a-r, a+r) \times (b, +\infty)) \cap X = \varnothing$ (similarly for points $\langle a, -\infty\rangle$).*

Note: *The sets $\{(a,b) \mid \langle a,b\rangle$ is normal $\}$, $\{a \mid \langle a, +\infty\rangle$ is normal $\}$, and $\{a \mid \langle a, -\infty\rangle$ is normal $\}$ are definable.*

Now, diverging from [vdD98-a], we define the sets

$$\mathcal{G} = \{x \in \mathcal{R} \mid \forall y \in \mathcal{R}_{\pm\infty}\ \langle x, y\rangle \text{ is normal}\}, \quad \mathcal{B} = \{x \mid x \notin \mathcal{G}\}$$

and call them the 'good' and 'bad' points, respectively, of $\mathcal{R}^1$ (the first coordinate). It is clear that the sets $\mathcal{G}$ and $\mathcal{B}$ are definable as well.

We will next show that $\mathcal{B}$ is discrete, and hence that $\mathcal{G}$ is a union of disjoint open intervals. But first:

Claim: *If $a \in \mathcal{B}$, then there is a least $b \in \mathcal{R}_{\pm\infty}$ such that $\langle a,b\rangle$ is not normal.*

Proof. This follows immediately since the fibre X_a is discrete and definable: we can define the set $\{b \in \mathcal{R}_{\pm\infty} \mid \langle a,b\rangle$ is normal$\}$, take its complement inside $X_a \cup \{\pm\infty\}$, and note that this is a discrete definable set. It thus has a least element and we are done. ∎

Claim: *$\mathcal{B}$ is discrete.*

Proof. This proof is similar to the proof in [vdD98-a] that $\mathcal{B}$, as defined there is not infinite.[22]

As a subset of $\mathcal{R}$, $\mathcal{B}$ not being discrete would imply the existence of a subinterval $(a,b) = I \subseteq \mathcal{B}$. Define $k(x)$ for $x \in I$ to be the unique least element of $\mathcal{R}_{\pm\infty}$ which is non-normal above x (which exists by the previous claim). We also define $j(x) \coloneqq p_{X_x}(k(x))$ and $l(x) \coloneqq s_{X_x}(k(x))$. Note that for some $x \in I$, either j or l may be undefined, but nonetheless,

[22] Note that in our case, a definable set not containing an interval is equivalent to discreteness, instead of finiteness.

$\operatorname{dom} j, \operatorname{dom} k, \operatorname{dom} l \subseteq I$. We claim, for a contradication, that the point $\langle x, k(x)\rangle$ is normal for some $x \in I$.

Since k is definable on I, by the Monotonicity Theorem, we may assume (by possibly shrinking I), that k is continuous on I. Now, one of the following four sets must contain an interval:

$$\operatorname{dom} l \cap \operatorname{dom} j,\ I - (\operatorname{dom} l \cup \operatorname{dom} j),\ \operatorname{dom} l - \operatorname{dom} j, \operatorname{dom} j - \operatorname{dom} l$$

This is because the union of these four sets is all of I, and the union of four discrete sets is discrete.

I will deal with the first case, since the others are simpler, and are proven similarly.

In the first case, we are supposing that there is an interval J contained inside $\operatorname{dom} l \cap \operatorname{dom} j$. In particular, the functions j, k and l are all defined on J. Now, again by the Monotonicity Theorem, we may shrink J so that j, k, and l are all continuous on J.

But since $j < k < l$ where the three functions are defined, in particular, their graphs do not intersect. Now note that $k(x)$ need not be an element of X: it is just the least b such that $\langle x, b\rangle$ is non-normal. So, we define the two sets

$$J_1 := \{x \in J \mid \langle x, k(x)\rangle \in X\} \quad \text{and} \quad J_2 := \{x \in J \mid \langle x, k(x)\rangle \notin X\}.$$

Since both are definable, and partition J, at least one contains an interval. Replacing J with that interval, we then have either $\Gamma(k \upharpoonright J) \subseteq X$ (if J_1 contained an interval) or $\Gamma(k \upharpoonright J) \cap X = \varnothing$ (in case J_2 contained an interval). In either case, it is clear that $\Gamma(k \upharpoonright J)$ contains only normal points (keeping in mind that $j < k < l$ and all three are continuous here. But this is a contradiction: for any $x \in J$, $\langle x, k(x)\rangle$ is normal. ■

Now so far, we have only used **(T)**, and the proof has largely mirrored the proof in the o-minimal case. Here though is where things change, and where we do not see how we could get by this obstacle without using an assumption like **(Unbounded Tally)**.

Claim: *If $a \in \mathcal{G}$, then the tally of X_x is constant for all x in the interval $I = (b, s_{\mathcal{B}}(b))$, where b is the greatest element of $\mathcal{B}$ less than a (i.e. $b = \gamma\mathcal{B}_{<a}$.)*

Proof. Let $J = [a, a']$, where $b < a < a' < s_{\mathcal{B}}(b)$, and let $\rho = a' - a$. What we will prove is that we can take the function defined locally near $z = \langle a, y\rangle$ for $y \in X_a$ by its normality and 'continue' it to the right (or left), in the sense of expanding its domain to include all of J by a series of extensions.

What we need to obtain is an ϵ so that for all $z \in X \cap (J \times \mathcal{R})$, the set $B_\epsilon(z)$ is the graph of a continuous function.

To do this, we start by defining D to be the set of 'bad distances':

$$D := \{r \in \mathcal{R}_{>0} \mid \exists z \in (X \cap (J \times \mathcal{R})),\ B_r(z) \cap X \text{ is } \textbf{not} \text{ the graph of a continuous function}\}.$$

We say that r is a *bad distance for $z \in X \cap (J \times \mathcal{R})$* if z witnesses that $r \in D$.

Let $\epsilon = \frac{1}{3} \cdot \lambda \operatorname{bd}(D)$. Since $D \subseteq \mathcal{R}_{>0}$ is definable, $\operatorname{bd}(D)$ is discrete, and so has a least element (by definition ≥ 0). We will show though, that this element is strictly greater than

0. If $\lambda := \lambda\mathrm{bd}(D) = 0$, then we would have an interval $(0,t) \subseteq D$ for some $t \in \mathcal{R}_{>0}$. To see that this implies a contradiction, first we define, for $\langle x,y\rangle \in X$,

$$D_{\langle x,y\rangle} := \{r \in \mathcal{R}_{>0} \mid r \text{ is a bad distance for } \langle x,y\rangle\}$$

and we define the following function on J:

$$h(x) := \lambda\{\inf D_{\langle x,y\rangle} \mid y \in X_x\}$$

This function is well-defined, since for each y in X_x, we have the set of bad distances, and clearly this set is an interval of the form $(b,+\infty)$ for some $b > 0$. Thus, it has an infimum, b. And since we have one such value for each element y of the fibre X_x, we have a discrete set of such values, and there is thus a least element. Note that the infimum b of a set $D_{\langle x,y\rangle}$ is always positive: normality forces there to be some non-bad, (positive) distances.

Now, since h is a definable function on an interval in $\mathcal{R}$, by the Monotonicity Theorem, we know that h is continuous on J except on a discrete subset. Furthermore, we may assume that since D contains an interval $(0,t)$, that on some subinterval, h is continuous and has image $(0,t')$ for some $0 < t' < t$. Shrink J to such a subinterval.

Then finally, we may again assume by the Monotonicity Theorem that there is a further subinterval where h is monotone. (h cannot be constant everywhere on this interval since we are assuming it has image $(0,t')$.) In particular, assume without loss of generality that h is monotonically decreasing here. Let d be the right endpoint of this interval. Then letting y be the unique value such that $h(d) := \inf D_{\langle d,y\rangle}$, we have that $\langle d,y\rangle$ is not normal since y, by the definition of d, has no non-bad distances.

What we have defined then, is the value $\epsilon = \epsilon(J)$, depending only on our initial choice of $J \subsetneq (b, s_{\mathcal{B}}(b))$ which, by construction, has the property that for all $z \in X \cap (J \times \mathcal{R})$, the box $B_\epsilon(z)$ intersects X in the graph of a function.

Now, what we do is define a discrete 'sampling' of the interval J, using ϵ: let $r = \rho/\epsilon$, and note that an evenly-spaced discrete set with spacing r and least element a would have to have greatest element greater than a'; thus, with this in mind, we let β be the least element of $\mathcal{N}$ such that $\beta \cdot \epsilon \geq \rho$ and then define

$$E = \{e \in J \mid e = a \ \vee\ \exists \alpha \in \mathcal{N}_{\leq\beta}\ (e = a + \alpha \cdot r)\}$$

The number β necessary to define this clearly must exist[23], and by **(UT)**, the set $\mathcal{N}_{\leq\beta}$ is definable (up to using a parameter for β).[24]

But now, we can define the set of fibres above E: let $E_X := (X_e)_{e\in E}$. With this set, we can then define certain discrete unions of ϵ-boxes, or 'tubes' that we can use to continue the functions defined locally near points of $X \cap (J \times \mathcal{R})$ by the normality of those points. To this

[23] We are not *defining* β, we are just choosing it.

[24] This is the key use of **(UT)**: if we do not assume **(UT)**, it could be that ϵ is so small, or that ρ is so large that r is greater than all the tallies of discrete definable sets in $\mathcal{R}$, in which case E would not be definable.

end, suppose that $z = \langle a, x\rangle$ has the property that $F(z) - F(\lambda X_a) = \beta$; that is, $t_E(z) = \beta$. Then we define

$$A = \{\langle u, v\rangle \in E_X \mid t_{E_X}(v) = \beta\}$$

and

$$T = \{\langle u, v\rangle \mid a \leq u \leq a' \ \wedge \ \exists\zeta \in A, \langle u, v\rangle \in B_\epsilon(\zeta)\}$$

T is one of the aforementioned 'tubes', and consists of a union of boxes $B_\epsilon(\zeta)$ for points $\zeta \in E_X$. It is clearly definable, since A is (and A is definable since t_{E_X} is). But then T is a definably connected set containing z, containing some point $z' = \langle a', x'\rangle$, and with the property that $T \cap X$ is the graph of a continuous function (if it were not, there would have to be a bad point in J). That is, we have 'continued' the function witnessing the normality of z to the whole interval J. In particular, it is easy to check that z' will have $F(z') - F(\lambda X_{a'}) = t_{E_X}(z') = \alpha$ as well, and that every z^* in $T \cap X$ will be the α^{th} element of the fibre above $\pi(z^*)$.

The goal from here is to use this ability to continue these locally defined functions to show that the size of the fibres of X above elements between successive bad points is constant.

(Recall: $I = (b, s_\mathcal{B}(b))$ for some $b \in \mathcal{B}$, and that $J = [a, a'] \subsetneq I$.)

To this end we start by showing that the tallies of the fibres above elements of J are all the same. If this were not the case, then suppose without loss of generality first that the there is some $\langle c, x'\rangle \in X$ such that under the tally map for X_c, x' has image β and $x' = \gamma X_c$; and secondly suppose that $\langle a, x\rangle \in X$ has image $\beta + 1$ under the tally map for X_a (i.e. $t_{E_X}(\langle a, x\rangle) = \beta + 1$).

Now, by the definition of ϵ, we know that in the neighbourhood $(a, a + \epsilon)$, we have can define $\beta + 1$ continuous functions, $f_1 < ... < f_{\beta+1}$. But by applying the continuation process to each of these functions, we have, by the construction, that the continuations, $\widetilde{f_\alpha}$ will have $\widetilde{f_1} < ... < \widetilde{f_{\beta+1}}$, and we will have that they are defined on an interval containing $(a, c]$. Then, we notice the contradiction: $f_{\beta+1}(c)$ is the $(\beta + 1)^{st}$ element of the fibre X_c. But then X_c has tally $\geq \beta + 1$, contradicting that $x' = \gamma X_c$, and that x' has tally β under the tally for X_c.

Now, let d be the midpoint between b and $s_\mathcal{B}(b)$, and take J_ρ to be the interval $(d - \rho/2, d + \rho/2)$. We notice that $\epsilon(J_\rho)$ really only depends on ρ, and thus, since

$$I = (b, s_\mathcal{B}(b)) = \bigcup_{\substack{\rho = |s_\mathcal{B}(b) - b| \\ b \in \mathcal{B}}} J_\rho,$$

and since there is a constant tally for all the fibres of X above elements of J_ρ by the above continuation argument, this same tally will work for all J_ρ, and hence for all $x \in I$. ■

With this under our belts, we can now complete the proof of the theorem:

Proof. (of Theorem 3.5.1) We have shown that for successive elements of $\mathcal{B}$, the tallies of fibres of X in between are constant. What we will show is that there is a way to go from this to a maximum for the tallies of all the fibres of X.

We define the **skeleton** of X, $\mathrm{Sk}(X)$, in the following way: first, let $\mathcal{G}_{1/2}$ be the set of midpoints between elements of $\mathcal{B} \cup \{\pm\infty\}$; that is,

$$\mathcal{G}_{1/2} = \left\{ x \in \mathcal{G} \;\middle|\; \exists b \in \mathcal{B} \text{ such that } x = \frac{b + s_{\mathcal{B}}(b)}{2} \;\vee\; x = \lambda\mathcal{B} - 1 \;\vee\; x = \gamma\mathcal{B} + 1 \right\}.$$

Then we define:

$$\mathrm{Sk}(X) := \{\langle x, y\rangle \in X \mid x \in \mathcal{G}_{1/2} \vee x \in \mathcal{B}\}$$

That is, the skeleton of X is the union of the fibers of X above only the elements of $\mathcal{B}$, and the midpoints between the elements of $\mathcal{B}$.

By definition, $\mathrm{Sk}(X)$ is discrete, and by the previous claim, we know that the maximum size of a fibre of X is the maximum size of a fiber of $\mathrm{Sk}(X)$. Thus, we can note at this point that we could take the tally of the collapse of the skeleton (it being a definable discrete set), and this would give us a definite bound on the size of the fibres of $\mathrm{Sk}(X)$, and thus of X. That is, we can already conclude that there is a uniform bound on the size of the fibres; we will further show that we can obtain a strict bound on the size of the fibres, in the sense of this maximum tally being obtained.

To get a strict bound, we start by taking the collapse of the skeleton of X,

$$\check{X} := \underline{\downarrow}\mathrm{Sk}(X).$$

Note that for each fiber of the skeleton, it is compressed into some interval $[q, q + \delta(q)]$ (see the definition of the collapse of a set: Definition 3.2.3).

Now since $\check{X}$ is a definable discrete set, by assumption, it has a tallymorphism $f_{\check{X}}$ onto an initial segment of an nn-part, with maximum element $N_{\check{X}}$. We define the set

$$M := \{x \in \check{X} \mid x = q + \delta(q) \text{ for some } q \in \pi(\check{X})\}$$

By definition, M is a discrete definable set that consists of the image under the collapse of the greatest element of each fibre in the skeleton of X.

We then let $\tilde{M} := f_{\check{X}}(M)$, and we define α to be the greatest element of the set

$$\{x \in \mathcal{R}_{>0} \mid x = |m - p_{\tilde{M}}(m)| \text{ for some } m \in \tilde{M} - \{\lambda\tilde{M}\}\}.$$

Since this set is definable, and since it consists of exactly the distances between successive elements of $\tilde{M}$, it gives the set of tallies of the fibers of X. That is, $|m - p_{\tilde{M}}(m)|$ is the distance in the nn-part that $\mathcal{R}$ is tallying with between $f_{\check{X}}$-images of successive elements of M. But such distances are, by definition, the tallies of sets $[q, q + \delta(q)] \cap \check{X}$, each of which is just the number of elements of the fibre of X above q.

Thus, since α is its maximum, α is the exact maximum size of the fibers of X. In particular, there is some fiber of X of size α, and there is no fiber of size greater. ■

3.6 Approximate Tallies and Overtallies

In this section we try to rectify the problem we noticed with tallies: there are some LOM+DC +dPHP structures with discrete definable sets that cannot be tallied in any LOM+DC+ dPHP expansion. In order to fix this, we introduce the idea of approxmiate tallies, and over- and undertallies. We are in the same sort of situation as in the previous sections: $\mathcal{R}$ is an expansion of real closed field satisfying LOM+DC+dPHP, and $\mathcal{N}$ is a fixed nn-part for $\mathcal{R}$.

Definition 3.6.1:

- *Let X be a discrete definable set in $\mathcal{R}$. We say that a map $f : X \to \mathcal{N}_{\leq N}$ is an* ***approximate tally of*** *X if it is a definable order-preserving injection.*
- *We say that X* ***has tally bounded above by*** *N, if there is an approximate tally f as above.*
- *We call a map $f : X \to Y$, for Y another discrete definable set, an* ***approximate tallymorphism*** *if it is an order-preserving injection which factors as $f = h^{-1} \circ g$, for g an approximate tallymorphism from X to $\mathcal{N}_{\leq N}$ for some N, and h an approximate tallymorphism from Y to the same initial segment.*

The definition of approximate tally is supposed to be a way to capture the idea that we can in some sense 'guess' at the number of elements of X. Before getting further into these, we record a quick lemma relating this notion to tallymorphisms:

Lemma 3.6.1: *If X has tally bounded above by $N \in \mathcal{N}$ via a map f, and there is a definable injection g from $\mathcal{N}_{\leq N}$ to X, then X is tallied by $\mathcal{N}_{\leq N}$. (Note that we do not assume that g is order-preserving).*

Proof. Compose the maps to get $F = g \circ f$. Then since F is a definable injective self-map of X, it is surjective as well by dPHP. But then for every $x' \in X$, there is some $x \in X$ such that $g(f(x)) = x'$. So in particular, if g were not surjective, we could take $x' \notin g(\mathcal{N}_{\leq x}) \subseteq X$, and derive a contradiction. So g is surjective. Similarly, if f were not surjective, we would then have $n \in \mathcal{N}_{\leq N}$ such that $n \notin f(X) \subseteq \mathcal{N}_{\leq N}$. But then since g is injective, the image $g(n) \in X$ would not be hit by F. Thus, f is a tallymorphism. ■

We can always define a number which is a discrete set's **overtally** (a bound on what the tally of the set 'could' be were it talliable):

Letting X be a definable discrete set in $\mathcal{R}^1$, we let w_X be the minimum distance between any two successive elements of X. (As we saw in the previous section, this set is definable and discrete, and so indeed has a least element.) We then call the following number **the (default) overtally for** X:

$$\mathrm{ot}(X) := 1 + \gamma\left\{\frac{1}{w_X} \cdot \left(x - \lambda X\right) + 1 \;\middle|\; x \in X\right\}$$

In particular, we define **the standard position of** X to be the set defined in the definition of $\mathrm{ot}(X)$:

$$\mathrm{St}(X) := \left\{ \frac{1}{w_X} \cdot \left(x - \lambda X \right) + 1 \;\middle|\; x \in X \right\}$$

Fact 3.6.1: $\mathrm{St}(X)$ *has least element 1, second element 2, and* $w_{\mathrm{St}(X)} = 1$.

Because of this fact, we know that $\mathrm{St}(X)$ has, at most, $\gamma\mathrm{St}(X)$ elements (possibly rounded up by 1) in the sense of tallies: this is because the 'largest' set with the restrictions listed in the Fact would be an evenly-spaced set with least element 1, greatest element $\gamma\mathrm{St}(X)$, and spacing 1. This set, by virtue of being evenly-spaced, would in fact have a tally,[25] and the tally would have to be $\gamma\mathrm{St}(X)$ or $\gamma\mathrm{St}(X) + 1$. This is the reasoning behind our definition of $\mathrm{ot}(X)$: it is the absolute largest that the tally of X could be if X were tallied in some expansion of $\mathcal{R}$.

Note that we can define the overtally of X without making *any* assumptions about what sets or maps are definable in $\mathcal{R}$. In particular, it does not require that any (interval) initial segments of $\mathcal{N}$ are definable in $\mathcal{R}$. However, even though we can make this definition, it lacks teeth without any such definable interval initial segments: we can see that $\mathrm{ot}(X)$ is in fact a bound, but we are not able to define an approximate tally for X. An approximate tally for X would tell us not only a bound on the size of X, but would, for each element of X, give its approximate relative position in the set X. But if we have that $\mathcal{N}_{\le \mathrm{ot}(X)}$ is definable, then we can define the following **(default) approximate tally for** X:

$$\mathrm{at}_X : X \to \mathcal{N}_{\le \mathrm{ot}(X)}, \quad \mathrm{at}_X(x) := \gamma((\mathcal{N}_{\le \mathrm{ot}(X)})_{\le \mathrm{St}(x)})$$

where $\mathrm{St}(x)$ is just the image of x under the standard position map, i.e. $\frac{1}{w_X} \cdot (x - \lambda X) + 1$. Essentially at_X takes x to its position in $\mathrm{St}(X)$, then takes the (unique) 'natural number-part' of x.

Also note that the definitions of $\mathrm{ot}(X)$, $\mathrm{St}(X)$, and at_X are all defined, so far, only for sets X in one ambient dimension. However, we can quickly extend them to higher ambient dimensions by defining $\mathrm{ot}(X) = \mathrm{ot}(\underline{\downarrow}(X))$, $\mathrm{St}(X) = \mathrm{St}(\underline{\downarrow}(X))$ and $\mathrm{at}_X(x)$ to be $\mathrm{at}_{\underline{\downarrow}(X)}(\underline{\downarrow}(x))$.

These concepts are quite weak (especially without defining any initial segments of $\mathcal{N}$) in that overtallies sometimes give drastic overestimates of the 'true' size of the set and can give very different answers for sets which only differ by a single element or small set. We note this only to make a point of how weak this notion is.

Example 3.6.1: *Let* X *be the set* $\{-1, 2, 8\}$. *Then* $w_X = 3$, $\mathrm{St}(X) = \{1, 2, 4\}$, *and* $\mathrm{ot}(X) = 5$. *However,* $Y = \{-1, 2, 8, 10^{10}\}$ *has only one more element, but we have* $\mathrm{ot}(Y) \approx 3 \times 10^9$, *a significant difference considering that we have only added 1 new element.*

The example points out that the value of $\mathrm{ot}(X)$ really only depends on the first and last element, combined with whatever is the least distance between elements. So even though

[25] Recall that the map would be given by subtracting $\lambda X - 1$ from the set, then dividing by the spacing.

that minimum distance might only be realized once in that set, and the spacing be very different throughout the rest of the set, it still affects the overtally. An overtally would certainly *not* be useful in a real-world application!

With the weakness of our notion duly noted, we proceed to prove an analogue of the Uniform Pseudofiniteness theorem from the previous section.

3.6.1 Approximate Uniform Pseudofiniteness (Arbitrary Ambient Dimension)

In this subsection, we will only be assuming the following much weaker verion of **(Tally)**:

(Approximate Tally): every discrete definable set $\mathcal{R}$ has a definable approximate tally.

Just to be clear, this is equivalent to assuming only that some (interval) initial segments of $\mathcal{N}$ are definable: once we have those, from what we saw in the previous subsection, we can then define default overtallies and so on. We mention a conjecture related to this assumption in the next (and last) subsection of this chapter.

First, we have a version of Proposition 3.5.1. And though we will not be using it we record it anyway for completeness:

Proposition 3.6.1 (Automatic Uniformity of Overtallies)**:** *If $D \subsetneq \mathcal{R}^2$ is discrete, infinite and definable, with infinitely many infinite fibres, then letting uat_D be the map sending $z = \langle q, d\rangle \in D$ to $\gamma((\mathcal{N}_{\leq \mathrm{ot}(D)})_{\leq \mathrm{St}(d)})$, we have that $\mathrm{uat}_D \upharpoonright D_q = \mathrm{at}_{D_q}$ for all $q \in \pi(D)$. Thus, uat_D collects all of the default approximate tally maps for the fibres of D, uniformly in $q \in \pi(D)$. In particular, we can define the set $\{q \in \pi(D) \mid D_q$ has tally bounded by $\beta\}$ for any $\beta \leq \mathrm{ot}(X)$.*

Note that uat_D maps into the same initial segment, $\mathcal{N}_{\leq \mathrm{ot}(D)}$, that F does; that is, it does not increase the overall approximate tally of the set. Nor does it increase the overtally of the individual fibres.

Theorem 3.6.1 (Approximate Uniform Pseudofiniteness (Arbitrary Dimension, Bounded Variation))**:** *Let $X \subsetneq \mathcal{R}^{1+n}$ be definable and suppose that for all $r \in \mathcal{R}^1$ the fibre X_r of X above r is discrete. Furthermore, suppose that X is of **bounded variation** in the following sense: there are $\mu, M \in \mathcal{R}_{>0}$ (depending on X) such that for any two $z, z' \in X_x$ for some $x \in \mathcal{R}$, we have $\mu < |x - x'| < M$.*[26] *Then assuming **(Unbounded Tally)** we can find $\alpha \in \mathcal{N}$ such that for every r, the fibre X_r has tally bounded above by α (in the sense of Definition 3.6.1).*

[26] Note that such a set need not be bounded (that is, contained in some open box): take X to be the graph of $x \mapsto 1/x$: each fibre has size 1, and so will satisfy the conditions so far set out. That is, the condition of bounded variation (by defintion) only affects places in X with fibres of size > 1.

Proof. The proof of this is actually quite different from the proof of the version for tallies. However, parts of that proof did not directly rely on the assumption **(Tally)**. In particular, by running the safe proofs here, we can again define *normality* and the sets of 'good' and 'bad' points, $\mathcal{G}$ and $\mathcal{B}$ in $\mathcal{R}^1$. In addition, we have the first two claims of that proof: that for every $b \in \mathcal{B}$, there is a least element of the fibre above b which is not normal, and secondly that the (definable) set $\mathcal{B}$ is discrete.

Now, our proof will actually be much quicker from hereon. We start by noting that we defined $\mathrm{St}(X)$ for higher-dimensional discrete sets X, and thus strictly, $\mathrm{St}(X)$ is not defined for our set X.

However, we will define the following related set S:

$$S := \{\langle q, y\rangle \in \mathcal{R}^2 \mid q \in \pi(X) \ \wedge \ y \in \mathrm{St}(X_q)\}$$

Ideally, we would like to say that $\alpha = \mathrm{ot}(S)$ is the desired bound. However, we notice immediately that there are two potential sorts of possible problems this could lead us to. We illustrate these with a simple example:

Let X be the union of the graphs of the three functions $x \mapsto 2 + x^{-2}$, $x \mapsto 0$ and $x \mapsto 1$. Then as $r \to 0$, we can see that $\gamma\mathrm{St}(X_r) \to +\infty$, since the minimum distance between elements, near 0 remains 1, while the greatest element of X_r goes to $+\infty$. Thus, in $\mathrm{St}(X_r)$, that top element will still go to $+\infty$.

Similarly, as $r \to \pm\infty$, we have that the minimum distance between elements of X goes to 0. But then the difference between the first and last elements of the fibre will go to ∞, so we have that $\gamma\mathrm{St}(X_r)$ again goes to $+\infty$.

However, we specifically have made the assumption of *bounded variation* to deal with these problems. We can actually immediately derive our result:

Since X is of bounded variation, we know that for any $a \in \mathcal{R}^m$, $\mu < w_{X_a}$ (where w_{X_a} is the minimum distance between successive elements of the set X_a) and $\gamma X_a - \lambda X_a < M$. Thus, since in general the image of γY under St is always $\frac{1}{w_Y} \cdot (\gamma Y - \lambda Y) + 1$, we have that

$$\mathrm{ot}_{X_a} = \gamma\mathrm{St}(X_a) = \frac{1}{w_{X_a}} \cdot (\gamma X_a - \lambda X_a) + 1 \leq \frac{M}{\mu} + 1$$

That is, the *default overtally* for X_a is bounded by the fixed constant $M/\mu + 1$. Since this value does not depend on $a \in \mathcal{R}^m$, we have that the number $M/\mu + 1$ is a uniform bound.

■

One might hope for a stronger statement of uniform finiteness (that is, which does not restrict the definable sets). However, we would then need to refine the definition of approximate tally to handle this: the examples given during the proof above show that there is, for general definable fibrewise-discrete sets X, no upper bound on the approximate tally. The way that this would have to work is that we would have to give a more nuanced definition of approximate tally which did not rely only on the two values w_X and $\gamma_X - \lambda_X$. Or else, we would have to ask for something slightly weaker in the statement of the theorem.

For example, we could ask that there be an α such that for $a \in \mathcal{R}^m$, there is a neighbourhood $B \ni a$ and a (not neccesarily $\mathcal{R}$-definable) 'deformation' $f_B : X \times [0,1] \to B \times \mathcal{R}$ such that the image has overtally less than α, and such that f preserves the 'abstract tally' of the individual fibres, except at fibres above bad points, in the sense being almost everywhere continuous (or something possibly some stronger condition). A proof would work in this case by showing that the set $\mathcal{G}$ of good points is always open in $\mathcal{R}^m$ (as in the o-minimal case), and then showing that for any point $a \in \mathcal{G}$ there is a function f_B as above on some neighbourhood of a. The latter part would work by piecing together the graphs of the local functions defined in neighbourhoods of points of X in the fibre above a.

3.6.2 A Conjecture on Approximate Tallies and nn-Parts

The following conjecture would be very important to settle. If it is true, then the result in the previous section would show that we could always extend $\mathcal{R}$ to a structure with approximate tallies for all of its definable sets.

Conjecture 3.6.1: *For any LOM+DC+dPHP/pseudo-o-minimal expansion of a field, $\mathcal{R}$, there is some nn-part $\mathcal{N}$ such that $(\mathcal{R}, (\mathcal{N}_{\leq N})_{N \in \mathcal{N}})$ is also LOM+DC+dPHP (respectively, pseudo o-minimal).*[27]

This conjecture has some good intuition behind it, since it works if $\mathcal{R}$ is assumed to be ultra-o-minimal. To be clear, we do not believe that the main roadblock to proving is in securing an nn-part: if no interval initial segments of nn-parts are definable in $\mathcal{R}$ already then we hope that *any* choice of nn-part in $\mathcal{R}$ leads to an LOM+DC+dPHP (resp. pseudo-o-minimal) expansion consisting of $\mathcal{R}$ together with all of this nn-part's (interval) initial segments, and that all of these structures are isomorphic. If $\mathcal{R}$ *does* define some interval initial segments of some nn-part (recall that pairwise, such sets either extend or are extended by the other), then we hope that any nn-part which extends $\langle N_{\leq N} \rangle$ for some such initial segment gives rise, again, to a nice expansion of $\mathcal{R}$ (and again, that the choice should be irrelevant, up to isomorphism).

[27] That is, we may need to assume pseudo-o-minimality to make this work; for now it is quite unclear if this will be the case.

Chapter 4

Some Ultra-O-Minimal Structures

4.1 Introduction

In this chapter, we will investigate examples of ultra-o-minimal structures. A few of these we will be able to show are actually *o-minimal*, including a function related to the exponential function. These are 'new' examples of o-minimal structures, in that they are not even mentioned elsewhere in the literature. And though we are not able to answer some of the natural questions that we might ask about the classes of examples we will introduce, we make a start at the study of them here.

First of all, we will consider the family of examples consisting of ultraproducts of polynomials over $R \vDash \text{RCF}$ considered as functions defined in ultrapowers of R. These functions can naturally be called *pseudopolynomials*. Some specific subfamilies of these will be studied, including the pseudopolynomials which are ultraproducts of Taylor polynomials for real total analytic functions[1] and total Pfaffian functions in particular (see Definition 4.2.3). It ends up that at least some such pseudopolynomials will be o-minimal (i.e. the ultraproduct we construct will be o-minimal). The most important example that we can show is o-minimal, is the ultraproduct of the Taylor polynomials for the exponential function, $\widetilde{\exp}$. Some of our examples will not be o-minimal: replacing exp with sin in the previous example, we will see - as we might expect - that we get a non-o-minimal structure. We will show that this is actually systemic: any pseudopolynomial consisting of Taylor polynomials for a real analytic function f with the property that $(\overline{\mathbb{R}}, f)$ is not o-minimal will result in a structure $(\mathcal{R}, \widetilde{f})$ which is not o-minimal. In fact, we will see that this can be generalized: I will introduce the notion of a *pseudopolynomial approximation*, of which these are all examples, and show that if we take such an approximation, $\widetilde{f}$, of a real function f with $(\overline{\mathbb{R}}, f)$ non-o-minimal, then the structure $(\mathcal{R}, \widetilde{f})$ will also be non-o-minimal. This result is not as straightforward as it may seem at first glace: the fact that $(\overline{\mathbb{R}}, f)$ is not o-minimal does not immediately show that $\widetilde{f}$ will be non-o-minimal: even if $\varphi(x)$ is an $\mathcal{L}_{RCF}(f)$-formula which defines a subset of $\mathcal{R}$ with infinitely many components, it is not clear prima facie that reinterpreting f as $\widetilde{f}$ in φ would necessarily result in a formula for a definable set with infinitely many components. So to prove this result, we apply some recent work of Marikova on o-minimal structures with an added 'standard part map' to derive our result.

A second family of examples will come from what I call *pseudodynamical functions*: they are infinite iterates of a fixed unary function f. This will rely on noting that in general ultra-o-minimal structures, there is a completely natural notion of the α^{th}-iterate of a function f. Some interesting examples of these functions will be put forward: when f is a polynomial or rational function, the theory is already difficult; and when f is exp, the question of its o-minimality is seemingly quite intractable at the moment. However, the latter function being o-minimal would answer a long open question about o-minimal structures.[2] We will

[1] That is, real functions with everywhere convergent Taylor series - in particular, those that are restrictions of *entire* complex analytic functions to the real axis.

[2] At least, it would answer it for general o-minimal structures: the case where the universe is the reals would still be an open question.

not, in the end, have that much to say about this example: as we expected from the outset, the study of this example is quite difficult: it involves considering the 'eventual' behaviour of sets defined by exponential polynomials, and since the latter are already quite complicated to deal with, this problem (even in low dimensions, and simplified particular cases) has resisted our efforts. Nonetheless, I present some thoughts about how to approach it.

In this chapter, the focus will be exclusively on ultra-o-minimal structures. As such, we will assume from now on that $\mathcal{R}$ is an ultra-o-minimal expansion of a real closed field. In particular, the default presentation for $\mathcal{R}$ will be $\prod_{i \in I} \mathcal{R}_i / \mathcal{U}$ for I an index set and $\mathcal{U}$ a (non-principal) ultrafilter on I. Most of the actual examples will have $I = \mathbb{N}$, so sometimes we will assume this without specifically reminding the reader. Most of our examples will be of the form $\mathcal{R} = (\overline{\mathbb{R}}^I / \mathcal{U}, G)$, in that the language will consist of $\mathcal{L}_{RCF}(G)$ for a new unary function symbol G, and the structure $\mathcal{R}$ will be, in the reduct to $\mathcal{L}_{RCF}$, just an ultrapower of $\overline{\mathbb{R}}$. In particular, all of our structures will at least be real closed fields (so if we write (R, P), we mean $(\overline{R}, P)$ implicitly).

4.2 Pseudopolynomials

In this section, I will discuss examples of the following form: let $R \vDash \text{RCF}$, and let $(p_i)_{i \in I}$ be an I-indexed sequence of polynomials $p_i \in R[x]$ (where x could be a tuple of variables), and consider the $\mathcal{L}_{RCF}(p)$-structure consisting of $\mathcal{R} = (R^{\mathcal{U}}, (p_n)_{n \in \mathbb{N}}/\mathcal{U})$, writing $\widetilde{p}$ for $(p_n)_{n \in \mathbb{N}}/\mathcal{U}$ in what follows.

Definition 4.2.1: *We call a function $\widetilde{p}$ as above a* ***pseudopolynomial (over** $R^{\mathcal{U}}$**)**.

- *The set of pseudopolynomials naturally has a ring structure, just like the ring of (ordinary polynomials), and we write $R^{\mathcal{U}}[x]_\infty$ for this ring.*[3] *See [BDLV79] for more on this topic. We note that we have the following natural system of embeddings (where $R[x]_\infty$ is the subring of $R^{\mathcal{U}}[x]_\infty$ consisting of those pseudopolynomials with coefficients only in R):*

$$\begin{array}{ccccc} R & \hookrightarrow & R[x] & \hookrightarrow & R[x]_\infty \\ \downarrow & & \downarrow & & \downarrow \\ R^{\mathcal{U}} & \hookrightarrow & R^{\mathcal{U}}[x] & \hookrightarrow & R^{\mathcal{U}}[x]_\infty. \end{array}$$

- *We define the* ***degree of*** $\widetilde{p} \in R^{\mathcal{U}}[x]_\infty$ *to be* $\alpha = \prod_{i \in \mathcal{I}} \deg(p_i)/\mathcal{U} \in \mathbb{N}^*$.

4.2.1 Taylor Pseudopolynomials

Let f be a real analytic function with Taylor polynomials $(T_n)_{n \in \mathbb{N}}$ defined on the set dom f. In particular, for every $x \in \operatorname{dom} f \subseteq \mathbb{R}$, we have $\lim_{n \to \infty}(T_n(x)) = f(x)$. Thus, in $\mathcal{R} = (\mathbb{R}^{\mathcal{U}}, \widetilde{T}) = (\mathbb{R}^{\mathcal{U}}, (T_n)_{n \in \mathbb{N}}/\mathcal{U})$, we have for all $x \in \operatorname{dom} f \subseteq \mathbb{R}$, that $\widetilde{T}(x) = f(x) + \epsilon_f(x)$ for some infinitesimal (not necessarily definable) function ϵ_f.[4] By definition, $\epsilon_f(x)$ equal to $\widetilde{T}(x) - f(x)$; i.e it is the 'error function' for the approximation of f by the Taylor polynomials $(T_n)_{n \in \mathbb{N}}$.

We call a function $\widetilde{f} = \widetilde{T}$ for some real analytic f with $(T_n)_{n \in \mathbb{N}}$ its sequence of Taylor polynomials the **(default) Taylor Pseudopolynomial for** f.

We use the word 'default' because, for example, letting σ be a permutation of $\mathbb{N}$, we have that $\widetilde{f}^* := \prod_{n \in \mathbb{N}} T_{\sigma(n)}/\mathcal{U}$ will also satisfy $\widetilde{f}^*(x) = f(x) + \epsilon^*_f(x)$ for some infinitesimal function ϵ^*_f. Similarly, the same property is also satisfies by $\widetilde{f}^{**} := \prod_{n \in \mathbb{N}} T_{n+1}/\mathcal{U}$, and many other such pseudopolynomials.[5]

Example 4.2.1:

- *$T_n \sim P$ for some polynomial P. Thus, $T_n = P$ for sufficiently large n. Thus, $\widetilde{P} = P$.*

---

[3] This generalizes the notation '$R[x]_n$' for the polynomials over R of degree $\leq n$.

[4] We are not assuming that $\epsilon_f(x)$ is positive, let alone non-zero everywhere.

[5] Note that these two examples are not the same: $n \mapsto n+1$ is *not* a permutation of $\mathbb{N}$. The most general statement of these examples is that if σ is a self-map of $\mathbb{N}$ with infinite image, we get a new Taylor pseudopolynomial with the same property.

- $T_n \sim \exp$. *We will see below that this structure is o-minimal.*

- $T_n \sim \sin$. *This structure is not o-minimal: since* $\widetilde{\sin}(x) = \sin(x) + \epsilon(x)$ *for all* $x \in \mathbb{R}$ *for some infinitesimal function* ϵ*, in particular letting*

$$Z(\widetilde{\sin}) := \{x \in \mathcal{R} \mid \widetilde{\sin}(x) = 0\},$$

we have that $Z \cap \mathbb{R}$ *has infinitely many components. That is,* $x \in Z \cap \mathbb{R}$ *if and only if there is a (unique)* $n \in \mathbb{Z}$ *such that* $x = n\pi + \epsilon(x)$*, and since* ϵ *is infinitesimal,* $Z \cap \mathbb{R}$ *is discrete (actually, since* Z *is an ultraproduct of discrete sets, it is discrete everywhere, not just on* $\mathbb{R}$*). Thus* Z*, a definable set in* $\mathcal{R}$*, has infinitely many components as well.*

A consequence of Theorem 4.2.1 below is that if f is real analytic and $(\overline{\mathbb{R}}, f)$ is not o-minimal (as with $f = \sin$), then $(\mathcal{R}, \widetilde{f})$ is also not o-minimal. Also, we will see later that if f is a special kind of real analytic function, called Pfaffian, then $(\overline{\mathbb{R}}, f)$ is o-minimal, and if $\widetilde{f}$ is *also* Pfaffian, then we obtain that $(\mathcal{R}, \widetilde{f})$ is o-minimal as well. At this point, this is the extent to which we can classify the examples of Taylor pseudopolynomial approximations.

4.2.2 General Pseudopolynomial Approximations

There is no reason why we need to specifically restrict to the cases where P is a sequence of Taylor polynomials for a function. In general:

Definition 4.2.2: *If* f *is a function, and* $\widetilde{f} = \prod_{i \in I} f_i / \mathcal{U}$ *a pseudopolynomial such that for all* $x \in \operatorname{dom} f \subseteq \mathbb{R}$*, we have* $\lim_{n\to} P_n(x) = f(x) + \epsilon_f(x)$ *for some infinitesimal function* ϵ_f*, then we say that* $\widetilde{f}$ *is a* ***pseudopolynomial approximation of*** f.

The study of pseudopolynomial approximations seems quite interesting due to the combined facts that (a) a pseudopolynomial approximation is, on a convex neighbourhood of 0 containing the finite elements of $\mathcal{R}$, only off from the given function f by an infinitesimal; and (b) that most likely, f will not be definable in $\mathcal{R}$. The latter is likely because for most functions f, $\widetilde{f}$ and f will diverge from each other eventually (i.e. away from the aforementioned convex set). For instance, with $f = \exp$, it is a consequence of Lemma 4.3.1 that for any pseudopolynomial approximation, $\widetilde{f}$, eventually $|\exp - \widetilde{f}| > r$ for any $r \in \mathbb{R}$.

Theorem 4.2.1: *If* f *is a function such that* $(\overline{\mathbb{R}}, f)$ *is not o-minimal, and if* $\widetilde{f} = \widetilde{P}$ *is some pseudopolynomial approximation of* f*, then* $(\mathcal{R}, \widetilde{f})$ *is not o-minimal.*

We note that as in the case where $f = \sin$, the conclusion is almost immediate. However, for general f, a formula which defines (with f) a subset of $\mathbb{R}$ witnessing that f is not o-minimal does not, seem to have to a priori, be a formula which (with $\widetilde{f}$) defines a subset of $\mathcal{R}$ witnessing that $\widetilde{f}$ is not o-minimal. In particular, the formula could contain many quantifiers and so geometrically be quite a mess of projections and complements etc.

So what we will do is apply some recent work of Marikova in [Ma10, Ma11]. In the second referenced paper, she proves the following theorem (which I preface with some notation from her paper):

Let R be an o-minimal expansion of a real closed field, V a proper convex subring of R, and let $\mathrm{st}: V \to \mathfrak{K}$ be the corresponding residue (i.e 'standard part') map with kernel $\mathfrak{m}$, and residue field $\mathfrak{K} = V/\mathfrak{m}$. Let $\mathrm{st}X := \mathrm{st}(X \cap V^n)$ for $X \subseteq R^n$, and let $\mathfrak{K}_{ind}$ denote the $\mathfrak{K}$ expanded by symbols for all the sets $\mathrm{st}X \subseteq \mathfrak{K}^n$ for definable $X \subseteq R^n$ for some n. Say that $(R, V) \vDash \Sigma(n)$ if for every definable $X \subseteq [-1,1]^{1+n}$ there is $\epsilon_0 \in \mathfrak{m}_{>0}$ such that $\mathrm{st}X_{\epsilon_0} = \mathrm{st}X_\epsilon$ for all $\epsilon \in \mathfrak{m}_{>\epsilon_0}$. And say that $(R, V) \vDash \Sigma$ if $(R, V) \vDash \Sigma(n)$ for every n.

Theorem 4.2.2 ([Ma11], Theorem 1.2)**:** *The following are equivalent:*

1. $\mathfrak{K}_{ind}$ *is o-minimal*
2. $(R, V) \vDash \Sigma$
3. $(R, V) \vDash \Sigma(1)$.

She also proved the following in the earlier paper:

Lemma 4.2.1 ([Ma10], Lemma 3.4)**:** *Let R be ω-saturated. Then letting V be the convex hull of $\mathbb{Q}$ in $\mathcal{R}$, we have $(R, V) \vDash \Sigma$.*

Proof. Using this result, the proof of our result is a quick application: if $\mathcal{R}' = (\mathcal{R}, \widetilde{f})$ were o-minimal, then since ultraproducts are ω-saturated, we let V be the convex hull of $\mathbb{Q}$ in $\mathcal{R}'$, and obtain that $(\mathcal{R}', V) \vDash \Sigma$. But then $\mathfrak{K}_{ind}$ is o-minimal. But since $(\overline{\mathbb{R}}, f)$ is a reduct of this structure, it must be o-minimal too, a contradiction. ∎

We would also like to eventually show the converse of this theorem. A proof of this would likely rely on the properties mentioned before the theorem: in some sense, $\widetilde{f}$ has two distinct pieces: the piece where it is infinitesimally close to f, and the part where they separate. If a set were definable in $\mathcal{R}$, (assuming that f is o-minimal), then we would like to show that it comes from a definable set in $(\mathbb{R}, f)$, by replacing the occurences of f with $\widetilde{f}$; and that in virtue of this, there is a way to break this set into the part defined where $\widetilde{f} \sim f$, and the part where $\widetilde{f} \sim c{\cdot}x^r$ for some c, r. (That $\widetilde{f}$ is eventually $\sim c{\cdot}x^r$ is again a consequence of Lemma 4.3.1 below.) The hope is that it would then be fairly striaghtforward to show that each piece is o-minimal.

4.2.3 Pfaffian Functions

There is much to know about Pfaffian functions and their position within the theory of o-minimality. For a short survey of the basics and their relevance to o-minimality, see [Ma97]. Pfaffian functions are always real analytic, and in particular, include the exponential function. The definition below specializes to the standard definition (i.e for when $\mathcal{R}$ below is o-minimal) but also works for our more general context:

Definition 4.2.3: *Let $\mathcal{R}$ be an LOM+DC expansion of an ordered field. Let $f_1, ..., f_s : \mathcal{R} \to \mathcal{R}$ be definable and $\mathcal{C}^\infty$. We say that $(f_1, ..., f_s)$ is a (one-variable)*[6] *Pfaffian chain in $\mathcal{R}$ if $f_i' \in \mathcal{R}[x, f_1, ..., f_i]$ for $i = 1, ..., s$. A definable map $F : \mathcal{R} \to \mathcal{R}$ is a (one-variable) Pfaffian function in $\mathcal{R}$ if $F \in \mathcal{R}[x, f_1, ..., f_s]$ for some Pfaffian chain $(f_1, ..., f_s)$ in $\mathcal{R}$.*

Crucially, the definition requires that the F and the functions f_i' be *ordinary polynomials* over $\mathbb{K}$. In particular, they each must have some finite degree. With this in mind, we define:

Definition 4.2.4: *The **complexity** of a Pfaffian function F as in the above definition is the sequence $(s, \deg q, (\deg p_j)_{j \le s})$, where $F(x) = q(x, f_1(x), ..., f_s(x))$, and $f_j'(x) = p(x, f_1(x), ..., f_j(x))$. (That is, the complexity of F is the sequence consisting of the length of the chain, and the degrees of all the polynomials witnessing the definition.)*

With that definition, we can record the following theorem. It is the tool that we need for our examples:

Theorem 4.2.3 ([FS10])**:** *Let $\mathcal{R}$ be an LOM+DC expansion of a field by a family of Pfaffian functions. Then $\mathcal{R}$ is o-minimal.*

In particular, this says that if a function in an ultra-o-minimal structure is Pfaffian in the above sense, then it is o-minimal.

In this particular case though, we do not need the full power of the theorem above. In fact, there is a significantly different and far shorter proof of the weaker version that we are using. But before we state and prove it, we record a couple of important facts we will use about Pfaffianity in the *o-minimal* setting, specifically:

Theorem 4.2.4 (see [Ma97], Corollary 6.1)**:** *Let $\varphi(x, y)$ be a quantifier-free $\mathcal{L}_{\mathrm{Pf}}$, which is the language $\mathcal{L}_{RCF}$ augmented by a symbol for all Pfaffian functions, and let $\mathbb{R}_{\mathrm{Pf}}$ be the structure consisting of the real field together with the standard interpretations of all of these functions. Then there is a number N such that $Z = \{x \in \mathbb{R}^n_{\mathrm{Pf}} \mid \mathbb{R}_{\mathrm{Pf}} \models \varphi(x, a)\}$ has at most N connected components for all $a \in \mathcal{R}$.*

This implies that for a quantifier-free formula, there is a bound on the number of connected components that the set that formula defines in $\mathbb{R}$ which is *independent of the parameters.* But we also have the following:

Theorem 4.2.5 ([Wi96])**:** *If $\mathcal{R}$ is an expansion of the real field by $\mathcal{C}^\infty$ functions such that the quantifier-free definable sets have finitely many connected components and such that this bound is independent of parameters, then $\mathcal{R}$ is o-minimal.*

This in particular, implies that the structure $\mathbb{R}_{\mathrm{Pf}}$ is o-minimal. We will not use that, but it is an important consequence. Instead we will use the combination of facts to prove the aforementioned weaker version of Theorem 4.2.3:

[6] The definition of Pfaffian functions can be extended to higher arity functions, but we will not require this for our examples.

Lemma 4.2.2: *If $\mathcal{R}$ is an o-minimal ultraproduct of o-minimal expansions of a real closed field, then its expansion by a family of Pfaffian functions is also o-minimal.*

Proof. Let $\mathcal{R} = \prod_{i\in I} \mathcal{R}_i/\mathcal{U}$ be an o-minimal expansion of a real closed field, and let $(F_j)_{j\in J}$ be a family of Pfaffian functions. We will show that any reduct $(\mathcal{R}, F_1, ..., F_n)$ of $(\mathcal{R}, (F_j)_{j\in J})$ is o-minimal. This suffices since if the latter were not o-minimal, there would be a formula $\varphi(x)$ (possibly in many variables), which defined a set with infinitely many components. However, φ could only mention finitely many of the functions F_j, and so $\mathcal{R}$ together with these functions would already be non-o-minimal. *In fact,* we will only prove it for $n = 1$, since the proof is the same, only requiring more indices to keep track of. Thus, in what follows the subscript n on F_n refers to the index in the ultrafilter.

Let $\varphi(x, y)$ be a quantifier-free formula. Now since each index model $\mathcal{R}_n := (\mathbb{R}, F_n)$ is an expansion of $\mathbb{R}$ by a $\mathcal{C}^\infty$ function, Khovanskii's theorem (Theorem 4.2.4) tells us that there is a number N_n depending only on the complexity of F_n which bounds the number of connected components of $\varphi(\mathcal{R}_n, a)$ independent of the parameters a that are chosen.

Since the function F is an ordinary polynomial in $\mathcal{R}[x, f_1, ..., f_s]$, it has some finite degree d. But then in $\mathcal{U}$-most index models, the function F_n also must have degree d as a polynomial in $\mathbb{R}[x, f_{n,1}, ..., f_{n,s}]$. We can make the same obvservation about the polynomials $q_{n,i}$ which witness $f'_{n,i} \in \mathbb{R}[x, f_{n,1}, ..., f_{n,i}]$. And finally, since s is finite, the length of the Pfaffian chain $(f_{n,1}, ..., f_{n,s_n})$ must be this finite number s in $\mathcal{U}$-most index models as well.

But the complexity of F is determined by exactly these finite quantities; thus, in $\mathcal{U}$-most index models, the F_n have this fixed complexity. But the bound on the components of $\varphi(\mathcal{R}_n, a)$ depends only on the complexity. Thus, there is a uniform bound N on the components of $\varphi(\mathcal{R}_n, a)$ that does not depend on the parameters. Thus, this is also a bound in $\mathcal{R}$. But then, Wilkie's theorem (Theorem 4.2.5) implies that $\mathcal{R}$ is o-minimal. ∎

4.2.4 $\widetilde{\exp}$ is O-Minimal

As the title suggests, we will show in this section that $(\mathbb{R}^{\mathcal{U}}, \widetilde{\exp})$ is o-minimal. We will prove this by showing that it is a reduct of an o-minimal structure, and we will show that using the lemma just proved in the previous section, Lemma 4.2.2.

Extend $\mathcal{L}_{RCF}$ by a new function symbol f, and two new constant symbols, c and d. In this extended language, let $\mathcal{R} = \prod_{n\in\mathbb{N}_{>0}} \mathcal{R}_n/\mathcal{U}$ for $\mathcal{U}$ any non-principal ultrafilter, where $\mathcal{R}_n$ consists of $\overline{\mathbb{R}}$ with the interpretations $f^{\mathcal{R}_n} = (x \mapsto x^n)$, $c^{\mathcal{R}_n} = n$ and $d^{\mathcal{R}_n} = n!$. These structures are clearly o-minimal. In $\mathcal{R}$, we write $N = c^{\mathcal{R}} = (1, 2, 3, ...)_{\mathcal{U}}$ and $\frac{x^{N-1}}{(N-1)!}$ for the function $\frac{1}{d^{\mathcal{R}}} \cdot f^{\mathcal{R}}$ (i.e. the function $x \mapsto \left(1, x, \frac{x^2}{2!}, \frac{x^3}{3!}, ...\right)_{\mathcal{U}}$.)

Corollary 4.2.1 (Of Lemma 4.2.2)**:** $(\mathcal{R}, \widetilde{\exp})$ *is o-minimal.*

Proof. Define the following restricted functions:

$$f_1 := \frac{1}{x} \upharpoonright (0, \infty),$$

$$f_2 := \frac{x^N}{N!} \restriction (0,\infty),$$

$$\text{and } f_3 = F := g^{\mathcal{R}'} \restriction (0,\infty) = \widetilde{\exp} \restriction (0,\infty).$$

Note that f_1, f_2 and f_3 form a Pfaffian chain for F (at least, relative to the domain $(0,\infty)$): by definition $F = f_3$, and we have

$$f_3' = \widetilde{\exp}' = (0, 1, 1+x, 1+x+\frac{x^2}{2}, \ldots)_{\mathcal{U}} = (1, 1+x, 1+x+\frac{x^2}{2}, \ldots)_{\mathcal{U}} - (1, x, \frac{x^2}{2}, \ldots) = f_3 - f_2,$$

$$f_2' = (N-1) \cdot \frac{x^{N-2}}{(N-1)!} = (N-1) \cdot f_1 f_2$$

$$\text{and } f_1' = -f_1^2$$

Thus, F is a Pfaffian function relative to the domain $(0,\infty)$.

Similarly, let $G = \widetilde{\exp} \restriction (-\infty, 0)$ and define g_1, g_2 and g_3 in analogy with f_1, f_2 and f_3 above, but with their domains restricted to $(-\infty, 0)$. By the same reasoning, g_1, g_2, and g_3 also form a (restricted) Pfaffian chain and thus G is also a (restricted) Pfaffian function.

Now, the structure $(\mathcal{R}, F, G)$ is an ultraproduct of o-minimal structures, expanded by a (two-element) family of Pfaffian functions.[7] Consequently, by Lemma 4.2.2, this structure is o-minimal. But since $\widetilde{\exp}$ is definable in this structure, we have that the reduct $(\mathcal{R}, \widetilde{\exp})$ is o-minimal as well. ∎

4.2.5 A Variant of $\widetilde{\exp}$ Is Also O-Minimal

There are other choices of pseudopolynomial approximations of exp which are Pfaffian as well.

It is well known that $\lim_{n\to\infty}(1+\frac{x}{n})^n = \exp x$. With this in mind, we define $P_n(x) = (1+\frac{x}{n})^n$. (Note that these polynomials are certainly not the Taylor polynomials for exp.) And we let $P = \prod_{n\in\mathbb{N}_{>0}} P_n/\mathcal{U}$.

Then the function P will also be Pfaffian, so:

Corollary 4.2.2: *$(\mathcal{R}, P)$ is o-minimal.*

Proof. Doing a proof similar to the one in the previous section, we compute

$$P' = N\left(1+\frac{x}{N}\right)^{N-1} \cdot \left(\frac{1}{N}\right) = \left(1+\frac{x}{N}\right)^{N-1} = P \cdot \left(\frac{1}{1+\frac{x}{N}}\right)$$

[7] The fact that F and G are technically restricted is unimportant since they are restricted to single intervals. (Though it would be important if we restricted them to a Cantor set, for example!) We could trivially extend them to total functions (along with the f_i and g_i) and $\widetilde{\exp}$ would still be definable in this structure, of course.

$$\text{and } \left(\frac{1}{1+\frac{x}{N}}\right)' = \frac{-\frac{1}{N}}{(1+\frac{x}{N})^2} = \frac{-1}{N(1+\frac{x}{N})^2} = -\frac{1}{N}\cdot\left(\frac{1}{(1+\frac{x}{N})}\right)^2 :$$

Thus, we can define $q_2 = Q = P \restriction (0,\infty)$, and $q_1 = \frac{1}{1+\frac{x}{N}}$. It is easy to see that Q is (restricted) Pfaffian, and that an analogous definition works to the left of 0. The rest is the same. ∎

An open question is whether this will be true for any pseudopolynomial approximation of exp:

Question: *If P is a pseudopolynomial approximation of* exp*, or in more generality, of any Pfaffian function, is P also Pfaffian, and in particular, is $(\mathcal{R}, P)$ also o-minimal?*

4.3 Pseudodynamical Functions

In this short section, we will discuss another family of examples of ultra-o-minimal structures. And though we will have less that we can specifically conclude about them, we will discuss the importance of one of the examples of this family, and the relationship of this family to other problems in model theory.

4.3.1 Notation and Some Definitions

Suppose that $\mathcal{R} = (R^{\mathcal{U}}, ...)$, and that $f : R \to R$ is some function on R. Then the ultraproduct $\prod_{\imath \in I}(f^{(\alpha_i)})/\mathcal{U}$ for some $\alpha \in \mathbb{N}^* \subsetneq R^{\mathcal{U}}$ is called the α^{th}**-iterate of** f.

Note that it is not necessarily the case that the function f will be definable in $(\mathcal{R}, f^{(\alpha)})$. So we will assume that f is definable already in $\mathcal{R}$, even though in some cases (for example when $f = \exp$), we could making this assumption.

Definition 4.3.1: *For $f : R \to R$ a unary function on a real closed field R, we call the family of functions $(f^{(\alpha)} : \mathcal{R} \to \mathcal{R})_{\alpha \in \mathbb{N}^*}$ a **pseudodynamical system**, and we refer to functions which are the α^{th} iterate of some function f as **pseudodynamical functions**.*

Note that if $\mathcal{U}$ were principal, and $R = \mathbb{R} = \mathcal{R}$, then a pseudodynamical system is just a dynamical system in the ordinary sense.[8]

4.3.2 Polynomial and Rational Examples

From now on, let $I = \mathbb{N}_{>0}$ and $R = \mathbb{R}$ (and thus $\mathcal{R} = \mathbb{R}^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$ on $\mathbb{N}_{>0}$). And fix an infinite $N := \prod_{k \in \mathbb{N}_{>0}} k/\mathcal{U} = (1,2,3,4,...)_{\mathcal{U}}$.

[8] This definition is interchangable with various other possible definitions of a 'dynamical system': for instance, is it similar to saying that a dynamical system consists of a set X together with a self-map f. In our case, we could also just as easily assumed that dom $f \subseteq R$, but since all our examples will be total we will just assume that f is too.

If f is chosen to be a polynomial, we will find already that our problem is quite difficult if we set as our task the classification of for which f, will $(\mathcal{R}, f^{(N)})$ be o-minimal.

Example 4.3.1:

1. *$f = rx^n$, for some $n \in \mathbb{N}$ and $r \in \mathbb{R}^{\neq 0}$.*
   - *if $n = 0$, $f^{(N)} \equiv r$.*
   - *otherwise, $f^{(N)}(x) = r^{n\cdot(N-1)} x^{n\cdot N}$.*
2. *$f = rx^{-n}$, for some $n \in \mathbb{N}_{>0}$ and $r \in \mathbb{R}^{\neq 0}$.*
   - *let $K = (1, 1-n, 1-n+2n, 1-n+2n-3n, ...)_{\mathcal{U}} = \sum_{k=1}^{N} (-1)^{k-1} \cdot N$.*
   - *then either $f^{(N)}(x) = r^K x^{-n\cdot N}$ (if $\{1,3,5,...\} \in \mathcal{U}$)*
   - *or $f^{(N)}(x) = r^K x^{n\cdot N}$ (if $\{2,4,6,...\} \in \mathcal{U}$).*
3. *$f = \frac{ax+b}{cx+d}$ for $a,b,c,d \in \mathbb{R}$, with either $c \neq 0$ or $d \neq 0$.*
   - *let $M = (m_{ij})_{1\le i,j\le 2}$ be the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. And let the number A_k for $k \in \mathbb{N}^*$ be the $(1,1)$-entry in M^k (matrix multiplication), and let $A = (A_1, A_2, A_3, ...)_{\mathcal{U}}$. Similarly define B_k, C_k, D_k using M^k and define $B, C, D \in \mathbb{N}^*$ analogously to A.*
   - *Then $f^{(N)}(x) = \frac{Ax+B}{Cx+D}$.*

     (Notice that f is just the action of an element of the group $\mathrm{PSL}(2,\mathbb{R})$ (if not of an even simpler form due to some coefficients being 0): thus, since multiplication of elements of $\mathrm{PSL}(2,\mathbb{R})$ correspond to composition of their corresponding actions, the family $(f^{(n)})_{n\in\mathbb{N}_{>0}}$ consists only of actions of elements of $\mathrm{PSL}(2,\mathbb{R})$. Thus, $f^{(N)}$, as an ultraproduct of them, will be the action of an element of $PSL(2,\mathcal{R})$, and in particular of the same form, but taking coefficients from $\mathcal{R}$ instead of $\mathbb{R}$.)
   - *The particular case when $c = 0$ gives us that $f = a'x + b'$, and thus, we will have $f^{(N)}(x) = Ax + B$ for some $A, B \in \mathcal{R}$.*
   - *In any case, all of these functions are o-minimal, since they are definable (with parameters) already in $\mathcal{R}$ just using the field language.*
4. *For f a general higher-degree polynomial $x \mapsto p(x)$ or $x \mapsto \frac{p(x)}{q(x)}$, the best we can say for now is that $f^{(N)}$ is a pseudopolynomial P or a ratio of pseudopolynomials $\frac{P}{Q}$, which we dub a* ***pseudorational function****.*
   - *In particular, when $f = ax^n + bx^k + ...$ for some $n > 1$, $k \in \mathbb{N}$, and $a, b \neq 0$, then $f^{(N)}$ will most likely be a pseudopolynomial with infinitely many different monomials of different degrees. For example, if $f = x^2 + 1$, then $f^{(N)}$ will have infinitely many non-zero even-degree terms.*

The last example is clearly not particularily satisfying: we have already seen how difficult it is to classify even the pseudopolynomials into the o-minimal and non-o-minimal ones; the case of 'pseudorationals' promises to be even more complicated. There is an obvious reason why these examples quickly become so difficult to handle: the model theoretic study of algebraic dynamics studies the eventual behaviour of polynomial and rational dynamical systems, and this field is relatively new, with a large space of open problems. Some progess has been made though. For instance, see [MeSc] for more on algebraic dynamics and in particular for the current limit of our understanding of polynomial algebraic dynamical systems.

However, there is at least one subfamily of these examples which are always o-minimal: the ones which have finitely many terms in the ultraproduct. That is:

Fact 4.3.1: *Let f be a pseudorational function given by a ratio $\frac{P}{Q}$ of pseudopolynomials. (In particular, f would just be a pseudopolynomial if Q were a factor of P.) Then if P and Q have finitely many terms, i.e $\deg(P), \deg(Q) \in \mathbb{N}$, then $(\mathcal{R}, f)$ is o-minimal.*

The proof is trivial, as such functions are either already definable in $\mathcal{R}$ with parameters just using the field language, or can be defined after adding finitely many power functions, each of which can be added to $\mathcal{R}$ o-minimally.

4.3.3 Powerboundedness

Recall the definition of powerboundedness from Chapter 1: an LOM+DC structure $\mathcal{R}$ is **powerbounded** if every unary definable function f is eventually bounded by some definable power function x^r. Recall that $\mathcal{R}$ is powerbounded or else defines the exponential function. We define $\mathcal{R}$ to be **exponentially bounded** if every unary definable function f is eventually bounded by $\exp^{(n)}$.

We record the following quick fact, before looking at the case where $f = \exp$.

Lemma 4.3.1: *If $f : \mathbb{R} \to \mathbb{R}$ is a function definable in a powerbounded o-minimal expansion of $\mathbb{R}$ (and hence polynomially bounded), then $(\mathcal{R}, f^{(N)})$ is powerbounded. And in general, if $\widetilde{f}$ is a pseudopolynomial, then $(\mathcal{R}, P)$ is powerbounded.*

Proof. Since each iterate $f^{(n)}$ of f is definable in $(\overline{\mathbb{R}}, f)$, we have that there is are $k_n \in \mathbb{N}$ and $x_n \in \mathbb{R}$ such that $f^{(n)}(x) < x^{k_n}$ for all $x > x_n$. But then with $K = \prod_{n \in \mathbb{N}_{>0}} k_n / \mathcal{U}$ and $x_N = \prod_{n \in \mathbb{N}_{>0}} x_n$, we have $f^{(N)}(x) < x^K$ for all $x > x_N$.

The second statement follows analogously. ■

In particular, the examples in the previous section are all powerbounded.

4.3.4 On $\exp^{(\alpha)}$

In this short section, we finish the chapter by mentioning the example of $f = \exp$.

The following has been an open question ever since Miller proved the Growth Dichotomy (first, for expansions of $\mathbb{R}$ in [Mi94]):

Question: *Are there any non-exponentially bounded o-minimal structures? In particular, are there any that expand* $\mathbb{R}$*?*

If $(\mathcal{R}, \exp^{(\alpha)})$ is o-minimal, then it would answer the first question affirmatively, while leaving the latter open.

Approaching this question is quite difficult given our current tools. Even in one variable, and with other simplifying assumptions, the functions involved are quite difficult to analyze.[9] However, we will end with some comments about a possible line of attack:

Wilkie showed in [Wi99] that the structure $\mathbb{R}_{\exp} = (\overline{\mathbb{R}}, \exp)$ is o-minimal, and in [Wi96] that it is model complete. A result of this is that the definable sets in this structure consist exactly of the projections of zerosets of *exponential polynomials*. An exponential polynomial is a function $f = P(\overline{x}, \exp\overline{x})$ for $P(\overline{X}, \overline{Y})$ a polynomial. The main steps we might hope to take to show the o-minimality of $(\mathcal{R}, \exp^{(\alpha)})$ are first showing that definable sets are just projections of zerosets of what we call 'pseudoexponential polynomials',[10] which are functions $f = P(\overline{x}, \exp^{(\alpha)}\overline{x})$, for $P(\overline{X}, \overline{Y})$ a polynomial; then second, to show that these sets are as well behaved as the zerosets of exponential polynomials. The idea we have in mind for the latter step would involve showing that exponential polynomials of the form $f = P(\overline{x}, \exp^{(k)}\overline{x})$ are well-behaved as $k \to \infty$; that is, that eventually in k, the number of definably connected components of the projection of the zeroset of such a function becomes constant.

[9]Note that $\exp^{(\alpha)}$ is not Pfaffian a priori, even though it is *pseudoPfaffian* in a certain sense. Thus we cannot directly apply Lemma 4.2.2.

[10]Unfortunately, the terminology 'pseudoexponential' is in use elsewhere in model theory; but it seems unlikely to cause much confusion here.

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