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Fractional Taylor Series Expansions for Classes of Generalized Fractal Strings

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Fractional Taylor Series Expansions for Classes of Generalized Fractal Strings

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Matthew Overduin

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ABSTRACT OF THE DISSERTATION

Fractional Taylor Series Expansions for Classes of Generalized Fractal Strings

by

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University of California, Riverside, June 2024
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The study of fractals and their associated complex dimensions has led to the development of a new form of calculus. In [LvF13], Michel Lapidus and Machiel van Frankenhuijsen introduce two kinds of fractals, ordinary fractal strings and generalized fractal strings. Generalized fractal strings are locally compact measures taking place over the positive real line with mass near zero. In [LvF13], Lapidus and van Frankenhuijsen derive a way of recovering a generalized fractal string from it known complex dimensions via an explicit formula.

In this thesis, we offer two key results which involve Taylor series expansions of fractals. The first result involves writing the explicit formula as a Taylor series, summing over the fractional derivatives of δ where the order is taken at the complex dimensions of the generalized fractal string. The second result is specific to more recent work done by Michel Lapidus and Claire David in [DL22a] and [DL22b]. This result involves the determination of coefficients of a fractal power series for the Weierstrass graph, mentioned in [DL22b]. Both results are important as they contribute to the development of fractal calculus.

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Outline of Chapter Contents

- Chapter 2 provides an introduction into the subject of fractal geometry in the context of sequences of real numbers. Concepts such as ordinary fractal strings, fractal tube formulae, Minkowski dimension, geometric zeta function, and complex dimensions are all defined. Examples of ordinary fractals are also provided.

- Chapter 3 discusses generalized fractal strings and how we can recover these fractal strings from their complex dimensions via an explicit formula under specific conditions (called languid conditions). Definitions and examples of generalized fractal strings are provided before the explicit formula is introduced. Several theorems are presented involving the explicit formula, presenting both pointwise cases and distributional cases. At the end, an example of the Cantor string is given as a means to show how the string can be recovered from its explicit formula via its complex dimensions.

- Chapter 4 provides the definition of the distributional fractional derivative of a distribution borrowed from Laurent Schwartz. This definition allows us to compute the fractional derivative of a distribution and to re-express the explicit formula as a fractional Taylor series involving complex-ordered derivatives of δ taken to the order of the complex dimensions. Certain techniques were used to do this including Hadamard regularization and meromorphic continuation.

- Chapter 5 introduces the subject of fractal cohomology. These concepts are used to derive a fractional Taylor series representation for classes of functions belonging to the Weierstrass curve.

Chapter 1

Introduction

1.1 Motivations

Early developments in Fractal geometry date back almost 150 years ago when George Cantor discovered the Cantor Set. The Cantor set was created iteratively: beginning with the interval $[0, 1]$ and recursively removing the middle third of the lengths in the previous iteration, then taking this iteration scheme to infinity. Interesting properties resulted from the limiting set: A compact set that had measure zero and contained an uncountable number of points. Mathematicians also became interested in how to quantify the dimension of such sets that were generated recursively. Minkowski discovered a mathematical definition for how to quantify the dimension of such a set by considering an ε-tubular neighborhood surrounding the set, multiplying by an appropriate power of ε and then taking the limit to zero: If the limit was a finite non-zero number, then one minus the power would be known as the Minkowski dimension of the set. It turns out that in the limit the Cantor set has dimension $\log_3(2)$, a non-integer dimension that intrigued many mathematicians. Such sets with a non-integer dimension became known as fractals. These fractals generated recursively came to be of great interest to mathematicians.

Mathematicians such as Michel Lapidus discovered that we could uniquely characterize a fractal as the sequence of lengths of the deleted intervals generated at each step in the iteration process. Following this characterization, a counting function outputting the number of reciprocal lengths less than a given number was created. Taking the Mellin trasform of this function gave rise to a complex-valued function known as the geometric zeta function for the given fractal. The poles of such a function became known as the complex dimensions of the fractal.

Mathematicians soon became interested in the complex dimensions of a given fractal and how they were scattered across the complex plane. Fractals where the complex dimensions lied on a sequence of vertical lines that were evenly spaced in the complex plane became known as lattice strings. On the other hand, fractals that gave rise to complex dimensions where this property did not hold became known as non-lattice strings.

Without surprise, the counting function could also be used to characterize fractals. This characterization motivated the class of fractals to be expanded from a sequence of real numbers (corresponding to the lengths of intervals in the iteration process) to include a class of measures which were locally compact, had zero mass at the origin, and were finite. These additional objects became known as generalized fractal strings.

In the case of generalized fractal strings, it became of interest if one could recover the fractal string and its distributional derivatives from the distribution of its complex dimensions in the complex plane. First, analysis began of whether this was possible in a given

window in the complex plane, a region in the complex plane bounded by two vertical lines (known as **screens**). An explicit formula was developed in order to undergo this procedure: expressing the fractal counting function (and its derivatives/anti-derivatives) as a formal sum of residues taking place at each of its complex dimensions in a given window. In the case where there was no screen present (when the entire window was the complex plane), there is no error term present. In the case where there was a window in the complex plane arose an error term dependent on the screen. This procedure was done in both the pointwise and distributional cases.

Soon, interest developed into how to express the explicit formula for generalized fractals as a generalized Taylor series involving fractional derivatives. In his famous book Des Distributions, Schwartz expressed the complex ordered fractional derivative of a distribution as the distribution convolved against a variable raised to the negative of that complex ordered power. Naturally, it is possible to view a measure as a distribution: a measure applied to a set is equivalent to integrating the characteristic function on that set against the measure. Thus, a measure induces a distribution. Using Schwartz's definition of the fractional derivative of a distribution applied in the case when our fractal is a measure, we can express the explicit formula of the measure as a sum involving the fractional derivatives of the measure at orders which are the complex dimensions of the fractal along with an error term. In this framework, we can interpret the explicit formula as a generalized Taylor series of the fractal, coefficients of which are residues of its geometric zeta function applied multiplied by the Mellin transform of an inputted test function. It is still unknown whether this error term can be interpreted as the error term associated with the normal Taylor series.

More recent work has been done on the explicit formula of generalized fractal strings, in particular, on how to compute the complex dimensions of these fractals. It was natural to first consider the case when our generalized fractal was simply a function that was smooth on the real line and had zero mass at the origin. Using Hadamard regularization, it was apparent that the complex dimensions were simply the set of positive integers including zero. Analysis quickly expanded to the case when our generalized fractal string was an n-differentiable function. In this situation, the difficulty arose on how to meromorphically continue the associated geometric zeta function past its abscissa of convergence. To analyze this situation from the proper perspective, analysis was first done on the Weierstrass curve, a nowhere differentiable curve which was also continuous. Claire David and Michel Lapidus were able to compute the complex dimensions of the curve by computing the poles its geometric zeta function viewed in two- dimensional space. In order to do this, very careful analysis was achieved by expressing the ε-tubular volume neighboorhood as a function of ε. It became important to express this tubular volume neighborhood formula as an infinite sum involving powers of ε. The infinite sum was plugged into the two-dimensional geometric zeta function and the poles were computed. The explicit formula for the curve was then found by computing the residues of the geometric zeta function at each of its poles. Various methods have been established to compute the numerical value of these coefficients. Michel Lapidus realized these coefficients could be expressed as residues of the geometric zeta function at its poles.

To better understand the fractal properties of the Weierstrass curve, fractal cohomology was used to analyze functions belonging to certain fractal cohomology groups. These functions satisfied the same Holder conditions of the Weierstrass function, belonging to the same class.

Naturally, the question arose as to how to express Weierstrass curve as fractional Taylor series, with orders taken at the complex dimensions of its associated fractal. In order to approach this problem, Claire David and Michel Lapidus defined the fractional derivative of a fractal in terms of cohomology and the fractional Laplacian. Quickly, the issue arose of computing these coefficients both theoretically and numerically.

1.2 Summary of Main Results

In this section, the main results in this thesis will be summarized. There are two main results: The first result involves rewriting the explicit formula as a distributional Taylor series and the second result involves the derivation of the fractional Taylor series expansion for the Weierstrass curve (and curves belonging to this same class). For details on the first result, we refer you to chapter four in this work. For details on the second result, we refer you to chapter five.

In the first result, we provide a theorem as a way of re-expressing the explicit formula as a fractional Taylor series. This Taylor series involves the summation of derivatives of the delta distribution taken to the complex order (the order of which are the complex dimensions of the string). The coefficients are explicitly determined and explained via Hadamard regularization.

The second result involves rewriting the Weierstrass curve (and those functions satisfying the same Holder condition) as a fractional Taylor series. In order to derive this representation, an iterated function system was developed so that the limiting object was the Weierstrass curve. With this iterated function system in place, a fractal cohomological chain complex was developed to analyze the system further. The fractional Taylor series is exact at all vertices derived from the IFS and is written in terms of $\varepsilon > 0$. The order of derivatives in the sum are taken to be the complex codimensions of the Weierstrass curve.

Chapter 2

Basic Fractal Geometry

An ordinary fractal string is defined as any bounded open subset of $\mathbb{R}$. These objects are important in fractal geometry since they possess extremely important properties. For one, one can define a geometric zeta function for these objects much in the same light as we can define a zeta function for the Riemann function. In other words, ordinary fractal strings completely generalize the Riemann zeta function to much more abstract settings. Interestingly, one can take an object such an ordinary fractal string (which lives on the real line) and completely characterize the string in a complex setting by understanding its associated geometric zeta function. The geometric zeta function gives a means of providing depth to our fractal by inserting it into a much more geometric type setting, which really was before, just a sequence of real numbers.

The discussion begins with some main definitions and theorems.

2.1 Main Definitions and Theorems

The modern definition of ordinary fractal strings is motivated by the notion of self-similar sets. Sets that when zooming in, the same images is generated over and over again. While it is possible to rigorously define self-similar sets, fractals are defined through a more general definition, that is, really any open set on the positive real line. Intrinsic to this definition is the notion of the fractal dimension for these kind of sets. The discussion begins with the mathematical definition for an ordinary fractal string. This definition is due to Michel Lapidus and Carl Pomerance and can be found in [LP93] and in their earlier work, [LP90].

Definition 2.1.1 *An ordinary fractal string $\mathcal{L}$ is defined as any bounded open set of $\mathbb{R}$. since any open set can be experessed as the disjoint union of a countable number of open intervals of decreasing length,*

$$\mathcal{L} = \bigcup_{i=1}^{\infty}(a_i, b_i)$$

We can also characterize a fractal string by the respective lengths of these intervals,

$$\mathcal{L} = \{l_1, l_2, ..., l_n, ...\}$$

There are certain key resulting properties that ordinary fractal strings possess. Since the lengths are summable, we have that the lengths tend to zero. Moreover the sum of the lengths is finite by definition. Mathematically, we can state these properties as,

1) $l_i \to 0$ as $i \to \infty$

2) $\sum_{i=1}^{\infty} l_i < \infty$

In order to capture information about our fractal, it is important to compute the ε-

tubular volume neighborhood. This neighborhood reveals crucial information including the Minkowski dimension as well as the oscillations (vibrations) associated with our fractals. The following definition can be found on p. 13 of [LvF13], Eq. (1.3).

Definition 2.1.2 *Let $\mathcal{L}$ be an ordinary fractal string, where $\mathcal{L} = \{(a_1, b_1), (a_2, b_2), ..., (a_n, b_n), ...\}$. Then we define the tubular volume as $V_{\mathcal{L}}(\epsilon) = |\{x \in \mathcal{L} | d(x, \partial\mathcal{L}) < \epsilon\}|$.*

Fortunately, there exists a fractal tube formula that allows one to compute the volume of our tubular neighborhood of a given fractal given an $\varepsilon > 0$. This formula is given in terms of the number of lengths less than 2ε and the number of lengths greater than 2ε. The derivation of this formula is not too difficult to obtain and is highly useful in computing the Minkowski dimension of our string. The mathematical theorem below is due to Michel Lapidus and Machiel van Frankenhuisjen, see p. 13 of [LvF13], Eq. (1.9).

Theorem 2.1.1 [LvF13, Eq. (1.9)] The tubular volume formula of a fractal string $\mathcal{L} = \cup(a_i, b_i)$ has the following form,

$$V_{\mathcal{L}}(\epsilon) = \sum_{j:\ell_j \geq 2\epsilon} 2\epsilon + \sum_{j:\ell_j < 2\epsilon} \ell_j$$

Proof. Geometrically, it is clear that, $V_{\mathcal{L}}(\epsilon) = \sum_{i=1}^{\infty} V_{(a_i,b_i)}(\epsilon)$. Thus, we can write the tubular volume on each interval and add these up. We break this into two cases,

Case 1: $l_i < 2\epsilon$. If $l_i < 2\varepsilon$ then $V_{(a_i,b_i)}(\epsilon) = (a_i, a_i + \epsilon) \cup (b_i - \varepsilon, b_i) = (a_i, b_i)$. And so, $V_{(a_i,b_i)}(\varepsilon) = |(a_i, b_i)| = l_i$.

Case 2: $l_i > 2\varepsilon$. If $l_i > 2\varepsilon$, then $V_{(a_i,b_i)}|(a_i, a_i + \varepsilon) \cup (b_i - \varepsilon, b_i)| = 2\varepsilon$.

Combining these cases, we arrive at the formula written in the theorem. ■

One important concept in the field of fractal geometry is that of Minkowski dimension. The Minkowski dimension coincides with our usual understanding of dimension. That is, the Minkowski dimension of a point is just 0, the Minkowski dimension of a countable number of points is 0, and the Minkowski dimension of a bounded interval is 1. The mathematical definition is below (see also [LvF13], p. 11):

Definition 2.1.3 *The Minkowski dimension of an ordinary fractal string $\mathcal{L}$ is defined as the inner Minkowski dimension of $\mathcal{L}$,*

$$D = D_{\mathcal{L}} = \inf\{\alpha \geq 0 : V(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}$$

All of these properties of a fractal stated above can be revealed by simple analysis on the real line. What is more interesting and motivating is that we the ability to understand our fractal in the complex plane, a much more geometric setting. Placing the fractal in this setting allows us to generate rich theory of our fractals. Allowing us to reveal more properties of our fractals including that of complex dimensions. The complex dimensions are defined as the set of poles of the fractal's associated geometric zeta function. The geometric zeta function is the Mellin transform of our fractal. We start with some key definitions which can also be found in chapter 1 of [LvF13].

Definition 2.1.4 *Let $\mathcal{L}$ be an ordinary fractal string with associated lengths $\{l_1, l_2, ...\}$. We define the geometric zeta function as the meromorphic continuation of the function $\zeta_{\mathcal{L}} : \mathbb{C} \to \mathbb{C} = \sum_{i=1}^{\infty} {l_i}^s$.*

From the meromorphic continuation of this geometric zeta function, we can define a fractals complex dimensions in the following way. The complex dimensions provide a formal way of characterizing our fractal string as we will see later via an explicit formula. This is of course assuming that our fractal satisfies very specific growth conditions.

Definition 2.1.5 *Let $\mathcal{L}$ be a ordinary fractal string with associated lengths $\{l_1, l_2, ...\}$. We define the complex dimensions as the poles of the meromorphic continuation of its associated geometric zeta function defined above.*

Definition 2.1.6 *Let $\mathcal{L}$ be an ordinary fractal string, then we define the abscissa of convergence for its associated geometric zeta function as,*

$$\inf_{s \in \mathbb{C}} \{\mathrm{Re}(s) : \sum_{i=1}^{\infty} l_i^{\mathrm{Re}(s)} < \infty\}$$

Geometrically, this corresponds to the the vertical line for which the geometric zeta function converges absolutely on the right everywhere. This also corresponds to the vertical line at which we can meromorphically continue our geometric zeta function to the left.

Since this takes place in the complex plane, we can discuss the poles lying in a particular rectangular region in the complex plane by defining a window as well as a screen. A screen is defined to be the left hand boundary of our window of interest. Formally, we can write this as,

$$S := S(t) + it$$

Where S is assumed to be bounded Lipschitz continuous function.

The abscissa of convergence lends itself to a very important theorem tying the concept of tubular volume neighborhood for a fractal with its associated geometric zeta function. This next theorem is due to Lapidus and van Frankenhuijsen.

Theorem 2.1.2 [LvF13, Thm. 1.10] *The abcissa of convergence for an ordinary fractal string is equal to its Minkowski dimension.*

We do not provide a detailed proof of this theorem, but the proof can be found on pp. 17-18 in [LvF13], Theorem 1.10.

2.2 Cantor String Example

One of the most prototypical examples that one comes across in the study of fractal strings is the Cantor string. This example is referred to extensively in [LvF13]. The Cantor string has lengths of $\frac{1}{3^n}$ with multiplicity 2^{n-1}. The motivation for the Cantor string comes from the Cantor set, which is obtained when we delete the middle one-third of the unit interval. From the remaining pieces, we remove the middle third of each piece. We continue this process to infinity, obtaining the Cantor set in the limiting process. Formally, the definition for this fractal is presented below. See also p. 13 of Sec. 1.1.2 in [LvF13]:

Definition 2.2.1 *The Cantor string is defined by the union of the disjoint open intervals,*

$$\left(\frac{1}{3},\frac{2}{3}\right) \cup \left[\left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right)\right] \cup \left[\left(\frac{1}{27},\frac{2}{27}\right) \cup \left(\frac{7}{27},\frac{8}{27}\right) \cup \left(\frac{19}{27},\frac{20}{27}\right) \cup \left(\frac{25}{27},\frac{26}{27}\right)\right] \cup .....$$

The corresponding lengths of the intervals above are, $\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{81}.....$ *That is,* $_k = \frac{1}{3^n}$ *when* $2^{n-1} \leq k \leq 2^n - 1$.

Below is a figure depicting the Cantor string and its associated tubular volume nbhd with $\varepsilon = 0.037$, see [LvF13] , p. 13.

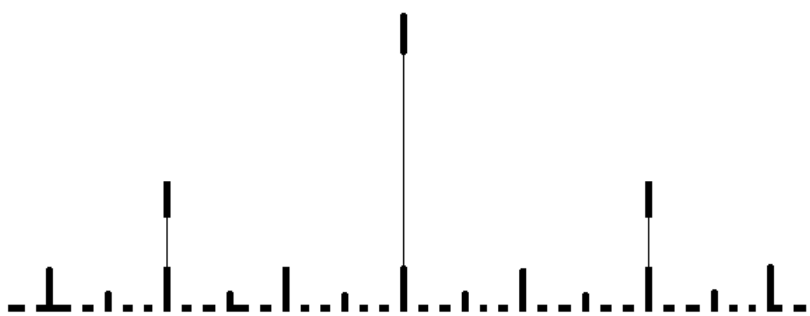

Figure 2.1: From "Fractal Geometry, Complex Dimensions, and Zeta Functions" by Michel Lapidus and Machiel van Frankenhuisjen, p. 13

On the next page is a theorem due to Lapidus and van Frankenhuisjen in [LvF13]. See p. 14, Eq. 1.10 for the tubular volume formula and Eq. 1.11 for the Minkowski dimension.

The proof is a reformulation of the proof given in this text.

Theorem 2.2.1 [LvF13, Eq. (1.10) and Eq. (1.11)] *The tubular volume of the Cantor string* Ω *is* $V_\Omega(\varepsilon) = (2^N - 1)(2\varepsilon) + \left(\frac{2}{3}\right)^N$ *where* $N = \log_3(2\varepsilon)$. *Furthermore, the Minkowski dimension of the Cantor string is* $\lfloor \log_3 2 \rfloor$.

Proof.

Take $\epsilon > 0$, then there exists a, $\frac{1}{3^{N+1}} < 2\epsilon < \frac{1}{3^N}$ for all $N \in \mathbb{N}$. And so,

$$V(\epsilon) = \sum_{l_j \geq 2\epsilon} (2\epsilon) + \sum_{l_j < 2\epsilon} l_j =$$

$$= (1 + 2 + ... + 2^{N-1})2\epsilon + \left(\frac{2^N}{3^{N+1}} + \frac{2^{N+1}}{3^{N+2}} + ... \right)$$

$$= (2^{N-1} - 1)2\varepsilon + 3\left(\frac{2^N}{3^{N+1}} \right)$$

This gives the tubular volume formula for the Cantor string where $N = \lfloor \log_3(2\varepsilon) \rfloor$. We can achieve a bound on the formula above by recasting in terms of ε. Since ε^α is a decreasing function when $0 < \varepsilon < 1$, we have that the above line becomes,

$$\sim \varepsilon^{1 - \frac{1}{\log_2 3}} - \varepsilon \leq C\varepsilon^{1-\alpha}$$

only when $\alpha \geq \frac{1}{\log_2 3} = \log_3 2$. Therefore the Minkowski dimension of the fractal is $\log_3 2$.

■

On the next page is a theorem again due to Lapidus and collaborators stated in [LvF13]. See Sec. 1.2.2, Eq. 1.29 which states the geometric zeta function and Eq. 1.30 which states the complex dimensions. The proof provided is a reformulation of the justification given in this text.

Theorem 2.2.2 [LvF13, Eq. (1.29) and Eq. (1.30)] *The geometric zeta function for the Cantor string has the form,*

$$\sum_{j=0}^{\infty} 2^j 3^{-(j+1)s}$$

Moreover, the complex dimensions take the form,

$$\left\{\log_3 2 + \frac{2\pi i n}{\log 3} | n \in \mathbb{N}\right\}$$

Which comprise a vertical line in the complex plane passing through the real part, $\log_3 2$*.*

Moreover, these poles are simple poles.

Proof.

Recall that the lengths of the Cantor string are of the form, $3^{-(j+1)}$ with multiplicity 2^j with $j \in \mathbb{N}_0$. And so the geometric zeta becomes,

$$\zeta_\eta(s) = \sum_{j=0}^{\infty} l_j^s = \sum_{j=0}^{\infty} 2^j 3^{-s(j+1)} = \sum_{j=0}^{\infty} 3^{-s} \left(\frac{2}{3^s}\right)^j$$

The sum of this geometric series simply becomes,

$$\frac{3^{-s}}{1 - \frac{2}{3^s}}$$

The poles of this function are located when $\frac{2}{3^s} = 1$ which lie on the set,

$$\left\{\log_3 2 + \frac{2\pi i n}{\log 3} | n \in \mathbb{N}\right\}$$

■

Another very important example in the study of ordinary fractal strings is that of the a-string. The a-string also possesses a non-integral dimension which is also computable by using the tubular volume neighborhood. We refer you to sec. 6.5.1, p. 198 of [LvF13]. A reformulation of the proof for a partial to Thm. 6.20 in the text is given here.

2.3 a-string Example

Theorem 2.3.1 [LvF13, Theorem 6.20] *Define the a-string with $a > 0$ to be the union,*

$$\mathcal{L} = \bigcup_{j=1}^{\infty} \left(\frac{1}{j+1}^{a}, \frac{1}{j}^{a} \right)$$

With the corresponding lengths satisfying the equality, $l_j = j^{-a} - (j+1)^{-a}$. Then the Minkowski dimension $D_{\mathcal{L}} = \frac{1}{a+1}$.

Proof.

Let $\epsilon > 0$ be given, then there exists a $N \in \mathbb{N}$ such that, $(N+1)^{-a} - (N+2)^{-a} > 2\epsilon > N^{-a} - (N+1)^{-a}$. Therefore, the tubular volume neighborhood becomes,

$$V_{\mathcal{L}}(\varepsilon) = \sum_{j:\ell_j \geq 2\varepsilon} 2\varepsilon + \sum_{j:\ell_j < 2\varepsilon} \ell_j$$

$$= N(2\varepsilon) + \sum_{j=N+1}^{\infty} \left(\frac{1}{j}^{a} - \frac{1}{j+1}^{a} \right)$$

$$= 2N\varepsilon + \left(\frac{1}{N+1} \right)^{a}$$

Now,

$$\varepsilon = (N+1)^{-a} - (N+2)^{-a} + O(N^{-a-2}) = \frac{(N+2)^{a} - (N+1)^{a}}{(N+1)^{a}(N+2)^{a}} + O(N^{-a-2})$$

$$\lesssim \frac{N^{a-1}}{N^{2a}} = (N^{-a-1})$$

Rearranging the above equation, we get that,

$$N = C\varepsilon^{-\frac{1}{a+1}}$$

Plugging into the tubular volume formula, we get that,

$$V_{\mathcal{L}}(\epsilon) \lesssim 2\varepsilon^{1-\frac{1}{1+a}} + \varepsilon^{-\frac{a}{1+a}}$$

It is clear from the above equation that this implies that the Minkowski dimension is exactly,

$$D_{\mathcal{L}} = \frac{1}{a+1}$$

■

Theorem 6.20 in [LvF13] also states that the complex dimensions of the a-string are exactly the set of numbers,

$$0, -\frac{1}{a+1}, -\frac{2}{a+1}, ... - \frac{m}{a+1}, -\frac{m+1}{a+1}, ...$$

Also from Theorem 6.20, the simple poles have residues equal to $ma^{D_{\mathcal{L}}}$. There are some ordinary fractal strings that have a natural boundary for which we cannot continue its associated geometric zeta function. Consider the fractal string in Example 1.17 given in ch. 1 of [LvF13]. This is the generalized fractal string whose geometric zeta function is,

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} j e^{Dj^2} e^{-j^2 s}$$

That is, the geometric zeta function for the fractal string with lengths e^{-j^2} each with repeated multiplicity je^{Dj^2}, for $0 < D < 1$. The geometric zeta function has a natural boundary at its Minkowksi dimension, D, and cannot be analytically continued past this point.

Chapter 3

Generalized Fractal Strings and the Explicit Formula

The main goal is this chapter is to show when we generalize our space of ordinary fractal strings to a set of measures, called generalized fractal strings, we are able to recover our fractal string from its associated complex dimensions via an explicit formula under specific conditions. This is an important characterization. For one, we are able to characterize classes of fractal strings via this formula. For two, it lends its way to the development of fractal calculus by rewriting the explicit formula as a fractional Taylor series. We begin this chapter by defining a generalized fractal string, it's geometric zeta function, as well as the definition of the screen and window (subsets of the complex plane).

3.1 Main Definitions: Generalized Fractal Strings

We begin with the definition of a generalized fractal string and its associated geometric zeta function, see the beginning of chapter 4 in [LvF13].

Definition 3.1.1 *Let η be a measure on $(0, \infty)$ such that $|\eta|(0, x_0) = 0$ for some $x_0 > 0$, and such that for every compact set $C \subset\subset \mathbb{R}^{\geq 0}$, $\eta(C) < \infty$, then η is defined to be a generalized fractal string.*

A simple example of a generalized fractal string is that of Dirac comb. $\eta = \Sigma_{i=1}^{\infty} \delta_{{l_i}^{-1}}$, where $\{l_i\}_{i \in \mathbb{N}}$ is a nonincreasing sequence of positive numbers tending to zero and $\sum_{i=1}^{\infty} l_i < \infty$. This has no mass around zero: choose $l_1^{-1} > \delta > 0$, then $\eta = \Sigma_{i=1}^{\infty} \delta_{{l_i}^{-1}}((0, \delta)) = 0$. Moreover, this measure is locally finite (there are only a finite number of reciprocal lengths in any bounded interval).

As in the case for ordinary fractal strings, we define the geometric zeta function for generalized fractal strings below.

Definition 3.1.2 *The geometric zeta function of a generalized fractal string η is defined to be the meromorphic continuation of the Mellin transform of η:*

$$\zeta_\eta(s) = \int_0^\infty x^{-s} \eta(dx)$$

A simple example of the connection between ordinary fractal strings and generalized fractal strings is through the computation of the geometric zeta function for the Dirac comb.

Theorem 3.1.1 *Let $\eta = \sum_{i=1}^{\infty} \delta_{l_i^{-1}}$ where $\sum_{i=1}^{\infty} l_i < \infty$, then its geometric zeta function is the meromorphic continuation of,*

$$\sum_{i=1}^{\infty} {l_i}^s$$

Thus, these special cases of generalized fractal strings show that generalized fractal strings generalize ordinary fractal strings.

Proof.

$$\zeta_\eta(s) = \int_0^\infty x^{-s}\eta(dx) = \int_0^\infty x^{-s}\left(\sum_{i=1}^{\infty}\delta_{l_i^{-1}}\right)(dx) =$$

$$\sum_{i=1}^{\infty}\int_0^\infty x^{-s}\delta_{l_i^{-1}}(dx) = \sum_{i=1}^{\infty}(l_i{}^{-1})^{-s} = \sum_{i=1}^{\infty} l_i{}^s$$

■

3.2 Screen and Window

The window is a region in the complex plane bounded to the left by a screen. It is unbounded to the right, and from below, and above. The set of visible complex dimensions are those complex dimensions contained in the window. These definitions are made more precise below. See also section 1.2.1 of [LvF13] for these definitions as well.

Definition 3.2.1 *A screen $S := S(t) + it$, where $S(t) : \mathbb{R} \to (-\infty, D]$ and is a bounded continuous Lipschitz function.*

Definition 3.2.2 *The window is defined as the set $W = \{s \in \mathbb{C} : \Re(s) \geq S(\Im(s))\}$ as a subset of $\mathbb{C}$.*

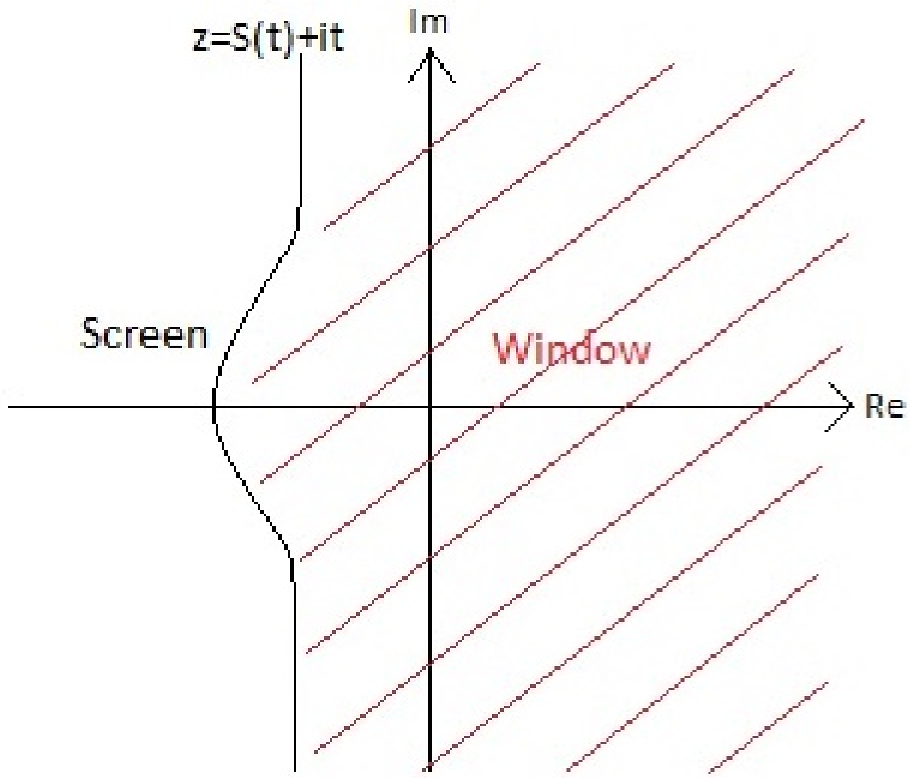

Figure 3.1: An example of window and screen in the complex plane.

Definition 3.2.3 *The set of visible complex dimensions, denoted by $D_\eta(W)$, are the set of complex dimensions belonging to the window W.*

3.3 Explicit Formula

For us to recover our generalized fractal string from its known complex dimensions in a given window, it is useful if the fractal satisfies certain growth conditions in the window. Loosely, if our geometric zeta function is bounded along any fixed horizontal line in our window (as $\Re(s) \to \infty$) and if our zeta function satisfies polynomial growth conditions vertically, then recovery is possible via an explicit formula. This is made precise by satisfying languid conditions. The following definition for languidity below is found in [LvF13], see p. 142-143, Eq. (5.19) and Eq. (5.20). In this text, these are known as the **L1** and **L2** conditions respectively.

Definition 3.3.1 *A generalized fractal string η is said to be languid if its geometric zeta function ζ_η satisfies the followings growth conditions: There exist real constants κ and $C > 0$ and a two-sided sequence $\{T_n\}_{n\in\mathbb{Z}}$ such that $T_{-n} < 0 < T_n$ for $n \geq 1$, and,*

$$\lim_{n\to\infty} T_n = \infty \ , \ \lim_{n\to\infty} T_{-n} = -\infty \ , \ \frac{T_n}{|T_{-n}|} = 1$$

The first two cases cover when η is assumed to be languid, the final case covers when η is assumed to be strongly languid.

1. *For all $n \in \mathbb{N}$ and all $\sigma \geq S(T_n)$,*

$$|\zeta_\eta(\sigma + iT_n)| \leq C(|T_n| + 1)^\kappa$$

2. *For all $t \in \mathbb{R}$, $|t| \geq 1$,*

$$|\zeta_\eta(S(t) + it)| \leq C|t|^\kappa$$

3. *There exists a sequence of screens $S_m : t \to S_m(t) + it$ for $m \geq 1, t \in \mathbb{R}$, with $\sup S_m \to -\infty$ as $m \to \infty$ and with uniform Lipchitz bound $\sup_{m\geq 1} ||S_m||_{lip} < \infty$ such that there exists constants $A, C > 0$ such that for all $t \in \mathbb{R}$ and $m \geq 1$,*

$$|\zeta_\eta(S_m(t) + it)| \leq CA^{|S_m(t)|}(|t| + 1)^\kappa$$

One important consequence of this is that being strongly languid implies that our string is languid. If our string is strongly languid, then there is no remainder term present in the explicit formula ($x > A$). There is though if we assume our fractal string to be just simply languid.

Proposition 3.3.1 *If η is strongly languid, then it is languid on each screen separately.*

Proof. Clearly, $CA^{|S_m(t)|}(|t| + 1)^\kappa \leq CA^{|S_m(t)|}2^\kappa |t|^\kappa = C|t|^\kappa$ whenever $|t| \geq 1$. ■

If our fractal string is languid, it is possible to recover the string via an explicit formula including an error term. If our fractal string is strongly languid, the error term identically vanishes given that $x > A$ (A appearing the strongly languid condition). We now state a lemma for approximating the Heaviside function and its k^{th} anti-derivatives, see pp. 139-142 of [LvF13] for a statement of the lemma as well as its proof (which we do not provide here).

Lemma 3.3.1 [LvF13, Lemma 5.1] *For $c > 0$, $T_- < 0 < T_+, x, y > 0$, and $k = 1, 2, ...,$ the k^{th} Heaviside function is approximated as follows,*

$$H^{[k]}(x - y) = \frac{1}{2\pi i}\int_{c+iT_-}^{c+iT^+} x^{s+k-1}y^{-s}\frac{ds}{(s)_k} + E$$

Putting $T_{min} = min\{T_+, |T_-|\}$ and $T_{max} = max\{T_+, |T_-|\}$, the error E of this approximation does not exceed in absolute value,

$$\begin{cases} x^{c+k-1}y^{-c}{T_{\min}}^{-k}\min\left\{T_{\max}, \frac{1}{|\log(x)-\log(y)|}\right\} & x \neq y \\ x^{k-1}y^{-c}{T_{\min}}^{-k}T_{\max} & x = y, \text{ for all } k \\ ((c+k-1)2^{k-1} + T_{\max} - T_{\min})x^{k-1}{T_{min}}^{-k} & x = y, \text{ for } k \text{ is odd} \end{cases}$$

Using this lemma, Lapidus and van Frankenhuijsen proved that it is possible to recover a fractal string from its complex dimensions via an explicit formula. In other words, the explicit formula allows us to completely characterize our languid fractal string from its known complex dimensions. There are two cases we consider here, when our string is languid (with error term) versus when our string is strongly languid (without error term). We consider both the distributional cases and the pointwise cases. The pointwise explicit formula for a fractal of languidity is given in the theorem on the next page in terms of $N^{[k]}$ (see also pp. 149-150 of [LvF13]). Here, ${N_\eta}^{[k]}$ is given by an iterated integral. More precisely, it is the k^{th} anti-derivative of η which with its first $(k-1)$ derivatives all vanish at zero. The formal definition is given below.

Definition 3.3.2 *Let η be a generalized fractal string, then we define the k^{th} anti-derivative of η as the following iterated integral,*

$${N_\eta}^{[k]}(x) = \int_0^x \frac{(x-y)^{k-1}}{(k-1)!}\eta(dy)$$

We now state the explicit formula in terms of the ${N_\eta}^{[k]}$ as defined above.

Theorem 3.3.1 [LvF13, Theorem 5.10] *Let η be a languid generalized fractal string, and let k be an integer such that $k \geq \max\{1, \kappa+1\}$ where κ is the exponent occuring in the statement of L1 and L2. Then, for all $x > 0$, the pointwise explicit formula is given by the following equality,*

$$N_\eta^{[k]}(x) = \sum_{\omega \in \mathcal{D}_\eta(W)} res\left(\frac{x^{s+k-1}\zeta_\eta(s)}{(s)_k}; \omega\right)$$

$$+\frac{1}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j x^{k-1-j}\zeta_\eta(-j) + R_\eta^{[k]}(x)$$

Here for $x > 0$, $R(x) = R_\eta^{[k]}(x)$ is the error term given by the absolutely convergent integral,

$$R(x) = R_\eta^{[k]}(x) = \frac{1}{2\pi i}\int_S x^{s+k+1}\zeta_\eta(s)\frac{ds}{(s)_k}$$

Further, for all $x > 0$, we have,

$$R(x) = R_\eta^{[k]}(x) \leq C(1 + ||S||_{Lip})\frac{x^{k-1}}{k-\kappa-1}\max\{x^{sup}, x^{inf}\} + C'$$

Where C is the positive constant occuring in L1 and C' is the positive constant occuring L1 and L2 and C' depend only on η and the screen, but not on k. (Here, inf(S) and sup(S)). In particular, we have the following pointwise error estimate:

$$R(x) = R_\eta^{[k]}(x) = O(x^{sup+k-1})$$

as $x \to \infty$. Moreover, if $S(t) < sup(S)$ for all $t \in \mathbb{R}$ (i.e.; if the screen lies strictly to the left of the line $\Re = sup(S)$), then $R(x)$ is of order less than $x^{sup(S)+k-1}$ as $x \to \infty$: $R(x) = R_\eta^{[k]}(x) = o(x^{sup(S)+k-1})$, as $x \to \infty$).

In the above formula there is no error term present when our generalized fractal string is strongly languid and when $x > A$ (constant in the definition of strong languidity).

Often times, we are concerned with the form the explicit formula will have when our string acts distributionally on test functions. A plus of doing it this way is that there are no limitations on k, it can be either a positive or negative integer. We can distributionally differentiate our distribution in both directions quite easily (this is referred to as the method of descent). The theorem is given below in terms of ${P_\eta}^{[k]}$. Note that ${P_\eta}^{[k]}$ is defined the k^{th} distributional anti-derivative of η which (along with its derivatives) vanish at infinity. The formal definition is given before the theorem.

Definition 3.3.3 *Let η be a generalized fractal string, we define ${P_\eta}^{[k]}$ to be the k^{th} distributional anti-derivative of η given by an iterated integral applied to a test function ϕ below,*

$$\langle {P_\eta}^{[k]}, \phi \rangle = \int_0^\infty \int_y^\infty \frac{(x-y)^{k-1}}{(k-1)!} \phi(x) dx \eta(dy)$$

In particular, ${P_\eta}^{[0]} = \eta$

Theorem 3.3.2 [LvF13, Theorem 5.18] *Let η be a languid generalized fractal string, i.e. it satisfies the hypotheses of languidity above. Then for every $k \in \mathbb{Z}$, the distribution ${P_\eta}^{[k]}$ is given by,*

$${P_\eta}^{[k]}(x) = \sum_{\omega \in \mathcal{D}_\eta(W)} res\left(\frac{x^{s+k-1}\zeta_\eta(s)}{(s)_k}; \omega\right) + \frac{1}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j x^{k-1-j}\zeta_\eta(-j) + R_\eta^{[k]}(x)$$

That is, the action of ${P_\eta}^{[k]}$ on a test function ϕ is given by,

$$\langle {P_\eta}^{[k]}, \phi \rangle = \sum_{\omega \in \mathcal{D}_\eta(W)} res\left(\frac{\zeta_\eta(s)\tilde{\phi}(s+k)}{(s)_k}; \omega\right) +$$

$$\frac{1}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j \zeta_\eta(-j)\tilde{\phi}(k-j) + \langle R_\eta^{[k]}, \phi \rangle$$

Here, the distribution $R = {R_\eta}^{[k]}$ is the error term, given by,

$$\langle R, \phi \rangle = \langle {R_\eta}^{[k]}, \phi \rangle = \frac{1}{2\pi i}\int_S \zeta_\eta(s)\tilde{\phi}(s+k)\frac{ds}{(s)_k}$$

When our fractal string η is assumed to be strongly languid this error term is identically equal to zero. Another restriction for the error term to be identically zero is that the space of test functions are restricted to those smooth compactly supported functions that have compact support bounded away from zero, that is, on the interval $[A, \infty]$, where $A > 0$ (Again, A is the constant appearing in the defintion of strong languidity). The theorem below has been shown to be true by Lapidus and van Frankenhuisjen in [LvF13], see p. 158, Thm. 5.2 for the statement of the theorem as well as the proof. It is reproved here in detail.

Theorem 3.3.3 [LvF13, Theorem 5.2] *Assume η is a strongly languid generalized fractal string. If $\phi \in C_c{}^{\infty}((A+\delta, \infty))$ for some $A > 0$, then $\langle R_\eta^{[0]}, \phi \rangle = 0$.*

Proof. Consider the error term,

$$R_\eta^{[0]} = \int_{S|_m} \zeta_\eta(s)\tilde{\phi}(s)ds$$

Here, $S|_m = S_m(t) + it$, where $S_m(t)$ is a bounded from above Lipschitz function. We can bound the Mellin transform of the test function ϕ in the following way for each $t \in \mathbb{R}$,

$$|\tilde{\phi}(s)| \leq \int_0^\infty |\phi(x)| x^{S_m(t)-1} dx = \int_{A+\delta}^\infty |\phi(x)| x^{S_m(t)-1} dx$$
$$\leq (A+\delta)^{S_m(t)} \int_0^\infty \frac{|\phi(x)|}{x} dx$$

The last inequality is true since $S_m(t)$ is a negative-valued function for large enough $m \in \mathbb{N}$. Finally, the integral is bounded since ϕ has zero mass at the origin. And so the error term can be bounded,

$$\left| \frac{1}{2\pi i} \int_{S|_m} \zeta_\eta(s)\tilde{\phi}(s)ds \right| \leq \frac{1}{2\pi} \int_{S|_m} |\zeta_\eta(s)||\tilde{\phi}(s)|ds$$

Now we can use the strong languidity assumption to bound the integral on the right.

$$\frac{1}{2\pi}\int_{S|_m} |\zeta_\eta(s)||\tilde{\phi}(s)|ds$$

$$\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} CA^{|S_m(t)|}(|t|+1)^{\kappa}(A+\delta)^{S_m(t)}\left[\int_0^{\infty}\frac{|\phi(x)|}{x}dx\right]dt$$

But the following integral above exists, and since as $m \to \infty$,

$$\frac{A^{|S_m(t)|}}{(A+\delta)^{|S_m(t)|}} \to 0$$

By the dominated convergence theorem, the right-most integral in the inequality tends to zero as well as $m \to \infty$. ■

In [LvF13], Lapidus and van Frankenhuisjen used the method of descent, and so we can take any order derivative of the error term and we quickly note that for every value $k \in \mathbb{N}$, we have that $\langle R_\eta^{[k]}, \phi\rangle = 0$.

With the explicit formula, we can recover our fractal string easily from its known complex dimensions. We consider other test function spaces.

3.4 Explicit Formula: Other Test Function Spaces

Interestingly, we can also recover our fractal string assuming that η is languid and by expanding our class of test functions to include those of polynomial decay near zero, not only those that are smooth and of compact support. A theorem for both pointwise and distributional cases by Michel Lapidus and van Frankenhuisjen illustrates this, see [LvF13], sec. 5.4.1.

Theorem 3.4.1 [LvF13, Theorem 5.26] *Let η be a languid generalized fractal string. Let $k \in \mathbb{Z}$ and let $q \in \mathbb{N}$ be such that $k+q > \kappa+1$, where κ is given in the definition of languidity. Further, let ϕ be a test function that is q-times continuously differentiable on $(0,\infty)$, and assume that the j-derivative satisfies, for each $0 \leq j \leq q$ and some $\delta > 0$,*

$$\phi^{(j)}(x) = O(x^{-k-j-D-\delta}) \text{ as } x \to \infty$$

and,

$$\phi^{(j)}(x) = \sum_{\alpha} a_{\alpha}^{(j)} x^{-\alpha-j} + O(x^{-k-j-inf(S)+\delta}) \text{ as } x \to 0^{+}$$

for a finite sequence of complex exponents α as above. We then have the following distributional formula explicit formula with error term for $P_{\eta}^{[k]}$,

$$P_{\eta}^{[k]}(x) = \sum_{\omega \in \mathcal{D}_{\eta}(W)} res\left(\frac{x^{s+k-1}\zeta_{\eta}(s)}{(s)_k};\omega\right) + \frac{1}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j x^{k-1-j}\zeta_{\eta}(-j)$$
$$+ \sum_{\substack{\alpha \in W\backslash D_{\eta} \\ \alpha \in \{1-k,\dots,0\}}} \tau_{\alpha}(x)\frac{\zeta_{\eta}(\alpha)}{(\alpha)_k} + R_{\eta}^{[k]}(x)$$

Note that the sum over α is finite by our assumption on the space of test functions. Applied to a test function ϕ, the distribution $P_{\eta}^{[k]}$ is given by,

$$\langle P_{\eta}{}^{[k]}, \phi\rangle = \sum_{\omega \in \mathcal{D}_{\eta}(W)} res\left(\frac{\zeta_{\eta}(s)\tilde{\phi}(s+k)}{(s)_k};\omega\right) + \frac{1}{(k-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\zeta_{\eta}(-j)\tilde{\phi}(k-j)$$

$$+ \sum_{\substack{\alpha \in W \setminus D_\eta \\ \alpha \in \{1-k,\ldots,0\}}} a_\alpha^{[0]} \frac{\zeta_\eta(\alpha)}{(\alpha)_k} + \langle R_\eta^{[k]}, \phi \rangle$$

In the above theorem, the distribution $\langle \tau_\alpha, \phi \rangle = a_\alpha^{[0]}$. Note that expanding our test function spaces lends its way to a proper generalization of the explicit formula. In the next section, we express the explicit formula as a fractional Taylor series.

Chapter 4

Fractional Derivatives of Distributions and Fractional Distributional Taylor Series

The first result that M. Lapidus and I obtained involves rewriting the explicit formula as a Taylor series involving a sum of fractional derivatives of orders taking place at the complex dimensions of our fractal. Thus we can actually represent our fractal as a fractional Taylor series. This in a sense leads to the birth of the subject of fractal calculus.

4.1 Schwartz's Definition of Fractional Derivative

For a distribution T with compact support on the positive real line, Schwartz's definition of the α-order fractional distributional derivative, $\alpha \in \mathbb{C}$, is as follows where f.p. is referred to as the finite part under Hadamard regularization,

Definition 4.1.1 *Let T be a distribution over the positive real line with compact support and let $\alpha \in \mathbb{C}$, the we can define the α-order derivative of the distribution applied to a test function ϕ as follows,*

$$\langle D^{\alpha}T, \phi \rangle = \langle Y_{-\alpha} * T, \phi \rangle$$

Where,

$$Y_{\alpha} = \begin{cases} \frac{1}{\Gamma(\alpha)} \text{f.p.}(x^{\alpha-1}) & \alpha \notin \{0, -1, -2, ...\} \\ \delta^{(-\alpha)} & \alpha \in \{0, -1, -2, -3...\} \end{cases}$$

It should be noted from the definition that Y_{α} depends holomorphically on α. See also, p. 43 of Ex. 2, *Pseudofonctions. Parties finies de Hadamard* in the text [Sch78] for the definition. f.p stands for the finite part (convergent part). Here, $(*)$ denotes the convolution between two distributions and is formally defined in the following way between two distributions S and T where at least one is assumed to have compact support.

$$\langle S * T, \phi \rangle = \int_0^{\infty} \int_0^{\infty} S(\eta)T(\xi)\phi(\eta + \xi) d\xi d\eta$$

In the beginning of this chapter, we assume that ϕ is taken to be a compactly supported smooth function of the positive real line. This enables us to regularize the Mellin transform of the test function ϕ under Hadamard regularization. We start with a theorem below in the work of Schwartz in [Sch78] that allows us to compute the finite part of the Mellin transform for a function at cetain values of $\mathbb{C}$. It is stated below in this next section.

4.2 Taylor Series Representation for Explicit Formula

Theorem 4.2.1 [Sch78, II,2;26] *Let $\tilde{\phi}$ be defined as the Mellin transform of ϕ, that is,*

$$\tilde{\phi} = \int_0^\infty x^{s-1}\phi(x)dx$$

Then the finite part of the integral is defined as,

$$\lim_{\varepsilon\to 0^+}\left[\int_\varepsilon^\infty x^{s-1}\phi(x)dx + \phi(0)\frac{\varepsilon^s}{s} + \phi'(0)\frac{\varepsilon^{s+1}}{s+1} + ... + \phi^{(k)}(0)\frac{\varepsilon^{s+k}}{s+k}\right]$$

With $k \geq -\Re(s)$ chosen. This provides the finite part whenever s is not zero or a negative integer $\geq -k$.

We are able to compute the finite part of the Mellin transform for ϕ using the above theorem. On p. 43 of [Sch78], he introduces Y_α as in the definition 4.1.1 above which enables us to regularize $\tilde{\phi}$ to all $\mathbb{C}$.

Thus we can regularize the product below as (which provides a meromorphic extension to all $\mathbb{C}$),

$$\zeta_\eta(s)\tilde{\phi}(s) = \zeta_\eta(s)(\Gamma(s)\langle Y_s, \phi\rangle)$$

The theorem below is due to the author of this dissertation and Michel Lapidus. It is a theorem that re-expresses the explicit formula as a fractional Taylor series.

Theorem 4.2.2 *Let η be a generalized fractal string, then we can rewrite the explicit formula when $k = 0$ in the following Taylor-like series way,*

$$\langle \eta, \phi \rangle = \sum_{\omega \in D_\eta(W)} res\left(\tilde{\phi}(s)\zeta_\eta(s); \omega\right) + \langle R_\eta^{[0]}, \phi \rangle$$

$$= \sum_{\omega \in D_\eta(W)} res(\zeta_\eta(s)\Gamma(s)\langle Y_s, \phi\rangle, \omega) + \langle R_\eta^{[0]}, \phi \rangle$$

Theorem 4.2.2 allows us to reformulate ${P_\eta}^{[k]}$ in the general form of the explicit formula given in [LvF13]. Also stated as Theorem 3.3.2 in this thesis. There are certain cases when we can take the residue of ζ_η out of the above expression in the theorem above and when we cannot: 1) If ω is a pole of Γ, that is $\omega \in \{0, -1, -2, ...\}$, then we have a multiple pole in the product $\zeta_\eta\Gamma$ (regardless of whether this is a simple pole of ζ_η), and one cannot pull out the expression from the initial residue. 2) If ω is not a pole of Γ, but if ω is a simple pole of ζ_η, we have the following corollary below. 3) If ω is not a pole of Γ, but ω is a multiple pole of ζ_η, then one can pull out $\Gamma(\omega)$ from the initial residue. Recall that $\Gamma(s)$ does not vanish on $\mathbb{C}$.

Corollary 4.2.1 *Let η be a generalized fractal string for which all of its complex dimensions are simple and do not belong to the set $\{0, -1, -2, -3, ...\}$, then we can rewrite the explicit formula when $k = 0$ in the following Taylor-like series way,*

$$\langle \eta, \phi \rangle = \sum_{\omega \in D_\eta(W)} \Gamma(\omega)\langle Y_\omega, \phi\rangle res\left(\zeta_\eta(s); \omega\right) + \langle R_\eta^{[0]}, \phi \rangle$$

The above corollary solves case 2 stated above, the simple pole case. To approach cases 1 and 3, that is, the cases of a multiple pole, we refer you to sec. 6.2 of the work of Lapidus in [LvF13]. Here, the authors explicitly write out the Laurent expansion for ζ_η about the pole. The residues can be explicitly determined in this case.

The theorem above is important since it allows us to re-express the Distributional formula as a fractal power series. This is important in the case of generalized fractal strings. Now we are interested in what this power series might look like when considering generalized fractal strings or measures that induce a smooth function as an integrand. Indeed, when a smooth function is induced, we have poles at $\{0, -1, -2, -3, ...\}$ of its associated geometric zeta function. In this particular case, we generate the original Taylor series of the smooth function.

In the next chapter, we consider what happens when our fractal belongs to a class of continuous, nowhere differentiable functions, called Weierstrass functions. In this case, we can also generate fractal power series.

Chapter 5

Fractal Cohomology and Taylor-like Expansions

In this chapter, we are first going to view the graph of the Weierstrass curve as a two-dimensional fractal. In [DL22a], Claire David and Michel Lapidus computed the Minkowski dimension as well as the complex dimensions for such a fractal. We will discuss their methods as well as their results in some level of detail in the beginning of this chapter. We refer extensively to both [DL22a] and [DL24a] in the beginning of this chapter. We will also present some numerical results that were generated in Matlab by the author which provide prefractal approximations of this fractal.

Next, we will discuss a cohomological complex associated with the Weierstrass fractal. The m^{th} element in the cohomological chain represents a module of **fermions** taking place over the vertices of the m^{th} pre-fractal complex defined by the integrated function system. The closure of such a limit is the graph of the Weierstrass curve. The importance of such is to define a fractional power series representing the Weierstrass function taking place at a prescribed set of points, called vertices. This comes from Claire David's and Michel Lapidus's work in [DL22b] and [DL24b]. We begin with the definition of the Weierstrass curve as well as its graph.

5.1 Weierstrass Curve and the Integrated Function System

Definition 5.1.1 *The Weierstrass curve is defined by the nowhere differentiable function,*

$$W(x) = \sum_{n=1}^{\infty} \lambda^n \cos(N_b^n x)$$

Where the parameters $N_b\lambda > 1$*, and* $0 < \lambda < 1$*. The graph, we define, is therefore the restriction to the interval* $[0,1]$*,* $\Gamma_{\mathcal{W}} = \{(x, W(x)) | x \in [0,1]\}$.

Following remark 2.1 in [DL24a], the authors considered an integrated function system for which we generate the graph of the Weierstrass graph given that $N_b \in \mathbb{N}$. We refer you to [Dav17] for the early developments of this ifs. The maps that define the ifs are as follows,

Definition 5.1.2 *We define the following maps from* $\mathbb{R}^2 \to \mathbb{R}^2$*, where* $i = 0, ..., N_b - 1$*,*

$$T_i(x, y) = \left(\frac{x+i}{N_b}, \lambda y + \cos(2\pi \left(\frac{x+i}{N_b} \right) \right)$$

In the work of David in [Dav17], the author showed that these maps generate an integrated function system for which the graph of the Weierstrass function is an attractor of. This is a key property. This is also restated as proposition 2.1 of David and Lapidus in [DL24a]. This is stated as a theorem below.

Theorem 5.1.1 [DL24a, Proposition 2.1]

The graph of the Weierstrass curve is an attractor for the IFS defined above. That is,

$$\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$$

Now with an integrated function system with the graph of the Weierstrass curve as its limit, David showed that there exists a sequence of prefractal approximations leading up to it. For more on this construction, we refer you to [Dav17]. This is defined more precisely below and again in [DL24a].

Definition 5.1.3 *Let V_0 denote the set of fixed points for the maps $T_i(x, y)$ for $i = 0, ..., N_b -$ 1. In particular,*

$$V_0 = \left\{ \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\pi i}{N_b - 1} \right) \right) \middle| i \in 0, 1, ...N_b - 1 \right\}$$

And define the set of vertices at the m^{th} stage for all $m \in \mathbb{N}$ as,

$$V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$$

From this set of vertices, we have that the union is dense in the graph of the Weierstrass curve. We refer you to [DL22b], also in press in Mathematische Zeitschrift. That is,

$$\Gamma_{\mathcal{W}} = \overline{\bigcup_{m=0}^{\infty} V_m}$$

In [Dav17], the author showed that the following properties for these subsets of vertices hold (Also see proposition 2.2 in [DL24a]):

1. $V_m \subset V_{m+1}$ for all $m \in \mathbb{N}_0$

2. $\#V_m = (N_b - 1){N_b}^m + 1$

We denote the set of vertices V_m as the m^{th}-prefractal approximation. On the next page are some numerical results which display the graph of vertices at each stage. These were obtained using Matlab by the author of this thesis. We refer you to [DL24a] for the same graphs obtained via Mathematica by Claire David and collaborators.

Below are some of the prefractal approximations when $N_b = 3$ and $\lambda = 0.5$ (which are the parameters also considered by Michel Lapidus and Claire David). These graphs were generated via Matlab software.

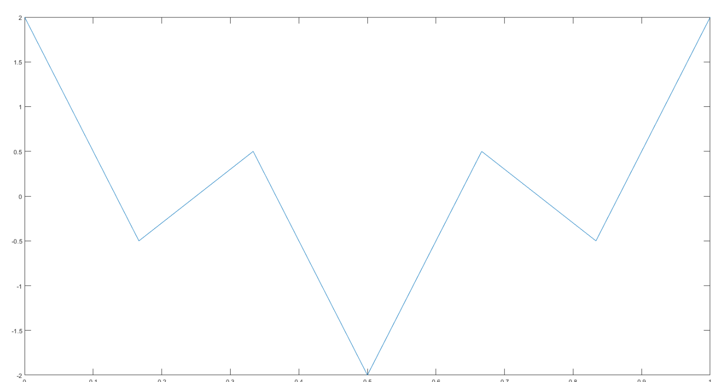

Figure 5.1: The first pre-fractal approximation at $N_b = 3$ and $\lambda = 0.5$

Note that in the graph above, there are a total of seven vertices. This number can be computed directly using the second property above. In the graph below, there are nineteen vertices, which can be computed directly using the formula in the property as well.

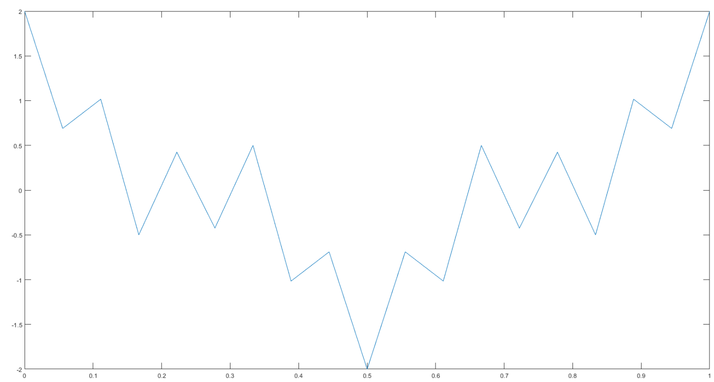

Figure 5.2: The second pre-fractal approximation at $N_b = 3$ and $\lambda = 0.5$

In [DL22a], Claire David and Michel Lapidus considered the ε-tubular neighborhood of prefractal approximations. This helped them in their computation of the complex dimensions for the graph of the Weierstrass function. For the same graph below, also see fig. 11 on p. 45 in [DL22a] (the graph below was recreated by the author of this paper).

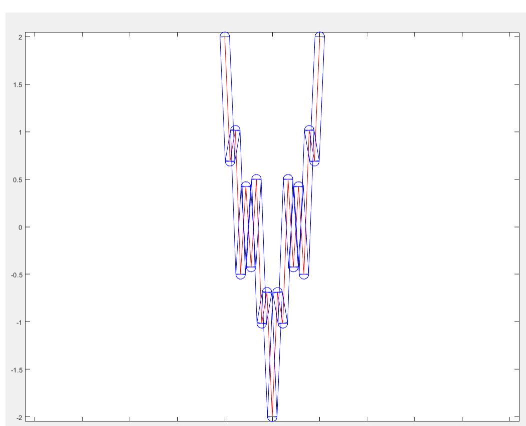

Figure 5.3: The ε-neighborhood at $N_b = 3$ and $\lambda = 0.5$ with an equal to $\varepsilon = 0.05$.

In the paper, [DL22a], the volume of the ε-nbhd was computed at each iteration by assuming that $\varepsilon = \frac{1}{(N_b)^m(N_b-1)}$. These computations were carried out by breaking the ε-neighborhood into various shapes including parallelograms, arcs of circles, and rectanges. Again, we refer you to [DL22a] for details on this. Claire David and Michel Lapidus found a way of constructing these shapes as well as computing the tubular volume. The graph below illustrates this point (figure was generated in Matlab by the author of this thesis).

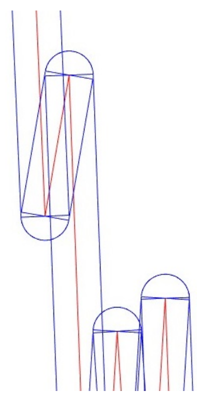

Figure 5.4: ε-nbhd showing the shapes: paralleograms, circles, and rectangles

On the next page is a table of values displaying the pre-fractal approximation number, along with its associated ε, as well as its tubular volume for the parameters $N_b = 3$, and $\lambda = 0.5$. In [DL22a], Michel Lapidus and Claire David provide a precise formula for computing the tubular volume via areas of rectangles, parallelograms, circles, and triangles. The author of this dissertation used Matlab to compute the tubular volume from this formula.

Table 5.1: Tubular Volume as a function of ε at each Iteration

Iteration	ε	Tubular Volume
1	$\varepsilon = 0.1667$	3.9493
2	$\varepsilon = 0.0556$	1.8975
3	$\varepsilon = 0.0185$	0.9307
4	$\varepsilon = 0.0062$	0.4603
5	$\varepsilon = 0.0021$	0.2287
6	$\varepsilon = 0.00069$	0.1139
7	$\varepsilon = 0.00023$	0.0568
8	$\varepsilon = 0.0000762$	0.0283

On the next page are the graphs of the ε-tubular volume neighborhood at the first two iterations. The volume of each matches the beginning of the table above. These were generated by the author using Matlab. For these same graphs, we refer you to Claire David's and Michel Lapidus's numerical results in [DL22a].

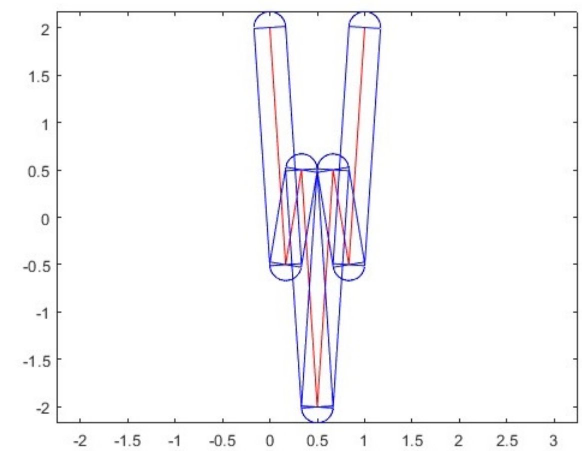

Figure 5.5: ε-neighborhood of the first approximation at $N_b = 3$ and $\lambda = 0.5$, $\varepsilon = 0.1667$.

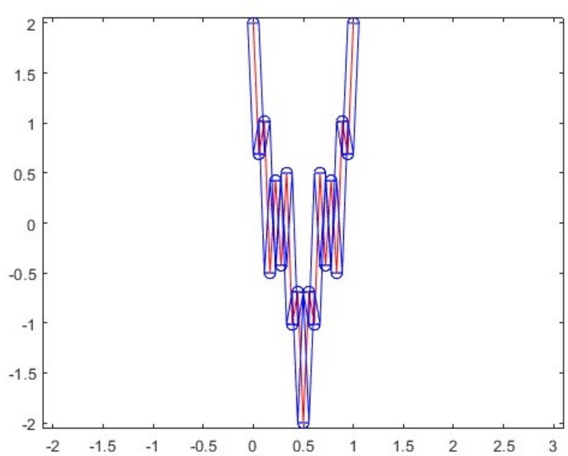

Figure 5.6: ε-neighborhood of the second approximation at $N_b = 3$ and $\lambda = 0.5$, $\varepsilon = 0.0556$.

In work of David in [Dav17], the author computed the Minkowski dimension to be exactly, $D_{\mathcal{W}} = 2 + \frac{\log(\lambda)}{\log(N_b)}$. We refer you to corollary 2.2 of this paper. We also refer to [DL22a]. The author of this thesis checked this numerically that this is indeed the Minkowski dimension of the fractal by computing the product of $V(\varepsilon)$ against $\varepsilon^{D_{\mathcal{W}}-2}$ and seeing that this product stabilizes.

Table 5.2: Convergence of $V(\varepsilon)\varepsilon^{D_{\mathcal{W}}-2}$ as a function of ε

Iteration	ε	$V(\varepsilon)$	$V(\varepsilon)\varepsilon^{(D_{\mathcal{W}}-2)}$
1	$\varepsilon = 0.1667$	3.9493	12.2314
2	$\varepsilon = 0.0556$	1.8975	11.7534
3	$\varepsilon = 0.0185$	0.9307	11.5303
4	$\varepsilon = 0.0062$	0.4603	11.4061
5	$\varepsilon = 0.0021$	0.2287	11.3328
6	$\varepsilon = 0.00069$	0.1139	11.4061
7	$\varepsilon = 0.00023$	0.0568	11.2865
8	$\varepsilon = 0.0000762$	0.0283	11.2539
9	$\varepsilon = 0.0000254$	0.0141	11.2318
10	$\varepsilon = 0.00000847$	0.0071	11.2167

5.2 Fractal Cohomology and the Weierstrass Curve

In [DL22b], Michel Lapidus and Claire David placed a cohomological structure on the integrated function system leading up to the Weierstrass fractal. In order to endow the

Weierstrass fractal with cohomological structure, we first need to find a topological invariant over the pre-fractal approximations leading up to the graph. This is the motivation for fermions. These are signature maps acting on the vertices (or points) of V_m. See p. 19 of their document, [DL22b].

Definition 5.2.1 *A fermion f is a map that is antisymmetric with respect to transpositions. In other words f is a fermion if,*

$$f(x_1, x_2, ..., x_i, x_{i+1}, ..., x_p) = -f(x_1, x_2, ...x_{i+1}, x_i, ..., x_p)$$

for any $(x_1, x_2, ..., x_p) \in \mathcal{F}^p(V_m, \mathbb{C})$. Here, $\mathcal{F}^p(V_m, \mathbb{C})$ is the vector space of p-fermions acting on the vertices (points) in V_m.

As mentioned in [DL22b], the motivation for the term fermion originates from quantum mechanics. Michel Lapidus and Claire David consider modules consisting of fermions and they created a cohomological chain linking these together. Moreover, they defined boundary maps mapping from one module to the next. These are defined on the next page.

For now on, we denote the points $M_{j,m}$ to be the j^{th} vertex of the graph at the m^{th} level pre-fractal approximation. At each stage of the approximation, we can compute the cardinality of these vertices by utilizing the second property on p. 41 of this thesis.

Definition 5.2.2 *We can define the cohomological chain of modules in the following way,*

$$\mathscr{F}^{N_b+1}(V_1,\mathbb{C}) \to \mathscr{F}^{{N_b}^2+1}(V_2,\mathbb{C}) \to .... \to \mathscr{F}^{{N_b}^m+1}(V_m,\mathbb{C}) \to .....$$

In between each element in this co-chain complex one can define boundary maps $\delta_{m-1,m}$ *acting on functions in* $\mathscr{F}^{{N_b}^m+1}(V,\mathbb{C})$. *The map* $\delta_{m-1,m}$ *is defined as follows for* $f \in \mathscr{F}^{{N_b}^m+1}(V_p,\mathbb{C})$,

$$\delta_{m-1,m}(f)(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, ..., M_{j+N_b-1,m}) = c_{m-1,m}\left(\sum_{q=0}^{N_b}(-1)^q f(M_{j+q,m})\right)$$

Here, $c_{m-1,m}$ *is a constant belonging to the complex numbers.*

Claire David and Michel Lapidus showed that this concept allows for fractional differentiation to take place and one can define an infinitesimal. The crux of this is to define a Taylor series for a function belonging to the p^{th} cohomology group and also of functions with the same type as the Weierstrass function. The natural representation for this is the following,

$$f(M_{j,m}) = \sum_{k=0}^{n} c_k(f, M_{j,m})\varepsilon^{k(2-D_{\mathcal{W}})}$$

We begin with a theorem from [DL22b], see Theorem 4.3. Here, we consider functions f belonging to the Holder class, $Hold(\Gamma_{\mathcal{W}})$ satisfying the definition stated below:

Definition 5.2.3 *Let* M *and* M' *be adjacent vertices at the* m^{th} *pre-fractal approximation with complexified affixes* z *and* z', *then* $f \in Hold(\Gamma_{\mathcal{W}})$ *if* f *satisfies the bi-Holder estimate given below,*

$$\tilde{C}_{inf}|z - z'|^{2-D_{\mathcal{W}}} \leq |f(z) - f(z')| \leq \tilde{C}_{sup}|z - z'|^{2-D_{\mathcal{W}}}$$

Where $\tilde{C}_{inf}$ *and* $\tilde{C}_{sup}$ *are positive constants.*

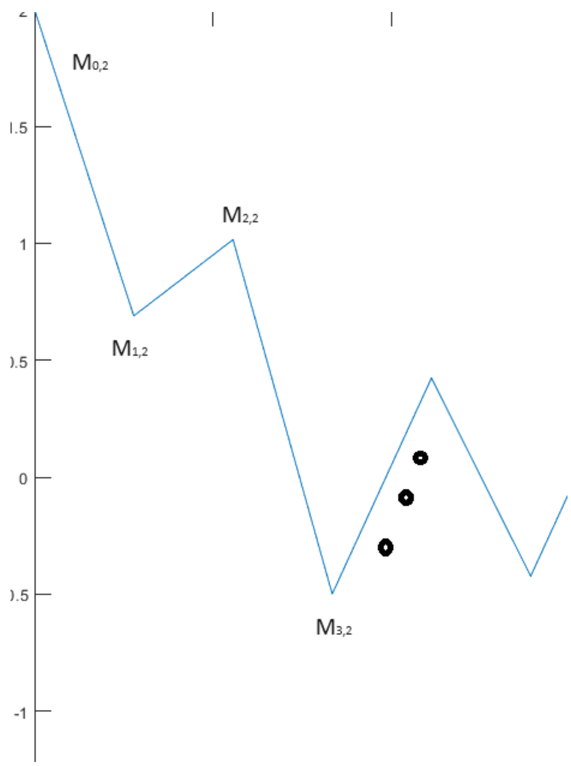

Figure 5.7: Vertices at 2nd pre-fractal approximation, parameter $N_b = 3$, $\lambda = 0.5$

The theorem in [DL22b] is stated in part below. The actual full theorem in [DL22b] provides a complete characterization of the elements of the cohomology groups H_m and refer you to [DL22b] for a complete statement.

Theorem 5.2.1 [DL22b, Theorem 4.3] *Let $m \in \mathbb{N}$ be arbitrary. Then within the set $Hold(\Gamma_{\mathcal{W}})$, then for any integer $m \geq 1$, and with the convention, $H_0 = Im(\delta_{-1,0}) = \{0\}$, the cohomology groups,*

$$H_m = ker(\delta_{m-1,m})/Im(\delta_{m-2,m-1})$$

which are comprised of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e. the restrictions to $V_m^{{N_b}^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables, involving the restrictions *to V_m of Holder continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$, the following Taylor-like expansion is satisfied,*

$$f(M_{j,m}) = \sum_{k=0}^{m} c_k(f, M_{j,m})\varepsilon^{k(2-D_{\mathcal{W}})}$$

where, for each integer k such that $0 \leq k \leq m$, the number $\varepsilon^k > 0$ is the k^{th} cohomology infinitesimal introduced above.

We do not provide the proof of this theorem here, but we refer you to the work of David and Lapidus [DL22b] for a detailed proof using induction. The extension of this theorem is due to the author of this dissertation as well as Michel Lapidus and Claire David. See also the preliminary version of the joint paper, "Fractional Taylor series: Conditions of existence, computation, and explicit formulas," which this theorem will be included in. It is currently in preparation, [DLO24].

Theorem 5.2.2 *In Taylor-like expansion stated in the theorem above,*

$$f(M_{j,m}) = \sum_{k=0}^{m} c_k(f, M_{j,m})\varepsilon^{k(2-D_{\mathcal{W}})}$$

where, for each integer k such that $0 \leq k \leq m$, the number $\varepsilon^k > 0$ is the k^{th} cohomology infinitesimal introduced above. It follows then that $c_k(f, M_{j,m}) =$

$$\sum_{s=0}^{j_k-(j_{k+1}N_b-1)} \delta_{m-k,m-k+1}(f_k)(M_{j_k-s,m-k}, ..., M_{(j_k-s)N_b-(N_b-1),m-k+1}, M_{j_k-s-1,m-k})\varepsilon^{k(D_{\mathcal{W}}-2)}$$

Where $M_{j_k,m-k}$ is the closest vertex to $M_{j_{k-1},m-k+1}$ in V_{m-k} for $1 \leq k \leq m$, where initially, $j_0 = j$. And where,

$$f_k(x) = \begin{cases} f \text{ on } V_{m-k} \setminus V_{m-k-1} \\ 0 \text{ elsewhere} \end{cases}$$

This is a natural formulation since the associated fermion of f, called g splits:

$$g \in H_m = ker(\delta_{m-1,m})/Im(\delta_{m-2,m-1})$$

$g = g_0 + g_0'$*, where* $g_0 \in ker(\delta_{m-1,m})$*, and* $g_0' = Im(\delta_{m-2,m-1})$*. Furthermore,* $g_0' = g_1 + g_1'$*, where* $g_1 \in ker(\delta_{m-2,m-1})$*, and* $g_1' = Im(\delta_{m-3,m-2})$*. Repeating this recursively, we get that,*

$$g = g_0 + g_1 + ... + g_{m-1}$$

Where $g_k \in ker(\delta_{m-k-1,m-k})$*. That is, an* $(m-k)$*-fermion. The restriction to each of these components are the continuous functions,* $f_0, f_1, ..., f_{m-1}$*, which satisfy,*

$$f = f_0 + f_1 + ... + f_{m-1}$$

Where,

$$f_k(x) = \begin{cases} f \ \textit{on} \ V_{m-k} \setminus V_{m-k-1} \\ 0 \ \textit{elsewhere} \end{cases}$$

Proof.

Let f_k be the set of functions that satisfy,

$$f_k(x) = \begin{cases} f \text{ on } V_{m-k} \setminus V_{m-k-1} \\ 0 \text{ elsewhere} \end{cases}$$

Let $M_{j_1,m-1}$ be the closest vertex to $M_{j,m}$ in V_{m-1}, then we can we write,

$$f(M_{j,m}) = [f(M_{j,m}) - f(M_{j_1,m-1})] + f(M_{j_1,m-1}) = ([f(M_{j,m}) - f(M_{j-1,m})]$$

$$+[f(M_{j-1,m}) - f(M_{j-2,m})]... + [f(M_{j_1 N_b+1,m}) - f(M_{j_1,m-1})]) + f(M_{j_1,m-1})$$

$$= \sum_{s=0}^{j-(j_1 N_b+1)} [f_0(M_{j-s,m}) - f_0(M_{j-s-1,m})] + f_1(M_{j_1,m-1})$$

$$= \sum_{s=0}^{j-(j_1 N_b+1)} [f_0(M_{j-s,m}) - f_0(M_{(j-s)N_b-1,m+1}) + f_0(M_{(j-s)N_b-2,m+1}) - ... + f_0(M_{(j-s-1,m)})]$$

$$+f_1(M_{j_1,m-1})$$

$$= \sum_{s=0}^{j-(j_1 N_b+1)} \delta_{m,m+1}(f_0)(M_{j-s,m}, M_{(j-s)N_b-1,m+1}, M_{(j-s)N_b-2,m+1}, ..., M_{(j-s)N_b-(N_b-1),m})$$

$$+f_1(M_{j_1,m-1})$$

The second equality is due to the fact that $f_0 \equiv f$ on V_m and $f_1 \equiv f$ on V_{m-1}. The third equality is due to the fact that, $f_0 \equiv 0$ on $V_{m+1} \setminus V_m$.

Now we can do the same process with the function f_1 to obtain the following expression,

$$= \sum_{s=0}^{j-(j_1 N_b+1)} \delta_{m,m+1}(f_0)(M_{j-s,m}, M_{(j-s)N_b-1,m+1}, M_{(j-s)N_b-2,m+1}, ..., M_{(j-s)N_b-(N_b-1),m})+$$

$$\sum_{s=0}^{j_1-(j_2 N_b+1)} \delta_{m-1,m}(f_1)(M_{j_1-s,m-1}, M_{(j_1-s)N_b-1,m}, M_{(j_1-s)N_b-2,m}, ..., M_{(j_1-s)N_b-(N_b-1),m-1})$$

$$+f_2(M_{j_2,m-2})$$

In the above expression, $M_{j_2,m-2}$ is the closest vertex to $M_{j_1,m-1}$ in V_{m-2}. We can keep doing this process recursively to obtain the following expression,

$$= \sum_{k=0}^{m} \sum_{s=0}^{j_k-(j_{k+1} N_b+1)} \delta_{m-k,m-k+1}(f_k)(M_{j_k-s,m-k}, ..., M_{j_k-s-1,m-k})$$

Now, we can introduce the power of ε in the expression to match the order of the differential in the inner-most sum,

$$\sum_{s=0}^{j_k-(j_{k+1} N_b+1)} \delta_{m-k,m-k+1}(f_k)(M_{j_k-s,m-k}, ..., M_{j_k-s-1,m-k})\varepsilon^{(k-m)(2-D_{\mathcal{W}})}$$

We can rewrite the double sum with this appropriate introduction of ε,

$$\sum_{k=0}^{m} \left(\sum_{s=0}^{j_k-(j_{k+1} N_b+1)} \delta_{m-k,m-k+1}(f_k)(M_{j_k-s,m-k}, ..., M_{j_k-s-1,m-k})\varepsilon^{(k-m)(2-D_{\mathcal{W}})} \right) \varepsilon^{(m-k)(2-D_{\mathcal{W}})}$$

It immediately follows that the coefficients can be written as,

$$c_k = \sum_{s=0}^{j_k-(j_{k+1}N_b+1)} \delta_{m-k,m-k+1}(f_k)(M_{j_k-s,m-k}, ..., M_{j_k-s-1,m-k})\varepsilon^{(k-m)(D_{\mathcal{W}}-2)}$$

Where the f_k functions are defined as above. ■

In the above theorems, we were able to generate a fractal Taylor series and its known coefficients by placing a cohomological structure on the Weierstrass fractal. It is then curious if we could generate the same kind of Taylor series for other similar functions such as the Hardy functions, see p. 48 of [JM96]. The complex dimensions of such kinds of graphs are still not yet known but are possible to compute using similar methods. Recall that the Hardy functions have a form similar to that of the Weierstrass function,

$$f(x) = \sum_k \frac{1}{n_k} \sin(n_k x + \psi_k)$$

where $\frac{n_{k+1}}{n_k} \geq q > 1$.

Bibliography

[Dav17] Claire David. Laplacian, on the graph of the Weierstrass function, arxiv:1703.03371. 2017.

[Dav18] Claire David. Bypassing dynamical systems: A simple way to get the box-counting dimension of the graph of the Weierstrass function. *Proceedings of the International Geometry Center*, 18(2), 2018.

[Dav19] Claire David. On fractal properties of Weierstrass-type functions. *Proceedings of the International Geometry Center*, 12(2), 2019.

[DL22a] Claire David and Michel L. Lapidus. Weierstrass fractal drums - i - a glimpse of complex dimensions,. URL: https://hal.science/hal-03797595v2/document, 2022.

[DL22b] Claire David and Michel L. Lapidus. Weierstrass fractal drums-ii towards a fractal cohomology. Mathematische Zeitschrift, in press, 2024; URL: https://hal.science/hal-03758820v4/document initially, 2022.

[DL24a] Claire David and Michel L Lapidus. Fractal complex dimensions and cohomology of the Weierstrass curve. In Patricia Alonso Ruiz, Michael Hinz, Kasso A. Okoudjou, Luke G. Rogers, and Alexander Teplyaev, editors, *From Classical Analysis to Analysis on Fractals: A Tribute to Robert Strichartz*, 2 of *Applied and Numerical Harmonic Analysis.* Springer, in press, 2024.

[DL24b] Claire David and Michel L Lapidus. Iterated fractal drums some new perspectives: Polyhedral measures, atomic decompositions and Morse theory, in press. In Hafedh Herichi,Maria Rosaria Lancia, Therese-Basa Landry, Anna Rozanova-Pierrat, and Steffen Winter, editors, *Fractal Geometry in Pure and Applied Mathematics.* New York, Contemporary Mathematics, Amer. Math. Society., [Providence], RI, in press, 2024.

[DLO24] Claire David, Michel L Lapidus, and Matthew Overduin. Fractional Taylor series, conditions of existence, computation, and explicit formula [In preparation]. 2024.

[GK06] Robert Greene and Steven Krantz. *Function Theory of One Complex Variable*, volume 40. American Mathematical Society, 3 edition, 2006.

[HS09] Gunther Hormann and Roland Steinbauer. Theory of distributions. URL: https://www.mat.univie.ac.at/ stein/lehre/SoSem09/distrvo.pdf, 2009.

[Hut82] John Hutchinson. Fractals and self-similarity. *Indiana University Mathematics Journal*, 30(5):713–747, 1982.

[JM96] Stephane Jaffard and Yves Meyer. Wavelet methods for pointwise regularity and local oscillations of functions. *Memoirs of the American Mathematical Society*, 123(587), 1996.

[Kat02] Yitzhak Katznelson. *An Introduction to Harmonic Analysis*. Cambridge Mathematical Library, 3 edition, 2002.

[LP90] Michel Lapidus and Carl Pomerance. Fonction zêta de Riemann et conjecture de Weyl–Berry pour les tambours fractals. *Comptes rendus de l'Académie des Sciences Paris Sér. I Math*, 310, 1990.

[LP93] Michel Lapidus and Carl Pomerance. Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums. *Proceedings of London Math Society*, 66(3), 1993.

[LRD17] Michel Lapidus, Goran Radunovic, and Zubrinic Darko. *Fractal Zeta Functions and Fractal Drums: Higher Dimensional Theory of Complex Dimensions*. Springer, 2017.

[LvF13] Michel L. Lapidus and Machiel van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions*. Springer, New York, 2 edition, 2013.

[OS02] Keith Oldham and Jerome Spanier. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Aribitrary Order*. Dover Publications, Inc, 2002.

[Rud64] Walter Rudin. *Principles of Real Analysis*. McGraw-Hill Inc, 3 edition, 1964.

[Sch78] Laurent Schwartz. *Théorie Des Distributions*. Hermann, Paris, 1978.

[Yun15] Po Lam Yung. Proof of Taylor's theorem. `https://www.math.cuhk.edu.hk/course_builder/1516/math1010c/Taylor.pdf`, 2015.