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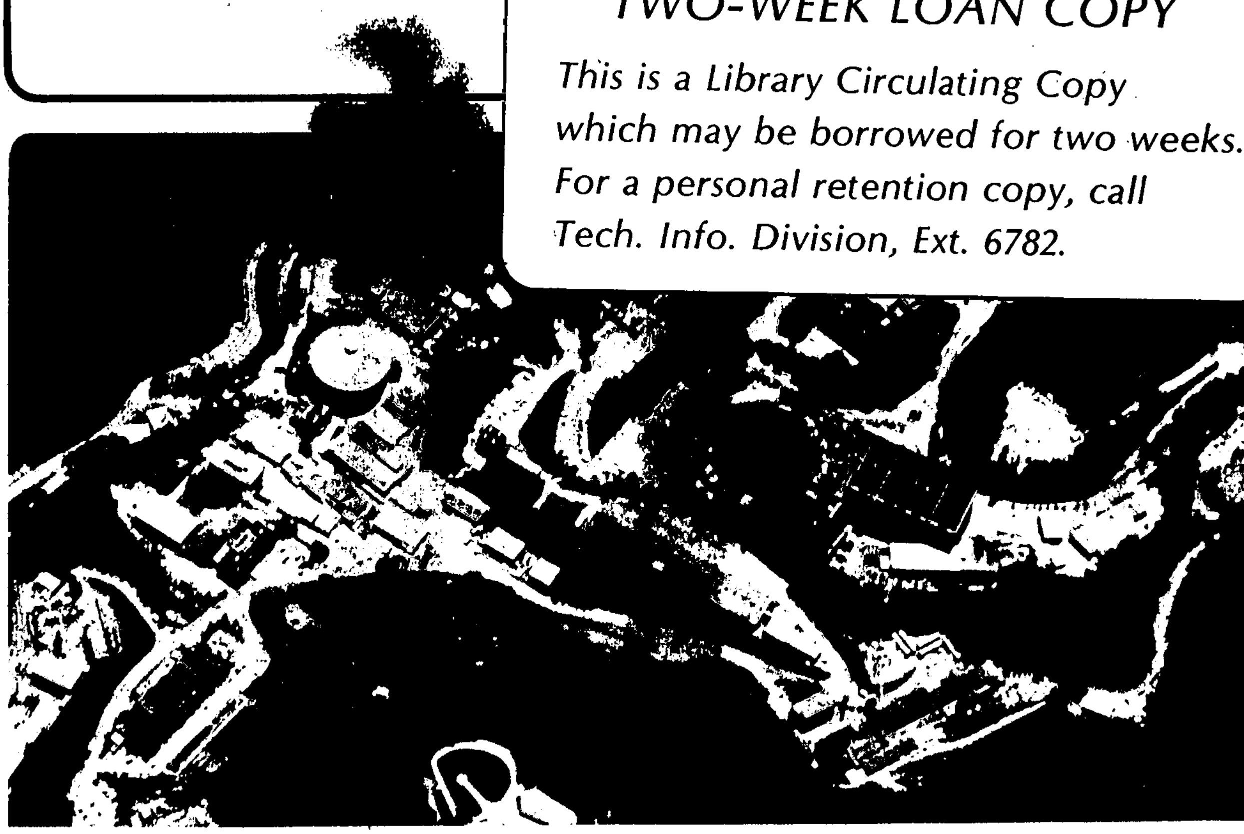

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CONTINUOUS AND DISCONTINUOUS DISAPPEARANCE OF CAPILLARY SURFACES[1]

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CONTINUOUS AND DISCONTINUOUS DISAPPEARANCE OF CAPILLARY SURFACES

Paul Concus and Robert Finn

This note relates to a series of recent papers [1-9] describing behavior of capillary free surfaces in the absence of external (gravity) force fields. Those papers were concerned with the characteristic property, that to every cylinder Z with (planar) section Ω there corresponds a critical angle γ_0, $0 \le \gamma_0 \le \pi/2$, such that a capillary surface S projecting simply onto Ω and meeting Z in a prescribed angle γ exists when $\gamma > \gamma_0$ (if $\gamma_0 \neq \pi/2$) and fails to exist when $\gamma < \gamma_0$ (if $\gamma_0 \neq 0$). The surface can be represented by a solution of the (nonlinear) equation

$$\operatorname{div} Tu = 2 \frac{\Sigma}{\Omega} \cos\gamma$$

in Ω, with

$$Tu = \frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u \ ,$$

under the boundary condition

$$\nu \cdot Tu = \cos\gamma \ ,$$

with ν the unit exterior normal on $\Sigma = \partial\Omega$. We use the symbols Σ, Ω, ... to denote alternatively a set or its measure.

The question of what happens when $\gamma = \gamma_0$ was dealt with in [1-9] only indirectly, in the sense that the answer was reduced to the question of whether a curve (or system of curves) $\Gamma_0 \subset \Omega$ can be found, cutting off a non-null subdomain $\Omega^* \subset \Omega$ bounded by Γ_0 and by subarcs $\Sigma^* \subset \Sigma$, for which the functional

$$\Phi[\Gamma;\gamma] \equiv \Gamma - \Sigma^* \cos\gamma + H_\gamma \Omega^* , \qquad (1)$$

with $H_\gamma = \frac{\Sigma}{\Omega} \cos\gamma$, satisfies $\Phi[\Gamma_0;\gamma_0] \le 0$ (cf. Figure 1). It was proved in [1] that the *existence* of such a Γ_0 implies the *nonexistence* of a solution to the capillary problem. We refer to this behavior as the *existence-nonexistence property*. Given a domain Ω, we define $\gamma_0 = \min\{\frac{\pi}{2}, \sup \gamma : \exists \Gamma \subset \Omega, \Phi[\Gamma;\gamma] \le 0\}$. If $\gamma_0 = \pi/2$ then the capillary surface $u \equiv$ const trivially satisfies the required conditions. Our interest here centers on the case $0 < \gamma_0 < \pi/2$. The existence of a solution when $\gamma_0 < \gamma \le \pi/2$ follows from the results of Giusti [9].

We prove here the following result.

Theorem. *Let $\Sigma = \partial\Omega$ be piecewise smooth, in the sense that $\Sigma \in C^{(2)}$ except perhaps at a finite number of vertices P_j at which two uniformly smooth boundary arcs meet to form an interior angle $2\alpha_j$. Let $\alpha = \min\{\pi/2, \min_j \alpha_j\}$. (If there are no vertices, set $\alpha = \pi/2$.) Then $\gamma_0 \ge \frac{\pi}{2} - \alpha$. If $\frac{\pi}{2} - \alpha < \gamma_0 < \pi/2$ there is no surface of the type described corresponding to $\gamma = \gamma_0$. If $\gamma_0 = \frac{\pi}{2} - \alpha$ then a surface may or may not exist at γ_0, depending on the remaining geometry of Ω.*

We may note the apparent conflict with intuition -- in the case $0 < \gamma_0 < \frac{\pi}{2}$ the surface fails to exist at $\gamma = \gamma_0$ when Σ is smooth but may exist if a corner appears. The matter may however also be considered from another point of view: if Σ is smooth the surface disappears continuously, while if a corner is present the surface may disappear discontinuously, as γ decreases past γ_0.

Proof of the theorem. Our principal tool will be the characterization of the extremal sets for Φ, given in [7]. Suppose first $\frac{\pi}{2} - \alpha < \gamma_0 < \frac{\pi}{2}$. Consider a sequence of values γ approaching γ_0 from below. To each such γ there corresponds a domain Ω^* bounded by Γ and by Σ^*, such that $\Phi[\Gamma;\gamma] \le 0$.

For fixed γ, we consider a minimizing sequence of such curves Γ, each of which may be assumed (as in [7]) to consist of a countable number of components. According to Theorem 1 (and the preceding material) of [7], a non-null minimizing set Γ exists in Ω, and consists of a finite number of disjoint subarcs of semicircles, each of radius $R_\gamma = H_\gamma^{-1}$. At any intersection point of such an arc interior to a smooth subarc of Σ, the two arcs meet with angle γ, as indicated in Figure 2. Intersections at the P_j can occur only if $\alpha_j > \pi/2$. At such points the angles β', β'' satisfy $\beta' \geq \gamma$, $\beta'' \geq \pi-\gamma$.

We now let γ approach γ_0 from below. If the arcs of Γ and of Σ^* bounding a given component of Ω^* are traversed simply, and then Σ is traversed in a specified sense to the next component, and so on until all components are included, we obtain a curve in $\bar{\Omega}$ in which Γ is traversed once and Σ at most twice. Since $\Phi \leq 0$, the lengths $\{\Gamma\}$ are equibounded as γ approaches γ_0 from below, hence we obtain a family of curves in $\bar{\Omega}$, equibounded in length. It follows there is a subsequence that converges uniformly and lower semicontinuously in length, to a curve in $\bar{\Omega}$ of finite length.

Each member of the subsequence contains at least one non-null circular arc Γ in Ω, and since $\gamma_0 > 0$ the length of each such arc is bounded from zero.(1) We restrict attention to these interior arcs $\{\Gamma\}$; we observe that they converge to a non-null set of circular arcs Γ_0, of radius R_{γ_0}, and that $\Phi[\Gamma_0;\gamma_0] = \lim \Phi[\Gamma;\gamma] \leq 0$.

This relation does not in itself exclude the possibility of a solution at γ_0, as Γ_0 could conceivably lie partly or wholly on Σ (cf. Figure 3), and hence would not be admissible for Φ as originally defined. We consider a component Ω_0^* determined by the limit set Γ_0, for which the functional Φ, defined in the sense implied by the limiting procedure, is non-positive. For simplicity we denote the boundary of Ω_0^* again by Γ_0, Σ_0^*. We write $\Gamma_0 = \Gamma_0' \cup \Gamma_0^*$, with $\Gamma_0' = \Gamma_0 \cap \Omega$, $\Gamma_0^* = \Gamma_0 \cap \Sigma$. We have

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(1)Note this assertion could fail if $\gamma_0 = \dfrac{\pi}{2} - \alpha$.

$$\Phi[\Gamma_0;\gamma_0] = \Gamma_0' + \Gamma_0^* - \Sigma_0^* \cos\gamma + H_{\gamma_0}\Omega_0^* \le 0\ , \qquad (3)$$

where $\Sigma_0^* = \lim_{\gamma\to\gamma_0} \Sigma^*$ may in part coincide with Γ_0^*.

Suppose $\Gamma_0' \neq 0$. Letting $\Sigma'^* = \partial\Omega_0^* \cap \Sigma$, we may write

$$\Sigma_0^* = \Sigma'^*_0 + \Gamma_0^* \cap \Sigma_0^*$$

and then from (3)

$$0 \ge \Gamma_0' - \Sigma'^*_0 \cos\gamma_0 + H_{\gamma_0}\Omega_0^* + \Gamma_0^* - (\Gamma_0^* \cap \Sigma_0^*)\cos\gamma$$

$$\ge \Gamma_0' - \Sigma'^*_0 \cos\gamma_0 + H_{\gamma_0}\Omega_0^* = \Phi[\Gamma_0';\gamma_0]$$

so that, by the existence-nonexistence property, no solution can exist at $\gamma = \gamma_0$.

Suppose $\Gamma_0' = 0$. Then either $\Omega_0^* = \phi$ or $\Omega_0^* = \Omega$. In the former case $\Sigma_0^* = \Gamma_0^*$, and we find from (3)

$$0 \ge \Phi[\Gamma_0;\gamma_0] = \Gamma_0^*(1-\cos\gamma) > 0\ ,$$

a contradiction. If $\Omega_0^* = \Omega$, (3) yields

$$0 \ge \Gamma_0^* - \Sigma_0^* \cos\gamma + \left(\frac{\Sigma}{\Omega}\cos\gamma\right)\Omega$$

$$= \Gamma_0^*(1-\cos\gamma) > 0\ ,$$

again a contradiction. We conclude $\Gamma_0' \neq \phi$ and hence no solution can exist, as was to be proved.

We consider finally the case $\gamma_0 = \frac{\pi}{2} - \alpha$. In this case at least one corner P_0 must occur, with interior half angle $\alpha_0 = \frac{\pi}{2} - \gamma_0$. When that happens, the arcs $\{\Gamma\}$ can degenerate to P_0 for $\gamma \to \gamma_0$ (cf. Figure 4). Various types of behavior can occur, depending on the remaining geometry. For example, it can happen that other arcs of $\{\Gamma\}$ converge to a non-null $\Gamma_0 \subset \Omega$, for which $\Phi[\Gamma_0;\gamma_0] \le 0$. In that event, no capillary surface over Ω will exist at γ_0. The particular situation in which all arcs of $\{\Gamma\}$ disappear at the corners has a special interest, and will be discussed as a special case of a more general result of the latter author that is now in preparation [10]. We remark here only that in the case just mentioned it can happen that a surface exists at γ_0. A simple example is obtained by choosing for Ω an equilateral triangle. Here $\alpha = \pi/6$; the lower hemisphere whose equatorial circle is the circumscribing circle for Ω yields a capillary surface over Ω that meets the vertical bounding walls in the constant angle $\gamma_0 = \frac{\pi}{2} - \alpha = \frac{\pi}{3}$.

REFERENCES

1. P. Concus and R. Finn, *On capillary free surfaces in the absence of gravity*, Acta Math., 132 (1974), pp. 177-198.
2. Jin-Tzu Chen, *On the existence of capillary free surfaces in the absence of gravity*, Pacific J. Math., 88 (1980), pp. 323-361.
3. E. Giusti, *On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions*, Invent. Math., 46 (1978), 111-137.
4. R. Finn, *Existence and non-existence of capillary surfaces*, Manuscripta Math., 28 (1979), pp. 1-11.
5. R. Finn and E. Giusti, *Non-existence and existence of capillary surfaces*, Manuscripta Math., 28 (1979), pp. 13-20.
6. R. Finn, *Existence criteria for capillary free surfaces without gravity*, Indiana Univ. Math. J., to appear.
7. R. Finn, *A subsidiary variational problem and existence criteria for capillary surfaces*, preprint, Math. Dept., Stanford Univ., Stanford, CA, 1982.
8. E. Giusti, *Generalized solutions for the mean curvature equation*, Pacific J. Math., 88 (1980), pp. 297-321.
9. E. Giusti, *Boundary value problems for non-parametric surfaces of prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 3 (1976), pp. 501-548.
10. R. Finn, *A limiting geometry for capillary surfaces*, in preparation.

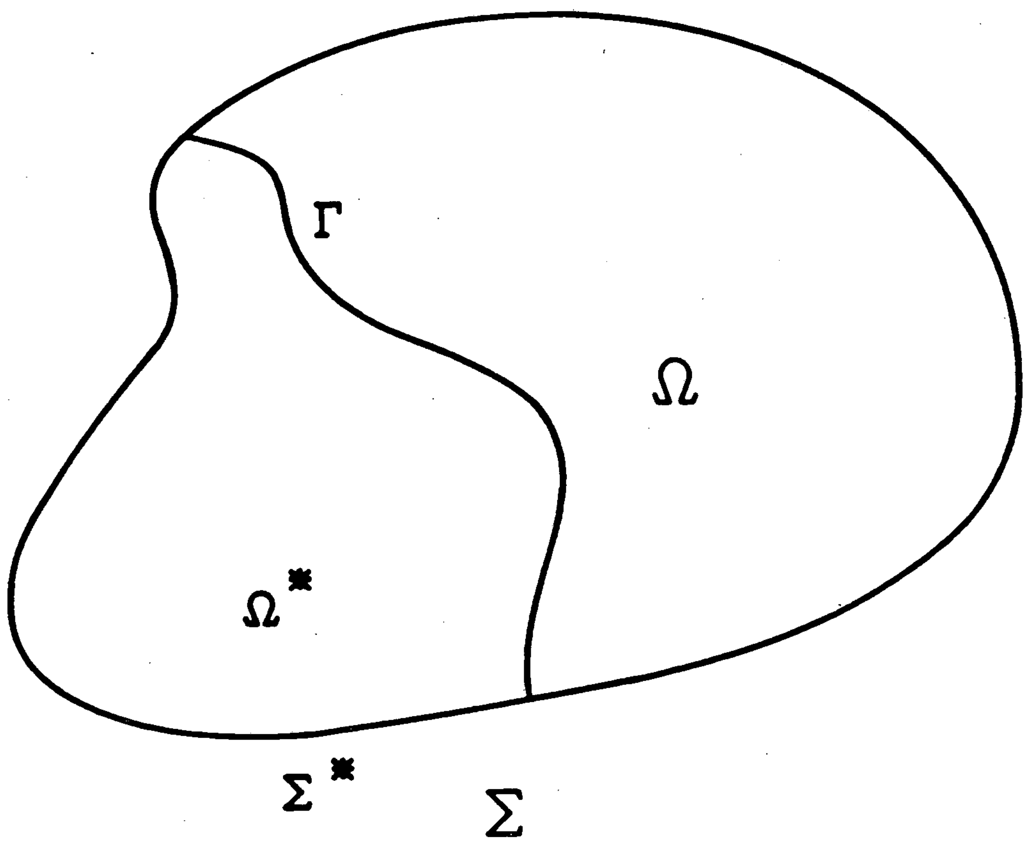

Figure 1

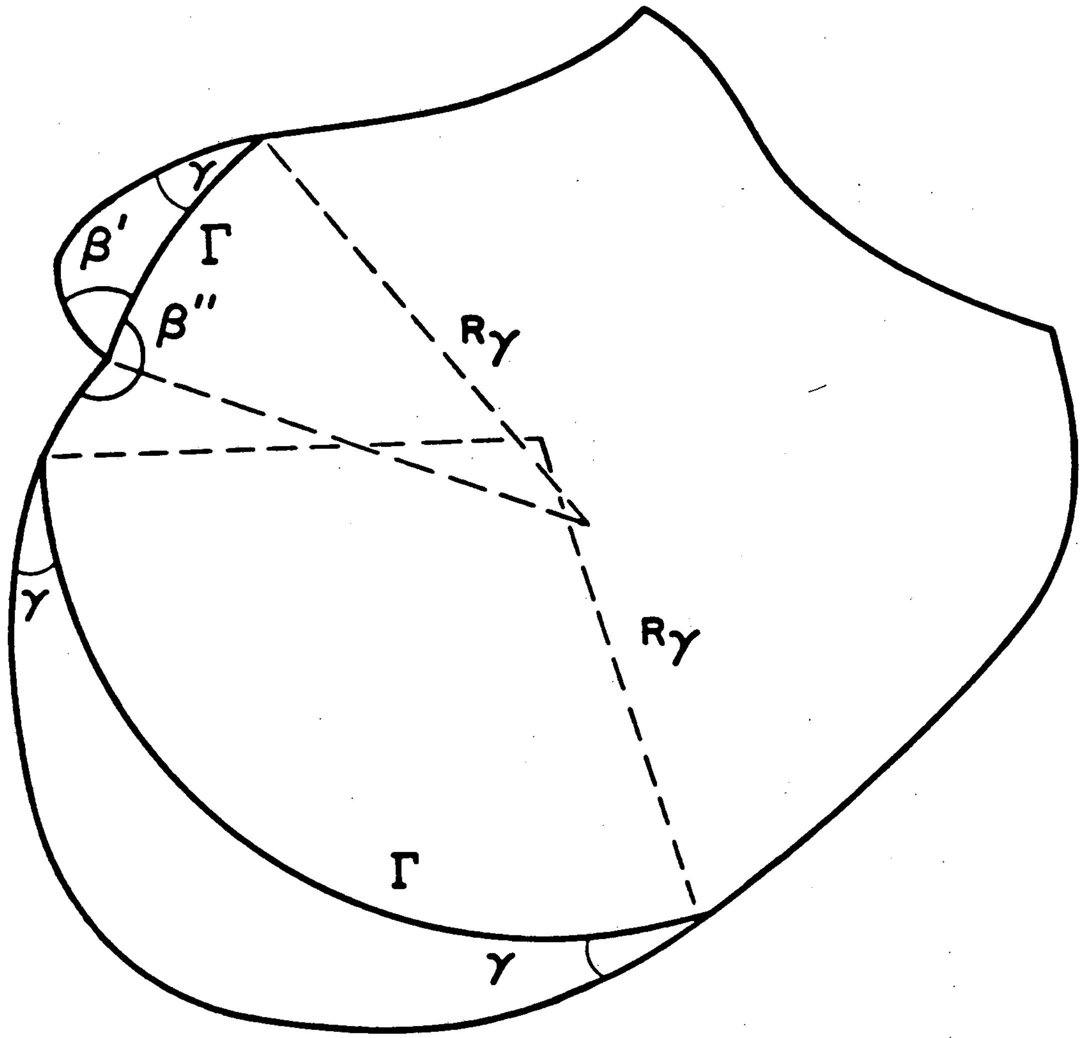

Figure 2

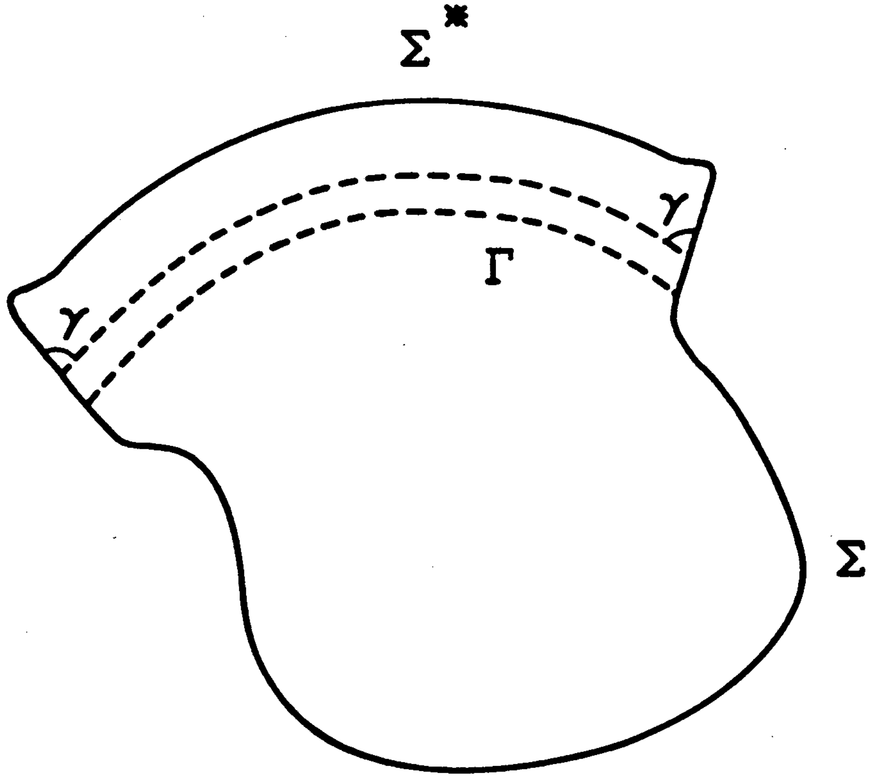

Figure 3

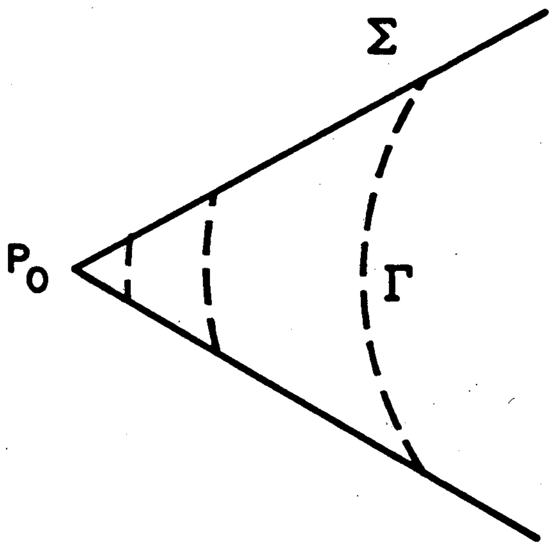

Figure 4