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1. Introduction

In this talk I will outline some proofs of dispersion relations for every order in perturbation theory. After this I will indicate some further topics that can be studied by perturbation methods.

The following dispersion relations (DR) have now been proved in perturbation theory:

Single-variable DR	(a)	Vertex parts
	(b)	Forward scattering
	(c)	Non-forward scattering (in a limited range)
	(d)	External-mass DR
	(e)	Internal-mass DR
Partial-wave DR	(a)	Equal masses
	(b)	General masses without anomalous thresholds
Mandelstam representation	(a)	Equal masses
	(b)	General masses without anomalous thresholds.

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The further topics in this talk will include some remarks about integral representations with anomalous thresholds, and representations of production amplitudes. I will also mention some points connected with experiments on final-state interactions in which production amplitudes are involved.

Before outlining the proofs I will indicate some of the mathematical methods on which they are based.

2. Methods

(a) Conditions for Singularities

An integral transform,

$$f(x) = \int_A^B g(x, y)dy \qquad (2.1)$$

along a given contour C from A to B, will be singular at $x = x_0$ if[1]

either a singularity $y_1(x)$ of g tends to an end point of the contour as x tends to x_0,

or two singularities $y_1(x)$, $y_2(x)$ tend to coincidence from opposite sides of the contour as x tends to x_0. The first condition is described as an "end point" singularity, and the second as a "coincident" singularity.

(b) Rules for Singularities of an Amplitude

These conditions lead to simple rules for the singularities of a function defined from a Feynman diagram. It is sufficiently general to consider only scalar particles. A scattering amplitude from a Feynman diagram will depend on two of the invariant energies squared, s, t, and u, where

$$s + t + u = \sum_1^4 M_i^2 .$$

Denote the four-momentum in any line by q_i , and the internal momentum variables by k_j. Then

$$F_\epsilon(s, t) = c_1 \int dk_1 \ldots dk_\ell \frac{1}{\prod_1^n (q_i^2 - m_i^2 + i\epsilon)} \qquad (2.2)$$

$$= c_1 \int_0^1 d\alpha_1 \ldots d\alpha_n \int dk_1 \ldots dk_\ell \frac{\delta(1 - \Sigma \alpha_i)}{[\psi(\alpha, q)]^n} \qquad (2.3)$$

$$= c_2 \int_0^1 d\alpha_1 \ldots d\alpha_n \frac{\delta(1 - \Sigma \alpha_i)[C(\alpha)]^{n-2\ell-1}}{[D_\epsilon(\alpha, s, t)]^{n-2\ell}} , \qquad (2.4)$$

where

$$\psi(\alpha, q) = \sum_1^n \alpha_i(q_i^2 - m_i^2 + i\epsilon) . \qquad (2.5)$$

The discriminant of ψ as a quadratic form in the internal momenta k_j is written $D_\epsilon(\alpha, s, t)$. Its form when ϵ is zero is $D(\alpha, s, t)$. The quantity $C(\alpha)$ is the discriminant of the quadratic part of ψ.

The rules for locating singularities of

$$F(s, t) = \lim_{\epsilon \to 0} F_\epsilon(s, t) \qquad (2.6)$$

can be stated in two equivalent forms. The first is[2,3,4]

<u>either</u> $\frac{\partial D}{\partial \alpha_i} = 0$, <u>or</u> $D = 0$ at $\alpha_i = 0$ for all i.

This includes the condition $D = 0$, since D is homogeneous in the α variables.

The second form is[2,5]

(1) Either $\alpha_i = 0$, or $q_i^2 = m_i^2$ for all $i = 1,\ldots,n$,

and

(2) $\frac{\partial\psi}{\partial k_j} = 0$, for all $j = 1,\ldots,\ell$.

An alternative form of (2) is

(2a) $\Sigma\, \alpha_j\, q_j = 0$, summed around any closed circuit in the diagram.

The second form of these rules provides the basis for electric circuit analogies[5,6] and for the method of dual diagrams.[2,7,8,9] In this talk most of my proofs will be based on the discriminant D and the first form of the conditions.

(c) Reduced Diagrams

If we have an end-point singularity, $D = 0$ for $\alpha_1 = 0$ say, there is no further condition on the momentum in the corresponding line. We can therefore consider, instead of the initial Feynman diagram, a reduced diagram in which the line α_1 is reduced to a point. It is sometimes convenient to classify singularities in terms of the set of all possible reduced diagrams.[10] For these we need consider only coincident singularities in the α variables.

(d) Surfaces of Singularities

The solution of the equations

$$\frac{\partial D}{\partial \alpha_i} = 0 , \quad \text{for } i = 1, 2,\ldots,n , \tag{2.7}$$

leads to a relation between s and t,

$$\sigma(s, t) = 0 . \tag{2.8}$$

When s and t are taken to be complex variables, this defines a two-dimensional surface in the four-dimensional space. If it has solutions with s and t real these will give curves or straight lines in the real s, t plane. It seems probable that some branch of $F(s, t)$ will be singular at any given point of the surface, but I do not think anyone has proved this. For this talk we will need to consider only singularities of the physical branch and its analytic continuation on the physical sheet.

(e) The Physical Branch and the Physical Sheet

The physical branch of the amplitude is defined in physical scattering regions by writing $(m^2 - i\epsilon)$ for the internal masses in $F_\epsilon(s, t)$ and letting ϵ tend to zero. The discriminant is

$$D_\epsilon(\alpha, s, t) = sf(\alpha) + tg(\alpha) - K(m, \alpha) + i\epsilon \Sigma \alpha_i C(\alpha) . \qquad (2.9)$$

In the physical scattering regions the α contours of integration are the real range $[0, 1]$. Then $C(\alpha)$ is positive, and hence D_ϵ is never zero on the contour, so the integral is well defined.

I will show later that near the physical scattering region where s is the energy, the discriminant

$$D(\alpha, s + i\epsilon, t) = (s + i\epsilon)f + tg - K \qquad (2.10)$$

has its zeros located relative to the contours of integration in the same way as those of D_ϵ.[11] This enables us to prove a dispersion relation in s, which defines the physical sheet for complex s.

In physical scattering regions the singularities of the physical branch of $F(s, t)$ can be identified by requiring the Feynman parameters α to be positive. This positive α condition cannot in general be assumed to be true elsewhere in the physical sheet. In fact, with anomalous thresholds of the second kind, the condition is not valid,[12] owing to distortions of the α contour that include both complex and negative values of these parameters.

3. Single-Variable Dispersion Relations

(a) Vertex Parts[13]

The discriminant for any diagram has the form

$$D_\epsilon(\alpha, z) = z P_1(\alpha) + M_2^2 P_2(\alpha) + M_3^2 P_3(\alpha) - \Sigma \alpha_i m_i^2 C(\alpha) + i\epsilon \Sigma \alpha_i C(\alpha) \tag{3.1}$$

$$= D(\alpha, z) + i \epsilon \Sigma \alpha_i C(\alpha) . \tag{3.2}$$

The coefficient of z is positive for α real. Hence $D(\alpha, z)$ is never zero for z in the upper half plane. It is real for z large and negative, and it has the same form as D_ϵ as z tends to the positive axis from the upper half plane. The vertex function defined from $D(\alpha, z)$ instead of D_ϵ is therefore real on the real axis and analytic in the upper half plane. It also is an analytic continuation of the vertex function defined for real z from D_ϵ, and it satisfies a dispersion relation.

(b) Forward Scattering

(c) Non-Forward Scattering[14,15]

I will assume equal masses for simplicity. The amplitude is defined from

$$D_\epsilon(\alpha, s, t) = sf + tg - K + i \epsilon \Sigma \alpha_i C(\alpha) . \tag{3.3}$$

Consider

$$D(\alpha, s, t) = (s - 4m^2)f + (4m^2 f + tg - K) . \tag{3.4}$$

For s in a physical region $s > 4m^2$, the first term in D is negative when f is negative. The second term is also negative or zero for

$(-4m^2 < t < 4m^2)$. When f is positive, D cannot be zero if s has a positive imaginary part. Hence $D(\alpha, s + i\epsilon, t)$ has its zeros located in the same way as those of D_ϵ relative to the α contours of integration. Further, $D(\alpha, s, t)$ is nonzero for s in the upper half plane and for $(-4m^2 < t < 4m^2)$, and it is real when s is real in the range $(-t < s < 4m^2)$. This proves that the amplitude $F(s, t)$ defined from $D(\alpha, s, t)$ satisfies a dispersion relation. It also tends to the physical branch of the amplitude, defined from D_ϵ, as s tends to the real axis from the upper half plane in the physical region $s > 4m^2$. This dispersion relation defines the physical sheet in the complex variable s.

(d) <u>External-Mass Dispersion Relation</u>

(e) <u>Internal-Mass Dispersion Relation</u>

The coefficient of any external mass M_j^2 in $D(\alpha, s, t)$ is positive when α is real. Hence we can replace $D_\epsilon(\alpha, s, t, M_j^2)$ by $D(\alpha, s, t, z_j)$ without changing $F(s, t)$ in a physical scattering region, provided z_j has a small positive imaginary part. Now we have

$$D(\alpha, s, t, z_j) = (z_j - M_j^2)\, P(\alpha) + sf + tg - K \,. \qquad (3.5)$$

This function is nonzero for real α and z_j in the upper half plane (s and t being real). It is real and nonzero for z_j real and sufficiently negative. Hence it satisfies a dispersion relation in the external mass variable z_j.

The coefficient in D of any internal mass m_i^2 is negative for real α. This leads to anlaytic continuation into the lower half plane and gives a dispersion relation in the internal mass variable.

4. Partial-Wave Dispersion Relations

(a) Equal Masses

Writing k for the three momentum and θ for the scattering angle in the center-of-mass system, we have

$$t = -2k^2(1 - \cos\theta) , \tag{4.1}$$

$$u = -2k^2(1 + \cos\theta) , \tag{4.2}$$

$$s = 4k^2 + 4m^2 . \tag{4.3}$$

The kinematics of the physical regions are indicated in Fig. 1.

The partial-wave amplitude is

$$A_\ell(s) = \int_{-1}^{1} d(\cos\theta)\, A(s, -2k^2(1 - \cos\theta))\, P_\ell(\cos\theta) \tag{4.4}$$

$$= -\frac{1}{2k^2} \int_{u=0}^{t=0} dt\, A(s, t)\, P_\ell\left(1 + \frac{t}{2k^2}\right) . \tag{4.5}$$

Now, we have

$$A(s, t) = \int d\alpha_1 \cdots \frac{n(\alpha)}{[D(\alpha, s, t)]^{n-2\ell}} , \tag{4.6}$$

where D is linearly dependent on t. Hence $A_\ell(s)$ cannot contain any coincident singularities in the integration over t. It follows that $A_\ell(s)$ can be singular only when the integrand in Eq. (4.5) gives end-point singularities in t. Hence for $A_\ell(s)$ to be singular either $A(s, t=0)$ or $A(s, u=0)$ must be singular. From the forward-scattering dispersion relations we know that these amplitudes are singular only when s is real. We also know that $A_\ell(s)$ is real when s is real and in the range $0 < s < 4m^2$. This proves a dispersion relation for the partial-wave amplitude,[16]

$$A_\ell(s) = \left\{ \int_{-\infty}^{0} ds' + \int_{4m^2}^{\infty} ds' \right\} \frac{A_\ell^{\,1}(s')}{(s' - s)} . \tag{4.7}$$

(b) Unequal Masses

For illustration I will consider pion-nucleon scattering. Take

$$s = (p_n + p_\pi)^2 , \tag{4.8a}$$

$$t = (p_n + p'_{\bar{n}})^2 , \tag{4.8b}$$

$$u = (p_n + p'_\pi)^2 . \tag{4.8c}$$

Using the same arguments as in the equal-mass case, we need to consider only the end-point singularities in the integration over $\cos\theta$. Making a change of variable to u, these end-point singularities occur at[17] $t = 0$ or

$$u = 2M^2 + 2m^2 - s , \tag{4.9}$$

and at $t = -4k^2$ or

$$u = \frac{(M^2 - m^2)^2}{s} . \tag{4.10}$$

The first surface $t = 0$ gives forward scattering for which we have a dispersion relation. However, the second end point is on a curved hypersurface on which s, t, and u can all be complex. Hence in addition to the known normal thresholds in u and t we have to consider complex singularities. It is possible to show that none exists, but the method is more complicated than first proving the Mandelstam representation and then deducing that there are no complex singularities on this hypersurface. I will therefore adopt the latter procedure.

For completeness I should mention the well-known fact that the (real) normal thresholds in t lead to complex singularities in s through the equation

$$t = -4k^2 = -\frac{[s-(M+m)^2][s-(M-m)^2]}{s} . \qquad (4.11)$$

5. The Mandelstam Representation

The Mandelstam representation[18] assumes that the amplitude $A(s, t)$ can be continued analytically so that there are no singularities in the physical sheets of the variables s, t, and u, where

$$s + t + u = \Sigma M_i^2 . \qquad (5.1)$$

The singularities are assumed to occur only on the boundaries of the physical sheets where the variables are real.

In my proof[19] of the Mandelstam representation I will assume that there are no anomalous thresholds. This condition is satisfied by terms of all orders when it is satisfied by the fourth-order term. I will indicate later how it may be possible to relax this condition to allow a proof of the Mandelstam representation when there is one type of anomalous threshold. In order to simplify the kinematics I will illustrate the proof by reference to the equal-mass case. I will begin by obtaining a few properties of the scattering amplitude that will be required for the proof. Then I will use the method of analytic completion[20] to extend the region of analyticity that is given by the single-variable dispersion relations. This will then permit a double application of Cauchy's theorem, which establishes the Mandelstam representation for every order in perturbation theory.

The main points in the proof are

(1) There are no anomalous thesholds. This follows from the result that D is negative for $s < 4m^2$, $t < 4m^2$, $u < 4m^2$. Hence any line of singularities $s =$ constant must intersect a physical region. In these regions the only singularities are at normal thresholds.

(2) The slope of a curve of singularities is given by

$$\frac{dt}{ds} = -\frac{f(\alpha)}{g(\alpha)} , \tag{5.2}$$

where the α variables satisfy

$$\frac{\partial D}{\partial \alpha_i} = 0 , \qquad \Sigma \alpha_i = 1 . \tag{5.3}$$

I will call these the critical values of the Feynman parameters.

(3) A curve of singularities on the boundary of the physical sheet can touch a normal threshold only at infinity. We have

$$D(\alpha, s, t) = sf + tg - m^2 K , \tag{5.4}$$

where g contains as a factor a product of at least two Feynman parameters that are zero at the normal threshold singularity. Let these be α_1 and α_2. As the curve of singularities tends to the normal threshold, α_1 and α_2 both give coincident singularities that tend to end-point singularities at the point of tangency. For example, on the curve, at the critical values of the α variables, we have

$$\frac{\partial D(\alpha, s, t)}{\partial \alpha_1} = s \frac{\partial f}{\partial \alpha_1} - m^2 \frac{\partial K}{\partial \alpha_1} + t \frac{\partial g}{\partial \alpha_1} = 0 . \tag{5.5}$$

On the normal threshold, $\alpha_1 = 0$ is an end-point singularity, and

$$s \frac{\partial f}{\partial \alpha_1} - m^2 \frac{\partial K}{\partial \alpha_1} \neq 0 . \tag{5.6}$$

If this term were zero, in the simplest case we could construct an anomalous threshold with $\alpha_1 \neq 0$ and $\alpha_2 = 0$. More generally we might be able to construct a curve of singularities with $\alpha_1 \neq 0$ and the line α_2 reduced to a point. Such a curve would be tangent to the threshold at the same point. By choosing the most fully reduced of these curves we can apply the preceding argument. Now, $\partial g/\partial \alpha_1$ contains a factor α_2, which tends to zero at the point of tangency. But $t \frac{\partial g}{\partial \alpha_1}$ is finite at the point of tangency. Hence t tends to infinity as the curve tends to the normal threshold.

(4) A singularity at a normal threshold in s can be avoided by continuation of $F(s, t)$, giving a small positive or negative imaginary part to s. This is independent of whether t has an imaginary part. Similarly a threshold in t can be avoided by $t \pm i\epsilon$.

(5) The curve of singularities that touches a normal threshold in t, say $t = 4m^2$, connects to a surface of singularities. This surface is not encountered when we continue $F(s, t)$ on the physical sheet near $(s_1 + i\epsilon,\ t_1 \pm i\epsilon')$ where s_1 and t_1 are real in the neighborhood of the curve of singularities. We have

$$D(\alpha, s_1 + i\epsilon, t_1 \pm i\epsilon') = (s + i\epsilon)f + (t \pm i\epsilon')g - K . \tag{5.7}$$

On the curve, g is positive. To be definite we take also s positive, so that f is positive but tends to zero near $t = 4m^2$ (the curve lies above this threshold). Then it is clear that the function F is analytic at $(s_1 + i\epsilon,\ t_1 + i\epsilon')$. Given any $\epsilon > \epsilon' > 0$, we can always choose a point s' on the curve such that

$$\left|\frac{dt}{ds}\right| < \frac{\epsilon'}{\epsilon}, \qquad \text{for} \quad s > s' . \tag{5.8}$$

Then using the relation $\frac{dt}{ds} = -\frac{f}{g}$, we obtain, for the critical values of α,

$$i\,\epsilon\, f - i\,\epsilon'\, g \neq 0, \qquad \text{for} \quad s > s' . \tag{5.9}$$

Hence F is analytic up to some point $t_1 > 4m^2$, also at $(s_1 + i\epsilon,\ t_1 - i\epsilon')$. This argument can be applied to any curve touching a higher normal threshold, by working with the suitably reduced diagram.

(6) The dispersion relation in s shows that for real values of t in the range $-4m^2 < t < 4m^2$, the amplitude $F(s, t)$ has no singularities either in the upper or lower half planes of the variables s.

(7) A curve of singularities in the real s, t plane has real slope $\frac{dt}{ds}$. If the slope is positive then at points on the neighboring complex surface Σ, s and t will have imaginary parts of the same sign. Thus

$$\frac{dt}{ds} > 0 \quad \text{leads to} \quad (s + i\epsilon,\ t + i\epsilon'), \text{ and } (s - i\epsilon,\ t - i\epsilon') \text{ on } \Sigma .$$

Similarly

$$\frac{dt}{ds} < 0 \quad \text{leads to} \quad (s + i\epsilon,\ t - i\epsilon'), \text{ and } (s - i\epsilon,\ t + i\epsilon') \text{ on } \Sigma .$$

However, it is important to note that these neighboring points are not necessarily on the physical sheet when the curve of singularities is on its boundary.

The next part of the proof[19] makes use of the method of analytic completion[20] for a function of two complex variables. We begin with the information given by the single-variable dispersion relations in s, and use this to provide a contour that can be moved past the threshold in t.

(8) For $-4m^2 < t < 4m^2$, $F(s, t)$ is analytic in the upper half s plane. Hence we can write

$$F(s, t) = \frac{1}{2\pi i} \int_C \frac{F(z, t)dz}{z - s} , \qquad (5.10)$$

where the contour C is an arbitrarily large semicircle in the upper half s plane.

The method of analytic completion now allows us to extend the region of analyticity of $F(s, t)$ by displacing the contour C parallel to itself (in the complex space of s and t), provided the contour moves entirely in an analytic region of $F(s, t)$.

(9) First we displace the contour to give the function

$$F(s, t_1 + i\epsilon') ,$$

which is analytic in the upper half s plane, by step (4) in this proof.

(10) Next we displace the contour C to a point where t_1 is greater than $4m^2$. This is possible by step (5).

(11) We now displace C by continuously increasing t_1; this is illustrated in Fig. 2. It will not meet any complex singularities, since these would correspond to horns projecting into the contour. Analytic completion shows that we can continue into horns. We therefore need consider only singularities that might distort the contour. Since these lie initially within $i\epsilon$ of real s, and $i\epsilon'$ of real t, they must come from surfaces associated with curves of singularities, if they exist at all. From step (7) only curves of positive slope can give trouble.

(12) A curve of singularities cannot have a minimum except possibly at a spurious turning point, since there are no anomalous thresholds. A spurious turning point has $dt/ds = 0$, with $f = 0$, but all its α singularities are coincident (and are not also end-point singularities as at an anomalous turning point). Thus a minimum in the curve as a function of t would lead to a horn projecting down into C. It therefore cannot exist. There cannot be a maximum of a curve of singularities, since by continuity it would also be associated with a minimum which would be encountered first by C. We can use continuity, since a curve of singularities can leave the boundary of the physical sheet only by touching another curve of the same slope, or asymptotically through a normal threshold. This proves that there are no spurious turning points.

(13) From step (7) the only curves that would not lead to horns extending into C, for $s = s_1 + i\epsilon$ (and with $t = t_1 + i\epsilon'$) on the straight edge of C, have negative slope and go asymptotically to the normal threshold. Thus there may be curves of singularities of $F(s_1 + i\epsilon,\ t_1 + i\epsilon')$ in the limit of ϵ, ϵ' tending to zero, only in the region $s > 4m^2$, $t > 4m^2$, and not in the region $u > 4m^2$, $t > 4m^2$. Any continuous curve in the latter region would lead to complex singularities that are disallowed by our analytic completion.

(14) Similarly there are no complex singularities of $F(s, t_1 - i\epsilon')$ in the upper half s plane, and no curves of singularities in the region $s_1 > 4m^2$, $t_1 > 4m^2$ that can be reached by allowing ϵ and ϵ' to tend to zero in $F(s_1 + i\epsilon,\ t_1 - i\epsilon')$. However, this function may have curves of singularities in $u > 4m^2$, $t > 4m^2$ in this limit. We must keep $\epsilon > \epsilon'$ during analytic completion to avoid normal thresholds in u.

(15) We can now deduce the Mandelstam representation by making a double application of Cauchy's theorem.[11,19] We have established a single dispersion relation,

$$F(s, t) = \frac{1}{2\pi i} \left\{ \int_{-\infty}^{-s} dt' + \int_{4m^2}^{\infty} dt' \right\} \left\{ \frac{F(s, t' + i\epsilon) - F(s, t' - i\epsilon)}{t' - t} \right\} . \tag{5.11}$$

We have shown that in the limit, as ϵ tend to zero, $F(s, t \pm i\epsilon)$ has no singularities in the upper half s plane. Also in this limit

$$\frac{1}{2i} \left\{ F(s, t' + i\epsilon) - F(s, t' - i\epsilon) \right\}$$

is real for $0 < s < 4m^2$, at least. This permits a second application of Cauchy's theorem from the upper contour t' in the range $4m^2$ to infinity. The integrand on the lower half contour can be expressed by a dispersion relation with the variable u kept constant. The contours used in this double application of Cauchy's theorem are shown in Fig. 3. Finally the oblique axes of integration[11] can be combined to give the Mandelstam representation.

The arguments on which this proof is based require only that there exist a single dispersion relation, and that the consequent analyticity can be continued by analytic completion throughout the physical sheet. This analytic completion depends on the properties of the amplitude near normal thresholds. The basic requirement is that there be no anomalous thresholds. Provided this condition is satisfied the proof applies also to the general mass case.

I will discuss next the properties of amplitudes with anomalous thresholds.

6. Anomalous Thresholds

(a) Anomalous Type I

I will consider first anomalous thresholds whose surfaces of singularities do not enter the physical sheet. The conditions on the masses under which this holds for fourth-order terms have been investigated by Tarski.[3] The same conditions will probably suffice to keep singularities from higher-order terms off the physical sheet, but this has not yet been proved.

One can prove, however, that the existence of anomalous thresholds can be determined from the fourth-order term. The proof is obtained by noting that removal of an internal line from a Feynman diagram will not raise the lowest threshold value.[21] We have[10]

$$\frac{\partial D(\alpha, st)}{\partial \alpha_i} = D(\alpha, \alpha_i^{-1} s, t) - m_i^2 C(\alpha) - \alpha_i m_i^2 C(\alpha, \alpha_i^{-1}) , \tag{6.1}$$

where α_i^{-1} denotes that the line $\underline{i}$ is removed. Thus, if $D(\alpha, \alpha_i^{-1}, s, t)$ is negative for all real α, so is $\frac{\partial D(\alpha, s, t)}{\partial \alpha_i}$. Since we have

$$D = \frac{1}{n} \Sigma \alpha_i \frac{\partial D}{\partial \alpha_i} , \tag{6.2}$$

it follows that if every fourth-order diagram has $D(\alpha, s, t) < 0$, (for s, t, u below the first threshold), then the same is true for every higher-order diagram. Hence the region of real $A(s, t)$ in the s, t plane is determined by the fourth-order term.

From my choice of definition of this type of anomalous threshold, the Mandelstam representation applies. The singular curves of a fourth-order diagram have the form shown in Fig. 4. The amplitude $A(s + i\epsilon, \quad t + i\epsilon')$ is singular as ϵ and ϵ' tend to zero, on the singular curves where their slope is negative. The amplitude $A(s + i\epsilon, \quad t - i\epsilon')$ is singular in the same limit, where the curves have positive slope.

(b) Anomalous Type II

The anomalous thresholds of the second type have singularity surfaces entering the physical sheet.[3] The intersection with the real s, t plane is shown in Fig. 5. The curves AB and CD are connected by complex singular curves in the upper half s and t planes, and in the lower half planes. Below CD the amplitude is real and the α integration is real in the range $0, 1$. On following the singularity from a point on CD where the contour is real to a point on AB, we find that the contour of integration becomes distorted so that on AB the singularities are due to a pinching of the contour at negative value of α.[12] Thus in this case both negative and complex values of α are relevant to singularities in the physical sheet.

I would like also to remark on an integral representation with anomalous thresholds of this type which is plausible but not yet proved in perturbation theory. The addition of internal lines quite clearly moves the normal thresholds to higher values. It almost obviously moves anomalous thresholds to higher values. It is plausible that this will result in the disappearance of anomalous thresholds of Type II for terms of sufficiently high order. It will not remove anomalous thresholds of Type I; this can be easily seen from ladder diagrams. This suggests that the amplitude can be expressed by a finite number of terms in perturbation theory with physical masses and coupling constants together with a remainder that satisfies the Mandelstam representation.

One further point about anomalous thresholds is that an anomalous-type diagram internal to a diagram not otherwise anomalous does not cause an anomalous threshold. This follows from the fact that the removal of an internal line cannot lower the leading threshold. By suitably reducing lines, we see that the lowest threshold is given by the appropriate fourth-order term.

(c) Partial-Wave Dispersion Relation with Anomalous Thresholds

These can be studied by perturbation methods.[22] The most useful approach appears to be analytic continuation in the external, or internal, masses from a situation in which no anomalous thresholds occur. I have no special results on this, but mention it only for completeness.

7. Production Amplitudes

(a) Complex Singularities

When we reduce lines in any closed-loop diagram as indicated in Fig. 6, it is at once obvious that the five-point function will in general have singularities in the complex parts of the physical sheet. It does not follow that they will be complex, say for the five-point function, if four of the independent variables are held at their physical values, but each individual case requires investigation on this point.[23]

(b) Closed-Loop Poles and Resonances

It has been shown[24] that the closed-loop diagram of the five-point function gives rise to a pole in each variable when the other four independent variables are held fixed. I wish to note here that an internal resonance may be approximated by a single line with complex mass. This will lead to a complex pole in the five-point closed-loop diagram. In particular cases this may lead to a resonance in a physical region. This may cause considerable complications in the interpretation of resonances in final-state interactions.

(c) Experiments on Final-State Interactions

I will mention two experiments in which the complication of closed-loop resonances may cause difficulties. One is the pion-production experiment[25] illustrated in Fig. 7. The simple pion resonance is shown in diagram (a). The closed-loop pole is shown in diagram (b) and the lowest closed-loop resonance is indicated in diagram (c). The problem of locating these poles is complicated, but should be completed for some cases in the near future.[26]

The second experiment is on the reaction[27]

$$p + d \rightarrow He^3 + \text{neutral system.}$$

The diagram that may be related to a simple pion-pion resonance is shown in Fig. 8(a). The corresponding closed-loop resonance is shown in Fig. 8(b). It seems probable that the closed-loop resonance will be below the two-pion threshold, but it will occur in both the s and p states of the outgoing pions. The simple pion-pion resonance may be above the threshold but is likely to be in the p wave only.[28]

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FIGURE LEGENDS

Fig. 1. Kinematics for equal masses.

Fig. 2. Displacement of the contour used for analytic completion.

Fig. 3. Contours used in the double application of Cauchy's theorem.

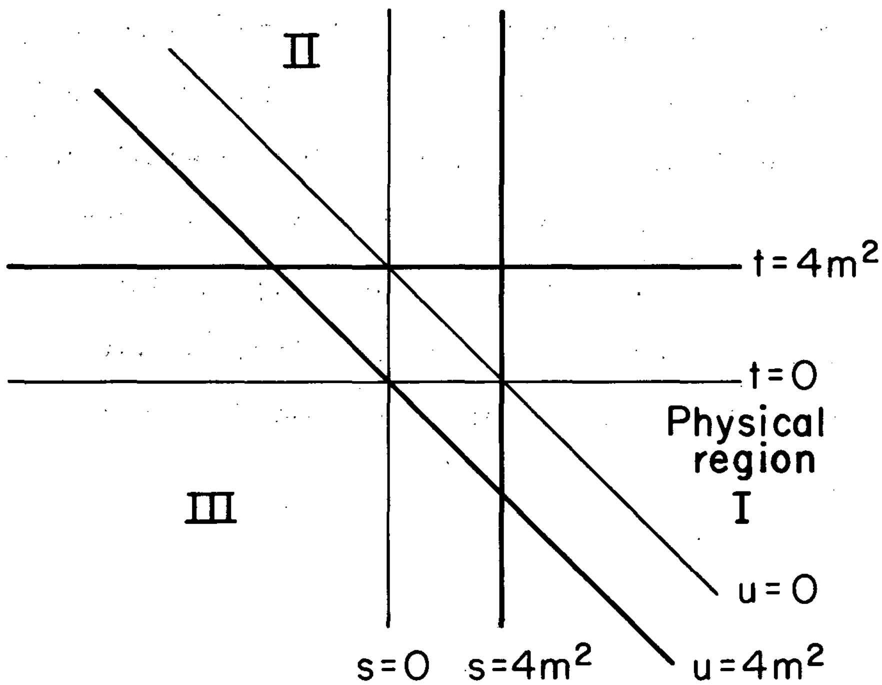
Fig. 4. Curve of singularities for anomalous thresholds of Type I.

Fig. 5. Curve of singularities for anomalous thresholds of Type II.

Fig. 6. Reduced diagrams giving complex singularities.

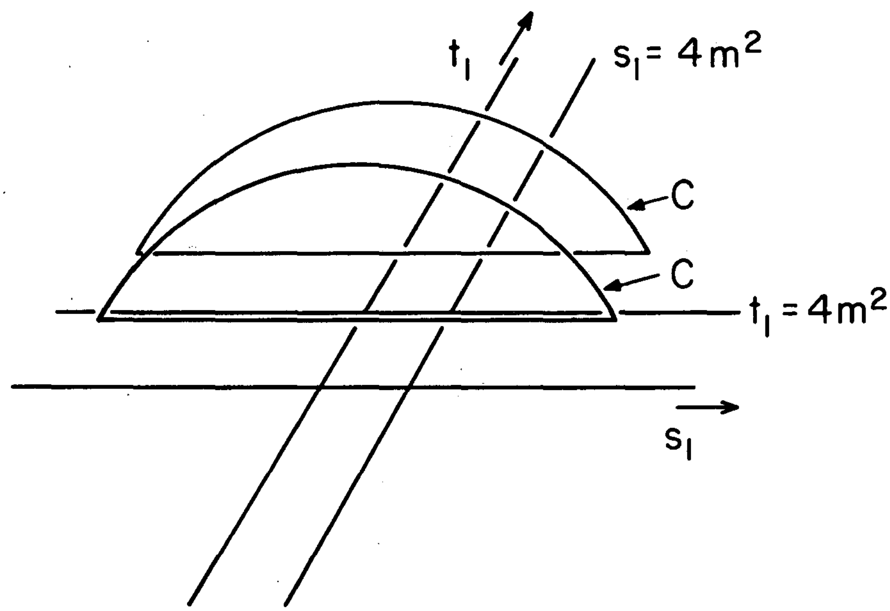
Fig. 7. Closed-loop poles and resonances in the pion-production experiment.

Fig. 8. A final-state resonance and a closed-loop resonance in the reaction $p + d \rightarrow He^3 + \pi^+ - \pi^-$.



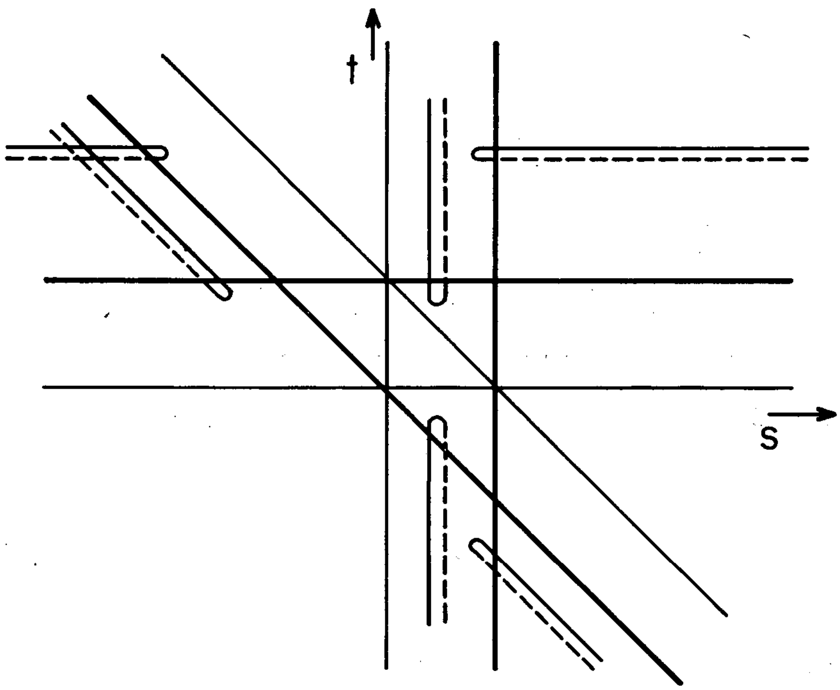
MU-21102

Fig. 1



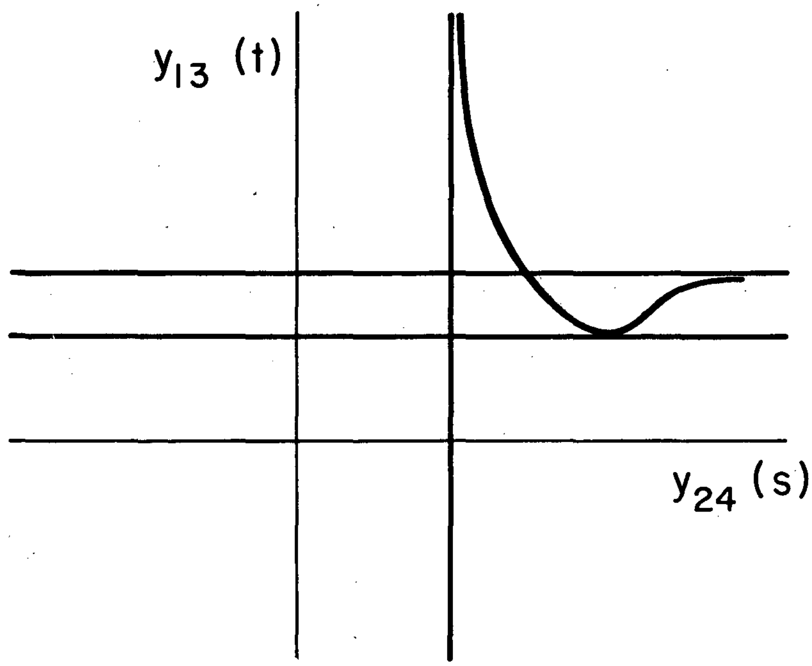
MU-21103

Fig. 2



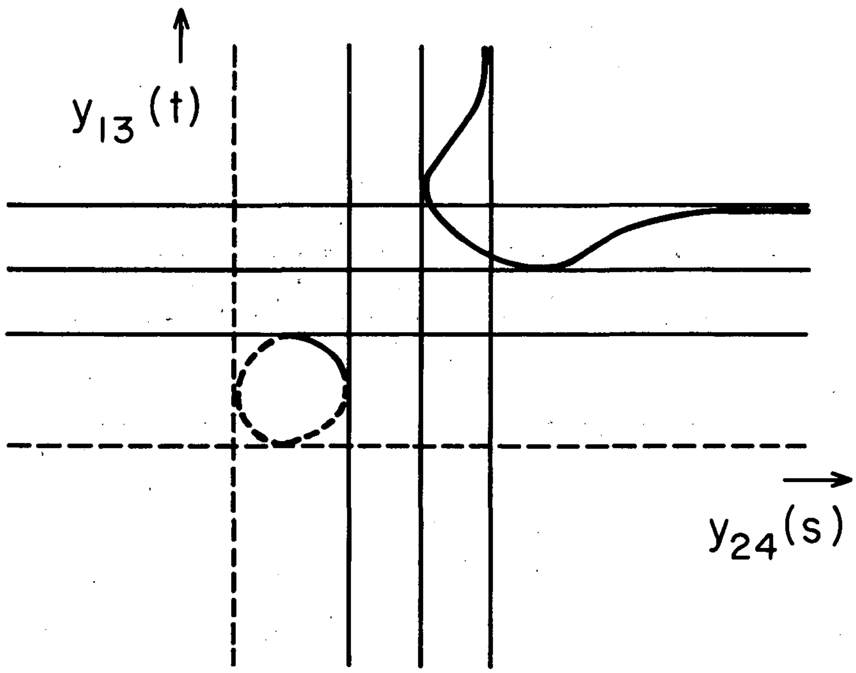
MU-21104

Fig. 3



MU – 21105

Fig. 4



MU-21106

Fig. 5

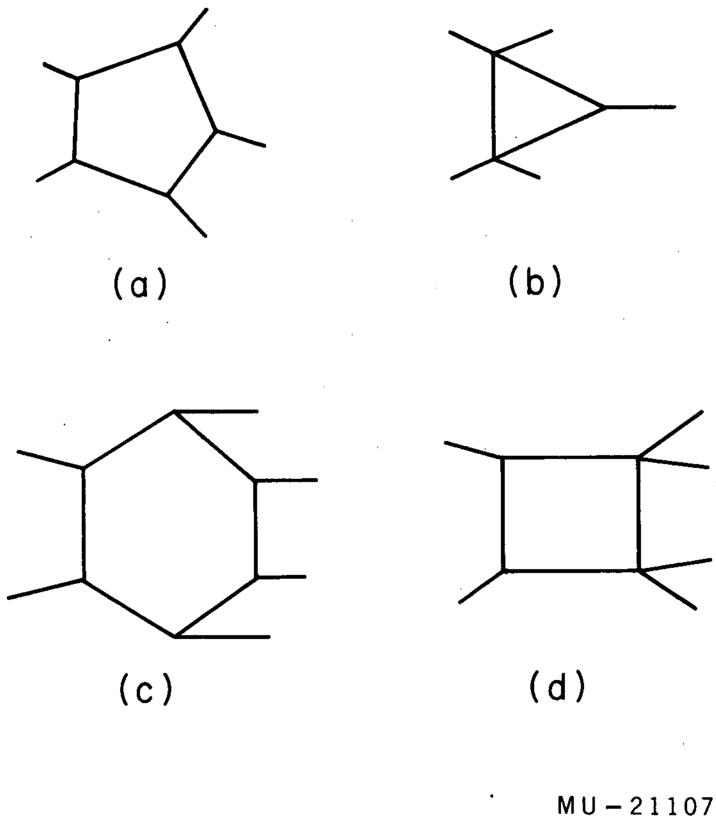
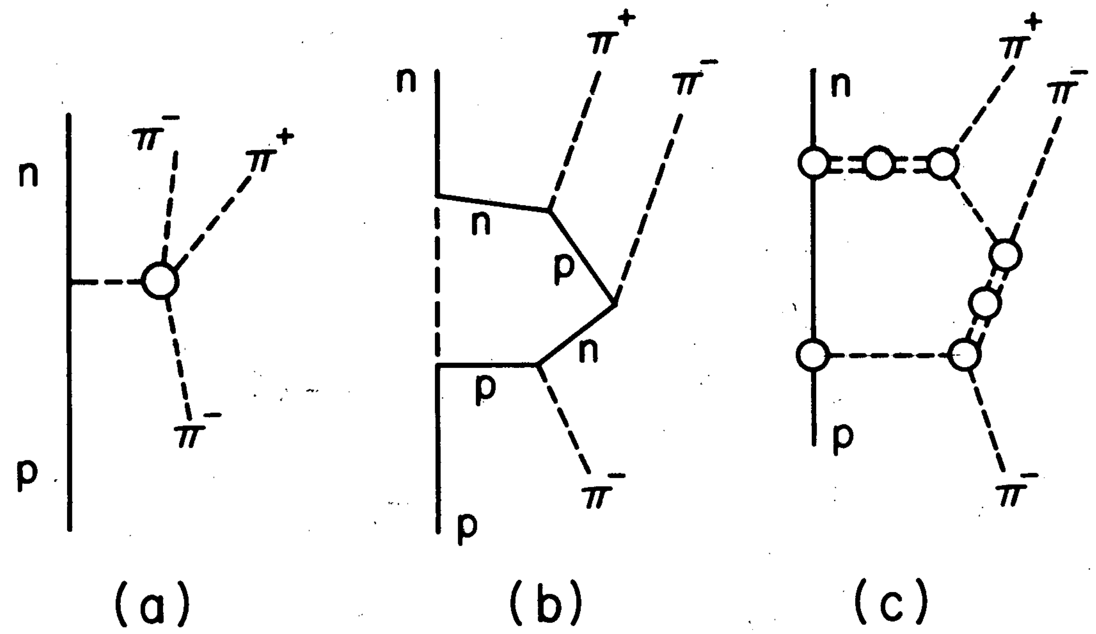


Fig. 6



MU-21108

Fig. 7

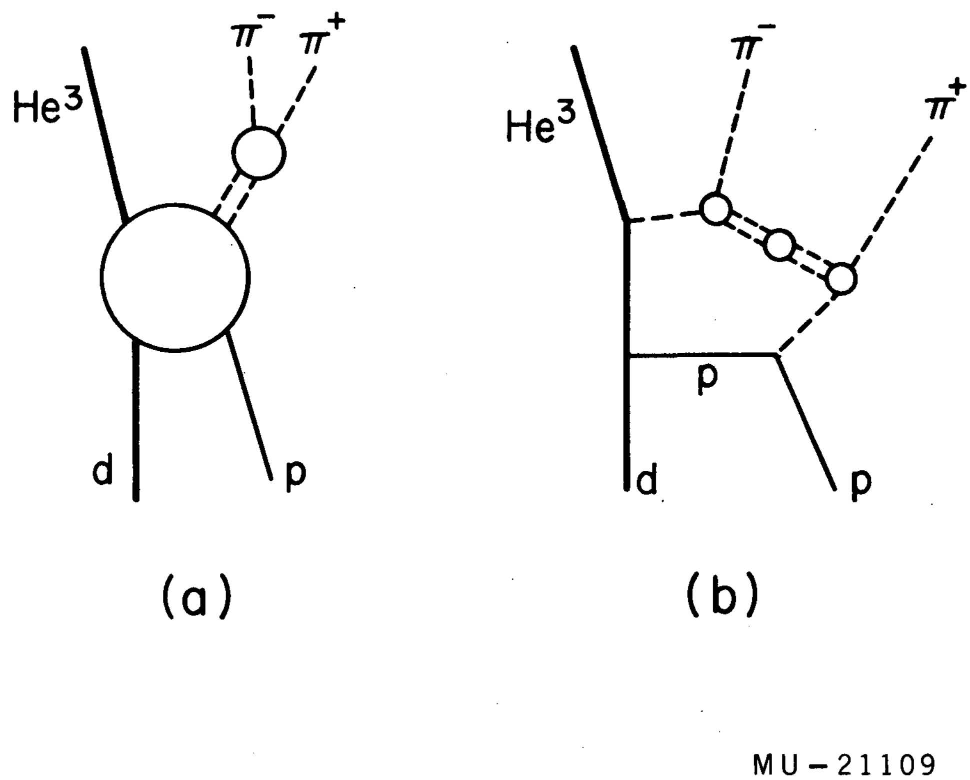


Fig. 8