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Authors
González Nogueras, María del Mar
Qing, Jie

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Fractional conformal Laplacians and fractional Yamabe problems

María del Mar González *
Univ. Politècnica de Catalunya

Jie Qing †
Univ. of California, Santa Cruz

Abstract

Based on the relations between scattering operators of asymptotically hyperbolic metrics and Dirichlet-to-Neumann operators of uniformly degenerate elliptic boundary value problems observed by Chang and González, we formulate fractional Yamabe problems that include the boundary Yamabe problem studied by Escobar. We observe an interesting Hopf type maximum principle together with interplays between analysis of weighted trace Sobolev inequalities and conformal structure of the underlying manifolds, which extend the phenomena displayed in the classic Yamabe problem and boundary Yamabe problem.

1 Introduction

In this paper, based on the relations between scattering operators of asymptotically hyperbolic metrics and Dirichlet-to-Neumann operators of uniformly degenerate elliptic boundary value problems observed in [11], we formulated and solved fractional order Yamabe problems that include the boundary Yamabe problem studied by Escobar in [16].

Suppose that X^{n+1} is a smooth manifold with smooth boundary M^n for $n \geq 3$. A function ρ is a defining function of the boundary M^n in X^{n+1} if

$$\rho > 0 \text{ in } X^{n+1}, \quad \rho = 0 \text{ on } M^n, \quad d\rho \neq 0 \text{ on } M^n.$$

We say that g^+ is conformally compact if, for some defining function ρ, the metric $\bar{g} = \rho^2 g^+$ extends to $\bar{X}^{n+1}$ so that $(\bar{X}^{n+1}, \bar{g})$ is a compact Riemannian manifold. This induces a conformal class of metrics $\hat{h} = \bar{g}|_{TM^n}$ on M^n when defining functions vary. The conformal manifold $(M^n, [\hat{h}])$ is called the conformal infinity of (X^{n+1}, g^+). A metric g^+ is said to be asymptotically hyperbolic if it is conformally compact and the sectional curvature approaches -1 at infinity.

In the recent work [27], Graham and Zworski introduced the meromorphic family of scattering operators $S(s)$, which is a family of pseudo-differential operators, for a given asymptotically hyperbolic manifold $(X^{n+1},\ g^+)$ and a choice of the representative $\hat{h}$ of the conformal infinity $(M^n,\ [\hat{h}])$. Often one instead considers the normalized scattering operators

$$P_\gamma[g^+, \hat{h}] = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S\left(\frac{n}{2} + \gamma\right).$$

The normalized scattering operators $P_\gamma[g^+, \hat{h}]$ are conformally covariant,

$$P_\gamma[g^+, w^{\frac{4}{n-2\gamma}}\hat{h}]\phi = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma[g^+, \hat{h}](w\phi),$$

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†Research of Qing is partially supported by NSF 0700535 and CNSF 10728103.

with principal symbol

$$\sigma(P_\gamma[g^+, \hat{h}]) = \sigma((-\Delta_{\hat{h}})^\gamma).$$

Hence they may be considered to be conformal fractional Laplacians for $\gamma \in (0,1)$ for a given asymptotically hyperbolic metric g^+. As proven in [27], [19], when g^+ is Poincaré-Einstein, P_1 is the conformal Laplacian, P_2 is the Paneitz operator, and in general P_k for $k \in \mathbb{N}$ are the conformal powers of the Laplacian discovered in [26].

When g^+ is a fixed asymptotically hyperbolic metric we may simply denote

$$P_\gamma^{\hat{h}} := P_\gamma[g^+, \hat{h}].$$

We will consider the associated "fractional order curvature"

$$Q_\gamma^{\hat{h}} = P_\gamma^{\hat{h}}(1),$$

and the normalized total curvature

$$I_\gamma[\hat{h}] = \frac{\int_{M^n} Q_\gamma^{\hat{h}} dv_{\hat{h}}}{\left(\int_{M^n} dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}}.$$

When a background metric $\hat{h}$ is fixed, we may write

$$I_\gamma[w, \hat{h}] = I_\gamma[w^{\frac{4}{n-2\gamma}}\hat{h}] = \frac{\int_{M^n} w P_\gamma^{\hat{h}} w dv_{\hat{h}}}{\left(\int_{M^n} w^{\frac{2n}{n-2\gamma}} dv_{\hat{h}}\right)^{\frac{n-2\gamma}{n}}}.$$

This functional $I_\gamma[\hat{h}]$ is clearly an analogue to the Yamabe functional. Hence one may ask if there is a metric which is the minimizer of I_γ among metrics in the class $[\hat{h}]$ and whose curvature Q_γ is a constant. We will refer to that problem as a fractional Yamabe problem when $\gamma \in (0,1)$. For the original Yamabe problem readers are refereed to [30], [40]. A similar question was studied in [39] for $\gamma > 1$ and g^+ being a Poincaré-Einstein metric. Because of the lack of a maximum principle these generalized Yamabe problems in general are difficult to solve. Yet this new window to the analytic aspects of conformal geometry remains fascinating. For example, it was proven in [28] that the location of the first scattering pole is dictated by the sign of the Yamabe constant and the Green's function of $P_\gamma^{\hat{h}}$ is positive for $\gamma \in (0,1)$ when the Yamabe constant is positive, at least in the case where g^+ is conformally compact Einstein.

It turns out that one may use the relations of scattering operators and the Dirichlet-to-Neumann operators to reformulate the above fractional Yamabe problems as degenerate elliptic boundary value problems. The correspondence between pseudo-differential equations and degenerate elliptic boundary value problems is inspired by the works in [10]. Interestingly, the corresponding degenerate elliptic boundary value problem is a natural extension of the boundary Yamabe problem raised and studied in [16].

Recall from [11] that, given an asymptotically hyperbolic manifold $(X^{n+1},\ g^+)$ and a representative $\hat{h}$ of the conformal infinity $(M^n,\ [\hat{h}])$, one can find a geodesic defining function ρ such that the compactified metric can be written as

$$\bar{g} := \rho^2 g^+ = d\rho^2 + h_\rho = d\rho^2 + \hat{h} + h^{(1)}\rho + h^{(2)}\rho^2 + o(\rho^2)$$

near infinity. One may consider the degenerate elliptic boundary value problem of $\bar{g}$ as follows:

$$\begin{cases} -\mathrm{div}\,(\rho^a \nabla U) + E(\rho)U = 0 & \text{in } (X^{n+1},\ \bar{g}), \\ U|_{\rho=0} = f & \text{on } M^n, \end{cases} \tag{1.1}$$

where

$$E(\rho) = \rho^{-1-s}\left(-\Delta_{g^+} - s(n-s)\right)\rho^{n-s},$$

$s = \frac{n}{2} + \gamma$, and $a = 1 - 2\gamma$.

Lemma 1.1 (Chang and Gonzalez [11]). *Let $(X^{n+1},\ g^+)$ be an asymptotically hyperbolic manifold. Suppose that U is the solution to the boundary value problem* (1.1). *Then*

1. *For $\gamma \in (0, \frac{1}{2})$ and $-\frac{n^2}{4} + \gamma^2$ not an L^2-eigenvalue for the Laplacian of g^+,*

$$P_\gamma[g^+, \hat{h}]f = -d^*_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho U, \tag{1.2}$$

where

$$d^*_\gamma = -\frac{2^{2\gamma-1}\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}. \tag{1.3}$$

2. *For $\gamma = \frac{1}{2}$,*

$$P_{\frac{1}{2}}[g^+, \hat{h}]f = -\lim_{\rho \to 0} \partial_\rho U + \tfrac{n-1}{2} H f,$$

where $H := \frac{1}{2n} Tr_{\hat{h}}(h^{(1)})$ is the mean curvature of M.

3. *For $\gamma \in \left(\frac{1}{2}, 1\right)$,* (1.2) *still holds if $H = 0$.*

In light of Lemma 1.1, consider, for $\gamma \in (0, 1)$,

$$I^*_\gamma[U, \bar{g}] = \frac{d^*_\gamma \int_{X^{n+1}} (\rho^a |\nabla U|^2 + E(\rho) U^2) dv_{\bar{g}}}{\int_{M^n} U^{\frac{2n}{n-2\gamma}} dv_{\hat{h}}}.$$

It is then a very natural variational problem for I^*_γ. For instance, right away one sees that a minimizer of I^*_γ is automatically nonnegative, which was a huge issue for the functional I_γ.

One key ingredient in our work here is the following Hopf type maximum principle. We drew inspiration from some version of Hopf's lemma for the Euclidean half space case (Proposition 4.11 in [9]).

Proposition 1.2. *Let $\gamma \in (0, 1)$. Suppose that U is a nonnegative solution to* (1.1) *in X^{n+1}. Let $p_0 \in M^n = \partial X^{n+1}$ and B_r be a geodesic ball of radius r centered at p_0 in M^n. Then, for sufficiently small r_0, if $U(q_0) = 0$ for $q_0 \in B_{r_0} \setminus \overline{B_{\frac{1}{2}r_0}}$ and $U > 0$ on $\partial B_{\frac{1}{2}r_0}$, then*

$$y^a \partial_y U|_{q_0} > 0. \tag{1.4}$$

It seems weaker than the original one, but it suffices for our purposes. A nice and immediate consequence of the above maximum principle is that the first eigenfunction of the fractional conformal Laplacian $P^{\hat{h}}_\gamma$ is always positive, which has been a rather challenging question in general for the pseudo-differential operators $P^{\hat{h}}_\gamma$ (cf. [28]). Hence one can produce a metric in the class $[\hat{h}]$ that has positive, negative, or zero Q_γ curvature when the first eigenvalue is positive, negative, or zero respectively.

Our approach to solve the γ-Yamabe problem is very similar to the one taken in [16], where one of the crucial steps is the understanding of a trace inequality. In our case, the relevant sharp weighted trace Sobolev inequality appeared in the works [31], [13], [37]:

Proposition 1.3. *Let $\gamma \in (0, 1)$ and $a = 1 - 2\gamma$. Suppose that $U \in W^{1,2}(\mathbb{R}^{n+1}_+, y^a)$ with trace $TU = w$. Then, for some constant $\bar{S}(n, \gamma)$,*

$$\|w\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \bar{S}(n, \gamma) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2 \, dxdy, \tag{1.5}$$

where $2^* = \frac{2n}{n-2\gamma}$. Moreover the equality holds if and only if

$$w(x) = c \left(\frac{\mu}{|x - x_0|^2 + \mu^2} \right)^{\frac{n-2\gamma}{2}}, \quad x \in \mathbb{R}^n,$$

for $c \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ fixed, and U is its Poisson extension of w as given in (2.13).

As in the case of original Yamabe problem, one can define the γ-Yamabe constant

$$\Lambda_\gamma(M^n, [\hat{h}]) = \inf_{h\in[\hat{h}]} I_\gamma[h].$$

It is then easily seen that

$$\Lambda_\gamma(S^n, [g_c]) = \frac{d_\gamma^*}{\bar{S}(n,\gamma)}$$

where $[g_c]$ is the canonical conformal class of metrics on the sphere S^n. Analogous to the cases of the original Yamabe problem we obtain

Theorem 1.4. *Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold. Suppose, in addition, that $H = 0$ when $\gamma \in (\frac{1}{2}, 1)$. Then, if*

$$-\infty < \Lambda_\gamma(M, [\hat{h}]) < \Lambda_\gamma(S^n, [g_c]), \tag{1.6}$$

then the γ-Yamabe problem is solvable for $\gamma \in (0, 1)$.

Remark. It is easily seen that $\Lambda_\gamma(M, [\hat{h}]) > -\infty$ in the light of (1.4) in Theorem 1.1 and Theorem 1.2 in [29] when $\gamma \in (0, \frac{1}{2}]$ or if some additional assumptions in Theorem 1.2 in [29] hold.

Based on computations similar to ones in [16], we have

Theorem 1.5. *Suppose that $(X^{n+1},\ g^+)$ is an asymptotically hyperbolic manifold and that*

$$\rho^{-2}\big(R[g^+] - Ric[g^+](\rho\partial_\rho) + n^2\big) \to 0 \quad \text{as } \rho \to 0. \tag{1.7}$$

If X^{n+1} has a non-umbilic point on ∂X^{n+1} and

$$-\frac{n+a-3}{1-a} 2^{2\gamma+1} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} + \frac{n-1+a}{a+1} < 0, \tag{1.8}$$

then

$$\Lambda_\gamma(M, [\hat{h}]) < \Lambda_\gamma(S^n, [g_c])$$

and hence the γ-Yamabe problem is solvable for $\gamma \in (0, 1)$.

We remark now that the $\frac{1}{2}$-Yamabe problem introduced in here reduces back to the boundary Yamabe problem consider in [16] in this way. Notice that, in this case, we have

$$I^*_{\frac{1}{2}}[U,\ \phi^{\frac{4}{n-1}}\bar{g}] = I^*_{\frac{1}{2}}[U\phi,\ \bar{g}] \tag{1.9}$$

for any positive function ϕ on $\bar{X}^{n+1}$ and therefore (1.7) is no longer needed. Also notice that the condition (1.8) becomes $n > 5$ when $\gamma = \frac{1}{2}$, which agrees with the conclusion in [16].

Suppose we start with a compact Riemannian manifold $(X^{n+1},\ \bar{g})$ and its boundary $(M^n,\ \hat{h})$. Then one can construct an asymptotically hyperbolic manifold $(X^{n+1},\ g^+)$ which is conformal to $(X^{n+1},\ \bar{g})$. For example, as observed in [11], one may require according to the works in [33], [2] that

$$R[g^+] = -n(n+1). \tag{1.10}$$

Then the induced degenerate equation becomes

$$-\operatorname{div}(\rho^a \nabla U) + \tfrac{n-1+a}{4n} R[\bar{g}]\rho^a U = 0 \quad \text{in } (X^{n+1}, \bar{g}) \tag{1.11}$$

whose associated variational functional becomes

$$F[U] = \int_X \rho^a |\nabla U|^2_{\bar{g}}\, dv_{\bar{g}} + \tfrac{n-1+a}{4n} \int_X R[\bar{g}]\rho^a |U|^2\, dv_{\bar{g}}. \tag{1.12}$$

In section 2 we recall the work from [11] to make possible the passage from pseudo-differential equations to second order elliptic boundary value problems as in [10]. In Section 3 we study regularity (L^∞ and Schauder estimates) for degenerate elliptic boundary value problems. And more importantly we establish the Hopf type maximum principle. In Section 4 we formulate the fractional Yamabe problem and obtain some properties for the fractional case that are analogous to the original Yamabe problem with the help of the Hopf-type maximum principle. In Section 5 we analyze sharp weighted Sobolev trace inequalities. We define, on any conformal manifold, the fractional Yamabe constant associated with an asymptotically hyperbolic metric and show that the one of the standard round spheres associated to the standard hyperbolic metric is the largest. In Section 6 we take a subcritical approximation and prove our Theorem 1.4. In the last section we adopt the calculation from [16] and prove our Theorem 1.5 by choosing a suitable test function.

We finally mention the two related works [4, 41] on nonlinearities with critical exponents for the fractional Laplacian.

2 Conformal fractional Laplacians

In this section we introduce the recent works in [11] to relate two equivalent definitions of conformal fractional Laplacians. Conformal fractional Laplacians are defined via scattering theory on asymptotically hyperbolic manifolds in [27], [19]. We also have seen fractional Laplacians defined as Dirichlet-to-Neumann operators for degenerate equations on compact manifolds with boundary in [10]. It turns out in some way these two fractional Laplacians are the same.

Let X^{n+1} be a smooth manifold of dimension $n+1$ with compact boundary $\partial X = M^n$. A function ρ is a *defining function* of ∂X in X if

$$\rho > 0 \text{ in } X, \quad \rho = 0 \text{ on } \partial X, \quad d\rho \neq 0 \text{ on } \partial X.$$

We say that g^+ is *conformally compact* if the metric $\bar{g} = \rho^2 g^+$ extends to $\bar{X}^{n+1}$ for a defining function ρ so that $(\bar{X}^{n+1}, \bar{g})$ is a compact Riemannian manifold. This induces a conformal class of metrics $\hat{h} = \bar{g}|_{TM^n}$ on M^n when the defining function varies, which is called the *conformal infinity* of (X^{n+1}, g^+). A metric g^+ is said to be *asymptotically hyperbolic* if it is conformally compact and the sectional curvature approaches to -1 at infinity.

Given an asymptotically hyperbolic manifold (X^{n+1}, g^+) and a representative $\hat{h}$ of the conformal infinity $(M^n, [\hat{h}])$, there is a uniquely geodesic defining function ρ such that, on a neighborhood $M \times (0, \delta)$ in X, g^+ has the normal form

$$g^+ = \rho^{-2}(d\rho^2 + h_\rho) \tag{2.1}$$

where h_ρ is a one parameter family of metrics on M such that

$$h_\rho = \hat{h} + h^{(1)}\rho + O(\rho^2). \tag{2.2}$$

From [34], [27] it follows that, given $f \in \mathcal{C}^\infty(M)$, $Re(s) > \frac{n}{2}$ and $s(n-s)$ is not a L^2-eigenvalue for $-\Delta_{g^+}$, the generalized eigenvalue problem

$$-\Delta_{g^+} u - s(n-s)u = 0, \quad \text{in } X \tag{2.3}$$

has a solution of the form

$$u = F\rho^{n-s} + G\rho^s, \quad F, G \in \mathcal{C}^\infty(\bar{X}), \quad F|_{\rho=0} = f. \tag{2.4}$$

The *scattering operator* on M is then defined as

$$S(s)f = G|_M.$$

It is shown in [27] that, by a meromorphic continuation, $S(s)$ is a meromorphic family of pseudo-differential operators in the whole complex plane. Instead, it is often useful to consider the normalized scattering operators $P_\gamma[g^+,\hat{h}]$ defined as:

$$P_\gamma[g^+,\hat{h}] := d_\gamma S\left(\frac{n}{2}+\gamma\right), \quad d_\gamma = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{2.5}$$

Note that $s=\frac{n}{2}+\gamma$. With this regularization the principal symbol of $P_\gamma[g^*,\hat{h}]$ is exactly the principal symbol of the fractional Laplacian $(-\Delta_{\hat{h}})^\gamma$. Hence we will call (assuming implicitly the dependence on the extension metric g^+)

$$P_\gamma^{\hat{h}} := P_\gamma[g^+,\hat{h}]$$

a conformal fractional Laplacian for each $\gamma\in(0,1)$ which is not a pole of the scattering operator, i.e. $\frac{n^2}{4}-\gamma^2$ is not a L^2-eigenvalue for $-\Delta_{g^+}$. It is a conformally covariant operator, in the sense that it behaves like

$$P_\gamma^{\hat{h}_w}\varphi = w^{-\frac{n+2\gamma}{n-2\gamma}}P_\gamma^{\hat{h}}(w\varphi) \tag{2.6}$$

for a conformal change of metric $\hat{h}_w = w^{\frac{4}{n-2\gamma}}\hat{h}$. We will call

$$Q_\gamma^{\hat{h}} = P_\gamma^{\hat{h}}(1)$$

the fractional scalar curvature associated to the conformal fractional Laplacian $P_\gamma^{\hat{h}}$. From the above (2.6) we have

$$P_\gamma^{\hat{h}}(w) = Q_\gamma^{\hat{h}_w} w^{\frac{n+2\gamma}{n-2\gamma}}. \tag{2.7}$$

The familiar case is $\gamma=1$, where

$$P_1^{\hat{h}} = -\Delta_{\hat{h}} + \frac{n-2}{4(n-1)}R[\hat{h}]$$

becomes the conformal Laplacian and the associated curvature is the scalar curvature $Q_1^{\hat{h}} = \frac{n-2}{4(n-1)}R[\hat{h}]$ of the metric $\hat{h}$ which undergoes the change

$$P_1^{\hat{h}}w = \frac{n-2}{4(n-1)}R[\hat{h}_w]w^{\frac{n+2}{n-2}}$$

when taking conformal change of metrics, provided that $(X^{n+1},\ g^+)$ is a Poincaré-Einstein as established in [27], [19]. The conformal fractional Laplacians and fractional scalar curvatures should also be compared to the higher order generalization of the conformal Laplacian and scalar curvature: the Paneitz operator $P_2^{\hat{h}}$ and its associated Q-curvature (see [38], [6], [39]).

It was observed by Chang and González in [11] that the generalized eigenvalue problem (2.3) on a non-compact manifold (X^{n+1},g^+) is equivalent to a linear degenerate elliptic problem on the compact manifold $(\bar{X}^{n+1},\bar{g})$, for $\bar{g}=\rho^2 g^+$. Hence Chang and González reconciled the definition of the fractional Laplacians given in the above as normalized scattering operators and the one given in the spirit of the Dirichlet-to-Neumann operators by Caffarelli and Silvestre in [10]. This observation in [11] plays a fundamental role in this paper and provides an alternative way to study the fractional partial differential equation (2.7). First, we know by the conformal covariance that

$$P_1^{g^+}u = \rho^{\frac{n+3}{2}}P_1^{\bar{g}}(\rho^{-\frac{n-1}{2}}u).$$

Let $a=1-2\gamma\in(-1,1)$, $s=\frac{n}{2}+\gamma$, and $U=\rho^{s-n}u$. Then we may write the equation (2.3) as

$$-\mathrm{div}(\rho^a\nabla_{\bar{g}}U)+E(\rho)U=0, \quad \text{in } (X^{n+1},\ \bar{g}),$$

where

$$E(\rho) := \rho^{\frac{a}{2}} P_1^{\bar g} \rho^{\frac{a}{2}} - \left(s(n-s) + \tfrac{n-1}{4n} R[g^+]\right) \rho^{a-2}, \tag{2.8}$$

or writing everything back in the metric g^+,

$$E(\rho) = \rho^{-1-s} \left(-\Delta_{g^+} - s(n-s)\right) \rho^{n-s}. \tag{2.9}$$

Notice that, in a neighborhood $M \times (0,\delta)$ where the metric g^+ is in the normal form,

$$E(\rho) = \tfrac{n-1+a}{4n} \left[R[\bar g] - (n(n+1) + R[g^+])\rho^{-2}\right] \rho^a \quad \text{in } M \times (0,\delta). \tag{2.10}$$

Proposition 2.1 (Chang and González [11]). *Let $(X^{n+1},\ g^+)$ be an asymptotically hyperbolic manifold. Then, given $f \in \mathcal{C}^\infty(M)$, the generalized eigenvalue problem* (2.3)-(2.4) *is equivalent to*

$$\begin{cases} -div\,(\rho^a \nabla U) + E(\rho) U = 0 & \text{in } (X, \bar g), \\ U|_{\rho=0} = f & \text{on } M, \end{cases} \tag{2.11}$$

where $U = \rho^{n-s} u$ and U is the unique minimizer of the energy

$$F[V] = \int_X \rho^a |\nabla V|_{\bar g}^2 \, dv_{\bar g} + \int_X E(\rho) |V|^2 \, dv_{\bar g}$$

among all the functions $V \in W^{1,2}(X, \rho^a)$ with fixed trace $V|_{\rho=0} = f$. Moreover,

1. *For $\gamma \in (0, \frac{1}{2})$,*
$$P_\gamma^{\hat h} f = -d_\gamma^* \lim_{\rho \to 0} \rho^a \partial_\rho U, \tag{2.12}$$
where the constant d_γ^* is given in (1.3).
2. *For $\gamma = \frac{1}{2}$, we have an extra term*
$$P_{\frac{1}{2}}^{\hat h} f = -\lim_{\rho \to 0} \partial_\rho U + \tfrac{n-1}{2} H f,$$
where $H := \frac{1}{2n} Tr_{\hat h}(h^{(1)})$ is the mean curvature of M.
3. *For $\gamma \in \left(\frac{1}{2}, 1\right)$,* (2.12) *still holds if and only if $H = 0$.*

Remark. It should be noted here that there are many asymptotically hyperbolic manifolds $(X^{n+1},\ g^+)$ whose conformal infinity is prescribed as $(M^n,\ [\hat h])$. If one insists $(X^{n+1},\ g^+)$ to be Poincaré-Einstein, then the normalized scattering operators $P_\gamma^{\hat h}$ are a bit more intrinsic, at least at positive integers as observed in [27], [19]. It should also be noted that one can simply start with a compact Riemannian manifold $(\bar X^{n+1},\ \bar g)$ with boundary $(M^n,\ \hat h)$ and easily build an asymptotically hyperbolic manifold whose conformal infinity is given by $(M^n,\ [\hat h])$. Please see the details of this observation in [11].

The simplest example of a conformally compact Einstein manifold is the hyperbolic space $(\mathbb{H}^{n+1}, g_{\mathbb{H}})$. It can be characterized as the upper half-space (with coordinates $x \in \mathbb{R}^n$, $y \in \mathbb{R}_+$), endowed with the metric:

$$g^+ = \frac{dy^2 + |dx|^2}{y^2}.$$

Then (2.11) with Dirichlet condition w reduces to

$$\begin{cases} -\text{div}\,(y^a \nabla U) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ U|_{y=0} = w & \text{on } \mathbb{R}^n, \end{cases}$$

and the fractional Laplacian at the boundary $\mathbb{R}^n$ is just

$$P_\gamma^{|dx|^2} w = (-\Delta_{|dx|^2})^\gamma w = -d_\gamma^* \lim_{y \to 0} (y^a \partial_y U).$$

This is precisely the Caffarelli-Silvestre extension [10]. Note that this extension U can be written in terms of the Poisson kernel K_γ as follows:

$$U(x,y) = K_\gamma *_x w = C_{n,\gamma} \int_{\mathbb{R}^n} \frac{y^{1-a}}{(|x-\xi|^2 + |y|^2)^{\frac{n+1-a}{2}}} \, w(\xi)\, d\xi, \tag{2.13}$$

for some constant $C_{n,\gamma}$. Moreover, given $w \in H^\gamma(\mathbb{R}^n)$, U is the minimizer of the functional:

$$F[V] = \int_{\mathbb{R}^{n+1}_+} y^a |\nabla V|^2 \, dxdy$$

among all the possible extensions in the set

$$\left\{ V : \mathbb{R}^{n+1}_+ \to \mathbb{R} \, : \, \int_{\mathbb{R}^{n+1}_+} y^a |\nabla V|^2 \, dxdy < \infty, \; V(\cdot, 0) = w \right\}.$$

Based on (2.9) it is observed in [11] that one may use

$$\rho^* = v^{\frac{1}{n-s}}$$

as a defining function, where v solves

$$-\Delta_{g^+} v - s(n-s)v = 0$$

and $\rho^{s-n} v = 1$ on M, to eliminate $E(\rho^*)$ from equation (2.11). It suffices to show that v is strictly positive in the interior. But this is true because, away from the boundary, it is the solution of an uniformly elliptic equation in divergence form, thus it cannot have a non-positive minimum. Hence we arrive at an improvement of Proposition 2.1 as follows:

Proposition 2.2. *The function ρ^* is a defining function of M in X such that $E(\rho^*) \equiv 0$. Hence $U = (\rho^*)^{s-n} u$ solves*

$$\begin{cases} -div((\rho^*)^a \nabla U) = 0 & in \; (X, \bar{g}^*), \\ U = w & on \; M, \end{cases} \tag{2.14}$$

with respect to the metric $\bar{g}^* = (\rho^*)^2 g^+$ and U is the unique minimizer of the energy

$$F[V] = \int_X (\rho^*)^a |\nabla V|^2_{\bar{g}^*} \, dv_{\bar{g}^*} \tag{2.15}$$

among all the extensions $V \in W^{1,2}(X, (\rho^*)^a)$ satisfying $V|_M = w$. Moreover,

$$\rho^*(\rho) = \rho \left[1 + \frac{Q^{\hat{h}}_\gamma}{(n-s) d_\gamma} \rho^{2\gamma} + O(\rho^2) \right]$$

near the infinity and

$$P^{\hat{h}}_\gamma w = -d^*_\gamma \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} U + w Q^{\hat{h}}_\gamma, \tag{2.16}$$

provided that $H = 0$ when $\gamma \in (\frac{1}{2}, 1)$.

We will sometimes use the defining function ρ^*, denoted by y unless explicitly stated otherwise, because it allows us to work with a pure divergence equation with no lower order terms.

We end this section by discussing the assumption that $H = 0$ for an asymptotically hyperbolic metric g^+. It turns out that this indeed is an intrinsic condition.

Lemma 2.3. *Suppose that* $(X^{n+1},\ g^+)$ *is an asymptotically hyperbolic manifold and that* ρ *and* $\tilde\rho$ *are the geodesic defining functions of* M *in* X *associated with representatives* $\hat h$ *and* $\tilde h$ *of the conformal infinity* $(M^n,\ [\hat h])$ *respectively. Hence*

$$g^+ = \rho^{-2}(d\rho^2 + h_\rho) = \tilde\rho^{-2}(d\tilde\rho^2 + \tilde h_{\tilde\rho})$$

where

$$h_\rho = \hat h + \rho h^{(1)} + O(\rho^2)$$

and

$$\tilde h_{\tilde\rho} = \tilde h + \tilde\rho \tilde h^{(1)} + O(\tilde\rho^2)$$

near the infinity. Then

$$\tilde h^{(1)} = h^{(1)} \quad \text{on } M.$$

In particular

$$H = \frac{\tilde\rho}{\rho}\bigg|_{\rho=0} \tilde H \quad \text{on } M.$$

Proof. This simply follows from the equations that define the geodesic defining functions. Let

$$\tilde\rho = e^w \rho$$

near the infinity. Then

$$1 = |d(e^w\rho)|^2_{e^{2w}\rho^2 g^+} = |d\rho|^2_{\rho^2 g^+} + 2\rho\langle dw, d\rho\rangle_{\rho^2 g^+} + \rho^2 |dw|^2_{\rho^2 g^+},$$

which implies

$$2\frac{\partial w}{\partial \rho} + \rho\left[\left(\frac{\partial w}{\partial\rho}\right)^2 + |\nabla w|^2_{h_\rho}\right] = 0.$$

Hence it is rather obvious that $\frac{\partial w}{\partial \rho} = 0$ at $\rho = 0$. Therefore the proof is complete in the light of the fact that

$$\tilde g = \tilde\rho^2 g^+ = e^{2w}\rho^2 g^+ = e^{2w}\bar g.$$

□

3 Uniformly degenerate elliptic equations

Considering the fractional powers of the Laplacian as Dirichlet-to-Neumann operators in Proposition 2.2 allows to relate the properties of non-local operators to those of uniformly degenerate elliptic equations in one more dimension. The same strategy has been used, for instance, in the recent work of Cabré-Sire [9].

Fix $\gamma \in (0,1)$. Let $y = \rho^*$ be the special defining function given in Proposition 2.2 and set $\bar g^* = y^2 g^+$. We are concerned with the uniformly degenerate elliptic equation

$$\begin{cases} -\operatorname{div}(y^a \nabla U) = 0 & \text{in } (X, \bar g^*), \\ U = w & \text{on } M. \end{cases} \tag{3.1}$$

For our purpose we will concentrate on the local behaviors of the solutions to (3.1) near the boundary. First, we write our equation in local coordinates near a fixed boundary point $(p_0, 0)$. More precisely, for some $R > 0$, we set

$$\begin{aligned} B_R^+ &= \{(x,y) \in \mathbb{R}^{n+1} : y > 0, |(x,y)| < R\}, \\ \Gamma_R^0 &= \{(x,0) \in \partial\mathbb{R}^{n+1}_+ : |x| < R\}, \\ \Gamma_R^+ &= \{(x,y) \in \mathbb{R}^{n+1} : y \geq 0, |(x,y)| = R\}. \end{aligned}$$

In local coordinates on Γ_R^0 the metric $\hat{h}$ is of the form $|dx|^2(1+O(|x|^2))$, where $x(p_0)=0$. Consider the matrix

$$A(x,y)=\sqrt{|\det \bar{g}^*|}y^a(\bar{g}^*)^{-1}.$$

Then the equation (3.1) is equivalent to

$$\sum_{i,j=1}^{n+1} \partial_i\left(A_{ij}\partial_j U\right)=0. \tag{3.2}$$

Moreover we know that

$$\frac{1}{c}y^a I \le A \le c y^a I. \tag{3.3}$$

This shows that (3.2) is a uniformly degenerate elliptic equation. For instance, the weight $\psi(y)=y^a$ is an $\mathcal{A}_2$ weight in the sense of [36]. Equation (3.2) has been well understood in a series of papers by Fabes, Jerison, Kenig, Serapioni ([18], [17]). Let us state a regularity result that is relevant to us. We will concentrate on problems of the form

$$\begin{cases} \operatorname{Div}(A(DU))=0 & \text{in } B_R^+,\\ \quad -y^a\partial_y U=F, & \text{on } \Gamma_R^0,\end{cases} \tag{3.4}$$

where, for the rest of the section, A satisfies the ellipticity condition (3.3) for $a\in(-1,1)$, the derivatives are Euclidean, that is, $D:=(\partial_{x_1},\dots,\partial_{x_n},y)$, and

$$\operatorname{Div}(A(DU)):=\sum_{i,j=1}^{n+1}\partial_i\left(A_{ij}\partial_j U\right).$$

Definition 3.1. *Given $R>0$ and a function $F\in L^1(\Gamma_R^0)$, we call U a weak solution of* (3.4) *if U satisfies*

$$(DU)^t A(DU)\in L^1(B_R^+)$$

and

$$\int_{B_R^+}(D\phi)^t A(DU)\,dxdy-\int_{\Gamma_R^0}F\phi\,dx=0$$

for all $\phi\in\mathcal{C}^1(\overline{B_R^+})$ such that $\phi\equiv 0$ on Γ_R^+ and $(D\xi)^t A(D\phi)\in L^1(B_R^+)$.

Hölder regularity for weak solutions was shown in [18], Lemma 2.3.12, for any A satisfying (3.3). Using this main result, regularity of weak solutions up to the boundary was carefully shown in [9], Lemma 4.3, at least when $A=y^aI$. However, their proof only depends on the divergence structure of the equation and the behavior of the weight. Hence we have

Proposition 3.2. *Let $\gamma\in(0,1)$, $\gamma=\frac{1-a}{2}$ and $\beta\in(0,\min\{1,1-a\})$. Let $R>0$ and $U\in L^\infty(B_{2R^+})\cap W^{1,2}(B_{2R}^+,y^a)$ be a weak solution of*

$$\begin{cases} Div(A(DU))=0 & \text{in } B_{2R}^+,\\ \quad -y^a\partial_y U=F(U) & \text{on } \Gamma_{2R}^0,\end{cases} \tag{3.5}$$

for A satisfying (3.3)*. If $F\in\mathcal{C}^{1,\beta}$, then $U\in\mathcal{C}^{0,\tilde\beta}(\overline{B_R^+})$ and $\partial_{x_i}U\in\mathcal{C}^{0,\tilde\beta}(\overline{B_R^+})$, $i=1,\dots,n$, for some $\tilde\beta\in(0,1)$.*

Particularly, when $F(x,t)=\alpha(x)t+\beta(x)t^{\frac{n+2\gamma}{n-2\gamma}}$, to get smoothness it is necessary to know the local boundedness of weak solutions U on $\overline{B_R^+}$. To get this local boundedness for weak solutions we employ the usual Moser's iteration scheme adapted to boundary valued problems (see Theorem 3.4 below). However, a new idea is required: we will perform two coupled iterations, one in the interior and one at the boundary, that need to be handled simultaneously. Note that in the linear case when $F\equiv 0$, local boundedness was shown in [18, Corollary 2.3.4], using the weighted Sobolev embeddings in the interior described in Proposition 3.3. However, when a non-linearity $F(U)$ is present at the boundary term, instead we need to use weighted trace Sobolev embeddings.

First, we recall a weighted Sobolev embedding theorem in the interior (c.f. [18, Theorem 1.3], see also [12]):

Proposition 3.3. *Let Ω be an open bounded set in $\mathbb{R}^{n+1}$. Take $1 < p < \infty$. There exist positive constants C_Ω and δ such that for all $u \in \mathcal{C}_0^\infty(\Omega)$ and all k satisfying $1 \le k \le \frac{n+1}{n} + \delta$,*

$$\|u\|_{L^{kp}(\Omega, y^a)} \le C_\Omega \, \|\nabla u\|_{L^p(\Omega, y^a)} \, .$$

C_Ω maybe taken to depend only on n, p, a and the diameter of Ω.

Now we can state the theorem. Note that we actually prove it in the flat case but it is straightforward to generalize it to the manifold setting:

Theorem 3.4. *Let U be a weak solution of the problem*

$$\begin{cases} div(y^a \nabla U) = 0 \quad in \ B_{2R}^+, \\ \quad -y^a \partial_y U = F(U) \quad on \ \Gamma_{2R}^0, \end{cases} \tag{3.6}$$

where $F(z)$ satisfies

$$F(z) = O\left(|z|^{\beta-1}\right), \quad when \ |z| \to \infty, \quad for\ some \quad 2 < \beta < 2^*.$$

Assume, in addition, that $\int_{\Gamma^0_{2r_0}} |U|^{2^*} \, dx =: V < \infty$. Then for each $\bar{p} > 1$, there exists a constant $C_{\bar{p}} = C(\bar{p}, V) > 0$ such that

$$\sup_{B_R^+} |U| + \sup_{\Gamma_R^0} |U| \le C_{\bar{p}} \left[\left(\frac{1}{R^{n+1+a}} \right)^{1/\bar{p}} \|U\|_{L^{\bar{p}}(B_{2R}, y^a)} + \left(\frac{1}{R^n} \right)^{1/\bar{p}} \|U\|_{L^{\bar{p}}(\Gamma^0_{2R})} \right].$$

Proof. Let $p \in \partial X$. Note that we can work with normal coordinates $x_1, \dots, x_n \in \mathbb{R}^n$, $y > 0$ near p. Without loss of generality, assume that $R = 1$. Then the general case is obtained by rescaling. Let $\eta = \eta(r)$, $r = (|x|^2 + y^2)^{1/2}$, be a smooth cutoff function such that $\eta = 1$ if $r < 1$, $\eta = 0$ if $r \ge 2$, $0 \le \eta \le 1$ if $r \in (1,2)$. Next, by working with $U^+ := \max\{U, 0\}$, $U^- := \max\{-U, 0\}$ separately, we can assume that U is positive.

A good reference for Moser iteration arguments in divergence structure equations is [22, chapter 8]. We generalize this method, considering a double iteration: one at the boundary, using Sobolev trace inequalities to handle the non-linear term $F(U)$, the other in the interior domain.

The first step is to use that U is a weak solution of (3.6) by finding a good test function. Formally we can write the following: multiply equation (3.6) by $\eta^2 U^\alpha$ and integrate by parts:

$$0 = 2 \int_{B_2^+} y^a \eta U^\alpha \nabla \eta \nabla U \, dxdy + \alpha \int_{B_2^+} y^a \eta^2 U^{\alpha-1} |\nabla U|^2 \, dxdy + \int_{\Gamma_2^0} \eta^2 U^\alpha F(U) \, dx. \tag{3.7}$$

This implies, using Hölder estimates to handle the crossed term,

$$\int_{B_2^+} y^a \eta^2 U^{\alpha-1} |\nabla U|^2 \, dxdy \le \frac{2}{\alpha} \int_{\Gamma_2^0} \eta^2 U^\alpha F(U) \, dx + \frac{4}{\alpha^2} \int_{B_2^+} y^a |\nabla \eta|^2 U^{\alpha+1} \, dxdy. \tag{3.8}$$

On the other hand, again using Hölder inequality, we have

$$\int_{B_2^+} y^a |\nabla(\eta U^\delta)|^2 \, dxdy \le 2\delta^2 \int_{B_2^+} y^a \eta^2 U^{2(\delta-1)} |\nabla U|^2 \, dxdy + 2 \int_{B_2^+} y^a U^{2\delta} \left|\nabla \eta\right|^2 \, dxdy.$$

If we insert formula (3.8) into the inequality above, for the choice $\alpha = 2\delta - 1$, we obtain

$$\begin{aligned} J &:= \int_{B_2^+} y^a |\nabla(\eta U^\delta)|^2 \, dxdy \\ &\le 2 \left(1 + \left(\tfrac{\alpha+1}{\alpha}\right)^2\right) \int_{B_2^+} y^a |\nabla \eta|^2 U^{2\delta} \, dxdy + \frac{(\alpha+1)^2}{\alpha} \int_{\Gamma_2^0} \eta^2 U^\alpha F(U) \, dx \\ &=: I_1 + I_2. \end{aligned} \tag{3.9}$$

For the left hand side above, recall the trace Sobolev embedding (Corollary 5.3):

$$J=\int_{B_2^+} y^a|\nabla(\eta U^\delta)|^2\,dxdy \gtrsim \left(\int_{\Gamma_2^0}(\eta U^\delta)^{2^*}\,dx\right)^{\frac{2}{2^*}}, \tag{3.10}$$

and the standard weighted Sobolev embedding from Proposition 3.3.

$$J=\int_{B_2^+} y^a|\nabla(\eta U^\delta)|^2\,dxdy \gtrsim \left(\int_{B_2^+} y^a(\eta U^\delta)^{k}\right)^{\frac{2}{k}} \tag{3.11}$$

for some $1<k<2\frac{n+1}{n}$.

Next, we estimate from above the terms I_1, I_2 in (3.9). I_1 can be easily handled since $|\nabla\eta|\le C$:

$$I_1=\int_{B_2^+} y^a|\nabla\eta|^2U^{2\delta}\,dxdy \lesssim \int_{B_2^+} y^aU^{2\delta}\,dxdy. \tag{3.12}$$

Now we consider the second term. To estimate I_2, if we write $U^{2\delta-2+\beta}=U^{\beta-2}U^{2\delta}$, then using Hölder inequality with $p=\frac{2^*}{\beta-2}$, $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\int_{\Gamma_2^0}\eta^2U^{2\delta-1}F(U)\,dx \le \left[\int_{\Gamma_2^0}U^{2^*}\,dx\right]^{\frac{1}{p}}\left[\int_{\Gamma_2^0}\eta^{2q}U^{2\delta q}\,dx\right]^{\frac{1}{q}} \le V^{\frac{1}{p}}\left[\int_{\Gamma_2^0}\eta^{2q}U^{2\delta q}\,dx\right]^{\frac{1}{q}}. \tag{3.13}$$

This last integral can be handled as follows. Call $\chi=\frac{2^*}{2}$, for simplicity. Because our hypothesis on β, we know that $q\in(1,\chi)$. Then, there exists $\lambda\in(0,1)$ such that $q=\lambda+(1-\lambda)\chi$, and an interpolation inequality gives:

$$\left[\int f^q\right]^{\frac{1}{q}} \le \left[\int f\right]^{\frac{\lambda}{q}}\left[\int f^\chi\right]^{\frac{1-\lambda}{q}} = \left[\int f^\chi\right]^{\frac{1}{\chi}}\left(\left[\int f\right]\left[\int f^\chi\right]^{-\frac{1}{\chi}}\right)^{\frac{\lambda}{q}}. \tag{3.14}$$

Since $\frac{\lambda}{q}<1$, Young's inequality reads

$$z^{\frac{\lambda}{q}}\le C_\epsilon z+\epsilon,$$

for ϵ small. If we substitute $z=\left[\int f\right]\left[\int f^\chi\right]^{-\frac{1}{\chi}}$ above, together with (3.14), we arrive at

$$\left[\int f^q\right]^{\frac{1}{q}} \le \epsilon\left[\int f^\chi\right]^{\frac{1}{\chi}}+C_\epsilon\int f.$$

Then from (3.13) it follows that

$$I_2\le V^{\frac{1}{p}}\left\{\epsilon\left(\int_{\Gamma_2^0}(\eta U^\delta)^{2^*}\,dx\right)^{\frac{2}{2^*}}+C_\epsilon\int_{\Gamma_2^0}\eta^2U^{2\delta}\,dx\right\}, \tag{3.15}$$

where ϵ will be chosen later and will depend on the value of α,δ.

We go back now to the main iteration formula (3.9). It is clear from (3.10), that the first integral of the right hand side of the formula for I_2 (3.15) can be absorbed into the left hand side of (3.9), and using (3.11) and (3.10) we get that

$$\left(\int_{\Gamma_1^0}U^{\delta 2^*}\,dx\right)^{\frac{2}{2^*}}+\left(\int_{B_1^+}U^{2k\delta}\,dxdy\right)^{\frac{1}{k}} \le C(\delta)\left[\int_{\Gamma_2^0}U^{2\delta}\,dx+\int_{B_2^+}U^{2\delta}\,dxdy\right],$$

for some suitable choice of ϵ. Or switching notation from 2δ to δ,

$$\left(\int_{\Gamma_1^0}U^{\delta\chi}\,dx\right)^{\frac{1}{\chi}}+\left(\int_{B_1^0}U^{k\delta}\,dxdy\right)^{\frac{1}{k}} \le C(\delta)\left[\int_{\Gamma_2^0}U^{\delta}\,dx+\int_{B_2^0}U^{\delta}\,dxdy\right]. \tag{3.16}$$

Next, because we will always have $\delta > 1$, we can use that

$$C_1(a^{\frac{1}{\delta}} + b^{\frac{1}{\delta}}) \le (a+b)^{\frac{1}{\delta}} \le C_2(a^{\frac{1}{\delta}} + b^{\frac{1}{\delta}}),$$

so from (3.16) we get that

$$\|U\|_{L^{\chi\delta}(\Gamma_1^0)} + \|U\|_{L^{k\delta}(B_1^+, y^a)} \le \|U\|_{L^{\delta}(\Gamma_2^0)} + \|U\|_{L^{\delta}(B_2^+, y^a)}.$$

For simplicity, we set

$$\theta := \min\{\chi, k\} > 1,$$

and

$$\Phi(\delta, R) := \left(\frac{1}{R^n}\right)^{\frac{1}{\delta}} \|U\|_{L^{\delta}(\Gamma_1^0)} + \left(\frac{1}{R^{n+1+a}}\right)^{\frac{1}{\delta}} \|U\|_{L^{\delta}(B_1^+, y^a)}.$$

Then, after explicitly writing all the constants involved, formula (3.16) simply reduces to

$$\Phi(\theta\delta, 1) \le [C(1+\delta)^{\sigma}]^{\frac{2}{\delta}}\, \Phi(\delta, 2),$$

for some positive number σ. It is clear that the same proof works if we replace B_1, B_2 by B_{R_1}, B_{R_2}. The only difference is in (3.12), where we need to estimate $|\nabla\eta| \le C(R_2 - R_1)^{-1}$. Thus we would obtain

$$\Phi(\theta\delta, R_1) \le \left[\frac{C(1+\delta)^{\sigma}}{R_2 - R_1}\right]^{\frac{2}{\delta}} \Phi(\delta, R_2). \tag{3.17}$$

Now we iterate equation (3.17): set $R_m = 1 + \frac{1}{2^m}$ and $\theta_m = \theta^m \bar{p}$. Then

$$\Phi(\theta_m, 1) \le \Phi(\theta_m, R_m) \le (c_1\theta)^{c_2 \sum_{i=0}^{m-1} \frac{i}{\theta^i}}\, \Phi(\bar{p}, 2) \le C\Phi(\bar{p}, 2), \tag{3.18}$$

for some constant C because the series $\sum_{i=0}^{\infty} \frac{i}{\theta^i}$ is convergent.

Finally, note that

$$\sup_{\Gamma_1^0} U = \lim_{\delta\to\infty} \|U\|_{L^{\delta}(\Gamma_1^0)}, \quad \sup_{B_1^+} U = \lim_{\delta\to\infty} \|U\|_{L^{\delta}(B_1^+, y^a)},$$

so that (3.18) is telling us that

$$\sup_{B_1^+} U + \sup_{\Gamma_1^0} U \le C\left[\|U\|_{L^{\bar{p}}(B_2, y^a)} + \|U\|_{L^{\bar{p}}(\Gamma_2^0)}\right].$$

Rescaling to a ball of radius R concludes the proof of the theorem. □

The next main ingredient is the proof of the positivity of a solution to (3.5). We observed that a Hopf lemma, some version of which was known for the Euclidean half space case (Proposition 4.10 in [9]), can be obtained for the uniformly degenerate elliptic equation (3.1). This nice Hopf's lemma turns out to be one of the keys for us in this paper. It is interesting to observe a different behavior between the cases $\gamma \in (0, 1/2)$ and $\gamma \in [1/2, 1)$ in our proof - this dichotomy does not seem to appear in the flat case in [9].

We continue to use the setting as in Proposition 2.2. Let $p_0 \in \partial X$ and (x, y) be the local coordinate at p_0 for $\bar{X}$ with $x(p_0) = 0$, where x is the normal coordinate at p_0 with respect to the metric $\hat{h}$ on the boundary M^n.

Theorem 3.5. *Suppose that U is a nonnegative solution to* (3.1) *in X^{n+1}. Then, for sufficiently small r_0, if $U(q_0) = 0$ for $q_0 \in \Gamma_{r_0}^0 \setminus \overline{\Gamma_{\frac{1}{2}r_0}^0}$ and $U > 0$ on $\partial\Gamma_{\frac{1}{2}r_0}^0$ on the boundary M^n, then*

$$y^a \partial_y U|_{q_0} > 0. \tag{3.19}$$

Proof. First we assume that $\gamma \in [1/2, 1)$, i.e., $a \in (-1, 0]$. We consider a positive function

$$W = y^{-a}(y + Ay^2)(e^{-B|x|} - e^{-Br_0}). \tag{3.20}$$

To calculate $\mathrm{div}(y^a \nabla W)$ in the metric $\bar{g}^*$ we first calculate from Proposition 2.2 that

$$\bar{g}^* = (1 + \alpha_1 y)\, dy^2 + (1 + \alpha_2 y)\, \hat{h} + o(y)$$

for some constants α_1, α_2 and

$$\det \bar{g}^* = \det \hat{h}\, (1 + \alpha_3 y) + o(y),$$

for some constant α_3. Then

$$\mathrm{div}(y^a \nabla W) = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{\det \bar{g}^*}} \partial_y \left(\sqrt{\det \bar{g}^*} (\bar{g}^*)^{yy} ((1-a) + (2-a)yA)(e^{-B|x|} - e^{-Br_0}) \right) \\ &= (\alpha_4 + (2-a)A + o(1))\, (e^{-B|x|} - e^{-Br_0}), \\ I_2 &= \frac{1}{\sqrt{\det \bar{g}^*}} \partial_{x^k} \left(\sqrt{\det \bar{g}^*} (\bar{g}^*)^{ky} ((1-a) + (2-a)yA)(e^{-B|x|} - e^{-Br_0}) \right) \\ &= o(1)(e^{-B|x|} - e^{-Br_0}) + o(y) B e^{-Br}, \end{aligned}$$

for some constant α_4,

$$I_3 = \frac{1}{\sqrt{\det \bar{g}^*}} \partial_y \left(\sqrt{\det \bar{g}^*} (\bar{g}^*)^{yk} (y + y^2 A) \partial_{x^k} (e^{-B|x|} - e^{-Br_0}) \right) = o(y) B e^{-Br},$$

and

$$\begin{aligned} I_4 &= \frac{y + y^2 A}{\sqrt{\det \bar{g}^*}} \partial_{x^k} \left(\sqrt{\det \bar{g}^*} (\bar{g}^*)^{kj} \partial_{x^j} (e^{-B|x|} - e^{-Br_0}) \right) \\ &= \frac{y + y^2 A}{\sqrt{\det \bar{g}^*}} \partial_{x^k} \left(\sqrt{\det \bar{g}^*} (\bar{g}^*)^{kj} \left(-\frac{x_j}{r} B e^{-Br} \right) \right) \qquad . \\ &= y B^2 e^{-Br} + o(y) B^2 e^{-Br} + y B^2 o(r^2) e^{-Br} + o(y) B e^{-Br}. \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{div}(y^a \nabla W) = {} & (\alpha_4 + (2-a)A + o(1))\, (e^{-B|x|} - e^{-Br_0}) \\ & + (B^2 + o(1)B) y e^{-Br}. \end{aligned}$$

We remark here that all constants α's can be explicit, but it would not be any more use. Take r_0 sufficiently small and A and B sufficiently large so that

$$\mathrm{div}(y^a \nabla W) \geq 0$$

provided that $a \leq 0$. Now we know

$$\mathrm{div}\, (y^a \nabla (U - \epsilon W)) \leq 0$$

in $\left(\Gamma^0_{r_0} \setminus \overline{\Gamma^0_{\frac{1}{2}r_0}}\right) \times (0, r_0)$ for all $\epsilon > 0$, and moreover

$$U - \epsilon W \geq 0$$

on $\partial \left\{ \left(\Gamma^0_{r_0} \setminus \overline{\Gamma^0_{\frac{1}{2}r_0}}\right) \times (0, r_0) \right\}$, provided we choose ϵ appropriately small. Therefore, due to the maximum principle we know that

$$U - \epsilon W > 0$$

in $\left(\Gamma^0_{r_0} \setminus \overline{\Gamma^0_{\frac{1}{2}r_0}}\right) \times (0, r_0)$. Thus, when $U(x(q_0), 0) = 0$, we have

$$y^a \partial_y (U - \epsilon W)|_{(x(q_0),0)} \geq 0,$$

which implies

$$y^a \partial_y U|_{(x(q_0),0)} \geq \epsilon y^a \partial_y W|_{(x(q_0),0)} = \epsilon(1-a)(e^{-B|x(q_0)|} - e^{-Br_0}) > 0,$$

as desired.

When $a \in (0, 1)$, or equivalently, $\gamma \in (0, \frac{1}{2})$, we instead use the function

$$W = y^{-a}(y + Ay^{2-a})(e^{-B|x|} - e^{-Br_0}).$$

Then a similar calculation will prove that the conclusion still holds. □

Positivity of solutions for (3.1) is now clear:

Corollary 3.6. *Suppose that $U \in \mathcal{C}^2(X) \cap \mathcal{C}(\bar{X})$ is a nonnegative solution to the equation*

$$\begin{cases} div(y^a \nabla U) = 0 & \text{in } (X,\ \bar{g}^*), \\ y^a \partial_y U = F(U) & \text{on } M, \end{cases}$$

where $F(0) = 0$. Then $U > 0$ on $\bar{X}$ unless $U \equiv 0$.

Proof. First, $U > 0$ in X, and U is not identically zero on the boundary if it is not identically zero on $\bar{X}$. Then, on the boundary, the set where U is positive is nonempty and open. Hence, if the set where U vanishes is not empty, then, for any small number r_0, there always exist points p_0 and q_0 as given in the assumptions of Theorem 3.5. Thus we would arrive at the contradiction from Theorem 3.5. □

4 The γ-Yamabe problem

Now we are ready to set up the fractional Yamabe problem for $\gamma \in (0, 1)$. On the conformal infinity $(M^n,\ [\hat{h}])$ of an asymptotically hyperbolic manifold $(X^{n+1},\ g^+)$, we consider a scale-free functional on metrics in the class $[\hat{h}]$ given by

$$I_\gamma[\hat{h}] = \frac{\int_M Q^{\hat{h}}_\gamma \, dv_{\hat{h}}}{(\int_M \, dv_{\hat{h}})^{\frac{n-2\gamma}{n}}}. \tag{4.1}$$

Or, if we set a base metric $\hat{h}$ and write a conformal metric

$$\hat{h}_w = w^{\frac{4}{n-2\gamma}} \hat{h},$$

then

$$I_\gamma[w, \hat{h}] = \frac{\int_M w P^{\hat{h}}_\gamma(w) \, dv_{\hat{h}}}{\left(\int_M w^{2^*} \, dv_{\hat{h}}\right)^{\frac{2}{2^*}}} \tag{4.2}$$

where $2^* = \frac{2n}{n-2\gamma}$. We will call I_γ the γ-Yamabe functional.

The γ-*Yamabe problem* is to find a metric in the conformal class $[\hat{h}]$ that minimizes the γ-Yamabe functional I_γ. It is clear that a metric $\hat{h}_w$, where w is a minimizer of $I_\gamma[w, \hat{h}]$, has a constant fractional scalar curvature $Q^{\hat{h}_w}_\gamma$, that is,

$$P^{\hat{h}}_\gamma(w) = cw^{\frac{n+2\gamma}{n-2\gamma}}, \quad w > 0, \tag{4.3}$$

for some constant c on M.

This suggests that we define the γ-Yamabe constant

$$\Lambda_\gamma(M,[\hat{h}]) = \inf\left\{I_\gamma[h] : h \in [\hat{h}]\right\}. \tag{4.4}$$

It is then apparent that $\Lambda_\gamma(M,[\hat{h}])$ is an invariant on the conformal class $[\hat{h}]$ when g^+ is fixed.

In the mean time, based on Proposition 2.1, we set

$$I_\gamma^*[U,\bar{g}] = \frac{d_\gamma^* \int_X \rho^a\, |\nabla U|_{\bar{g}}^2\, dv_{\bar{g}} + \int_X E(\rho)|U|^2\, dv_{\bar{g}}}{\left(\int_M |U|^{2^*}\, dv_{\hat{h}}\right)^{\frac{2}{2^*}}}, \tag{4.5}$$

or similarly, using Proposition 2.2, we may set

$$I_\gamma^*[U,\bar{g}^*] = \frac{d_\gamma^* \int_X y^a\, |\nabla U|_{\bar{g}^*}^2\, dv_{\bar{g}^*} + \int_M Q_\gamma^{\hat{h}}|U|^2\, dv_{\hat{h}}}{\left(\int_M |U|^{2^*}\, dv_{\hat{h}}\right)^{\frac{2}{2^*}}}. \tag{4.6}$$

It is obvious that it is equivalent to solve the minimizing problems for I_γ and I_γ^*. But a very pleasant surprising is that this immediately tells us that

$$\Lambda_\gamma(X,\ [\hat{h}]) = \inf\left\{I_\gamma^*[U,\bar{g}] : U \in W^{1,2}(X,y^a)\right\} \tag{4.7}$$

(please see the definitions and discussions of the weighted Sobolev spaces in Section 5). Note that one has that $I_\gamma^*[|U|] \leq I_\gamma^*[U]$, to handle positivity issues. Therefore we have

Lemma 4.1. *Suppose that U is a minimizer of the functional $I_\gamma^*[\cdot,\bar{g}]$ in the weighted Sobolev space $W^{1,2}(X,y^a)$ with $\int_M |TU|^{2^*}\, dv_{\hat{h}} = 1$. Then its trace $w = TU \in H^\gamma(M)$ solves the equation*

$$P_\gamma^{\hat{h}}(w) = \Lambda_\gamma(X,\ [\hat{h}])w^{\frac{n+2\gamma}{n-2\gamma}}.$$

To resolve the γ-Yamabe problem is to verify I_γ has a minimizer w, which is positive and smooth. But before launching our resolution to the γ-Yamabe problem we are first due to discuss the sign of the γ-Yamabe constant. These statements are familiar and easy ones for the Yamabe problem but not so easy at all for the γ-Yamabe problem, where the conformal fractional Laplacians are just pseudo-differential operators. One knows that eigenvalues and eigenfunctions of the conformal fractional Laplacians are even more difficult to study than the differential operators. There are some affirmative results analogous to the conformal Laplacian proven in [28] when the Yamabe constant of the conformal infinity is assumed to be positive. Here we will take the advantage of our Hopf's Lemma and the interpretation of the conformal fractional Laplacians through extensions provided in Proposition 2.2.

For each $\gamma \in (0,1)$ we know that each conformal fractional Laplacian is self-adjoint (cf. [27], [20]). Hence we may look for the first eigenvalue λ_1 by minimizing the quotient

$$\frac{\int_M wP_\gamma^{\hat{h}}w\, dv_{\hat{h}}}{\int_M w^2\, dv_{\hat{h}}}. \tag{4.8}$$

Moreover, again in the light of Proposition 2.2, it is equivalent to minimizing

$$\frac{d_\gamma^* \int_X y^a\, |\nabla U|_{\bar{g}^*}^2\, dv_{\bar{g}^*} + \int_M Q_\gamma^{\hat{h}}|U|^2\, dv_{\hat{h}}}{\int_M |U|^2\, dv_{\hat{h}}}. \tag{4.9}$$

We arrive at the eigenvalue equation:

$$P_\gamma^{\hat{h}}w = \lambda_1 w, \quad \text{on } M.$$

Or, equivalently,

$$\begin{cases} \operatorname{div}(y^a\nabla U) = 0 \quad \text{in } (X,\ \bar{g}^*), \\ -d_\gamma^* \lim_{y\to 0} y^a\partial_y U + Q_\gamma^{\hat{h}}U = \lambda_1 U \quad \text{on } M, \end{cases} \tag{4.10}$$

As a consequence of Proposition 2.2 and Theorem 3.5 we have:

Theorem 4.2. *Suppose that* $(X^{n+1},\ g^+)$ *is an asymptotically hyperbolic manifold. For each* $\gamma \in (0,1)$ *there is a smooth, positive first eigenfunction for* $P_\gamma^{\hat h}$ *and the first eigenspace is of dimension one, provided* $H = 0$ *when* $\gamma \in (\frac{1}{2}, 1)$.

Proof. We use the variational characterization (4.9) of the first eigenvalue. We first observe that one may always assume there is a nonnegative minimizer for (4.9). Then regularity and the maximum principle in Section 3 insure that such a first eigenfunction is smooth and positive. To show that the first eigenspace is of dimension 1, we suppose that ϕ and ψ are positive first eigenfunctions for $P_\gamma^{\hat h}$. Then

$$\begin{aligned} P_\gamma^{\hat h_\phi} \frac{\psi}{\phi} &= \phi^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{\hat h} \psi = \lambda_1 \phi^{-\frac{n+2\gamma}{n-2\gamma}} \psi \\ &= (\phi^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{\hat h} \phi) \frac{\psi}{\phi} \\ &= Q_\gamma^{\hat h_\phi} \frac{\psi}{\phi}, \end{aligned}$$

where $\hat h_\phi = \phi^{\frac{4}{n-2\gamma}} \hat h$. That is, there is a function U satisfying

$$\begin{cases} \operatorname{div}(y_\phi^a \nabla U) = 0 & \text{in } (X,\ \bar g_\phi^*), \\ \lim_{y_\phi \to 0} y_\phi^a \dfrac{\partial U}{\partial y_\phi} U = 0 & \text{on } M, \end{cases}$$

and $U = \frac{\psi}{\phi}$ on M, where y_ϕ and $\bar g_\phi^*$ are associated with $\hat h_\phi$ as y and $\bar g^*$ are associated with $\hat h$ in Proposition 2.2 respectively. Replace U by $U - U_m$ for $U_m = \min_{\bar X} U$ and apply Theorem 3.5 and Corollary 3.6 to conclude that U has to be a constant. □

Consequently, we get the following.

Corollary 4.3. *Suppose that* $(X^{n+1},\ g^+)$ *is an asymptotically hyperbolic manifold. Assume that* $\gamma \in (0,1)$ *and that* $H = 0$ *when* $\gamma \in (\frac{1}{2}, 1)$. *Then there are three mutually exclusive possibilities for the conformal infinity* $(M^n,\ [\hat h])$:

1. *The first eigenvalue of* $P_\gamma^{\hat h}$ *is positive, the* γ*-Yamabe constant is positive, and* M *admits a metric in* $[\hat h]$ *that has pointwise positive fractional scalar curvature.*
2. *The first eigenvalue of* $P_\gamma^{\hat h}$ *is negative, the* γ*-Yamabe constant is negative, and* M *admits a metric in* $[\hat h]$ *that has pointwise negative fractional scalar curvature.*
3. *The first eigenvalue of* $P_\gamma^{\hat h}$ *is zero, the* γ*-Yamabe constant is zero, and* M *admits a metric in* $[\hat h]$ *that has vanishing fractional scalar curvature.*

Proof. First of all it is obvious that the sign of the first eigenvalue of the conformal fractional Laplacian $P_\gamma^{\hat h}$ does not change within the conformal class due to the conformal covariance property of the conformal fractional Laplacian. The three possibilities are distinguished by the sign of the first eigenvalue λ_1 of the conformal fractional Laplacian $P_\gamma^{\hat h}$. Because, if ϕ is the positive first eigenfunction of $P_\gamma^{\hat h}$, then

$$Q_\gamma^{\hat h_\phi} = \lambda_1^{\hat h} \phi^{-\frac{4\gamma}{n-2\gamma}}$$

where $\hat h_\phi = \phi^{\frac{4}{n-2\gamma}} \hat h$. □

5 Weighted Sobolev trace inequalities

Let us continue in the setting provided by Proposition 2.2. On the compact manifold M^n, for $\gamma \in (0,1)$, we recall the fractional order Sobolev space $H^\gamma(M)$, with its usual norm

$$\|w\|^2_{H^\gamma(M)} := \|w\|^2_{L^2(M)} + \int_M w(-\Delta_{\hat{h}})^\gamma w \, dv_{\hat{h}}.$$

An equivalent norm on this space is

$$\|w\|^2_{H^\gamma(M)} := A\,\|w\|^2_{L^2(M)} + \int_M w P_\gamma^{\hat{h}} w \, dv_{\hat{h}},$$

for some appropriately large number A, since $P_\gamma^{\hat{h}}$ is an elliptic pseudo-differential operator of order 2γ with its principal symbol being the same as that of $(-\Delta_{\hat{h}})^\gamma$.

Note that in $\mathbb{R}^n$, this Sobolev norm can be easily written in terms of the Fourier transform as

$$\|w\|^2_{H^\gamma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^\gamma \hat{w}^2(\xi)\, d\xi. \tag{5.1}$$

We would also like to recall the definition of the weighted Sobolev spaces. For $\gamma \in (0,1)$ and $a = 1-2\gamma$, consider the norm

$$\|U\|^2_{W^{1,2}(X,y^a)} = \int_X y^a |\nabla U|^2_{\bar{g}^*}\, dv_{\bar{g}^*} + \int_X y^a U^2\, dv_{\bar{g}^*}.$$

The following is then known.

Lemma 5.1. *There exists a unique linear bounded operator*

$$T : W^{1,2}(X, y^a) \to H^\gamma(M)$$

such that $TU = U|_M$ for all $U \in \mathcal{C}^\infty(\bar{X})$, which is called the trace operator.

Lemma 5.1 was explored by Nekvinda [37] in the case when X is a subset of $\mathbb{R}^{n+1}$ and M^n a piece of its boundary; see also [32]. It then takes some standard argument to derive the Lemma 5.1 from, for instance, [37].

The classical Sobolev trace inequality on Euclidean space is well known (see, for instance, Escobar [15]), and reads:

$$\left(\int_{\mathbb{R}^n} |Tu|^{\frac{2n}{n-1}}\, dx\right)^{\frac{n-1}{2n}} \le C(n) \left(\int_{\mathbb{R}^{n+1}_+} |\nabla u|^2\, dxdy\right)^{\frac{1}{2}} \tag{5.2}$$

where the constant $C(n)$ is sharp and the equality case is completely characterized. This corresponds to $a = 0$ for our cases. The same result is true for any other real $a \in (-1,1)$. Indeed there are general Weighted Sobolev trace inequalities. Let us first recall the well known fractional Sobolev inequalities. They were considered first in the remarkable paper by Lieb [31] (see also the more recent [21], [13], or the survey [14]):

Lemma 5.2. *Let $0 < \gamma < n/2$, $2^* = \frac{2n}{n-2\gamma}$. Then, for all $w \in H^\gamma(\mathbb{R}^n)$ we have*

$$\|w\|^2_{L^{2^*}(\mathbb{R}^n)} \le S(n,\gamma)\|(-\Delta)^{\frac{\gamma}{2}} w\|^2_{H^\gamma(\mathbb{R}^n)} = S(n,\gamma)\int_{\mathbb{R}^n} w(-\Delta)^\gamma w\, dx, \tag{5.3}$$

where

$$S(n,\gamma) = 2^{-2\gamma}\pi^{-\gamma}\frac{\Gamma\left(\frac{n-2\gamma}{2}\right)}{\Gamma\left(\frac{n+2\gamma}{2}\right)}\left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right]^{\frac{2\gamma}{n}} = \frac{\Gamma\left(\frac{n-2\gamma}{2}\right)}{\Gamma\left(\frac{n+2\gamma}{2}\right)}|vol(S^n)|^{-\frac{2\gamma}{n}}.$$

We have equality in (5.3) if and only if

$$w(x) = c\left(\frac{\mu}{|x-x_0|^2+\mu^2}\right)^{\frac{n-2\gamma}{2}}, \quad x \in \mathbb{R}^n,$$

for $c \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ fixed.

Note that we may interpret the above inequality as a calculation of the best γ-Yamabe constant on the standard sphere as the conformal infinity of the Hyperbolic space. Namely, if g_c is the standard round metric on the unit sphere,

$$\|w\|^2_{L^{2^*}(S^n)} \leq S(n,\gamma) \int_{S^n} w P_\gamma^{g_c} w \, dv_{g_c}. \tag{5.4}$$

Such an inequality for the sphere case was also considered independently by Beckner [5], Branson [6], and Morpurgo [35], in the setting of interwining operators. Indeed, we have the following explicit expression for $P_\gamma^{S^n}$:

$$P_\gamma^{S^n} = \frac{\Gamma\left(B+\gamma+\frac{1}{2}\right)}{\Gamma\left(B-\gamma+\frac{1}{2}\right)}, \quad \text{where} \quad B := \sqrt{-\Delta_{S^n} + \left(\tfrac{n-1}{2}\right)^2}.$$

It is clear from (5.4) that

$$\Lambda_\gamma(S^n,[g_c]) = \frac{1}{S(n,\gamma)}. \tag{5.5}$$

Sobolev trace inequalities can be obtained by the composition of the trace theorem and the Sobolev embedding theorem above. There have been some related works that deal with these types of energy inequalities, for instance, Nekvinda [37], González [23], and Cabré-Cinti [7]. In particular, in the light of the work of Caffarelli and Silvestre [10] and Lemma 5.2, we easily see the more general form of (5.2) as follows:

Corollary 5.3. *Let $w \in H^\gamma(\mathbb{R}^n)$, $\gamma \in (0,1)$, $a = 1-2\gamma$, and $U \in W^{1,2}(\mathbb{R}^{n+1}_+, y^a)$ with trace $TU = w$. Then*

$$\|w\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \bar{S}(n,\gamma) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2 \, dxdy, \tag{5.6}$$

where

$$\bar{S}(n,\gamma) := d_\gamma^* S(n,\gamma). \tag{5.7}$$

Equality holds if and only if

$$w(x) = c\left(\frac{\mu}{|x-x_0|^2+\mu^2}\right)^{\frac{n-2\gamma}{2}}, \quad x \in \mathbb{R}^n,$$

for $c \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ fixed, and U is its Poisson extension of w as given in (2.13).

In the following lines we take a closer look at the extremal functions that attain the best constant in the inequality above. On $\mathbb{R}^n$ we fix

$$w_\mu(x) := \left(\frac{\mu}{|x|^2+\mu^2}\right)^{\frac{n-2\gamma}{2}}, \tag{5.8}$$

these correspond to the conformal diffeomorphisms of the sphere. We set

$$U_\mu = K_\gamma *_x w_\mu \tag{5.9}$$

as given in (2.13). Then we have the equality

$$\|w_\mu\|^2_{L^{2^*}(\mathbb{R}^n)} = \bar{S}(n,\gamma) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\mu|^2 \, dxdy.$$

It is clear that

$$w_\mu(x) = \frac{1}{\mu^{\frac{n-2\gamma}{2}}} w_1\left(\frac{x}{\mu}\right), \quad \text{and} \quad U_\mu(x,y) = \frac{1}{\mu^{\frac{n-2\gamma}{2}}} U_1\left(\frac{x}{\mu},\frac{y}{\mu}\right). \tag{5.10}$$

Moreover, U_μ is the (unique) solution of the problem

$$\begin{cases} \operatorname{div}(y^a \nabla U_\mu) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{y\to 0} y^a \partial_y U_\mu = c_{n,\gamma}(w_\mu)^{\frac{n+2\gamma}{n-2\gamma}} & \text{on } \mathbb{R}^n, \end{cases} \tag{5.11}$$

On the other hand, if we multiply equation (5.11) by U_μ and integrate by parts,

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\mu|^2 \,dxdy = c_{n,\gamma} \int_{\mathbb{R}^n} (w_\mu)^{2^*}\,dx. \tag{5.12}$$

Now we compare (5.12) with (5.6). Using (5.5) we arrive at

$$\Lambda(S^n, [g_c]) = c_{n,\gamma} d^*_\gamma \left[\int_{\mathbb{R}^n} (w_\mu)^{2^*}\,dx \right]^{\frac{2\gamma}{n}}. \tag{5.13}$$

Before the end of this section we calculate the general upper bound of the γ-Yamabe constants. Indeed there is a complete analogue to the case of the usual Yamabe problem (cf. [3], [30]). Namely, the following.

Proposition 5.4. *Let $\gamma \in (0,1)$. Then*

$$\Lambda_\gamma(M, [\hat{h}]) \le \Lambda_\gamma(S^n, [g_c]).$$

Proof. First of all we will instead use the functional (4.6) to estimate the γ-Yamabe constant for a good reason. The approach is rather the standard method of gluing a "bubble" (5.8) to the manifold M (see, for instance, [30], Lemma 3.4).

For any fixed $\epsilon > 0$, let B_ϵ be the ball of radius ϵ centered at the origin in $\mathbb{R}^{n+1}$ and B^+_ϵ be the half ball of radius ϵ in $\mathbb{R}^{n+1}_+$. Choose a smooth radial cutoff function η, $0 \le \eta \le 1$ supported on $B_{2\epsilon}$, and satisfying $\eta \equiv 1$ on B_ϵ. Then, consider the function $V = \eta U_\mu$ with its trace $v = \eta w_\mu$ on $\mathbb{R}^n$. We have that

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla V|^2\,dxdy \le (1+\epsilon) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\mu|^2\,dxdy + C(\epsilon) \int_{B^+_{2\epsilon}\setminus B^+_\epsilon} U_\mu^2\,dxdy. \tag{5.14}$$

Note that $w_\mu = O(\mu^{\frac{n-2\gamma}{2}} |x|^{2\gamma-n})$ in the annulus $\epsilon \le |x| \le 2\epsilon$ and U_μ is $O(\mu^{\frac{n-2\gamma}{2}})$ in the annulus $B^+_{2\epsilon}\setminus B^+_\epsilon$. This allows to estimate the second term in right hand side of (5.14) by $O\left(\mu^{n-2\gamma}\right)$ as $\mu \to 0$, for ϵ fixed. For the first term in the right hand side of (5.14) we first use the fact that w_μ attains the best constant in the Sobolev inequality, so

$$\bar{S}(n,\gamma) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U_\mu|^2\,dxdy = \left(\int_{\mathbb{R}^n} w_\mu^{2^*}\,dx \right)^{\frac{2}{2^*}} \le \left(\int_{\mathbb{R}^n} v^{2^*}\,dx \right)^{\frac{2}{2^*}} + O(\mu^n). \tag{5.15}$$

Now we need to transplant the function V to the manifold $(\bar{X}, \bar{g}^*)$. Fix a point on the boundary M and use normal coordinates $\{x_1, \dots, x_n, y\}$ around it, in a half ball $B^+_{2\epsilon}$ where V is supported. Two things must be modified: when $\epsilon \to 0$,

$$|\nabla V|^2_{\bar{g}^*} = |\nabla V|^2 (1 + O(\epsilon)),$$

and

$$dv_{\bar{g}^*} = (1 + O(\epsilon)) dxdy,$$

so that

$$\begin{aligned} I_{\epsilon,\mu} &:= d^*_\gamma \int_{B^+_{2\epsilon}} y^a |\nabla V|^2_{\bar{g}^*}\,dv_{\bar{g}^*} + \int_{|x|\le 2\epsilon} Q^{\hat{h}}_\gamma v^2\,dv_{\hat{h}} \\ &\le (1+O(\epsilon)) \left(\int_{B^+_{2\epsilon}} y^a |\nabla V|^2\,dxdy + C \int_{|x|<2\epsilon} v^2\,dx \right). \end{aligned}$$

It is easily seen that

$$\int_{|x|<2\epsilon} w_\mu^2 \, dx = o(1).$$

This is a small computation that can be found in Lemma 3.5 of [30]. Then, from (5.15), fixing ϵ small and then μ small, we can get that

$$I_{\epsilon,\mu} \leq (1+C\epsilon)\left(\frac{1}{S(n,\gamma)}\|v\|^2_{L^{2^*}(M)} + C\mu\right)$$

which implies

$$\Lambda_\gamma(M,[\hat{h}]) \leq \frac{1}{S(n,\gamma)} = \Lambda_\gamma(S^n,[g_c]).$$

□

We end this section by remarking that, although most of the results mentioned here were already known in different contexts, it is certainly very interesting to put all the analysis and geometry together in the context of conformal fractional Laplacians and the associated γ-Yamabe problems in a way that is analogous to what has been done on the subject of the Yamabe problem, which becomes fundamental to the development of geometric analysis.

6 Subcritical approximations

In this section we take a well known subcritical approximation method to solve the γ-Yamabe problem and prove Theorem 1.4. There does not seem to be any more difficulty than usual after our discussions in previous sections. But, for the convenience of the readers, we present a brief sketch of the proof. Similar to the case of the usual Yamabe problem we consider the following subcritical approximations to the functionals I_γ and I_γ^* respectively. Set

$$I_\beta[w] = \frac{\int_M w P_\gamma^{\hat{h}} w \, dv_{\hat{h}}}{\left(\int_M w^\beta \, dv_{\hat{h}}\right)^{\frac{2}{\beta}}}$$

and

$$I_\beta^*[U] = \frac{d_\gamma^* \int_X y^a \, |\nabla U|^2_{\bar{g}^*} \, dv_{\bar{g}} + \int_M Q_\gamma^{\hat{h}} U^2 \, dv_{\hat{h}}}{\left(\int_M U^\beta \, dv_{\hat{h}}\right)^{\frac{2}{\beta}}}.$$

for $\beta \in [2, 2^*)$, where $2^* = \frac{2n}{n-2\gamma}$ and $\gamma \in (0,1)$. These are subcritical problems and can be solved through standard variational methods. For clarity we state the following:

Proposition 6.1. *For each $2 \leq \beta < 2^*$, there exists a smooth positive minimizer U_β for $I_\beta^*[U]$ in $W^{1,2}(X, y^a)$, which satisfies the equations*

$$\begin{cases} div\,(y^a \nabla U_\beta) = 0 \quad in\ (X, \bar{g}^*), \\ -d_\gamma^* \lim_{y\to 0} y^a U_\beta + Q_\gamma^{\hat{h}} U_\beta = c_\beta U_\beta^{\beta-1} \quad on\ M, \end{cases}$$

where the derivatives are taken with respect to the metric $\bar{g}^*$ in X and $c_\beta = I_\beta^*[U_\beta] = \min I_\beta^*$. And the boundary value w_β of U_β, which is a positive smooth minimizer for $I_\beta[w]$ in $H^\gamma(M)$, satisfies

$$P_\gamma^{\hat{h}} w_\beta = c_\beta w_\beta^{\beta-1}.$$

Using a similar argument as in the proof of Lemma 4.3 in [30] (see also [3]) we have the following.

Lemma 6.2. *If $vol(M,\hat{h}) = 1$, then $|c_\beta|$ is non-increasing as a function of $\beta \in [2, 2^*]$; and if $\Lambda_\gamma(M,[\hat{h}]) \geq 0$, then c_β is continuous from the left at $\beta = 2^*$.*

We now start the proof of Theorem 1.4. Readers are referred to [16], [30], [40] for more details. Instead of applying the standard Sobolev embedding in the Yamabe problem we apply the weighted trace ones discussed in the previous section. To ensure that U_β as $\beta \to 2^*$ produces a minimizer for the γ-Yamabe problem, we want to establish the a priori estimates for U_β. In the light of the discussions in Section 3, we only need to have a uniform L^∞ bound for w_β. We will establish the L^∞ bound for w_β by the so-called blow-up method.

Otherwise, assume there exist sequences $\beta_k \to 2^*$, $w_k := w_{\beta_k}$ and $U_k := U_{\beta_k}$, $x_k \in M$ such that $w_k(x_k) = \max_M\{w_k\} = m_k \to \infty$ and $x_k \to x_0 \in M$ as $k \to \infty$. Take a normal coordinate system centered at x_0, and rescale

$$V_k(x, y) = m_k^{-1} U_k(\delta_k x + x_k, \delta_k y),$$

with the boundary value

$$v_k(x) = m_k^{-1} w_k(\delta_k x + x_k),$$

where $\delta_k = m_k^{\frac{1-\beta_k}{2\gamma}}$. Then V_k is defined in a half ball of radius $R_k = \frac{1-|x_k|}{\delta_k}$ and is a solution of

$$\begin{cases} \operatorname{div}(\rho^a \nabla V_k) = 0 \quad \text{in } B^+_{R_k}, \\ -d^*_\gamma \lim_{y\to 0} y^a \partial_y V_k + (Q^{\hat{h}}_\gamma)_k v_k = c_k v_k^{\beta-1} \quad \text{on } B_{R_k}, \end{cases} \tag{6.1}$$

with respect to the metric $\bar{g}^*(\delta_k x + x_k, \delta_k y)$, where

$$(Q^{\hat{h}}_\gamma)_k = \delta_k^{1-a} Q^{\hat{h}}_\gamma(\delta_k x + x_k) \to 0.$$

Due to, for example, $\mathcal{C}^{2,\alpha}$ a priori estimates for the rescaled solutions V_k, to extract a subsequence if necessary, we have $V_k \to V_0$ in $C^{2,\alpha}_{\text{loc}}$. Moreover the metrics $\bar{g}^*(\delta_k x + x_k, \delta_k y)$ converge to the Euclidean metric. Hence V_0 is a non-trivial, non-negative solution of

$$\begin{cases} -\operatorname{div}(y^a \nabla V_0) = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ -d^*_\gamma \lim_{y\to 0} y^a \partial_y V_0 = c_0 V_0^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on } \mathbb{R}^n, \end{cases} . \tag{6.2}$$

Let $v_0 = TV_0$. It is easily seen that

$$\int_{\mathbb{R}^n} v_0^{2^*}(x)\, dx \le 1. \tag{6.3}$$

Theorem 3.5 and Corollary 3.6 then assure that $V_0 > 0$ on $\overline{\mathbb{R}^{n+1}_+}$. Therefore we can obtain

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla V_0|^2\, dxdy = c_0 d^*_\gamma \int_{\mathbb{R}^n} v_0^{2^*}(x)\, dx. \tag{6.4}$$

It is then obvious that $c_0 > 0$, that is, $c_0 = \Lambda_\gamma(M, [\hat{h}])$ in the light of Lemma 6.2. Moreover, by the trace inequalities from Lemma 5.3, we have

$$\left(\int_{\mathbb{R}^n} v_0^{2^*}(x)\, dx\right)^{\frac{2}{2^*}} \le \bar{S}(n, \gamma) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla V_0|^2\, dxdy. \tag{6.5}$$

Then (6.3), (6.4) and (6.5), together with the definition of $\Lambda_\gamma(S^n, [g_c])$ in (5.5) contradict the initial hypothesis (1.6).

Once we have a uniform L^∞ estimate, by the regularity theorems in Section 3 we may extract a subsequence if necessary and pass to a limit U_0, whose boundary value w_0 satisfies

$$P^{\hat{h}}_\gamma w_0 = \Lambda w_0^{2^*-1}, \quad I_\gamma[w_0] = \Lambda, \quad \Lambda = \lim c_\beta. \tag{6.6}$$

Theorem 3.5 and Corollary 3.6 also ensure that $w_0 > 0$ on M. It remains to check that $\Lambda = \Lambda_\gamma(M, [\hat{h}])$. However, this is a direct consequence of Lemma 6.2 when $\Lambda_\gamma(M, [\hat{h}]) \ge 0$. Meanwhile it is easily seen that by the definition of the γ-Yamabe constants and (6.6) that Λ can not be less than $\Lambda_\gamma(M, [\hat{h}])$. Hence it is also implied that $\Lambda = \Lambda_\gamma(M, [\hat{h}])$ by Lemma 6.2 when $\Lambda_\gamma(M, [\hat{h}]) < 0$. Thus, in any case, w_0 is a minimizer of I_γ, as desired. □

7 A sufficient condition

In this section we give the proof of Theorem 1.5, which provides a sufficient condition for the resolution of the γ-Yamabe problem. Here the precise structure of the metric will play a crucial role since a careful computation of the asymptotics is required, following the calculation in [16]. The section is divided into two parts: the first contains the necessary estimates on the Euclidean case, while in the second we go back to the geometry setting and finish the proof of the theorem.

7.1 Some preliminary results on $\mathbb{R}^{n+1}_+$

Here we consider the divergence equation (2.11) on $\mathbb{R}^{n+1}_+$, as understood in [10], [23]. The main point is that by using the Fourier transform, a solution to this problem can be written in terms on its trace value on $\mathbb{R}^n$ and the well known Bessel functions. Indeed, let U be a solution of

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(x,0) = w & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{7.1}$$

or equivalently, $U = K_\gamma *_x w$, where K_γ is the Poisson kernel as given in (2.13).

The main idea is to reduce (7.1) to an ODE by taking Fourier transform in x. We obtain

$$\begin{cases} -|\xi|^2 \hat{u}(\xi,y) + \frac{a}{y}\hat{u}_y(\xi,y) + \hat{u}_{yy}(\xi,y) = 0, \\ \hat{U}(\xi,0) = \hat{w}(\xi), \end{cases}$$

that is an ODE for each fixed value of ξ.

On the other hand, consider the solution $\varphi : [0,+\infty) \to \mathbb{R}$ of the problem

$$-\varphi(y) + \frac{a}{y}\varphi_y(y) + \varphi_{yy}(y) = 0, \tag{7.2}$$

subject to the conditions $\varphi(0) = 1$ and $\lim_{t\to+\infty} \varphi(t) = 0$. This is a Bessel function and its properties are summarized in Lemma 7.1. Then we have that

$$\hat{U}(\xi,y) = \hat{w}(\xi)\varphi(|\xi|\, y). \tag{7.3}$$

For a review of Bessel functions (see, for instance, Lemma 5.1 in [23], or section 9.6.1. in [1]):

Lemma 7.1. *Consider the following ODE in the variable $y > 0$:*

$$-\varphi(y) + \frac{a}{y}\varphi_y(y) + \varphi_{yy}(y) = 0,$$

with boundary conditions $\varphi(0) = 1$, $\varphi(\infty) = 0$. Its solution can be written in terms of Bessel functions:

$$\varphi(y) = c_1 y^\gamma \mathcal{K}_\gamma(y),$$

where $\mathcal{K}_\gamma$ is the modified Bessel function of the second kind that has asymptotic behavior

$$\mathcal{K}_\gamma(y) \sim \frac{\Gamma(\gamma)}{2}\left(\frac{2}{y}\right)^\gamma, \quad \text{when } y \to 0^+,$$

$$\mathcal{K}_\gamma(y) \sim \sqrt{\frac{\pi}{2y}}\, e^{-y}, \quad \text{when } y \to +\infty,$$

for a constant

$$c_1 = \frac{2^{1-\gamma}}{\Gamma(\gamma)}.$$

Now we are ready to prove the main technical lemmas in the proof of Theorem 1.5. More precisely, we will explicitly compute several energy terms through Fourier transforms, thanks to expression (7.3). Such precise computation is needed in order to obtain the exact value of the constant (1.8). For the rest of the section, we denote $|\nabla U|^2 = (\partial_{x_1} U)^2 + \ldots + (\partial_{x_n} U)^2 + (\partial_y U)^2$, and $|\nabla_x U|^2 = (\partial_{x_1} U)^2 + \ldots + (\partial_{x_n} U)^2$.

Lemma 7.2. *Given $w \in H^\gamma(\mathbb{R}^n)$, let $U = K_\gamma * w$ defined on $\mathbb{R}^{n+1}_+$. Then*

$$\mathcal{A}_1(w) := \int_{\mathbb{R}^{n+1}_+} y^{a+2} |\nabla U|^2 \, dxdy = d_1 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{2(\gamma-1)} \, d\xi, \tag{7.4}$$

$$\mathcal{A}_2(w) := \int_{\mathbb{R}^{n+1}_+} y^{a+2} |\nabla_x U|^2 \, dxdy = d_2 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{2(\gamma-1)} \, d\xi, \tag{7.5}$$

$$\mathcal{A}_3(w) := \int_{\mathbb{R}^{n+1}_+} y^{a} U^2 \, dxdy = d_3 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{2(\gamma-1)} \, d\xi, \tag{7.6}$$

where

$$d_2 = \frac{-a+3}{6} d_1, \quad d_3 = \frac{1}{a+1} d_1.$$

Proof. We write $\mathcal{A}_i := \mathcal{A}_i(w)$, $i = 1, 2, 3$, for simplicity. Note that the integrals in the right hand side of (7.4), (7.5), (7.6) are finite because $w \in H^\gamma(\mathbb{R}^n) \hookrightarrow H^{\gamma-1}(\mathbb{R}^n)$, and because of the definition of the Sobolev norm (5.1).

Thanks to (7.3) we can easily compute, using the properties of the Fourier transform,

$$\begin{aligned}
\mathcal{A}_1 :&= \int_{\mathbb{R}^n_+} y^{a+2} |\nabla U|^2 \, dxdy = \int_{\mathbb{R}^n_+} y^{a+2} \left(|\nabla_x U|^2 + |\partial_y U|^2 \right) \, dxdy \\
&= \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} \left(|\xi|^2 |\hat{U}|^2 + |\partial_y \hat{U}|^2 \right) \, dyd\xi \\
&= \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} \, |\hat{w}(\xi)|^2 \, |\xi|^2 \left(|\varphi(|\xi| \, y)|^2 + |\varphi'(|\xi| \, y)|^2 \right) \, dyd\xi \\
&= \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{-1-a} \int_0^\infty t^{a+2} \left(|\varphi(t)|^2 + |\varphi'(t)|^2 \right) \, dtd\xi \\
&= d_1 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{-1-a} \, d\xi
\end{aligned} \tag{7.7}$$

for a constant

$$d_1 := \int_0^\infty t^{a+2} \left(|\varphi(t)|^2 + |\varphi'(t)|^2 \right) \, dt. \tag{7.8}$$

Similarly,

$$\begin{aligned}
\mathcal{A}_2 :&= \int_{\mathbb{R}^n_+} y^{a+2} |\nabla_x U|^2 \, dxdy = \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} |\xi|^2 |\hat{U}|^2 \, dyd\xi \\
&= \int_{\mathbb{R}^n} \int_0^\infty y^{a+2} \, |\hat{w}(\xi)|^2 \, |\xi|^2 \, |\varphi(|\xi| \, y)|^2 \, dyd\xi \\
&= \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{-1-a} \int_0^\infty t^{a+2} \, |\varphi(t)|^2 \, dtd\xi \\
&= d_2 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, |\xi|^{-1-a} \, d\xi
\end{aligned}$$

for

$$d_2 := \int_0^\infty t^{a+2} \, |\varphi(t)|^2 \, dt. \tag{7.9}$$

And finally,

$$
\begin{aligned}
\mathcal{A}_3 := \int_{\mathbb{R}^{n+1}_+} y^a U^2\, dxdy &= \int_{\mathbb{R}^n}\int_0^\infty y^a |\hat{U}|^2\, dyd\xi = \int_{\mathbb{R}^n}\int_0^\infty y^a |\hat{w}(\xi)|^2 |\varphi(|\xi|\, y)|^2\, dyd\xi \\
&= \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2\, |\xi|^{-1-a} \int_0^\infty t^a |\varphi(t)|^2\, dtd\xi = d_3 \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2\, |\xi|^{-1-a}\ d\xi,
\end{aligned}
\tag{7.10}
$$

for

$$
d_3 = \int_0^\infty t^a |\varphi(t)|^2\, dt.
$$

In the next step, we find the relation between the constants d_1,d_2,d_3. All the integrals will be evaluated between zero and infinity in the following. Multiply (7.2) by $\varphi_t t^{a+3}$ and integrate by parts:

$$
-\int \varphi\varphi_t t^{a+3} + a\int \varphi_t^2 t^{a+2} + \int \varphi_{tt}\varphi_t t^{a+3} = 0. \tag{7.11}
$$

In the above formula, we estimate the first term by

$$
\int t^{a+3}\varphi\varphi_t = \tfrac{1}{2}\int t^{a+3}\partial_t\left(\varphi^2\right) = -\tfrac{a+3}{2}\int t^{a+2}\varphi^2,
$$

and the last one by

$$
\int t^{a+3}\varphi_{tt}\varphi_t = \tfrac{1}{2}\int t^{a+3}\partial_t\left(\varphi_t^2\right) = -\tfrac{a+3}{2}\int t^{a+2}\varphi_t^2,
$$

so from (7.11) we obtain

$$
(a+3)\int t^{a+2}\varphi^2 = (-a+3)\int t^{a+2}\varphi_t^2.
$$

Together with (7.8) and (7.9) this gives

$$
d_1 = \frac{6}{-a+3} d_2,
$$

as desired.

Now, multiply equation (7.2) by φt^{a+2} and integrate:

$$
-\int t^{a+2}\varphi\varphi_t + a\int t^{a+1}\varphi_t^2 + \int t^{a+2}\varphi_{tt}\varphi = 0. \tag{7.12}
$$

The third term above is computed as

$$
\int t^{a+2}\varphi_{tt}\varphi = -\int t^{a+2}\varphi_t^2 - (a+2)\int t^{a+1}\varphi_t\varphi,
$$

so (7.12) becomes

$$
d_1 = -2\int t^{a+1}\varphi_t\varphi = (a+1)\int t^a\varphi^2 = (a+1)d_3. \tag{7.13}
$$

This completes the proof of the lemma. ∎

In the following, we continue the estimates of the different error terms, although now we only need the asymptotic behavior and not the precise constant.

Lemma 7.3. *Let w be defined on $\mathbb{R}^n$ and $U = K_\gamma *_x w$. Then*

1. *For each $k \in \mathbb{N}$, if $w \in H^{\gamma-k/2}(\mathbb{R}^n)$,*

$$
\mathcal{E}_k := \int_{\mathbb{R}^{n+1}_+} y^{a+k}|\nabla U|^2\, dxdy < \infty. \tag{7.14}
$$

2. If $w \in H^{\gamma-3/2}(\mathbb{R}^n)$ *and* $(|x|w) \in H^{-1/2+\gamma}(\mathbb{R}^n)$*, then*

$$\tilde{\mathcal{E}}_3 := \int_{\mathbb{R}^{n+1}_+} y^a \left|(x,y)\right|^3 |\nabla U|^2 \, dxdy < \infty. \tag{7.15}$$

Proof. Taking into account (7.3), we can proceed as in the calculation for $\mathcal{A}_1$ in (7.7), easily arriving at

$$\mathcal{E}_k = c_k \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 |\xi|^{1-k-a} \, d\xi,$$

where

$$c_k := \int_0^\infty t^{a+k} \left(\varphi^2(t) + \varphi_t^2(t)\right) \, dt < \infty,$$

and this last integral is finite for all $k \in \mathbb{N}$ because of the asymptotics of the Bessel functions from Lemma 7.1. The second conclusion of the lemma is a little more involved. To show that the integral (7.15) is finite, first note that (7.14) with $k = 3$ gives

$$\int_{\mathbb{R}^{n+1}_+} y^{a+3} |\nabla U|^2 \, dxdy < \infty.$$

It is clear that it only remains to prove

$$\int_{\mathbb{R}^{n+1}_+} y^a |x|^3 |\nabla U|^2 \, dxdy < \infty.$$

Since the computation of the previous integral can be made component by component, it is clear that is enough to restrict to the case $n = 1$. Then we just need to show that

$$J := \int_0^\infty \int_{\mathbb{R}} y^a |x|^3 (\partial_x U)^2 \, dxdy < \infty. \tag{7.16}$$

This is an easy but tedious calculation using Fourier transform. Without loss of generality, we will drop all the constants 2π appearing in the Fourier transform. First notice that

$$\begin{aligned}\int_{\mathbb{R}} |x|^3 (\partial_x U)^2 \, dx &= \| \{|x|^{3/2} \partial_x U\} \|^2_{L^2(\mathbb{R})} = \| D_\xi^{3/2} \widehat{\partial_x U} \|^2_{L^2(\mathbb{R})} = \| D^{3/2} (|\xi| \, \hat{U}) \|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |\xi| \, \hat{U} D_\xi^3 (|\xi| \, \hat{U}) \, d\xi.\end{aligned} \tag{7.17}$$

At this point we go back to (7.3) to substitute the explicit expression for $\hat{U}$. We will need to compute

$$\begin{aligned} D_\xi^3 \left(|\xi| \hat{w}(\xi) \varphi(|\xi| y) \right) &= \hat{w}''' \left[|\xi| \varphi \right] + \hat{w}'' \left[3\varphi + 3|\xi| \varphi' y \right] \\ &\quad + \hat{w}' \left[6\varphi' y + 3|\xi| \varphi'' y^2 \right] + \hat{w} \left[|\xi| \varphi''' y^3 + 3\varphi'' y^2 \right] \\ &= \hat{w}''' \left[|\xi| \varphi \right] + \hat{w}'' \left[3\varphi + 3t\varphi' \right] \\ &\quad + \hat{w}' \left[6|\xi|^{-1} t\varphi' + 3|\xi|^{-1} t^2 \varphi'' \right] + \hat{w} \left[|\xi|^{-2} \varphi''' t^3 + 3|\xi|^{-2} t^2 \varphi'' \right], \end{aligned}$$

after the change $|\xi| y = t$. When we substitute the above expression into (7.17) and then back into (7.16), taking into account the change of variables, we obtain:

$$\begin{aligned} J = &\int_0^\infty t^a \varphi^2 \, dt \int_{\mathbb{R}} \hat{w}''' \hat{w} |\xi|^{1-a} \, d\xi \\ &+ \int_0^\infty t^a \left[\varphi^2 + 3t\varphi\varphi' \right] \, dt \int_{\mathbb{R}} \hat{w}'' \hat{w} |\xi|^{-a} \, d\xi \\ &+ \int_0^\infty t^a \left[6t\varphi' \varphi + 3t^2 \varphi'' \varphi \right] dt \int_{\mathbb{R}} \hat{w}' \hat{w} |\xi|^{-a-1} \, d\xi \\ &+ \int_0^\infty t^a \left[t^3 \varphi''' \varphi + 3t^2 \varphi'' \varphi \right] dt \int_{\mathbb{R}} \hat{w}^2 |\xi|^{-a-2} \, d\xi \\ &=: c_1 J_1 + c_2 J_2 + c_3 J_3 + c_4 J_4. \end{aligned}$$

It is clear, looking at the asymptotic behavior of φ from Lemma 7.1 that the constants c_i, $i = 1, 2, 3, 4$, are finite. On the other hand, by an straightforward integration by parts argument, we can write each of the terms J_i, $i = 1, 2, 3, 4$, as a linear combination of just

$$\int_{\mathbb{R}} \hat{w}^2(\xi)|\xi|^{-a-2}\, d\xi \quad \text{and} \quad \int_{\mathbb{R}} \hat{w}'(\xi)^2|\xi|^{-a}\, d\xi. \tag{7.18}$$

Finally, the proof is completed because the initial hypotheses show that both integrals in (7.18) are finite. In particular, these hypothesis show that all the derivations are rigorous. □

Lemma 7.4. *Let w be defined on $\mathbb{R}^n$ and $U = K_\gamma *_x w$.*

1. *For each $k \in \mathbb{N}$, if $w \in H^{\gamma-k/2-1}(\mathbb{R}^n)$,*

$$\mathcal{F}_k := \int_{\mathbb{R}^{n+1}_+} y^{a+k} U^2\, dxdy < \infty. \tag{7.19}$$

2. *If $w \in H^{\gamma-5/2}(\mathbb{R}^n)$ and $(|x|w) \in H^{\gamma-3/2}(\mathbb{R}^n)$,*

$$\tilde{\mathcal{F}}_3 := \int_{\mathbb{R}^{n+1}_+} y^a\, |x|^3\, U^2\, dxdy < \infty. \tag{7.20}$$

Proof. The first assertion (7.19) follows as in (7.10):

$$\begin{aligned}\mathcal{F}_k := &\int_{\mathbb{R}^{n+1}_+} y^{a+k} U^2\, dxdy = \int_{\mathbb{R}^n}\int_0^\infty y^{a+k}|\hat{U}|^2\, dyd\xi = \int_{\mathbb{R}^n}\int_0^\infty y^{a+k}|\hat{w}(\xi)|^2|\varphi(|\xi|\, y)|^2\, dyd\xi \\ &= \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2\, |\xi|^{-1-a-k} \int_0^\infty |\varphi(t)|^2 t^{a+k}\, dtd\xi = c_k \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2\, |\xi|^{-1-a-k}\, d\xi,\end{aligned}$$

for

$$c_k := \int_0^\infty |\varphi(t)|^2 t^{a+k}\, dt < \infty.$$

For the second assertion, under the light of our previous discussions, it is enough to show that in the one-dimensional case,

$$\int_{\mathbb{R}} |x|^3\, U^2\, dx = \|\{|x|^{3/2}U\}\|^2_{L^2(\mathbb{R})} = \|D^{3/2}\widehat{U}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \hat{U} D^3_\xi(\hat{U})\, d\xi.$$

Substitute the expression for $\hat{U}$ from (7.3). Then

$$\int_{\mathbb{R}} |x|^3\, U^2\, dx = \int \hat{w}'''\hat{w}\varphi^2\, d\xi + 3\int \hat{w}''\hat{w}\varphi'\varphi y\, d\xi + 3\int \hat{w}'\hat{w}\varphi'\varphi y^2\, d\xi + \int \hat{w}^2\varphi'''\varphi y^3\, d\xi,$$

so when we change variables $t = |\xi|\, y$,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} y^a\, |x|^3\, U^2\, dxdy &= \int_0^\infty t^a\varphi^2\, dt \int_{\mathbb{R}} \hat{w}'''\hat{w}\, |\xi|^{-1-a}\, d\xi \\ &+ 3\int_0^\infty t^{1+a}\varphi'\varphi\, dt \int_{\mathbb{R}} \hat{w}''\hat{w}\, |\xi|^{-2-a}\, d\xi \\ &+ 3\int_0^\infty t^{2+a}\varphi''\varphi\, dt \int_{\mathbb{R}} \hat{w}'\hat{w}\, |\xi|^{-3-a}\, d\xi \\ &+ \int_0^\infty t^{3+a}\varphi'''\varphi dt \int_{\mathbb{R}} \hat{w}^2\, |\xi|^{-4-a}\, d\xi \\ &= \tilde{c}_1\tilde{J}_1 + \tilde{c}_2\tilde{J}_2 + \tilde{c}_3\tilde{J}_3 + \tilde{c}_4\tilde{J}_4.\end{aligned}$$

Clearly, from the asymptotics of the Bessel functions from Lemma 7.1, the constants $\tilde{c}_i$, $i = 1, 2, 3, 4$ are finite. At the same time, each of the four integrals $\tilde{J}_i$, $i = 1, 2, 3, 4$, can be written as a linear combination of two:

$$\int (\hat{w}')^2\, |\xi|^{-2-a}\, d\xi \quad \text{and} \quad \int (\hat{w})^2\, |\xi|^{-4-a}\, d\xi,$$

which are finite because of the hypothesis on w. □

Next, we check what happens with the previous two lemmas under rescaling. Here $f = o(1)$ means

$$\lim_{\epsilon/\mu \to 0} f = 0.$$

Given any function w defined on $\mathbb{R}^n$, we consider its extension to $\mathbb{R}^{n+1}_+$ as $U = K_\gamma *_x w$, and the rescaling, for each $\mu > 0$,

$$U_\mu(x, y) := \frac{1}{\mu^{\frac{n-2\gamma}{2}}} U\left(\frac{x}{\mu}, \frac{y}{\mu}\right). \tag{7.21}$$

Corollary 7.5. *Fix $\epsilon, \mu > 0$ and let the hypotheses be as in Lemma 7.3 (in each of the two cases).*

1. *For each $k \in \mathbb{N}$,*

$$\int_{B_\epsilon^+} y^{a+k} |\nabla U_\mu|^2 dxdy = \mu^k \int_{B^+_{\epsilon/\mu}} y^{a+k} |\nabla U|^2 dxdy = \mu^k \left[\mathcal{E}_k + o(1)\right] \tag{7.22}$$

2. *Also*

$$\int_{B_\epsilon^+} y^a |(x, y)|^3 |\nabla U_\mu|^2 \, dxdy = \mu^3 \int_{B^+_{\epsilon/\mu}} y^{a+k} |\nabla U|^2 \, dxdy = \mu^3 \left[\tilde{\mathcal{E}}_3 + o(1)\right], \tag{7.23}$$

where U_μ is the rescaling (7.21), *and $\mathcal{E}_k, \tilde{\mathcal{E}}_3 < \infty$ are defined as in Lemma 7.3.*

Corollary 7.6. *Fix $\epsilon, \mu > 0$ and let the hypotheses be as in Lemma 7.4 (in each of the two cases).*

1. *For each $k \in \mathbb{N}$,*

$$\int_{B_\epsilon^+} y^{a+k} (U_\mu)^2 \, dxdy = \mu^{k+2} \int_{B^+_{\epsilon/\mu}} y^{a+k} U^2 \, dxdy = \mu^{k+2} \left[\mathcal{F}_k + o(1)\right], \tag{7.24}$$

2. *Also,*

$$\int_{B_\epsilon^+} y^a \, |(x, y)|^3 \, (U_\mu)^2 \, dxdy = \mu^5 \int_{B^+_{\epsilon/\mu}} y^a \, |x|^3 \, U^2 \, dxdy = \mu^5 \left[\tilde{\mathcal{F}}_3 + o(1)\right], \tag{7.25}$$

where U_μ is the rescaling (7.21), *and $\mathcal{F}_k, \tilde{\mathcal{F}}_3 < \infty$ are defined as in Lemma 7.4.*

7.2 Proof of Theorem 1.5

We first need to choose a very particular background metric for X near a non-umbilic point on M. We will follow the steps as Escobar did in Lemmas 3.1 - 3.3 of [16]. But our situation is a little different. Our freedom of choice of metrics is restricted to the boundary. Hence we will make some assumptions on the behavior of the asymptotically hyperbolic manifolds in order to allow us to see clearly what we can get for a good choice of representative from the conformal infinity.

Lemma 7.7. *Suppose that $(X^{n+1}, \ g^+)$ is an asymptotically hyperbolic manifold and ρ is a geodesic defining function associated with a representative $\hat{h}$ of the conformal infinity $(M^n, [\hat{h}])$. Assume that*

$$\rho^{-2} \big(R[g^+] - Ric[g^+](\rho\partial_\rho) + n^2\big) \to 0 \quad \text{as } \rho \to 0. \tag{7.26}$$

Then, at $\rho = 0$,

$$H := Tr_{\hat{h}} h^{(1)} = 0 \tag{7.27}$$

and

$$Tr_{\hat{h}} h^{(2)} = \frac{1}{2}(\|h^{(1)}\|^2_{\hat{h}} + \frac{1}{2(n-1)} R[\hat{h}]), \tag{7.28}$$

where

$$g^+ = \frac{d\rho^2 + h_\rho}{\rho^2}, \quad h_\rho = \hat{h} + h^{(1)}\rho + h^{(2)}\rho^2 + o(\rho^2).$$

Proof. This simply follows from the calculations in [25]. Recall (2.5) from [25]

$$\begin{aligned}\rho h''_{ij} + (1-n)h'_{ij} - h^{kl}h'_{kl}h_{ij} - \rho h^{kl}h'_{ik}h'_{jl} + \frac{1}{2}\rho h^{kl}h'_{kl}h'_{ij} - 2\rho R_{ij}[\hat{h}] \\ = \rho(R_{ij}[g^+] + ng^+_{ij}),\end{aligned} \tag{7.29}$$

where we use h to stand for h_ρ for simplicity. Taking its trace with respect to the metrics h, we have

$$\begin{aligned}\rho \mathrm{Tr}_h h'' + (1-2n)\mathrm{Tr}_h h' - \rho\|h'\|^2_{\hat{h}} + \frac{1}{2}\rho(\mathrm{Tr}_h h')^2 - 2\rho R[\hat{h}] \\ = \rho^{-1}(R[g^+] - Ric[g^+](x\partial_x) + n^2)\end{aligned} \tag{7.30}$$

Immediately from (7.26) we see that

$$\mathrm{Tr}_h h' = 0 \quad \text{at } \rho = 0.$$

Then, dividing ρ in both sides of the equation (7.30) and taking $\rho \to 0$, we have (7.28), under the assumption (7.26), because

$$(\mathrm{Tr}_h h')' = \mathrm{Tr}_{\hat{h}} h'' - \|h'\|^2_{\hat{h}}$$

at $\rho = 0$. □

Notice that (7.26) is an intrinsic curvature condition of an asymptotically hyperbolic manifold, which is independent of the choice of geodesic defining functions. Consequently we have the following.

Lemma 7.8. *Suppose that* $(X^{n+1},\ g^+)$ *is an asymptotically hyperbolic manifold and* (7.26) *holds. Then, given a point* p *on the boundary* M*, there exists a representative* $\hat{h}$ *of the conformal infinity such that,*

i. $H =: Tr_{\hat{h}} h^{(1)} = 0$ *on* M*,*

ii. $Ric[\hat{h}](p) = 0$ *on* M*,*

iii $Ric[\bar{g}](\partial_\rho)(p) = 0$ *on* M*,*

iv. $R[\bar{g}](p) = \|h^{(1)}\|^2_{\hat{h}}$ *on* M*.*

Proof. The proof, like the proof of Lemma 3.3 in [16], uses Theorem 5.2 in [30]. Therefore we may choose a representative of the conformal infinity whose Ricci curvature vanishes at any given point $p \in M$. In the light of Lemma 7.7 we get *i.* and *ii.* right away. We then calculate

$$Ric[\bar{g}](\partial_x) = -\frac{1}{2}\mathrm{Tr}_{\hat{h}} h^{(2)} + \frac{1}{4}\|h^{(1)}\|^2_{\hat{h}} = 0$$

at $p \in M$ from (7.28). Finally we recall that

$$R[\bar{g}] = 2Ric[\bar{g}](\partial_\rho) + R[\hat{h}] + \|h^{(1)}\|^2_{\hat{h}} - (\mathrm{Tr}_{\hat{h}} h^{(1)})^2 = \|h^{(1)}\|^2_{\hat{h}}.$$

The proof is complete. □

Assume that $0 \in M = \partial\bar{X}$ is a non-umbilic point. Choose normal coordinates $x_1, \ldots, x_n$ around 0 on M and let $(x_1, \ldots, x_n, \rho)$ be the Fermi coordinates on X around 0. In particular, we can write

$$g^+ = \rho^{-2}(d\rho^2 + h_{ij}(x,\rho)dx_i dx_j), \quad \bar{g} = d\rho^2 + h_{ij}(x,\rho)dx_i dx_j.$$

In order to simplify the later notation, we denote the coordinate ρ by y. The only risk of confusion comes from the fact that we have previously used y for the special defining function ρ^* from Proposition 2.2, but we will not need it any longer. In the new notation we have

$$\bar{g} = dy^2 + h_{ij}(x,y)dx_i dx_j$$

for some functions $h_{ij}(x,y)$, $i,j = 1,\ldots,n$. From what we have in the above two lemmas we get from Lemma 3.1 and 3.2 of [16] the following.

Lemma 7.9. *Suppose that* $(X^{n+1},\ g^+)$ *is an asymptotically hyperbolic manifold satisfying* (7.26). *Given a non-umbilic point* p *on the boundary* M, *i.e.* $\|h^{(1)}\|_{\hat{h}}(p) \neq 0$ *for* $p \in M$, *where* $\hat{h}$ *is chosen as in Lemma 7.8. Then:*

1. $\sqrt{|\bar{g}|} = 1 - \frac{1}{2}\|\pi\|^2 y^2 + O(|(x,y)|^3)$.
2. $\bar{g}^{ij} = \delta_{ij} + 2\pi^{ij}y - \frac{1}{3}R^i{}_{kl}{}^j[\hat{h}]\,x_k x_l + \bar{g}^{ij}{}_{,ym}yx_m + \left(3\pi^{im}\pi_m{}^j + R^i{}_{y}{}^j{}_y[\bar{g}]\right)y^2 + O(|(x,y)|^3)$,

where, for simplicity, we set $\pi = h^{(1)}$.

As in Proposition 5.4, we try to find a good test function for the Sobolev quotient given by

$$I^*_\gamma[U,\bar{g}] = \frac{d^*_\gamma \int_X y^a |\nabla U|^2_{\bar{g}}\,dv_{\bar{g}} + \int_X E(y)U^2\,dv_{\bar{g}}}{\left(\int_M |U|^{2^*}\,dv_{\hat{h}}\right)^{\frac{2}{2^*}}},$$

where $E(y)$ is given by (2.8), with respect to the metric $\bar{g}$:

$$E(y) = \frac{n-1-a}{4n}\left[R[\bar{g}] - (n(n+1) + R[g^+])y^{-2}\right]y^a. \tag{7.31}$$

We need to perform a careful computation of the lower order terms in order to find an estimate for $\Lambda_\gamma(M,[\hat{h}])$. For simplicity, we introduce the following notation: for a subset $\Omega \subset \mathbb{R}^{n+1}_+$, we consider the energy functional restricted to Ω given by

$$\mathcal{K}(U,\Omega) := d^*_\gamma \int_\Omega y^a |\nabla U|^2_{\bar{g}}\,dv_{\bar{g}} + \int_\Omega E(y)U^2\,dv_{\bar{g}}$$

Given any $\epsilon > 0$, let B_ϵ be the ball of radius ϵ centered at the origin in $\mathbb{R}^{n+1}$ and B^+_ϵ be the half ball of radius ϵ in $\mathbb{R}^{n+1}_+$. Choose a smooth radial cutoff function η, $0 \le \eta \le 1$, supported on $B_{2\epsilon}$, and satisfying $\eta = 1$ on B_ϵ. We recall here the conformal diffeomorphisms of the sphere w_μ given in (5.8) and their extension U_μ as in (5.9). Our test function is simply

$$V_\mu := \eta U_\mu.$$

Step 1: Computation of the energy in B^+_ϵ.

It is clear that in the half ball B^+_ϵ, $V_\mu = U_\mu$, so that $\mathcal{K}(V_\mu, B^+_\epsilon) = \mathcal{K}(U_\mu, B^+_\epsilon)$. We compute the first term in the energy $\mathcal{K}(U_\mu, B^+_\epsilon)$. Using the asymptotics for $\bar{g}$ from Lemma 7.9 (here the indexes i,j run from 1 to n),

$$\begin{aligned}
\int_{B^+_\epsilon} y^a |\nabla U_\mu|^2_{\bar{g}}\,dv_{\bar{g}} &= \int_{B^+_\epsilon} y^a\left[\bar{g}^{ij}(\partial_i U_\mu)(\partial_j U_\mu) + (\partial_y U_\mu)^2\right]dv_{\bar{g}} \\
&= \int_{B^+_\epsilon} y^a |\nabla U_\mu|^2\,dv_{\bar{g}} \\
&+ 2\pi^{ij}\int_{B^+_\epsilon} y^{a+1}(\partial_i U_\mu)(\partial_j U_\mu)\,dv_{\bar{g}} \\
&+ \int_{B^+_\epsilon} y^{a+2}\left(3\pi^{im}\pi_m{}^j + R^i{}_{y}{}^j{}_y[\bar{g}]\right)(\partial_i U_\mu)(\partial_j U_\mu)\,dv_{\bar{g}} \\
&+ \int_{B^+_\epsilon} y^{a+1}\bar{g}^{ij}{}_{,tk}x_k(\partial_i U_\mu)(\partial_j U_\mu)\,dv_{\bar{g}} \\
&- \frac{1}{3}\int_{B^+_\epsilon} y^a R^i_{kl}{}^j[\bar{g}]x_k x_l(\partial_i U_\mu)(\partial_j U_\mu)\,dv_{\bar{g}} \\
&+ c\int_{B^+_\epsilon} y^a|(x,y)|^3|\nabla U_\mu|^2\,dv_{\bar{g}} \\
&=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned} \tag{7.32}$$

We estimate the first integral J_1 in the right hand side of (7.32), using the estimate for the volume element $\sqrt{|\bar g|}$ from Lemma 7.9:

$$
\begin{aligned}
J_1 &= \int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dv_{\bar g} \\
&\le \int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dxdy - \tfrac{1}{2}\left\|\pi\right\|^2 \int_{B_\epsilon^+} y^{2+a}\left|\nabla U_\mu\right|^2 \, dxdy \\
&+ c\int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \left|(x,y)\right|^3 \, dxdy \\
&\le \int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dxdy - \tfrac{1}{2}\left\|\pi\right\|^2 \mu^2 \mathcal{A}_1 + \mu^2 o(1) + c\mu^3\left[\tilde{\mathcal{E}}_3 + o(1)\right],
\end{aligned}
\tag{7.33}
$$

if we take into account the notation from (7.4) and Corollary 7.5.

Now we look closely at the equation for U_μ. Multiply expression (5.11) by U_μ and integrate by parts:

$$
\int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dxdy = c_{n,\gamma}\int_{\Gamma_\epsilon^0} w_\mu^{2^*}\, dx + \int_{\Gamma_\epsilon^+} U_\mu\left(\partial_\nu U_\mu\right) d\sigma \le c_{n,\gamma}\int_{\Gamma_\epsilon^0} w_\mu^{2^*}\, dx, \tag{7.34}
$$

where ν is the exterior normal to B_ϵ^+. Here we have used the properties of the convolution with a radially symmetric, nonincreasing kernel K_γ. More precisely, since w_μ is radially symmetric and non-increasing, $U_\mu = K_\gamma *_x w_\mu$ also satisfies $\partial_\nu U_\mu \le 0$ on Γ_ϵ^+ (c.f. [8], Lemma 2.3, for instance).

From (7.34), using (5.13), we arrive at

$$
\int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dxdy \le \Lambda(S^m,[g_c])(d_\gamma^*)^{-1}\left[\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dx\right]^{\frac{n-2\gamma}{n}}. \tag{7.35}
$$

For simplicity, we set $\Lambda_1 := \Lambda(S^m,[g_c])(d_\gamma^*)^{-1}$. Equations (7.33) and (7.35) tell us that

$$
J_1 = \int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dv_{\bar g} \le \Lambda_1\left[\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dx\right]^{\frac{2}{2^*}} - \tfrac{1}{2}\left\|\pi\right\|^2\mu^2\mathcal{A}_1 + \mu^2 o(1) + c\mu^3. \tag{7.36}
$$

On the other hand, the asymptotics for the metric $\hat h = \bar g|_{y=0}$ near the origin are explicit. Indeed, from Lemma 7.8 we know that

$$
\sqrt{|\hat h|} = 1 + O(|x|^3). \tag{7.37}
$$

Moreover, we can compute from (5.10)

$$
\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\left|x\right|^3 \, dx = \mu^3 \int_{\Gamma_{\epsilon/\mu}^0} (w_1)^{2^*}\left|x\right|^3 \, dx \le c\mu^3.
$$

Consequently, from (7.37) we are able to relate the integrals in $dv_{\hat h}$ and dx:

$$
\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dx \le \int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dv_{\hat h} + c\mu^3.
$$

And substituting the above expression into (7.36) we get

$$
J_1 = \int_{B_\epsilon^+} y^a \left|\nabla U_\mu\right|^2 \, dv_{\bar g} \le \Lambda_1\left[\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dv_{\hat h}\right]^{\frac{2}{2^*}} - \tfrac{1}{2}\left\|\pi\right\|^2\mu^2\mathcal{A}_1 + \mu^2 o(1) + c\mu^3.
$$

Now we go back to (7.32), and try to estimate the second term J_2 in the right hand side. If we again use the asymptotics of the metric $\bar{g}$ given in Lemma 7.9, then

$$\int_{B_\epsilon^+} y^{a+1} \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dv_{\bar{g}} \leq \int_{B_\epsilon^+} y^{a+1} \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dxdy + \mathcal{B}, \tag{7.38}$$

for

$$\mathcal{B} \leq c \int_{B_\epsilon^+} y^{a+3} \left|\nabla U_\mu\right|^2 dxdy + c \int_{B_\epsilon^+} y^{a+1} |\nabla U_\mu|^2 \left|(x,y)\right|^3 \, dxdy.$$

We notice here that $\mathcal{B}$ can be easily estimated from Corollary 7.5:

$$\mathcal{B} \leq c\mu^3(\mathcal{E}_3 + o(1)) + c\mu^3\epsilon \left(\tilde{\mathcal{E}}_3 + o(1)\right) \leq c\mu^3 + \mu^3 o(1). \tag{7.39}$$

Let us look at the cross terms $(\partial_i U_\mu)(\partial_j U_\mu)$, $1 \leq i,j \leq n$ in (7.38). We note that $\partial_i U_\mu = K_\gamma *_x (\partial_i w_\mu)$, just by taking the derivatives in the convolution. This last derivative can be explicitly written, and in particular, $\partial_i w_\mu$ is an odd function in the variable x_i. By the properties of the convolution, we know that $\partial_i U_\mu$ is also an odd function in the variable x_i. Then, using the symmetries of the half ball, the integral $\int_{B_\epsilon^+} y^{a+1}(\partial_i U_\mu)(\partial_j U_\mu)\, dxdy$ is zero if $i \neq j$. If $i = j$, we use that the mean curvature at the point vanishes, i.e., $\pi_i^i = 0$ by Lemma 7.8. Then, when we substitute formula (7.38) in the expression for J_2, only the error term remains, and by (7.39) we conclude that

$$J_2 = 2\pi^{ij} \int_{B_\epsilon^+} y^{a+1} \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dv_{\bar{g}} \leq \mathcal{B} \leq \mu^3(c + o(1)). \tag{7.40}$$

Now we estimate the next term in (7.32), J_3. Again using the asymptotics for the volume element $dv_{\bar{g}}$ from Lemma 7.9, we have that

$$\int_{B_\epsilon^+} y^{a+2} \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dv_{\bar{g}} \leq \int_{B_\epsilon^+} y^{a+2} \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dxdy + \mathcal{B}', \tag{7.41}$$

for

$$\begin{aligned} \mathcal{B}' &\leq c \int_{B_\epsilon^+} y^{a+4} \left|\nabla U_\mu\right|^2 dxdy + c \int_{B_\epsilon^+} y^{a+2} \left|(x,y)\right|^3 \left|\nabla U_\mu\right|^2 dxdy \\ &\leq \mu^4(\mathcal{E}_4 + o(1)) + \mu^3\epsilon^2(\tilde{\mathcal{E}}_3 + o(1)) \leq c\mu^3, \end{aligned}$$

where the last estimate follows thanks to Corollary 7.5 again.

Notice again that, for $i \neq j$ the first integral in the right hand side of (7.41) vanishes - thanks to the symmetries of the half ball and the discussion above on the oddness of the derivatives of U_μ. Then, we recall the definition of $\mathcal{A}_2$ from (7.5) and the estimate (7.22). When we put all these ingredients together:

$$\begin{aligned} J_3 &= \left(3\pi^{im}\pi_m{}^j + R^i{}_y{}^j{}_y[\bar{g}]\right) \int_{B_\epsilon^+} y^{a+2} \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dv_{\bar{g}} \\ &= \frac{1}{n}\left[3\left\|\pi\right\|^2 + Ric(\nu)\right] \mu^2 \mathcal{A}_2 + c\mu^3 \\ &= \frac{3}{n}\left\|\pi\right\|^2 \mu^2 \mathcal{A}_2 + \mu^2 o(1) + c\mu^3. \end{aligned}$$

if we take into account that $Ric(\nu)(0)[\hat{h}] = 0$ because of Lemma 7.8.

Next, the calculation for J_4 is very similar to the previous one. Indeed,

$$\int_{B_\epsilon^+} y^{a+1} x_k \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dv_{\bar{g}} \leq \int_{B_\epsilon^+} y^{a+1} x_k \left(\partial_i U_\mu\right) \left(\partial_j U_\mu\right) \, dxdy + \mathcal{B}'',$$

and because of symmetries on the unit ball, the first integral in the right hand side above vanishes for all i,j,k, while $\mathcal{B}'' \leq c\mu^3$. Thus

$$J_4 = \bar{g}^{ij}{}_{,tk} \int_{B_\epsilon^+} y^{a+1} x_k \left(\partial_i V_\mu\right) \left(\partial_j V_\mu\right) \, dv_{\bar{g}} \leq c\mu^3.$$

And finally J_5, J_6 can be estimated in a similar manner.

Putting all the estimates together for the J_j, $j = 1, \ldots, 6$, we have shown that (7.32) reduces to

$$\int_{B_\epsilon^+} y^a |\nabla U_\mu|^2_{\bar{g}}\, dv_{\bar{g}} \leq \Lambda_1 \left[\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dv_{\hat{h}} \right]^{\frac{2}{2^*}} + \left[-\tfrac{1}{2}\mathcal{A}_1 + \tfrac{3}{n}\mathcal{A}_2 \right] \|\pi\|^2 \mu^2 + \mu^2 o(1) + c\mu^3. \quad (7.42)$$

Finally, we are able to complete the computation of the energy $\mathcal{K}(U_\mu, B_\epsilon^+)$. Note that in the half ball B_ϵ^+, we have a very precise behavior for the lower order term (7.31). In particular, Lemma 7.8 gives that $R[\bar{g}](p) = \|\pi\|^2$, so

$$E(y) = \frac{n-1+a}{4n} \|\pi\|^2 y^a + O(y^{1+a}). \quad (7.43)$$

Then, again using the asymptotics for the volume element $dv_{\bar{g}}$,

$$\int_{B_\epsilon^+} E(y)(U_\mu)^2\, dv_{\bar{g}} = \frac{n-1+a}{4n} \|\pi\|^2 \int_{B_\epsilon^+} y^a (U_\mu)^2\, dxdy + \mathcal{B}''', \quad (7.44)$$

where

$$\mathcal{B}''' \leq c \int_{B_\epsilon^+} y^{a+1} (U_\mu)^2\, dxdy + c \int_{B_\epsilon^+} y^a |x|^3 (U_\mu)^3\, dxdy$$

can be estimated from Corollary 7.6 as

$$\mathcal{B}''' \leq c\mu^3 + o(1). \quad (7.45)$$

Summarizing, from (7.44) and (7.45), and using the scaling properties of U_μ as given in (5.10), we have

$$\begin{aligned} \int_{B_\epsilon^+} E(y)(U_\mu)^2\, dv_{\bar{g}} &\leq \tfrac{n-1+a}{4n} \|\pi\|^2 \mu^2 \int_{B_{\epsilon/\mu}^+} y^a (U_1)^2\, dxdy + c\mu^3 \\ &= \tfrac{n-1+a}{4n} \|\pi\|^2 \mu^2 \mathcal{A}_3 + c\mu^2 o(1) + c\mu^3, \end{aligned} \quad (7.46)$$

where for the last inequality we have used Corollary 7.6 and the definition of $\mathcal{A}_3$ from (7.6).

The energy of V_μ in the half ball B_ϵ^+ is computed from (7.42) and (7.46), noting that $\Lambda_1 = \Lambda(S^n, [g_c]) d_\gamma^*$, and the relation between $\mathcal{A}_1$,$\mathcal{A}_2$,$\mathcal{A}_3$ from Lemma 7.2:

$$\begin{aligned} &\mathcal{K}(V_\mu, B_\epsilon^+) = d_\gamma^* \int_{B_\epsilon^+} y^a |\nabla U_\mu|^2\, dv_{\bar{g}} + \int_{B_\epsilon^+} E(y)(U_\mu)^2\, dv_{\bar{g}} \\ &\leq \Lambda(S^n, [g_c]) \left[\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*} dv_{\hat{h}} \right]^{\frac{2}{2^*}} + \left[d_\gamma^* \left(-\tfrac{1}{2}\mathcal{A}_1 + \tfrac{3}{n}\mathcal{A}_2 \right) + \tfrac{n-1+a}{4n}\mathcal{A}_3 \right] \|\pi\|^2 \mu^2 + \mu^2 o(1) + c\mu^3 \\ &\leq \Lambda(S^n, [g_c]) \left[\int_{\Gamma_\epsilon^0} (w_\mu)^{2^*}\, dv_{\hat{h}} \right]^{\frac{2}{2^*}} + \theta_{n,\gamma} \|\pi\|^2 \mu^2 \int_{\mathbb{R}^n} |\xi|^{2(\gamma-1)} |\hat{w}_1(\xi)|^2\, d\xi + \mu^2 o(1) + c\mu^3 \end{aligned}$$

for

$$\theta_{n,\gamma} = \frac{1}{4n} \left[-\frac{n+a-3}{1-a} 2^{2\gamma+1} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} + \frac{n-1+a}{a+1} \right] d_1. \quad (7.47)$$

Finally, we note that the $w_1 \in H^\gamma(\mathbb{R}^n)$ and $(|x|w) \in H^\gamma(\mathbb{R}^n)$, so that all our computations are well justified.

Step 2: Computation of the energy in the half-annulus $B_{2\epsilon}^+ \backslash B_\epsilon^+$.

In order to compute $\mathcal{K}(V_\mu, B_{2\epsilon}^+ \backslash B_\epsilon^+)$, note that

$$|\nabla V_\mu|^2_{\bar{g}} \leq c|\nabla V_\mu|^2 \leq c\left(\eta^2 |\nabla U_\mu|^2 + (U_\mu)^2 |\nabla \eta|^2 \right)$$

so that, because of the structure of the cutoff function η,

$$|\nabla V_\mu|_{\bar g}^2 \le c|\nabla U_\mu|^2 + \frac{c}{\epsilon}(U_\mu)^2. \tag{7.48}$$

Moreover,

$$\int_{B_{2\epsilon}^+\setminus B_\epsilon^+} y^a\,(U_\mu)^2\,dxdy \le \mu^2\int_{B_{2\epsilon/\mu}^+\setminus B_{\epsilon/\mu}^+} y^a (U_1)^2\,dxdy = \mu^2 o(1), \tag{7.49}$$

because the integral $\int_{\mathbb{R}^n} y^a(U_1)^2\,dxdy$ is finite and $\epsilon/\mu\to\infty$. On the other hand, we know that

$$\left(\frac{\epsilon}{\mu}\right)^3\int_{B_{2\epsilon/\mu}^+\setminus B_{\epsilon/\mu}^+} y^a|\nabla U_1|^2\,dxdy \le \int_{B_{2\epsilon/\mu}^+\setminus B_{\epsilon/\mu}^+} y^a\,|(x,y)|^3\,|\nabla U_1|^2\,dxdy \le \tilde{\mathcal{E}}_3 < \infty$$

because of Lemma 7.4. As a consequence,

$$\int_{B_{2\epsilon}^+\setminus B_\epsilon^+} y^a|\nabla U_\mu|^2\,dxdy = \int_{B_{2\epsilon/\mu}^+\setminus B_{\epsilon/\mu}^+} y^a|\nabla U_1|^2\,dxdy \le \left(\frac{\mu}{\epsilon}\right)^3\tilde{\mathcal{E}}_3. \tag{7.50}$$

If we put together formulas (7.48), (7.49) and (7.50) we arrive at

$$\mathcal{K}(V_\mu, B_{2\epsilon}^+\setminus B_\epsilon^+) = \int_{B_{2\epsilon}^+\setminus B_\epsilon^+} y^a|\nabla U_\mu|^2\,dxdy + \int_{B_{2\epsilon}^+\setminus B_\epsilon^+} E(y)(U_\mu)^2\,dxdy \le \mu^2 o(1)$$

when $\mu/\epsilon\to 0$.

Step 3: Completion of the proof.

We have very carefully computed

$$\begin{aligned}\mathcal{K}(V_\mu, X) &= d_\gamma^*\int_X y^a\,|\nabla V_\mu|^2\,dvol_{\bar g} + \int_X E(y)(V_\mu)^2\,dvol_{\bar g}\\ &\le \Lambda(S^n,[g_c])\left[\int_{\Gamma_\epsilon^0}(w_\mu)^{2^*}\,dv_{\hat h}\right]^{\frac{2}{2^*}} + \theta_{n,\gamma}\,\|\pi\|^2\,\mu^2\int_{\mathbb{R}^n}|\hat w_1(\xi)|^2|\xi|^{2(\gamma-1)}\,d\xi + \mu^2 o(1) + c\mu^3,\end{aligned}$$

where $\theta_{n,\gamma}$ is given in (7.47).

If there is a non-umbilic point, $\|\pi\|^2\neq 0$ at that point. In the case that $\theta_{n,\gamma}<0$, we are done, because fixing ϵ small and then choosing μ much smaller, then

$$\mathcal{K}(V_\mu, X) < \Lambda(S^n,[g_c])\left[\int_M (w_\mu)^{2^*}\,dv_{\hat h}\right]^{\frac{2}{2^*}},$$

as desired. □

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References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] L. Andersson, P. T. Chruściel, and H. Friedrich. On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations. *Comm. Math. Phys.*, 149(3):587–612, 1992.

[3] T. Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

[4] B. Barrios, E. Colorado, A. De Pablo, U. Sánchez. On some critical problems for the fractional Laplacian operator. *Journal of Differential Equations*, 252 (11): 6133-6162, 2012.

[5] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.

[6] T. P. Branson. Sharp inequalities, the functional determinant, and the complementary series. *Trans. Amer. Math. Soc.*, 347(10):3671–3742, 1995.

[7] X. Cabré and E. Cinti. Sharp energy estimates for nonlinear fractional diffusion equations. Calculus of Variations and Partial Differential Equations, online 2012.

[8] X. Cabré and J.-M. Roquejoffre. The influence of fractional diffusion in Fisher-KPP equations. Comm. Math. Phys. 320 (2013), no. 3, 679722.

[9] X. Cabré and Y. Sire. Non-linear equations for fractional Laplacians I: regularity, maximum principles and Hamiltoniam estimates. To appear in Transactions of AMS.

[10] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[11] S.-Y. A. Chang and M. d. M. González. Fractional Laplacian in conformal geometry. *Adv. Math.*, doi:10.1016/j.aim.2010.07.016, 2010.

[12] F. Chiarenza and M. Frasca A note on a weighted Sobolev inequality. *Proceedings of the AMS* 93, no 4 (1985), 703–704.

[13] A. Cotsiolis and N. K. Tavoularis. Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.*, 295(1):225–236, 2004.

[14] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Preprint.

[15] J. F. Escobar. Sharp constant in a Sobolev trace inequality. *Indiana Univ. Math. J.*, 37(3):687–698, 1988.

[16] J. F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.

[17] E. Fabes, D. Jerison, and C. Kenig. The Wiener test for degenerate elliptic equations. *Ann. Inst. Fourier (Grenoble)*, 32(3):vi, 151–182, 1982.

[18] E. B. Fabes, C. E. Kenig, and R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[19] C. Fefferman and C. R. Graham. The ambient metric. Preprint. arXiv:0710.0919.

[20] C. Fefferman and C. R. Graham. Q-curvature and Poincaré metrics. *Math. Res. Lett.*, 9(2-3):139–151, 2002.

[21] R. Frank and E. Lieb. A new, rearrangement-free proof of the sharp Hardy-Littlewood-Sobolev inequality. Preprint, arXiv:1010.582, 2010.

[22] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[23] M. d. M. González. Gamma convergence of an energy functional related to the fractional Laplacian. *Calc. Var. Partial Differential Equations*, 36(2):173–210, 2009.

[24] M. d. M. González and R. Monneau. Motion of particle systems as a limit of a reaction-diffusion equation with fractional laplacian. To appear in Discrete and Continuous Dynamical Systems - A.

[25] C. R. Graham. Volume and Area Renormalizations for Conformally Compact Einstein Metrics. *math.DG: arXiv:9909042*, The Proceedings of the 19th Winter School "Geometry and Physics" (Srn, 1999). *Rend. Circ. Mat. Palermo* (2) Suppl. No. 63 (2000), 31–42.

[26] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling. Conformally invariant powers of the Laplacian. I. Existence. *J. London Math. Soc. (2)*, 46(3):557–565, 1992.

[27] C. R. Graham and M. Zworski. Scattering matrix in conformal geometry. *Invent. Math.*, 152(1):89–118, 2003.

[28] C. Guillarmou and J. Qing. Spectral characterization of Poincaré-Einstein manifolds with infinity of positive Yamabe type. *Int. Math. Res. Not. IMRN*, (9):1720–1740, 2010.

[29] T. Jin and J. Xiong, Sharp constant in weighted trace inequalities on Riemannian manifolds, *To appear in Var. Calc and PDE.*

[30] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[31] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)*, 118(2):349–374, 1983.

[32] V. G. Maz'ja. *Sobolev spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[33] R. Mazzeo. Regularity for the singular Yamabe problem. *Indiana Univ. Math. J.*, 40(4):1277–1299, 1991.

[34] R. R. Mazzeo and R. B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987.

[35] C. Morpurgo. Sharp inequalities for functional integrals and traces of conformally invariant operators. *Duke Math. J.*, 114(3):477–553, 2002.

[36] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[37] A. Nekvinda. Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d_M^\epsilon)$ on M. *Czechoslovak Math. J.*, 43(118)(4):695–711, 1993.

[38] S. M. Paneitz. A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary). *SIGMA Symmetry Integrability Geom. Methods Appl.*, 4:Paper 036, 3, 2008.

[39] J. Qing and D. Raske. On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds. *Int. Math. Res. Not.*, pages Art. ID 94172, 20, 2006.

[40] R. Schoen and S.-T. Yau. *Lectures on differential geometry.* Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, 1994.

[41] R. Servadei. The Yamabe equaiton in a non-local setting. Preprint, 2012.