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by

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Abstract

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In this dissertation we investigate two questions in the subject of algorithmic randomness. The first question we address is "Given a real, is there a probability measure for which the real is not an atom, but relative to which the real is algorithmically random?" This question was originally studied by Reimann and Slaman with respect to Martin-Löf randomness, and this research continues their investigation by considering the question with respect to stronger notions of randomness and by providing metamathematical analysis of Reimann and Slaman's methods.

The second question we investigate is "What are the first-order consequences of the existence of 2-random reals?" Conidis and Slaman showed that the consequences lie somewhere between $\mathrm{I}\Sigma^0_1$ and $\mathrm{B}\Sigma^0_2$, but left open the question of further classification. We show that the existence of 2-random reals does not imply $\mathrm{B}\Sigma^0_2$, and thus the consequences lie strictly between $\mathrm{I}\Sigma^0_1$ and $\mathrm{B}\Sigma^0_2$. Furthermore, by utilizing the methods in this proof we are able to construct a κ-like model of $\neg\mathrm{B}\Sigma^0_2$ and thereby answer an open question posed by Kaye in 1995.

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I would also like to thank my family for not only being immensely supportive during the trials and tribulations of graduate school, but also in all the years leading up to it. Although this dissertation was produced in only the last few years, my tenure as a student has spanned over two decades. This work is the culmination of all those years of study, and the encouragement and support I received has carried me through from the beginning to the end.

Finally, I want to thank my friends. Often they have helped me stay motivated and focused, and — some may say too often — they have helped to provide me with necessary distraction. My friends have been a constant and integral part of my life, and I am grateful that they choose to share in the adventure with me. *amicitia magia est.*

Chapter 1

Introduction

The premise of this dissertation is to study algorithmic randomness and thereby better understand what it means for any infinite string of digits (usually referred to as a "real" in the context of recursion theory) to be "random." This is a subject that dates back many years, with a number of approaches defined by Kolmogorov, Church, and Martin-Löf in the mid 20th century. If one considers "random" to simply mean unpredictable, one's definition may be that it is impossible to reliably predict the digits a random real. From another point of view, one may say that a random real should have no simple description and therefore has no better way of describing it than just writing out the entire string itself. Yet another point of view would suggest that if reals are known to have some property P with probability 1, then for a real to be considered random it must also have property P.

Each of the above notions translates into a well-studied definition of randomness, and it turns that — when the definitions are made precise — they are actually all equivalent. Furthermore, they each have a common feature of referring to a certain notion of "effectivity." For example, how powerful should our process of predicting digits be allowed to be? When we speak of reproducing a string from short descriptions, how powerful are the algorithms which can extract the original string from a short definition? When we say a real should satisfy properties occurring with probability 1, this will only make sense if we restrict the scope to of such properties, usually to a "describable" subset. When we enter into this context, we are in the realm of "algorithmic randomness" or "algorithmic information theory."

There are many texts which cover the area of algorithmic randomness extensively, and we refer the reader to books by Rod Downey and Denis Hirschfeldt [6] and André Nies [18] for a full treatment on background material.

This dissertation focuses on two independent questions from the area of algorithmic randomness. The first is "Which reals are random for *some* probability distribution?" This question was originally studied by Reimann and Slaman in 2008 [19] for a specific definition of randomness, and we will expand upon their work. The second question considered is "What are the implications of the existence of random reals?" This question was considered by Conidis and Slaman in 2013 [3], and this work continues that investigation.

These two questions are considered in chapters 2 and 3 respectively. The independent nature of these questions allows the chapters to be read independently of one another. The rest of this introductory chapter provides more specific background relevant to both chapters, and introduces the notion used throughout. Each of the chapters also has its own introduction of background material relevant to that chapter specifically.

1.1 Notation

The notation $2^{<n}$ refers to the set of binary strings of length strictly less n. Similarly, $2^{\leq n}$ refers to the set of strings of length at most n. $2^{<\omega}$ refers to the set of all finite binary strings, and 2^{ω} represents the set of infinite binary strings (also known as Cantor space).

Lower case Greek letters σ, τ, ρ will be used to indicate finite binary strings, i.e. elements of $2^{<\omega}$. The empty string will be represented by $\langle\rangle$. $|\sigma|$ denotes the length of the string σ. The notation $\sigma^\frown i$ for $i \in \{0,1\}$ represents the string of length $|\sigma|+1$ which is σ followed by the bit i. If $n \leq |\sigma|$ we use $\sigma \upharpoonright n$ to represent the string that is the first n bits of σ. If $|\sigma| > 0$ we let $\sigma^* = \sigma \upharpoonright (|\sigma| - 1)$. We will say $\sigma \subseteq \tau$ if τ is an extension of σ, i.e. $|\tau| \geq |\sigma|$ and $\tau \upharpoonright |\sigma| = \sigma$.

The notation $\langle \cdot, \cdot \rangle$ represents the standard uniformly recursive pairing function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, i.e. a recursive bijection from $\mathbb{N}^2$ to $\mathbb{N}$. The $\oplus$ operator refers to the recursive join. That is, for reals X, Y the real $X \oplus Y$ is defined by $(X \oplus Y)(2n) = X(n)$ and $(X \oplus Y)(2n+1) = Y(n)$. For finite strings σ, τ where $|\sigma| = |\tau|$ or $|\sigma| = |\tau| + 1$ the finite string $\sigma \oplus \tau$ is defined as analogously as the finite string with length $|\sigma| + |\tau|$.

Similarly, we will use the notation $\bigoplus\limits_{n<\omega} X_n$ to denote the uniform recursive join of countably many reals. Specifically, this denotes the real X such that $X(\langle n, i\rangle) = X_n(i)$.

Upper case Latin letters from the end of the alphabet X, Y, Z will be used to represent reals, i.e. elements of 2^{ω}. Upper case letters from the beginning of the alphabet A, B, C will be used to denote sets of reals, i.e. subsets of 2^{ω}. In a slight abuse of notation, we will write $\sigma \subset X$ if σ is an initial segment of X. Similarly, $X \upharpoonright n$ represents the finite string equal to the first n bits of X, so $\sigma \subset X$ iff $\sigma = X \upharpoonright |\sigma|$. We will use $[\sigma]$ to denote the set of reals extending σ, i.e. $[\sigma] = \{X \in 2^{\omega} : X \supset \sigma\}$.

We let $\Phi_0, \Phi_1, \ldots$ be a uniform listing of all partial Turing functionals, and let $W_0, W_1, \ldots$ be a uniform listing of all recursively enumerable (r.e.) sets. We write $\Phi_{e,s}(x)$ to denote the computation of $\Phi_e(x)$ up to stage s, and write $\Phi_e(x)\downarrow$ to indicate that the functional Φ_e converges on argument x. Given a real X, X' refers to the Turing jump relative to X, i.e. $X' = \{e : \Phi_e^X(e)\downarrow\}$. The notation $X^{(n)}$ refers to the nth iterate of the Turing jump.

Given two sets A, B, $A \setminus B = \{x : x \in A \wedge x \notin B\}$. We let $\overline{A} = \mathbb{N} \setminus A$.

If $\mathfrak{M}$ is a model, $|\mathfrak{M}|$ represents the (first-order) domain of the model.

1.2 Background on Algorithmic Randomness

As mentioned in the introduction, there are a number of equivalent definitions of algorithmically random reals. Throughout this dissertation we will be using the definition coined by Per Martin-Löf. His definition captures the notion that a random real should satisfy any effectively described "almost surely" property. For example, a real satisfies the Law of Large Numbers with probability 1. Since the Law of Large Numbers is easily described, any random real should satisfy this property.

Definition 1. *A Martin-Löf test* $\{V_n\}_{n\in\omega}$ *is a uniformly r.e. sequence of sets* $V_n \subseteq 2^{<\omega}$ *such that for all* n, $\sum_{\sigma\in V_n} 2^{-|\sigma|} \leq 2^{-n}$.

Definition 2. *A real* X *passes a Martin-Löf test* $\{V_n\}_{n\in\omega}$ *if* $X \notin \bigcap_{n\in\omega} V_n$.

Definition 3. *A real* X *is Martin-Löf random if* X *passes every Martin-Löf test.*

As is often the case in recursion theory, these definitions can be relativized.

Definition 4. *A Martin-Löf test relative to* Z *is a sequence of sets* $V_n \subseteq 2^{<\omega}$ *for* $n \in \omega$ *which is uniformly r.e. relative to* Z*, and for which* $\sum_{\sigma\in V_n} 2^{-|\sigma|} \leq 2^{-n}$ *for all* n*. A real* X *is Martin-Löf random relative to* Z *if* $X \notin \bigcap_{n\in\omega} V_n$ *for all Martin-Löf tests* $\{V_n\}_{n\in\omega}$ *relative to* Z*.*

Chapter 3 specifically discusses 2-random reals, which are reals that are Martin-Löf random relative to $0'$.

An important result from literature is the existence of a universal Martin-Löf test, a fact that is used throughout chapters 2 and 3. Consequently, we present it here as necessary background for both chapters which follow.

Theorem 1. *There is a total recursive function* f *such that for any oracle* Z*,* $\{W^Z_{f(n)}\}_{n\in\omega}$ *is a universal Martin-Löf test relative to* Z*. That is, a real* X *is Martin-Löf random relative to* Z *iff* $X \notin \bigcap_n W^Z_{f(n)}$*.*

We will write U^Z_n to represent the nth Π^0_1 class of the universal Martin-Löf test relative to Z. That is, $U^Z_n = \overline{W^Z_{f(n)}}$. Hence a real X is 2-random relative to Z iff there is an n such that $X \in U^Z_n$.

Chapter 2

Randomizing Reals

2.1 Introduction

Although our introductory definition of Martin-Löf randomness makes no mention of probability measure, the familiar reader will realize that the requirement $\sum_{\sigma \in V_n} 2^{-|\sigma|} \leq 2^{-n}$ is the same as requiring $\lambda(V_n) \leq 2^{-n}$ where λ is the Lebesgue measure on 2^ω. However, Martin-Löf's original definition of an algorithmically random sequence was defined relative to any probability measure on 2^ω, and Levin extended many of his results to arbitrary probability measures. In particular, Levin proved the existence of *neutral measures*, i.e. a measure relative to which every real is Martin-Löf random. We expand upon the topic of neutral measures below.

One of the fundamental questions in randomness is simply "What reals are random?" This question can be interpreted a number of ways. For example, one of the first challenges in algorithmic randomness was to provide an explicit example a Martin-Löf random sequence. One such example was provided by Gregory Chaitin in the form of a real defined as the *halting probability*, also called Chaitin's Ω. Another way of interpreting this question is to ask where in the Turing degrees random reals can lie. Antonin Kučera [14] and Péter Gács [9] answered this question by showing that every real is bounded above by a Martin-Löf random.

In 2008 Reimann and Slaman [19] investigated a reversal of this question and asked, "Given a real X, is there a measure μ such that $\mu(\{X\}) = 0$ and X is Martin-Löf random relative to μ?" They studied this question for both arbitrary measures and for continuous measures. In their paper, they were able to show that for arbitrary measures, such a μ exists iff X is non-recursive. For continuous measures, it is sufficient to be non-hyperarithmetic, but for hyperarithmetic reals there are both examples and counter-examples. Thus they defined the class NCR of not-continuously-random reals. Since we will be generalizing several of their results and analyzing their methods, the reader should be familiar with their paper.

This chapter is primarily concerned with expanding Reimann and Slaman's question beyond Martin-Löf randomness. That is, given a real X and one of the alternate definitions of randomness (defined below), is there a measure μ such that X is random relative to μ? The main results of this chapter will be demonstrating a class of non-recursive reals for which no such μ exists (thus differentiating this case from the one studied by Reimann and Slaman), and then providing a metamathematical analysis of Reimann and Slaman's result showing that some of their methods are necessary to their proof, and are so even in the context of these alternate notions of randomness.

In the remainder of this section, we discuss measures and their representations. In section 2 we will give the definitions of randomness used throughout the chapter and also make some preliminary observations. In the section 3 we show the non-existence results, contrasting with the case of Martin-Löf randomness. In section 4 we discuss the construction of measures which make reals appear random, providing a result showing that reals which can be made to appear strongly random relative to some μ have a certain density within the Turing degrees. We conclude by proving a theorem which provides the aforementioned metamathematical analysis.

Measure Representations

Given a topological space $\mathcal{T}$ with σ-ring $\mathcal{S} \subseteq \mathcal{P}(\mathcal{T})$, a measure μ on $\mathcal{S}$ is a mapping from $\mathcal{S}$ to non-negative real numbers with the properties that it is monotone and countably additive. That is, if $A \subseteq B$ then $\mu(A) \leq \mu(B)$, and if $A_0, A_1, \dots$ are disjoint elements of $\mathcal{S}$ then $\sum\limits_{i=0}^{\infty} \mu(A_i) = \mu(\bigcup\limits_{i=0}^{\infty} A_i)$.

A measure μ is a probability measure if $\mu(\mathcal{S}) = 1$. Given an ring $\mathcal{R} \subseteq \mathcal{S}$ that generates $\mathcal{S}$, Carathéodory's extension theorem implies that μ is uniquely determined by the values it takes on $\mathcal{R}$ so long as the measure is σ-finite (which in particular holds if μ is a probability measure). On the other hand, a function μ from $\mathcal{R}$ to non-negative real numbers is called a pre-measure if it is countably additive on $\mathcal{R}$ and if $\mu(\emptyset) = 0$. In such a case, it is a well known theorem of measure theory that μ can be extended to a measure on $\mathcal{S}$. Hence considering measures on all of $\mathcal{S}$ is equivalent to considering pre-measures on $\mathcal{R}$.

In our context, the topological space will be 2^ω, $\mathcal{S}$ will be the Borel subsets of 2^ω, and the generating ring will be the basic clopen sets, i.e. finite unions of $\{[\sigma] : \sigma \in 2^{<\omega}\}$. Hence a measure is completely determined by its values on each set $\mu([\sigma])$. We will frequently abuse notation and simply write this as $\mu(\sigma)$. The above paragraph implies that so long as $\mu(\langle\rangle) = 1$ and $\mu(\sigma^\frown 0) + \mu(\sigma^\frown 1) = \mu(\sigma)$, μ can be extended uniquely to a probability measure on the Borel subsets of 2^ω.

We will let λ denote the Lebesgue measure on 2^ω, i.e. the measure for which $\lambda(\sigma) = 2^{-|\sigma|}$ for all σ. Given a measure μ, we call a real X an atom of the measure if $\mu(\{X\}) > 0$. A measure is called continuous if it has no atoms.

If we wish to extend the definition of Martin-Löf randomness to arbitrary measures μ, we need to take care if μ is not recursive. We would like to provide Martin-Löf tests with the ability to calculate the measure μ and we will do so by relativizing the definition to a real R which represents the measure. We define what it means for a real to represent a measure below, but it is worth noting that this is by no means the only valid definition. For a detailed discussion on the representations of measures (including alternate definitions), we refer to the reader to Day and Miller's paper [5].

Fix an effective enumeration $\sigma_0, \sigma_1, \ldots$ of all finite strings. Given a real R and string σ, let n be the index such that $\sigma = \sigma_n$ and let $R_\sigma = \sum\limits_{i=0}^{\infty} 2^{-i} \cdot R(\langle n, i\rangle)$. Then R_σ is a number in $[0,1]$ for every σ. Define by induction the pre-measure μ_R as

$$\begin{aligned} \mu_R(\langle\rangle) &= 1 \\ \mu_R(\sigma^\frown 0) &= \mu_R(\sigma) \cdot R_\sigma \\ \mu_R(\sigma^\frown 1) &= \mu_R(\sigma) \cdot (1 - R_\sigma) \end{aligned}$$

It is clear that μ_R is a well-defined pre-measure. In this way every real R represents some probability measure μ_R on 2^ω, and it is clear that for every probability measure has a real which represents it. Of course, given a measure μ there is not necessarily a unique representation R (if $\frac{\mu(\sigma^\frown 0)}{\mu(\sigma)}$ is a dyadic rational then there are two choices for R_σ).

If R is a representation of a measure μ, we write $R_s(\sigma)$ to be an approximation of $\mu(\sigma)$ to within 2^{-s}. That is to say, we require $R_s(\sigma) - 2^{-s} \leq \mu(\sigma) \leq R_s(\sigma) + 2^{-s}$. It is clear that the function $R_s(\sigma)$ is uniformly recursive in s, σ relative to R. That $R_s(\sigma)$ is uniformly recursive is the only feature of our definition of measure representations that we will use going forward, and is a feature common to any reasonable definition of measure representations.

Given this definition of R_s, it is useful to note the following facts about representations of measures, as they will be used later on.

- $\mu(\sigma) < q$ iff $\exists s (R_s(\sigma) < q - 2^{-s})$.
- $\mu(\sigma) > q$ iff $\exists s (R_s(\sigma) > q + 2^{-s})$.
- $\mu(\sigma) = x$ iff $\mu(\sigma) \not< x$ and $\mu(\sigma) \not> x$ iff $\forall s (R_s(\sigma) \leq x + 2^{-s})$ and $\forall s (R_s(\sigma) \geq x - 2^{-s})$.

Put another way, inequality is a $\Sigma^0_1(R)$ property and equality is a $\Pi^0_1(R)$ property.

2.2 Formal Definitions and Preliminary Observations

We begin by making explicit all notions and generalizations of randomness used in this chapter, relavitized to arbitrary measures.

Definition 5. *A real X is μ-$(n+1)$-random relative to Y if there is some representation R of μ such that $X \notin \bigcap_i V_i$ for every test $\{V_i\}$ where V_i is uniformly r.e. in $(R \oplus Y)^{(n)}$ and $\mu(V_i) \leq 2^{-i}$ for all i.*

Note that a real being 1-random is equivalent to a real being Martin-Löf random.

Definition 6. *[7] A real X μ-difference-random relative to Y if there is some representation R of μ such that $X \notin \bigcap_i V_i$ for every test $\{V_i\}$ where $\mu(V_i) \leq 2^{-i}$ the sets $\{V_i\}$ are uniformly 2-r.e. in $R \oplus Y$.*

Franklin and Ng defined this notion of randomness and showed that the above definition is equivalent to allowing tests to be uniformly k-r.e. for any $k \geq 2$. They further characterized the difference randoms as a subset of the 1-randoms:

Theorem 2. *[7] A real X is μ-difference-random iff there is a representation R of μ such that X is μ-1-random relative to R and $X \oplus R \not\geq_T R'$.*

Definition 7. *A real X is μ-weak-$(n{+}2)$-random relative to Y if there is some representation R of μ such that $X \notin \bigcap_i V_i$ for every test $\{V_i\}$ where $\lim_{i\to\infty} \mu(V_i) = 0$ and the sets $\{V_i\}$ are uniformly r.e. in $(R \oplus Y)^{(n)}$*

In the above definitions we relativize our tests to $(R \oplus Y)^{(n)}$. One may ask why we do not instead relativize to $R^{(n)} \oplus Y^{(n)}$ or $R \oplus Y^{(n)}$. The convention used by Reimann and Slaman in their definition of NCR_n (reals which are not n-random relative to any continuous measure) allows for the use of the nth iterate of jump of R, so we follow that convention (thus ruling out the last option). The results of this chapter don't pursue relativization beyond iterates of the Turing jump itself and would therefore apply to either of the former definitions.

For a fixed real X and measure μ we have the following implications:

$$(n+1)\text{-random} \Rightarrow \text{weak-}(n+1)\text{-random} \Rightarrow \text{difference-}(n+1)\text{-random} \Rightarrow n\text{-random}$$

For Lebesgue measure, it is known that none of these arrows reverse. The reader is referred to Downey and Hirschfeldt [6] for proofs and further discussion, and to Franklin and Ng [7] for proofs about difference-randomness.

We can now precisely state the main question of this chapter: for which reals X is there a measure μ such that $\mu(\{X\}) = 0$ and such that X is [weak-n, n, difference]-random relative to μ?

It will be a bit cumbersome to rewrite the above phrase throughout everything which follows, and so we adopt a the term *randomizable* to refer to such reals.

Definition 8. *A real X is [weak-n, n, difference]-randomizable if there is a measure μ such that $\mu(\{X\}) = 0$ and X is [weak-n, n, difference]-random relative to μ.*

In this language, the principal question of this chapter is simply "What reals are randomizable?"

As aforementioned, Reimann and Slaman [19] answered this question for 1-randomness by showing that a real is 1-randomizable iff it is not recursive. Thus, the goal of this chapter is to shed some light on the problem for the notions of randomness stronger than 1-randomness.

Preliminary Observations

In 1970 Levin proved that there exists a measure μ such that every real is μ-Martin-Löf random [16]. That is, for every X there is a representation R of μ such that X μ-Martin-Löf random relative to R. Gács [10] later termed such measures *neutral measures*.

The existence of neutral measures is a peculiarity of Martin-Löf randomness, and it stems from Martin-Löf tests lacking the computational power to handle multiple representations of measures. To make this statement more precise, observe that given any measure representation R there is a real X_R such that X_R is not μ_R-Martin-Löf random relative to R (the proof of this is not difficult; see Lemma 4.1 of [5]). We are therefore able to conclude (by the existence of neutral measures) that different representations R necessarily produce different reals X_R, and there is no test which can uniformly capture the same X for every representation.

Day and Miller [5] studied neutral measures and provided an alternate construction of a neutral measure which yielded an entire Π^0_1 class of representations of neutral measures. As a corollary, they provide an alternative proof to Reimann and Slaman's result that every non-recursive real is 1-randomizable.

Given this direction of study of Martin-Löf randomness with respect to arbitrary measures by Levin *et al.*, and given that Day and Miller's investigation yielded results directly related to the primary question of this chapter, we first wish to investigate neutral measures in the context of higher randomness. However, our first result shows that difference-randomness (the weakest notion of randomness stronger than 1-randomness which we consider) is able to overcome the ambiguity introduced by representations of measures, and hence no neutral measure can exist for difference-randomness. Therefore a deeper investigation of neutral measures will not aid in our discussion.

Theorem 3. *There are no neutral measures for difference-randomness. That is, for any measure μ there is a real X such that X is not μ-difference-random.*

Proof. Fix a measure μ and define X inductively and follows. Suppose we have defined $X \upharpoonright n$ and let $\sigma = X \upharpoonright n$. If $\mu(\sigma^\frown 0) \leq \mu(\sigma^\frown 1)$ let $X(n) = 0$ and otherwise let $X(n) = 1$. Note that this implies $\mu(X \upharpoonright (n+1)) \leq \frac{1}{2}\mu(X \upharpoonright n)$ and hence $\mu(X \upharpoonright n) \leq 2^{-n}$. In particular this also means $\mu(\{X\}) = 0$.

It is claimed that X is not μ-difference-random. To see this, fix a representation R of μ. Define the sets $D_{n,0}, D_{n,1}$ inductively as follows. Let $D_{0,0} = \{\langle\rangle\}$ and $D_{0,1} = \emptyset$.

Let

$$D_{n,0} = \left\{ \tau^\frown i \in 2^n : \tau \in D_{n-1,0} \wedge \left(\left(i = 0 \wedge R_{n+1}(\tau^\frown i) \leq \frac{1}{2} R_n(\tau) + 2^{-n} \right) \right.\right.$$
$$\left.\left. \vee \left(\exists s \left(R_{s+1}(\tau^\frown i) < \frac{1}{2} R_s(\tau) - 2^{-s} \right) \right) \right) \right\}$$

$$D_{n,1} = \Big\{\tau^\frown i \in 2^n : (\exists \sigma \subseteq \tau \; \exists k < n \, (\sigma \in D_{k,1})) \\ \vee \left(\exists s \left(R_{s+1}(\tau^\frown i) > \frac{1}{2} R_s(\tau) + 2^{-s} \right) \right) \Big\}$$

It is clear from the definition that each $D_{n,i}$ is r.e. It is first claimed that $X \upharpoonright n \in D_{n,0} \setminus D_{n,1}$ for each n, and hence $X \in \bigcap_n (D_{n,0} \setminus D_{n,1})$. We show this by induction on n. The case of $n = 0$ is immediate. In the inductive case let $X \upharpoonright n = \tau^\frown i$. By induction $\tau \in D_{n-1,0}$. First take the case that $\mu(\tau^\frown i) < \frac{1}{2}\mu(\tau)$. Fix s large enough so that $2^{-s} < \frac{1}{4}\left(\frac{1}{2}\mu(\tau) - \mu(\tau^\frown i)\right)$. Then we have $\mu(\tau^\frown i) < \frac{1}{2}\mu(\tau) - 4 \cdot 2^{-s}$ and we get

$$R_{s+1}(\tau^\frown i) \leq \mu(\tau^\frown i) + 2^{-(s+1)} < \frac{1}{2}\mu(\tau) - 3 \cdot 2^{-s} \leq \frac{1}{2}(R_s(\tau) + 2^{-s}) - 3 \cdot 2^{-s} \\ < \frac{1}{2} R_s(\tau) - 2^{-s}$$

Hence $\tau^\frown i \in D_{n,0}$. In the other case we have $\mu(\tau^\frown i) = \frac{1}{2}\mu(\tau)$ and hence by definition of X we have $i = 0$. Then we have $R_{n+1}(\tau^\frown i) \leq \mu(\tau^\frown i) + 2^{-(n+1)} = \frac{1}{2}\mu(\tau) + 2^{-(n+1)} \leq \frac{1}{2}(R_n(\tau) + 2^{-n}) + 2^{-(n+1)} = \frac{1}{2}R_n(\tau) + 2^{-n}$ and hence $\tau^\frown i \in D_{n,0}$.

Finally note that it cannot be the case that $\tau^\frown i$ is in $D_{n,1}$ or else this would imply that there is a stage s such that $R_{s+1}(\tau^\frown i) > \frac{1}{2}R_s(\tau) + 2^{-s}$ and hence

$$\mu(\tau^\frown i) \geq R_{s+1}(\tau^\frown i) - 2^{-(s+1)} > \frac{1}{2} R_s(\tau) + 2^{-s} - 2^{-(s+1)} \\ \geq \frac{1}{2}(\mu(\tau) - 2^{-s}) + 2^{-s} - 2^{-(s+1)} = \frac{1}{2}\mu(\tau)$$

a contradiction.

Now we want to verify that $\mu(D_{n,0} \setminus D_{n,1}) \leq 2^{-n}$ for all n. To do this we show that $\sigma \in D_{n,0} \setminus D_{n,1}$ iff $\sigma = X \upharpoonright n$, which suffices since by our definition of X, $\mu(X \upharpoonright n) \leq 2^{-n}$ for all n. Again we proceed by induction, and the case of $n = 0$ is trivial. Suppose that $\tau^\frown i \in D_{n,0} \setminus D_{n,1}$. Then $\tau \notin D_{n-1,1}$ or else $\tau^\frown i$ would be in $D_{n,1}$ by definition. Since $\tau^\frown i \in D_{n,0}$ it must also be the case that $\tau \in D_{n-1,0}$. Thus $\tau \in D_{n-1,0} \setminus D_{n-1,1}$, so $\tau = X \upharpoonright (n-1)$ by induction.

Suppose for a contradiction that $\tau^\frown i \neq X \upharpoonright n$. First suppose $\tau^\frown i \in D_{n,0}$ because there is an s such that $R_{s+1}(\tau^\frown i) < \frac{1}{2}R_s(\tau) - 2^{-s}$. However we then have

$$\mu(\tau^\frown i) \leq R_{s+1}(\tau^\frown i) + 2^{-(s+1)} < \frac{1}{2} R_s(\tau) - 2^{-s} + 2^{-(s+1)} \\ \leq \frac{1}{2}(\mu(\tau) + 2^{-s}) - 2^{-s} + 2^{-(s+1)} = \frac{1}{2}\mu(\tau)$$

Then $\tau^\frown i = X \upharpoonright n$ contradicting our assumption. Thus it must be the case that $i = 0$ and $R_{n+1}(\tau^\frown i) \leq \frac{1}{2}R_n(\tau) + 2^{-n}$. In this case we claim that $\tau^\frown i \in D_{n,1}$, which will achieve a contradiction since we assumed $\tau^\frown i \in D_{n,0} \setminus D_{n,1}$. Since we are assuming that $\tau = X \upharpoonright (n-1)$ and $\tau^\frown i \neq X \upharpoonright n$, this implies $X(n) \neq 0$ and hence $\mu(\tau^\frown 0) > \mu(\tau^\frown 1)$. Fix an s such that $2^{-s} < \frac{1}{4}\left(\mu(\tau^\frown 0) - \frac{1}{2}\mu(\tau)\right)$, so that we have $\mu(\tau^\frown 0) > \frac{1}{2}\mu(\tau) + 4 \cdot 2^{-s}$. Then we get

$$\begin{aligned} R_{s+1}(\tau^\frown i) &\geq \mu(\tau^\frown i) - 2^{-(s+1)} > \frac{1}{2}\mu(\tau) + 3 \cdot 2^{-s} \geq \frac{1}{2}(R_s(\tau) - 2^{-s}) + 3 \cdot 2^{-s} \\ &> \frac{1}{2}R_s(\tau) + 2^{-s} \end{aligned}$$

and hence $\tau^\frown i \in D_{n,1}$ as desired.

Therefore $V_n = D_{n,0} \setminus D_{n,1}$ is a μ-difference-randomness test relative to R which captures X. Since such a test exists for any R representing μ, we have that X cannot be μ-difference-random. □

2.3 Non-randomizable Reals

Since Reimann and Slaman showed that a real is 1-randomizable iff it is not recursive, our first foray into generalizing their result to higher notions of randomness will be asking if every non-recursive real is difference-randomizable, weak-n-randomizable, or even n-randomizable for $n > 1$. However, this section provides a number of counter-examples, showing that their result will not generalize.

Recall that Reimann and Slaman defined the class NCR of not-continuously-random reals, i.e. the reals X for which there is no continuous μ such that X is μ-1-random. More generally they defined NCR_n, the set of reals X for which there is no continuous μ such that X is μ-n-random. In their paper they showed that NCR_n is a subset of HYP for each n, though an earlier result by Kjos-Hanssen and Montalbán showed that the NCR reals are unbounded within HYP.

Theorem 4. *[13] For every $\beta < \omega_1^{CK}$ there is a real $X \equiv_T 0^{(\beta)}$ such that $X \in NCR$.*

Our first result helps to bridge the gap between being not-continuously-random and being not randomizable, taking 2 jumps to do so. Consequently, we show that the reals which are not 3-randomizable are a superset of NCR and in particular contain more than just the recursive reals. Before we prove the theorem, we prove some lemmas, the first of which is a fairly simple and well-known fact about measures.

Lemma 1. *Let μ be a probability measure. Then μ has only countably many atoms.*

Proof. Suppose that μ had uncountably many atoms. Let $B_n = \{X \in 2^\omega : \mu(\{X\}) > \frac{1}{n}\}$. Since $\bigcup_n B_n$ contains all atoms of μ, there is a single n such that B_n is uncountable. Let $X_1, X_2, \ldots, X_{n+1}$ be $n+1$ many reals from B_n. Then we have by countable additivity $\mu(B_n) \geq \sum_{i=1}^{n+1} \mu(\{X_i\}) > \frac{n+1}{n} > 1$, a contradiction. □

The next lemma is another well-known result which says that all atoms of a measure are computable from a representation of that measure.

Lemma 2. *Suppose X is an atom of μ. Then for any representation R of μ, $X \leq_T R$.*

Proof. Since $\mu(\{X\}) > 0$, fix a rational q such that $q < \mu(\{X\}) < 2q$. Fix some $\sigma \subset X$ such that $\mu(\sigma) < 2q$. Fix s such that $2^{-s} < q$. Then R can compute initial segments of X by looking for $\tau \supseteq \sigma$ and $t \geq s$ such that $R_t(\tau) > q + 2^{-t}$. For every length $l > |\sigma|$ there is at most one such τ, for if there were two strings τ_1, τ_2 then we would have

$$\mu(\sigma) \geq \mu(\tau_1) + \mu(\tau_2) \geq R_{t_1}(\tau_1) - 2^{-t_1} + R_{t_2}(\tau_2) - 2^{-t_2} > q + q > 2q$$

which contradicts our choice of σ. □

Now fix a real X in NCR and a measure μ such that $\mu(\{X\}) = 0$. We will show that X cannot be μ-3-random.

Since μ has countably many atoms, list them as $Y_1, Y_2, \ldots$. Let $A(\sigma) = \sum_i \{\mu(\{Y_i\}) : Y_i \in [\sigma]\}$. That is, $A(\sigma)$ is the sum of the weight of atoms in $[\sigma]$. Fix a representation R of μ. Our next lemma establishes the complexity of the relation $A(\sigma) > q$.

Lemma 3. *The relation $A(\sigma) > q$ is uniformly recursive in R''.*

Proof. First we define the relation $M(\sigma, q)$ which holds iff there is an atom $Y \in [\sigma]$ with $\mu(\{Y\}) \geq q$. Then we have

$$M(\sigma, q) \Leftrightarrow (\forall m \geq |\sigma|)(\exists \tau \in 2^{\leq m})(|\tau| = m \wedge \tau \supseteq \sigma \wedge R_m(\tau) \geq q - 2^{-m})$$

Thus $M(\sigma, q)$ is a $\Pi^0_1(R)$ relation.

Returning to analyzing the complexity of $A(\sigma)$, first suppose it is the case that $A(\sigma) > q$. Then there is a finite collection of atoms $Z_1, \ldots, Z_n$ in $[\sigma]$ such that $\sum_i \mu(Z_i) > q$. Pick a length l large enough so that the strings $Z_i \upharpoonright l$ are pairwise distinct, and let $\sigma_i = Z_i \upharpoonright l$. Fix rationals q_i such that $\mu(Z_i) > q_i$ and $\sum_i q_i > q$. Then $M(\sigma_i, q_i)$ holds for each i. Hence we have:

> $\exists n \exists \sigma_1, \ldots, \sigma_n \exists q_1, \ldots, q_n$ such that the σ_i all have the same length and extend σ, the σ_i are pairwise distinct, $\sum_i q_i > q$, and $M(\sigma_i, q_i)$ holds for each i. $(*)$

Conversely, suppose $(*)$ holds. Since $M(\sigma_i, q_i)$ holds for each i, there are n distinct atoms $Z_1, \ldots, Z_n$ with $\mu(Z_i) \geq q_i$. Hence $\mu(\sigma) \geq \sum_i q_i > q$. Hence $(*)$ is equivalent to $A(\sigma) > q$. Since $(*)$ is $\Sigma^0_2(R)$, the predicate $A(\sigma) > q$ is recursive in R''. □

We will also need to be able to decide if $A(\sigma) = 0$ and therefore analyze the complexity of that relation.

Lemma 4. *The relation $A(\sigma) = 0$ is $\Pi^0_2(R)$ and is therefore decidable in R''.*

Proof. Note that

$$A(\sigma) = 0 \Leftrightarrow \forall n \exists l \forall \tau \in 2^{\leq l} \cap [\sigma](|\tau| = l \Rightarrow R_{n+1}(\tau) \leq 2^{-n})$$

and by observation this is Π^0_2. □

The last lemma before we proceed to the main theorem uses the above lemmas to show that R'' can compute a finite listing of atoms in $[\sigma]$ in order to approximate $A(\sigma)$.

Lemma 5. *Suppose q is a rational and $A(\sigma) > q$. Then uniformly in σ, q R'' can compute a finite list of indices $e_0, \ldots, e_k$ such that $\Phi^R_{e_i}$ is an atom of μ in $[\sigma]$ and such that $\sum_{i=0}^{k} \mu(\{\Phi^R_{e_i}\}) > q$.*

Proof. R'' can compute the set of total Turing functionals $\Phi^R_{e_0}, \Phi^R_{e_1}, \ldots$, and since every atom of μ is recursive in R, every atom contained in $[\sigma]$ appears in this list.

R'' can therefore compute indices from this list such that each index j satisfies $\Phi^R_{e_j} \supseteq \sigma$ and $\exists q_j \exists s \forall k R_s(\Phi^R_{e_j} \upharpoonright k) > q_j - 2^{-s}$. R'' can continue computing such indices until $\sum_j q_j > q$, and then output those indices.

If this program halts, it is clear that the indices output satisfy the lemma. To verify that the program will halt, fix atoms $Y_1, \ldots, Y_n$ in $[\sigma]$ such that $\sum \mu(Y_j) > q$. Then we can fix rationals $q_1, \ldots, q_n$ such that $\mu(\{Y_j\}) > q_j$ and $\sum_j q_j > q$. Fixing indices e_j such that $Y_j = \Phi^R_{e_j}$, the above program will halt by the time it discovers the indices $e_1, \ldots, e_n$. □

We are now ready to prove the first main result.

Theorem 5. *If X is NCR, then X is not 3-randomizable.*

Proof. We non-uniformly split into two cases. First suppose that $A(\langle\rangle) = 1$. By Lemma 5 R'' can for every n uniformly compute atoms $Z_1, \ldots, Z_m$ such that $\sum \mu(Z_i) > 1 - 2^{-n}$. Let $V_n = \{\sigma : \sigma \text{ is incomparable to each } Z_i\}$. Then clearly $\mu(V_n) < 2^{-n}$ and R'' can compute each V_n. Since X is not an atom of μ, $X \in \bigcap_n V_n$, so X is not Martin-Löf random relative to R''.

Now suppose that $A(\langle\rangle) < 1$ and define the measure ν by

$$\nu(\sigma) = \frac{\mu(\sigma) - A(\sigma)}{1 - A(\langle\rangle)}$$

Note that ν is a well-defined measure, since $\nu(\langle\rangle) = 1$ and for any σ we have

$$\begin{aligned} \nu(\sigma^\frown 0) + \nu(\sigma^\frown 1) &= \frac{\mu(\sigma^\frown 0) - A(\sigma^\frown 0) + \mu(\sigma^\frown 1) - A(\sigma^\frown 1)}{1 - A(\langle\rangle)} \\ &= \frac{\mu(\sigma) - A(\sigma)}{1 - A(\langle\rangle)} = \nu(\sigma) \end{aligned}$$

Since R'' can recursively compute $\mu(\sigma)$ and $A(\sigma)$, R'' can compute a representation S of ν.

Now observe that ν is continuous, for suppose there is a Y such that $\nu(\{Y\}) > 0$. Since $\frac{\mu(Y\upharpoonright n)-A(Y\upharpoonright n)}{1-A(\langle\rangle)} = \nu(Y \upharpoonright n)$ we can take the limit of both sides as $n \to \infty$ and get

$$\frac{\mu(\{Y\}) - \lim_{n\to\infty} A(Y \upharpoonright n)}{1 - A(\langle\rangle)} = \nu(\{Y\})$$

But since $\lim_{n\to\infty} A(Y \upharpoonright n) = \mu(\{Y\})$ we have $0 = \nu(\{Y\}) > 0$, a contradiction. Hence ν is continuous.

Since ν is continuous and X is NCR, there is some ν-Martin-Löf test $\{V_n\}$ uniformly r.e. in S (hence uniformly r.e. in R'') such that $\{V_n\}$ captures X. Define V'_k as follows.

List the strings enumerated into V_{k+1} as $\sigma_1, \sigma_2, \ldots$. When σ_i is enumerated into V_{k+1}, if $A(\sigma_i) = 0$ then put σ_i in V'_k. Otherwise search for a rational q_i such that $q_i < A(\sigma_i) < q_i + 2^{-(k+i+1)}$, and by Lemma 5 pick atoms $Z_1, \ldots, Z_m$ such that $\sum_j \mu(\{Z_j\}) > q_i$. Enumerate into V'_k all extensions of σ_i which are incomparable to each Z_j.

Since $X \in \bigcap_n V_n$ and X is not an atom of μ, it is clear that $X \in \bigcap_n V'_n$. So the only thing that needs to be checked is that $\mu(V'_k) \leq 2^{-k}$ for each k. Observe

$$\begin{aligned}
\mu(V'_k) &\leq \sum_{i=1}^{\infty} (\mu(\sigma_i) - q_i) \\
&\leq \sum_{i=1}^{\infty} \big(\mu(\sigma_i) - (A(\sigma_i) - 2^{-(k+i+1)})\big) \\
&= 2^{-(k+1)} + \sum_{i=1}^{\infty} (\mu(\sigma_i) - A(\sigma_i)) \\
&= 2^{-(k+1)} + (1 - A(\langle\rangle)) \cdot \nu(V_{k+1}) \\
&\leq 2^{-(k+1)} + (1 - A(\langle\rangle)) \cdot 2^{-(k+1)} \\
&\leq 2^{-k}
\end{aligned}$$

□

Note that the above proof relativizes to the more general statement that if X is NCR_n then X is not $(n+2)$-randomizable.

We next show that the reals which are not difference-randomizable contain the r.e. reals, and hence Reimann and Slaman's result does not even generalize to difference-randomness.

Proposition 1. *Suppose that X is a real with a total recursive function $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $\forall n \lim_s f(n, s) = X(n)$ and*

$$\forall n \, (|\{s : f(n, s) \neq f(n, s + 1)\}| < 2)$$

Then X is not difference-randomizable.

The above categorization of X in particular captures both r.e. and co-r.e. reals.

Proof. Fix a measure μ and a representation R of μ. Let

$$D_{n,0} = \{\sigma : \exists s(R_s(\sigma) \leq 2^{-n} - 2^{-s} \wedge \forall i < |\sigma|(\sigma(i) = f(i,s))\}$$
$$D_{n,1} = \{\sigma : \exists s \exists s' > s \exists i < |\sigma|(f(i,s') \neq f(i,s))\}$$

It is clear that each $D_{n,i}$ is r.e. We first claim that $\mu(D_{n,0} \setminus D_{n,1}) \leq 2^{-n})$. Suppose $\sigma \in D_{n,0} \setminus D_{n,1}$. Fix the stage s at which σ enters $D_{n,0}$. Then $R_s(\sigma) \leq 2^{-n} - 2^{-s}$ and hence $\mu(\sigma) \leq 2^{-n}$. Furthermore, for all $i < |\sigma|$ it must be the case that $f(i,s) = \lim_{s'} f(i,s') = X(i)$ or else there would be a stage $s' > s$ such that $f(i,s') \neq f(i,s)$ and hence $\sigma \in D_{n,1}$. Hence $\sigma \subset X$, so the only members of $D_{n,0} \setminus D_{n,1}$ are initial segments of X with measure less than 2^{-n}.

Finally we claim that if $\mu(\{X\}) = 0$ then $X \in \bigcap_n (D_{n,0} \setminus D_{n,1})$. Fix n and pick k large enough so that $\mu(X \upharpoonright k) < 2^{-(n+1)}$. Let $\sigma = X \upharpoonright k$. Fix s large enough so that for all $i < k$, $f(i,k) = \lim_s f(i,s)$ and large enough such that $2^{-(s-1)} < 2^{-(n+1)}$. Then we have

$$\begin{aligned} R_s(\sigma) &\leq \mu(\sigma) + 2^{-s} < 2^{-(n+1)} + 2^{-s} = 2^{-(n+1)} + 2^{-(s-1)} - 2^{-s} \\ &< 2^{-(n+1)} + 2^{-(n+1)} - 2^{-s} = 2^{-n} - 2^{-s} \end{aligned}$$

Hence $\sigma \in D_{n,0}$, and since there is no stage $s' > s$ such that $f(i,s') \neq f(i,s)$ for any $i < |\sigma|$, we have $\sigma \notin D_{n,1}$. □

This result has an interesting implication in the analysis of Reimann and Slaman's method for showing non-recursive reals are 1-randomizable. Suppose that X is an r.e. real and μ is a measure such that X is μ-1-random. Since X cannot be μ-difference-random, Theorem 2 implies that for any representation R of μ for which X is μ-1-random relative to R, $X \oplus R \geq_T R'$. In a sense, every measure which randomizes X is generic relative to X. We will make reference to this fact below, so we state this result as a corollary.

Corollary 1. *Suppose X is r.e., μ is a measure such that $\mu(\{X\}) = 0$, and R is a representation of μ such that X is μ-1-random relative to R. Then $R \oplus X \geq_T R'$.*

We next investigate those reals which are not weak-2-randomizable. To do so we will use an extremely useful characterization of weak-n-randomness. A proof of this characterization is presented in Downey and Hirschfelt's book [6], though only with respect to Lebesgue measure. In order to carefully track the generalization to arbitrary measures, we present a fully generalized version of the proof below.

Theorem 6. *Fix $n \geq 1$, a measure μ, and a representation R of μ. The following are equivalent:*

1. *X is μ-weak-$(n+1)$-random relative to R.*

2. *X is μ-n-random relative to R and $X \oplus R^{(n-1)}$ forms a minimal pair with $R^{(n)}$ over $R^{(n-1)}$. That is, if $A \leq_T X \oplus R^{(n-1)}$ and $A \leq_T R^{(n)}$ then $A \leq_T R^{(n-1)}$.*

Proof. First suppose X is μ-weak-$(n+1)$-random relative to R. Suppose that there is a real A such that $X \oplus R^{(n-1)} \geq_T A$ and $R^{(n)} \geq_T A$. Let $f(m,s)$ be the $\Delta^0_2(R^{(n-1)})$ approximation of A. That is, f is recursive in $R^{(n-1)}$ and $\lim_s f(m,s) = A(m)$ for all m. Fix e such that $\Phi_e^{X \oplus R^{(n-1)}} = A$. Define

$$T = \{Y : \forall m \exists s (\Phi_{e,s}^{Y \oplus R^{(n-1)}}(m) \downarrow) \\ \wedge \forall m \forall s (\Phi_{e,s}^{Y \oplus R^{(n-1)}}(m) \downarrow \neq f(m,s) \Rightarrow (\exists s' > s)(f(m,s') \neq f(m,s)))\}$$

Then T is a $\Pi^0_2(R^{(n-1)})$ class containing X. Since X is μ-weak-$(n+1)$-random relative to R, it must be the case that $\mu(T) > 0$. Fix a rational $q > 0$ such that $\frac{3}{4}\mu(T) < q < \mu(T)$. By the definition of μ as an outer measure on the basic open sets, there is a finite set of strings F such that $T \subseteq \bigcup\{[\sigma] : \sigma \in F\}$ and $\mu(F) < \mu(T) + \frac{q}{3}$. Let $[F]$ denote $\bigcup\{[\sigma] : \sigma \in F\}$.

The following is a program in $R^{(n-1)}$ computing A, showing $A \leq_T R^{(n-1)}$ as desired.

Given m and $i = 0, 1$ define

$$T_i = \{\sigma \in [F] : \exists s (\Phi_{e,s}^{\sigma \oplus R^{(n-1)}}(m) \downarrow = i\}$$

For $i = A(m)$, $T_i \supseteq T$, so $\mu(T_i) > q$. Hence there is a stage s such that $R_s(T_{i,s}) > q$ for at least one i. On the other hand since $q > \frac{3}{4}\mu(T)$ we have $\mu(F) < \mu(T) + \frac{q}{3} < \frac{4}{3}q + \frac{q}{3} = \frac{5}{3}q < 2q$. Since $T_0 \cup T_1 \subseteq [F]$ we thus have $\mu(T_i) > q$ for exactly one i, which as noted above is the i such that $i = A(m)$. Hence this computes the kth bit of A.

For the reverse, we demonstrate the contrapositive. Suppose X is not μ-weak-$(n+1)$-random relative to R, but X is μ-n-random relative to R. We will build an $A \not\leq_T R^{(n-1)}$ such that $A \leq_T X \oplus R^{(n-1)}$ and $A \leq_T R^{(n)}$.

Since X is not weak-$(n+1)$-random, there is a uniformly $\Sigma^0_1(R^{(n-1)})$ sequence $\{V_m\}$ such that $X \in \bigcap_m V_m$ and $\lim_m \mu(V_m) = 0$. We construct A as an r.e. in $R^{(n-1)}$ set, and hence $A \leq_T R^{(n)}$.

For each e pick a fresh large n_e. If we ever see a stage s such that $R_s(V_{n_e,s}) > 2^{-(e+1)}$ (and we have not yet stopped action for e) choose a new n_e and restart. Otherwise if we see a stage $s \geq e$ such that $n_e \in W_{e,s}^{R^{(n-1)}}$ then we wait for a stage $s' > s$ such that $R_{s'}(V_{n_e,s}) < 2^{-e}$. When this happens, put n_e into A and stop the action of e.[†]

Note that for all e, each n_e eventually reaches a final value. If we reach a final value of n_e because $\mu(V_n) \leq 2^{-(e+1)}$ and $n_e \notin W_e^{R^{(n-1)}}$, then we never put n_e into A, so $\overline{A} \neq W_e^{R^{(n-1)}}$. On the other hand, we may reach a final value of n_e because we find a stage s such that $n_e \in W_{e,s}^{R^{(n-1)}}$ and eventually see that $\mu(V_{n_e,s}) < 2^{-e}$. In this case we have $n_e \in A$ so again $\overline{A} \neq W_e^{R^{(n-1)}}$. Hence $A \not\leq_T R^{(n-1)}$.

[†]The choice of $2^{-(e+1)}$ in the first step and then 2^{-e} is the second is intentional and necessary. If we see n_e enter A at stage s, then if $\mu(V_{n_e,s}) = 2^{-e}$ we may never see this fact and hence never choose a new value for n_e nor put n_e into A. But in this case we would see a stage such that $R_s(V_{n_e,s}) > 2^{-(e+1)}$ and pick a new n_e.

Now it just remains to show $A \leq_T X \oplus R^{(n-1)}$. Define the Martin-Löf test relative to $R^{(n-1)}$ by $V'_e = \{\sigma \in V_{n_{e+1},s} : n_{e+1}$ enters A at stage $s\}$. By our definition of putting n_{e+1} into A, we have $\mu(V'_e) = \mu(V_{n_{e+1},s}) < 2^{-(e+1)} + 2^{-(e+1)} = 2^{-e}$. Furthermore, since we can enumerate A using $R^{(n-1)}$, V'_e is uniformly r.e. in $R^{(n-1)}$. Hence it is Martin-Löf test relative to $R^{(n-1)}$.

Let $V''_e = \bigcup_{k=e+1}^{\infty} V'_e$ and observe that V''_e is also a Martin-Löf test relative to $R^{(n-1)}$, so there is some e such that $X \notin V''_e$. Hence for all but finitely many e, $X \notin V'_e$. Since we were assuming $X \in \bigcap_m V_m$, let s_{n_e} be the least stage such that $X \in V_{n,s_{n_e}}$, which is uniformly recursive in $X \oplus R^{(n-1)}$. Then for all but finitely many e, $n_e \in A$ iff $n_e \in A_{s_{n_e}}$. Hence $A \leq_T X \oplus R^{(n-1)}$.

□

The requirement that X form a (relativized) minimal pair with R' for some representation R of a measure μ makes it particularly difficult to directly construct a measure such that X is μ-weak-2-random. The next theorem shows that this requirement is enough to narrow down the class of reals for which such a μ can exist. Furthermore, it actually provides sharp strengthening of observations made by Reimann and Slaman during their investigation of NCR.

Proposition 2. *[19] If $n \geq 2$, then for all $k \geq 0$, $0^{(k)}$ is not n-random with respect to a continuous measure.*

Proposition 3. *[19] For $n \geq 3$, $0^{(\omega)}$ is not n-random with respect to a continuous measure.*

In the proof that follows, we will use a lemma by Sacks [20] which shows the hyperjump sets are uniformly Π^0_2 definable. We use the notation of Sacks in which H_a is the ath iterate of the halting problem for a, a notation for a recursive ordinal. The reader is referred to Sacks's book for additional background on the hyperarithmetic hierarchy.

Lemma 6. *[20] There is a Π^0_2 formula $H(a, Z)$ such that if a is a notation for a recursive ordinal, $H(a, Z)$ holds iff $Z = H_a$.*

Theorem 7. *Fix $k \geq 1$ and let X be a real such that $0^{(\alpha)} \leq_T X \leq_T 0^{(\alpha+k)}$ for any recursive ordinal $\alpha < \omega_1^{CK}$. Then X is not weak-$(k+1)$-randomizable.*

Proof. Suppose for a contradiction that X is μ-weak-$(k+1)$-random for some measure μ with $\mu(\{X\}) = 0$. Fix an ordinal representation a such that $|a|_{\mathcal{O}} = \alpha$ and fix e such that $\Phi_e^X = H_a$. Let $\mathcal{C} = \{Z : \Phi_e^Z$ is total $\wedge H(a, \Phi_e^Z)\}$. Then by Lemma 6, $\mathcal{C}$ is a Π^0_2 class, and by assumption $X \in \mathcal{C}$. Since we are assuming that X is μ-weak-$(k+1)$-random, this implies that $\mu(\mathcal{C}) > 0$.

Let R be a representation of μ. It is claimed that $R \geq_T H_a$. We first select a string σ such that $\mu([\sigma] \cap \mathcal{C}) > \frac{3}{4}\mu(\sigma)$. Such a σ exists by the general density theorem (a generalization of the Lebesgue Density Theorem; see e.g. Mattila [15] for reference). Define $Z(n, s, i) =$

$\{X \in [\sigma] : \Phi^X_{e,s}(n) \downarrow= i\}$. Fix t large enough so that $2^{-t} < \frac{1}{5}\mu(\sigma)$. Note that for any $s > t$ we have $2^{-t} < \frac{1}{5}(R_s(\sigma) + 2^{-s}) < \frac{1}{5}R_s(\sigma) + \frac{1}{5}2^{-t}$ and hence $2^{-t} < \frac{1}{4}R_s(\sigma)$.

We can now compute the nth bit of H_a from R by waiting for a stage $s > t$ such that $R_{s+1}(Z(n, s, i)) \geq \frac{1}{2}R_s(\sigma)$ and outputting i.

We note that such a stage exists since $\lim_{s\to\infty} Z(n, s, H_a(n)) \supseteq [\sigma] \cap \mathcal{C}$, so we can pick s_1 such that $\mu([\sigma] \cap \mathcal{C}) - \mu(Z(n, s_1, H_a(n))) < \frac{1}{4}\mu(\sigma)$. We then pick s_2 large enough so that $2^{-s_2} < \mu([\sigma] \cap \mathcal{C}) - \frac{3}{4}\mu(\sigma)$ and let $s = \max(s_1, s_2)$. We then have

$$\begin{aligned} R_{s+1}(Z(n, s, H_a(n))) &\geq \mu(Z(n, s, H_a(n))) - 2^{-(s+1)} \\ &\geq \mu([\sigma] \cap \mathcal{C}) - \frac{1}{4}\mu(\sigma) - 2^{-(s+1)} \\ &> \frac{3}{4}\mu(\sigma) + 2^{-s} - \frac{1}{4}\mu(\sigma) - 2^{-(s+1)} \\ &= \frac{1}{2}\mu(\sigma) + 2^{-s} - 2^{-(s+1)} \\ &\geq \frac{1}{2}(R_s(\sigma) - 2^{-s}) + 2^{-s} - 2^{-(s+1)} \\ &= \frac{1}{2}R_s(\sigma) \end{aligned}$$

Furthermore, we note that if $i \neq H_a(n)$ then $\mu(Z(n, s, i)) \leq \frac{1}{4}\mu(\sigma))$ and hence

$$\begin{aligned} R_{s+1}(Z(n, s, i)) &\leq \mu(Z(n, s, i)) + 2^{-(s+1)} \leq \frac{1}{4}\mu(\sigma) + 2^{-(s+1)} \\ &\leq \frac{1}{4}(R_s(\sigma) + 2^{-s}) + 2^{-(s+1)} < \frac{1}{4}R_s(\sigma) + 2^{-s} \\ &< \frac{1}{4}R_s(\sigma) + 2^{-t} < \frac{1}{2}R_s(\sigma) \end{aligned}$$

Thus whenever our procedure outputs an answer, it will be equal to $H_a(n)$.

Since we are assuming that X is μ-weak-$(k+1)$-random, there is a representation R of μ such that X is μ-weak-$(k+1)$-random relative to R. In particular this means $R^{(k-1)} \not\geq_T X$. However, $R^{(k)} \geq_T 0^{(\alpha+k)} \geq_T X$. This contradicts the minimal pair property of Theorem 6: $X \leq_T X \oplus R^{(k-1)}$ and $X \leq_T R^{(k)}$, but $X \not\leq_T R^{(k-1)}$. Hence X could not have been μ-weak-$(k+1)$-random. □

Since this theorem implies that no Δ^0_2 real is weak-2-randomizable and Proposition 1 implies there are no r.e. difference-randomizable reals, it is interesting to note that there are ω-r.e. difference-randomizable reals. Indeed, with respect to Lebesgue measure one can choose a low ω-r.e. path through the first level of the universal Martin-Löf test $\overline{U_1}$. This path will be Martin-Löf random and Turing incomplete and hence is difference-random by Theorem 2. By contrast, there are no weak-2-randomizable Δ^0_2 reals. It is still open as to whether or not there are n-r.e. difference-randomizable reals. These results are summarized in the below table, where each column represents reals not belonging the columns to the left.

Which Δ^0_2 reals are randomizable?

	recursive	r.e.	n-r.e.	ω-r.e.
1-randomizable	None	All	All	All
difference-randomizable	None	None	?	Some (All?)
weak-2-randomizable	None	None	None	None

In a slight tangent, we would like to show that there are no n-r.e. 1-random reals for Lebesgue measure (and hence neither are there such difference-randoms). This eliminates the possibility of using such an example to demonstrate n-r.e. difference-randomizable reals, as we did above to show that there is a ω-r.e difference-randomizable real.

Proposition 4. *Suppose X is n-r.e. for some $n \geq 1$. Then X is not Martin-Löf random with respect to Lebesgue measure.*

Proof. We proceed by induction on n. Define $f(i, s)$ to be a recursive approximation of X. Let $c_i = |\{s : f(i, s) \neq f(i, s + 1)\}|$ be the number of times the approximation to i changes. If there are only finitely i such that $c_i = n$, then there is a recursive approximation $\hat{f}$ which changes at most $n - 1$ times for each i. Hence X is $(n - 1)$-r.e. so we are done by induction.

Otherwise for each k define s_k as the least number such that $\exists i_1, \dots, i_k < s_k \forall j \leq k(|\{s < s_k : f(i_j, s) \neq f(i_j, s + 1)\}| = n)$ Note that each of s_k and $i_1, \dots, i_k$ are uniformly recursive. Define the test

$$V_k = \{\sigma : \forall 1 \leq j \leq k(\sigma(i_j) = f(i_j, s_k))\}$$

Since the approximation $f(i_j, s_k)$ has already changed n times, it cannot change again, so $f(i_j, s_k) = X(i_j)$. Hence $X \in V_k$ for each k. Further, it is clear by definition that $\lambda(V_k) = 2^{-k}$. Thus X is not Martin-Löf random. □

2.4 Constructing Measures for Randomizable Reals

The set of reals which are not n-randomizable is contained in HYP (since $\text{NCR}_n \subseteq \text{HYP}$), so the task of identifying which reals are not randomizable can be constrained to this context. With this in mind, Theorem 7 provides a large coverage of the domain in question. In the results which follow, we show that this theorem is, in a sense, as strong as it can be made since if we widen the interval between jumps we *can* always find a randomizable real.

However, before we show that result we provide a more general context for the construction of measures, as it will aid in the discussion which follows. Given a continuous function $f : 2^\omega \to 2^\omega$, we can define the image measure μ_f as $\mu_f(A) = \lambda(f^{-1}(A))$. It is easy to verify that this is a well-defined measure, and in fact it can be shown that every probability measure on 2^ω is equal to μ_f for some f (though we won't make use of this fact).

Since a Turing reduction Φ is a (partial) map from $2^{<\omega}$ to $2^{<\omega}$, we can extend it to a continuous (partial) map from 2^ω to 2^ω and define the measure μ_Φ in the same way. In our context of pre-measures, we can equivalently define $\mu_\Phi(\sigma) = \sum_{\tau:\Phi(\tau)\supseteq\sigma} 2^{-|\tau|}$ where we assume the sum ranges over a prefix-free set of τ. We immediately have $\mu_\Phi(\langle\rangle) = 1$, but to get additivity (and thus have μ_Φ be a well-defined measure) Φ must be total on all oracles. Equivalently, Φ must be a truth-table reduction.

Given a probability measure μ, suppose there is a constant $c > 0$ such that for all σ, $\mu(\sigma) \geq c\lambda(\Phi^{-1}(\sigma))$ (which in particular will be true for $c = 1$ if μ_Φ is well-defined and $\mu = \mu_\Phi$). Then μ preserves randomness in the sense that if X is random for Lebesgue measure then $\Phi(X)$ is random for μ. This is made precise by the next proposition:

Proposition 5. *Let X be a real and Φ a Turing functional with $X \in \mathrm{dom}(\Phi)$. Let μ be a measure with representation recursive in R such that X is [difference, weak-n, n]-random relative to R. Further suppose that there is a constant $c > 0$ such that for all σ, $\mu(\sigma) \geq c\lambda(\Phi^{-1}(\sigma))$. Then $\Phi(X)$ is μ-[difference, weak-n, n]-random relative to R.*

Proof. Suppose the conclusion does not hold, i.e. there is a test $\{V_n\}$ relative to R which has the appropriate μ measure and captures $\Phi(X)$. We will show that X is not actually random, thus reaching a contradiction. We verify this independently for each of the three types of randomness, although each of the cases are analogous to one another.

n-randomness The test $\{V_n\}$ is uniformly r.e. in $R^{(n-1)}$ and satisfies $\mu(V_n) \leq 2^{-n}$. Fix k large enough so that $2^k > \frac{1}{c}$ and define $V'_n = \bigcup\{\Phi^{-1}(\sigma) : \sigma \in V_{n+k}\}$. Then clearly V'_n is also uniformly r.e. in $R^{(n-1)}$, captures X, and

$$\lambda(V'_n) = \sum_{\sigma\in V_{n+k}} \lambda(\Phi^{-1}(\sigma)) \leq \sum_{\sigma\in V_{n+k}} \frac{1}{c}\mu(\sigma) < 2^k 2^{-(n+k)} = 2^{-n}$$

Weak-n-randomness The test $\{V_n\}$ is uniformly r.e. in $R^{(n-1)}$ and satisfies the limit property $\lim_{n\to\infty} \mu(V_n) = 0$. Define $V'_n = \bigcup\{\Phi^{-1}(\sigma) : \sigma \in V_n\}$. Then clearly V'_n is also uniformly r.e. in $R^{(n-1)}$, captures X, and

$$\lim_{n\to\infty} \lambda(V'_n) \leq \lim_{n\to\infty} \frac{1}{c}\mu(V_n) = \frac{1}{c} \lim_{n\to\infty} \mu(V_n) = 0$$

Difference-randomness The test $\{V_n\}$ can be written as $V_n = D_{n,0} \setminus D_{n,1}$ where each set $D_{n,i}$ is uniformly r.e. in R and $\mu(V_n) \leq 2^{-n}$. We can assume without loss of generality that $D_{n,1} \subseteq D_{n,0}$ for all n, and hence $\mu(V_n) = \mu(D_{n,0}) - \mu(D_{n,1})$. Fix k large enough so that $2^k > \frac{1}{c}$ and define $D'_{n,i} = \bigcup\{\Phi^{-1}(\sigma) : \sigma \in D_{n+k,i}\}$. Then clearly each $D'_{n,i}$ is uniformly r.e. in R. Furthermore, $D'_{n,0} \setminus D'_{n,1} = \Phi^{-1}(D_{n+k,0}) \setminus \Phi^{-1}(D_{n+k,1})) =$

$\Phi^{-1}(D_{n+k,0} \setminus D_{n+k,1}) = \Phi^{-1}(V_{n+k})$, so X is captured by the sequence $D'_{n,0} \setminus D'_{n,1}$. Finally,

$$\begin{aligned}\lambda(D'_{n,0} \setminus D'_{n,1}) &= \lambda(\Phi^{-1}(D_{n+k,0})) - \lambda(\Phi^{-1}(D_{n+k,1})) \\ &\leq \frac{1}{c}\left(\mu(D_{n+k,0}) - \mu(D_{n+k,1})\right) \\ &= \frac{1}{c}\mu(V_{n+k}) \\ &\leq \frac{1}{c}2^{-(n+k)} \\ &< 2^{-n}\end{aligned}$$

□

This then gives us a tool for showing reals are randomizable. Suppose $X \equiv_{tt} Y$ and X is random relative to some real R. Let Φ be the truth-table Turing functional such that $\Phi(X) = Y$. Then Y is μ_Φ-random relative to R. Furthermore, because the equivalence is truth-table, the resulting measure μ_Φ is actually continuous. Indeed, this statement reverses showing that this is the only way to find reals which are random relative to a continuous measure. This was made precise by Reimann and Slaman for n-randomness, but also applies to difference-randomness and weak-n-randomness.

Proposition 6. *[19] Fix reals X and R. Then the following are equivalent:*

1. *X is random relative to a continuous measure μ with a representation recursive in R.*
2. *There is a real Y which is Lebesgue random relative to R and is truth-table in R equivalent to X.*

We now extend this idea to find randomizable reals arbitrarily high in the Turing degrees, and contrast this with Theorem 7.

Proposition 7. *Suppose Z is [difference, weak-n, n]-random relative to X. Then there is a continuous measure μ such that $Z \oplus X$ is μ-[difference, weak-n, n]-random relative to X.*

Proof. Define the Turing functionals Φ, Ψ relative to X as follows:

$$\begin{aligned}\Phi(\sigma \oplus \tau) &= \sigma \\ \Psi(\sigma) &= \sigma \oplus X \upharpoonright |\sigma|\end{aligned}$$

Both Φ and Ψ are truth-table reductions, so $Z \equiv_{tt(X)} Z \oplus X$. Hence μ_Ψ is well defined and has a representation recursive in X. Since Z is random relative to X, by Proposition 5 $\Psi(Z) = Z \oplus X$ is μ_Ψ-random relative to X as desired. □

Corollary 2. *For any $n \geq 1$ and real X, there is a real Z such that $X \leq_T Z \leq_T X^{(n+1)}$ such that Z is continuously n-randomizable.*

In particular this shows that there are reals above $0'$ which are difference-random. Since Franklin and Ng characterized the Lebesgue difference-randoms as those Martin-Löf randoms not above $0'$, this contrasts by showing that — for arbitrary measures — there are arbitrarily high difference-randomizable reals.

At this point, it is worth reviewing the specific methods used in Reimann and Slaman's proof that all non-recursive reals are 1-randomizable. The intent is to show that these methods are required features of any proof which shows a real is randomizable.

In Reimann and Slaman's proof, they started with a non-recursive real X and first appealed to a Posner-Robinson argument to find a generic real G such that $X \equiv_{T(G)} G'$. Then by applying a result by Kučera, they found a real Z such that $Z \equiv_T G'$ and Z is 1-random relative to G. So X is Turing equivalent in G to a 1-random.

Although the Turing equivalence was not total (and therefore they could not just appeal to the image measure μ_Φ), they were able to define a Π^0_1 class of measure representations that were consistent with the image measure of total functions extending Φ. They were then able to finish their proof by finding a path in the Π^0_1 class relative to which Z was still 1-random. Since this path represents a measure consistent with the image measure and has a representation relative to which Z is random, the argument of Proposition 5 implies X is random relative to this measure.

As briefly mentioned in the previous section, the step of finding a generic G seems to be a requisite step of the proof, at least for r.e. reals. Recalling Corollary 1, if X is an r.e. real and μ a measure with representation R relative to which X is μ-1-random, it must be the case that $X \oplus R \equiv_T R'$.

Continuing this analysis we could ask whether the other steps are necessary. That is, must the proof find an intermediate random Z such that $X \equiv_{T(G)} Z$. We answer this question in the affirmative by providing a direct analogue of Proposition 6 for 1-randomness.

Theorem 8. *Fix reals X and R. Then the following are equivalent:*

1. *R computes a representation of a measure μ such that $\mu(\{X\}) = 0$ and X is μ-1-random relative to a R.*
2. *There is a real Y which is Lebesgue 1-random relative to R and is Turing equivalent in R to X.*

Proof. ($2 \Rightarrow 1$) follows from Reimann and Slaman's proof, as outlined above. ($1 \Rightarrow 2$) will follow from Theorem 9 below. □

In showing ($1 \Rightarrow 2$) we will not be able to show that the μ in question is the image measure of some Turing equivalence (as Reimann and Slaman's proof suggests), but instead we will show that μ induces a Turing functional Φ with the measure-preserving requirements of Proposition 5. The Turing functional will not in general be total (unless μ is continuous, in which case we could appeal to Proposition 6 directly).

In the Theorem 8, there is nothing special about Lebesgue measure other than that it is continuous. Thus the fully generalized version of the theorem below defines Y as a μ_2-random real for any continuous measure μ_2.

Before proceeding to the proof of Theorem 9, we will need to refine our representation of measures in order to keep track of the multiplicative effects of errors. Consequently it will be necessary to use approximations which are within a multiplicative constant of the correct value, rather than being within an additive constant as in our definition of $R_s(\sigma)$. Thus we define some new notation.

If M is a representation of μ, we want $M(\sigma, \epsilon)$ be the estimation of $\mu(\sigma)$ to within an relative error of ϵ; that is, we would like to have $M(\sigma, \epsilon) \leq \mu(\sigma) \leq M(\sigma, \epsilon)(1 + \epsilon)$. Our particular definition of $M(\sigma, \epsilon)$ will also need some monotonicity properties expanded upon below.

First define the approximation $M^*(\sigma, 2^{-n})$ for $n \geq 0$. Let s_1 be least such that $M_{s_1}(\sigma) > 2^{-s_1}$ if it exists, and pick s_2 least such that

$$2^{-s_2} < \left(1 + \frac{2^{-n}}{2 + 2^{-n}}\right)^{-1} \frac{1}{2 + 2^{-n}} 2^{-n}(M_{s_1}(\sigma) - 2^{-s_1})$$

Then define $M^*(\sigma, 2^{-n}) = M_{s_2}(\sigma) - 2^{-s_2}$. Note that the computation of $M^*(\sigma, 2^{-n})$ does not converge if $\mu(\sigma) = 0$ and otherwise such an s_1 does exist, so the computation converges.

By definition $M^*(\sigma, 2^{-n}) = M_{s_2}(\sigma) - 2^{-s_2} \leq \mu(\sigma)$ as desired, and furthermore we have

$$\begin{aligned} 2^{-s_2} &\leq \left(1 + \frac{2^{-n}}{2 + 2^{-n}}\right)^{-1} \frac{1}{2 + 2^{-n}} 2^{-n} \left(M_{s_1}(\sigma) - 2^{-s_1}\right) \\ &\leq \left(1 + \frac{2^{-n}}{2 + 2^{-n}}\right)^{-1} \frac{1}{2 + 2^{-n}} 2^{-n} \left(M_{s_2}(\sigma) + 2^{-s_2}\right) \end{aligned}$$

Thus

$$\begin{aligned} \left(1 + \frac{2^{-n}}{2 + 2^{-n}}\right)(2 + 2^{-n})2^{-s_2} &\leq 2^{-n}(M_{s_2}(\sigma) + 2^{-s_2}) \\ (2 + 2^{-n})2^{-s_2} + 2^{-n}2^{-s_2} &\leq 2^{-n}M_{s_2}(\sigma) + 2^{-n}2^{-s_2} \\ (2 + 2^{-n})2^{-s_2} &\leq 2^{-n}M_{s_2}(\sigma) \\ (1 + 2^{-n})2^{-s_2} + 2^{-s_2} &\leq (1 + 2^{-n})M_{s_2}(\sigma) - M_{s_2}(\sigma) \\ M_{s_2}(\sigma) + 2^{-s_2} &\leq (1 + 2^{-n})(M_{s_2}(\sigma) - 2^{-s_2}) \\ \mu(\sigma) &\leq (1 + 2^{-n})M^*(\sigma, 2^{-n}) \end{aligned}$$

Hence we have $M^*(\sigma, 2^{-n}) \leq \mu(\sigma) \leq M^*(\sigma, 2^{-n})(1 + 2^{-n})$. Define inductively $M(\sigma, 2^{-n})$ as follows. For $n = 0$ we let $M(\sigma, 2^{-n}) = M^*(\sigma, 2^{-n})$. For $n > 0$, if $M^*(\sigma, 2^{-n})(1 + 2^{-n}) >$

$M(\sigma, 2^{-(n-1)})(1 + 2^{-(n-1)})$ let $M(\sigma, 2^{-n}) = M(\sigma, 2^{-(n-1)})\frac{1+2^{-(n-1)}}{1+2^{-n}}$. Otherwise let $M(\sigma, 2^{-n})$ be the larger of $M^*(\sigma, 2^{-n})$ and $M(\sigma, 2^{-(n-1)})$.

We now verify a number of facts about $M(\sigma, 2^{-n})$, proceeding by induction on n.

1. $M(\sigma, 2^{-n}) \leq \mu(\sigma)$: First suppose $M(\sigma, 2^{-n}) = M^*(\sigma, 2^{-n})$. Then by the above verification, $M^*(\sigma, 2^{-n}) \leq \mu(\sigma)$ so we are done. If $n > 0$ and $M(\sigma, 2^{-n}) = M(\sigma, 2^{-(n-1)})$ then by induction we have $M(\sigma, 2^{-(n-1)}) \leq \mu(\sigma)$ so again we are done. Otherwise it must be the case that

$$M^*(\sigma, 2^{-n})(1 + 2^{-n}) > M(\sigma, 2^{-(n-1)})(1 + 2^{-(n-1)}) = M(\sigma, 2^{-n})(1 + 2^{-n})$$

   and hence $\mu(\sigma) \geq M^*(\sigma, 2^{-n}) > M(\sigma, 2^{-n})$.

2. $M(\sigma, 2^{-n})(1 + 2^{-n}) \geq \mu(\sigma)$: If $M(\sigma, 2^{-n}) = M(\sigma, 2^{-(n-1)})\frac{1+2^{-(n-1)}}{1+2^{-n}}$ then by induction we have $M(\sigma, 2^{-n})(1 + 2^{-n}) = M(\sigma, 2^{-(n-1)})(1 + 2^{-(n-1)}) \geq \mu(\sigma)$. Otherwise suppose $M(\sigma, 2^{-n}) = M^*(\sigma, 2^{-n})$. Then by the above verification, $\mu(\sigma) \leq M^*(\sigma, 2^{-n})(1+2^{-n})$. Finally, suppose $n > 0$ and $M(\sigma, 2^{-n}) = M(\sigma, 2^{-(n-1)})$ in which case $M^*(\sigma, 2^{-n}) \leq M(\sigma, 2^{-(n-1)})$. Then we have by the above verification that

$$\begin{aligned}\mu(\sigma) &\leq M^*(\sigma, 2^{-n})(1 + 2^{-n}) \\ &\leq M(\sigma, 2^{-(n-1)})(1 + 2^{-n}) = M(\sigma, 2^{-n})(1 + 2^{-n})\end{aligned}$$

3. $M(\sigma, 2^{-(n+1)}) \geq M(\sigma, 2^{-n})$: First suppose that

$$M(\sigma, 2^{-(n+1)}) = M(\sigma, 2^{-n})\frac{1 + 2^{-n}}{1 + 2^{-(n+1)}}$$

   Then we immediately have $M(\sigma, 2^{-(n+1)}) > M(\sigma, 2^{-n})$. Otherwise we have by definition that $M(\sigma, 2^{-(n+1)})$ is the larger of $M^*(\sigma, 2^{-(n+1)})$ and $M(\sigma, 2^{-n})$ so again we are done.

4. $M(\sigma, 2^{-(n+1)})(1 + 2^{-(n+1)}) \leq M(\sigma, 2^{-n})(1 + 2^{-n})$: First suppose it is the case that $M(\sigma, 2^{-(n+1)}) = M(\sigma, 2^{-n})\frac{1+2^{-n}}{1+2^{-(n+1)}}$. Then $M(\sigma, 2^{-(n+1)})(1 + 2^{-(n+1)}) = M(\sigma, 2^{-n})(1 + 2^{-n})$ so we are done. Otherwise it must be the case that $M^*(\sigma, 2^{-(n+1)})(1 + 2^{-(n+1)}) \leq M(\sigma, 2^{-n})(1+2^{-n})$, so if $M(\sigma, 2^{-(n+1)}) = M^*(\sigma, 2^{-(n+1)})$ we are done. Otherwise we are in the case that $M(\sigma, 2^{-(n+1)}) = M(\sigma, 2^{-n})$. Then we have $M(\sigma, 2^{-(n+1)})(1 + 2^{-n+1}) = M(\sigma, 2^{-n})(1 + 2^{-(n+1)}) < M(\sigma, 2^{-n})(1 + 2^{-n})$, so we have the desired conclusion.

In general define $M(\sigma, \epsilon)$ as $M(\sigma, 2^{-n})$ where n is least such that $2^{-n} < \epsilon$. Then we have in summary that $M(\sigma, \epsilon) \leq \mu(\sigma) \leq M(\sigma, \epsilon)(1 + \epsilon)$, $M(\sigma, \epsilon)$ is monotonically increasing as ϵ goes to zero, and $M(\sigma, \epsilon)(1 + \epsilon)$ is monotonically decreasing as ϵ goes to zero.

We now present the main lemma in the proof of Theorem 9.

Lemma 7. *Let μ_1, μ_2 be measures with representations M_1, M_2 respectively. Then there are Turing functionals Φ, Ψ relative to $M_1 \oplus M_2$ such that* $\mathrm{dom}(\Phi) \supseteq \{X \in 2^\omega : \mu_1(\{X\}) = 0 \wedge \forall n\ \mu_1(X \upharpoonright n) > 0\}$*, for all $X \in \mathrm{dom}(\Phi)$ $\Psi(\Phi(X)) = X$, and there is a $c > 0$ such that for all σ $\mu_2(\sigma) \geq c\mu_1(\Phi^{-1}(\sigma))$.*

Proof. We will construct the Turing functional Φ recursively in $M_1 \oplus M_2$ as follows, simultaneously constructing its inverse functional Ψ. Let $\epsilon_s = \frac{1}{5}4^{-(s+1)}$ and note $\epsilon_s \leq \frac{1}{20}$ for all s. The construction will use a sequence of machines $\{T_s\}_{s\in\omega}$ which run in parallel. Each machine will be simultaneously acting on tuples $\langle \sigma, \{\tau_1, \ldots, \tau_n\}, \delta \rangle$ which have the following properties:

1. $\mu_1(\sigma) > 0$ and $\mu_2(\tau_i) > 0$ for each τ_i
2. $(1 + \epsilon_s)\, M_1(\sigma, \epsilon_s) \leq \sum_{i=1}^{n} M_2(\tau_i, \epsilon_s) + \delta$
3. $\frac{\delta}{\sum_{i=1}^{n} M_2(\tau_i, \epsilon_s)} \leq \frac{1}{3}\left(1 - 4^{-s}\right)$

Intuitively, the machine T_s is acting to put extensions of σ into the domain of Φ, with the range being restricted to $\bigcup_{i=1}^{n} [\tau_i]$. The rational δ represents the "over-commitment" of measure, as made precise by restriction (2) above.

At the beginning of the construction we add $\langle \langle\rangle, \{\langle\rangle\}, \epsilon_1 \rangle$ to the machine T_1.

When a machine T_s receives a tuple $\langle \sigma, \{\tau_1, \ldots, \tau_n\}, \delta \rangle$ it acts as follows. First set variables $A_1 = \cdots = A_n = 0$ and $B_1 = \cdots = B_n = \emptyset$ which will be local to this tuple. The variables A_i will keep track of the μ_1-measure assigned to each τ_i while the B_i will be used to keep track of the extensions of τ_i already put into the range of Φ. Let $\alpha = \epsilon_s \min(\{M_2(\tau_i, \epsilon_s) : 1 \leq i \leq n\})$. Begin searching for $\hat{\sigma} \supset \sigma$ such that $M_1(\hat{\sigma}, \epsilon_s) < \frac{\alpha}{2}$.

When such a $\hat{\sigma}$ is found, proceed as follows. Pick i least such that $A_i \leq M_2(\tau_i, \epsilon_s) + \delta \frac{M_2(\tau_i, \epsilon_s)}{\sum_j M_2(\tau_j, \epsilon_s)}$ (it will be shown below that such an i always exists). Define

$$\hat{\delta} = (1 + \epsilon_s)\left(\delta \frac{M_2(\tau_i, \epsilon_s)}{\sum_{j=1}^{n} M_2(\tau_j, \epsilon_s)} + \alpha\right)$$

Let $\epsilon^* = \frac{1}{2}\epsilon_s M_1(\hat{\sigma}, \epsilon_s) \min(1, \frac{M_2(\tau_i, \epsilon_s)}{\hat{\delta}})$. Let k be large enough so that $\mu_2(\tau) < \epsilon^*$ for all strings τ of length k (that such a k can be found effectively is the only use of the fact that μ_2 is continuous). Let $\tau'_1, \ldots, \tau'_{n'}$ be the strings extending τ_i of length k extending τ_i not already contained in B_i.

Fix t such that $2^{-t} < \frac{\alpha}{2n'}$ and let $\hat{\tau}_1, \ldots, \hat{\tau}_{\hat{n}}$ be those τ'_i such that $M_{2,t}(\tau'_i) > 0$. Note that

$$\sum_{i=1}^{\hat{n}} \mu_2(\hat{\tau}_i) \geq \sum_{i=1}^{n'} \mu_2(\tau'_i) - n' \cdot 2^{-t} > \sum_{i=1}^{n'} \mu_2(\tau'_i) - \frac{\alpha}{2}$$

Pick $m \leq \hat{n}$ such that

$$\sum_{j=1}^{m}\left(M_2(\hat{\tau}_j,\epsilon_s)+\hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right) < (1+\epsilon_s)\, M_1(\hat{\sigma},\epsilon_s)$$
$$\leq \sum_{j=1}^{m}\left(M_2(\hat{\tau}_j,\epsilon_s)+\hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right) + \epsilon_s M_1(\hat{\sigma},\epsilon_s)$$

It will be shown below that we can always find such an m.

Increase A_i by $(1+\epsilon_s)\, M_1(\hat{\sigma},\epsilon_s)$, add $\{\hat{\tau}_1,\ldots,\hat{\tau}_m\}$ to B_i, set $\Phi(\hat{\sigma})=\tau_i$, and set $\Psi(\hat{\tau}_j)=\hat{\sigma}$ for $j=1,\ldots,m$. Define

$$\delta_{s+1} = \hat{\delta}\cdot\frac{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)} + \epsilon_s M_1(\hat{\sigma},\epsilon_s)$$

and add $\langle\hat{\sigma},\{\hat{\tau}_1,\ldots,\hat{\tau}_m\},\delta_{s+1}\rangle$ to machine T_{s+1}. This ends the action upon finding $\hat{\sigma}$ and ends the construction. We now verify that all the requirements of the construction are met.

First we check that (2) is maintained for the new tuple added to T_{s+1}. This is immediate from the above observation:

$$(1+\epsilon_{s+1})\, M_1(\hat{\sigma},\epsilon_{s+1}) \leq (1+\epsilon_s)\, M_1(\hat{\sigma},\epsilon_s)$$
$$\leq \sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s) + \hat{\delta}\cdot\frac{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)} + \epsilon_s M_1(\hat{\sigma},\epsilon_s)$$
$$= \sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s) + \delta_{s+1}$$

To verify (3) we first observe using the inequality above that:

$$(1+\epsilon_s)\, M_1(\hat{\sigma},\epsilon_s) \leq \sum_{j=1}^{m}\left(M_2(\hat{\tau}_j,\epsilon_s)+\hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right) + \epsilon_s M_1(\hat{\sigma},\epsilon_s)$$
$$M_1(\hat{\sigma},\epsilon_s) \leq \sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s) + \hat{\delta}\frac{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}$$
$$\frac{M_1(\hat{\sigma},\epsilon_s)}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)} \leq 1 + \frac{\hat{\delta}}{M_2(\tau_i,\epsilon_s)}$$

Then

$$\begin{aligned}
\frac{\hat{\delta}}{M_2(\tau_i,\epsilon_s)} &= (1+\epsilon_s)\left(\frac{\delta}{\sum_{j=1}^{n} M_2(\tau_j,\epsilon_s)} + \frac{\alpha}{M_2(\tau_i,\epsilon_s)}\right) \\
&\leq (1+\epsilon_s)\left(\frac{1}{3}\left(1-4^{-s}\right)+\epsilon_s\right) \\
&\leq (1+\epsilon_s)\left(\frac{1}{3}+\epsilon_s\right) \\
&\leq (1+\frac{1}{20})(\frac{1}{3}+\frac{1}{20}) < \frac{1}{2}
\end{aligned}$$

so we can achieve the crude upper-bound of $\frac{M_1(\hat{\sigma},\epsilon_s)}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)} < 1 + \frac{1}{2} < 2$.

Using this we have

$$\begin{aligned}
\frac{\delta_{s+1}}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_{s+1})} &\leq \frac{\delta_{s+1}}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)} \\
&= \frac{\hat{\delta}}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)} \cdot \frac{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)} + \frac{\epsilon_s M_1(\hat{\sigma},\epsilon_s)}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)} \\
&= \frac{\hat{\delta}}{M_2(\tau_i,\epsilon_s)} + \frac{\epsilon_s M_1(\hat{\sigma},\epsilon_s)}{\sum_{j=1}^{m} M_2(\hat{\tau}_j,\epsilon_s)} \\
&\leq (1+\epsilon_s)\left(\frac{1}{3}\left(1-4^{-s}\right)+\epsilon_s\right) + 2\epsilon_s \\
&= \frac{1}{3}\left(1-4^{-s}\right) + \epsilon_s + \frac{1}{3}\left(1-4^{-s}\right)\epsilon_s + \epsilon_s^2 + 2\epsilon_s \\
&\leq \frac{1}{3}\left(1-4^{-s}\right) + 5\epsilon_s \\
&= \frac{1}{3} - \frac{4}{3}4^{-s-1} + 4^{-s-1} \\
&= \frac{1}{3}\left(1-4^{-s-1}\right)
\end{aligned}$$

Next we check the claims made during the construction. The first is that we can always find an i as above. That is to say, that there is always some i such that $A_i \leq M_2(\tau_i,\epsilon_s) + \delta\frac{M_2(\tau_i,\epsilon_s)}{\sum_j M_2(\tau_j,\epsilon_s)}$. Suppose otherwise for a contradiction. Recall that A_i is increased by $(1+\epsilon_s)\,M_1(\hat{\sigma},\epsilon_s)$ each time the machine acts on some $\hat{\sigma}$, so list such strings $\hat{\sigma}_1,\ldots,\hat{\sigma}_k$ acted on so far. Then we have

$$\begin{aligned}
(1+\epsilon_s)\,M_1(\sigma,\epsilon_s) &\geq (1+\epsilon_s)\sum_{i=1}^{k} M_1(\hat{\sigma}_i,\epsilon_s)\\
&= \sum_{i=1}^{n} A_i\\
&> \sum_{i=1}^{n}\left(M_2(\tau_i,\epsilon_s) + \delta\frac{M_2(\tau_i,\epsilon_s)}{\sum\limits_{j=1}^{n} M_2(\tau_j,\epsilon_s)}\right)\\
&= \sum_{i=1}^{n} M_2(\tau_i,\epsilon_s) + \delta\frac{\sum\limits_{i=1}^{n} M_2(\tau_i,\epsilon_s)}{\sum\limits_{j=1}^{n} M_2(\tau_j,\epsilon_s)}\\
&= \sum_{i=1}^{n} M_2(\tau_i,\epsilon_s) + \delta
\end{aligned}$$

This contradicts the requirement described by (2).

Secondly we claim we can always find an m. Let $m' \leq \hat{n}$ be largest such that

$$\sum_{j=1}^{m'}\left(M_2(\hat{\tau}_j,\epsilon_s) + \hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right) < (1+\epsilon_s)\,M_1(\hat{\sigma},\epsilon_s)$$

First take the case that $m' < \hat{n}$. In this case we have

$$\begin{aligned}
(1+\epsilon_s)\,M_1(\hat{\sigma},\epsilon_s) &\leq \sum_{j=1}^{m'}\left(M_2(\hat{\tau}_j,\epsilon_s) + \hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right)\\
&\qquad + M_2(\hat{\tau}_{m'+1},\epsilon_s)\left(1 + \frac{\hat{\delta}}{M_2(\tau_i,\epsilon_s)}\right)\\
&\leq \sum_{j=1}^{m'}\left(M_2(\hat{\tau}_j,\epsilon_s) + \hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right) + \epsilon^* + \epsilon^*\frac{\hat{\delta}}{M_2(\tau_i\,\epsilon_s)}\\
&\leq \sum_{j=1}^{m'}\left(M_2(\hat{\tau}_j,\epsilon_s) + \hat{\delta}\frac{M_2(\hat{\tau}_j,\epsilon_s)}{M_2(\tau_i,\epsilon_s)}\right) + \epsilon_s M_1(\hat{\sigma},\epsilon_s)
\end{aligned}$$

Hence m' works as the desired m. Now suppose $m' = \hat{n}$ but m' does not work as the desired m, i.e.

$$\sum_{j=1}^{\hat{n}} \left(M_2(\hat{\tau}_j, \epsilon_s) + \hat{\delta} \frac{M_2(\hat{\tau}_j, \epsilon_s)}{M_2(\tau_i, \epsilon_s)} \right) + \epsilon_s M_1(\hat{\sigma}, \epsilon_s) < (1 + \epsilon_s) M_1(\hat{\sigma}, \epsilon_s)$$

Let $\rho_1, \ldots, \rho_p$ be the strings extending σ that the machine T_s has already acted on and for which $\Phi(\rho_j) = \tau_i$. Let $B_i(\rho_j)$ denoted the set of strings put into B_i when the machine acted on ρ_j. Then we have (from the choice of m when T_s acted on ρ_j) that $\sum\limits_{\tau \in B_i(\rho_j)} \left(M_2(\tau, \epsilon_s) + \hat{\delta} \frac{M_2(\tau, \epsilon_s)}{M_2(\tau_i, \epsilon_s)} \right) < (1 + \epsilon_s) M_1(\rho_j, \epsilon_s)$.

Hence we have that

$$\begin{aligned}
M_2(\tau_i, \epsilon_s) &= \sum_j \sum_{\tau \in B_i(\rho_j)} M_2(\tau, \epsilon_s) + \sum_{j=1}^{n'} M_2(\tau'_j, \epsilon_s) \\
&\leq \sum_j \sum_{\tau \in B_i(\rho_j)} M_2(\tau, \epsilon_s) + \sum_{j=1}^{\hat{n}} M_2(\hat{\tau}_j, \epsilon_s) + \frac{\alpha}{2} \\
&< \sum_j \left[(1 + \epsilon_s) M_1(\rho_j, \epsilon_s) - \hat{\delta} \frac{\sum\limits_{\tau \in B_i(\rho_j)} M_2(\tau, \epsilon_s)}{M_2(\tau_i, \epsilon_s)} \right] \\
&\qquad + (1 + \epsilon_s) M_1(\hat{\sigma}, \epsilon_s) - \hat{\delta} \frac{\sum\limits_{j=1}^{\hat{n}} M_2(\hat{\tau}_j, \epsilon_s)}{M_2(\tau_i, \epsilon_s)} - \epsilon_s M_1(\hat{\sigma}, \epsilon_s) + \frac{\alpha}{2} \\
&= (1 + \epsilon_s) \left(M_1(\hat{\sigma}, \epsilon_s) + \sum_j M_1(\rho_j, \epsilon_s) \right) \\
&\qquad - \frac{\hat{\delta}}{M_2(\tau_i, \epsilon_s)} \left(\sum_{j=1}^{\hat{n}} M_2(\hat{\tau}_j, \epsilon_s) + \sum_j \sum_{\tau \in B_i(\rho_j)} M_2(\tau, \epsilon_s) \right) \\
&\qquad - \epsilon_s M_1(\hat{\sigma}, \epsilon_s) + \frac{\alpha}{2} \\
&\leq (1 + \epsilon_s) \left(M_1(\hat{\sigma}, \epsilon_s) + \frac{1}{1 + \epsilon_s} A_i \right) \\
&\qquad - \hat{\delta} \frac{M_2(\tau_i, \epsilon_s)}{M_2(\tau_i, \epsilon_s)} - \epsilon_s M_1(\hat{\sigma}, \epsilon_s) + \frac{\alpha}{2} \\
&= M_1(\hat{\sigma}, \epsilon_s) + A_i - \hat{\delta} + \frac{\alpha}{2} \\
&\leq M_1(\hat{\sigma}, \epsilon_s) + M_2(\tau_i, \epsilon_s) + \delta \frac{M_2(\tau_i, \epsilon_s)}{\sum\limits_{j=1}^{n} M_2(\tau_j, \epsilon_s)}
\end{aligned}$$

$$- (1+\epsilon_s)\left(\delta\frac{M_2(\tau_i,\epsilon_s)}{\sum_{j=1}^{n} M_2(\tau_j,\epsilon_s)}+\alpha\right)+\frac{\alpha}{2}$$
$$\leq M_2(\tau_i,\epsilon_s)+M_1(\hat{\sigma},\epsilon_s)-\frac{\alpha}{2}$$
$$< M_2(\tau_i,\epsilon_s)$$

Thus, we arrive at a contradiction, and so we are always able to find an m as above.

Having verified that the construction of Φ is well-defined and satisfies the requirements above, we now make a few remarks about Φ. Note that for any real Y, if $\mu_1(\{Y\}) = 0$ and $\mu_1(Y \restriction n) > 0$ for each n then every machine will find a $\sigma \subset Y$ to act on, and hence $Y \in \mathrm{dom}(\Phi)$. Suppose $Y \in \mathrm{dom}(\Phi)$ and $Z = \Phi(Y)$. It is claimed that $\Psi(Z) = Y$. To see this, fix n and pick $\sigma \subset Y$ such that $|\sigma| \geq n$ and σ was the first component in some tuple belonging to a machine T_s. When T_s defined $\Phi(\sigma) = \tau_i$, it added $\Psi(\hat{\tau}_j) = \sigma$ for each $j = 1, \ldots, m$. Hence it will suffice to see that Z extends one of the $\hat{\tau}_j$. Additionally, when T_s defined $\Phi(\sigma) = \tau_i$, it added $\langle \sigma, \langle \hat{\tau}_1, \ldots, \hat{\tau}_m \rangle, \delta \rangle$ to T_{s+1}. Hence when T_{s+1} acts on some $\hat{\sigma}$ extending σ, it will set $\Phi(\hat{\sigma}) = \hat{\tau}_j$ for some j. In particular this holds of $\hat{\sigma} \subset Y$ which must have been added to the domain of Φ by T_{s+1} since we assumed $Y \in \mathrm{dom}(\Phi)$.

Finally, suppose τ is in the range of Φ, so $\tau = \tau_i$ in some tuple in a machine T_s. Then

$$\begin{aligned}
\mu_1(\Phi^{-1}(\tau_i)) &\leq (1+\epsilon_s)\, M_1(\Phi^{-1}(\tau_i),\epsilon_s) \\
&\leq A_i \\
&\leq M_2(\tau_i,\epsilon_s)+\delta\frac{M_2(\tau_i,\epsilon_s)}{\sum_j M_2(\tau_j,\epsilon_s)}+(1+\epsilon_s)\alpha \\
&\leq M_2(\tau_i,\epsilon_s)\left(1+\frac{\delta}{\sum_j M_2(\tau_j,\epsilon_s)}+(1+\epsilon_s)\epsilon_s\right) \\
&\leq M_2(\tau_i,\epsilon_s)\left(1+\frac{1}{3}\left(1-4^{-s}\right)+(1+\epsilon_s)\epsilon_s\right) \\
&< \mu_2(\tau_i)\left(1+\frac{1}{3}+1\right) \\
&< 3\mu_2(\tau_i)
\end{aligned}$$

Hence for any $\tau \in \mathrm{ran}(\Phi)$, $\mu_2(\tau) \geq \frac{1}{3}\mu_1(\Phi^{-1}(\tau))$. □

Theorem 9. *Let μ_1, μ_2 be measures and X a real such that $\mu_1(\{X\}) = 0$ and is μ_1-[difference, weak-n, n]-random relative to $M_1 \oplus M_2$ where M_1, M_2 are representations of μ_1, μ_2 respectively. Furthermore suppose μ_2 is continuous. Then X is Turing equivalent in $M_1 \oplus M_2$ to a real Y which is μ_2-[difference, weak-n, n]-random relative to $M_1 \oplus M_2$.*

Proof. Fix Φ, Ψ as in the above lemma so that there is a $c > 0$ such that $\mu_2(\sigma) \geq c\mu_1(\Phi^{-1}(\sigma))$ for all σ. Since X is μ_2-random, $\mu(X \restriction n) > 0$ for all n, and $\mu(\{X\}) = 0$ by assumption, so

$X \in \text{dom}(\Phi)$ and $\Psi(\Phi(X)) = X$. Then by Proposition 5 $Y = \Phi(X)$ is μ_2-random relative to $M_1 \oplus M_2$. $\square$

The above theorem uses the hypothesis that μ_2 is continuous, so this leaves open the case that μ_2 is not continuous. One direction holds trivially. Suppose that $\mu_2(\{Y\}) > 0$ for some Y. Then $M_2 \geq_T Y$ so $X \geq_{T(M_1 \oplus M_2)} Y$. However, the other direction does not hold in general. Suppose $\mu_2(0^\omega) = 1$. Then 0^ω is the only μ_2-random and there is a recursive representation M_2, so $0^\omega \geq_{T(M_1 \oplus M_2)} X$ would imply $M_1 \geq_T X$ which is impossible if X is μ_1-random.

Since we usually consider $\mu(\{Y\}) > 0$ a degenerate instance of randomness, one might ask under which conditions can a μ_1-random real X compute a μ_2-random real Y such that $\mu_2(\{Y\}) = 0$. However the following propositions demonstrate examples of measures μ_2 for which this cannot hold. Thus any generalization of Theorem 9 would require stricter hypotheses.

Proposition 8. *Suppose μ_2 has finitely many atoms $Y_1, \ldots, Y_n$ and $\sum_{i=1}^{n} \mu_2(Y_i) = 1$. Then the only μ_2-1-randoms are $Y_1, \ldots, Y_n$.*

Proof. Fix a representation M_2 of μ_2. By Lemma 2 $Y_1, \ldots, Y_n$ are recursive in M_2, so fix indices $e_1, \ldots, e_n$ such that $\Phi_{e_i}^{M_2} = Y_i$. Then the set of strings which are incomparable to each $\Phi_{e_i}^{M_2}$ is recursive in M_2 and has measure 0. $\square$

Proposition 9. *Suppose μ_2 has atoms $Y_1, Y_2, \ldots$ and $\sum_{i=1}^{\infty} \mu_2(Y_i) = 1$. Then the only μ_2-weak-2-randoms are $Y_1, Y_2, \ldots$.*

Proof. Let $V_n = \{\sigma : \mu_2(\sigma) < 2^{-n}\}$. If $2^{-n} < \mu_2(Y_i)$ then $Y_i \notin V_n$, so $\lim_{n\to\infty} \mu_2(V_n) = 0$, so $\{V_n\}$ is a weak-2 test. If $X \neq Y_i$ for any i then X is in each V_n, so X is not random. $\square$

On the other hand, if μ_2 has finitely many atoms with mass less than 1, any representation of μ_2 can recursively compute a continuous measure in a manner analogous to Theorem 5, and hence in this case Theorem 9 will hold.

Proposition 10. *Suppose μ_2 has finitely many atoms $Y_1, \ldots, Y_n$ and $\sum_{i=1}^{n} \mu_2(Y_i) < 1$. Suppose μ_1 is a measure, X is a real such that $\mu_1(\{X\}) = 0$, and X is μ_1-[difference, weak-n, n]-random relative to $M_1 \oplus M_2$ where M_1, M_2 are representations of μ_1, μ_2 respectively. Then X is Turing equivalent in $M_1 \oplus M_2$ to a real Y which is μ_2-[difference, weak-n, n]-random relative to $M_1 \oplus M_2$ and such that $\mu_2(\{Y\}) = 0$.*

Proof. By Lemma 2 fix indices $e_1, \ldots, e_n$ such that $\Phi_{e_i}^{M_2} = Y_i$. Let $A(\sigma) = \sum_{Y_i : Y_i \supset \sigma} \mu_2(Y_i)$ and note that $A(\sigma)$ is recursive in M_2. Define

$$\nu(\sigma) = \frac{\mu_2(\sigma) - A(\sigma)}{1 - A(\langle\rangle)}$$

As in the proof of Theorem 5, ν is a continuous measure with representation recursive in M_2. Hence by Theorem 9 X is Turing equivalent in $M_1 \oplus M_2$ to a ν-random relative to $M_1 \oplus M_2$ real Y. Note that $Y \neq Y_i$ for any i or else $Y \leq_T M_2$ contradicting that Y is ν-random relative to M_2.

Suppose for a contradiction that Y is not μ_2-random relative to $M_1 \oplus M_2$. Let $\{V_n\}$ be a test capturing Y and fix k large enough so that $2^{-k} < 1 - A(\langle\rangle)$. Let $V'_n = V_{n+k}$. Then

$$\begin{aligned}\nu(V'_n) &= \sum_{\sigma \in V_{n+k}} \nu(\sigma) = \frac{1}{1 - A(\langle\rangle)} \sum_{\sigma \in V_{n+k}} (\mu_2(\sigma) - A(\sigma)) \\ &\leq \frac{1}{1 - A(\langle\rangle)} 2^{-(n+k)} < 2^{-n}\end{aligned}$$

Hence $\{V'_n\}$ is a test capturing X, contradicting that X is ν-random. □

In light of these results, an outstanding case of interest is when μ_2 has infinitely many atoms of total mass less than 1.

2.5 Open Questions

This chapter begins an exploration of randomizable reals for higher randomness, but leaves many interesting questions open. Although we have highlighted a number of reals which can and cannot be made to appear random, the general classification problem still exists:

1. Is there a "natural" description of those reals which are randomizable with respect to difference, weak-n, or n-randomness?

With Martin-Löf randomness, the description was "non-recursive" which is of course arithmetic. The natural definition of randomizable is Σ^1_1, so another way of framing a similar question would be:

2. Is there an arithmetic sentence $\varphi(X)$ such that $\varphi(X)$ holds iff X is randomizable with respect to difference, weak-n, or n-randomness?

This chapter only made a few remarks about the effect of constraining μ to be continuous, but this case still remains mostly unstudied.

3. What can be said about the classification of reals which are difference, weak-2, or n-randomizable with respect to *continuous* measures?

In Proposition 1 we showed that r.e. reals cannot be difference-randomizable, but as described the case of n-r.e. reals for $n \geq 2$ is open.

4. Is there a difference-randomizable n-r.e. real for $n \geq 2$?

It is worth noting that Theorem 7 implies no n-r.e. real can be weak-2-randomizable, so a positive answer to the previous question would be an interesting difference between weak-2 and difference-randomness.

In Proposition 9 we showed that a measure with all of its mass on atoms has no weak-2-random reals except for its atoms. Whether or not this applies to 1-randoms or difference-randoms in open.

5. Suppose μ is a measure with infinitely many atoms $Y_1, Y_2, \ldots$ and $\sum_{i=1}^{\infty} \mu(Y_i) = 1$. Can there be a μ-1-random real X such that $\mu(\{X\}) = 0$?

The end of section 4 leaves open whether Theorem 9 can be generalized to the case that μ_2 is not continuous, but has infinitely many atoms with a total mass less than 1.

6. Suppose μ_2 has infinitely many atoms with total mass less than 1. Suppose μ_1, μ_2 have representations M_1, M_2 and that X is μ_1-random relative to $M_1 \oplus M_2$ with $\mu_1(\{X\}) = 0$. Is X always Turing equivalent in $M_1 \oplus M_2$ to a real Y such that $\mu_2(\{Y\}) = 0$ and Y is μ_2-random relative to $M_1 \oplus M_2$?

Chapter 3

First-Order Consequences of Randoms

3.1 Introduction

In this chapter, we turn our attention to a subject of mathematical logic not directly related to algorithmic randomness: reverse mathematics. Reverse mathematics studies the question "which set-theoretic existence axioms are necessary to prove traditional theorems of mathematics," and formulates this is the language of second-order arithmetic. Specifically, the program of reverse mathematics defines a weak base theory and asks which set-theoretic axioms are implied by traditional theorems over this base theory. Reverse mathematics has a rich body of literature, and the reader is referred to the book by Stephen Simpson [21] for background.

The base theory over which we work is referred to as RCA_0, and is defined formally below. In our discussion, the set-theoretic existence axiom of interest will be 2RAN, the existence of 2-random reals. This axiom requires special consideration in order to be formalized within second-order arithmetic, so we take care to define it precisely below.

In 2009 Csima and Mileti [4] studied the reverse mathematics of the Rainbow Ramsey Theorem (RRT), showing that RRT is strictly weaker than Ramsey's Theorem for pairs (RT^2_2). Part of this proof included the result that 2RAN implies RRT, opening the door for a study on the reverse mathematical strength of 2RAN.

In 2013 Conidis and Slaman [3] followed up on this paper with an investigation into the first-order consequences of 2RAN. The existence of 1-random reals is implied by Weak König's Lemma (WKL_0) which is Π^1_1-conservative over RCA_0 and hence has first-order consequences no stronger than $\mathrm{P}^- + \mathrm{I}\Sigma^0_1$. By contrast, Conidis and Slaman showed that 2RAN implies the cardinality schema for Σ^0_2 sentences ($\mathrm{C}\Sigma^0_2$), a first-order axiom schema defined below. Consequently, $\mathrm{RCA}_0 + \mathrm{2RAN}$ has first-order consequences strictly stronger than $\mathrm{P}^- + \mathrm{I}\Sigma^0_1$. On the other hand, they showed 2RAN is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$, so the first-order strength of $\mathrm{RCA}_0 + \mathrm{2RAN}$ is at most that of $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$.

This chapter, in research conducted jointly with Ted Slaman, continues that investigation and provides a partial solution to the open problem at the end of Conidis and Slaman's paper: "Characterize the set of first-order consequences of 2RAN." In section 2 we establish that $\mathrm{RCA}_0 + 2\mathrm{RAN}$ does not imply $\mathrm{B}\Sigma^0_2$, so the first-order strength of $\mathrm{RCA}_0 + 2\mathrm{RAN}$ lies *strictly* between $\mathrm{I}\Sigma^0_1$ and $\mathrm{B}\Sigma^0_2$. In our proof we develop a general method of building models of $\neg\mathrm{B}\Sigma^0_2$, and in section 3 we apply this method to further characterize the reverse mathematical strength of the cardinality schema (CARD) thereby answering an open question of Kaye [11].

Axiom Schemas and Background

We let P^- denote the first order theory of Peano Arithmetic without induction in the language of $(+, \cdot, 0, 1, <)$. To be precise and self-contained, these axioms are the universal closure of

- $n + 1 \neq 0$
- $m + 1 = n + 1 \Rightarrow n = m$
- $m + 0 = m$
- $m + (n + 1) = (m + n) + 1$
- $m \cdot 0 = 0$
- $m \cdot (n + 1) = (m \cdot n) + m$
- $\neg(m < 0)$
- $m < n + 1 \Leftrightarrow (m < n \vee m = n)$

$\mathrm{I}\Sigma^0_n$ denotes the axiom schema of induction for all Σ^0_n formulas. That is, for every Σ^0_n formula $\varphi(x)$ (possibly with first and second-order parameters), $\mathrm{I}\Sigma^0_n$ contains the universal closure of

$$[\varphi(0) \wedge \forall x(\varphi(x) \Rightarrow \varphi(x + 1))] \Rightarrow \forall x\varphi(x)$$

The schemas $\mathrm{I}\Pi^0_n$ are defined analogously. $\mathrm{I}\Delta^0_n$ is similarly defined: if $\varphi(x), \psi(x)$ are Σ^0_n formulas (possibly with first and second-order parameters), $\mathrm{I}\Delta^0_n$ contains the universal closure of

$$[\forall x(\varphi(x) \Leftrightarrow \neg\psi(x)) \wedge \varphi(0) \wedge \forall x(\varphi(x) \Rightarrow \varphi(x + 1))] \Rightarrow \forall x\varphi(x)$$

Let PA denote the theory of Peano Arithmetic, i.e. $\mathrm{P}^- + \bigcup_n \mathrm{I}\Sigma^0_n$.

$\mathrm{B}\Sigma^0_n$ denotes the bounding axiom schema for Σ^0_n formulas. That is, for every Σ^0_n formula $\varphi(x, y)$ (possibly with first and second-order parameters), $\mathrm{B}\Sigma^0_n$ contains the universal closure of

$$\forall b\, [\forall x < b \exists y \varphi(x, y) \Rightarrow \exists a \forall x < b \exists y < a \varphi(x, y)]$$

The schemas $\mathrm{B}\Pi^0_n$ are defined analogously.

The following are some of the known results about the relative strength between $B\Sigma^0_n$ and $I\Sigma^0_n$:

$$I\Sigma^0_{n+1} \Rightarrow B\Sigma^0_{n+1} \Rightarrow I\Sigma^0_n$$
$$B\Sigma^0_n \Leftrightarrow I\Delta^0_n$$

The reader is referred for Slaman's paper [22] on the subject for proofs of these facts.

The cardinality schema for Σ^0_n sentences is denoted by $C\Sigma^0_n$ and represents a restricted version of the bounding axiom schema. This axiom schema says that there is no Σ^0_n injection from all numbers to a bounded initial segment. That is, for every Σ^0_n formula $\varphi(x, y)$ (possibly with first and second-order parameters) $C\Sigma^0_n$ includes the universal closure of

$$\forall x \exists y \varphi(x, y) \Rightarrow [\exists x_1, x_2, y(x_1 \neq x_2 \wedge \varphi(x_1, y) \wedge \varphi(x_2, y)) \vee \forall b \exists x \exists y > b \varphi(x, y)]$$

It is easy to see that $B\Sigma^0_n \Rightarrow C\Sigma^0_n$. The schema CARD represents $\bigcup_n C\Sigma^0_n$.

As is traditional in reverse mathematics, we will work over the base theory of RCA_0, the second-order theory containing the theory $P^- + I\Sigma^0_1$ along with Δ^0_1-comprehension. Δ^0_1-comprehension is the second-order axiom schema for all Σ^0_1 formulas $\varphi(x), \psi(x)$:

$$\forall x(\varphi(x) \Leftrightarrow \neg\psi(x)) \Rightarrow \exists X \forall x(x \in X \Leftrightarrow \varphi(x))$$

The axiom 2RAN asserts the existence of a 2-random real relative to any other real. That is,

$$\forall Y \exists R \exists n \forall \tau(\tau = Y' \upharpoonright |\tau| \Rightarrow R \in U^\tau_n)$$

Here U^Z_n is the uniformly-defined universal Martin-Löf test as defined previously. Note that the above axiom refers to Y', a real which may not exist in the absence of arithmetic comprehension. So for the 2RAN to be well-defined, the axiom intentionally refer to Y' only in terms of finite initial segments that are consistent with Y' and does not assert the existence of Y' as a second-order object.

3.2 Building Models of $\neg B\Sigma^0_2$

In 2001 Chong, Slaman, and Yang [2] showed that the Stable Ramsey's Theorem for pairs does not imply $I\Sigma^0_2$ over RCA_0. In their proof, they worked in a non-standard model of set theory $\mathcal{V}$ and created a $\mathcal{V}$-finite chain of models of PA. The direct limit of these models satisfied $\neg I\Sigma^0_2$. Our proof will use a similar tactic, although instead of a chain we will build a $\mathcal{V}$-finite tree of models of PA and take a generic path through that tree. Because our methods are similar, it would be useful (although not necessary) for the reader to be familiar with their paper.

As stated, we start by letting $\mathcal{V}$ be a model of set theory in which $\mathbb{N}^{\mathcal{V}}$ is non-standard. Fix $b \in \mathbb{N}^{\mathcal{V}} \setminus \mathbb{N}$ a non-standard $\mathcal{V}$-finite number with countably many predecessors.

To describe how we will build a failure of $B\Sigma^0_2$, suppose $\mathfrak{M}$ is a model of P^- containing b and that f is a binary-valued function definable in $\mathfrak{M}$, with downward-closed domain (i.e. if $n \in \mathrm{dom}(f)$ and $m \leq n$ then $m \in \mathrm{dom}(f)$). Define $g_{f,0}$ and $g_{f,1}$ by recursion as follows. We will maintain by induction that $g_{f,0}(n) \leq g_{f,1}(n)$, $g_{f,0}(n) \leq g_{f,0}(n+1)$, and $g_{f,1}(n) \geq g_{f,1}(n+1)$.

Let $g_{f,0}(0) = 0$ and $g_{f,1}(0) = b$. For $n > 0$, if $n \notin \mathrm{dom}(f)$ we leave $g_{f,0}(n), g_{f,1}(n)$ undefined. Otherwise, if $g_{f,0}(n-1) = g_{f,1}(n-1)$ then let $g_{f,0}(n) = g_{f,0}(n-1)$ and $g_{f,1}(n) = g_{f,1}(n-1)$. Otherwise if $f(n) = 0$ define $g_{f,0}(n) = g_{f,0}(n-1)+1$ and $g_{f,1}(n) = g_{f,1}(n-1)$. Finally if $f(n) = 1$ define $g_{f,0}(n) = g_{f,0}(n-1)$ and $g_{f,1}(n) = g_{f,0}(n-1) - 1$.

Given $g_{f,0}, g_{f,1}$ define the interval $I_f \subseteq [0, b]$ as $I_f = \{n \leq b : \exists m(n < g_{f,0}(m))\}$.

We will frequently refer to the difference $g_{f,1}(n) - g_{f,0}(n)$, so we write $\tilde{g}_f(n) = g_{f,1}(n) - g_{f,0}(n)$. For strings $\sigma \in 2^{\leq b}$ we will also frequently refer to $\tilde{g}_\sigma(|\sigma|)$, so in a slight abuse of notation we also define $\tilde{g}(\sigma)$ to be $\tilde{g}_\sigma(|\sigma|)$.

Suppose it is the case that for all $n \leq b$ there is an m such that either $n < g_{f,0}(m)$ or $n \geq g_{f,1}(m)$. Then I_f is recursive in f (even if the domain of f is not). In particular, if f is Δ^0_2-definable in $\mathfrak{M}$, so is I_f. This leads us to the following lemma:

Lemma 8. *Suppose f is Δ^0_2-definable, has a downward closed domain, and satisfies the following properties:*

1. *For every $n \in \mathrm{dom}(f)$, $\tilde{g}_f(n) > 0$.*
2. *For every $n < b$ there is an $m < b$ such that $n < g_{f,0}(m)$ or $n \geq g_{f,1}(m)$.*
3. *For every $n \in \mathrm{dom}(f)$ there is some $m > n$ such that $f(m) = 0$.*

Then $\mathfrak{M} \models \neg B\Sigma^0_2$.

Proof. As noted above, condition 2 above ensures that I_f is Δ^0_2-definable. Since $\mathrm{I}\Delta^0_2 \Leftrightarrow \mathrm{B}\Sigma^0_2$, it suffices to see that I_f is non-principal, i.e. that I_f has no greatest element. Suppose for a contradiction that there is some greatest $n \in I_f$. Fix $m < b$ such that $n < g_{f,0}(m)$. Since $n+1 \notin I_f$, we know $g_{f,0}(m) = n+1$. By condition 3, there is some $m' > m$ such that $f(m') = 0$. By condition 1 $g_{f,1}(m'-1) - g_{f,0}(m'-1)$ is greater than zero, so $g_{f,0}(m') = g_{f,0}(m'-1) + 1 \geq g_{f,0}(m) + 1 > g_{f,0}(m) = n+1$. Hence $n+1 \in I_f$, a contradiction. □

Our goal then will be to construct a model with a definable function satisfying the above conditions. To do so, we will create a tree of models of height b, such that any path through the tree is Δ^0_2-definable in the direct limit of that path. To establish branching at each node in the tree, we will utilize the following result by Friedman [8]. His original result is actually stronger, but we reproduce this version directly relevant to our current context.

In what follows, we let $\mathrm{Tr}(1)$ abbreviate the Π^0_1-definable set of (the Gödel numbers of) all true Π^0_1 sentences. The notation $\pm\varphi$ is short-hand for either φ or $\neg\varphi$.

Lemma 9. *[8] Let T be a consistent, Π^0_1-definable theory extending $PA + Tr(1)$. Then there exist mutually independent Π^0_2 sentences φ_0, φ_1 over T. That is, for any boolean combination $\pm\varphi_0 \wedge \pm\varphi_1$ we have $PA + Tr(1) \vdash \mathrm{Con}(T) \Rightarrow \mathrm{Con}(T + \pm\varphi_0 \wedge \pm\varphi_1)$.*

Proof. Using Gödel self-reference, define φ_0 as the formula which says the following:

"For every proof of φ_0 from T there is a shorter proof of $\neg\varphi_0$ from T."

Similarly, having defined φ_0, define φ_1 as:

"For every proof of $\pm\varphi_0 \vee \varphi_1$ from T there is a shorter proof of $\pm\varphi_0 \vee \neg\varphi_1$ from T."

Note that both formulas are of the form $\forall n(\psi(n) \to \exists m < n(\psi'(m)))$, where both ψ and ψ' are Π^0_1 (since T is Π^0_1-definable). Hence the formulas φ_0, φ_1 are Π^0_2.

First we show that $T \nvdash \pm\varphi_0$. Suppose for a contradiction that $T \vdash \varphi_0$. If $\mathbb{N} \models \varphi_0$ then by definition of φ_0, $T \vdash \neg\varphi_0$, contradicting our assumption that T is consistent. On the other hand, if $\mathbb{N} \models \neg\varphi_0$ then since φ_0 is Π^0_2 we have $\mathrm{PA} + \mathrm{Tr}(1) \vdash \neg\varphi_0$, which again contradicts the consistency of T.

Next suppose $T \vdash \neg\varphi_0$. Let k be the length of the shortest proof. If $\mathbb{N} \models \varphi_0$ then $\mathrm{PA} + \mathrm{Tr}(1) \vdash \varphi_0$, by simply showing that there are no proofs of φ_0 from T of length less than or equal to k. This contradicts the consistency of T. If $\mathbb{N} \models \neg\varphi_0$ then there is a proof of φ_0 from T, so $T \vdash \varphi_0$, again contradicting the consistency of T.

Now we want to show that $T \nvdash \pm\varphi_0 \vee \pm\varphi_1$. Suppose this is not to case, so for some boolean combination we have $T \vdash \pm\varphi_0 \vee \pm\varphi_1$. It is first claimed that $\mathrm{PA} + \mathrm{Tr}(1) \vdash \pm\varphi_1$. To see this, first suppose $T \vdash \pm\varphi_0 \vee \varphi_1$. If there is no shorter proof of $\pm\varphi_0 \vee \neg\varphi_1$ from T, then this is a counter-example to φ_1 and hence $\mathrm{PA} + \mathrm{Tr}(1) \vdash \neg\varphi_1$. Otherwise we have $T \vdash \pm\varphi_0 \vee \neg\varphi_1$, and this is a case similar to the above; let k be the length of the shortest such proof. Then either there is a shorter proof of $\pm\varphi_0 \vee \varphi_1$ and hence $\mathrm{PA} + \mathrm{Tr}(1) \vdash \neg\varphi_1$ or else $\mathrm{PA} + \mathrm{Tr}(1)$ proves φ_1 is true just by showing that there are no proofs of length less than or equal to k of $\pm\varphi_0 \vee \varphi_1$.

Given that $\mathrm{PA} + \mathrm{Tr}(1) \vdash \pm\varphi_1$, we now take the case that $\mathrm{PA} + \mathrm{Tr}(1) \vdash \varphi_1$. Then $\mathbb{N} \models \varphi_1$, so $T \vdash \pm\varphi_0 \vee \neg\varphi_1$. Hence $T \vdash \pm\varphi_0$, contradicting our above proof that $T \nvdash \pm\varphi_0$. On the other hand, suppose $\mathrm{PA} + \mathrm{Tr}(1) \vdash \neg\varphi_1$. Then $\mathbb{N} \models \neg\varphi_1$, so $T \vdash \pm\varphi_0 \vee \varphi_1$ and again we have $T \vdash \pm\varphi_0$ contradicting the above. □

It is important to note that (the Gödel numbers for) the sentences φ_0, φ_1 are uniformly recursive in (the Gödel number for) T, as this will allow us to recursively refer to those formulas as we try to define our failure of $\mathrm{B}\Sigma^0_2$.

In order to ensure our tree can be extended at each node, we will require that each model satisfies a certain amount of consistency. Define $T_0 = \mathrm{PA} + \mathrm{Tr}(1)$ and inductively define $T_{n+1} = \mathrm{PA} + \mathrm{Tr}(1) + \mathrm{Con}(T_n)$. Inductively, $\mathbb{N}^{\mathcal{V}} \models T_n$ for all $n \leq b$, so each T_n is consistent. By Lemma 9, for each n there are Π^0_2 sentences φ^n_0, φ^n_1 such that $\mathrm{Con}(T_n) \Rightarrow \mathrm{Con}(T_n + \pm\varphi^n_0 \wedge \pm\varphi^n_1)$ for any boolean combination of $\pm\varphi^n_0 \wedge \pm\varphi^n_1$.

Let $\mathfrak{M}_{\langle\rangle}$ be a model of T_b. We will inductively define $\mathfrak{M}_\sigma$ for $|\sigma| \leq b$ with the induction hypothesis that $\mathfrak{M}_\sigma \models T_{b-|\sigma|}$. Suppose $\mathfrak{M}_\sigma$ is already defined. Since $\mathfrak{M}_\sigma \models \mathrm{Con}(T_{b-|\sigma|-1})$, we have $\mathfrak{M}_\sigma \models \mathrm{Con}(T_{b-|\sigma|-1} + \pm\varphi^{|\sigma|}_0 \wedge \pm\varphi^{|\sigma|}_1)$ for any boolean combination $\pm\varphi^{|\sigma|}_0 \wedge \pm\varphi^{|\sigma|}_1$. Hence for $i = 0, 1$ there are models $\mathfrak{N}_i$ such that $\mathfrak{N}_i \models T_{b-|\sigma|-1} + \neg\varphi^{|\sigma|}_i \wedge \varphi^{|\sigma|}_{1-i}$. Since $T_{b-|\sigma|-1}$ contains $\mathrm{Tr}(1)$, we also have that $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{N}_i$. Let $\mathfrak{M}_{\sigma^\frown i} = \mathfrak{N}_i$ for $i = 0, 1$.

Since $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{M}_{\sigma^\frown i}$ for $i = 0, 1$, we can apply induction to see $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{M}_\tau$ for any $\tau \supseteq \sigma$.

Furthermore, note that for each σ and $i = 0, 1$, the model $\mathfrak{M}_{\sigma^\frown i}$ is definable in $\mathfrak{M}_\sigma$, and thus if $A \subseteq \mathfrak{M}_\sigma$ is definable in $\mathfrak{M}_{\sigma^\frown i}$, then it is also definable in $\mathfrak{M}_\sigma$. Applying induction to the preceding fact yields that if $\tau \supseteq \sigma$ and $A \subseteq \mathfrak{M}_\sigma$ is definable in $\mathfrak{M}_\tau$, then A is also definable in $\mathfrak{M}_\sigma$. We formalize this as follows and will reference it later:

Lemma 10. *Suppose $A \subseteq \mathfrak{M}_\sigma$ is definable in $\mathfrak{M}_\tau$ for $\tau \supseteq \sigma$. Then A is definable in $\mathfrak{M}_\sigma$.*

Working in our outer model (in which b is recognized as non-standard), we now develop a notion of forcing on this tree of models. Our poset $\mathbb{P}$ will be $\{\sigma : \tilde{g}(\sigma) \text{ is non-standard}\}$. Comparability is defined by $\sigma \prec_{\mathbb{P}} \tau$ precisely when $\sigma \supseteq \tau$. Given a generic $G \subseteq \mathbb{P}$, we let our generic model $\mathfrak{M}_G$ be $\bigcup\{\mathfrak{M}_\sigma : \sigma \in G\}$.

We claim that $\mathfrak{M}_G$ is a Π^0_1-elementary extension of each $\mathfrak{M}_\sigma$ for $\sigma \in G$. Suppose not; then there is a formula $\varphi(x)$ with only bounded quantifiers such that $\mathfrak{M}_G \models \neg\forall x\varphi(x)$ while for some $\sigma \in G$ we have $\mathfrak{M}_\sigma \models \forall x\varphi(x)$. Fix $a \in \mathfrak{M}_G$ such that $\mathfrak{M}_G \models \neg\varphi(a)$. Fix $\tau \in G$ such $a \in \mathfrak{M}_\tau$ and assume without loss of generality that $\tau \supseteq \sigma$ (if $\tau \subset \sigma$ then $a \in \mathfrak{M}_\sigma$ and we could instead have chosen $\tau = \sigma$). Since φ has only bounded quantifiers and $\mathfrak{M}_\tau$ is an initial segment of $\mathfrak{M}_G$, we have $\mathfrak{M}_\tau \models \neg\varphi(a)$ and hence $\mathfrak{M}_\tau \models \neg\forall x\varphi(x)$. But this contradicts the fact that $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{M}_\tau$.

We can extend this argument to show that Σ^0_2 facts are forced by conditions in G in the following sense.

Lemma 11. *Suppose φ is a Σ^0_2 sentence. Then $\mathfrak{M}_G \models \varphi$ iff there is a condition $\sigma \in G$ such that $\mathfrak{M}_\sigma \models \varphi$.*

Proof. Write φ as $\exists x\psi(x)$ where $\psi(x)$ is a Π^0_1 formula. First suppose σ is a condition in G such that $\mathfrak{M}_\sigma \models \exists x\psi(x)$. Then we can fix $a \in \mathfrak{M}_\sigma$ such that $\mathfrak{M}_\sigma \models \psi(a)$. Since $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{M}_G$ we have $\mathfrak{M}_G \models \varphi(a)$. Thus $\mathfrak{M}_G \models \exists x\varphi(x)$.

For the reverse, now suppose $\mathfrak{M}_G \models \exists x\varphi(x)$. Fix $a \in \mathfrak{M}_G$ a witness such that $\mathfrak{M}_G \models \psi(a)$. Fix $\sigma \in G$ such that $a \in \mathfrak{M}_\sigma$. Then by downward absoluteness of Π^0_1 formulas we have $\mathfrak{M}_\sigma \models \psi(a)$. Hence $\mathfrak{M}_\sigma \models \exists x\varphi(x)$. $\square$

Next observe that $\mathfrak{M}_G \models \mathrm{I}\Sigma^0_1$.

Lemma 12. *If G is generic then $\mathfrak{M}_G \models I\Sigma^0_1$.*

Proof. Suppose $\mathrm{I}\Sigma^0_1$ does not hold. Fix a Σ^0_1 formula $\varphi(x)$ such that $\mathfrak{M}_G \models \exists x\varphi(x) \wedge \neg\exists x(\varphi(x) \wedge \forall y < x\neg\varphi(y))$. Fix some $a \in \mathfrak{M}_G$ such that $\mathfrak{M}_G \models \varphi(a)$. Fix $\sigma \in G$ such that $a \in \mathfrak{M}_\sigma$. Note $\mathfrak{M}_\sigma \models \varphi(a)$ since φ is Σ^0_1. Since $\mathfrak{M}_\sigma \models \mathrm{PA}$, $\mathfrak{M}_\sigma \models \exists x(\varphi(x) \wedge \forall y < x\neg\varphi(y))$. Let $\hat{a}$ be the witness for this x in $\mathfrak{M}_\sigma$, so $\mathfrak{M}_\sigma \models \varphi(\hat{a}) \wedge \forall y < \hat{a}\neg\varphi(y)$. Since $\varphi(\hat{a})$ is Σ^0_1, $\mathfrak{M}_G \models \varphi(\hat{a})$. Similarly since $\forall y < \hat{a}\neg\varphi(y)$ is Π^0_1, $\mathfrak{M}_G \models \forall y < \hat{a}\neg\varphi(y)$. But this contradicts our assumption that $\mathfrak{M}_G \models \neg\exists x(\varphi(x) \wedge \forall y < x\neg\varphi(y))$. $\square$

Let $f_G = \bigcup G$. Given $n \in \mathrm{dom}(f_G)$ let $\sigma = f_G \upharpoonright n$, so $\sigma \in G$. Then $\mathfrak{M}_\sigma \models \neg\varphi^n_{\sigma(n)} \wedge \varphi^n_{1-\sigma(n)}$. Since every structure $\mathfrak{M}_\sigma$ is a Π^0_1-elementary substructure of $\mathfrak{M}_G$, the least witness in $\mathfrak{M}_G$ of $\neg\varphi_0^n \vee \neg\varphi_1^n$ is a witness for $\neg\varphi_{\sigma(n)}^n$. Hence $\mathfrak{M}_G$ has a Δ_2^0 definition of the function f defined by $f(n) = i$ where $i = 0, 1$ is chosen such that the least witness to $\neg\varphi_0^n \vee \neg\varphi_1^n$ is a witness to $\neg\varphi_i^n$. The domain of f is equal to that of f_G, and $f = f_G$ on its domain, so f_G is Δ_2^0-definable in $\mathfrak{M}_G$.

Lemma 13. *Let G be a generic and let $f_G = \bigcup G$. Then f_G satisfies the following:*

1. *$\tilde{g}_{f_G}(n) > 0$ for all $n \in \mathrm{dom}(f_G)$.*
2. *For all $n < b$ there is some m such that $n < g_{f_G,0}(m)$ or $n \geq g_{f_G,1}(m)$.*
3. *For every $n \in \mathrm{dom}(f_G)$ there is some $m > n$ such that $f_G(m) = 0$.*

Proof. The first condition is immediate, since $\tilde{g}(\sigma)$ is non-standard for all $\sigma \in \mathbb{P}$, and $f_G(n)$ is defined precisely when there is a string $\sigma \in G$ of length n.

We wish to show that meeting the second condition is dense for each $n < b$. Fix σ and n. The gap $\tilde{g}(\sigma) = g_{\sigma,1}(|\sigma|) - g_{\sigma,0}(|\sigma|)$ is non-standard, so at least one of the gaps $g_{\sigma,1}(|\sigma|) - n$ and $n - g_{\sigma,0}(|\sigma|)$ is non-standard. If $g_{\sigma,1}(|\sigma|) - n$ is non-standard, let $\tau = \sigma^\frown 0^{n - g_{\sigma,0}(|\sigma|)+1}$. Then $g_{\tau,0}(|\tau|) = n + 1$ and $g_{\tau,1}(|\tau|) = g_{\sigma,1}(|\sigma|)$, so τ is a valid condition and $n < g_{\tau,0}(|\tau|)$. Likewise if $g_{\sigma,1}(|\sigma|) - n$ is standard, let $\tau = \sigma^\frown 1^{g_{\sigma,1}(|\sigma|)-n}$. Then $g_{\tau,0}(|\tau|) = g_{\sigma,0}(|\sigma|)$ and $g_{\tau,1}(|\tau|) = n$, so again τ is a valid condition and now $n \geq g_{\tau,1}(|\tau|)$.

Finally, we want to know that meeting condition 3 is dense. Given $n \in \mathrm{dom}(f_G)$ and $\sigma \in \mathbb{P}$, let τ be any string extending σ of length at least n. Then the condition $\tau^\frown 0$ satisfies $f_G(|\tau| + 1) = 0$ and hence it is dense to satisfy 3. □

This result, combined with Lemma 8 implies that if G is generic, then $\mathfrak{M}_G \models \neg\mathrm{B}\Sigma_2^0$. The next step is to show that we can simultaneously build a real which is 2-random. First, we would like to point out that finding a single 2-random is sufficient.

Lemma 14. *Suppose $\mathfrak{M}$ is a first-order model satisfying $P^- + I\Sigma_1^0$ and $\mathfrak{M}$ models that $X \subseteq |\mathfrak{M}|$ is 2-random. Then $\mathfrak{M}$ can be extended to a second-order model which satisfies RCA_0 + 2RAN.*

Proof. Let $X = \bigoplus_{n<\omega} X_n$. Since X is 2-random in $\mathfrak{M}$, by Van Lambalgen's Theorem the model also satisfies that every finite join of columns of X is 2-random relative to any other finite join of columns from X. Hence we can let the second-order part of $\mathfrak{M}$ be $\{Y : \exists i_1, \ldots, i_k(Y \leq_T X_{i_1} \oplus \cdots \oplus X_{i_k})\}$.

The second-order model constructed clearly satisfies both 2RAN and Δ_1^0-comprehension, so we just need to verify that adding the second-order predicates doesn't cause a failure of $\mathrm{I}\Sigma_1^0$. To do so, we show that even adding X as a predicate would not cause a failure of $\mathrm{I}\Sigma_1^0$.

Suppose $\mathrm{I}\Sigma_1^0$ does not hold, so there is a $\Sigma_1^0(X)$ sentence $\varphi(x, X)$ such that $I = \{x : \mathfrak{M} \models \varphi(x, X)\}$ is bounded by some $m \in \mathfrak{M}$ but is non-principal. Write $\varphi(x, X)$ as $\exists s \psi(x, s, X)$ where ψ has only bounded quantifiers. Consider the set

$$V_s = \{Y : \exists x \leq m\, (\exists s' > s(\psi(x, s', Y)) \wedge \forall s' \leq s(\neg\psi(x, s', Y)))\}$$

Since $\bigcap_s V_s = \emptyset$ and the sets V_s are uniformly Σ^0_1, $0'$ can compute the function $f(n)$ where for each n, $f(n)$ is the smallest s such that $\lambda(V_s) < 2^{-n}$. Hence $\{V_{f(n)}\}_{n\in\omega}$ is a Martin-Löf test relative to $0'$. Since X is 2-random, $X \notin V_{f(n)}$ for some n. Let $a = f(n)$ and fix σ such that $a \in \mathfrak{M}_\sigma$. Then for $x \leq m$ we have $x \in I$ iff $\mathfrak{M} \models \varphi(x, X)$ iff $\mathfrak{M} \models \exists s \leq a(\psi(x, s, X \upharpoonright a))$ iff $\mathfrak{M}_\sigma \models \exists s \leq a(\psi(x, s, X \upharpoonright a))$. Hence I is definable in $\mathfrak{M}_\sigma$, but this contradicts $\mathfrak{M}_\sigma \models$ PA. □

We are now ready to prove the main theorem:

Theorem 10. *There is a model of $RCA_0 + \neg B\Sigma^0_2 +$ 2RAN.*

Proof. By the prior lemma, it suffices to find an $\mathfrak{M}$-real X such that X is 2-random in $\mathfrak{M}$ (noting that the constructed counter-example of BΣ^0_2 remains a counter-example regardless of adding second-order predicates).

We work again inside our model in which we view b as standard. Recall that $\{U^Z_k\}_{k\in\omega}$ represents the universal relativized Martin-Löf test. Let J_σ represent the real $0'$ as calculated by the model $\mathfrak{M}_\sigma$. Suppose $\sigma \subseteq \tau$; since $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{M}_\tau$, calculation of $0'$ must agree on the common domain of the two models. Put another way, $J_\sigma \subseteq J_\tau$. Similarly, $U^{J_\sigma}_k$ is a subset of $U^{J_\tau}_k$, since if $\rho \in U^{J_\sigma}_k$ this is a $\Delta^0_1(J_\sigma)$ fact and J_τ agrees with J_σ on all the bits used in this computation. Hence $\rho \in U^{J_\tau}_k$.

It will also be useful to note that if ρ is a finite string in $\mathfrak{M}_\sigma$, $\sigma \subseteq \tau$, and $\mathfrak{M}_\tau \models \lambda([\rho] \cap U^{0'}_k) \geq q$ then $\mathfrak{M}_\sigma \models \lambda([\rho] \cap U^{0'}_k) \geq q$. This is because witnessing the measure of a $\Pi^0_1(0')$ class dropping below a rational value q is a Σ^0_2 event, and so the stage witnessing this event would also occur in $\mathfrak{M}_\tau$.

To define our 2-random real X, we will fix a number k and define for each string σ a $\mathfrak{M}_\sigma$-string X_σ. For σ with $|\sigma| > 0$, X_σ will have length equal a finite number from $\mathfrak{M}_\sigma \setminus \mathfrak{M}_{\sigma^*}$. If $\sigma \subseteq \tau$ then we will have $X_\sigma \subseteq X_\tau$. Finally, we will have $\mathfrak{M}_\sigma \models X_\sigma \in U^{0'}_k$.

Assuming we have done this, we will return to our external model and consider a generic G. Let $X_G = \bigcup\{X_\sigma : \sigma \in G\}$. Note that X_G is a real defined on the entire domain of $\mathfrak{M}$. It is also claimed that $\mathfrak{M} \models X_G \in U^{0'}_k$, for suppose not. Then there is some $n \in \mathfrak{M}$ such the tree $U^{0'}_k$ above $X_G \upharpoonright n$ becomes empty. However, this is a Σ^0_2, so it is forced by some $\sigma \in G$ (and assume without loss of generality that σ is large enough so that $n \in \mathfrak{M}_{\sigma^*}$). Then $\mathfrak{M}_\sigma \models [X_G \upharpoonright n] \cap U^{0'}_k = \emptyset$. However, since $X_G \supseteq X_\sigma \supseteq X_G \upharpoonright n$ and $\mathfrak{M}_\sigma \models X_\sigma \in U^{0'}_k$ we have a contradiction. Thus $\mathfrak{M}$ models that X_G is 2-random.

We now return to our construction of the strings X_σ. We will define X_σ inductively. For the base case we let $X_{\langle\rangle} = \langle\rangle$. Let $\epsilon_0 = 2^{-b^2-1}$. For $s < b$ define $\epsilon_{s+1} = 2^b\epsilon_s$ and note that $\epsilon_b = \epsilon_0 \prod_{s=1}^b 2^b = \epsilon_0 2^{b^2} = \frac{1}{2}$. In particular this implies $1 - \epsilon_s > 0$ for all $s \leq b$. Pick $k > b^2+1$; then for every σ we have $\mathfrak{M}_\sigma \models \lambda(U^{0'}_k) \geq 1-\epsilon_0$. In our construction will maintain the induction hypothesis that $\mathfrak{M}_\tau \models \frac{\lambda([X_\sigma]\cap U^{0'}_k)}{\lambda([X_\sigma])} \geq 1-\epsilon_{|\sigma|}$ for all $\tau \supseteq \sigma$. Note that in the base case for $\sigma = \langle\rangle$, this holds by our selection of k and ϵ_0. Besides aiding in the construction, since $1 - \epsilon_s > 0$ for all s, this implies $\mathfrak{M}_\tau \models X_\sigma \in U^{0'}_k$, our desired conclusion.

Given σ with $|\sigma| > 0$, let $s = |\sigma|$, fix some $l \in \mathfrak{M}_\sigma \setminus \mathfrak{M}_{\sigma^*}$, and let $m = |X_{\sigma^*}|$, which is $\mathfrak{M}_\sigma$-finite. Consider all $\mathfrak{M}_\sigma$-strings Y with length l extending X_{σ^*}. If there is a Y such that $\mathfrak{M}_\tau \models \frac{\lambda([Y]\cap U_k^{0'})}{\lambda([Y])} \geq 1 - \epsilon_s$ for all $\tau \supseteq \sigma$ of length b, let X_σ be equal to this Y. By our above remark, this holding for all τ of length b is enough to satisfy the induction hypothesis.

We would like to show that there always exists such a Y. For each τ of length b let $B_\tau = \{Y \supset X_{\sigma^*} : |Y| = l \wedge \mathfrak{M}_\tau \models \frac{\lambda([Y]\cap U_k^{0'})}{\lambda([Y])} < 1 - \epsilon_s\}$. The first claim is that $\mathfrak{M}_\tau \models |B_\tau| < 2^{l-m-b}$, for otherwise — working inside $\mathfrak{M}_\tau$ — we have the following computation:

$$
\begin{aligned}
1 - \epsilon_{s-1} &\leq \frac{\lambda([X_{\sigma^*}] \cap U_k^{0'})}{\lambda([X_{\sigma^*}])} \\
&= 2^m \sum_{Y \supset X_{\sigma^*}, |Y| = l} \lambda([Y] \cap U_k^{0'}) \\
&= 2^{m-l} \sum_{Y \supset X_{\sigma^*}, |Y| = l} \frac{\lambda([Y] \cap U_k^{0'})}{\lambda([Y])} \\
&= 2^{m-l} \left[\sum_{Y \supset X_{\sigma^*}, |Y| = l, Y \notin B_\tau} \frac{\lambda([Y] \cap U_k^{0'})}{\lambda([Y])} + \sum_{Y \in B_\tau} \frac{\lambda([Y] \cap U_k^{0'})}{\lambda([Y])} \right] \\
&< 2^{m-l} \left[(2^{l-m} - |B_\tau|) \cdot 1 + |B_\tau| \cdot (1 - \epsilon_s) \right] \\
&= 2^{m-l} \left[2^{l-m} - |B_\tau| \epsilon_s \right] \\
&\leq 2^{m-l} \left[2^{l-m} - 2^{l-m-b} \epsilon_s \right] \\
&= 1 - 2^{-b} \epsilon_s \\
&= 1 - \epsilon_{s-1}
\end{aligned}
$$

Hence we arrive at a contradiction. Although we defined each B_τ in terms of $\mathfrak{M}_\tau$, each set B_τ is a $\mathfrak{M}_\sigma$-finite set of $\mathfrak{M}_\sigma$-strings and hence is coded by a parameter from $\mathfrak{M}_\sigma$. Since $\mathfrak{M}_\tau \models |B_\tau| < 2^{l-m-b}$ which is a Δ_1^0 fact, we have $\mathfrak{M}_\sigma \models |B_\tau| < 2^{l-m-b}$. Hence

$$\mathfrak{M}_\sigma \models |\{Y : |Y| = l \wedge Y \supset X_{\sigma^*}\}| = 2^{l-m} = 2^b \cdot 2^{l-m-b} > \sum_\tau |B_\tau|$$

so there is some string Y of length l extending X_{σ^*} not belonging to any B_τ. By definition this implies that $\mathfrak{M}_\tau \models \frac{\lambda([Y])\cap U_k^{0'})}{\lambda([Y])} \geq 1 - \epsilon_s$ for each τ as desired. □

3.3 Building Models of CARD

The framework of the previous section gives us a more general tool of building models of $\neg \mathrm{B}\Sigma_2^0$. In this section we explore an additional application, showing that CARD does not imply $B\Sigma^0_2$. To do so, we first extend the forcing of the previous section to a formal forcing language.

Let $\mathbb{Q}$ be a downwards-closed subset of $2^{\leq b}$. For $\sigma \in \mathbb{Q}$ and a sentence φ (possibly with parameters from $\mathfrak{M}_\sigma$) define $\sigma \Vdash_{\mathbb{Q}} \varphi$ by induction on the complexity of φ.

In the base case that φ is a Σ^0_2 formula, define $\sigma \Vdash_{\mathbb{Q}} \varphi$ iff $\mathfrak{M}_\sigma \models \varphi$.

Inductively, given forumals φ, ψ, define $\sigma \Vdash_{\mathbb{Q}} \varphi \wedge \psi$ iff $\sigma \Vdash_{\mathbb{Q}} \varphi$ and $\sigma \Vdash_{\mathbb{Q}} \psi$. Similarly define $\sigma \Vdash_{\mathbb{Q}} \varphi \vee \psi$ iff $\sigma \Vdash_{\mathbb{Q}} \varphi$ or $\sigma \Vdash_{\mathbb{Q}} \psi$.

Define $\sigma \Vdash_{\mathbb{Q}} \exists x \psi(x)$ iff there is some $a \in \mathfrak{M}_\sigma$ such that $\sigma \Vdash_{\mathbb{Q}} \psi(a)$.

Finally, define $\sigma \Vdash_{\mathbb{Q}} \neg\varphi$ iff for every $\tau \in \mathbb{Q}$ extending σ, $\tau \nVdash_{\mathbb{Q}} \varphi$.

Having defined this notion of forcing, we prove a few basic facts:

Lemma 15. *(Monotonicity) Let φ be a formula and suppose $\sigma \Vdash_{\mathbb{Q}} \varphi$. If $\tau \supseteq \sigma$ then $\tau \Vdash_{\mathbb{Q}} \varphi$.*

Proof. We proceed by induction on the complexity of φ. If φ is Σ^0_2 then write φ as $\exists x \psi(x)$. Since $\mathfrak{M}_\sigma \models \varphi$ there is some $a \in \mathfrak{M}_\sigma$ such that $\mathfrak{M}_\sigma \models \psi(a)$. Since $\mathfrak{M}_\sigma \prec_{\Pi^0_1} \mathfrak{M}_\tau$ we have $\mathfrak{M}_\tau \models \psi(a)$. Hence $\mathfrak{M}_\tau \models \varphi$, so $\tau \Vdash_{\mathbb{Q}} \varphi$.

Inductively, assume we have shown the lemma for forumlas φ, ψ. Then $\sigma \Vdash_{\mathbb{Q}} \varphi \wedge \psi$ implies $\sigma \Vdash_{\mathbb{Q}} \varphi$ and $\sigma \Vdash_{\mathbb{Q}} \psi$, so by induction $\tau \Vdash_{\mathbb{Q}} \varphi$ and $\tau \Vdash_{\mathbb{Q}} \psi$, so by definition $\tau \Vdash_{\mathbb{Q}} \varphi \wedge \psi$. Similarly if $\sigma \Vdash_{\mathbb{Q}} \varphi \vee \psi$ then $\sigma \Vdash_{\mathbb{Q}} \varphi$ or $\sigma \Vdash_{\mathbb{Q}} \psi$ so by induction $\tau \Vdash_{\mathbb{Q}} \varphi$ or $\tau \Vdash_{\mathbb{Q}} \psi$, so $\tau \Vdash_{\mathbb{Q}} \varphi \vee \psi$.

If $\sigma \Vdash_{\mathbb{Q}} \exists x \varphi(x)$ then there is an $a \in \mathfrak{M}_\sigma$ such that $\sigma \Vdash_{\mathbb{Q}} \varphi(a)$, so by induction $\tau \Vdash_{\mathbb{Q}} \varphi(a)$ and hence $\tau \Vdash_{\mathbb{Q}} \exists x \varphi(x)$.

Lastly, if $\sigma \Vdash_{\mathbb{Q}} \neg\varphi$ then there is no extension of σ in $\mathbb{Q}$ which forces φ. In particular there is no extension of τ in $\mathbb{Q}$ which forces φ, so $\tau \Vdash_{\mathbb{Q}} \neg\varphi$. □

We now prove the usual forcing lemma:

Lemma 16. *(Forcing Lemma) Let G be a generic and suppose φ is a sentence, possibly with parameters from $\mathfrak{M}_G$. Then $\mathfrak{M}_G \models \varphi$ iff there is a condition $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \varphi$.*

Proof. We proceed by induction on the complexity of φ. First suppose φ is Σ^0_2. Then the lemma holds by Lemma 11.

Inductively suppose we have shown the lemma for φ, ψ and suppose $\mathfrak{M}_G \models \varphi \wedge \psi$. Then by induction there are $\sigma, \sigma' \in G$ such that $\sigma \Vdash_{\mathbb{P}} \varphi$ and $\sigma' \Vdash_{\mathbb{P}} \psi$. Since both σ, σ' are in G they are compatible, so let τ be the longer of the two strings. By Lemma 15, $\tau \Vdash_{\mathbb{P}} \varphi$ and $\tau \Vdash_{\mathbb{P}} \psi$. Hence $\tau \Vdash_{\mathbb{P}} \varphi \wedge \psi$. For the reverse, if there is a condition $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \varphi \wedge \psi$ then $\sigma \Vdash_{\mathbb{P}} \varphi$ and $\sigma \Vdash_{\mathbb{P}} \psi$, so by induction $\mathfrak{M}_G \models \varphi$ and $\mathfrak{M}_G \models \psi$, so $\mathfrak{M} \models \varphi \wedge \psi$.

Similarly for disjunction, if $\mathfrak{M}_G \models \varphi \vee \psi$, assume without loss of generality that $\mathfrak{M}_G \models \varphi$. Then by induction there is a $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \varphi$ and hence $\sigma \Vdash_{\mathbb{P}} \varphi \vee \psi$. For the reverse, if there is a $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \varphi \vee \psi$ then $\sigma \Vdash_{\mathbb{P}} \varphi$ or $\sigma \Vdash_{\mathbb{P}} \psi$. Without loss of generality assume it is the former, so by induction $\mathfrak{M}_G \models \varphi$. Hence $\mathfrak{M}_G \models \varphi \vee \psi$.

For the existential case, if $\mathfrak{M}_G \models \exists x \varphi(x)$ then there is some $a \in \mathfrak{M}_G$ such that $\mathfrak{M}_G \models \varphi(a)$. By induction there is some $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \varphi(a)$. For the reverse, if $\sigma \Vdash_{\mathbb{P}} \exists x \varphi(x)$ then there is a $a \in \mathfrak{M}_\sigma$ such that $\sigma \Vdash_{\mathbb{P}} \varphi(a)$, so by induction $\mathfrak{M}_G \models \varphi(a)$, so $\mathfrak{M}_G \models \exists x \varphi(x)$.

Finally, take the case that there is some $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \neg\varphi$. Suppose for a contradiction that $\mathfrak{M}_G \models \varphi$. Then by induction there is some $\tau \in G$ such that $\tau \Vdash_{\mathbb{P}} \varphi$. If $\tau \subseteq \sigma$, then Lemma 15 implies $\sigma \Vdash_{\mathbb{P}} \varphi$, contradicting that $\sigma \Vdash_{\mathbb{P}} \neg\varphi$. On the other hand if $\sigma \subseteq \tau$ then again by Lemma 15 $\tau \Vdash_{\mathbb{P}} \neg\varphi$, which is contradicted by $\tau \Vdash_{\mathbb{P}} \varphi$.

For the reverse, suppose $\mathfrak{M} \models \neg\varphi$, and suppose for a contradiction that there is no $\sigma \in G$ such that $\sigma \Vdash_{\mathbb{P}} \neg\varphi$. Then by definition, for every $\sigma \in G$ there is a $\tau \in \mathbb{P}$ extending σ such that $\tau \Vdash_{\mathbb{P}} \varphi$. Hence the set of such τ is dense, so there must be such a τ in G. But then by induction $\mathfrak{M}_G \models \varphi$, a contradiction. $\square$

To prove the target theorem, we would like to apply a counting argument to show that there can be no one-to-one mapping from $\mathfrak{M}_G$ into a $\mathfrak{M}_G$-finite interval. Exploiting the finite-ness of the interval requires implementing the counting argument inside the models $\mathfrak{M}_\sigma$ themselves. Since $\mathbb{P}$ is not definable in the models $\mathfrak{M}_\sigma$, it is not possible to refer to the forcing relation directly. We therefore define the following subsets of $2^{\leq b}$:

Definition 9. *For $k \leq b$ let $\mathbb{P}_k = \{\tau \in 2^{\leq b} : \tilde{g}(\tau) \geq k\}$.*

Note that $\mathbb{P}_0 = 2^{\leq b}$ and $\mathbb{P} = \bigcap_{k \in \mathbb{N}} \mathbb{P}_k$. Since each $\mathbb{P}_k$ is definable inside $\mathfrak{M}_\sigma$, we are able to refer to our formal forcing notion when $\Vdash_{\mathbb{P}}$ and $\Vdash_{\mathbb{P}_k}$ coincide. The following lemma captures such instances:

Lemma 17. *(Overspill) Fix a Σ^0_3 sentence φ and $\sigma \in \mathbb{P}$. Then $\sigma \Vdash_{\mathbb{P}} \varphi$ iff there is some $k \in \mathbb{N}$ such that $\sigma \Vdash_{\mathbb{P}_k} \varphi$.*

Proof. The first claim is if φ is a Σ^0_2 formula, $\sigma \Vdash_{\mathbb{P}} \varphi$ iff for every $k \in \mathbb{N}$, $\sigma \Vdash_{\mathbb{P}_k} \varphi$. However, this is immediate by definition since both hold iff $\mathfrak{M}_\sigma \models \varphi$.

The next claim is if φ is a Π^0_2 formula, $\tau \Vdash_{\mathbb{P}} \varphi$ iff there is some $k \in \mathbb{N}$ such that $\sigma \Vdash_{\mathbb{P}_k} \varphi$. First suppose that there is a standard k such that $\sigma \Vdash_{\mathbb{P}_k} \varphi$. Then there is no $\tau \in \mathbb{P}_k$ extending σ such that $\mathfrak{M}_\tau \models \neg\varphi$. Since $\mathbb{P}_k \supseteq \mathbb{P}$ this implies $\sigma \Vdash_{\mathbb{P}} \varphi$.

For the reverse, suppose $\sigma \Vdash_{\mathbb{P}} \varphi$ but for every $k \in \mathbb{N}$, $\sigma \nVdash_{\mathbb{P}_k} \varphi$. Then inside $\mathfrak{M}_\sigma$ define $A = \{k \leq b : \sigma \nVdash_{\mathbb{P}_k} \varphi\}$. As stated, $\mathbb{N} \subseteq A$. On the other hand, if $k \notin \mathbb{N}$ then $\mathbb{P}_k \subseteq \mathbb{P}$, so for every $\tau \in \mathbb{P}_k$ extending σ we have $\mathfrak{M}_\tau \not\models \neg\varphi$. Hence $\sigma \Vdash_{\mathbb{P}_k} \varphi$, so $k \notin A$. But this implies $A = \mathbb{N}$. Since A was definable in $\mathfrak{M}_\sigma$, this is a contradiction.

Finally we suppose φ is Σ^0_3, and write φ as $\exists x \psi(x)$. First suppose $\sigma \Vdash_{\mathbb{P}} \varphi$. Then there is an $a \in \mathfrak{M}_\sigma$ such that $\sigma \Vdash_{\mathbb{P}} \psi(a)$. By the above, this implies there is a $k \in \mathbb{N}$ such that $\sigma \Vdash_{\mathbb{P}_k} \psi(a)$ and hence by definition $\sigma \Vdash_{\mathbb{P}_k} \varphi$. For the reverse, if there is a $k \in \mathbb{N}$ such that $\sigma \Vdash_{\mathbb{P}_k} \varphi$ then again there is an $a \in \mathfrak{M}_\sigma$ such that $\sigma \Vdash_{\mathbb{P}_k} \psi(a)$ and again by the above we have $\sigma \Vdash_{\mathbb{P}} \psi(a)$, so $\sigma \Vdash_{\mathbb{P}} \varphi$. $\square$

We are now ready to prove the first theorem of this section.

Theorem 11. *For G sufficiently generic, $\mathfrak{M}_G \models C\Sigma^0_3$. Hence there is a model of $P^- + I\Sigma^0_1 + \neg B\Sigma^0_2 + C\Sigma^0_3$*

Proof. Fix $\sigma \in \mathbb{P}$, a Σ^0_3 formula $\varphi(x,y)$, and a number $n \in \mathfrak{M}_\sigma$. To show that φ is not a counter-example of $\mathrm{C}\Sigma^0_3$, we want to show that φ is not total, φ is not one-to-one, or its range is not bounded by n. Write $\varphi(x,y)$ as $\exists z\psi(z,x,y)$.

Suppose there is some $\tau \in \mathbb{P}$ extending σ and $x \in \mathfrak{M}_\tau$ such that for every $\tau' \in \mathbb{P}$ extending τ, $\tau' \not\Vdash_{\mathbb{P}} \exists y\varphi(x,y)$. Then τ forces φ to be partial, so we can extend our condition to τ and we are done.

Otherwise, for each $x \in \mathfrak{M}_\sigma$ it is dense above σ to make φ converge at x. Let ψ' be the statement that φ is not a one-to-one function, i.e. $\exists x_1, x_2, y(x_1 \neq y_2 \wedge \varphi(x_1,y) \wedge \varphi(x_2,y))$. If there is a τ extending σ which forces ψ', we can extend to that condition and we are done. Otherwise for every $\tau \in \mathbb{P}$ extending σ, $\tau \not\Vdash_{\mathbb{P}} \psi'$. So by Lemma 17, for every $\tau \in \mathbb{P}$ extending σ and every $k \in \mathbb{N}$, $\tau \not\Vdash_{\mathbb{P}_k} \psi'$. In particular this holds of $k = 0$.

Let $\mathbb{Q} = \{\tau \in \mathbb{P}_0 : \tau \not\Vdash_{\mathbb{P}_0} \psi'\}$. The above paragraph implies $\mathbb{Q} \supseteq \mathbb{P}$.

For each $y \leq n$ and $\tau \in \mathbb{Q}$ extending σ, let $B_{\tau,y} = \{x \leq n \cdot (2^{b+1}+1) : \tau \Vdash_{\mathbb{P}_0} \varphi(x,y)\}$. By Lemma 10 each $B_{\tau,y}$ is definable in $\mathfrak{M}_\sigma$. Since $\tau \not\Vdash_{\mathbb{P}_0} \psi'$, it must be the case that $\mathfrak{M}_\sigma \models |B_{\tau,y}| \leq 1$, for suppose otherwise. Then there is a y and distinct $a, a' \in \mathfrak{M}_\sigma$ with $\mathfrak{M}_\sigma \models (\tau \Vdash_{\mathbb{P}_0} \varphi(a,y)) \wedge (\tau \Vdash_{\mathbb{P}_0} \varphi(a',y))$. Hence $\tau \Vdash_{\mathbb{P}_0} \varphi(a,y) \wedge \varphi(a',y)$. But then this implies $\tau \Vdash_{\mathbb{P}_0} \exists x_1, x_2(x_1 \neq x_2 \wedge \varphi(x_1,y) \wedge \varphi(x_2,y))$, contradicting that $\tau \in \mathbb{Q}$.

Given this we have $\mathfrak{M}_\sigma \models \sum\limits_{\tau\in\mathbb{Q}} \sum\limits_{y\leq n} |B_{\tau,y}| \leq n \cdot 2^{b+1}$, so $\mathfrak{M}_\sigma \models \exists x \leq n \cdot (2^{b+1}+1)(x \notin \bigcup\limits_{\tau\in\mathbb{Q}} \bigcup\limits_{y\leq n} B_{\tau,y})$. Fixing a witness for x we thus have that there is no $\tau \in \mathbb{Q}$ and $y \leq n$ such that $\tau \Vdash_{\mathbb{P}_0} \varphi(x,y)$. Since we are assuming it is dense to make φ converge, fix $\tau \in \mathbb{P}$ extending σ such that $\tau \Vdash \exists y\varphi(x,y)$. Since $\mathbb{Q} \supseteq \mathbb{P}$, $\tau \in \mathbb{Q}$ and hence the witness $y \in \mathfrak{M}_\tau$ of this must satisfy $y > n$. Thus $\tau \Vdash \exists y > n\varphi(x,y)$, so τ satisfies our goal of forcing φ to not have range bounded by n. □

The argument above heavily relies on the Overspill Lemma (Lemma 17), which doesn't seem to support extension beyond Σ^0_3 formulas. In what follows we produce a model of $\mathrm{I}\Sigma^0_1 + \neg\mathrm{B}\Sigma^0_2 + \mathrm{CARD}$, but appeal to much larger fragment of set theory. It is open whether or not it is possible to produce a model of $\mathrm{I}\Sigma^0_1 + \neg\mathrm{B}\Sigma^0_2 + \mathrm{C}\Sigma^0_n$ for $n > 3$ without using such strong set theoretic axioms.

In what follows, we will produce a model satisfying CARD by producing a κ-like model.

Definition 10. *[12] A model $\mathfrak{M}$ is said to be κ-like if* $\mathrm{card}(\mathfrak{M}) = \kappa$*, but for all $a \in \mathfrak{M}$,* $\mathrm{card}(\{n \in \mathfrak{M} : n < a\}) < \kappa$*.*

It is easy to see that all κ-like models satisfy CARD since there are no injections into a proper initial segment (and hence there are in particular no definable such injections). In order to produce such a model, we will appeal to the well-known MacDowell-Specker theorem for constructing end extensions of PA of arbitrary cardinality.

Theorem 12. *[17] Every model of PA has a proper elementary end extension.*

Corollary 3. *Given a cardinal κ and a model $\mathfrak{M}$ of PA, there is an elementary end extension of $\mathfrak{M}$ with cardinality at least κ.*

To build models with proper cardinality, we must construct them in our outer model of set theory rather than taking the direct limit of the models constructed in $\mathcal{V}$. To do so, we will need stronger consistency results than we had in the previous section. We let the formulas φ_i^k be as we defined them before, but write φ_i^k as $\forall x \hat{\varphi}_i^k(x)$ where $\hat{\varphi}_i^k$ is Σ_1^0. For $k \leq b$ and $i = 0, 1$ let $\psi_{k,i}$ be the statement $(\neg\varphi_0^k \vee \neg\varphi_1^k) \wedge \forall x(\neg\hat{\varphi}_{1-i}^k(x) \Rightarrow \exists y < x \neg\hat{\varphi}_i^k(y))$. This precisely captures that $\mathfrak{M}_G \models f_G(k) = i$. Note that for any $\sigma \in 2^{\leq b}$ and $k < |\sigma|$, $\mathfrak{M}_\sigma \models \psi_{k,\sigma(k)}$.

For $\sigma \in 2^{\leq b}$ let the theory S_σ be $\bigwedge_{k<|\sigma|} \psi_{k,\sigma(k)}$. Then as just noted, $\mathfrak{M}_\sigma \models S_\sigma$. Furthermore, if $\tau \supseteq \sigma$ then $S_\tau \Rightarrow S_\sigma$, so we also have $\mathfrak{M}_\tau \models S_\sigma$. However, we will more generally use the following:

Lemma 18. *Suppose $\mathfrak{M} \models S_\sigma$ and $\mathfrak{N}$ is a Π_1^0-elementary end extension of $\mathfrak{M}$. Then $\mathfrak{N} \models S_\sigma$.*

Proof. Suppose not, so there is a $k < |\sigma|$ such that $\mathfrak{N} \models \neg\psi_{k,\sigma(k)}$. Since $\mathfrak{N}$ is a Π_1^0-elementary end extension of $\mathfrak{M}$, all Σ_2^0 facts in $\mathfrak{M}$ also hold in $\mathfrak{N}$. In particular this means $\mathfrak{N} \models \neg\varphi_0^k \vee \neg\varphi_1^k$. So it must be the case that $\mathfrak{N} \models \neg\forall x(\neg\hat{\varphi}_{1-\sigma(k)}^k(x) \Rightarrow \exists y < x \neg\hat{\varphi}_{\sigma(k)}^k(y))$. Fix the witness x for this, and note that $x \in \mathfrak{N} \setminus \mathfrak{M}$. Then we have $\mathfrak{N} \models \forall y < x \hat{\varphi}_{\sigma(k)}^k(y)$. Since this in particular holds for all $y \in \mathfrak{M}$, we have $\mathfrak{M} \models \forall y \hat{\varphi}_{\sigma(k)}^k(y)$. However, since $\mathfrak{M} \models \psi_{k,\sigma(k)}$, by contrapositive this means $\mathfrak{M} \models \forall x \hat{\varphi}_{1-\sigma(k)}^k(y)$ contradicting $\mathfrak{M} \models \neg\varphi_0^k \vee \neg\varphi_1^k$. ☐

The more general consistency result we will utilize is as follows:

Lemma 19. *Fix $\sigma, \tau \in 2^{\leq b}$ with $\tau \supset \sigma$. Suppose $\mathfrak{N} \models T_{b-|\sigma|} + S_\sigma$. Then $\mathfrak{N} \models \mathrm{Con}(T_{b-|\tau|} + S_\tau)$.*

Proof. Fix σ and proceed by induction on τ. In the base case that $\tau = \sigma^\frown i$ for $i = 0, 1$ we follow the procedure used in the previous section. That is, we have by Lemma 9 that $\mathfrak{N} \models \mathrm{Con}(T_{b-|\sigma|-1} + \neg\varphi_i^{|\sigma|} + \varphi_{1-i}^{|\sigma|})$. Take $\mathfrak{N}'$ an end extension such that $\mathfrak{N}' \models T_{b-|\sigma|-1} + \neg\varphi_i^{|\sigma|} + \varphi_{1-i}^{|\sigma|}$, and since $\mathrm{Tr}(1) \subseteq T_{b-|\sigma|-1}$ we have $\mathfrak{N} \prec_{\Pi_1^0} \mathfrak{N}'$. By definition we have $\mathfrak{N}' \models \psi_{|\sigma|,i}$ and so Lemma 18 implies we have $\mathfrak{N}' \models S_\tau$. Hence by the completeness theorem, $\mathfrak{M}_\sigma \models \mathrm{Con}(T_{b-|\tau|} + S_\tau)$.

In the induction case, let $\tau \supset \sigma$ and let $i = 0, 1$. By the induction hypothesis, $\mathfrak{N} \models \mathrm{Con}(T_{b-|\tau|} + S_\tau)$. Let $\mathfrak{N}'$ be an end extension of $\mathfrak{N}$ which models $T_{b-|\tau|} + S_\tau$. As before, since $\mathrm{Tr}(1) \subseteq T_{b-|\tau|}$ we know $\mathfrak{N}'$ is a Π_1^0-elementary end extension of $\mathfrak{N}$. By the definition of $\varphi_0^{|\tau|}, \varphi_1^{|\tau|}$, $\mathfrak{N}' \models \mathrm{Con}(T_{b-|\tau|-1} + \neg\varphi_i^{|\tau|} + \varphi_{1-i}^{|\tau|})$. Let $\mathfrak{N}''$ be an end extension of $\mathfrak{N}'$ which models $T_{b-|\tau|-1} + \neg\varphi_i^{|\tau|} + \varphi_{1-i}^{|\tau|}$, which again is Π_1^0-elementary. By definition $\mathfrak{N}'' \models \psi_{|\tau|,i}$. Again by Lemma 18 we have $\mathfrak{N}'' \models S_\tau$, so $\mathfrak{N}'' \models T_{b-|\tau|-1} + S_{\tau^\frown i}$. Hence $\mathfrak{N}' \models \mathrm{Con}(T_{b-|\tau|-1} + S_{\tau^\frown i})$.

Suppose for a contradiction that $\mathfrak{N} \models \neg\,\mathrm{Con}(T_{b-|\tau|-1} + S_{\tau^\frown i})$. Then there is a proof $p \in \mathfrak{N}$ of a contradiction from $T_{b-|\tau|-1} + S_{\tau^\frown i}$. Since this theory is Π_1^0 and $\mathfrak{N}'$ is a Π_1^0-elementary end extension, p is in $\mathfrak{N}'$ and is also a valid proof in $\mathfrak{N}'$. Hence $\mathfrak{N}' \models \neg\,\mathrm{Con}(T_{b-|\tau|-1} + S_{\tau^\frown i})$, a contradiction. Thus $\mathfrak{N} \models \mathrm{Con}(T_{b-|\tau|-1} + S_{\tau^\frown i})$ as desired. ☐

We are now ready to prove the final result of this section.

Theorem 13. *There is a model* $\mathfrak{M}$ *of* $P^- + I\Sigma_1^0 + \neg B\Sigma_2^0 + CARD$

Proof. Let G be a generic from our poset $\mathbb{P}$. Since b was countable, G is countable and we can choose a sequence $\langle \sigma_n : n \in \mathbb{N} \rangle$ which is increasing and unbounded inside G. Define $\mathfrak{N}_{\sigma_n}$ by induction. We will maintain that $\mathfrak{N}_{\sigma_n} \models T_{b-|\sigma_n|} + S_{\sigma_n}$.

In the base case let $\mathfrak{N}_{\sigma_0} = \mathfrak{M}_{\sigma_0}$.

In the induction case, let $\sigma = \sigma_n$ and let $\tau = \sigma_{n+1}$. We have by induction that $\mathfrak{N}_\sigma \models T_{b-|\sigma|} + S_\sigma$, so by Lemma 19 we have $\mathfrak{N}_\sigma \models \mathrm{Con}(T_{b-|\tau|} + S_\tau)$. Let $\mathfrak{N}$ be an end extension of $\mathfrak{N}_\sigma$ such that $\mathfrak{N} \models T_{b-|\tau|} + S_\tau$. By Theorem 12 let $\mathfrak{N}_\tau$ be a conservative end extension of $\mathfrak{N}$ of cardinality at least $\mathcal{P}(\mathfrak{N}_\sigma)$.

Finally we let $\mathfrak{M}$ be the direct limit of the models $\mathfrak{N}_{\sigma_n}$ for $n \in \mathbb{N}$. As before, $\mathfrak{M}$ is a Π_1^0-elementary end extension of each $\mathfrak{N}_{\sigma_n}$, which implies $\mathfrak{M} \models \mathrm{I}\Sigma_1^0$. Additionally, by Lemma 18 we have $\mathfrak{M} \models S_{\sigma_n}$ for each n. Hence the function f_G is Δ_2^0-definable in $\mathfrak{M}$, so $\mathfrak{M} \models \neg\mathrm{B}\Sigma_2^0$ as desired. Furthermore since the cardinality of $\mathfrak{M}$ is strictly greater than the cardinality of any initial segment of $\mathfrak{M}$ (since all such initial segments are contained in $\mathfrak{N}_{\sigma_n}$ for some $n \in \mathbb{N}$), $\mathfrak{M}$ is a κ-like model. □

Returning once more to κ-like models, we answer an open question posed by Kaye [11] in 1995. Kaye describes an axiom schema INDISC which precisely characterizes κ-like models for strong limit cardinals κ. He also defines the axiom schema IB as

$$\mathrm{I}\Sigma_n^0 \Rightarrow \mathrm{B}\Sigma_{n+1}^0$$

for all n. He ends his paper with the open question "Is 'IB + exp' an axiomatization of the theory of κ-like models for singular strong limit cardinals κ?" or equivalently, is it the case that INDISC $\Rightarrow$ IB? Since $\mathfrak{M}$ has singular strong limit cardinality, it is a model of INDISC and we can answer his question in the negative.

Corollary 4. *INDISC* $\not\Rightarrow$ *IB*

Proof. Since $\mathfrak{M}$ is a κ-like model for a singular strong limit cardinal κ, we know $\mathfrak{M} \models$ INDISC. Furthermore, $\mathfrak{M} \models \mathrm{I}\Sigma_1^0 + \neg\mathrm{B}\Sigma_2^0$, so $\mathfrak{M} \models \neg\mathrm{IB}$. □

3.4 Open Questions

This chapter has continued the investigation into the first-order consequences of 2RAN. Although we have narrowed that characterization, the original question posed by Conidis and Slaman remains open:

1. Characterize the first-order consequences of 2RAN.

In section 3 we showed that $\mathrm{C}\Sigma_3^0 \not\vdash \mathrm{B}\Sigma_2^0$ using only a small fragment of set theory. We then showed the stronger statement that CARD $\not\vdash \mathrm{B}\Sigma_2^0$ but had to appeal to countably many iterates of the power set axiom. Whether or not this was necessary is an interesting open question.

2. Is there a proof of $I\Sigma_1^0 + \text{CARD} \not\vdash B\Sigma_2^0$ that can be carried out in just second-order arithmetic? Is this even possible for $I\Sigma_1^0 + C\Sigma_4^0$?

As section 3 demonstrated, this framework for creating models of $\neg B\Sigma_2^0$ can be used to show that $B\Sigma_2^0$ is not a consequence of other theorems and axioms. This chapter has only provided a few examples, but we suspect there are many others. Some potential candidates include $TS(2)$, the Thin Set Theorem for pairs, and the Carlson-Simpson Lemma.

3. Can the methods used in this chapter be used to show other theorems or axioms do not prove $B\Sigma_2^0$?

Avigad, Dean, and Rute investigated the strength of relativizing Weak Weak König's Lemma (WWKL) and showed that n-WWKL is equivalent to $B\Sigma_n^0 + n$-RAN over RCA_0 [1]. It remains unclear if there is a sufficient level of randomness such that n-RAN implies $B\Sigma_2^0$.

4. Is there an n such that $P^- + I\Sigma_1^0 + n$-RAN implies $P^- + B\Sigma_2^0$?

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