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LECTURES ON

GENERALIZED SOLUTIONS TO THE MINIMAL SURFACE EQUATION[1]

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GENERALIZED SOLUTIONS TO THE MINIMAL SURFACE EQUATION

1. SINGULAR MINIMAL HYPERSURFACES

In 1969 E. Bombieri-E. DeGiorgi-E. Giusti (B-DeG-G) proved that the $(2k-1)$-area of the cones

$$M = \{(x,y) \mid x \in R^k, y \in R^k, |x| = |y|\}$$

is a minimum as $k \geq 4$. That was already known as being false for $k < 4$. It was so proven that the codimension one Plateau problem admitted singular solutions for higher dimensions.

B-DeG-G original proof consisted in approximating M by smooth hypersurfaces minimizing the $(2k-1)$-area. Those approximating hypersurfaces could not be graphs of functions, as M is not. That made their determination not too simple.

What I want to show here is a 1983 result due to U. Massari and myself, consisting in proving that the cylinder $M \times R$ minimizes the $2k$-area, for $k \geq 4$, as a consequence of the existence of a convenient family of graphs approximating it.

To this problem consider the fourth degree polynomial

$$P(x,y) = (|x|^2 - |y|^2)(|x|^2 + |y|^2) = |x|^4 - |y|^4, \quad x \in R^k, y \in R^k .$$

The minimal surface operator $\mathbf{M}$ applied to P gives

$$\begin{aligned}\mathbf{M}P(x,y) &= div\left[\frac{\nabla P(x,y)}{\sqrt{1+|\nabla P(x,y)|^2}}\right] \\ &= (|x|^2 - |y|^2)\,\frac{(4k+8)\{1+16|x|^6+16|y|^6\} - 16\cdot 12(|x|^4+|y|^4)(|x|^2+|y|^2)}{\sqrt{1+16|x|^6+16|y|^6}^{\,3}} .\end{aligned}$$

Since

$$2(|x|^6+|y|^6) = 2(|x|^4-|x|^2|y|^2+|y|^4)(|x|^2+|y|^2) \geq (|x|^4+|y|^4)(|x|^2+|y|^2) ,$$

we get, for $|x| > |y|$,

$$\mathbf{M}P(x,y) \geq \frac{|x|^2-|y|^2}{\sqrt{1+16|x|^6+16|y|^6}^{\,3}}\left\{(4k+8)+[8(4k+8)-16\cdot 12](|x|^4+|y|^4)(|x|^2+|y|^2)\right\}$$

and the inequality goes into the opposite direction if $|x|<|y|$. Since $k \geq 4$ is equivalent to $8(4k+8)-16\cdot 12 \geq 0$, we obtain, for $k \geq 4$:

$$\mathbf{M}P(x,y) > 0 \quad \text{if} \quad |x| > |y| ,$$

$$\mathbf{M}P(x,y) < 0 \quad \text{if} \quad |x| < |y| .$$

In other words, the polynomial P is a subsolution to the minimal surface equation in the open set $|x|>|y|$ and a supersolution in the open set $|x|<|y|$, as $k \geq 4$!

This differential property of the polynomial P is equivalent to the following variational property of its graph (see [3], pp. 101-103): any $2k$-dimensional hypersurface sitting below the graph P and coinciding with it outside a compact subset of

$$\{(x,y) \mid |x|>|y|\} \times R ,$$

has a $2k$-area greater than the $2k$-area of graph P. (Surely, the two areas are infinite, what I mean comparing them has to be intended as the area comparison of the two portions of hypersurfaces contained in the compact region where they are different.) The same fact is true for $2k$-dimensional hypersurfaces sitting above the graph P and coinciding with it outside a compact subset of

$$\{(x,y) \mid |x|<|y|\} \times R .$$

consider now, for $\rho > 0$, the omotethie

$$\rho : R^{2k+1} \to R^{2k+1}$$

$$(x,y,t) \quad (\rho x, \rho y, \rho t) .$$

The hypersurface graph P is transformed by ρ into the hypersurface

$$\text{graph } P_\rho$$

where P_ρ is the polynomial

$$P_\rho(x,y) = \rho^{-1}P(\rho x,\rho y) = \rho^3 P(x,y) .$$

The hypersurfaces graph P_ρ have the same variational property as graph P and they tend to the hypersurface

$$M \times R \,, \quad \text{as } \rho \to +\infty \,.$$

Therefore, the $2k$-area of $M \times R$ has the following minimum property: any hypersurface, sitting in

$$\{(x,y) \mid |x| \geq |y|\} \times (0,+\infty)$$

and coinciding with $M \times R$ outside a compact subset of $R^{2k} \times (0,+\infty)$, has a $2k$-area greater than the $2k$-area of $M \times R$. Similarly, any hypersurface, sitting in

$$\{(x,y) \mid |x| \leq |y|\} \times (-\infty,0)$$

and coinciding with $M \times R$ outside a compact subset of

$$R^{2k} \times (-\infty,0) \,,$$

has a $2k$-area greater than the $2k$-area of $M \times R$.

Since $M \times R$ is a cylinder, that proves that $M \times R$ minimizes the $2k$-area, which surely means that M minimizes the $(2k-1)$-area.

Another approach for the approximation problem is the following: the "finite" cylinder

$$(M \cap B) \times R = \{(x,y) \mid x \in R^k, |y| \in R^k, |x|^2+|y|^2 < 1, |x| = |y|\} \times R$$

can be approximated by graphs of solutions to the minimal surface equation.

Consider in fact the Dirichlet problem for the minimal surface equation:

$$\begin{cases} \mathbf{M}f(x,y) = 0 & \text{for } |x|^2+|y|^2 < 1 \\ f(x,y) = P_\rho(x,y) & \text{for } |x|^2+|y|^2 = 1 \,. \end{cases}$$

This problem has a unique solution f_ρ satisfying

$$f_\rho(x,y) \geq P_\rho(x,y) \geq 0 \quad \text{if} \quad |x| > |y| ,$$

$$f_\rho(x,y) \leq P_\rho(x,y) \leq 0 \quad \text{if} \quad |x| < |y| .$$

Therefore

$$\text{graph}\, f_\rho \to (M \cap B) \times R \quad \text{as} \quad \rho \to +\infty ,$$

which is sufficient to conclude that M minimizes the $(2k-1)$-area.

2. GENERALIZED SOLUTIONS

In order to make as natural as possible the definition of generalized solutions, it is convenient to look at regular solutions to the minimal surface equation (m.s.e.) as codimension one minimal manifolds. That is, if Ω is an open set of R^n and $f \in C^2(\Omega)$ is a solution to the m.s.e., we consider hypersurface (codimension one manifold) in R^{n+1} given by the graph of f

$$\text{graph} f = \{(x, f(x) \mid x \in \Omega\} .$$

As we already said in Lecture 1, graph f is not only a manifold of zero mean curvature but it minimizes the n-area with respect to local modifications contained in $\Omega \times R$.

In Lecture 1 we have also proved the existence of singular minimal hypersurfaces, as are the cones

$$\{(x,y) \mid x \in R^k, y \in R^k, |x| = |y|\} \quad , \text{ for } k \geq 4 .$$

It is then useful to refer to a definition of minimal hypersurface, covering the singular cases as well as the case of regular manifolds. A definition satisfying such a requirement was given by DeGiorgi in 1960.

(DeGIORGI) DEFINITION OF MINIMAL HYPERSURFACES: an n dimensional minimal hypersurface is a closed subset M of an open set A of R^{n+1} with the following property. for all $z \in M$ there exists a positive real number ρ and a Lebesgue measurable set E of R^{n+1} s.t.

$$M \cap B_\rho(z) = \partial E \cap B_\rho(z) , \tag{1}$$

where ∂E is the boundary of E and $B_\rho(z)$ is the ball $\{\zeta \in R^{n+1} \mid |\zeta - z| < \rho\}$,

$$\Lambda_n(\partial E \cap B_\rho(z)) \leq \Lambda_n(\partial G \cap B_\rho(z)) , \tag{2}$$

where G is any other Lebesgue measurable set with $(G-E) \cup (E-G) \subset\subset B_\rho(z)$, and Λ_n is defined by

$$\Lambda_n(\partial E \cap B_\rho(z)) = sup\{\int_E div\,\phi(\zeta)d\zeta \mid \phi \in [C_0^1(B_\rho(z))]^{n+1}, |\phi(\zeta)| \leq 1 \text{ for all } \zeta\}. \qquad (3)$$

To understand where Λ_n comes out from, recall that if ∂E is sufficiently regular for the divergence theorem formula to hold

$$\int_E div\,\phi(\zeta)d\zeta = \int_{\partial E} \phi \cdot \nu(\zeta)dH_n(\zeta),$$

where ν is the exterior unit normal vector to ∂E and H_n is the n-dimensional Hausdorff measure in R^{n+1}, then

$$\Lambda_n(\partial E \cap B_\rho(z)) = H_n(\partial E \cap B_\rho(z)).$$

What is actually true for graph f, when f is a solution to m.s.e., and the cones $\{(x,y) \mid x \in R^k, y \in R^k, |x|=|y|\}$ when $k \geq 4$, is that they are minimal hypersurfaces in DeGiorgi's sense (see [3], pp. 101-103 and pp. 146-150).

The aim of this lecture is to show how one can get the singular minimal hypersurfaces as limits of the regular ones.

To this purpose, remember what was done at the end of Lecture 1: the finite cylinders

$$\{(x,y) \mid x \in R^k, y \in R^k, |x|=|y|, |x|^2+|y|^2 < 1\} \times R \quad , \text{ for } k \geq 4,$$

were proven to be limits of graphs of solutions to the m.s.e. Essentially the same argument can be used to prove the following

THEOREM. The cylinder constructed over any minimal hypersurface is locally the limit of minimal graphs.

PROOF. Being that our statement is a local one, we can assume the minimal hypersurface to be the boundary of a Lebesgue measurable set E, all over a given open set $\Omega \subset R^n$. Let us consider now any ball $B \subset\subset \Omega$, s.t.

$$H_{n-1}[\partial B \cap \partial E] = 0. \qquad (4)$$

This condition does not bother us, since the $H_{n-1}\big|_{\partial E}$ is locally finite in Ω. Consider now the Dirichlet problem for the m.s.e. in B with boundary values

$$\rho \text{ on } \partial B \cap E \quad , \quad -\rho \text{ on } \partial B - E \,.$$

The family $\{f_\rho\}$ of solutions to this problem is compact with respect to the convergence a.e. in B (we do accept $-\infty$ and $+\infty$ as limit values). A detailed proof of this fact and of other facts used here will be presented in Lecture 3. There exist then sequences $\rho_j \uparrow +\infty$ s.t. $\{f_{\rho_j}\}_j$ has a limit a.e. in B.

Put $f_j = f_{\rho_j}$ and

$$P = \{x \in B \,\big|\, f_j(x) \to +\infty\} \,.$$

We claim that the boundary of P is a minimal hypersurface in B and that P coincides with E on ∂B. Therefore also the set

$$(E-B) \cup P$$

has minimal boundary in Ω. Since a minimal boundary is analytic except for closed sets which cannot disconnect it, the analytic continuation applies to get $P = E \cap B$. One similarly proves that

$$B-E = \{x \in B \,\big|\, f_j(x) \to -\infty\} \,.$$

These two facts give to

$$\text{graph } f_j \to (\partial E \cap B) \times R \,. \qquad \blacksquare$$

What we shall do now is to investigate the properties of:

GENERALIZED SOLUTIONS TO THE M.S.E.: a Lebesgue measurable function

$$g : \Omega \to [-\infty, +\infty] \,,$$

is called a generalized solution to the minimal surface equation if the set

$$E = \{(x,t) \mid x \in \Omega, t < g(x)\}$$

has minimal boundary in $\Omega \times R$.

We shall refer to the part of the boundary of E contained in $\Omega \times R$ as the graph of g (graph g).

3. MINIMAL HYPERSURFACES

We recall the main properties of minimal hypersurfaces that will be used to study generalized solutions.

MONOTONICITY THEOREM. If E is a measurable set whose boundary is a minimal hypersurface in the open set Ω of R^{n+1}, then the real function

$$\rho \to \rho^{-n} \Lambda_n(\partial E \cap B_\rho(x))$$

is non decreasing for $\rho \in (0, dist\ (x, \partial\Omega))$.

BOUNDS FOR THE n-AREA. Since ∂E is a minimal hypersurface, the following inequalities must hold:

$$\Lambda_n(\partial E \cap B_\rho(x)) \leq H_n(E \cap \partial B_\rho(x)) \ , \tag{5}$$

$$\Lambda_n(\partial E \cap B_\rho(x)) \geq H_n(\partial B_\rho(x) - E) \ . \tag{6}$$

If ω_{n+1} is the measure of the unit ball of R^{n+1}, we have

$$H_n(\partial B_\rho(x)) = (n+1)\omega_{n+1}\rho^n \ ;$$

therefore, (5) and (6) imply

$$\rho^{-n} \Lambda_n(\partial E \cap B_\rho(x)) \leq \frac{n+1}{2} \omega_{n+1} \ . \tag{7}$$

If x is a regular point of $\partial E \cap \Omega$, then

$$\lim_{\rho \to 0} \rho^{-n} \Lambda_n(\partial E \cap B_\rho(x)) = \omega_n \ . \tag{8}$$

This will give

$$\rho^{-n} \Lambda_n(\partial E \cap B_\rho(x)) \geq \omega_n \ , \tag{9}$$

for all regular points $x \in \partial E \cap \Omega$ and $\rho \in (0, dist\ (x, \partial\Omega))$. The inequality (9) remains valid for singular points of $\partial E \cap \Omega$, because they all are limit points of regular ones.

Inequalities (5) and (9) imply

$$\rho^{-(n+1)} meas\,(E \cap B_\rho(x)) \geq \frac{\omega_n}{n+1}, \tag{10}$$

for $x \in \partial E \cap \Omega$ and $\rho \in (0, dist\,(x, \partial\Omega))$. Similarly (6) and (9) imply

$$\rho^{-(n+1)} meas\,(B_\rho(x) - E) \geq \frac{\omega_n}{n+1}. \tag{11}$$

DeGIORGI'S REGULARITY THEOREM

If ∂E is a minimal hypersurface in Ω and there exists a tangent cone at $x \in \partial E \cap \Omega$,(*) then x is a regular point for ∂E.(**)

EXISTENCE OF TANGENT CONES

Geometric measure theory arguments prove the existence of tangent cones for Λ_n--almost all points of $\partial E \cap \Omega$. This, together with DeGiorgi's regularity theorem, implies that the singular part of $\partial E \cap \Omega$ must be closed and have Λ_n-measure equal to zero. Inequality (9) will then imply that the singular part of $\partial E \cap \Omega$ must have Hausdorff n-dimensional measure equal to zero.

FURTHER REMARKS ABOUT SINGULAR POINTS

If there exists a minimal hypersurface in R^{n+1} with a singular point, then there must exist a singular minimal cone in R^{n+1}.

If a minimal singular cone of R^{n+1} has singularities outside its vertex, then there must exist minimal singular cones in R^n.

SIMONS' THEOREM

There exists no minimal singular cone in R^7.

Footnotes (*) and (**) appear at the end of this paper.

FEDERER'S REMARK

The singular part of a minimal hypersurface must have codimension greater than or equal to 7. That is, the Hausdorff s-dimensional measure of the singular part of an n-dimensional minimal hypersurface, is zero for all real $s > n-7$.

BOMBIERI-GIUSTI'S STRONG MAXIMUM PRINCIPLE

Assume E to be the set below the graph of a generalized solution $g:\Omega \to [-\infty,+\infty]$. If ν is the exterior unit normal vector to the regular points of $\partial E \cap (\Omega \times R)$, we have

$$\nu_{n+1} \geq 0 \,, \tag{12}$$

$$\Delta \nu_{n+1} + c^2 \nu_{n+1} = 0 \,, \tag{13}$$

where Δ is the Laplace operator over ∂E and c^2 is the sum of the squares of the principal curvatures of ∂E. Bombieri and Giusti were able to show that (12) and (13) imply

$$\inf_{B_\rho(x)} \nu_{n+1} > 0 \quad \text{or} \quad \nu_{n+1}\big|_{B_\rho(x)} = 0 \,,$$

whenever $x \in \Omega \times R$ and $\rho < \gamma(n)\, dist\,(x, \partial\Omega \times R)$, for a convenient choice of $\gamma(n) \in (0,1)$.

All results stated in this lecture can be found in [3]. Their first proofs appeared in [4],[5],[6],[7], and [8].

4. FIRST APPLICATIONS OF GENERALIZED SOLUTIONS

Let g be a generalized solution to the m.s.e. in an open set of $\Omega \subset R^n$. As a consequence of the strong maximum principle due to Bombieri and Giusti, the following theorem can be proven (see [5] or [3] pp. 218-220).

STRUCTURE THEOREM: For any given generalized solution $g: \Omega \to [-\infty, +\infty]$, there exist three subsets G, P, N of Ω, with the following properties:

(i) G is an open set, $g \big|_G$ is real analytic and a solution to the m.s.e.,

(ii) P is a Lebesgue measurable set with minimal boundary in Ω and $g \big|_P = +\infty$,

(iii) N is a Lebesgue measurable set with minimal boundary in Ω and $g \big|_N = -\infty$,

(iv) $\Omega = G \cup P \cup N \cup (\partial P \cap \partial N \cap \Omega)$,

(v) g is continuous at all points of $\Omega - \partial P \cap \partial N$.

This structure theorem has many quite interesting consequences when applied together with the following compactness property of generalized solutions:

COMPACTNESS THEOREM: For any increasing sequence $\{\Omega_j\}$ of open sets and any sequence $g_j: \Omega_j \to [-\infty, +\infty]$ of generalized solutions to the m.s.e., there exist an increasing sequence of integers $j(s)$ and a generalized solution

$$g: \Omega = \bigcup_j \Omega_j \to [-\infty, +\infty],$$

with

$$g_{j(s)}(x) \to g(x) \quad , \quad \text{for almost all } x \in \Omega .$$

This compactness property is a straightforward consequence of the analogous property for hypersurfaces with equally bounded measure (see [3], pp. 70-71).

1st APPLICATION: DIRICHLET PROBLEM FOR THE M.S.E. IN AN UNBOUNDED CONVEX DOMAIN DIFFERENT FROM A HALF SPACE

Let Ω be the intersection of at least two half spaces and $\varphi\colon \partial\Omega \to R$ a continuous function.

For any $\rho > 0$ let g_ρ be a solution to the m.s.e. in $\{x \in \Omega \,|\, |x| < \rho\}$ satisfying

$$g_\rho(x) = \varphi(x) \ , \ \text{whenever} \ x \in \partial\Omega \ , \ |x| = \rho \ .$$

Consider a sequence $\rho_j \uparrow +\infty$ such that

$$g_{\rho_j}(x) \underset{j}{\to} g(x) \ , \ \text{for almost all} \ x \in \Omega \ .$$

Consider now the set $P = \{x \in \Omega \,|\, g(x) = +\infty\}$. We want to show that P must be the empty set. If not, P should have not only minimal boundary in Ω, but also minimal boundary in R^n, as a consequence of the convexity of Ω. But this would imply P = halfspace, as we will show in the next lecture. This would imply Ω = halfspace, which is the case we have excluded by hypothesis. Then $P = \phi$ and the same can be said for N. The structure theorem will then say

$$\Omega = G$$

or g is real analytic in Ω, and a solution to our Dirichlet problem.

2nd APPLICATION: REMOVABLE SINGULARITIES FOR CLASSICAL SOLUTIONS TO THE M.S.E.

Assume Ω to be an open set of R^n and X a closed subset of Ω with

$$H_{n-1}(X) = 0 \ .$$

Assume

$$f : \Omega - X \to R$$

to be a classical solution to the m.s.e.

It makes sense to ask whether f is a generalized solution to the m.s.e. in Ω itself, f being defined almost everywhere in it.

A non difficult calculation using the hypothesis

$$H_{n-1}(X) = 0$$

and the definition of Hausdorff measure, shows that f is actually a generalized solution to the m.s.e. in Ω. The values $+\infty$ or $-\infty$ can be given to f only in points $x \in X$; therefore, the sets P and N must have measure zero. But a set with minimal boundary is either of positive measure or empty, then

$$P = N = \phi \, .$$

The structure theorem applied to the present case will then give

$$\Omega = G \, ,$$

that is, f is analytic in Ω and a solution to the m.s.e. over there.

5. BERNSTEIN'S THEOREM

In 1912 Sergei Bernstein published the following result:

$$\text{if } f : R^2 \to R \text{ solves the m.s.e. then } \nabla f = \text{constant} .$$

In 1957 J.C.C. Nitsche was able to connect the validity of Bernstein's statement to that of Liouville's theorem, so after 1957 Bernstein's theorem is as true as the fact that a bounded harmonic function defined in the whole R^2 is necessarily constant.

In 1962 a new story, having Bernstein's theorem as protagonist, was started by W.H. Fleming. As J.C.C. Nitsche had given the most elementary and most two-dimensional proof of Bernstein's theorem, so W.H. Fleming gave the most sophisticated one, apparently almost independent of dimension.

Fleming asked the question: are there measurable sets E with minimal boundary in R^n, other than the halfspaces?

FLEMING FUNDAMENTAL REMARK. If E has minimal boundary in R^n, then also

$$E_\rho = \{x \in R^n \,|\, \rho x \in E\}$$

has minimal boundary in R^n for all $\rho > 0$.

Since the family of measurable sets

$$\{E_\rho\}_{\rho > 0}$$

is compact, there exist sequences $\rho_j \uparrow +\infty$ such that

$$E_{\rho_j} \underset{j}{\to} E_\infty ,$$

where E_∞ is again a measurable set with minimal boundary in R^n. Because of its origin E_∞ must be a cone, say

$$E_\infty = (E_\infty)_\rho = \{x \in R^n \,|\, \rho x \in E_\infty\} \ , \ \text{for all } \rho > 0 .$$

A cone that has to be regular at its vertex is necessarily a halfspace.

If E_∞ is a halfspace, then

$$H_{n-1}(\partial E_\infty \cap B_\rho) = \omega_{n-1}\rho^{n-1},$$

for all positive ρ, and

$$\lim_j H_{n-1}(\partial E_{\rho_j} \cap B_\rho) = \lim_j \rho_j^{-n+1} H_{n-1}(\partial E \cap B_{\rho\rho_j}) = \omega_{n-1}\rho^{n-1}.$$

We would then have

$$\lim_j (\rho\rho_j)^{-n+1} H_{n-1}(\partial E \cap B_{\rho\rho_j}) = \omega_{n-1}.$$

Assuming $0 \in \partial E$, and recalling that

$$t^{-n+1} H_{n-1}(\partial E \cap B_t)$$

is non decreasing and greater than or equal to ω_{n-1}, we can conclude

$$t^{-n+1} H_{n-1}(\partial E \cap B_t) = \omega_{n-1},$$

for all positive t. This implies E itself must be a cone. If 0 is a regular point for ∂E then E must be a halfspace.

What we have proven is: if E and E_∞ are regular at 0, then E must be a halfspace.

W.H. Fleming was able to prove that there are no minimal singular cones in R^3. Therefore, for $n = 3$ E_∞ is always regular at 0. The above argument proves then that there are no subsets $E \subset R^3$ with minimal boundary, regular at 0 and different from a halfspace. In particular, there cannot be non-trivial solutions to the m.s.e. in R^2 (Bernstein's theorem).

We will see in Lecture 6 how the non existence of minimal singular cones in R^3 actually implies the validity of Bernstein's theorem for functions of three variables. F.J. Almgren, Jr., proved the non existence of minimal singular cones in R^4, J. Simons extended such result to R^7. Therefore, Bernstein's theorem in the stronger form we have presented is valid in the ambient space R^7.

The validity of BERNSTEIN'S STRONG THEOREM is stopped in R^8 by the fact that

$$\{x \in R^8 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is a minimal boundary.

6. EXISTENCE OF NON-LINEAR SOLUTIONS TO THE M.S.E. IN THE WHOLE SPACE

Assume

$$g : R^n \to [-\infty, +\infty]$$

to be a generalized solution to the m.s.e. For all $\rho > 0$ the function

$$g_\rho(x) = \rho^{-1} g(\rho x) \ , \ x \in R^n$$

is a generalized solution too.

Assume now $\rho_j \uparrow +\infty$ to be chosen in order to have

$$g_{\rho_j}(x) \underset{j}{\to} g_\infty(x) \ , \text{ for almost all } x \in R^n \ .$$

Assume $g(0) = 0$ and 0 to be a regular point for graph g; Fleming's remark would immediately give, for $n \leq 6$

$$\text{graph } g = \text{hyperplane} .$$

Observe that, in order to get 0 to be a singular point for graph g_∞, we must have graph g_∞ = cylinder; therefore, its horizontal cross-section would provide a minimal singular cone in R^8. This proves, in particular, that the existence of a non trivial generalized solution to the m.s.e. in R^n implies the existence of a singular minimal cone in R^n itself. This remark, together with Simons' result about minimal cones, extends the validity of Bernstein's theorem to generalized solutions of seven independent variables.

An equivalent formulation of this result is the following: for $n \leq 7$, given any sequence

$$g_j : B_{\rho_j} \to [-\infty, +\infty] \ , \ B_{\rho_j} \subset R^n$$

of generalized solutions to the m.s.e., with

$$g_j(0) = 0 \ , \ \rho_j \uparrow +\infty \ ,$$

then

$$\liminf_j g_j(x) \geq 0$$

for almost all x belonging to a convenient halfspace. On the contrary, in R^8 we can try the following: for any given $\rho > 1$, consider the unique solution f_ρ to the m.s.e. in

$$\{(x,y) \mid x \in R^4, y \in R^4, |x|^2 + |y|^2 < \rho^2\} ,$$

equal to a convenient positive real number $M(\rho)$ on

$$\{(x,y) \mid |x|^2 + |y|^2 = \rho^2, |x| > |y|\} ,$$

and to $-M(\rho)$ on

$$\{(x,y) \mid |x|^2 + |y|^2 = \rho^2, |x| < |y|\} .$$

The value $M(\rho)$ is determined by the requirement

$$f_\rho(1,0, \ldots ,0) = 1 .$$

The function f_ρ must be, for symmetry reasons, strictly positive for $|x| > |y|$ and strictly negative for $|x| < |y|$. If $\rho_j \uparrow +\infty$ is chosen in order to have the existence of

$$f_\infty(x,y) = \lim_j f_{\rho_j}(x,y) ,$$

for almost all $(x,y) \in R^8$, we get

$$f_\infty(x,y) \geq 0 \quad , \quad \text{for } |x| > |y| .$$

We cannot have

$$f_\infty(x,y) = 0 \quad , \quad \text{for all } (x,y) ,$$

nor

$$|f_\infty(x,y)| = +\infty \quad , \quad \text{for all } (x,y) ,$$

because

$$f_\infty(1,0, \ldots ,0) = 1 .$$

We have so obtained a non trivial generalized solution to the m.s.e. in R^8. Such a solution is a classical analytic solution over a convenient open non empty set Ω that is not necessarily equal to R^8. In other words the solution f_∞ can assume the values $+\infty$ and $-\infty$.

In order to have a classical solution in the whole R^8, recall the way indicated by Bombierii-DeGiorgi-Giusti: the function

$$F(x,y) = (|x|^2-|y|^2)(|x|^2+|y|^2)^{1/2}$$

is a subsolution to the m.s.e. where $|x|>|y|$ and a supersolution where $|x|<|y|$. There exists now another function G: $R^8 \to R$ that is a supersolution if $|x|>|y|$ and a subsolution if $|x|<|y|$. Moreover, G satisfies

$$0 \le F(x,y) \le G(x,y) \quad , \quad \text{for } |x|>|y| \, ,$$

$$G(x,y) \le F(x,y) \le 0 \quad , \quad \text{for } |x|<|y| \, .$$

The solution F_ρ to the m.s.e. in B_ρ with the boundary values F on ∂B_ρ, satisfies

$$|F(x,y)| \le |F_\rho(x,y)| \le |G(x,y)| \quad , \quad \text{for all } (x,y) \, .$$

Therefore, any convergent sequence $\{F_{\rho_j}\}$, with $\rho_j \uparrow +\infty$, provides a classical non trivial solution to the m.s.e. in R^8.

FOOTNOTES

(*)

A tangent cone at $x \in \partial E \cap \Omega$ is a Lebesgue measurable subset C of R^{n+1}, such that

(i) $y \in C, \lambda > 0 \Rightarrow \lambda y \in C$,

(ii) there exists a sequence of positive real numbers ρ_j, converging to zero, such that

$$E_j = \{y \in R^{n+1} \mid x + \rho_j y \in E\} \underset{j}{\to} C ,$$

in the measure theoretic sense, i.e.,

$$\lim_j \text{meas}[(E_j - C) \cup (C - E_j)] \cap K = 0 ,$$

for all compact subsets K of R^{n+1}.

(**)

x is regular for ∂E if there exists a positive number ρ such that

$$\partial E \cap B_\rho(x)$$

is the graph of a real analytic function defined over an open set of R^n.

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