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Viscoelasticity of Granular Materials

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Abstract

In a former work (Granik and Ferrari,1993) a micromechanical theory for elastic granular media was deduced on the basis of the identification of the constituent grains with the nodes of a Bravais lattice. The transition from the discrete structure to the continuum level was achieved through assumptions on the kinematical fields and through a variational formulation establishing the relationship between microstresses and macrostresses. In the present paper we extend to the linear viscoelasticity domain. We formulate the general linear viscoelastic relations in terms of microstresses and microdeformations, deducing the boundary value problems for the microstresses and for macrostresses. Then we focus our attention on the links between constitutive equations governing the microscopic level (formulated following the general methods of linear viscoelasticity) and their macroscopic counterparts. Thus we directly relate microstructural informations to the macroscopic constitutive laws,deducing classical macroscopic properties in terms of structural parameters. Finally the dynamical problem of plane waves propagation in a semi-infinite granular medium is analyzed and the influence that the arrangement of the particles exerts on the pulse propagation is discussed.

1 Introduction

The macroscopic (phenomenological) behavior of a given material derives from its discrete microstructure. To bring microstructural informations into the continuum description, in a consistent a fruitful way, is the objective of a vast research area in mechanics. Major contributes in this field over the last decade are due to Nemat-Nasser and his school (Oda, 1978; Konishi, 1978; Nemat-Nasser and Tobita, 1982; Mehrabadi and Nemat-Nasser, 1983; Balendran and Nemat-Nasser, 1993).

In a former work (Granik and Ferrari, 1993) a general theory of linear elastic granular media was deduced starting from a micromechanical approach. The constituent grains were identified with the nodes of a Bravais lattice and the transition to the continuum level was achieved through approximation conditions relating kinematical fields, and through a variational formulation establishing the relationship between microstresses and macrostresses. In the present paper we extend to the linear viscoelasticity domain. The need of microstructural informations, in formulating viscoelastic constitutive equations, arises for different classes of materials (Poirier,1983; Feda, 1992), whenever the internal structural features (spatial arrangement of particles, voids, clusters, etc.) cannot be ignored even in a first gross approximation. Therefore it is not surprising that predictions based on the classical viscoelasticity theory often significantly differ from experimental results (Feda,1989 refers that a prediction of experimental creep deformation with an accuracy of 30% is not discarded as a serious disagreement between the theory and the experiment). In order to characterize the anisotropic geometric microstructure of the material, the concept of *fabric tensor* (a second order tensor) was introduced (Oda,1972; Cowin, 1978; Nemat-Nasser,1980). Then the dipendence of the stress upon the fabric tensor is made explicit through formal arguments based on tensor functions theory, involving a notion of *transformed* stress tensor (Boehler,1987). By contrast to the *inductive* character inherent to this approach, we claim a purely *deductive* point of view , for in the present approach the macroscopic constitutive equations directly follow from microscopic properties. Indeed thanks to the relationship between microstrsses and macrostresses, we directly relate microstructural informations to the macroscopic constitutive laws.

In section 2 we recall the general relations of the thory of granular materials as well as introduced by Granik and Ferrari,1993, limiting ourself to the case of axial doublet microdeformation, in the first order approximation (non-scale theory). In section 3 linear viscoelastic constitutive equations, of differential and integral type, are formulated at the microstructural level. Then their macroscopic counterparts are derived; as a result the macroscopic description accounts for the microscopic parameters. In particular, internal stability and dissipation depend on the microgeometry of the material. In section 4 the general boundary value problem is formulated and it is shown how the solution can be characterized in terms of *microfields* and *macrofields*. Finally, as an application, we study the one-dimensional problem of plane shear waves in a semi-infinite medium, noting the influence of the microstructure on the propagation phenomenon.

Notation Troughout this paper boldface lowercase letters represent vectors, boldface uppercase letters represent second-order and fourth-order tensors. The inner product of two second-order tensors is $\mathbf{A}\cdot\mathbf{B} \equiv \mathrm{tr}(\mathbf{A}\mathbf{B}^T)$. The tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the second-order tensor $\mathbf{a}\otimes\mathbf{b}$ whose action on any vector $\mathbf{c}$ is $(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$. The tensor product of two second-order tensors is a fourth-order tensor defined as follows (Del Piero, 1979): $(\mathbf{A}\otimes\mathbf{B})\mathbf{C} = (\mathbf{B}\cdot\mathbf{C})\mathbf{A}$. If $\mathbf{u}$ is a unitary vector, the tensor $\mathbf{P}\equiv\mathbf{u}\otimes\mathbf{u}$ defines the projection in the $\mathbf{u}$ direction, it has the property $\mathbf{P}^2 = \mathbf{P}$.

2 General Relations

In what follows a granular body is thought as an assembly of elastic spherical particles arranged in a Bravais lattice Γ (Fig.1). The neighbor of each particle a, denoted $T_m(a)$, is the set of the m vectors joining a with the adjacent particles. Its elements $\boldsymbol{\zeta}_\alpha(a)$ are doublet vectors, with magnitude η_α and direction $\boldsymbol{\tau}_\alpha$ (Fig.2). Because $T_m(a)$ can be split in two disjointed subsets, equivalent via the center of inversion operation at the node $a\in\Gamma$, it is sufficient to restrict the attention to the set $T_n(a)$, with $m = 2n$, n being the valence of the Bravais lattice. The configuration of the granular body Γ can be approximated by a smooth region V of the Euclidean space, so in the the paper of Granik and Ferrari,(1993) the relation:

$$\sum_{\forall a \in \Gamma} \sum_{\alpha=1}^{n} F_\alpha(a) \approx \sum_{\alpha=1}^{n} \int_V F_\alpha(\mathbf{x}) dV, \tag{1}$$

was shown to hold for an arbitrary function $F_\alpha(a)$ defined on $T_n(a)$.

Let $\mathbf{u}(\mathbf{x}, t)$ be a smooth displacement field defined on V whose values are assumed to coincide with the particle displacements when $\mathbf{x} = \mathbf{x}_a$ ($\mathbf{x}$ is the position vector of an arbitrary point in V, and $\mathbf{x}_a$ is the position of the particle a). Moreover, the increment function:

$$\Delta \mathbf{u}_\alpha \equiv \mathbf{u}(\mathbf{x} + \boldsymbol{\zeta}_\alpha, t) - \mathbf{u}(\mathbf{x}, t), \tag{2}$$

is assumed to be expandable in a convergent Taylor series in the neighborhood of $\mathbf{x}_\alpha$. The kinematics of the granular structure is characterized by the microstrain scalar measure ε_α,representing the axial deformation of the doublet vector ,it is given by:

$$\varepsilon_\alpha = \frac{\boldsymbol{\tau}_\alpha \cdot \Delta \mathbf{u}_\alpha}{\eta_\alpha}. \tag{3}$$

Introducing the microstrain vector $\boldsymbol{\varepsilon}_\alpha = \varepsilon_\alpha \boldsymbol{\tau}_\alpha$, it follows, from (3):

$$\boldsymbol{\varepsilon}_\alpha = \frac{1}{\eta_\alpha}(\boldsymbol{\tau}_\alpha \otimes \boldsymbol{\tau}_\alpha) \Delta \mathbf{u}_\alpha. \tag{4}$$

Now, recalling Taylor's representation formula for the increment function:

$$\Delta \mathbf{u}_\alpha = \sum_{\chi=1}^{M} \frac{(\eta_\alpha)^\chi}{\chi!} (\boldsymbol{\tau}_\alpha \cdot \nabla)^\chi \mathbf{u}(\mathbf{x}, t) + O(|\ \eta_\alpha\ |^{M+1}). \tag{5}$$

In the first order approximation,($\chi = 1$; or non-scale)the microstrain vector takes the form:

$$\boldsymbol{\varepsilon}_\alpha = (\boldsymbol{\tau}_\alpha \otimes \boldsymbol{\tau}_\alpha) \mathbf{E} \boldsymbol{\tau}_\alpha \tag{6}$$

or, in components form :

$$\varepsilon_{\alpha i} = \tau_{\alpha i} \tau_{\alpha j} E_{jk} \tau_{\alpha k},$$

where

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$$

is the linear strain tensor. Thus, at the lowest order approximation, the microstrain measure (6) is unaffected by the scale parameter η_α; by contrast, at the second order, the strain vector is

$$\varepsilon_{\alpha i} = \tau_{\alpha i}\tau_{\alpha j}(\frac{\partial u_j}{\partial x_k} + \frac{1}{2}\eta_\alpha \frac{\partial^2 u_j}{\partial x_k \partial x_l}\tau_{\alpha l})\tau_{\alpha k}. \tag{7}$$

If not otherwise specified, we refer to (6) as microstrain measure.

The force system associated with the above kinematic fields is expressed by the microstress $\mathbf{p}_\alpha$ conjugate to $\boldsymbol{\varepsilon}_\alpha$,the volume force $\mathbf{b}$ and the inertial force $\rho\mathbf{a}$ defined in V, (ρ is the bulk density of the medium, $\mathbf{a}$ is the acceleration field), the surface force $\mathbf{t}$ defined on ∂V. The principle of virtual work states:

$$\sum_{\alpha=1}^{n} \int_V \mathbf{p}_\alpha \cdot \delta\boldsymbol{\varepsilon}_\alpha dV = \int_V (\mathbf{b} - \rho\mathbf{a}) \cdot \delta\mathbf{u} dV + \int_{\partial V} \mathbf{t} \cdot \delta\mathbf{u} dS, \tag{8}$$

for all admissible $\delta\mathbf{u}$. By virtue of Gauss-Green theorem, and with localization arguments, we obtain from (8), the first order

Equations of motion in V

$$\sum_{\alpha=1}^{n} \tau_{\alpha i}\tau_{\alpha j}\tau_{\alpha k}\frac{\partial p_{\alpha k}}{\partial x_j} + b_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \tag{9}$$

as well as the first order

Natural boundary conditions on ∂V

$$\sum_{\alpha=1}^{n} \tau_{\alpha i}\tau_{\alpha j}\tau_{\alpha k}p_{\alpha k}n_j = t_i. \tag{10}$$

Here n_j is the component of the outward unit normal to ∂V. Equation (10) establishes the relationship between micro and macrostress, upon interpreting

$$\sum_{\alpha=1}^{n} \tau_{\alpha i}\tau_{\alpha j}\tau_{\alpha k}p_{\alpha k} = T_{ij}, \tag{11}$$

as the Cauchy stress tensor.More compactly, this may be expressed as

$$\mathbf{T} = \sum_{\alpha=1}^{n} (\boldsymbol{\tau}_\alpha \cdot \mathbf{p}_\alpha)(\boldsymbol{\tau}_\alpha \otimes \boldsymbol{\tau}_\alpha).$$

3 Linear Viscoelastic Constitutive Equations

The formulation of (linear) constitutive equations modelling viscoelastic behavior usually follows two different approaches. The first relates the value of the stress at the time t with the complete past history of the strain, originating a constitutive equation in *integral form.* The second one,generalizing elementary rheological models,states a linear relation between the stress derivatives with respect to time, up to a fixed order, and the time strain derivatives, this yields a constitutive law in *differential form.* Though the two formulations are not mathematically independent (Gurtin and Sternberg, 1962), because every constitutive differential law can be put in integral form, but not conversely, the distinction is somewhat useful. Indeed, in deducing viscoelastic proprties of materials, from experimental data, both the integral and differential formulation are fruitfully used. Here we do not deal with a particular constitutive equation, neither our intention is to discuss the appropriateness of various models.

Moreover, in what follows we will formulate general linear viscoelastic equations for the discrete structure Γ, with the perspective to analyze their macroscopic continuum counterparts. Both the integral and the differential approaches will be discussed.

3.1 Constitutive Laws of Integral Type

We assume linear dependence between the generic doublet microstress at the time t and the microstrain histories concerning the doublets belonging to $T_m(a)$,therefore, granted to the *principle of fading memory*,the constitutive relation:

$$\mathbf{p}_\alpha(t) = \sum_{\beta=1}^{n} (A^0_{\alpha\beta}\boldsymbol{\varepsilon}_\beta(t) + \int_0^\infty \dot{A}_{\alpha\beta}(s)\boldsymbol{\varepsilon}_\beta(t-s)ds) \tag{12}$$

performs the behavior of the material at hand. The scalar valued mappings $A_{\alpha\beta}(s)$ are the relaxation functions with $A^0_{\alpha\beta} = A_{\alpha\beta}(0)$ representing the *instantaneous elasticities.* Moreover, the limit of $A_{\alpha\beta}(s)$, as $s \to \infty$, is assumed to exist and $A^{\infty}_{\alpha\beta} = \lim_{s\to\infty} A_{\alpha\beta}(s)$ is termed, for α and β fixed, *equilibrium elastic modulus.*

Classical thermodynamic restrictions [1] require that

$$A^{0}_{\alpha\beta} = A^{0}_{\beta\alpha}, \tag{13}$$

$$A^{\infty}_{\alpha\beta} = A^{\infty}_{\beta\alpha}, \tag{14}$$

$$\dot{A}_{\alpha\beta}(0) < 0. \tag{15}$$

The next goal is to deduce the macroscopic response function associated with (12). Defining

$$\mathbf{P}_{\gamma} \equiv (\boldsymbol{\tau}_{\gamma} \otimes \boldsymbol{\tau}_{\gamma}), \qquad \text{for } \gamma{=}\alpha,\beta \tag{16}$$

from the equation (12),in view of (11) and (6), with some algebra follows:

$$\mathbf{T}(t) = \mathbf{G}^{0}\mathbf{E}(t) + \int_{0}^{\infty} \dot{\mathbf{G}}(s)\mathbf{E}(t-s)ds, \tag{17}$$

with

$$\mathbf{G}(s) \equiv \sum_{\alpha,\beta=1}^{n} (\boldsymbol{\tau}_{\alpha} \cdot \boldsymbol{\tau}_{\beta}) A_{\alpha\beta}(s)(\mathbf{P}_{\alpha} \otimes \mathbf{P}_{\beta}). \tag{18}$$

The fourth order tensor valued mapping $\mathbf{G}(s)$ ($\mathbf{G}^{0} = \mathbf{G}(0)$) plays the role of the *relaxation function* of the classical linear viscoelasticity,its components given by:

$$G_{ijkl}(s) = \sum_{\alpha,\beta=1}^{n} \tau_{\alpha t}\tau_{\beta t} A_{\alpha\beta}(s)\tau_{\alpha i}\tau_{\alpha j}\tau_{\beta k}\tau_{\beta l}.$$

It may be elementary shown that $\mathbf{G}(s)$ meets the usual symmetry properties, i.e.

$$G_{ijkl}(s) = G_{jikl}(s) = G_{ijlk}(s), \tag{19}$$

$$G_{ijkl}(s) = G_{klij}(s), \tag{20}$$

[1]In the very general context of thermodynamics, the second law is imposed on constitutive functions,to obtain a priori characterizations. For a modern treatement of these aspects of the linear viscoelasticity see Fabrizio and Morro,1992

for all positive s. Finally the restriction of positive definiteness on $\mathbf{G}(s)$ produces the condition (Ferrari and Maddalena,1994):

$$\sum_{\alpha,\beta=1}^{n} (\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta) > 0. \tag{21}$$

This relates *internal stability* of the material with the *geometrical arrangement* of the particles.

In view of a discussion of the dissipation characteristics, of the microstructured medium, consider, without loss of generality, oscillating microdeformation histories, starting at time zero:

$$\boldsymbol{\varepsilon}_\alpha(t) = \boldsymbol{\varepsilon}_\alpha^0 \sin(\omega t). \tag{22}$$

Here $\omega \neq 0$ is the frequency and $T = \frac{2\pi}{\omega}$ the period of oscillation. The corresponding microstress is computed via (12):

$$\mathbf{p}_\alpha(t) = \sin(\omega t) \sum_{\beta=1}^{n} (A_{\alpha\beta}^0 + A_{\alpha\beta}^c(\omega)\boldsymbol{\varepsilon}_\beta^0) - \cos(\omega t) \sum_{\beta=1}^{n} A_{\alpha\beta}^s(\omega)\boldsymbol{\varepsilon}_\beta^0, \tag{23}$$

with

$$A_{\alpha\beta}^c(\omega) = \int_0^\infty \dot{A}_{\alpha\beta}(s)\cos(\omega s)ds, \tag{24}$$

$$A_{\alpha\beta}^s(\omega) = \int_0^\infty \dot{A}_{\alpha\beta}(s)\sin(\omega s)ds. \tag{25}$$

From (22) the *phase lag* of the microstress becomes evident, $A_{\alpha\beta}^s(\omega)$ being the constitutive feature governing the out-of-phase term,which in turn is responsible of the energy dissipation. Cauchy stress generated from (22) assumes the form:

$$\mathbf{T}(t) = \sin(\omega t)(\mathbf{G}^0 + \dot{\mathbf{G}}_c(\omega))\mathbf{E}^0 - \cos(\omega t)(\dot{\mathbf{G}}_s(\omega))\mathbf{E}^0, \tag{26}$$

with

$$\dot{\mathbf{G}}_c(\omega) \equiv \sum_{\alpha,\beta=1}^{n} \int_0^\infty (\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta)\dot{A}_{\alpha\beta}(s)(\mathbf{P}_\alpha \otimes \mathbf{P}_\beta)\cos(\omega s)ds, \tag{27}$$

$$\dot{\mathbf{G}}_s(\omega) \equiv \sum_{\alpha,\beta=1}^{n} \int_0^\infty (\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta)\dot{A}_{\alpha\beta}(s)(\mathbf{P}_\alpha \otimes \mathbf{P}_\beta)\sin(\omega s)ds. \tag{28}$$

$\dot{\mathbf{G}}_s(\omega)$ is termed *loss modulus*, since the energy dissipated in one period $[0, T]$, is

$$\int_0^T \mathbf{T}(\mathbf{t}) \cdot \dot{\mathbf{E}}(t)dt = -\pi \dot{\mathbf{G}}_s(\omega)\mathbf{E}^0 \cdot \mathbf{E}^0. \tag{29}$$

Specializing, for the sake of simplicity, to materials with $\dot{A}_{\alpha\beta}(s) = \dot{C}(s)\delta_{\alpha\beta}$, we obtain:

$$\dot{G}_{sijkl}(\omega) = \sum_{\alpha=1}^{n} \tau_{\alpha i}\tau_{\alpha j}\tau_{\alpha k}\tau_{\alpha l} \int_0^\infty \dot{C}(s)\sin(\omega s)ds. \tag{30}$$

Owing to the presence of the unit doublet vectors $\boldsymbol{\tau}_\alpha$ in this expression, it is concluded that the capacity of a granular material to dissipate energy depends on the spatial organization of its particles, and not only on their physical properties.

3.2 Constitutive Laws of Differential Type

Micro constitutive equations of differential type can be formulate. In the very general form:

$$M(\mathbf{p}_\beta) = N(\boldsymbol{\varepsilon}_\beta), \tag{31}$$

where M, N are differential operators so defined:

$$M = \sum_{r=0}^{k} M_{\alpha\beta}^r \frac{\partial^r}{\partial t^r}, \tag{32}$$

$$N = \sum_{r=0}^{k} N_{\alpha\beta}^r \frac{\partial^r}{\partial t^r}, \tag{33}$$

and $M_{\alpha\beta}^r, N_{\alpha\beta}^r$ are material constants. Equation (31), together with the appropriate initial conditions, describe the time dependent behavior of the doublets interactions. Therefore, the unit cell can be thought as formed by n doublets which constitutive behavior we can perform through elementary rheological models. The simplest in this class can be built imagining the connections, between each pair of doublet particles, consist of springs and dashpots,this corresponds to the case of purely axial interactions,(Fgg.3,4).Thus we will concern with *Micro Kelvin-Voigt Materials* when the generic doublet pair is jointed by a spring and a dashpot put in parallel. Otherwise we will deal with *Micro Maxwell Materials* for the case in which the spring and dashpot are put in series. Various combinations of these can be (not arbitrary) thought to represent more complex time dependent behavior. [2]

Micro Kelvin-Voigt Materials

Choosing the coefficients of (30) in the following way:

$$M^0_{\alpha\beta} = \delta_{\alpha\beta}, \qquad M^r_{\alpha\beta} = 0, \quad \text{for } r > 0, \tag{34}$$

$$N^r_{\alpha\beta} = 0 \qquad \text{for } r > 1, \tag{35}$$

the constitutive equation follows:

$$\mathbf{p}_\alpha(t) = N^0_{\alpha\beta}\boldsymbol{\varepsilon}_\beta(t) + N^1_{\alpha\beta}\dot{\boldsymbol{\varepsilon}}_\beta(t). \tag{36}$$

Equation (35) expresses the microstress in the α-doublet as a linear function of the microstrains and the rate of microstrains occuring in the n doublets of the unit cell. It corresponds to assume that each interaction is of *Kelvin-Voigt* type.

From (35), for the macroscopic level:

$$\mathbf{T}(t) = \mathbf{C}(\mathbf{E}(t)) + \mathbf{H}(\dot{\mathbf{E}}(t)), \tag{37}$$

with

$$\mathbf{C} = \sum_{\alpha,\beta=1}^{n} (\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta) N^0_{\alpha\beta}(\mathbf{P}_\alpha \otimes \mathbf{P}_\beta), \tag{38}$$

$$\mathbf{H} = \sum_{\alpha,\beta=1}^{n} (\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta) N^1_{\alpha\beta}(\mathbf{P}_\alpha \otimes \mathbf{P}_\beta), \tag{39}$$

representing,respectively, the elasticity and viscosity tensor. In components form:

$$C_{ijkl} = \sum_{\alpha,\beta=1}^{n} \tau_{\alpha t}\tau_{\beta t} N^0_{\alpha\beta}\tau_{\alpha i}\tau_{\alpha j}\tau_{\beta k}\tau_{\beta l}, \qquad H_{ijkl} = \sum_{\alpha,\beta=1}^{n} \tau_{\alpha t}\tau_{\beta t} N^1_{\alpha\beta}\tau_{\alpha i}\tau_{\alpha j}\tau_{\beta k}\tau_{\beta l}.$$

[2] It is well known that the elementary models of Kelvin-Voigt and Maxwell cannot represent viscoelastic behavior adequately (Tschoegl ,1989). Nevertheless, for the present aim, they well illustrate the problem of transition micro-macro.

Thus we note that the assumptions made at the microstructure level, give rise a macroscopic equation which model a Kelvin-Voigt like material. Then, thanks to the relation microstrss-macrostress it is possible to formulate constitutive equations in one dimension to obtain three dimensional constitutive laws.

Micro Maxwell Materials

In this case we specialize the coefficients of (30) to be:

$$M^r_{\alpha\beta} = 0 \qquad \text{for } r > 1, \tag{40}$$

$$N^1_{\alpha\beta} = \delta_{\alpha\beta}, \qquad N^r_{\alpha\beta} = 0, \quad \text{for } r \neq 1. \tag{41}$$

Thus, the governing equation for the microlevel is:

$$\dot{\boldsymbol{\varepsilon}}_\alpha(t) = M^0_{\alpha\beta}\mathbf{p}_\beta(t) + M^1_{\alpha\beta}\dot{\mathbf{p}}_\beta(t) \tag{42}$$

Unlike Kelvin-Voigt materials, it is not trivial to deduce the macroscopic counterpart of(41); indeed, in order to compute the macrostresses, we have to solve (41) with respect to $\mathbf{p}_\alpha$ and in doing this we are forced to reduce to purely axial interactions. Assume:

$$M^0_{\alpha\beta} = \mu^{-1}\delta_{\alpha\beta}, \tag{43}$$

$$M^1_{\alpha\beta} = k^{-1}\delta_{\alpha\beta}, \tag{44}$$

where μ is the viscosity and k is the elastic modulus, we obtain upon integration of (41)

$$\mathbf{p}_\alpha(t) = c(0)\boldsymbol{\varepsilon}_\alpha(t) + \int_0^\infty \dot{c}(s)\boldsymbol{\varepsilon}_\alpha(t-s)ds, \tag{45}$$

with

$$c(s) = k\exp\left(-\frac{ks}{\mu}\right). \tag{46}$$

Finally, for the macroscopic level, the equation

$$\mathbf{T}(t) = \mathbf{C}[c(0)\mathbf{E}(t) + \int_0^\infty \dot{c}(s)\mathbf{E}(t-s)ds], \tag{47}$$

with

$$\mathbf{C} = \mathbf{P}_\alpha \otimes \mathbf{P}_\alpha, \tag{48}$$

follows. We notice that the conditions under which it has been possible to deduce the macro constitutive equation of Maxwell like material, from the analogous formulated at the micro level,are quite strong. They regard either both the specific interaction between doublets,and the geometric structure of the unit cell. The last one is a *recurring theme* in our discussion which relies on understanding how microscopic features influence the macroscopic behavior of materials.

4 Boundary Value Problems

In this section the general problem of the equilibrium of viscoelastic granular bodies will be formulated. First, the equations governing the micro problem are stated, and their relationship with the corresponding continuum-mechanical problem is addressed. In section 4.2 the problem of plane shear waves in a semi-infinite medium is solved.

4.1 General Formulation

Let be given a granular body immersed in a regular region V of the Euclidean space(Fig.5). The boundary ∂V is decomposed in two disjoint surface elements $\partial_1 V$ and $\partial_2 V$.On V the body field force $\mathbf{b}(\mathbf{x}, t)$ is assigned.

Let us define a *microscopic admissible viscoelastic process* m_{VP}, to be the set of $3n$ functions:

$$m_{VP} \equiv [\mathbf{u}_\alpha(\mathbf{x},t); \boldsymbol{\varepsilon}_\alpha(\mathbf{x}, t); \mathbf{p}_\alpha(\mathbf{x},t)], \tag{49}$$

such that,

$$\varepsilon_{\alpha i} = \frac{1}{2}\tau_{\alpha i}\tau_{\alpha j}\tau_{\alpha k}(\frac{\partial u_{\alpha j}}{\partial x_k} + \frac{\partial u_{\alpha k}}{\partial x_j})_{x=x_\alpha}, \tag{50}$$

$$\sum_{\alpha=1}^{n} \tau_{\alpha i}\tau_{\alpha j}\tau_{\alpha k}\frac{\partial p_{\alpha k}}{\partial x_j} + b_i = \rho\frac{\partial^2 u_{\alpha i}}{\partial t^2} \tag{51}$$

$$p_{\alpha i} = \mathcal{F}_t(\varepsilon_{\beta j}), \tag{52}$$

where $\mathcal{F}_t$ is a viscoelastic constitutive functional.

Furthermore, let us define a *macroscopic admissible viscoelastic process* M_{VP}, to be the triplet of functions:

$$M_{VP} \equiv [\mathbf{u}(\mathbf{x},t); \mathbf{E}(\mathbf{x},t); \mathbf{T}(\mathbf{x},t)], \tag{53}$$

We say that M_{VP} is *compatible* with m_{VP} if the following are satisfied:

$$\mathbf{u} = \mathbf{u}_\alpha \qquad \text{at } \mathbf{x} = \mathbf{x}_\alpha, \tag{54}$$

$$\mathbf{P}_\alpha \mathbf{E} \boldsymbol{\tau}_\alpha = \boldsymbol{\varepsilon}_\alpha, \tag{55}$$

$$\mathbf{T} = \sum_{\alpha=1}^{n} (\boldsymbol{\tau}_\alpha \cdot \mathbf{p}_\alpha)\mathbf{P}_\alpha. \tag{56}$$

Let be given the boundary data $(\mathbf{u}^o, \mathbf{t}^o)$, then m_{VP} is a solution of the corresponding dynamic viscoelastic boundary value problem if and only if it satisfies the boundary conditions:

$$u_{\alpha i} = u_i^o \qquad \text{on } \partial_1 V, \tag{57}$$

$$\sum_{\alpha=1}^{n} \tau_{\alpha i} \tau_{\alpha j} \tau_{\alpha k} p_{\alpha k} n_j = t_i^o \qquad \text{on } \partial_2 V. \tag{58}$$

Finally, if M_{VP} is compatible with the solution m_{VP}, then it is a solution of the differential system:

$$\frac{\partial T_{ij}}{\partial x_i} + b_j = \rho \frac{\partial^2 u_j}{\partial t^2}, \tag{59}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{60}$$

$$T_{ij} n_j = t_i^o. \tag{61}$$

Thus we have obtained a characterization of the solution of the dynamic boundary value problem, for the body V, in terms of micro-fields and macro-fields.

4.2 Plane Shear Waves

We suppose that a semi-infinite viscoelastic granular body occupies the region $y > 0$. The geometry of the microstructure is assumed to be a face centered cubic lattice (Fig.6) which valence is $n = 3$. The unit doublet vectors are:

$$\boldsymbol{\tau}_1 = (-\cos\gamma, -\sin\gamma), \quad \boldsymbol{\tau}_2 = (\cos\gamma, -\sin\gamma), \quad \boldsymbol{\tau}_3 = (1, 0). \tag{62}$$

We are interested in studing the propagation into the body of a disturbance caused by a pulse velocity in the $\mathbf{e}_1$-direction at the boundary $y = 0$. The governing equations of the problem ,in the absence of body forces,are:

$$\sum_{\alpha=1}^{3} \tau_{\alpha 2}(\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\alpha)\frac{\partial \mathbf{p}_\alpha(y,t)}{\partial y} = \frac{\partial^2 \mathbf{u}(y,t)}{\partial t^2}, \tag{63}$$

$$\mathbf{p}_\alpha(t) = \sum_{\beta=1}^{3}(A^0_{\alpha\beta}\boldsymbol{\varepsilon}_\beta(t) + \int_0^\infty \dot{A}_{\alpha\beta}(s)\boldsymbol{\varepsilon}_\beta(t-s)ds), \tag{64}$$

$$\dot{\mathbf{u}}(0,t) = v_0 h[t]\mathbf{e}_1, \qquad \lim_{y\to\infty} \mathbf{u}(y,t) = 0, \quad t \geq 0, \tag{65}$$

where $h[t]$ is the unit step function.

Introducing the Laplace transform of a generic function:

$$\overline{f}(s) = \int_0^\infty f(t)\exp(-st)dt, \tag{66}$$

the above problem reduces, in the space of Laplace transforms, to the ordinary differential equation:

$$\sin^2(2\gamma)\overline{\phi}(s)\frac{\partial^2 \overline{u}_1(y,s)}{\partial y^2} - 2\rho s \overline{u}_1(y,s) = 0 \tag{67}$$

and the initial condition:

$$s\overline{u}_1(0,s) = s^{-1}v_0, \tag{68}$$

where

$$\overline{\phi}(s) \equiv (\overline{A}_{11}(s) + \overline{A}_{22}(s) - 2\overline{A}_{12}(s)(1 - 2\cos^2\gamma)). \tag{69}$$

The solution of (67),(68),in view of the second of (65), is:

$$\overline{u}_1(y,s) = s^{-1}v_0 \exp\left[\frac{-y}{\sin(2\gamma)}\Omega(s)\right], \tag{70}$$

and the related velocity field is:

$$\overline{v}_1(y,s) = v_0 \exp\,[\frac{-y}{\sin(2\gamma)}\Omega(s)], \tag{71}$$

with

$$\Omega(s) = (\frac{2\rho s}{\overline{\phi}(s)})^{\frac{1}{2}}. \tag{72}$$

By specializing to the case of *micro Maxwell material*, 64 takes the form:

$$\mathrm{p}_\alpha(t) = \int_0^t \dot{c}(s)\varepsilon_\alpha(t-s)ds, \tag{73}$$

where

$$c(t) = k \exp\,(-\frac{k}{\mu}t), \tag{74}$$

k and μ being respectively the elastic and viscosity modulus. Besides, from (69) and (72), the relations

$$\overline{\phi}(s) = 2\overline{c}(s), \tag{75}$$

$$\Omega(s) = s[\frac{\rho}{k}(1+\frac{k}{\mu s})]^{\frac{1}{2}} \tag{76}$$

follow. In view of the transform inversion, we look for an asymptotic solution applying the binomial expansion to (76)[3]:

$$\Omega(s) = 1 + \frac{1}{2}\frac{k}{\mu s} + \cdots \tag{77}$$

Finally, by sostituting (77) in (70), the solution has the form

$$u_1 = v_0 \exp\,[-(\frac{\eta}{2\lambda})y]h[t-\frac{y}{\lambda}], \tag{78}$$

with

$$\lambda \equiv \sin(2\gamma)(\frac{k}{\rho})^{\frac{1}{2}}, \qquad \eta \equiv \frac{k}{\mu}. \tag{79}$$

Then we conclude, from (78) that the rate of propagation of the disturbance cannot be greater than $\sin(2\gamma)(\frac{k}{\rho})^{\frac{1}{2}}$. Furthermore (78), at $t =$

[3] By neglecting the terms involving positive powers of $\frac{1}{s}$ we represent the behavior near $t = 0$.

$(\frac{\rho}{k})^{\frac{1}{2}}\frac{y}{\sin(2\gamma)}$ gives the amplitude of the propagating discontinuity and its decay is governed by $-(\frac{\rho}{k})^{\frac{1}{2}}\frac{k}{2\mu\sin(2\gamma)}$. As a matter of fact, we notice that the main features of the phenomenon are affected, at the same time, by the moduli k, μ and by the structural parameter γ.

Thus, the microstresses (Fig.7), are given by

$$\mathbf{p}_1 = (\frac{v_0\rho\lambda\eta}{2\sin 2\gamma}(\exp[-\eta(t-\frac{y}{\lambda})] + \exp[-\eta(\frac{y}{2\lambda}])h[t-\frac{y}{\lambda}])\boldsymbol{\tau}_1, \qquad (80)$$

$$\mathbf{p}_2 = -(\frac{v_0\rho\lambda\eta}{2\sin 2\gamma}(\exp[-\eta(t-\frac{y}{\lambda})] + \exp[-\eta(\frac{y}{2\lambda}])h[t-\frac{y}{\lambda}])\boldsymbol{\tau}_2, \qquad (81)$$

$$\mathbf{p}_3 = 0. \qquad (82)$$

The non zero component of Cauchy stress is

$$T_{12} = 2(\frac{v_0\rho\lambda\eta}{2\sin 2\gamma}(\exp[-\eta(t-\frac{y}{\lambda})] + \exp[-\eta(\frac{y}{2\lambda}])h[t-\frac{y}{\lambda}]). \qquad (83)$$

5 Conclusion

A micromechanical approach, based on the concept of doublet of particles, was employed in this article,to investigate general properties of linear viscoelastic materials. Both integral and differential constitutive laws were formulated for the microstructure and their macroscopic counterparts were derived on the basis of a relation connecting microstresses and macrostresses. As a result the macroconstitutive equations are characterized by the presence of projection tensors representing the spatial arrangement of the unit doublet vectors. Remarkable consequences are that dissipation as well as stability depend on the geometrical organization of the particles and not only on their physical properties. The general boundary value problem was formulated enlightening the corrispondence between microfields and macrofields and the problem of shear waves propagation in a semispace was addressed. It was shown how microstress waves contribute to originate the macroscopic shear wave.

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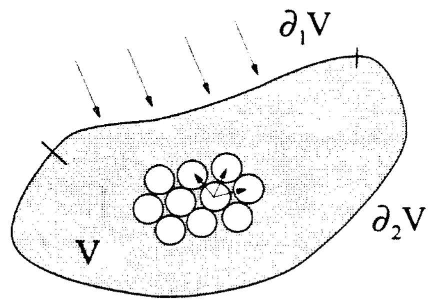

Fig. 5

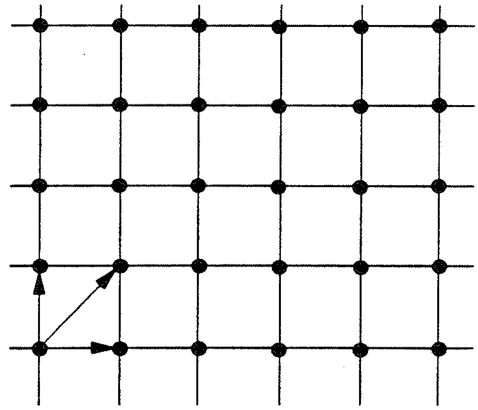

Fig. 1

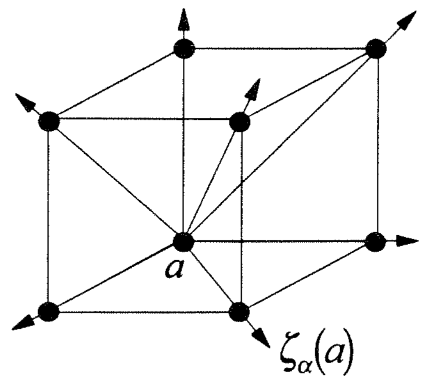

Fig. 2

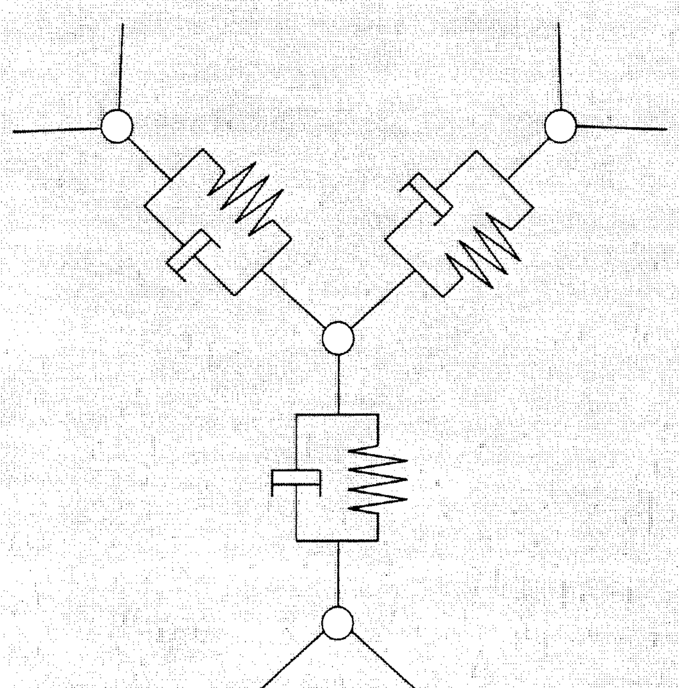

Fig. 3

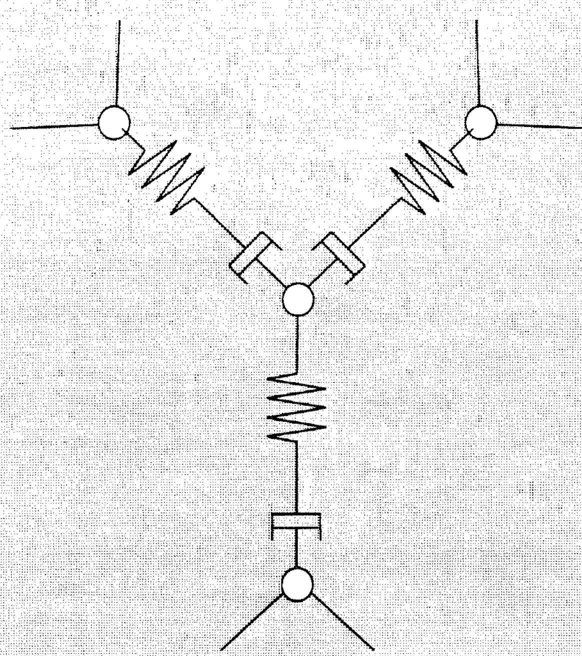

Fig. 4

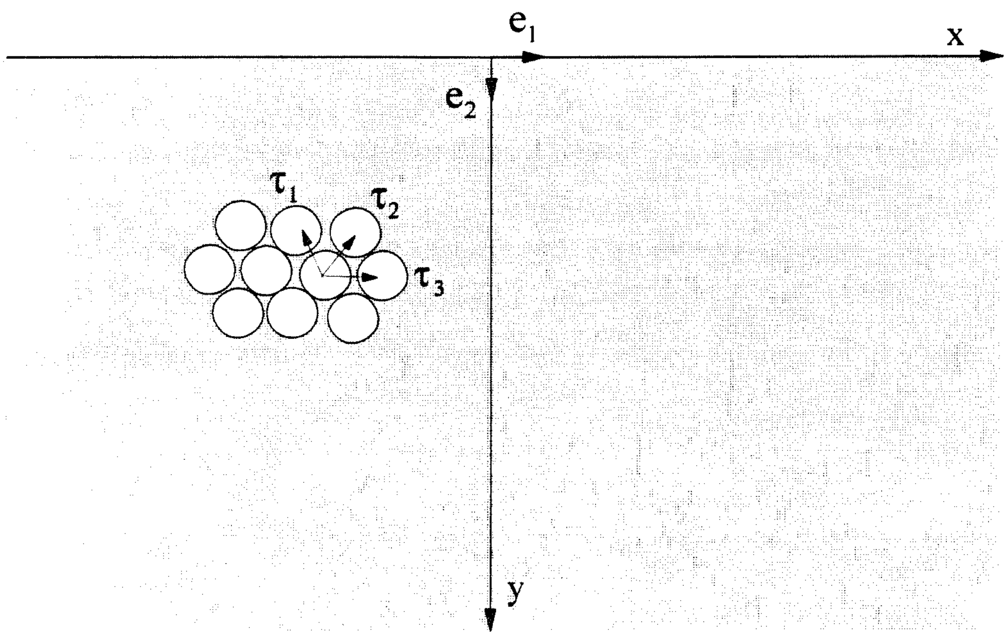

Fig. 6

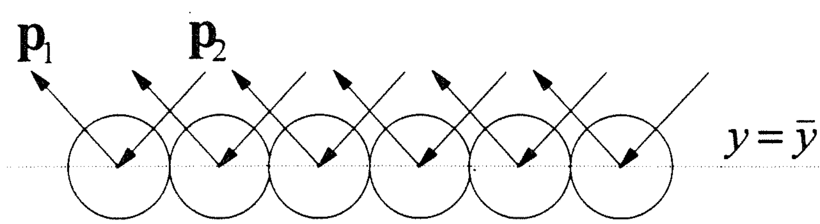

Fig. 7