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FERMION DETERMINANTS, CHIRAL SYMMETRY, AND THE WESS-ZUMINO ANOMALY[1]

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Abstract

A general method for constructing exactly solvable fermion determinants is discussed. A two dimensional determinant is solved exactly. A new class of four dimensional fermion models is presented. These theories are non-renormalizable yet the fermion determinant can be calculated and there is an analogue of the Adler-Bell-Jackiw anomaly.

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I. Introduction

In this paper we attempt to gain a better understanding of the fermion determinant in gauge theories by replacing the Dirac equation in the gauge theory with a suitable modification. The criteria for the replacement is that the fermion determinant should be exactly solvable, and that the modified Dirac operator sould retain the chiral symmetries of the original theory.

As a warm up we show that in two dimensions one can define a non-abelian generalization of the Schwinger model which has an exactly solvable fermion determinant. The most interesting feature is that the determinant consists of two terms. The first is the non-abelian extension of Schwinger's mass term. The second term is the two dimensional version of the Wess-Zumino anomaly term [1]. Balachandran, Nair and Trahern [2], Novikov [3] and Witten [4] have emphasized that the coefficient of such a term must be quantized due to global topological configurations.

Later we discuss the four dimensional analog of the two dimensional non-abelian Schwinger model. We show that the determinant is in principle exactly solvable by writing down an ordinary first order differential equation which the determinant must satisfy. The fly in the ointment is that one has to evaluate the heat kernel $\langle x|exp(-\epsilon D_t^2)|x\rangle$ for a certain operator D_t^2 at very small ϵ. General methods [5] imply that the calculation can be performed in a finite number of steps, but the algebra seems to be intractable.

After this work was completed we have seen that Polyakov and Wiegman [6] have found the same expression for the two dimensional fermion determinant as the one found by the present author. The S-matrix and β-function questions answered by Polyakov and Wiegman were not considered by the present author. Previous use of the methods of Wess and Zumino [1] in trying to understand two dimensional models may be found in the work of D'Adda, Davis, and DiVecchia [7].

The method of solution is to exploit the anomalies of the theory. The same approach is used in [6], [7] and [8] except that the spirit of the method in this paper is different. The methods in this paper can be used to study the behavior of the four dimensional fermionic determinant in a gauge theory as a function of the chiral phases [9].

This paper is organized as follows. In Sec. II, we discuss a theorem about fermion determinants [10] which is the main tool in the analysis. In Sec. III we discuss the two dimensional model, and in Sec. IV we discuss the four dimensional model.

II. A Theorem About Functional Determinants

In certain special situations one can show that the determinant of an operator is determined by the short distance properties of the theory [10]. We discuss a special case of this theorem. For simplicity we will neglect the existence of zero modes. It is easily shown that the zero modes lead to a determinental interaction of the 't Hooft type [11]. A more complicated version of the theorem and the inclusion of the zero modes is discussed in Sec. III of [12].

We will be studying a family of self-adjoint Dirac like operators parametrized by a parameter t. The D_t will anticommute with γ_5 and therefore for every positive eigenvalue there is a corresponding negative one. We define the determinant of D_t to be

$$det\, D_t = [det\, D_t^2]^{1/2}. \tag{2.1}$$

The latter is regulated by the proper time method:

$$ln\ det\ D_t^2 = Tr\ ln\ D_t^2$$

$$= -\int_\epsilon^\infty \frac{ds}{s}\, Tr[exp(-sD_t^2)], \tag{2.2}$$

where $\epsilon = \Lambda^{-2}$ is an ultraviolet cutoff on the proper time integration.

The operators we will consider have the property that

$$\dot{D}_t = fD_t + D_t f \tag{2.3}$$

where the dot denotes differentiation with respect to t, and f is a function independent of t. By using the cyclicness of the trace one can easily show that (2.2) satisfies the following differential equation:

$$\frac{d}{dt} Tr\, ln\, D_t^2 = 4\int_\epsilon^\infty ds\, Tr[f\ exp(-s\, D_t^2)]$$

$$= -4\int_\epsilon^\infty ds\, \frac{d}{ds}\, Tr[f\ exp(-s\, D_t^2)]$$

$$= 4\, Tr[f\ exp(-\epsilon\, D_t^2)] \tag{2.4}$$

Since f is a function we only need the diagonal part of the heat kernel for D_t^2. Seeley has shown [5] that there is an asymptotic small ϵ expansion given by

$$\langle x|exp(-\epsilon D_t^2)|x\rangle \rightarrow \frac{1}{(4\pi\epsilon)^{d/2}}[\,a_0^t(x) + \epsilon a_1^t(x) + \epsilon^2 a_2^t(x) + \cdots] \qquad (2.5)$$

where d is the dimensionality of space time.

The insertion of asymptotic expansion (2.5) into (2.4) leads to a differential equation for the determinant.

III. The Non-abelian Schwinger Model

In this section we show how one can modify two dimensional QCD in such a way that the fermion determinant is exactly solvable and given by an elementary expression. Nielsen, Rothe, and Schroer [8] have developed a recursive scheme for calculating fermion determinants. Our modification of the theory circumvents their scheme. The approach we will employ uses a well known way of solving the Schwinger model [13].

We work in Euclidean space with γ-matrices given by $\gamma_0 = \sigma_x$, $\gamma_1 = \sigma_y$, $\gamma_5 = \sigma_z$. Of particular importance is the identity

$$\epsilon_{\mu\nu}\gamma_\nu = i\gamma_\mu\gamma_5 \tag{3.1}$$

It is also important to remember that the generator of the rotation group is $\sigma_{01} = \gamma_5$. The Dirac operator is hermitian and it is given by

$$i\gamma_\mu(\partial_\mu + iC^a_\mu t_a), \tag{3.2}$$

where C^a_μ are the color gauge fields and $\{it_a\}$ are anti-hermitian generators of the color gauge group $SU(N_c)$. We work in the Lorentz gauge $\partial_\mu C^a_\mu = 0$. There exists functions ξ^a such that

$$C^a_\mu = \epsilon_{\mu\nu}\partial_\nu\xi^a. \tag{3.3}$$

Inserting (3.3) in (3.2) and using (3.1) one finds that the Dirac operator may be written as

$$i\gamma_\mu[\partial_\mu + \gamma_5 t_a(\partial_\mu\xi^a)]. \tag{3.4}$$

We will now perform our first modification on (3.4). Consider the differential operator D defined by

$$D = i\gamma_\mu e^{-\xi\cdot t\gamma_5}\partial_\mu e^{\xi\cdot t\gamma_5} \tag{3.5}$$

where $\xi \cdot t = \xi^a t_a$. This operator agrees with (3.4) to first order in ξ. In an abelian theory (3.5) and (3.4) would coincide. Using the properties of γ-matrices one can rewrite (3.5) in the form

$$D = e^{\xi\cdot t\gamma_5} i\gamma_\mu\partial_\mu e^{\xi\cdot t\gamma_5}\ . \tag{3.6}$$

Operator D is of the type that we can apply the theorem of Sec. II.

Before proceeding it is important to emphasize that (3.5) defines a non-linear sigma model. This is easily seen by using the Callan, Coleman, Wess, and Zumino formalism [14]. Consider the noncompact group G defined by the generators

$$T_j = it_j \tag{3.7a}$$

$$X_a = t_a \gamma_5 \,. \tag{3.7b}$$

The compact subgroup H generated by the T's is $SU(N_c)$. The symmetric space G/H is a noncompact version of $SU(N_c) \otimes SU(N_c)/SU(N_c)$. Equation (3.6) describes the coupling of fermions to a nonlinear sigma model with values in G/H.

The noncompact nature of G/H is related to the fact that we are working in Euclidean space where the rotation group is compact. Under a rotation $(t + ix) \to e^{i\alpha}(t + ix)$ one has $\psi \to e^{i\alpha\gamma_5/2}\psi$. The exponential factor in (3.6) may be viewed as a local chiral transformation. The latter may be defined to be noncompact in Euclidean space. The opposite is true in Minkowski space. Under the analytic continuation to Minkowski space, $\xi \to -i\xi$ since γ_5 is the generator of the rotation group. One can verify that the analytic continuation of (3.3) requires $\xi \to -i\xi$.

Let $g_t(x)$ be defined by

$$g_t(x) = exp[t\, \xi^a(x) X_a] \,. \tag{3.8}$$

The parameter t is the one introduced in Sec. II. Define a vector V^t_μ and an axial vector A^t_μ by

$$g_t^{-1} \partial_\mu g_t = V^t_\mu + A^t_\mu \tag{3.9}$$

where $V^t_\mu = V^t_{\mu i} T_i$ and $A^t_\mu = A^t_{\mu a} X_a$. Under a change of g by a group element of G one has that V_μ transforms as a gauge field with gauge group H, and A_μ tranforms under the adjoint representation of H. This means that derivatives should enter in the gauge covariant way

$$D^t_\mu = \partial_\mu + V^t_\mu \,. \tag{3.10}$$

For completeness we note that the covariant derivative of A^t_μ and the field strength V^t_μ are given by

$$D^t_\mu A^t_\nu = \partial_\mu A^t_\nu + [V^t_\mu, A^t_\nu] \,, \tag{3.11}$$

$$V^t_{\mu\nu} = \partial_\mu V^t_\nu - \partial_\nu V^t_\mu + [V^t_\mu, V^t_\nu] \,. \tag{3.12}$$

The integrability condition of (3.9) requires

$$D^t_\mu A^t_\nu - D^t_\nu A^t_\mu = 0\,, \tag{3.13a}$$

$$V^t_{\mu\nu} + [A^t_\mu, A^t_\nu] = 0\,. \tag{3.13b}$$

The Lagrangian corresponding to (3.6) may be written as

$$L = \psi^\dagger i\gamma_\mu D^t_\mu \psi + 1 \cdot \psi^\dagger i\gamma_\mu A^t_\mu \psi\,. \tag{3.14}$$

The model under study is a system of fermions coupled to a nonlinear sigma model with an axial vector coupling of unit strength. The differential operator of interest is

$$D_t = g_t i\gamma_\mu \partial_\mu g_t\,. \tag{3.15}$$

This operator is of the type discussed in Sec. II. A simple differentiation yields the result

$$\dot{D}_t = \xi \cdot X\, D_t + D_t\, \xi \cdot X\,. \tag{3.16}$$

Also note that D_t anti-commutes with γ_5. Differential equation (2.4) becomes

$$\frac{d}{dt} Tr\, ln\, D_t^2 \;=\; 4\, Tr\, [\xi \cdot X\, exp(-\epsilon D_t^2)]\,. \tag{3.17}$$

The operator D_t^2 may be written in the form

$$D_t^2 = -(\partial_\mu + G^t_\mu)^2 + E^t \tag{3.18}$$

where

$$G^t_\mu = V^t_\mu + i\epsilon_{\mu\nu}\gamma_5 A^t_\nu \tag{3.19}$$

$$E^t = -D^t_\mu A^t_\mu + i\gamma_5 \epsilon_{\mu\nu} [A^t_\mu, A^t_\nu] \tag{3.20}$$

In deriving the above we have used integrability conditions (3.13). It is important to note that E^t may be written as

$$E^t = -\frac{i}{2}\gamma_5 \epsilon_{\mu\nu} G^t_{\mu\nu} \tag{3.21}$$

where

$$G^t_{\mu\nu} = \partial_\mu G^t_\nu - \partial_\nu G^t_\mu + [G^t_\mu, G^t_\nu]\,. \tag{3.22}$$

The short time expansion of the diagonal element of the heat kernel for (3.18) is tabulated [15,16].

$$\langle x|exp(-\epsilon D_t^2)|x\rangle = \frac{1}{4\pi\epsilon}[1-\epsilon E^t + O(\epsilon^2)]\,. \tag{3.23}$$

Substituting (3.23) into (3.17) we find that

$$\frac{d}{dt}\, Tr\, ln\, D_t^2 = \frac{1}{\pi}\, Tr\, [\xi\cdot X\, E^t]\,. \tag{3.24}$$

It is simple to integrate the above equation with the result:

$$Tr\, ln\, D_t^2 = constant - \frac{1}{2\pi}\int d^2x\, Tr(A_\mu^t A_\mu^t) - \frac{i}{\pi}\epsilon_{\mu\nu}\int d^2x \int_0^t d\tau\, Tr\,([A_\mu^\tau, A_\nu^\tau]\,\xi\cdot X\,\gamma_5)\,. \tag{3.25}$$

An explicit expression for A_μ^t is

$$A_\mu^t = \frac{sinh[ad(t\xi\cdot X)]}{ad(t\xi\cdot X)}\partial_\mu(t\xi\cdot X) \tag{3.26}$$

where the ad operation on matrices Y and Z is defined by $(ad\,Y)(Z) = [Y, Z]$.

The $A_\mu A_\mu$ term is the nonabelian extension of Schwinger's result. It originates in the integration of the $D_\mu A_\mu$ term in E^t. The second term of (3.25) is a Chern-Simons secondary characteristic class term [17,18]. This term is the two dimensional analogue of the Wess-Zumino anomaly [1]. To better understand this term let us analytically continue G/H such that it becomes the compact symmetric space $SU(N_c)\otimes SU(N_c)/SU(N_c)$. This is simply done by choosing the X's to be the anti-hermitian matrices

$$X_a = it_a\gamma_5\,. \tag{3.27}$$

In the case of compact G/H, the authors of [2,3,4] have emphasized that the coefficient of the Chern-Simons term must be quantized because of global topological considerations. The theory with such an interaction can only be defined in a space such that its third cohomology class is the integers. This is true in the present example for $N_c \geq 2$. Note that the trace in (3.25) is proportional to the number of flavors, this is a consequence of the quantization of the coefficient of the Chern-Simons term. There is an additional result that follows from the structure of D_t^2. The coefficient of the A^2 term is also quantized since the term is inextricably tied to the $[A_\mu, A_\nu]$ term in (3.20) due to (3.21).

IV. The Four Dimensional Model

In this section we construct a four dimensional model with an exactly solvable fermion determinant. This model respects the flavor chiral symmetries but it has a different renormalizability structure. These models are a new class of theories based on anti-symmetric tensor fields with values in the Lie algebra of $SU(N_c)$.

We motivate the model by modifying the Dirac operator in the presence of a gauge. For simplicity we consider an abelian gauge theory and afterwards discuss the non-abelian generalization. Unlike two dimensions, the abelian model already involves a major modification of the original theory. The complications are due to the non-abelian nature of the rotation group SO(4). Consider the Dirac operator in the presence of an abelian gauge field C_μ:

$$i\gamma_\mu(\partial_\mu - iC_\mu) \tag{4.1}$$

in the gauge $\partial_\mu C_\mu = 0$. There exists an an anti-symmetric tensor field $\omega_{\mu\nu}$ such that

$$C_\mu = \partial_\nu \omega_{\nu\mu} \tag{4.2}$$

For our purposes it is convenient to introduce $\tilde{\omega}_{\mu\nu}$, the dual of $\omega_{\mu\nu}$ by

$$\tilde{\omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\omega_{\rho\sigma} \, . \tag{4.3}$$

Using the γ-matrices identity

$$\gamma_\nu\gamma_\rho\gamma_\sigma = \delta_{\nu\rho}\gamma_\sigma - \delta_{\nu\sigma}\gamma_\rho + \delta_{\rho\sigma}\gamma_\nu - \epsilon_{\nu\rho\sigma\tau}\gamma_\tau\gamma_5 \, , \tag{4.4}$$

and the definition $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + i\sigma_{\mu\nu}$, one can show that

$$\gamma_\mu C_\mu = (\partial_\mu \tilde{\omega}_{\mu\nu})\gamma_\nu\gamma_5 + \frac{1}{2} i\gamma_\rho(\partial_\rho \omega_{\mu\nu})\sigma_{\mu\nu} \, . \tag{4.5}$$

There is a gauge invariance of $\omega_{\nu\mu}$ which leaves C_μ invariant: $\tilde{\omega}_{\rho\sigma} \to \tilde{\omega}_{\rho\sigma} + \partial_\rho\eta_\sigma - \partial_\sigma\eta_\rho$ for any vector field η_ρ. One can use this gauge transformation to require $\partial_\nu\tilde{\omega}_{\nu\gamma} = 0$. We reach the exact result

$$\gamma_\mu C_\mu = \frac{1}{2} i\gamma_\rho(\partial_\rho \omega_{\mu\nu})\sigma_{\mu\nu} \, . \tag{4.6}$$

The Dirac operator (4.1) may be written as

$$i\gamma_\mu[\partial_\mu + \frac{1}{2}(\partial_\mu \omega_{\rho\tau})\sigma_{\rho\tau}] \, . \tag{4.7}$$

This equation is the analogue of (3.4). The first modification is to replace the above by

$$i\gamma_\mu e^{-\frac{1}{2}\omega\cdot\sigma}\partial_\mu e^{\frac{1}{2}\omega\cdot\sigma} \tag{4.8}$$

where $\omega\cdot\sigma = \omega_{\mu\nu}\sigma_{\mu\nu}$. This is analogous to the passage from (3.4) to (3.5). Equations (4.7) and (4.8) agree to first order in ω. Unfortunately equation (4.8) is not suitable for the theorem of Sec. II and one has to do a further modification. We will be interested in the operator D defined by

$$D \equiv e^{\frac{1}{2}\omega\cdot\sigma} i\gamma_\mu \partial_\mu e^{\frac{1}{2}\omega\cdot\sigma} . \tag{4.9}$$

This operator is of the type that we can apply the theorem of Sec. II. Unfortunately, (4.9) is not equal to (4.8) since the $\sigma_{\mu\nu}$'s do not anti-commute with the γ_λ's. In two dimensions we had that (3.6) was equal to (3.5). The difficulty is due to the nonabelian nature of $SO(4)$. In fact $\sigma_{\mu\nu} = \epsilon_{\mu\nu}\gamma_5$, in two dimensions. If one identifies ξ with $\omega_{\mu\nu}$ via $\omega_{\mu\nu} = \xi\epsilon_{\mu\nu}$ then one sees that (4.9) is the four dimensional analogue of the two dimensional model (3.6).

There are several precautionary remarks one should make about (4.9). We will make the remarks on the nonabelian version $SU(N_c)$ version of (4.9). Define D_t by

$$D_t = g_t i\gamma_\mu \partial_\mu g_t \tag{4.10}$$

where

$$g_t(x) = exp[\frac{1}{2} t\, \omega^a_{\mu\nu}(x)\, \sigma_{\mu\nu} \otimes t_a] . \tag{4.11}$$

One now has an $\omega_{\nu\mu}$ for each generator of the "color" group. The operator (4.10) does not agree with the Dirac operator (4.7) to first order in ω. The ultraviolet structure of the theory defined by D_t is different from the Dirac Yang-Mills case. At high energies the behavior of D_t is governed by $g_t i\gamma_\mu g_t \partial_\mu$ while the Yang-Mills case is governed by $i\gamma_\mu\partial_\mu$.

The key to the analysis that follows is the asymptotic expansion for the heat kernel associated with D_t^2.

$$\langle x|exp(-\epsilon D_t^2)|x\rangle \rightarrow \frac{1}{(4\pi\epsilon)^2}[a_0^t(x) + \epsilon a_1^t(x) + \epsilon^2 a_2^t(x) + O(\epsilon^3)] . \tag{4.12}$$

The coefficients $a_k^t(x)$ have not been tabulated for the operator in question. Work is currently taking place in attempting to tabulate the above. The above expansion can actually be used to prove some general theorems about the determinant without knowledge of the coefficients in (4.12). The differential equation for the determinant is

$$\frac{d}{dt} Tr\, ln\, D_t^2 = 2\, Tr[\omega \cdot \sigma\, exp(-\epsilon D_t^2)]\,. \tag{4.13}$$

Inserting (4.12) into the above leads to the expression

$$Tr\, ln\, D_t^2 - Tr\, ln\, D_0^2$$

$$= \frac{1}{8\pi^2} \int d^4x \int_0^t d\tau\, Tr(\omega \cdot \sigma [a_0^\tau(x)/\epsilon^2 + a_1^\tau(x)/\epsilon + a_2^\tau(x)]) \tag{4.14}$$

The above expression has terms that diverge as Λ^4 and as Λ^2. There is no logarithmic divergent term in the above. This does not mean that there is no logarithmic divergence. In fact, one can substitute (4.12) into (2.2) and extract the divergent pieces of the determinant. One finds

$$Tr\, ln\, D_t^2 = \frac{1}{16\pi^2} \int d^4x\, [-\frac{1}{2\epsilon^2} Tr\, a_0^t(x) - \frac{1}{\epsilon} Tr\, a_1^t(x)$$

$$+ ln\, \epsilon\, Tr\, a_2^t(x) + (finite\ as\ \epsilon \to 0)]\,. \tag{4.15}$$

Comparing (4.15) and (4.14) we learn some important relations:

$$\frac{d}{dt} \int d^4x\, Tr\, a_0^t(x) = -4 \int d^4x\, Tr\, [\omega \cdot \sigma\, a_0^t(x)] \tag{4.16a}$$

$$\frac{d}{dt} \int d^4x\, Tr\, a_1^t(x) = -2 \int d^4x\, Tr\, [\omega \cdot \sigma a_1^t(x)] \tag{4.16b}$$

$$\frac{d}{dt} \int d^4x\, Tr\, a_2^t(x) = 0\,. \tag{4.16c}$$

Note that a possible candidate for a_0^t is a constant times $exp(-4t\omega \cdot \sigma)$. This term has the correct scaling behavior. Another possible candidate is $a_0^t = 1$. Of particular interest is the a_2^t term. According to (4.16c),

$$\int d^4x\, Tr\, a_2^t(x) \tag{4.17}$$

is independent of t; in other words, it is independent ω. Such a term could be a topological invariant. In two dimensions one finds that the logarithmic divergence in curved space time is given by the Euler characteristic, see for example [12]. In the two dimensional example of Section 3, the logarithmically divergent term vanishes. We will see that when we prove an index theorem [19] for the axial current, the term that enters involves a_2^t. The index theorem term could possibly be a topological invariant.

In this theory the logarithmicaly divergent term does not affect the dynamics of the ω field. This is very different from the Yang-Mills case where the logarithmic divergence contributes a screening correction to the Yang-Mills Lagrangian.

There are several very desirable features of D_t. The most important are the chiral symmetries of the model defined by the Lagrangian

$$L = \psi^\dagger D_t \psi \,. \tag{4.18}$$

If there are N_f flavors then at the classical level there is a $U(N_f) \otimes U(N_f)$ flavor symmetry. We show that the $U(1)$ axial vector current has a potential anomaly. According to the classical equations of motion the axial current

$$J_5^\mu = \psi^\dagger g_t \gamma_\mu \gamma_5 g_t \psi \tag{4.19}$$

is conserved. Quantum effects can modify the conservation. Let $\{\phi_\alpha\}$ be the complete orthonormal set of eigenfunctions for D_t with respective eigenvalues $\{\lambda_\alpha\}$. We now take the zero modes into account. Since D_t anticommutes with γ_5 we have that if ϕ_α is an eigenfunction with eigenvalue $\lambda_\alpha \neq 0$ then $\gamma_5 \phi_\alpha$ has eigenvalue $-\lambda_\alpha$ and it is orthogonal to ϕ_α.

The regulated induced current is given by

$$J^\mu_{5,reg}(x) = -\sum_\alpha{}' \frac{\phi^\dagger_\alpha(x)\, g_t \gamma_\mu \gamma_5\, g_t\, \phi_\alpha(x)}{\lambda_\alpha} e^{-\epsilon\lambda_\alpha^2} \tag{4.20}$$

The prime in the summation symbol denotes the omission of the zero modes in the sum. The divergence of the above is given by

$$\partial_\mu J^\mu_{5,reg} = -2i \sum_\alpha{}' \phi^\dagger_\alpha(x)\, \gamma_5\, \phi_\alpha(x)\, e^{-\epsilon\lambda_\alpha^2} \,. \tag{4.21}$$

Remember that ϕ_α is orthogonal to $\gamma_5\phi_\alpha$ if $\lambda_\alpha \neq 0$, therefore, the integral of the right hand side of (4.21) is zero.

$$0 = \int d^4x \sum_\alpha{}' \phi^\dagger_\alpha(x)\, \gamma_5\, \phi_\alpha(x)\, e^{-\epsilon\lambda_\alpha^2}$$

$$= Tr[\gamma_5\, exp(-\epsilon D_t^2)] - Tr[P\gamma_5] \,, \tag{4.22}$$

where P is the projector onto the space of zero modes. $Tr(P\gamma_5) = n_+ - n_-$ is the difference in the number of right handed zero modes n_+ and the number of left handed zero modes n_-. Equation (4.22) may be written as

$$n_+ - n_- = Tr\,[\gamma_5\, e^{-\epsilon D_t^2}] \,. \tag{4.23}$$

The answer must be independent of ϵ therefore

$$n_+ - n_- = \frac{1}{16\pi^2} \int d^4x\; Tr\,[\gamma_5 a_2^t(x)] \,, \tag{4.24a}$$

$$0 = \int d^4x\; Tr[\gamma_5\, a_1^t(x)] \,, \tag{4.24b}$$

$$0 = \int d^4x\; Tr[\gamma_5\, a_0^t(x)] \,. \tag{4.24c}$$

The precise form of D_t, see (4.10), tell us that n_+ and n_- should be independent of t even though the eigenfunctions do depend on t. In particular we have derived an integrated anomaly equation, (4.24a), which tell us that the divergence of the axial current could possibly be given by a local expression just as the Adler-Bell-Jackiw anomaly.

What makes D_t attractive is that the determinant is explicitly calculable once asymptotic expansion (4.12) is known. In particular the flavor anomalies are very similar in structure to the gauge theory case and these models could provide a way of better understanding chiral symmetries.

The geometrical setting for the model described by Lagrangian (4.18) is not understood. To get the algebra spanned by $\sigma_{\mu\nu} \otimes t_a$ to close one also has to include $i\sigma_{\mu\nu} \otimes t_a$, $i\sigma_{\mu\nu} \otimes 1$, $\sigma_{\mu\nu} \otimes 1$, $i1 \otimes t_a$, $1 \otimes t_a$. This leads to the group $SL(2N_c) \otimes SL(2N_c)$. The g_t live on a submanifold of one of the coset spaces associated with $SL(2N_c) \otimes SL(2N_c)$. I do not have a candidate Lagrangian for the ω fields. It would be very interesting if someone could provide a better understanding of these toy models.

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