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# Dynamics of zonal flow saturation in strong collisionless drift wave turbulence

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Generalized Kelvin–Helmholtz (GKH) instability is examined as a mechanism for the saturation of zonal flows in the collisionless regime. By focusing on strong turbulence regimes, GKH instability is analyzed in the presence of a background of finite-amplitude drift waves. A detailed study of a simple model with cold ions shows that nonlinear excitation of GKH modes via modulational instability can be comparable to their linear generation. Furthermore, it is demonstrated that zonal flows are likely to grow faster than GKH mode near marginality, with insignificant turbulent viscous damping by linear GKH. The effect of finite ion temperature fluctuations is incorporated in a simple toroidal ion temperature gradient model, within which both zonal flow and temperature are generated by modulational instability. The phase between the two is calculated self-consistently and shown to be positive. Furthermore, the correction to nonlinear generation of GKH modes appears to be small, being of order  $O(\rho_i^2 k^2)$ . Thus, the role of linear GKH instability in the saturation of collisionless zonal flows, in general, seems dubious. © 2002 American Institute of Physics.

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## I. INTRODUCTION

The elimination of turbulent transport in fusion devices is critical in utilizing magnetic fusion as a realistic future energy source. For this reason, the origin and suppression of turbulent transport has been a hot research topic in magnetically confined plasmas. A classical paradigm for turbulent transport in tokamaks is drift wave (DW) turbulence, which arises due to the inhomogeneity in the background density, temperature, magnetic field, etc.<sup>1</sup> On the other hand, recent works have indicated that zonal flows play an important role in regulating turbulent transport.<sup>2,3</sup> As toroidally symmetric, mainly poloidal ( $\mathbf{E} \times \mathbf{B}$ ) flows with finite  $k_r$ , zonal flows reduce the radial transport by shearing eddies associated with the underlying turbulence.<sup>4</sup> Thus, the dynamics of a coupled system of DW and zonal flows is central to the understanding of turbulent transport. In fact, this coupled system is self-regulating since zonal flows, generated by DW through modulational instability, backreact on DW by shearing, thus weakening the very source of their generation (i.e., DW). This shall be explained in more detail in the following.

In simple terms, the generation of zonal flows by DW can be viewed as a decay instability. It is because zonal flow shearing preserves action of DW so that the wave quanta density  $N_k$ , proportional to  $E_k/\omega_k$ , is conserved. Here,  $E_k$  and  $\omega_k$  are the energy and frequency of DW. The decay instability occurs when DW has more modes with higher energy than the ones with lower energy, i.e., when DW spectrum  $N_k$  has more population for higher energy mode. A mode with higher energy can then decay into a DW with slightly lower energy and a zonal flow. As the energy of DW  $E_k \propto 1/(1+k_\perp^2 \rho_s^2)$ , this decay is entailed when  $dN_k/dk_r < 0$ ,

which is the criterion for modulational instability. When zonal flows grow, they shear turbulent eddies of DW and thus generate modes of DW with higher  $k_r$ , flattening  $N_k$  spectrum, which initially satisfies  $dN_k/dk_r < 0$ . Thus, the modification of the DW spectrum leads to the saturation of zonal flows as  $dN_k/dk_r \rightarrow 0$ . The main feature of this nonlinear feedback can be captured in a simple “predator–prey” model by employing a quasilinear closure.<sup>5,6</sup>

The aforementioned nonlinear spectral feedback is one of the mechanisms for the saturation of collisionless zonal flows. In the presence of an effective collision of ions ( $\nu_{\text{eff}} \neq 0$ ), zonal flows are directly subject to collisional damping.<sup>7</sup> Note that in tokamak core,  $\nu_{\text{eff}}$  arises from collisions between trapped and passing ions, due to high temperature. As zonal flows damp more for larger  $\nu_{\text{eff}}$ , the amplitude of DW increases with  $\nu_{\text{eff}}$ . So,  $\nu_{\text{eff}}$  regulates the amplitude of underlying DW turbulence, and thus the turbulent transport.

In addition to  $\nu_{\text{eff}}$ , the amplitude of DW turbulence is also set by its deviation from the marginal stability  $\epsilon = (1/L_T - 1/L_c)/(1/L_c)$ , which determines the strength of turbulent drive. Here,  $L_T$  is the scale length of the background ion temperature ( $L_T^{-1} = -\partial_x \ln T_{i0}$ ) and  $L_c$  is the critical temperature gradient scale length for the onset of instability. Therefore, the state of the coupled system of zonal flows and DW hinges on both  $\nu_{\text{eff}}$  and  $\epsilon$ . That is, to characterize the state of the coupled system, the dependence on these two parameters should be explored in the two-dimensional space formed by  $\nu_{\text{eff}}$  and  $\epsilon$ . It is interesting to note that previous works focused on the dependence on only one parameter. For instance, Lin *et al.*<sup>8</sup> studied the dependence of  $\chi_i$  on  $\nu_{\text{eff}}$  near marginality while Rogers *et al.*<sup>9</sup> investigated  $\chi_i$  as a function of  $\epsilon$  in the purely collisionless limit (i.e., for  $\nu_{\text{eff}}=0$ ). A systematic scan of the two-

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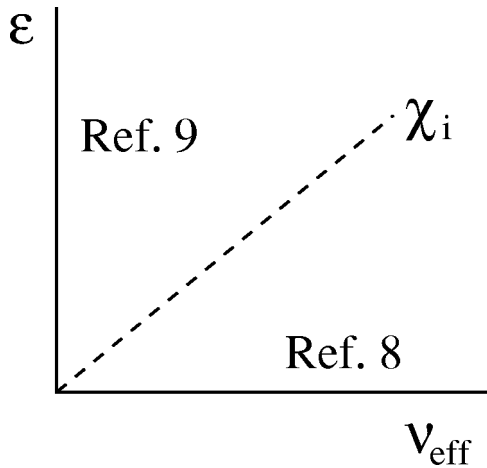


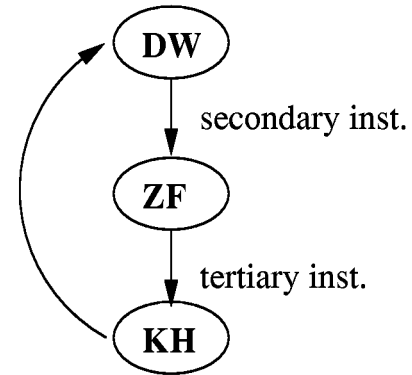
FIG. 1. A required systematic scan of the two-dimensional space of  $\epsilon$  and  $\nu_{\text{eff}}$ . Reference 8 explores  $\chi_i$  as a function of  $\nu_{\text{eff}}$  while Ref. 9 focuses on the dependence of  $\chi_i$  as a function of  $\epsilon$ .

dimensional space of  $\epsilon$  and  $\nu_{\text{eff}}$  remains to be done (see Fig. 1).

It is illuminating to consider the behavior of the system in detail as  $\nu_{\text{eff}}$  varies. In the limit of  $\nu_{\text{eff}}\Delta\omega/\gamma^2 \gg 1$ , zonal flow damping is efficient, so that DW saturates by wave-wave nonlinearity (mixing processes).<sup>5</sup> Here  $\gamma$  and  $\Delta\omega$  are linear growth and nonlinear damping rates of DW. In the opposite and more relevant limit of  $\nu_{\text{eff}}\Delta\omega/\gamma^2 \ll 1$ , the damping of DW by shearing becomes effective, resulting in the scaling relation  $\langle N_k \rangle \sim \nu_{\text{eff}}/\gamma$ .<sup>5</sup> Here  $\langle N_k \rangle$  is the mean wave number density of DW, representative of its amplitude. That is, the amplitude of DW is directly proportional to  $\nu_{\text{eff}}$ . Consequently, in the collisionless limit  $\nu_{\text{eff}}=0$  and near marginal stability  $\epsilon \ll 1$ , the zonal flow shearing can damp DW to a very small amplitude in certain cases, thereby leading to a pure flow (or, Dimits upshift) regime.<sup>10</sup> Note, however, that there are cases when DW amplitude are not significantly reduced.<sup>11</sup> In the case of a pure flow regime with very weak DW turbulence, an interesting question is what happens to this pure flow regime as  $\epsilon$  is further increased especially in the collisionless limit (where  $\nu_{\text{eff}}=0$ ), in other words, whether and how this pure flow regime terminates. When  $\nu_{\text{eff}}=0$ ,  $\chi_i$  is numerically observed to rapidly increase from almost zero beyond this pure flow regime upon the increase of  $\epsilon$ , suggesting the reappearance of DW upon the termination of this regime. As the zonal flow shearing is responsible for a pure flow regime via damping of DW, the reappearance of DW (as  $\epsilon$  increases) must be due to the saturation (or damping) of zonal flows.

In fact, the recent work by Rogers *et al.*<sup>9</sup> invoked a linear Kelvin–Helmholtz (KH)-type (shear) instability as a damping mechanism for zonal flows, which breaks up zonal flows and thus quenches the pure flow regime. Their analysis is linear in the sense that (1) they neglect the presence of DW and (2) they envision KH as a tertiary, linear instability about a zonal flow state. Even if their analysis is appropriate for the dynamics of zonal flows in the pure flow regime, it is not likely universally valid in general. This is particularly true since the incorporation of a weak but experimentally relevant

### Hierarchical



### Non-hierarchical

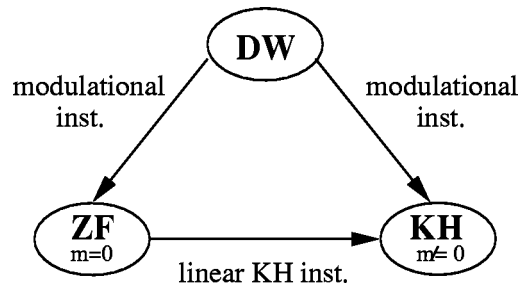


FIG. 2. The contrast of the linear view of the GKH modes to a more general case where GKH modes are generated by both linear and nonlinear modulational instabilities. The linear view is hierarchical in that GKH is generated by the linear instability of zonal flows (ZF), which are already generated by DW. In general, GKH modes can, however, be generated directly from DW by modulational instability.

amount of collisional damping can lead to sufficient damping of zonal flows, removing this pure flow regime.<sup>8</sup> That is, a pure flow regime is very unusual, occupying a vanishingly small measure of the entire parameter space.

Therefore, one has to face the problem of understanding a *generalized* KH (GKH) in the background of DW. In particular, when DWs are adiabatically modulated by GKH mode (i.e., when  $\gamma_{\text{KH}} < \omega$ , where  $\gamma_{\text{KH}}$  and  $\omega$  are the growth rate of GKH and real frequency of DW, respectively), GKH can extract energy from the DW spectrum (see Fig. 2). This nonlinear generation of GKH is similar to that of zonal flows (or streamers). Here, the distinction between the two is that GKH is a nonaxisymmetric mode ( $m \neq 0$ ) while zonal flows are axisymmetric ( $m = 0$ ). Thus, in the following discussion, nonaxisymmetric and GKH modes will be used interchangeably; a KH mode will refer to only a *linear* KH mode that results from linear KH (shear) instability while GKH mode represents a nonaxisymmetric mode, due to both linear and nonlinear generation. The nonlinear excitation is especially important as the linear KH instability is easily quenched or

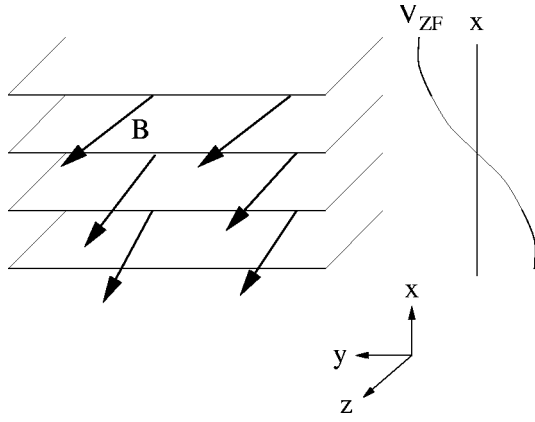


FIG. 3. A schematic picture showing the magnetic shear and zonal flow profile; the exchange of two vortices requires the alignment of those vortices with tilted magnetic field lines.

reduced by magnetic shear.<sup>12,13</sup> This is because the interchange of two vortices, which is energetically favorable in the absence of a magnetic shear, needs to pay an energy penalty to maintain alignment of vortices with the magnetic field in the presence of shear (see Fig. 3). Note that the effect of magnetic shear was not incorporated in the linear analysis in Ref. 9. Furthermore, KH tends to have a narrow mixing region,<sup>13</sup> even if excited so that the effective viscosity, as estimated by  $\gamma\Delta^2$ , is weak.

The consideration of nonlinear generation of GKH as well as zonal flow then points to the possibility that there may be no clear distinction between the secondary zonal flows and tertiary KH modes (see Fig. 2). That is, the treatment of KH as a tertiary instability of a secondary mode (zonal flows) is valid only when zonal flows evolve on time scale much longer than that of the KH. However, if the growth rate of zonal flows exceeds that of KH, this time ordering cannot be satisfied.

In this paper, we study these issues in detail by considering GKH in the background of DW, specifically focusing on a strong turbulence regime, excluding a pure flow regime. In the background of DW, GKH modes are nonlinearly excited by modulational instability of DW turbulence. This nonlinear excitation of GKH is due to the nonaxisymmetric modulation of DW spectrum. Specifically, we examine the validity of the linear picture of KH instability and clarify the role of linear KH instability in the saturation of collisionless zonal flows. (Note that as zonal flows are not subject to Landau damping due to  $k_\theta = k_\parallel = 0$ , their collisionless damping is especially important.) To this end, we address the following specific questions:

- (i) When does the linear generation of GKH modes dominate their nonlinear generation?
- (ii) When is the nonlinear generation of zonal flow negligible, so as to justify the treatment of zonal flow as a fixed background for the linear KH instability?
- (iii) How efficiently is momentum transported by the linear KH mode?

The linear picture of GKH is valid only if the linear generation of GKH dominates nonlinear generation [in (i)] and also

if the nonlinear growth rate of zonal flow is much smaller than the linear growth rate of KH mode [in (ii)]. And, the damping of zonal flows by linear KH instability is insignificant if it is much smaller than the nonlinear growth rate of zonal flows [in (iii)]. To study these issues in detail, we first consider a simple model with cold ions. After that, the effect of the finite ion temperature fluctuations will be incorporated in a simple, two-dimensional toroidal ion temperature gradient model.

The remainder of the paper is organized as follows. In Sec. II, we adopt a simple model with cold ions and compute the linear growth rate of GKH by inflection point instability and nonlinear growth rate through modulational instability. The comparison of these two growth rates then illustrates that the nonlinear excitation can become comparable to the linear excitation of the nonaxisymmetric mode for parameter values typical of tokamaks. We also establish the momentum transport by KH by computing the flux of zonal flow momentum via quasilinear analysis. Results indicate that the momentum transport by GKH is likely insignificant near marginality. In Sec. III, we extend our model to two-dimensional toroidal ion temperature gradient (ITG) mode,<sup>14</sup> by incorporating ion temperature fluctuations. In this model, zonal temperature as well as zonal flow are excited by modulational instability. We determine the amplitude of zonal temperature and its phase with zonal flow. This phase relation is then used for the linear stability analysis for KH. Nonlinear generation of GKH is revisited in this extended model. The conclusions of this paper are discussed in Sec. IV.

## II. MINIMAL MODEL WITH COLD IONS

We treat a GKH instability as a flute-like-mode with a small  $k_\parallel$ , and assume nonadiabatic electrons. This is in contrast to the work by Rogers *et al.*, who considered  $k_\parallel \sim 1/qR$ . As the adiabaticity of electrons reduces a linear KH instability via enhanced inertia, the assumption of nonadiabatic electrons allows us to obtain an upper bound on linear KH instability. In this section, we first study a simple model with cold ions to elucidate some of the key features of GKH in the simplest context. After studying this model in detail, we incorporate the effect of ion temperature fluctuation in Sec. III.

The governing equations for the large-scale electric potential can be obtained from the gyrokinetic equations for ions and drift kinetic equation for electrons. By using  $T_{i0} = 0$  ( $T_{i0}$  is the ion background temperature) and  $k_\parallel = 0$ , together with a quasineutrality, and by keeping terms up to first order in  $O(\rho_i^2 k^2)$ , we obtain

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \nabla_\perp^2 \phi = (\text{NL}), \quad (1)$$

where

$$\begin{aligned} (\text{NL}) &= \langle \mathbf{v}' \cdot \nabla \nabla_\perp^2 \phi' \rangle \\ &= (\partial_{xx} - \partial_{yy}) \langle v'_x v'_y \rangle + \frac{1}{2} \partial_{xy} \langle v_y'^2 - v_x'^2 \rangle. \end{aligned} \quad (2)$$

Our notation is as follows:  $\phi'$  is the electric potential perturbation associated with DW while  $\phi$  contains both zonal flow

and GKH mode, i.e.,  $\phi = \phi_{ZF} + \phi_{KH}$ , where  $\phi_{ZF}$  and  $\phi_{KH}$  are zonal and GKH components;  $\mathbf{v} = -\nabla \times \phi \hat{z}$  and  $\mathbf{v}' = -\nabla \times \phi' \hat{z}$ ; (NL) represents the Reynolds stress (nonlinear contribution) from DW;  $x$  and  $y$  represent radial ( $r$ ) and poloidal ( $\theta$ ) directions, respectively. In Eqs. (1) and (2), the length, velocity, time, and  $\phi$  are measured in units of  $\rho_s$ ,  $c_s$ ,  $1/\Omega_i$ , and  $T_{e0}/e$ , respectively; angular brackets denote the average over DW. Here,  $\rho_i$ ,  $c_s$ ,  $\Omega_i$ , and  $T_{e0}$  are ion gyro-radius, sound speed, ion gyro-frequency, and electron background temperature. Note that Eq. (2) can also be obtained from  $\langle \nabla \cdot \mathbf{J}_p \rangle = 0$  ( $\mathbf{J}_p$  polarization current).

The dynamics of the DW in the background of slowly varying  $\phi$  can be described by the wave-kinetic equation for wave quanta density (potential enstrophy)  $N_k = (1 + k_\perp^2)^2 |\phi_k'|^2$ :<sup>15</sup>

$$\begin{aligned} \frac{\partial}{\partial t} N_k + \frac{\partial}{\partial \mathbf{k}} (\mathbf{k} \cdot \mathbf{v} + \omega) \frac{\partial}{\partial \mathbf{x}} N_k - \frac{\partial}{\partial \mathbf{x}} (\mathbf{k} \cdot \mathbf{v} + \omega) \frac{\partial}{\partial \mathbf{k}} N_k \\ = \gamma N_k - \Delta \omega N_k^2, \end{aligned} \quad (3)$$

where  $\gamma$  and  $\Delta \omega$  are linear growth and nonlinear damping rates of DW. Note here that the wave-kinetic equation is valid in a strong turbulence regime.<sup>16</sup> For this minimal model, the frequency of DW is given by  $\omega = \omega_* / (1 + k^2)$  with  $\omega_* = v_* k_y$ . Here,  $v_* = \rho_s / L_n$  is the electron diamagnetic velocity, and  $L_n = -(\partial_x \ln n_{i0})^{-1}$  is the scale length of the background density.

If the zonal flow  $\phi_{ZF}$  evolves very slowly compared to  $\phi_{KH}$ ,  $\phi_{ZF}$  serves as the equilibrium background for  $\phi_{KH}$ . In that case, the equation for  $\phi_{KH}$  is obtained by considering the linear perturbation of Eq. (1) around  $\phi = \phi_{ZF}(x)$

$$\left( \frac{\partial}{\partial t} + \mathbf{v}_{ZF} \cdot \nabla \right) \nabla_\perp^2 \phi_{KH} + \mathbf{v}_{KH} \cdot \nabla_\perp^2 \phi_{ZF} = (\text{NL})_{\text{NA}}, \quad (4)$$

where the nonlinear term  $(\text{NL})_{\text{NA}}$  represents the nonaxisymmetric ( $m \neq 0$ ) part. In the absence of DW ( $(\text{NL})_{\text{NA}} = 0$ ), the previous equation reduces to the Rayleigh equation, which has been extensively studied as a paradigm for linear KH instability in hydrodynamics. This shear instability is a classical problem<sup>17,18</sup> and is well-known to arise because of an inflection point where the gradient of a shear flow vanishes (i.e., vorticity maximum). It is because the interchange of two vortices around an inflection point is energetically favorable, due to the lack of restoring force around this point. However, the presence of an inflection point is a necessary, but not sufficient condition for instability as the KH instability depends sensitively on the global geometry of a shear flow and the boundary conditions. Note that in Ref. 9, the linear KH instability of zonal flows was treated by a local analysis, by implicitly assuming a parabolic profile for a zonal flow, thereby overlooking these important issues.

In the presence of DW ( $(\text{NL})_{\text{NA}} \neq 0$ ), the Rayleigh criterion (inflection theorem) is no longer applicable due to the  $(\text{NL})_{\text{NA}}$  term. Thus, the presence of an inflection point is not necessarily required for instability. This is clear since GKH can be generated solely by the nonlinear contribution from

DW (see Sec. II B). Indeed, it will be demonstrated there that nonlinear generation of GKH can be comparable to the linear generation.

The simultaneous growth of both zonal flow and GKH by the same mechanism then makes the treatment of zonal flows as background for GKH mode dubious, especially when the nonlinear growth rate of zonal flow is larger than that of GKH, as noted in Sec. I. Thus, in general, the linearization of KH around zonal flow in Eq. (4) may not be justified in the presence of DW. For this reason, in the following, we shall look into linear and nonlinear generation of GKH, separately. First, by assuming stationary zonal flow, we estimate the linear growth rate of KH in Sec. II A. Then, in Sec. II B, we focus on the nonlinear generation of GKH, by treating GKH and zonal flows on an equal footing. That is, we shall consider the modulation of the DW spectrum by a large-scale flow, consisting of the axisymmetric part (zonal flow) and nonaxisymmetric part (GKH).

### A. Linear instability

In this section, we ignore the presence of DW and estimate the linear growth rate of the KH mode. To this end, it is useful to consider a simple sinusoidal profile for the mean flow,<sup>17</sup> which reveals some valuable information. We assume that the background zonal flow takes the form

$$\phi_{ZF} = \bar{\phi} \cos \bar{p} x, \quad (5)$$

and substitute it in Eq. (4) with  $(\text{NL}) = 0$ . As  $\phi_{ZF}$  is periodic in  $x$ , we seek a Floquet solution for  $\phi_{KH}$  in the following form:

$$\phi_{KH} = \sum_{n=-\infty}^{\infty} \hat{\phi}_n \exp\{i[(p_0 + n\bar{p})x + qy] + \gamma_L t\}. \quad (6)$$

Upon the substitution of Eq. (6) in (4), we truncate the equations by keeping the mode coupling among three adjacent modes ( $n=0, \pm 1$ ) to obtain the growth rate:

$$\begin{aligned} \gamma_L^2 = -\frac{1}{4} \bar{\phi}^2 \bar{p}^2 q^2 \frac{p_0^2 + q^2 - \bar{p}^2}{p_0^2 + q^2} \\ \times \left[ \frac{p_{-1}^2 + q^2 - \bar{p}^2}{p_{-1}^2 + q^2} + \frac{p_1^2 + q^2 - \bar{p}^2}{p_1^2 + q^2} \right], \end{aligned} \quad (7)$$

where  $p_n = n\bar{p} + p_0$ . It is interesting to note that the instability is possible (i.e.,  $\gamma_L^2 > 0$ ) when  $p_0^2 \leq 3\bar{p}^2$ . In fact, the maximum  $\gamma_L$  is obtained for  $p_0 = 0$  and  $q/\bar{p} \sim 0.5$ . Thus, for simplicity, we approximate Eq. (7) with  $p_0 \sim 0$  as

$$\gamma_L^2 = \frac{1}{2} \bar{\phi}^2 \bar{p}^2 q^2 \frac{\bar{p}^2 - q^2}{\bar{p}^2 + q^2}. \quad (8)$$

Therefore,  $\bar{p}^2 > q^2$  is required for the instability; the scale of the KH mode in the poloidal direction ( $y$ ) should be larger than the characteristic radial scale of zonal flows. For instance, in the limit of  $\bar{p} \rightarrow \infty$ , Eq. (8) recovers the well-known result that all  $q$  modes are unstable for a discontinuous profile in the absence of dissipation. In order to compare the linear growth rate with nonlinear growth rate (Sec. II B), we further approximate Eq. (8) as



$$\gamma_L \sim |qv_{ZF}|. \quad (9)$$

We remark that the effect of magnetic shear, which tends to localize a mode, may be captured by taking  $p_0^2 \gg \bar{p}^2$  within this model. In this limit, Eq. (7) becomes  $\gamma_L^2 \sim -\bar{\phi}^2 \bar{p}^2 q^2 / 2 < 0$ , consistent with the stabilization by magnetic shear.

## B. Nonlinear generation

In this section, we study the nonlinear generation of both zonal flow and GKH by modulation of DW spectrum. As zonal flow and GKH are generated by axisymmetric and nonaxisymmetric parts of the DW spectrum, one of the key issues is to calculate the branching ratio between the two. To this end, it is convenient to consider the equation for a large-scale electric potential, consisting of both components, which satisfies

$$\frac{\partial}{\partial t} \nabla_{\perp}^2 \langle \phi \rangle = (\partial_{xx} - \partial_{yy}) \langle v_x' v_y' \rangle + \frac{1}{2} \partial_{xy} \langle v_y'^2 - v_x'^2 \rangle. \quad (10)$$

Note that the second term on the right-hand side of Eq. (10) vanishes for isotropic DW turbulence with  $\langle v_x'^2 \rangle = \langle v_y'^2 \rangle$ , and thus shall be neglected compared to the first term in this paper.

When the generation of  $\phi = \phi_{ZF} + \phi_{KH}$  occurs on a time scale which is much larger than the characteristic time scale of the underlying DW turbulence, DW turbulence is adiabatically modulated by  $\phi$ , which is growing. Thus, we let  $N_k = \langle N_k \rangle + \tilde{N}_k$ , where  $\langle N_k \rangle$  and  $\tilde{N}_k$  are background and modulated parts of the DW spectrum. In the absence of zonal flows and GKH mode, the background DW turbulence is assumed to be stationary and homogeneous, satisfying  $\gamma \langle N_k \rangle = \Delta \omega \langle N_k \rangle^2$ . By using this leading order balance in Eq. (3) and by assuming harmonic modulation  $\tilde{N}_k \sim \exp\{-i(\Omega t - px - qy)\}$ , we obtain

$$\frac{\delta \tilde{N}_k}{\delta \phi} = -i \frac{(pk_y - qk_x)(p\partial_{k_x} + q\partial_{k_y}) \langle N_k \rangle}{\Omega - (pv_{gx} + qv_{gy}) + i\gamma}. \quad (11)$$

To obtain the nonlinear growth rate, we relate the Reynolds stress term in Eq. (10) to  $\tilde{N}_k$  as:

$$\langle v_x' v_y' \rangle = - \int d^2 k \frac{k_x^2 k_y^2}{(1+k_{\perp}^2)^2} \tilde{N}_k. \quad (12)$$

Then, by using Eqs. (11) and (12) with  $\phi \sim \exp\{-i(\Omega t - px - qy)\}$ , Eq. (10) gives us

$$\Omega = \frac{p^2 - q^2}{p^2 + q^2} \int d^2 k \frac{k_x k_y}{(1+k_{\perp}^2)^2} \frac{(pk_y - qk_x)(p\partial_{k_x} + q\partial_{k_y}) \langle N_k \rangle}{[\Omega - (pv_{gx} + qv_{gy}) + i\gamma]}. \quad (13)$$

In the limit as  $q \rightarrow 0$ , the imaginary part of Eq. (13) reproduces the growth rate of zonal flow,

$$\gamma_{ZF} \sim \int d^2 k \frac{k_x^2 k_y^2}{2(1+k^2)^2} \frac{p^2 \gamma}{(pv_{gx})^2 + \gamma^2} \left( - \frac{\partial \langle N_k \rangle}{\partial k_x^2} \right). \quad (14)$$

It manifests the generation of zonal flow for  $\partial_{k_x} \langle N \rangle < 0$ , which is satisfied for virtually all DW turbulence. To esti-

mate  $\gamma_{ZF}$ , we let  $\gamma = \omega \delta$  ( $\delta < 1$  near marginality) and use  $\gamma / |v_{gx} p| \sim \delta k_x / p(\rho_s k)^2 \geq 1$  (by assuming  $\rho_s k \leq 1$ ) to obtain

$$\gamma_{ZF} \sim \rho_s^2 k_0^2 c_s^2 p^2 \langle N_k \rangle / \gamma \sim \rho_s c_s p^2 / \delta L_n k_0. \quad (15)$$

Here,  $k_0$  is the characteristic scale of DW; we used  $\omega \sim k_0 \rho_s c_s / L_n$ ,  $\langle N_k \rangle \sim (k_x^2)^{-\xi}$  with  $\xi \sim 0(1)$ , and the mixing-length estimate  $\langle N_k \rangle \sim 1 / (k_0 L_n)^2$ . Note that we assumed a strong turbulence for this estimate, excluding the pure flow regime where  $\langle N_k \rangle$  can be much smaller due to the efficient damping of DW by zonal flow shearing. An accurate description of this regime requires the incorporation of spectral feedback.

On the other hand, by using  $v_{gx} \propto k_x k_y$  and  $|v_{gx} p| \gg \Omega_R$ , a zero real frequency of zonal flow ( $\Omega_R = 0$ ) can be shown to be a consistent solution to Eq. (13) if DW turbulence has reflectional symmetry in  $y$ , i.e.,  $\langle N_k(k_x, -k_y) \rangle = \langle N_k(k_x, k_y) \rangle$ .

In contrast, GKH mode has nonvanishing real frequency, as can easily be shown from Eq. (13). And, its imaginary part can be somewhat simplified as follows. First, we employ the following ordering:  $v_{gy} > v_{gx}$ ,  $\gamma / |v_{gy} q| \sim \delta k_y / q < 1$ ,  $\gamma / |v_{gx} p| \sim \delta k_x / p(\rho_s k)^2 \geq 1$ , and  $p \sim q$  ( $(p-q)^2 \sim (p+q)^2$ ). Then, the assumption of isotropy of the background DW turbulence ( $k_x \approx k_y$ ) reduces Eq. (13) to

$$\gamma_{NL} \sim \int d^2 k \frac{k_x^2 k_y^2}{2(1+k^2)^2} \frac{\gamma}{v_{gy}^2} \left( - \frac{\partial \langle N_k \rangle}{\partial k_x^2} \right). \quad (16)$$

To compare this with linear KH growth rate, we further simplify  $\gamma_{NL}$  in Eq. (16) by using  $\gamma = \delta \omega$ ,  $\langle N_k \rangle \sim (k_x^2)^{-\xi}$  with  $\xi \sim 0(1)$ , and  $\langle N_k \rangle \sim 1 / (k L_n)^2$  to obtain

$$\gamma_{NL} \sim k_0 c_s \rho_s \delta / L_n. \quad (17)$$

Thus, from Eqs. (9) and (17),

$$\frac{\gamma_{NL}}{\gamma_L} \sim \frac{k_0}{q} \frac{c_s}{v_{ZF}} \frac{\rho_s}{L_n} \delta. \quad (18)$$

For  $v_{ZF} \leq 10^{-2} c_s$  (e.g., see Ref. 19) and  $\rho_s / L_n \sim 0.01$ ,  $\gamma_{NL} / \gamma_L \geq (\delta k_0 / q)$ . That is, nonlinear generation of GKH may become comparable to linear generation, i.e., for  $v_{ZF} < 10^{-2} c_s$ . Note that if the above-assumed orderings are not valid, for instance, when  $v_{gy} < v_{gx} < \gamma / q$  such that  $\delta k_y / q > 1$ ,  $\gamma_{NL}$  becomes comparable to  $\gamma_{ZF}$  as given in Eq. (15).

To calculate the branching ratio between axisymmetric (zonal flow) and non axisymmetric modes (GKH), we take the ratio of Eqs. (15) and (17)

$$\frac{\gamma_{ZF}}{\gamma_{NL}} \sim \left( \frac{v_{gy} q}{\gamma} \right)^2 \sim \left( \frac{p}{k_0 \delta} \right)^2. \quad (19)$$

It reveals the following two important points. First, zonal flow can grow faster than GKH near marginality where  $\delta < p / k_0$ . Second, away from the marginality, say  $\delta > p / k_0$ , the generation of GKH can be comparable to that of zonal flow [recall that Eqs. (17) and (19) are valid only when  $v_{gy} q / \gamma \sim p / \delta k_0 > 1$  and that  $\gamma_{NL} \sim \gamma_{ZF}$  for  $\delta > p / k_0$ ]. However, considering that GKH modes are more easily damped (via Landau damping) than zonal flows, GKH modes are likely weaker than zonal flows. It is important to note that unlike zonal flows, GKH contains a radial flow and is capable of transporting energy in the radial direction, possibly contributing to  $\chi_i$ . We thus speculate that the excitation of

GKH with the amplitude comparable to zonal flow may provide another mechanism for the increase in  $\chi_i$  away from marginality.

A rapidly growing zonal flow near marginality implies that the assumption of stationary zonal flow for linear KH mode may not be justified in a strong turbulence regime. In order to find out when this is the case, we compare  $\gamma_{ZF}$  and  $\gamma_L$  by using Eqs. (18) and (19):

$$\frac{\gamma_{ZF}}{\gamma_L} \sim \frac{P}{\delta k_0} \frac{c_s}{v_{ZF}} \frac{\rho_s}{L_n}. \quad (20)$$

Thus, for  $v_{ZF} \sim 10^{-2} c_s$  and  $\rho_s/L_n \sim 0.01$ ,  $\gamma_{ZF}/\gamma_L \sim p/\delta k_0$ , which suggests that near marginality with  $\delta < p/k_0$ , zonal flows grow faster than a linear KH mode. Therefore, near marginality, zonal flow cannot be assumed to be stationary for the evolution of GKH instability.

To summarize, we have shown that in strong turbulence regimes, (1) the nonlinear generation of zonal flow can be more effective than both nonlinear and linear generation of GKH (i.e.,  $\gamma_{ZF} \geq \gamma_{NL}$  and  $\gamma_{ZF} \geq \gamma_L$ ) near marginality with  $\delta < p/k_0$ , and (2) the nonlinear generation of GKH can be comparable to its linear generation (i.e.,  $\gamma_{NL} \sim \gamma_L$ ). That is, near marginality (excluding a very weak turbulence regime), satisfying  $\delta < p/k_0$ , a linear shear instability analysis of zonal flow is invalid and the nonlinear generation of GKH should be taken into account.

### C. Momentum transport by KH mode

To quantify the damping of zonal flows by linear KH, it is necessary to calculate the momentum transport (or mixing) by KH. To this end, we shall assume zonal flow to be fixed and compute the momentum flux induced by KH. This stress represents momentum transport by KH. We then compare the damping of zonal flow by KH with the nonlinear growth rate of zonal flow to demonstrate that the former is likely small compared to the latter near marginality (i.e., for  $\delta \ll 1$ ).

In the presence of both linear KH mode and DW, the evolution equation for zonal flows takes the following form:

$$\frac{\partial}{\partial t} \nabla^2 \phi_{ZF} + \langle \mathbf{v}_{KH} \cdot \nabla \nabla_{\perp}^2 \phi_{KH} \rangle + \langle \mathbf{v}' \cdot \nabla \nabla_{\perp}^2 \phi' \rangle = 0, \quad (21)$$

where angular brackets in the second term are to be interpreted as the averages in the poloidal direction. The second term in Eq. (21) represents the stress by KH, and thus possible damping of zonal flows, while the third term is the Reynolds stress by DW, responsible for the growth of zonal flows. Note that, as  $\partial_y = 0$  for zonal flows, the stress by KH can be written as

$$\langle \mathbf{v}_{KH} \cdot \nabla \nabla_{\perp}^2 \phi_{KH} \rangle = \partial_x \langle v_{KH,x} \nabla_{\perp}^2 \phi_{KH} \rangle.$$

To estimate this term, we first compute quasilinear response of  $\phi_{KH}$  from Eq. (4):

$$\nabla^2 \delta \phi_{KH} = -\tau_c [v_{ZF} \partial_y \nabla^2 \phi_{KH} + v_{KH,x} \partial_x \nabla^2 \phi_{ZF}], \quad (22)$$

where  $\tau_c^{-1} = -i\omega + \gamma_L$ . By using Eq. (22), the stress can be shown to be (see the Appendix for details)

$$\begin{aligned} \langle v_{KH,x} \nabla_{\perp}^2 \phi_{KH} \rangle &= \langle \delta v_{KH,x} \nabla_{\perp}^2 \phi_{KH} + v_{KH,x} \nabla_{\perp}^2 \delta \phi_{KH} \rangle \\ &\sim -v_T \partial_{xx} v_{ZF} = -v_T \partial_x \nabla_{\perp}^2 \phi_{ZF}, \end{aligned} \quad (23)$$

where

$$\begin{aligned} v_T &\sim \int_{q < \bar{p}} dq \frac{1}{\gamma_L} [ |v_{KH,x}(0,q)|^2 + |v_{KH,x}(|\bar{p}|,q)|^2 ] \\ &\sim \frac{\langle v_{KH}^2 \rangle}{qv_{ZF}}. \end{aligned} \quad (24)$$

In obtaining Eq. (24),  $\tau_c = \gamma_L$  with  $\omega = 0$  and the conditions for KH instability  $\bar{p} > q$  have been used. Equations (21) and (23) clearly show that the stress ( $v_T > 0$ ) associated with KH leads to the damping of KH, as expected. The damping rate of zonal flow due to KH can be estimated to be

$$D_{ZF} \sim \frac{\langle v_{KH}^2 \rangle}{qv_{ZF}} \bar{p}^2. \quad (25)$$

To appreciate how effective this damping is, it is necessary to compare  $D_{ZF}$  with  $\gamma_{ZF}$ . The maximum damping rate  $D_{ZF}$  is estimated from Eq. (25) to be  $D_{ZF} \sim qv_{ZF}$ , by using  $v_{KH} \sim v_{ZF}$  and  $\bar{p} \sim q$ . Upon using Eq. (15),  $\rho_s/L_n \sim 0.01$ , and  $v_{ZF}/c_s \sim 0.01$ , the ratio of these two becomes

$$\frac{\gamma_{ZF}}{D_{ZF}} \sim \frac{1}{\delta} \left( \frac{p}{k_0} \right) \left( \frac{\rho_s c_s}{L_n v_{ZF}} \right) \sim \frac{p}{k_0 \delta}. \quad (26)$$

Thus, near marginality where  $\delta < p/k_0$ , the momentum transport by KH (or damping of zonal flows) is insignificant compared to the growth rate of zonal flows. This is consistent with the result obtained in Sec. II B.

On the other hand, Eq. (26) indicates that the damping of the zonal flow by linear KH instability may be important away from the marginality. However, this effect is likely to be weakened by Landau damping which damps KH modes, but not zonal flows. Furthermore, in this region, the nonlinear generation of GKH must be incorporated (see Sec. II B).

### III. THE EFFECT OF ION TEMPERATURE

We extend the simple model in Sec. II by including the ion temperature fluctuations. As is well known, ion temperature fluctuations lead to ITG modes, which can become unstable due to the gradient of background ion temperature. To keep the analysis tractable, we choose a simple toroidal ITG model in two dimensions. In this model, the instability of DW originates from the bad curvature where  $\nabla p \cdot \nabla B > 0$ .

The main governing equations for large-scale electric potential  $\phi$  and temperature  $T$  with  $k_{\parallel} = 0$  can be obtained by taking moments of the gyrokinetic equations for ions and drift equations for electrons and then using quasineutrality. By keeping the FLR effect for  $\phi$  to first order in  $(\rho_i^2 k^2) \ll 1$ , we can obtain

$$\begin{aligned} \partial_t \nabla_{\perp}^2 \phi + [\phi + \tau T, \nabla_{\perp}^2 \phi] - \tau [\partial_i \phi, \partial_i T] \\ = -\langle [\phi' + \tau p', \nabla_{\perp}^2 \phi'] \rangle + \tau \langle [\partial_i \phi', \partial_i p'] \rangle, \end{aligned} \quad (27)$$

$$\partial_t T + [\phi, T] + \langle [\phi', p'] \rangle = 0. \quad (28)$$

Here,  $\tau = T_{i0}/T_{e0}$ ;  $\phi'$  and  $p'$  are perturbations associated with DW, and angular brackets denote the average over DW;  $\phi = \phi_{ZF} + \phi_{KH}$  and  $T = T_{ZF} + T_{KH}$ , where  $T_{ZF}$  and  $T_{KH}$  are temperature of zonal flow and GKH; square brackets denote Poisson brackets, i.e.,  $[A, B] = \partial_x A \partial_y B - \partial_y A \partial_x B$ . In Eqs. (27) and (28), ion temperature is measured in unit of  $T_{i0}$ , and  $p$  is measured in unit of  $p_{i0}$ . Note that when  $\tau$  is small ( $< 1$ ), Eq. (27) states the conservation of total potential vorticity,  $\nabla_{\perp}^2(\phi + \tau T/2)$ , to first order in  $\tau$ . Compared to Eq. (1), Eq. (27) contains additional nonlinear terms due to pressure fluctuations of DW (i.e., diamagnetic effects). In principle, there should also be a drift term due to magnetic curvature and gradient in Eq. (27) for KH mode ( $\partial_y \neq 0$ ), which was neglected. The ballooning effect due to this term cannot be treated in the framework of the local analysis employed in the present paper and will be addressed in future works. Furthermore, the density fluctuation of KH is neglected for simplicity, i.e.,  $n_{i0} T \approx p$ . Note that the density fluctuation of zonal flow is very small ( $\sim O(\rho_s^2 k^2)$ ).

In the following, we shall first look at the linear dispersion relation for DW. We then examine the effect of  $\tau$  on the generation of zonal flows, while establishing the phase relation between zonal flow and temperature. Linear and nonlinear growth rates of GKH shall be revisited by incorporating  $\tau \neq 0$ .

### A. Linear dispersion

In this model, the parallel dynamics is not critical for the instability of DW. Thus, it is used only to ensure the adiabaticity of electrons for DW;  $k_{\parallel} \approx 0$  is assumed otherwise.

The governing equation for DW is obtained from the gyro-kinetic equation for ions by using adiabatic electron response as

$$\begin{aligned} \partial_t(1 - \nabla_{\perp}^2)\phi' - [\phi, \phi'] - [\phi + \tau p, \nabla_{\perp}^2\phi'] + \tau[\partial_i\phi, \partial_i p] \\ + v_*[1 - 2\epsilon_n + \tau(1 + \eta_i)\nabla^2]\partial_y\phi' - 2\epsilon_n v_*\tau\partial_y p' = 0, \end{aligned} \quad (29)$$

$$\partial_t p' + [\phi, p] + v_*(1 + \eta_i)\partial_y\phi' = 0. \quad (30)$$

Here,  $\epsilon_n = L_n/R$ ,  $\eta_i = L_n/L_T$ ,  $L_n = -(\partial_x \ln n_0)^{-1}$ ,  $R = -(\partial_x \ln B_0)^{-1}$ ,  $L_T = -(\partial_x \ln T_{i0})^{-1}$ , and  $v_* = \rho_s/L_n$ .  $\phi'$  and  $p'$  are electric potential and pressure perturbation associated with DW while  $\phi$  and  $p$  contain contributions from all three components of zonal flow, GKH, and DW. The linear part of Eqs. (29) and (30) leads to the following dispersion relation:

$$\begin{aligned} \omega_T = \frac{v_* k_y}{2(1+k^2)} [1 - 2\epsilon_n - \bar{\tau}k^2 \pm [(1 - 2\epsilon_n - \bar{\tau}k^2)^2 \\ - 8\bar{\tau}\epsilon_n(1+k^2)]^{1/2}], \end{aligned} \quad (31)$$

where  $\omega_T = \omega + i\gamma$  is the total frequency of DW and  $\bar{\tau} = \tau(1 + \eta_i)$ . As expected, this dispersion relation manifests that the instability sets in only when  $\epsilon_n \propto (\partial_x p_{i0})(\partial_x B) > 0$ . The group velocity follows from Eq. (31) as

$$v_{gx} = -\frac{v_* k_x k_y}{(1+k^2)^2} [1 - 2\epsilon_n + \bar{\tau}]. \quad (32)$$

### B. Zonal flow and temperature

In this section, we revisit modulational instability for the generation of both zonal flow and temperature, by including the effect of finite ion temperature ( $\tau \neq 0$ ). As we are in a strong turbulence regime, we envision that zonal flows and temperatures are continuously generated by DW, and thus ignore the transit time damping<sup>20</sup> in establishing phase relation between zonal flows and temperatures. We note that in a pure flow regime (with no collisional damping and no modulational drive), the transit time damping may be critical in determining the phase between zonal flow and temperature, since in this limit, the system chooses the phase between the two to minimize linear transit time damping. However, this is unlikely true in a more general case where a small but realistic collisional damping of zonal flow is present, and when finite amplitude drift wave turbulence is actively exciting the zonal flow. It is because collisional damping, even if very weak, will damp out even residual flows which survive the transient time damping.<sup>7</sup> Thus, in strong turbulence regimes, the phase between zonal flow and temperature is dynamically determined, as shall be shown.

By neglecting the contribution from linear KH, we rewrite Eqs. (27) and (28) as

$$\partial_t \nabla_{\perp}^2 \phi_{ZF} = -\langle [\phi' + \tau p', \nabla_{\perp}^2 \phi'] \rangle + \tau \langle [\partial_i \phi', \partial_i p'] \rangle, \quad (33)$$

$$\partial_t T_{ZF} = -\langle [\phi', p'] \rangle. \quad (34)$$

To compute the modulation of stresses on the right-hand sides of Eqs. (33) and (34), we use the wave-kinetic equation (3) for wave quanta density  $N_k = (1+k^2)^2 |\phi'_k|^2$ . Note that  $N_k$  does not involve the contribution from the pressure perturbation. This is because in two dimensions (with  $v_{\parallel} = 0$ ), both potential enstrophy and pressure are conserved separately,<sup>21</sup> allowing us to use the conservation of  $N_k = (1+k^2)^2 |\phi'_k|^2$  (cf. Ref. 22). Consequently, the shearing by zonal flows, which increases the  $k_x$  of DW (i.e.,  $N_k$ ), is explicitly incorporated in Eq. (3), while the effect of growing zonal temperature appears only implicitly through the frequency  $\omega$  and  $\gamma$  there. With growing zonal temperature  $T_{ZF}$ , both  $\omega$  and  $\gamma$  are modulated. However, since  $\gamma < \omega$  near marginality, only the modulation of  $\omega$  will be incorporated as

$$\delta\omega = \xi k_y \partial_x T_{ZF}, \quad (35)$$

where  $\xi = \tau k^2 / 2(1+k^2)$ .

The pressure perturbation  $p'$  appearing in Eqs. (27) and (28) is calculated by using the linear response from Eq. (30) as

$$p' = -\frac{i}{\omega_T} v_* (1 + \eta_i) \partial_y \phi', \quad (36)$$

where  $\omega_T$  is given by Eq. (31). By using this relation, it can easily be shown that the two terms containing  $p'$  on the right-hand side of Eq. (33) cancel each other. Therefore, the zonal flow is driven by the Reynolds stress term alone by Eq. (12), similar to the case with  $\tau = 0$ . On the other hand, by using Eq. (36), the thermal flux in Eq. (34) reduces to



$$\langle [\phi', p'] \rangle = v_*(1 + \eta_i) \partial_x \int d^2k \frac{\gamma k_y^2}{\omega^2 + \gamma^2} \frac{\tilde{N}_k}{(1 + k^2)^2}. \quad (37)$$

We incorporate the modulation of  $\tilde{N}_k$  by  $\phi_{ZF}$  and  $T_{ZF}$  as  $\tilde{N}_k = (\delta\tilde{N}_k/\delta\phi_{ZF})\tilde{N}_k + (\delta\tilde{N}_k/\delta T_{ZF})\tilde{N}_k$ . Then, by assuming  $\tilde{N}_k \propto \exp\{-i(\Omega t - \bar{p}x)\}$  and using Eq. (35), Eq. (3) gives us

$$\frac{\delta\tilde{N}_k}{\delta\phi_{ZF}} = i\Gamma, \quad \frac{\delta\tilde{N}_k}{\delta T_{ZF}} = i\xi\Gamma, \quad (38)$$

where

$$\Gamma = -\frac{\bar{p}^2 k_y}{\Omega - \bar{p}v_{gx} + i\gamma} \frac{\partial \langle N_k \rangle}{\partial k_x}.$$

By using Eq. (38) in Eqs. (33) and (34), together with  $\phi_{ZF} = \bar{\phi}_{ZF} \exp\{-i(\Omega t - \bar{p}x)\}$  and  $T_{ZF} = \bar{T}_{ZF} \exp\{-i(\Omega t - \bar{p}x)\}$ , we obtain

$$\Omega \bar{\phi}_{ZF} = -\Sigma_1 \Gamma (\bar{\phi}_{ZF} + \xi \bar{T}_{ZF}), \quad (39)$$

$$\Omega \bar{T}_{ZF} = i\bar{p}v_p \Sigma_2 \Gamma (\bar{\phi}_{ZF} + \xi \bar{T}_{ZF}). \quad (40)$$

Here,  $v_p = v_*(1 + \eta_i)$  and  $\Sigma_1$  and  $\Sigma_2$  are integral operators:

$$\Sigma_1 = \int d^2k \frac{k_x k_y}{(1 + k^2)^2},$$

$$\Sigma_2 = \int d^2k \frac{\gamma}{\omega^2 + \gamma^2} \frac{k_y^2}{(1 + k^2)^2}.$$

The coupled equations (39) and (40) are easily solved by dividing Eq. (40) by (39) to obtain

$$\frac{\bar{T}_{ZF}}{\bar{\phi}_{ZF}} = -i v_p \bar{p} \frac{\Sigma_2 \Gamma}{\Sigma_1 \Gamma}. \quad (41)$$

The previous equation establishes the phase relation between zonal flows and temperature as follows. In the presence of reflectional symmetry of DW turbulence in  $y$ , the real part of  $\Sigma_1 \Gamma$  and imaginary part of  $\Sigma_2 \Gamma$  vanish. Thus, Eq. (41) becomes

$$\begin{aligned} \frac{\bar{T}_{ZF}}{\bar{\phi}_{ZF}} &= -v_p \bar{p} \frac{(\Sigma_2 \Gamma)_R}{(\Sigma_1 \Gamma)_I} \\ &\sim v_*^2 (1 + \eta_i) (1 - 2\epsilon_n + \bar{\tau}) \bar{p}^2 \frac{k_{0y}^2}{\omega^2 + \gamma^2}, \end{aligned} \quad (42)$$

where the subscripts  $R$  and  $I$  denote real and imaginary parts, respectively. After restoring dimensions, this leads to the estimate  $\bar{T}_{ZF}/e\bar{\phi}_{ZF} \sim \tau(1 + \eta_i)(1 - 2\epsilon_n + \bar{\tau})(\rho_s \bar{p})^2$ . Furthermore, Eq. (42) reveals that the phase relation between zonal flow and temperature is likely positive (as  $\epsilon_n < 1/2$  in most cases) when the underlying DW turbulence is reflectionally symmetric in  $y$ . This positive phase was obtained self-consistently by considering modulational instability of zonal flow and temperature, and is one of our main results.

Finally, the substitution of Eq. (41) in Eq. (39) then gives us the frequency of modulational instability as

$$\Omega = -\Sigma_1 \Gamma \left[ 1 - \frac{i\xi v_p \bar{p} \Sigma_2 \Gamma}{\Sigma_1 \Gamma} \right] \sim -\Sigma_1 \Gamma \left[ 1 + \xi \frac{\bar{T}_{ZF}}{\bar{\phi}_{ZF}} \right]. \quad (43)$$

As previously shown, for  $\langle N(-k_y) \rangle = \langle N(k_y) \rangle$ ,  $\bar{T}_{ZF}/\bar{\phi}_{ZF}$  is real and positive while  $-(\Sigma_1 \Gamma)_I > 0$  is the growth rate of zonal flows with cold ions. Therefore, the effect of ion temperature fluctuation appears to increase the growth rate of modulational instability, even if by only a small amount.

The zonal temperature provides the extra source of free energy for KH mode in addition to zonal flow. Thus, the phase relation between zonal flow and temperature obtained here is critical in determining whether finite ion temperature promotes or inhibits linear KH instability. This issue will be addressed in the next section.

### C. Linear KH instability with $\tau \neq 0$

As mentioned in Sec. III B, the zonal temperature provides the extra source of free energy for the instability of linear KH mode. To see this clearly, it is advantageous to recast Eqs. (27) and (28) in terms of  $\psi = \phi + \tau T/2$  with  $(NL) = 0$ , as follows:

$$\partial_t \nabla_\perp^2 \psi + [\psi, \nabla_\perp^2 \psi] = \frac{\tau^2}{4} [T, \nabla_\perp^2 T], \quad (44)$$

$$\partial_t T + [\psi, T] = 0. \quad (45)$$

For  $\tau \ll 1$ , Eq. (44) becomes identical to Eq. (1) up to first order in  $\tau$ , implying the conservation of total potential vorticity  $\nabla_\perp^2 (\phi + \tau T/2)$  to that order. Note that in this case  $T_{KH}$  is weakly coupled to  $\psi_{KH}$ . Consequently, if  $\tau \ll 1$ , the leading order term in the linear KH growth rate is simply given by Eqs. (7)–(9) with  $\bar{\phi} \rightarrow \bar{\phi} + \tau \bar{T}/2$  when the zonal temperature has the same spatial variation as zonal flow in Eq. (5), i.e.,  $T_{ZF} = \bar{T} \cos \bar{p}x$ . For instance, Eq. (8) becomes ( $p_0 = 0$ )

$$\gamma_L^2 = \frac{1}{2} (\bar{\phi} + \tau \bar{T}/2)^2 \bar{p}^2 q^2 \frac{\bar{p}^2 - q^2}{\bar{p}^2 + q^2}. \quad (46)$$

Without having to repeat a similar analysis to that in Sec. II, Eq. (46) already implies one important effect of  $\tau$ . That is, since both zonal flow and temperature appear together as a simple sum in growth rate (46), the phase relation between the two is critical in determining the effect of  $\tau$  on instability. *As shown in Sec. III B, this phase is a dynamical quantity, determined by the generation of zonal flow and temperature by modulational instability (or more generally, the zonal flow generation mechanism), and thus cannot be treated as a free parameter.* Since the analysis in Sec. III B suggests that zonal flow and temperature are likely to have the same phase when  $\langle N_k(k_x, -k_y) \rangle = \langle N_k(k_x, k_y) \rangle$ , the ion temperature fluctuation ( $\tau \neq 0$ ) enhances the growth rate of the unstable mode (for  $\bar{p} > q$ ). On the other hand, stable modes (for  $\bar{p} < q$ ) remain stable. Note, however, that Eq. (42) implies that  $T_{ZF}/\phi_{ZF} \propto \tau k_y^2 \approx (\rho_i/k)^2 < 1$ . Thus, the correction due to  $\tau$  is small.

## D. Nonlinear generation of GKH with $\tau \neq 0$

In this section, we demonstrate that the effect of  $\tau$  on nonlinear generation GKH appears only as a small correction of order  $\tau k^2 = \rho_i^2 k^2 \ll 1$ . This will be sufficient to justify the order of estimates in  $\gamma_{NL}$  and  $\gamma_L$  in Sec. II.

As GKH modes are generated by the nonaxisymmetric part of DW spectrum, they satisfy the following:

$$\partial_t \nabla_{\perp}^2 \phi_{KH} = -\langle [\phi' + \tau p', \nabla_{\perp}^2 \phi'] \rangle_{NA} + \tau \langle [\partial_i \phi', \partial_i p'] \rangle_{NA}, \quad (47)$$

$$\partial_t T_{KH} = -\langle [\phi', p'] \rangle_{NA}, \quad (48)$$

where the subscript NA denotes the nonaxisymmetric part. To compute the stresses and thermal flux, we again use the quasilinear response of  $p'$  given by Eq. (36). Then, the two terms containing  $p'$  can be shown to cancel out each other. Thus,  $\phi_{KH}$  is driven by Reynolds stress alone, while  $T_{KH}$  is by thermal flux. Since  $T_{KH}$  is weakly coupled to  $\phi_{KH}$  only through the modulation of DW frequency (35), the effect of  $\tau$  is bound to appear in  $\gamma_{NL}$  as a small correction of order  $\tau k^2 = (\rho_i k)^2$ . Note that this is similar to the case for  $\phi_{ZF}$  and  $T_{ZF}$ . Thus, the effect of finite ion temperature would not fundamentally change the conclusions based upon estimates obtained in Sec. II.

## IV. CONCLUSIONS

In view of the crucial role that zonal flows play in regulating turbulent transport, the understanding of the nature of zonal flow damping remains as a critical issue. In particular, as zonal flows do not undergo Landau damping (unlike streamers), a detailed study of the saturation of collisionless zonal flows is especially important in quantifying the dynamics of a coupled system of DW turbulence and zonal flows. One of the possible mechanisms for collisionless saturation of zonal flows is linear KH instability. Seemingly plausible, this mechanism was shown to be ineffective in damping zonal flows for realistic tokamak parameters, i.e., magnetic shear.<sup>12</sup> The recent work by Rogers *et al.*,<sup>9</sup> however, revived the interest in this instability, suggesting its potential importance in the pure flow (Dimits upshift) regime near marginal stability of DW turbulence. While their linear analysis (neglecting the background DW turbulence and treating zonal flow as a fixed background) is valid in a pure flow regime, it needs be generalized outside this regime. This is important as a pure flow regime is atypical and can easily be eliminated by a weak collisional damping.

In this paper, we generalize KH instability in a strong turbulence regime, by taking into account the DW turbulence background. In the background of DW, GKH modes are nonlinearly excited by modulational instability of DW turbulence. This nonlinear excitation of GKH is due to the nonaxisymmetric modulation of DW spectrum. In particular, we focused on: (i) the comparison between the linear and nonlinear growth rates of GKH to determine when nonlinear generation of GKH can be neglected, (ii) the comparison between the linear generation of GKH with the growth of zonal flows to see whether the linear picture of KH instability, which assumes a fixed background zonal flow, can be

justified, and (iii) the momentum transport by the linear KH to compute the damping of zonal flow, which is then compared with the growth of zonal flow.

First, through a detailed study of a simple model with cold ions, we have shown that: (1) the nonlinear generation of zonal flow can be more effective than both linear and nonlinear generation of GKH near marginality for  $\delta < q/k_0$  ( $\delta = \gamma/\omega$ ); (2) the nonlinear generation of GKH can be comparable to its linear generation away from marginality  $\delta > p/k_0$ ; (3) near marginality where  $\delta < p/k_0$ , the momentum transport by KH (or damping of zonal flows) is insignificant compared to the nonlinear growth of zonal flows.

These findings imply the following. (i) The linear analysis of KH, which treats a zonal flow as an equilibrium background for the evolution of GKH, is invalid near marginal stability, where DW turbulence is strong. (ii) There is no clear distinction between secondary (zonal flow) and tertiary mode (KH). (iii) The excitation of GKH modes with the amplitude comparable to zonal flow away from marginality may contribute to  $\chi_i$ , if GKH modes are not efficiently damped via Landau damping, etc. (iv) Considering that GKH is more subject to damping than zonal flows (due to the presence of Landau damping and magnetic shear), it is possible that zonal flows dominate over GKH, even away from marginality.

By extending this simple model to the two-dimensional toroidal ITG to incorporate finite ion temperature fluctuations, zonal temperature as well as zonal flow are shown to be generated simultaneously by modulational instability. The phase between the two is set dynamically, and was found to be positive for DW turbulence with a reflectional symmetry in the poloidal direction. Furthermore, the effect of ion temperature appears as a small correction of order  $O(\rho_i^2 k^2)$  in nonlinear growth rate of GKH, and thus does not qualitatively change the estimates obtained from the simple model with cold ions. However, although small, it enhances the growth rate of the zonal flow.

We note that, conventionally, the dynamics of DW wave turbulence has been described by small-scale ITG modes and large-scale zonal flows or streamers (radially extended and poloidally localized structures). Here, ITG modes with finite  $m$  and  $k_{\parallel}$  are responsible for transport while zonal flows with  $m = k_{\parallel} = 0$  suppress it. However, in general, GKH modes with finite  $m$  but with  $k_{\parallel} = 0$  should also be taken into account. These GKH modes seem interesting, being generated by small-scale ITG modes, but also then contributing to transport. In this sense, they are somewhat similar to streamers, which can also be excited nonlinearly and then play an important role in radial transport. Recall that for various estimates in the paper, GKH modes are assumed to be nonlinearly generated by nonresonant interaction only. As GKH modes have finite frequency, their resonant generation may be significant, as in the case of streamers. Thus, it is plausible that GKH modes (with comparable wavelength in both radial and poloidal directions, for instance) may constitute a large-scale nonlinear structure in ITG turbulence, in addition to zonal flows and streamers. The crucial question would then be what structure is selected in what circumstances and also how they interact with each other.

In conclusion, the role of shear flow instability in collisionless zonal flow saturation seems quite unclear. Perhaps, the nonlinear spectral feedback may be more robust in saturating zonal flows,<sup>5</sup> and should be investigated in more detail. Note that nonlinear spectral feedback is likely to result in  $\langle N_k \rangle < (kL_n)^2$ , thus reducing  $\gamma_{ZF}/\gamma_L$ . Furthermore, given the complexity of the problem, involving linear and nonlinear aspects of GKH modes and the interplay among DW, zonal flows, and GKH, further studies of the following issues would be worthwhile. First, the distinction between linear and nonlinear generation of GKH modes observed in computer simulations may be made possible by using bispectral analysis which has proved to be successful in capturing mode couplings between zonal flows and DW.<sup>23</sup> Second, a rigorous study on the linear KH instability is necessary, including realistic profiles of zonal flows. Also, the momentum transport by linear KH modes and DW should also be computed by using these realistic profiles. Third, although DWs are thought to be almost completely damped in the pure flow (Dimits shift) regime near marginality, a recent study indicates that the amplitude of DW is not negligible compared to GKH modes.<sup>11</sup> Thus, one should precisely quantify how weak the amplitude of DWs is, in this regime. Fourth, nonlinear shear instability and the effect of high order perturbation should also be incorporated. Finally, the ballooning effect due to varying  $\epsilon_n$ , which was neglected in this work, may have a potentially important effect on GKH mode, and should be investigated in detail.

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## APPENDIX: DERIVATION OF EQS. (23) AND (24)

As  $\phi_{ZF} = \bar{\phi}_{ZF} \cos \bar{p}x = (\bar{\phi}_0/2)(e^{\bar{p}x} + e^{-\bar{p}x})$ , we need to compute both of the following:

$$I_1 = \langle v_{KH,x} \delta \nabla^2 \phi_{KH} + \delta v_{KH,x} \nabla^2 \phi_{KH} \rangle |_{\bar{p},q=0}, \quad (\text{A1})$$

$$I_2 = \langle v_{KH,x} \delta \nabla^2 \phi_{KH} + \delta v_{KH,x} \nabla^2 \phi_{KH} \rangle |_{-\bar{p},q=0}.$$

To compute these, we express both Eq. (A1) and Eq. (22) in Fourier space as:

$$2I_1 = i \int dp dq q [\bar{p}^2 - 2p\bar{p}] \tilde{\phi}(p,q) \delta \tilde{\phi}(\bar{p}-p,-q), \quad (\text{A2})$$

$$2I_2 = i \int dp dq q [\bar{p}^2 + 2p\bar{p}] \tilde{\phi}(p,q) \delta \tilde{\phi}(-\bar{p}-p,-q),$$

$$\begin{aligned} \delta \tilde{\phi}_{KH}(\bar{p}-p,-q) &= \frac{\tau(\bar{p}-p,-q)}{(\bar{p}-p)^2 + q^2} [\widetilde{v_{ZF}}(\bar{p}) \partial_y \widetilde{\nabla^2 \phi_{KH}}(-p,-q) \\ &\quad + \widetilde{v_x}(-p,-q) \partial_x \widetilde{\nabla^2 \phi_0}(\bar{p})], \quad (\text{A3}) \end{aligned}$$

where a tilde denotes the Fourier transform. By using Eq. (A3) in (A2), we obtain

$$\begin{aligned} 2I_1 &= - \int dp dq [\bar{p}^2 - 2p\bar{p}] |v_{KH,x}(p,q)|^2 \\ &\quad \times \frac{\tau(\bar{p}-p,-q)}{(\bar{p}-p)^2 + q^2} [(p^2 + q^2) \widetilde{v_{ZF}}(\bar{p}) + \partial_x \widetilde{\nabla^2 \phi_{ZF}}(\bar{p})], \quad (\text{A4}) \end{aligned}$$

$$\begin{aligned} 2I_2 &= - \int dp dq [\bar{p}^2 + 2p\bar{p}] |v_{KH,x}(p,q)|^2 \\ &\quad \times \frac{\tau(-\bar{p}-p,-q)}{(\bar{p}+p)^2 + q^2} [(p^2 + q^2) \widetilde{v_{ZF}}(-\bar{p}) \\ &\quad + \partial_x \widetilde{\nabla^2 \phi_{ZF}}(-\bar{p})]. \end{aligned}$$

To compute  $I_1$  and  $I_2$ , we replace  $\int dp = \sum_{n=-1}^{n=1}$  with  $p = n\bar{p}$ , and then use Eq. (8), i.e.,  $\tau(p,q) = 1/\gamma_L(p,q) \sim \sqrt{(\bar{p}^2 + q^2)/(\bar{p}^2 - q^2)}/|qv_{ZF}|$ . As  $\bar{p} > q$  for instability, we keep the terms to leading order in  $q/\bar{p}$  to obtain Eqs. (23) and (24).

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