Title

Weak Variations Optimal Boundary Control of Hyperbolic PDEs with Application to Traffic Flow and Delay Systems

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Abstract— **We investigate optimal boundary control of first-order hyperbolic PDEs. These equations are ubiquitous in engineered systems, such as traffic flows, fluid flows, heat exchangers, chemical reactors, and oil production systems. We derive linear quadratic regulator (LQR) results using a weak variations approach, recently developed for parabolic PDEs. The distinguishing characteristic of this approach is that it provides a systematic procedure for deriving LQR control laws without semi-group theoretic concepts. Ultimately, these control laws are given by the solution of an associated Riccati PDE. We demonstrate the applicability of these results on two case studies: traffic flow control and input-delayed systems. Finally, we extend the LQR results to solve the output reference tracking problem. Unlike motion planning, these reference tracking equations do not require state trajectory generation.**

I. Introduction

This paper develops linear quadratic regulator (LQR) results for boundary controlled first-order hyperbolic partial differential equations (PDEs). These equations describe several physical problems of interest, including traffic flows [1], shallow water flow dynamics [2], heat exchangers [3], chemical reactors [4], oil production systems [5], thermostatically controlled loads [6], [7], and as we shall see, input-delayed systems [8]. In addition to deriving LQR results, we seek a constructive method which is easily applicable and generalizable to physically relevant engineering systems. To this end, we consider the weak-variations approach, recently developed for parabolic PDEs [9], [10].

Several results already exist for hyperbolic equations, including [8], [11]–[13], which utilize concepts from geometic control, Riemann invariants, semi-group theory, and infinite-dimensional backstepping. The current work focuses on a weak-variations approach to deriving LQR results for first-order hyperbolic PDEs. This approach has the unique advantage of providing a constructive approach to deriving Riccati equations without approximating the system as finite-dimensional, while requiring relatively simple mathematical concepts. Ultimately, the control laws require the solution of Riccati PDEs, derived from the original model. We first focus on finite-time LQR results for a general class of hyperbolic PDEs. Both open-loop and closed-loop control laws are provided. Second, we apply these results to a standard problem in traffic flow control. Third, the results are extended to solve the stabilization problem in input-delayed finite-dimensional systems. Finally, we provide an output reference tracking algorithm for hyperbolic PDEs. Although several of these results are known, our approach provides a general and systematic procedure to handle linear quadratic regulation in hyperbolic equations, without discretization or semi-group theoretic concepts.

Throughout this paper, we consider models from the following class of linear hyperbolic PDEs:

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy \quad (1)$$

defined over the domain $(x,t) \in (0,1) \times \mathbb{R}^+$ with initial condition $u(x,0) = u_0(x)$. Assume the functions g, f are continuous. We consider the controllable boundary condition

$$u(1,t) = U(t). \quad (2)$$

We consider the particular class of systems represented by (1)-(2) for two reasons. First, this system is unstable for sufficiently large and positive g and f. Secondly, this model often arises from the reduced model of a singularly perturbed hyperbolic-parabolic system [14].

Our goal is to develop a state-feedback controller that optimally regulates the system to the origin. Specifically, we wish to minimize the following quadratic objective over a finite time-horizon $t \in [0, t_f]$:

$$J = \frac{1}{2}\int_0^{t_f} \left[\langle u(x,t), Q(u(x,t))\rangle + RU^2(t)\right] dt + \frac{1}{2}\langle u(x,t_f), P_f(u(x,t_f))\rangle. \quad (3)$$

The symbols Q, R, and P_f are weighting kernels that respectively weight the state, control, and terminal state of the closed loop system. We assume that $Q \geq 0$, $R > 0$, $P_f \geq 0$, where $Q, P_f \in \mathcal{C}([0,1] \times [0,1])$ and $R \in \mathbb{R}$, thus producing a convex cost functional. First, we derive the necessary conditions for optimality of the open-loop finite-horizon control problem using weak variations. These conditions form coupled PDEs with split initial conditions. Next, we derive the associated Riccati equation for the feedback linear operator. This Riccati equation is a 2-D spatial, 1-D temporal PDE. We then consider applications to traffic flows and input-delayed systems. Finally, we extend the LQR results to solve the output reference tracking problem.

This paper is organized as follows. Section II provides the main LQR results for a general class of hyperbolic PDEs. Section III considers the application of these results to control traffic flow. Section IV demonstrates how these results can be used to stabilize input-delayed systems. Section V provides results for the output reference tracking problem,

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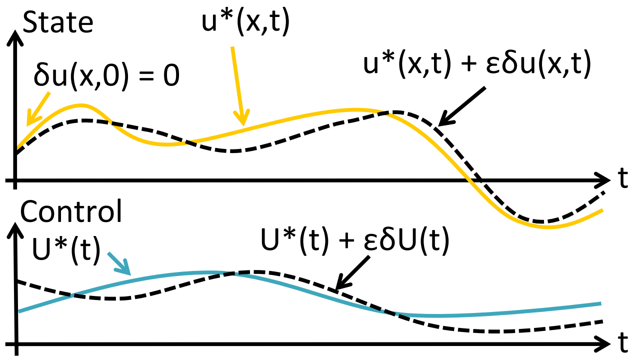

Fig. 1. A visualization of the weak variations concept for optimal state and control trajectories.

for hyperbolic PDEs. Finally, Section VI summarizes the key results.

II. LQR for First-Order Hyperbolic PDEs

A. Open-Loop Control

We start by stating the first order necessary conditions for the open loop finite-time horizon problem.

Lemma 1: Consider the linear first-order hyperbolic PDE described by (1)-(2) defined on the finite-time horizon $t \in [0, t_f]$ with quadratic cost criterion (3). Let $u^*(x,t)$, $U^*(t)$, and $\lambda(x,t)$ respectively denote the optimal state, control, and co-state that minimize the quadratic cost. Then the first order necessary conditions for optimality are

$$u_t^*(x,t) = u_x^*(x,t) + g(x)u^*(0,t) + \int_0^x f(x,y)u^*(y,t)dy, \quad (4)$$

$$\lambda_t(x,t) = \lambda_x(x,t) - \int_x^1 \lambda(\xi,t)f(\xi,x)d\xi - Q(u^*(x,t)) \quad (5)$$

with boundary conditions

$$u^*(1,t) = U^*(t), \quad (6)$$

$$\lambda(0,t) = \int_0^1 g(y)\lambda(y)dy, \quad (7)$$

and split initial/final conditions

$$u^*(x,0) = u_0(x), \qquad \lambda(x,t_f) = P_f(u^*(x,t_f)), \quad (8)$$

and the optimal control input is

$$U^*(t) = -\frac{1}{R}\lambda(1,t). \quad (9)$$

Proof: The necessary conditions are derived via weak variations [9], [10]. Suppose $u^*(x,t)$ and $U^*(t)$ are the optimal state and control inputs. Let $u(x,t) = u^*(x,t) + \epsilon\delta u(x,t)$, $U(t) = U^*(t) + \epsilon\delta U(t)$ and $\delta u(x,0) = 0$ represent perturbations from the optimal solutions. See Fig. 1 for a visualization of the weak variations concept. Consequently, the cost is

$$J(u^* + \epsilon\delta u, U^* + \epsilon\delta U) = \frac{1}{2}\int_0^{t_f} \left[\langle u^* + \epsilon\delta u, Q(u^* + \epsilon\delta u)\rangle + R(U^* + \epsilon\delta U)^2\right] dt + \frac{1}{2}\langle u^*(t_f) + \epsilon\delta u(t_f), P_f(u^*(t_f) + \epsilon\delta u(t_f))\rangle. \quad (10)$$

Define $g(\varepsilon)$ to be the cost functional above combined with the system dynamics constraint (1), using the method of Lagrange multipliers as follows

$$g(\epsilon) := \frac{1}{2}\int_0^{t_f} \left[\langle u^* + \epsilon\delta u, Q(u^* + \epsilon\delta u)\rangle + R(U^* + \epsilon\delta U)^2\right] dt + \frac{1}{2}\langle u^*(t_f) + \epsilon\delta u(t_f), P_f(u^*(t_f) + \epsilon\delta u(t_f))\rangle + \int_0^{t_f} \langle \lambda(x,t), u_x^* + \epsilon\delta u_x + g(x)u^*(0) + g(x)\epsilon\delta u(0) + \int_0^x f(x,y)\left[u^*(y) + \epsilon\delta u(y)\right] dy - \frac{\partial}{\partial t}(u^* + \epsilon\delta u)\rangle dt, \quad (11)$$

where $\lambda(x,t)$ is the Lagrange multiplier (a.k.a. the co-state in the context of optimal control). Then the necessary condition for optimality is $dg(\epsilon)/d\epsilon|_{\epsilon=0} = 0$. Differentiating $g(\epsilon)$ and a series of computations involving integration by parts produces

$$\frac{dg}{d\epsilon}(0) = \int_0^{t_f} \left[\langle Q(u^*) - \lambda_x + \lambda_t + \int_x^1 \lambda(\xi)f(\xi,x)d\xi, \delta u\rangle\right] dt + \int_0^{t_f} \left[-\lambda(0,t) + \langle \lambda, g(x)\rangle\right] \delta u(0,t)dt + \int_0^{t_f} \left[RU^* + \lambda(1)\right] \delta U + \langle P_f(u^*(x,t_f)) - \lambda(x,t_f), \delta u(x,t_f)\rangle = 0. \quad (12)$$

For the previous equation to hold true for arbitrary $\delta u(x,t), \delta U(t), \delta u(x,t_f)$, the following conditions are sufficient

$$\lambda_t(x,t) = \lambda_x(x,t) - \int_x^1 \lambda(\xi)f(\xi,x)d\xi + Q(u^*(x,t)), \quad (13)$$

$$\lambda(0,t) = \int_0^1 g(y)\lambda(y)dy, \quad (14)$$

$$\lambda(x,t_f) = P_f(u^*(x,t_f)), \quad (15)$$

$$U^*(t) = -\frac{1}{R}\lambda(1,t). \quad (16)$$

These conditions represent the co-state's PDE dynamics, boundary condition, final condition, and the optimal boundary control, respectively. Coupled together with the plant model (1)-(2), these conditions verify the first order necessary conditions of optimality, which completes the proof. ∎

B. State-Feedback Control

Now let us consider the state-feedback problem. That is, let us postulate that the co-state λ is related to the states according to the time-varying linear transformation:

$$\lambda(x,t) = \int_0^1 P(x,y,t)u^*(y,t)dy. \tag{17}$$

Theorem 2: The optimal control in state-feedback form is

$$U^*(t) = -\frac{1}{R}\int_0^1 P(1,y,t)u^*(y,t)dy. \tag{18}$$

The time-varying kernel $P(x,y,t)$ must satisfy the following Riccati PDE

$$P_t = P_x + P_y - \int_x^1 P(\xi,y)f(\xi,x)d\xi - \int_y^1 P(x,\xi)f(\xi,y)d\xi + \frac{1}{R}P(x,1)P(1,y) - Q, \tag{19}$$

with boundary conditions

$$P(x,0,t) = \int_0^1 P(x,y,t)g(y)dy, \tag{20}$$

$$P(0,y,t) = \int_0^1 P(x,y,t)g(x)dx, \tag{21}$$

and final condition

$$P(x,y,t_f) = P_f(x,y). \tag{22}$$

Proof: The proof consists of evaluating each λ term in (5), (7), and (8) using the postulated form in (17). The computations involve integration by parts and charging the order of integration in double integrals. ■

Remark 3 (Time-Invariant Control Law): The infinite-time horizon optimal controller is given by the steady-state solution of the Riccati PDE. Namely,

$$P_x^\infty + P_y^\infty - \int_x^1 P(\xi,y)f(\xi,x)d\xi - \int_y^1 P(x,\xi)f(\xi,y)d\xi + \frac{1}{R}P(x,1)P(1,y) - Q = 0, \tag{23}$$

$$P^\infty(x,0) = \int_0^1 P^\infty(x,y)g(y)dy, \tag{24}$$

$$P^\infty(0,y) = \int_0^1 P^\infty(x,y)g(x)dx. \tag{25}$$

The solution of this Riccati PDE, denoted $P^\infty(x,y)$, produces the time-invariant state-feedback control law

$$U^*(t) = -\frac{1}{R}\int_0^1 P^\infty(1,y)u^*(y,t)dy \tag{26}$$

Next we consider an application of this result to a prototypical problem encountered in traffic flow control research.

III. Application to Traffic Flow Control

A. Model and LQR Control Design

Consider the modified Lighthill-Whitham-Richards (LWR) model of highway and air traffic flows [1]

$$\frac{\partial\rho}{\partial t}(\xi,t) = \frac{\partial}{\partial\xi}\left[v(\xi)\rho(\xi,t)\right], \tag{27}$$

$$v(L)\rho(L,t) = U(t) + d(t), \tag{28}$$

$$\rho(\xi,0) = \rho_0(\xi), \tag{29}$$

defined over the domain $(\xi,t) \in (0,L)\times\mathbb{R}^+$. The variable $\rho(\xi,t)$ is the density of vehicles, $v(\xi)$ is the spatially-dependent mean velocity profile, $U(t)$ is a controllable flux of vehicles, and $d(t)$ is an exogenous disturbance that models an uncontrollable flux of vehicles at the boundary. We assume $v(\xi) > 0\ \forall\xi$. Define the flux of vehicles $F(\xi,t) = v(\xi)\rho(\xi,t)$. Then the LWR model can be written as

$$\frac{\partial F}{\partial t}(\xi,t) = v(\xi)\frac{\partial F}{\partial\xi}(\xi,t), \tag{30}$$

$$F(L,t) = U(t) + d(t), \tag{31}$$

$$F(\xi,0) = v(\xi)\rho_0(\xi). \tag{32}$$

Suppose we wish to stabilize the traffic flow around the equilibrium flux of vehicles $F(\xi,t) = F^0$, $\forall\xi,t$. Define the error variable $\tilde{F}(\xi,t) = F(\xi,t) - F^0$. Then the PDE of interest becomes

$$\frac{\partial\tilde{F}}{\partial t}(\xi,t) = v(\xi)\frac{\partial\tilde{F}}{\partial\xi}(\xi,t), \tag{33}$$

$$\tilde{F}(L,t) = U(t) + d(t) - F^0. \tag{34}$$

Our goal is to design a feedback control law for the influx of vehicles $U(t)$ which regulates traffic flow to the desired equilibrium profile. We mathematically formulate this using the objective function (3).

First, we apply the following invertible transformation $\tilde{F}(\xi,t) \leftrightarrow u(x,t)$

$$\tilde{F}(\xi,t) = u(x,t), \tag{35}$$

$$x = v_0\int_0^\xi \frac{ds}{v(s)}, \tag{36}$$

$$v_0 = \left[\int_0^L \frac{ds}{v(s)}\right]^{-1} \tag{37}$$

and assume $v(\xi)$ and L are selected such that $v_0 = 1$. This renders the $\tilde{F}$ system (33)-(34) into the form

$$u_t(x,t) = u_x(x,t), \tag{38}$$

$$u(1,t) = U(t) + d(t) - F^0. \tag{39}$$

This PDE fits within the class of PDEs considered in (1)-(2). Hence, we are in position to apply the weak variation optimal control results.

The time-invariant optimal control law in the original coordinates is given by

$$U(t) = F^0 - \frac{1}{R}\int_0^1 P^\infty(1,y)\left[F\left(\int_0^y v(s)ds, t\right) - F^0\right]dy, \tag{40}$$

where $F(\xi,t) = v(\xi)\rho(\xi,t)$ and $P^\infty(x,y)$ verifies the Riccati PDE

$$P_x^\infty + P_y^\infty + \frac{1}{R}P^\infty(x,1)P^\infty(1,y) - Q = 0, \tag{41}$$

$$P^\infty(x,0) = 0, \tag{42}$$

$$P^\infty(y,0) = 0. \tag{43}$$

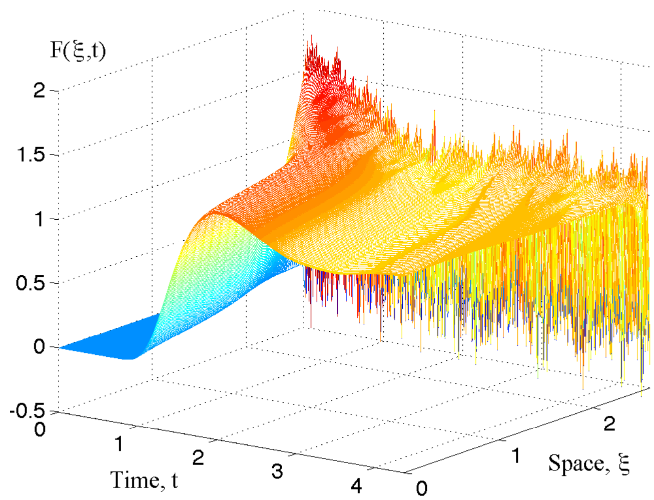

Fig. 2. Evolution of flux of vehicles, $F(\xi,t)$, regulated by the LQR controller (40). Note that $F(\xi,t)$ stabilizes around the desired value of $F^0 = 1$. A zero-mean normally distributed disturbance enters at the controlled boundary $F(L,t)$.

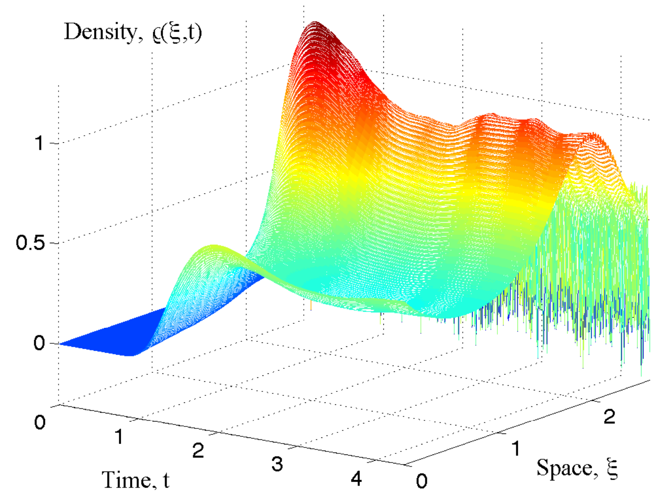

Fig. 3. Evolution of vehicle density, $\rho(\xi,t)$, regulated by the LQR controller (40). A zero-mean normally distributed disturbance enters at the controlled boundary $v(L)\rho(L,t)$.

B. Simulations

Next we present simulation examples for $v(\xi) = 2 + \sin(2\pi\frac{\xi}{L})$, $F^0 = 1$, $d(t) \sim \mathcal{N}[0, 0.2^2]$, $\rho_0(\xi) = 0$, $Q(x,y) = 1$, $R = 1$. All equations are discretized and solved numerically using the Lax-Friedrichs method [15]. We can see in Fig. 2 the controller stabilizes the flux $F(\xi,t)$ around the desired equilibrium profile of $F^0 = 1$. The evolution of the state, vehicle density $\rho(\xi,t)$, is provided in Fig. 3. The control gain $P^\infty(1,y)$, obtained from solving (41)-(43), is shown in Fig. 4(a). The controlled influx of vehicles at the boundary, $U(t)$, is shown in Fig. 4(b). Finally, the evolution of the spatial 2-norm of vehicle flux tracking error, $\|\tilde{F}(\xi,t)\|$, decays towards zero in Fig. 4(c).

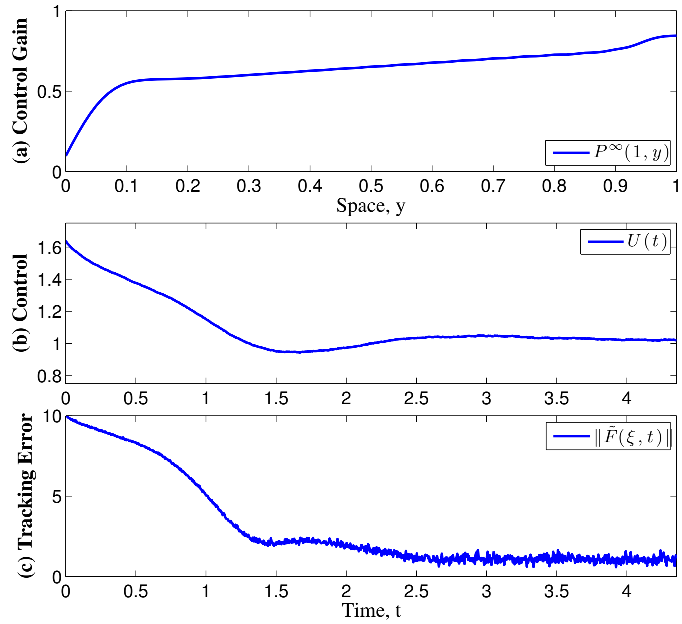

Fig. 4. (a) Control gain $P^\infty(1,y)$ for LQR control law in traffic flow problem. Plots (b) and (c) depict the evolution of the boundary control signal $U(t)$ and the spatial 2-norm of the flux tracking error $\|\tilde{F}(\xi,t)\|$, respectively. The LQR controller regulates tracking error to zero.

IV. Application to ODEs with Input Delay

We now consider the application of weak-variations optimal boundary control to ODEs with arbitrarily long actuator delay. In particular, consider a linear finite-dimensional system described by the ODE

$$\dot{X} = AX + BU(t-D), \tag{44}$$

where $X \in \mathbb{R}^n$, A is possibly non-Hurwitz, (A,B) is controllable, and the input signal $U(t)$ is delayed by a constant D units of time. Following the idea exploited in [8], [16] and demonstrated visually in Fig. 5, we model the delay as a first-order hyperbolic PDE

$$u_t(x,t) = u_x(x,t), \tag{45}$$
$$u(D,t) = U(t). \tag{46}$$

such that the output $u(0,t) = U(t-D)$ provides the delayed input to the ODE. We now write the ODE as

$$\dot{X} = AX + Bu(0,t). \tag{47}$$

Equations (45)-(47) form a PDE-ODE cascade driven by the input U. The key advantage of this representation is that the cascade is linear and amenable to our weak-variations optimal control techniques for PDEs.

A. State Regulation

We seek the optimal control which minimizes the following criterion

$$J = \frac{1}{2}\int_0^{t_f} \left[X^TQX + RU^2\right] dt + \frac{1}{2}X(t_f)^T P_f X(t_f), \tag{48}$$

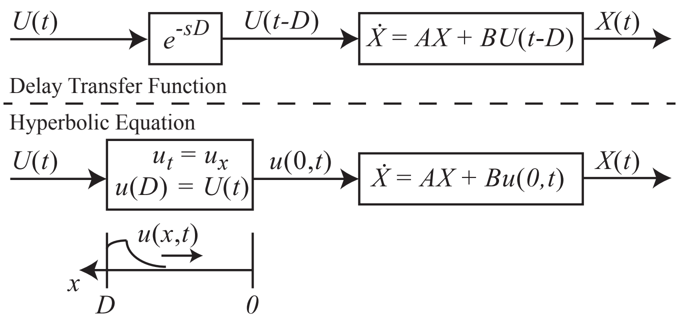

Fig. 5. Block diagrams for the two equivalent models of an actuator delayed system: the delay transfer function and a hyperbolic equation.

where $Q = Q^T > 0 \in \mathbb{R}^{n\times n}, R > 0 \in \mathbb{R}, P_f = P_f^T \geq 0 \in \mathbb{R}^{n\times n}$. First, we state the open-loop optimal control result.

Lemma 4: Consider the input delayed ODE system described by (45)-(47) defined on the finite-time horizon $t \in [0, t_f]$ with quadratic cost criterion (48). Let $X^*(t), u^*(x,t)$, and $U^*(t)$, respectively denote the optimal ODE state, actuator state, and control that minimize the quadratic cost. Moreover, let $\lambda(t)$ and $\mu(x,t)$ represent the co-states for $X(t)$ and $u(x,t)$, respectively. Then the first order necessary conditions for optimality are

$$\dot{X}^* = AX^* + Bu^*(0,t), \tag{49}$$
$$-\dot{\lambda} = A^T\lambda + QX^*, \tag{50}$$
$$X^*(0) = X_0, \quad \lambda(t_f) = P_f X^*(t_f), \tag{51}$$

for the ODE states. The PDE state equations are

$$u_t^* = u_x^*, \tag{52}$$
$$u^*(1,t) = U^*(t), \tag{53}$$
$$\mu_t = \mu_x, \tag{54}$$
$$\mu(0,t) = B^T\lambda, \tag{55}$$
$$u^*(x,0) = u_0(x), \tag{56}$$

and the optimal control input is

$$U^*(t) = -\frac{1}{R}\mu(D,t). \tag{57}$$

Proof: Suppose $X^*(t), u^*(x,t)$ and $U^*(t)$ are the optimal states and control input. Let $X(t) = X^*(t) + \delta X(t), u(x,t) = u^*(x,t) + \epsilon\delta u(x,t)$, $U(t) = U^*(t) + \epsilon\delta U(t), \delta u(D,t) = \delta U(t)$, and $\delta u(x,0) = 0$ represent perturbations from the optimal solutions. Consequently, we can write the cost using the method of Lagrange multipliers as

$$\begin{aligned} g(\epsilon) :=& \\ &\frac{1}{2}\int_0^{t_f} \left[(X^* + \epsilon\delta X)^T Q(X^* + \epsilon\delta X) + R(U^* + \epsilon\delta U)^2\right] dt \\ &+\frac{1}{2}(X^*(t_f) + \epsilon\delta X(t_f))^T P_f(X^*(t_f) + \epsilon\delta X(t_f)) \\ &+\int_0^{t_f} \lambda(t)\left[A(X^* + \epsilon\delta X) + B(u^*(0,t) + \epsilon\delta u(0,t)) \right.\\ &\left. -\frac{d}{dt}(X^* + \epsilon\delta X)\right] dt \\ &+\int_0^{t_f} \langle\mu(x,t), u_x^* + \epsilon\delta u_x - \frac{\partial}{\partial t}(u^* + \epsilon\delta u)\rangle dt, \end{aligned}$$

where $\lambda(t)$ and $\mu(x,t)$ are the co-states for $X(t)$ and $u(x,t)$, respectively. Then the necessary condition for optimality is $dg(\epsilon)/d\epsilon|_{\epsilon=0} = 0$. After differentiating $g(\epsilon)$ and applying a series of computations involving integration by parts, we find the conditions (49)-(57) are necessary to satisfy $dg(\epsilon)/d\epsilon|_{\epsilon=0} = 0$. ■

Now we seek to determine the state-feedback law from the first order necessary conditions in Lemma 4. As before, we postulate the co-state λ is related to the optimal state according to

$$\lambda(t) = P(t)X^*(t). \tag{58}$$

Under this postulation, we are in position to state the state-feedback control law for input delayed systems.

Proposition 5: The optimal state-feedback control law is

$$U^*(t) = -\frac{1}{R}B^T P(t+D) \times \left[e^{AD}X^*(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU^*(\theta)d\theta\right]. \tag{59}$$

The time-varying matrix $P(t)$ must satisfy the Riccati ODE

$$-\dot{P} = PA + A^TP - PB\frac{1}{R}B^TP + Q, \tag{60}$$
$$P(t_f) = P_f. \tag{61}$$

Remark 6: Notice that control law (59) is defined recursively, where $U^*(t)$ depends on previous values of $U^*(t)$.

Remark 7: The controller (59) is a predictor-based law. That is, it advances the measured state by D units of time and applies the corresponding optimal feedback gain. This result is not new. In fact, it is a variation of the venerable Smith Predictor [17]. However, the weak variations procedure provides a completely new and constructive method for control of input delayed systems, in an optimal control context.

Now we supply the proof for Proposition 5.

Proof: First we substitute (58) into (50) to obtain

$$-P_tX^* = PAX^* + PBu^*(0) + A^TPX^* + QX^*. \tag{62}$$

The term $u^*(0,t)$ can be written as

$$\begin{aligned} u^*(0,t) = U^*(t-D) &= -\frac{1}{R}\mu(D,t-D) \\ &= -\frac{1}{R}\mu(0,t) = -\frac{1}{R}B^T\lambda(t) \\ &= -\frac{1}{R}B^TPX^*(t), \end{aligned} \tag{63}$$

using (52)-(53), (54), (55), and (58) respectively. Consequently, (60) must be satisfied for any value of $X^*(t)$. Substituting (58) into final condition (51) produces the final condition (61) for the Riccati ODE. To derive (59), note that

$$\begin{aligned} U^*(t) &= -\frac{1}{R}\mu(D,t) = -\frac{1}{R}\mu(0,t+D) \\ &= -\frac{1}{R}B^T\lambda(t+D) = -\frac{1}{R}B^T P(t+D)X^*(t+D), \end{aligned} \tag{64}$$

using (57), (52), (55), and then (58). The term $X^*(t+D)$ can be written explicitly in terms of $X^*(t)$ and $U^*(t)$ using the exponential matrix as follows

$$\begin{aligned} X^*(t+D) &= e^{AD}X^*(t) + \int_t^{t+D} e^{A(t+D-\tau)}Bu(0,\tau)d\tau \\ &= e^{AD}X^*(t) + \int_t^{t+D} e^{A(t+D-\tau)}BU^*(\tau-D)d\tau \\ &= e^{AD}X^*(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU^*(\theta)d\theta \end{aligned}$$

This furnishes (59) and completes the proof. ■

B. Simulations

For demonstration we consider an LTI system with system matrices

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{65}$$

and an input time delay of $D = 0.3$. The open-loop system is unstable and its eigenvalues are 2 and $-1.5 \pm 1.4j$. We consider an infinite-time horizon LQR controller with unity weighting matrices. This example is adopted from [8]. The simulation results in Fig. 6 demonstrate how a non-predictor-based LQR controller fails to stabilize the input-delayed system, whereas the controller (59)-(61) succeeds. Note that reducing the transient during the initial D time units is impossible, due to the input delay.

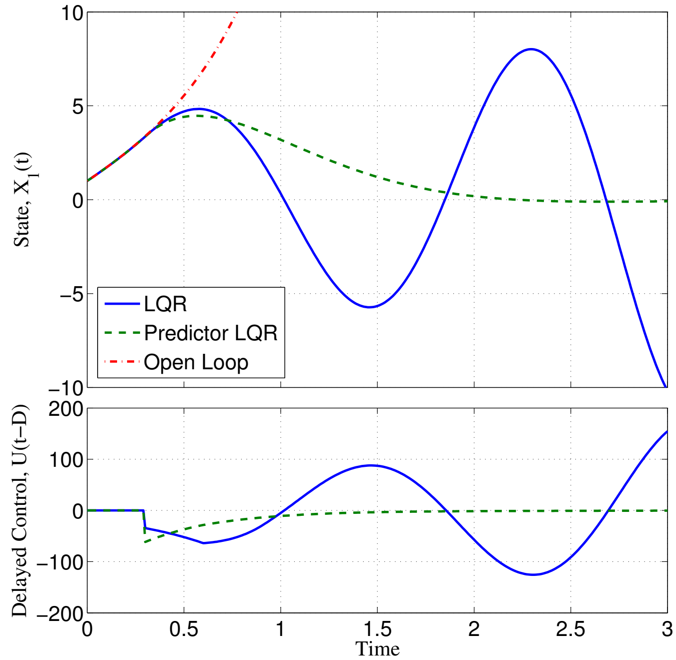

Fig. 6. State X_1 and delayed control $U(t-D)$ evolution for an input-delayed system. The open-loop system (dashed-dotted) is unstable. An LQR controller (solid) fails to stabilize the system, whereas the predictor-based controller (dashed) from (59)-(61) succeeds.

V. Output Reference Tracking

Next we consider the output reference tracking problem. For simplicity of presentation, we shall consider a subclass of the benchmark hyperbolic PDE (1), given by

$$u_t(x,t) = u_x(x,t), \tag{66}$$
$$u(1,t) = U(t), \tag{67}$$

with output function

$$y(t) = \int_0^1 h(x)u(x)dx. \tag{68}$$

It is relatively straight-forward to extend these results to the broader class in (1). Our goal is to derive a state feedback boundary control law such that output $z(t)$ asymptotically tracks the reference signal $z^r(t)$. To this end, define the error variable $e(t) = z^r(t) - z(t)$ and consider the objective functional

$$J = \frac{1}{2}\int_0^{t_f} \left[qe(t)^2 + RU(t)^2\right]dt + \frac{1}{2}q_f e(t_f)^2 \tag{69}$$

where $q \geq 0, R > 0, q_f \geq 0 \in \mathbb{R}$.

Proposition 8: The reference tracking controller is given by a feedback and feed forward term as follows:

$$U^*(t) = -\frac{1}{R}\int_0^1 P(1,y,t)u^*(y,t)dy + \frac{1}{R}G(1,t). \tag{70}$$

The time-varying kernel $P(x,y,t)$ must satisfy the following Riccati PDE

$$P_t = P_x + P_y + \frac{1}{R}P(x,1)P(1,y) - qh(x)h(y), \tag{71}$$

with boundary conditions

$$P(x,0,t) = 0, \tag{72}$$
$$P(0,y,t) = 0, \tag{73}$$

and final condition

$$P(x,y,t_f) = q_f h(x)h(y). \tag{74}$$

The time-varying feed forward term $G(x,t)$ must satisfy the following PDE

$$G_t = G_x + P(x,1,t)G(1,t) - qz^r(t)h(x), \tag{75}$$
$$G(0,t) = qz^r(t), \tag{76}$$
$$G(x,t_f) = q_f h(x)z^r(t_f). \tag{77}$$

Notice that the PDE for $G(x,t)$ depends on $z^r(t)$ and is coupled with $P(x,y,t)$. The proof is provided in the Appendix.

Remark 9: Notice that we do not need to generate a reference trajectory for the state - a key advantage. In contrast, alternative methods, such as differential flatness or backstepping [18], [19], require one to generate the reference state trajectory for the feed forward term. These approaches then stabilize the system around this reference trajectory. Here, the output reference $z^r(t)$ is incorporated via the feed forward term $G(x,t)$ and the associated PDE (75)-(77).

VI. Conclusions

This paper presents a new approach to linear quadratic regulation of first-order hyperbolic PDEs, via weak-variations. Ultimately, the control gains are obtained from the solution of a Riccati PDE. Two interesting applications are considered, including traffic flow control and input-delayed systems. The key benefit of this approach is that it provides a systematic procedure to derive Riccati equations, via an accessible set of mathematical tools. Consequently, the results are useful for a wide spectrum of engineered systems, including traffic flows, chemical reactors, oil production systems, shallow water fluid flows, and delay systems.

The generalizability of this approach creates many interesting opportunities for future work. Throughout, we have assumed full-state feedback. Consequently, optimal observers using boundary measurements are of interest [9]. One might also consider adaptive versions [20] of the controllers presented here. Other classes of PDEs can be considered as well, such as wave, beam, Navier-Stokes, and nonlinear hyperbolic PDEs. Finally, systems of multiple coupled hyperbolic PDEs also provide a particularly relevant and interesting system to study [21]. Ultimately, this paper provides a systematic procedure for deriving LQR control laws for physical systems described by first-order linear hyperbolic PDEs.

VII. Appendix

A. Proof of Proposition 8 [Reference Tracking]

The necessary conditions are derived via weak variations [9], [10]. Suppose $u^*(x,t)$ and $U^*(t)$ are the optimal state and control inputs. Let $u(x,t) = u^*(x,t) + \epsilon\delta u(x,t)$, $U(t) = U^*(t) + \epsilon\delta U(t)$ and $\delta u(x,0) = 0$ represent perturbations from the optimal solutions. Consequently, the cost is

$$\begin{aligned} J(u^* + \epsilon\delta u, U^* + \epsilon\delta U) = \\ \frac{1}{2}\int_0^{t_f} \left[q(z^r(t) - \langle h, u^* + \epsilon\delta u\rangle)^2 + R(U^* + \epsilon\delta U)^2\right] dt \\ +\frac{1}{2}q_f(z^r(t_f) - \langle h, u^*(t_f) + \epsilon\delta u(t_f)\rangle)^2. \end{aligned} \tag{78}$$

Define $g(\varepsilon)$ to be the cost functional above combined with the system dynamics constraint (1), using the method of Lagrange multipliers as follows

$$\begin{aligned} g(\epsilon) := \\ \frac{1}{2}\int_0^{t_f} \left[q(z^r(t) - \langle h, u^* + \epsilon\delta u\rangle)^2 + R(U^* + \epsilon\delta U)^2\right] dt \\ +\frac{1}{2}q_f(z^r(t_f) - \langle h, u^*(t_f) + \epsilon\delta u(t_f)\rangle)^2 \\ + \int_0^{t_f} \langle \lambda(x,t), u_x^* + \epsilon\delta u_x - \frac{\partial}{\partial t}(u^* + \epsilon\delta u)\rangle dt, \end{aligned}$$

where $\lambda(x,t)$ is the Lagrange multiplier. Then the necessary condition for optimality is $dg(\epsilon)/d\epsilon|_{\epsilon=0} = 0$. Differentiating $g(\epsilon)$ produces

$$\begin{aligned} \frac{dg}{d\epsilon}(\epsilon) = \\ \int_0^{t_f} [-q\langle h, \delta u\rangle(z^r(t) - \langle h, u^* + \epsilon\delta u\rangle) \\ +R\delta U(U^* + \epsilon\delta U)]\, dt \\ -q_f\langle h, \delta u(x,t_f)\rangle(z^r(t_f) - \langle h, u^*(x,t_f) - \epsilon\delta u(x,t_f)\rangle) \\ + \int_0^{t_f} \langle \lambda, \delta u_x - \frac{\partial}{\partial t}\delta u\rangle dt, \end{aligned} \tag{79}$$

Using integration by parts we can show

$$\langle \lambda(x), \delta u_x(x)\rangle = \lambda(1)\delta U - \lambda(0)\delta u(0) - \langle \lambda_x, \delta u\rangle, \tag{80}$$

$$\int_0^{t_f} \langle \lambda, \frac{\partial}{\partial t}\delta u\rangle dt = \langle \lambda(x,t_f), \delta u(x,t_f)\rangle - \int_0^{t_f} \langle \lambda_t, \delta u\rangle dt. \tag{81}$$

Now we plug (80), (81) into (79), set $\epsilon = 0$, and group like terms

$$\begin{aligned} 0 = \frac{dg}{d\epsilon}(0) = \\ \int_0^{t_f} \langle \lambda_t - \lambda_x - qh(x)(z^r - \langle h, u^*\rangle), \delta u\rangle \\ + \int_0^{t_f} -\lambda(0)\delta u(0) \\ + \int_0^{t_f} \left[RU^* + \lambda(1)\right]\delta U \\ + \langle -q_f h(x)(z^r(t_f) - \langle h, u^*(x,t_f)\rangle) - \lambda(x,t_f), \delta u(x,t_f)\rangle \end{aligned} \tag{82}$$

For the previous equation to hold true for all arbitrary $\delta u(x,t), \delta U(t), \delta u(x,t_f)$, the following conditions are sufficient

$$\lambda_t(x,t) = \lambda_x(x,t), \tag{83}$$

$$\lambda(0,t) = 0, \tag{84}$$

$$\lambda(x,t_f) = -q_f h(x)\left[z^r - \langle h, u^*(x,t_f)\rangle\right], \tag{85}$$

$$U^*(t) = -\frac{1}{R}\lambda(1,t). \tag{86}$$

Postulate that the co-state λ is related to the states according to the time-varying linear transformation

$$\lambda(x,t) = \int_0^1 P(x,y,t)u^*(y,t)dy - G(x,t). \tag{87}$$

After evaluating each term with this postulated form, we arrive at the control law and corresponding equations given in (70)-(77).

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