On the infinity Laplacian and Hrushovski's fusion

by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2010

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Abstract

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We study viscosity solutions of the partial differential equation

$$-\Delta_\infty u = f \quad \text{in } U,$$

where $U \subseteq \mathbb{R}^n$ is bounded and open, $f \in C(U) \cap L^\infty(U)$, and

$$\Delta_\infty u := |Du|^{-2} \sum_{ij} u_{x_i} u_{x_i} u_{x_i x_j}$$

is the *infinity Laplacian*.

Our first result is the Max-Ball Theorem, which states that if $u \in USC(U)$ is a viscosity subsolution of

$$-\Delta_\infty u \leq f \quad \text{in } U$$

and $\varepsilon > 0$, then the function $v(x) := \max_{\bar{B}(x,\varepsilon)} u$ satisfies

$$2v(x) - \max_{\bar{B}(x,\varepsilon)} v - \min_{\bar{B}(x,\varepsilon)} v \leq \max_{\bar{B}(x,2\varepsilon)} f,$$

for all $x \in U_{2\varepsilon} := \{x \in U : \operatorname{dist}(x, \partial U) > 2\varepsilon\}$. The left-hand side of this latter inequality is a monotone finite difference scheme that is comparatively easy to analyze. The Max-Ball Theorem allows us to lift results for this finite difference scheme to the continuum equation. In particular, we obtain a new proof of uniqueness of viscosity solutions to the Dirichlet problem when $f \equiv 0$, $\inf f > 0$, or $\sup f < 0$. The results mentioned so far are joint work with S. Armstrong.

The Max-Ball Theorem is also useful in the analysis of numerical methods for the infinity Laplacian. We obtain a rate of convergence for the numerical method of Oberman [32]. We also present a new adaptive finite difference scheme.

We also prove some results in Model Theory. We study rank-preserving interpretations of theories of finite Morley rank in strongly minimal sets. In particular, we partially answer a question posed by Hasson [20], showing that definable degree is not necessary for such interpretations. We generalize Ziegler's fusion of structures of finite Morley rank [38] to a class of theories without definable degree. Our main combinatorial lemma also allows us to repair a mistake in [23].

Contents

Chapter 1

Overview

This thesis comprises two disjoint parts. Chapters 2 and 3 study a problem in nonlinear partial differential equations and Chapter 4 studies a problem mathematical logic. This strange state of affairs reflects the unusual path of the author in graduate school. He initially studied model theory with Leo Harrington, but then switched to studying partial differential equations with Lawrence C. Evans. As both stages were important to the author's career, they are both represented here.

1.1 The infinity Laplacian

The archetypical problem in the L^∞ Calculus of Variations is to find a minimizer of the functional

$$\operatorname{Lip}(u, U) := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{u(x) - u(y)}{|x - y|},$$

subject to $u = g$ on ∂U, where $U \subseteq \mathbb{R}^n$ is bounded and open and $g \in C(\partial U)$ satisfies $\operatorname{Lip}(g, \partial U) < \infty$. A classical theorem of Kirszbraun [28] implies that g has a least one extension $u \in C(\bar{U})$ that satisfies

$$\operatorname{Lip}(u, \bar{U}) = \operatorname{Lip}(g, \partial U).$$

In fact, there are infinitely many such extensions in general [30, 36]. To obtain a *uniquely* optimal extension, we look for an extension $u \in C(\bar{U})$ that is *absolutely minimizing Lipschitz*. That is, it satisfies

$$\operatorname{Lip}(u, \bar{V}) = \operatorname{Lip}(u, \partial V) \quad \text{for every open } V \subset\subset U.$$

It is known [10] that a function $u \in C(U)$ is absolutely minimizing Lipschitz if and only if it is *infinity harmonic*. That is, a viscosity solution (see Chapter 2) of the partial differential equation

$$-\Delta_\infty u = 0 \quad \text{in } U,$$

where

$$\Delta_\infty u := |Du|^{-2} \sum_{ij} u_{x_i} u_{x_j} u_{x_i x_j}$$

is the *infinity Laplacian*.

Infinity harmonic extensions were first studied by Aronsson [5]. Existence and uniqueness appeared ten years later in a famous paper of Jensen [27]. Aronsson's famous example,

$$u(x,y) := |x|^{4/3} - |y|^{4/3},$$

of an infinity harmonic function on $\mathbb{R}^2$ showed that $C^{1,\alpha}$ is the best regularity one could hope for. Evans and Savin [17] proved that every infinity harmonic function on $\mathbb{R}^2$ is $C^{1,\alpha}_{loc}$. Recently, Evans and the author [15, 14] showed everywhere differentiability in higher dimensions.

Chapters 2 and 3 concern new techniques for the basic existence and uniqueness theory of infinity harmonic functions. The most significant is the Max-Ball Theorem, which states that if $u \in C(U)$ is a subsolution of

$$-\Delta_\infty u \leq 0 \quad \text{in } U \tag{1.1.1}$$

and we define

$$v(x) := \max_{|y-x|\leq\varepsilon} u(y),$$

then

$$2v(x) - \max_{|y-x|\leq\varepsilon} v(y) - \min_{|y-x|\leq\varepsilon} v(y) \leq 0, \tag{1.1.2}$$

for all $x \in U_{2\varepsilon} := \{x \in U : \operatorname{dist}(x, \partial U) > 2\varepsilon\}$. Informally, subsolutions of (1.1.1) perturb to subsolutions of the finite difference scheme (1.1.2). The idea for this theorem was derived from a paper by Peres, Schramm, Sheffield, and Wilson [34], who studied a two-player random-turn game called *tug-of-war*.

We use the Max-Ball Theorem in several applications. Among these are a new proof of uniqueness of infinity harmonic extensions, a rate-of-convergence analysis for Oberman's [32] numerical scheme for the infinity Laplacian, and a new adaptive finite difference scheme.

We remark that the results in Chapter 2, with the exception of the graph-theoretic interpretation in Section 2.6 and Proposition 2.7.2 are joint work with S. Armstrong. Indeed, the author has collaborated with a number people on "max-ball" projects [3, 4, 1]. We give here a new presentation of the highlights of [2] together with a number of new applications.

1.2 Rank preserving interpretations

A great deal of the progress in model theory in the last thirty years was made in an attempt to classify all strongly minimal theories. It was famously conjectured by Zilber that there were only three kinds of strongly minimal theories: trivial, vector space-like, and field-like. This idea was put to rest by Hrushovski [26], who constructed a strongly minimal theory that did not fit into the above classification. Since then, Hrushovski's proof technique has been adapted to produce more theories with a host of interesting properties [21].

Using Hrushovski's techniques, Hasson proved [20] that every complete first-order theory with finite definable Morley rank and Morley degree has a rank preserving interpretation in a strongly minimal set. He also proved a partial converse, showing that every theory that admits an interpretation (not necessarily rank preserving) in a strongly minimal set has finite definable Morley rank and definably bounded Morley degree. This left open the question of how much definable degree one needs to build a rank-preserving interpretation in a strongly minimal theory.

In Chapter 4, we show that definable degree is not necessary. Unfortunately, we do not show that definably bounded degree is sufficient. Instead, we show that a class of theories derived from a test case proposed by Hasson [20] admit such interpretations. We actually prove something slightly more general. We show that every pair of theories in our class have a *fusion*. A result of Ziegler [38] then implies that all theories in our class have a rank preserving interpretation in a strongly minimal set.

We also correct an error in the amalgamation construction of [23]. There, Remark 1.7 states that there are $2^{co(B'/A')}$ atomic types extending $atp_S(B', A') \cup atp_L(A')$. This is indeed the case. However, some may conflict with the earlier multiplicity rules and therefore are not admissible. Worse, the total number of admissible extensions may not be a power of 2. In particular, the theory T_μ defined by Hasson and Hrushovski is not consistent. Fixing this requires a definable way of detecting the number of admissible extensions. This is provided by the main combinatorial lemma in Chapter 4.

Chapter 2

The Max-Ball Theorem and some applications

This chapter concerns viscosity solutions of the boundary value problem

$$\begin{cases} -\Delta_\infty u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{2.0.1}$$

where $U \subseteq \mathbb{R}^n$ is a bounded and open set, $f \in C(U) \cap L^\infty(U)$, $g \in C(\partial U)$, and

$$\Delta_\infty u := |Du|^{-2} \sum_{ij} u_{x_i} u_{x_j} u_{x_i x_j}$$

is the *infinity Laplacian*. See Crandall [10] for an introduction to the theory of this equation.

Our main result is the Max-Ball Theorem, which states that subsolutions of (2.0.1) perturb to subsolutions of a certain finite difference scheme. The finite difference scheme is comparatively easy to analyze, and we use the Max-Ball Theorem to transfer the results of this analysis back to the continuum equation. Notably, we obtain a new proof of uniqueness of viscosity solutions of (2.0.1) when $f < 0$, $f > 0$, or $f \equiv 0$. Our proof is remarkable in that it is completely elementary. In particular, it avoids Alexandrov's theorem on the almost everywhere twice differentiability of convex functions used in [7, 6, 27, 11] and the probabilistic arguments of [34].

Using additional analysis of the finite difference scheme, we obtain an estimate on how the solution of (2.0.1) changes as the right-hand side varies. We also obtain a proof of convergence for the finite difference scheme that is stronger than what the famous theorem of Barles and Souganidis [8] on monotone schemes provides.

Also important is our graph-theoretic interpretation of the finite difference scheme in Section 2.6. Here we translate the ideas of [34] and [2] into a language suitable for the analysis of finite difference schemes in Chapter 3. These graph-theoretic ideas and Proposition 2.7.2 are the only parts of this chapter that are *not* joint work with S. Armstrong.

2.1 Preliminaries

Throughout this chapter U, f, and g will be as above unless otherwise stated. We let $C^k(U)$, $USC(U)$, $LSC(U)$ and $L^\infty(U)$ denote respectively the k-times continuously differentiable, upper semiconintuous, lower semincontinuous, and bounded measurable functions on U. We write $\bar{U}$ for the closure of U and $\partial U := \bar{U} \setminus U$ for the boundary of U. We write $|x|$ for the Euclidean norm of a point $x \in \mathbb{R}^m$. If $u \in C^1(U)$ and $x \in U$, then $Du(x) \in \mathbb{R}^n$ denotes the gradient of u at x. If $u \in C^2(U)$, then $D^2u(x) \in S_n$ denotes the $n \times n$ symmetric matrix of second derivatives at x.

We recall the notion of viscosity solution [12]. Given an upper semicontinuous function $u \in USC(U)$ and a function $f : U \to \mathbb{R}$, we say that the differential inequality

$$-\Delta_\infty u \le f \quad \text{in } U \tag{2.1.1}$$

holds in the *viscosity sense* if and only if the following condition holds.

$$\begin{array}{l}\text{If } \varphi \in C^\infty(U) \text{ and } x \mapsto (u-\varphi)(x) \text{ has a strict local} \\ \text{maximum at } y \in U, \text{ then } -\Delta_\infty^+\varphi(x) \le f(x).\end{array} \tag{2.1.2}$$

Here we have used the notation

$$\Delta_\infty^+\varphi(x) := \begin{cases} \Delta_\infty\varphi(x) & \text{if } D\varphi(x) \ne 0, \\ \max_{|v|=1}\langle D^2\varphi(x)v, v\rangle & \text{if } D\varphi(x) = 0, \end{cases} \tag{2.1.3}$$

which is necessary since $\Delta_\infty\varphi$ may not be everywhere defined.

We call a function $u \in USC(U)$ that satisfies (2.1.1) a *subsolution of* $-\Delta_\infty u = f$. Negating u and f, we obtain the dual notion of supersolution. That is, $v \in LSC(U)$ is a *supersolution* of $-\Delta_\infty v = f$ if and only if $u := -v$ is a subsolution of $-\Delta_\infty u = f$.

A *viscosity solution* of (2.0.1) is a function $u \in C(\bar{U})$ that satisfies $u = g$ on ∂U and is both a viscosity subsolution and a viscosity supersolution of $-\Delta_\infty u = f$ in U.

Remark 2.1.1. We drop the word viscosity in the sequel and assume that differential inequalities are to be interpreted in the viscosity sense. We also note that the symmetry between the notion of subsolution and supersolution allows the transfer of many results. We often use the symmetric versions of results without further comment in the sequel.

2.2 The Max-Ball Theorem

2.2.1 The finite difference infinity Laplacian

Given a bounded function $u : U \to \mathbb{R}$ and $\varepsilon > 0$, we define the functions $T^\varepsilon u : U_\varepsilon \to R$ and $T_\varepsilon u : U_\varepsilon \to \mathbb{R}$ by

$$T^\varepsilon u(x) := \sup_{\bar{B}(x,\varepsilon)} u \tag{2.2.1}$$

and

$$T_\varepsilon u(x) := \inf_{\bar{B}(x,\varepsilon)} u, \tag{2.2.2}$$

where

$$U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

We then define $S_\varepsilon^+ u, S_\varepsilon^- u, \Delta_\infty^\varepsilon u : U_\varepsilon \to \mathbb{R}$ by

$$S_\varepsilon^- u = \frac{1}{\varepsilon}(u - T_\varepsilon u),$$

$$S_\varepsilon^+ u = \frac{1}{\varepsilon}(T^\varepsilon u - u),$$

and

$$-\Delta_\infty^\varepsilon u := \frac{1}{\varepsilon}(S_\varepsilon^- u - S_\varepsilon^+ u) = \frac{1}{\varepsilon^2}(2u - T^\varepsilon u - T_\varepsilon u). \tag{2.2.3}$$

We call $\Delta_\infty^\varepsilon$ the *finite difference infinity Laplacian.*

2.2.2 Comparison with cones

The first step in the proof of the Max-Ball Theorem is the following *comparison with cones* lemma. The idea, originating in [27], is that one can restrict the test functions in the definition of viscosity solution to cones. We prove something slightly stronger than is necessary for the sequel. The proof is elementary and uses an interesting perturbation argument to handle the gradient zero case.

Lemma 2.2.1. *Suppose $U \subseteq \mathbb{R}^n$ is bounded and open, $c \in \mathbb{R}$, and $u \in USC(\bar{U})$ satisfies*

$$-\Delta_\infty u \le c \quad \text{in } U.$$

If $\varphi \in C(\bar{U}) \cap C^\infty(U)$ is given by

$$\varphi(x) := a|x - x_0| - \frac{c}{2}|x - x_0|^2, \tag{2.2.4}$$

for some $a \in R$ and $x_0 \in \mathbb{R}^2$, then

$$\max_{\bar{U}}(u - \varphi) = \max_{\partial U}(u - \varphi). \tag{2.2.5}$$

Proof. Suppose first that $x_0 \in U$. In this case $\varphi \in C^\infty(U)$ implies that $a = 0$. If (2.2.5) fails, then by continuity we may select a small $\varepsilon > 0$ and a $y_0 \in U$ such that

$$(u - \psi)(y_0) = \max_{\bar{U}}(u - \psi) > \max_{\partial U}(u - \psi),$$

where

$$\psi(x) := \varphi(x) - \frac{\varepsilon}{2}|x - x_0|^2.$$

The definition of subsolution then yields

$$c + \varepsilon = -\Delta_\infty \psi(y_0) \leq c,$$

a contradiction.

Now suppose $x_0 \notin U$ and (2.2.5) fails. We may again select $\varepsilon > 0$, $y_0 \in U$, and ψ as above. We may assume that $\varepsilon < |c|$ if $c \neq 0$. Now, if $c \leq 0$ or $D\psi(y_0) \neq 0$, then we again compute

$$-\Delta_\infty^+ u(y_0) \geq -\max\{0, c + \varepsilon\} > c.$$

Thus we need only worry about the case $c > 0$ and $D\psi(y_0) = 0$. Note that $D\psi(y_0) = 0$ implies that $|y_0 - x_0| = r := a/(c + \varepsilon) > 0$.

Consider the functions

$$\psi_1(x) = \psi(x) - \varepsilon||x - x_0| - r|,$$

and

$$\psi_2(x) = \psi(x) - \varepsilon||x - x_0| - r - \varepsilon| + a\varepsilon^2.$$

Assuming $\varepsilon > 0$ is small enough, we still have

$$\max_{\bar{U}}(u - \psi_i) > \max_{\partial U}(u - \psi_i),$$

for $i = 1, 2$. Observe that $\psi_1 \leq \psi_2$ and that $\psi_1(x) = \psi_2(x)$ when $|x - x_0| \leq r$.

Select $y_0 \in U$ such that $(u - \psi_1)(y_0) = \max_{\bar{U}}(u - \psi_1)$. If $|y_0 - x_0| \neq r$, then we again compute

$$c + \varepsilon = -\Delta_\infty \psi_1(y_0) \leq c.$$

On the other hand, if $|y_0 - x_0| = r$, then we in fact have $(u - \psi_2)(y_0) = \max_{\bar{U}}(u - \psi_2)$ and compute

$$c + \varepsilon = -\Delta_\infty \psi_2(y_0) \leq c.$$

Thus we have a contradiction in either case. $\square$

2.2.3 Slope estimates

The next step in the proof the max-ball theorem is the following *slope estimates*. Again, we prove more than is strictly necessary. These are a natural generalization of the slope estimates in [9, 10], adapted to the case of non-zero right-hand side.

If a function $u : U \to \mathbb{R}$ is locally Lipschitz and $x \in U$, we define $Lu : U \to \mathbb{R}$ by

$$Lu(x) := \inf_{r \to 0} \operatorname{Lip}(u, B(0, r)).$$

Observe that if $u \in C^1(U)$, then $Lu = |Du|$. We use Lu instead of $|Du|$ because it is upper semicontinuous and everywhere defined. In fact, the two are equal by a new result of the Evans and the author [15].

Lemma 2.2.2. *Suppose $u \in USC(U)$ satisfies*

$$-\Delta_\infty u \le c \quad \text{in } U,$$

for some $c \in \mathbb{R}$. If $\bar{B}(x, \varepsilon) \subseteq U$ and $y \in \bar{B}(x, \varepsilon)$ satisfies $u(y) = \max_{\bar{B}(x,\varepsilon)} u$, then

$$Lu(x) \le S_\varepsilon^+ u(x) + \frac{c}{2}\varepsilon \tag{2.2.6}$$

and

$$Lu(y) \ge S_\varepsilon^+ u(x) - \frac{c}{2}\varepsilon. \tag{2.2.7}$$

In particular, u is locally Lipschitz.

Proof. Given $z \in B(x, \varepsilon)$, define

$$\varphi_z(w) := u(z) + \left(\frac{u(y) - u(z)}{\varepsilon - |x - z|} + \frac{c}{2}(\varepsilon + |x - z|) \right) |w - z| - \frac{c}{2}|w - z|^2,$$

and observe by Lemma 2.2.1 that $\varphi_z \ge u$ on $\bar{B}(x, \varepsilon)$. In particular, if $w \in B(x, \varepsilon)$, then

$$u(w) \le u(z) + \left(\frac{u(y) - u(z)}{\varepsilon - |x - z|} + \frac{c}{2}(\varepsilon + |x - z|) \right) |w - z| - \frac{c}{2}|w - z|^2.$$

This rearranges to

$$\frac{u(w) - u(z)}{|w - z|} \le \left(\frac{u(y) - u(z)}{\varepsilon - |x - z|} + \frac{c}{2}(\varepsilon + |x - z|) \right) - \frac{c}{2}|w - z|.$$

Now, if $z \in B(x, \varepsilon/2)$, then $u(x) \le \phi_z(x)$ and $u(z) \le u(y)$ together imply that $|u(z)| \le C$ for some constant $C > 0$. Thus

$$\frac{u(y) - u(z)}{\varepsilon - |x - z|} \le \frac{u(y) - u(z)}{\varepsilon} + C|x - z| = S_\varepsilon^+ u(x) + C|x - z|.$$

Thus we obtain

$$\frac{u(w)-u(z)}{|w-z|} \leq S_\varepsilon^+ u(x) + \frac{c}{2}\varepsilon + C\max\{|w-x|,|z-x|\},$$

for all $w, z \in B(x, \varepsilon/2)$. This implies

$$\mathrm{Lip}(u, B(x,\delta)) \leq S_\varepsilon^+ u(x) + \frac{c}{2}\varepsilon + C\delta,$$

for all $\delta > 0$. Sending $\delta \to 0$ gives (2.2.6).

To prove (2.2.7), we may assume that the right-hand side is positive. In particular, $z \mapsto \varphi_x(z)$ is increasing in $|z|$ when $|z| = 1$.

We claim that $y \in \partial B(x, \varepsilon)$. If $c \geq 0$ this is obvious because $u(y) = \varphi_x(z)$ implies $|z - x| = \varepsilon$. When $c < 0$, use Lemma 2.2.1 to obtain

$$\max_{|z-x|\leq\varepsilon}(u(z) + \frac{c}{2}|z-x|^2) = \max_{|z-x|=\varepsilon}(u(z) + \frac{c}{2}|z-x|^2).$$

From this it follows that $u(z) < \max_{\partial B(x,\varepsilon)} u$ for all $z \in B(x, \varepsilon)$.

Now consider the maps $f, g : (0, \mathrm{dist}(x, \partial U)) \to \mathbb{R}$ given by

$$f(t) := u(x + \varepsilon^{-1}(y-x)t) \quad \text{and} \quad g(t) := \varphi(x + \varepsilon^{-1}(y-x)t).$$

Since $f(t) \leq g(t)$ on $(0, \varepsilon]$ and $f(\varepsilon) = g(\varepsilon)$, we have

$$Lu(y) \geq Lf(\varepsilon) \geq Lg(\varepsilon) \geq S_\varepsilon^+ u(x) - \frac{c}{2}\varepsilon,$$

which is precisely (2.2.7). □

2.2.4 Statement and proof of the Max-Ball Theorem

We are now ready to state and prove the max-ball theorem. The proof is a nearly trivial consequence of the slope estimates above.

Theorem 2.2.3 (Max-Ball Theorem). *If $U \subseteq \mathbb{R}^n$ is bounded open, $f : U \to \mathbb{R}$ is bounded, and $u \in USC(U)$ satisfies*

$$-\Delta_\infty u \leq f \quad \text{in } U,$$

then $u^\varepsilon \in USC(U_\varepsilon)$ satisfies

$$-\Delta_\infty^\varepsilon T^\varepsilon u \leq T^{2\varepsilon} f \quad \text{in } U_{2\varepsilon}.$$

Proof. Choose and arbitrary $x \in U_{2\varepsilon}$ and then select $y \in \bar{B}(x,\varepsilon)$ and $z \in \bar{B}(x,\varepsilon)$ such that $u(y) = T^\varepsilon u(x)$ and $u(z) = T^\varepsilon u(y)$. The slope estimates (2.2.6) and (2.2.7) give

$$u(y) - u(x) \leq \varepsilon L u(y) - \frac{\varepsilon^2}{2} T^{2\varepsilon} f(x)$$

and

$$u(z) - u(y) \geq \varepsilon L u(y) + \frac{\varepsilon^2}{2} T^{2\varepsilon} f(x).$$

Since $T_\varepsilon T^\varepsilon u(x) \geq u(x)$, we compute

$$\begin{aligned} -\varepsilon^2 \Delta_\infty^\varepsilon T^\varepsilon u(x) &= (T^\varepsilon u(x) - T_\varepsilon T^\varepsilon u(x)) - (T^{2\varepsilon} u(x) - T^\varepsilon u(x)) \\ &\leq (u(y) - u(x)) - (u(z) - u(y)) \\ &= \varepsilon^2 T^{2\varepsilon} f(x). \end{aligned}$$

Now divide by ε^2. □

2.3 Le Gruyer's comparison argument

Part of what gives the Max-Ball theorem its power is that the finite difference infinity Laplacian is a particularly easy to analyze. As a first example of this phenomenon, we give an easy proof of comparison. This proof technique is originally due to Le Gruyer [29], although our comparison result is stronger.

If $U \subseteq \mathbb{R}^n$ is bounded and open and $u \in USC(U)$, then an *ε-thick local maximum of u in U* is a closed set $F \subseteq U_\varepsilon$ such that u is constant on F and

(2.3.1) $$u(y) < u(F) \text{ for every } y \in U \setminus F \text{ such that } \operatorname{dist}(y, F) \leq \varepsilon.$$

Symmetrically, an ε-thick local minimum of a function $v \in LSC(U)$ is an ε-thick local maximum of $-v$.

Lemma 2.3.1. *Suppose $\varepsilon > 0$ and $u, -v \in USC(U)$ satisfy*

(2.3.2) $$-\Delta_\infty^\varepsilon u \leq -\Delta_\infty^\varepsilon v \quad \text{in } U_\varepsilon.$$

If u has no ε-thick local maximum in U, then

(2.3.3) $$\sup_U (u - v) = \sup_{U \setminus U_\varepsilon} (u - v).$$

Proof. Suppose for contradiction that (2.3.3) fails. In this case, $\sup_U (u - v) < \infty$. Define

$$E := \{x \in U : (u - v)(x) = \sup_U (u - v)\},$$

and

$$F := \{x \in E : u(x) = \max_E u\}.$$

Observe that $E \subseteq U_\varepsilon$ is closed and non-empty by the upper semicontinuity of $u-v$. Therefore the definition of F makes sense. We claim that F is an ε-thick local maximum of u in U.

To check (2.3.1), suppose for condradiction that there is a $y \in U \setminus F$ such that $|y-x| \leq \varepsilon$ for some $x \in F$ and $u(y) \geq u(F)$. Observe that if $z \in \bar{B}(x,\varepsilon)$ and $u(z) > u(y)$ then $z \notin F$. Thus, possibly selecting a different y, we may assume that

$$\varepsilon S^+ u(x) = u(y) - u(x).$$

Since $u(y) \geq \max_E u$ and $y \notin F$, we must have $y \notin E$. Thus $u(y) - v(y) < u(x) - v(y)$ and we compute

$$\varepsilon S^+ u(x) = u(y) - u(x) < v(y) - v(x) \leq \varepsilon S^+ v(x).$$

However, the definition of $x \in E$ implies that

$$S_\varepsilon^- u(x) \geq S_\varepsilon^- v(x),$$

so we have $-\Delta_\infty^\varepsilon u(x) > -\Delta_\infty^\varepsilon v(x)$, contradicting (2.3.2). □

It is useful to state a weaker comparison result that avoids the additional distraction of the ε-thick local maxima.

Lemma 2.3.2. *If $\varepsilon > 0$, $u, -v \in USC(U)$, and either*

$$-\Delta_\infty^\varepsilon u < -\Delta_\infty^\varepsilon v \quad \text{in } U_\varepsilon,$$

or

$$-\Delta_\infty^\varepsilon u \leq \min\{0, -\Delta_\infty^\varepsilon v\} \quad \text{in } U_\varepsilon,$$

then

$$\sup_U (u-v) = \sup_{U \setminus U_\varepsilon} (u-v). \tag{2.3.4}$$

Proof. In the case of strict inequality, suppose there is an $x \in U_\varepsilon$ such that

$$(u-v)(x) = \sup_U (u-v).$$

The above equality immediately implies

$$S_\varepsilon^+ u(x) \leq S_\varepsilon^+ v(x) \quad \text{and} \quad S_\varepsilon^- u(x) \geq S_\varepsilon^- v(x),$$

which contradicts $-\Delta_\infty^\varepsilon u(x) < -\Delta_\infty^\varepsilon v(x)$.

Otherwise, we note that u can not have an ε-thick local maximum and apply Lemma 2.3.1. Indeed, if $F \subseteq U_\varepsilon$ were an ε-thick local maximum and $x \in \partial F$, then we would have

$$S^+ u(x) = 0 \quad \text{and} \quad S^- u(x) > 0,$$

and therefore $-\Delta_\infty^\varepsilon u(x) > 0$. □

2.4 Uniqueness of viscosity solutions

Using the max-ball theorem together with Le Gruyer's argument, we easily obtain a comparison result for viscosity solutions.

Theorem 2.4.1. *Suppose $u, -v \in USC(\bar{U})$ satisfy*

$$-\Delta_\infty u \le f \le g \le -\Delta_\infty v \quad \text{in } U, \tag{2.4.1}$$

for some $f, g \in C(U) \cap L^\infty(U)$. If either $f < g$, $f \equiv 0$, $f < 0$, or $g > 0$, then

$$\sup_U (u - v) = \sup_{\partial U} (u - v). \tag{2.4.2}$$

Proof. First observe that if (2.4.2) fails, then by the upper semicontinuity of $u - v$ it still fails if we replace U with U_ε for some small $\varepsilon > 0$. In particular, we may assume that f and g are uniformly continuous and that either $\sup_U (f - g) < 0$, $f \equiv 0$, $\sup_U f < 0$, or $\inf_U g > 0$.

If $\sup_U (f - g) < 0$, then Theorem 2.2.3 gives

$$-\Delta_\infty T^\varepsilon u \le T^{2\varepsilon} f \quad \text{in } U_{2\varepsilon},$$

and

$$-\Delta_\infty T_\varepsilon v \ge T_{2\varepsilon} g \quad \text{in } U_{2\varepsilon},$$

By uniform continuity, we have

$$T^{2\varepsilon} f < T_{2\varepsilon} g \quad \text{in } U_{2\varepsilon},$$

for all sufficiently small $\varepsilon > 0$. Thus Lemma 2.3.2 implies that

$$\sup_{U_\varepsilon} (T^\varepsilon u - T_\varepsilon v) = \sup_{U_\varepsilon \setminus U_{2\varepsilon}} (T^\varepsilon u - T_\varepsilon v),$$

for all sufficiently small $\varepsilon > 0$. Sending $\varepsilon \to 0$ yields (2.4.2).

When $f \equiv 0$, then Theorem 2.2.3 gives

$$-\Delta_\infty T^\varepsilon u \le 0 \le -\Delta_\infty T_\varepsilon v \quad \text{in } U_{2\varepsilon}.$$

Thus Lemma 2.3.2 yields

$$\sup_{U_\varepsilon} (T^\varepsilon u - T_\varepsilon v) = \sup_{U_\varepsilon \setminus U_{2\varepsilon}} (T^\varepsilon u - T_\varepsilon v),$$

and sending $\varepsilon \to 0$ yields (2.4.2).

When $\sup_U f < 0$, we replace u with $(1 + \varepsilon)u$ for some small $\varepsilon > 0$. Since the infinity Laplacian is 1-homogeneous, we obtain $-\Delta_\infty((1 + \varepsilon)u) \le (1 + \varepsilon)f < g$. Thus (2.4.2) follows as above. When $\inf_U g > 0$ we replace v with $(1 + \varepsilon)v$. □

Corollary 2.4.2. *If satisfies either $f \equiv 0$, $\sup f < 0$, or $\inf f > 0$, then* (2.0.1) *has at most one viscosity solution.*

2.5 Convergence

As a second application of the max-ball theorem, we prove a convergence result. This result is interesting because it works in the absence of a comparison principal for the limiting equation. In particular, this result is *not* implied by the famous result of Barles and Souganidis [8] on monotone finite difference schemes for second-order equations. In fact, one can use this result to prove existence and stability of solutions for (2.0.1) for arbitrary $f \in C(U) \cap L^\infty(U)$, although we do not do that here. See [2] for more details.

The proof uses a perturbed test function argument [16]. That is, when $u - \varphi$ attains its maximum at x_0, we use the Max-Ball Theorem to deduce things about $T_\varepsilon \varphi$ and then send $\varepsilon \to 0$.

Theorem 2.5.1. *Suppose for each $n > 0$ that $\varepsilon_n > 0$ and $u_n : U \to \mathbb{R}$ are bounded and satisfy*

$$-\Delta_\infty^{\varepsilon_n} u_n \leq f \quad \text{in } U_\varepsilon,$$

for some $f \in C(U) \cap L^\infty(U)$. If $\varepsilon_n \to 0$ and $u_n \to u \in C(U)$ as $n \to \infty$, then

$$-\Delta_\infty u \leq f \quad \text{in } U.$$

Proof. Suppose $\varphi \in C^\infty(U)$ is a smooth test function and the map $x \mapsto (u - \varphi)(x)$ has a strict maximum in U at some point $y \in U$.

Since φ is smooth, we have

$$-\Delta_\infty \varphi \geq -\Delta_\infty^+ \varphi \quad \text{in } U,$$

in the sense of viscosity. Therefore Theorem 2.2.3 implies

$$-\Delta_\infty^\varepsilon T_\varepsilon \varphi \geq T_{2\varepsilon}(-\Delta_\infty^+ \varphi) \quad \text{in } U_{2\varepsilon},$$

for every $\varepsilon > 0$.

Since $u - \varphi$ has a strict maximum at y, we know that the function $u_n - T_{\varepsilon_n} \varphi$ attains its maximum on U_{ε_n} near y for all sufficiently large n. Thus we may select points $y_n \in U_{\varepsilon_n}$ such that

$$(u_n - T_{\varepsilon_n} \varphi)(y_n) = \sup_{U_{\varepsilon_n}} (u_n - T_{\varepsilon_n} \varphi).$$

This equality immediately implies that

$$-\Delta_\infty^{\varepsilon_n} u_n(y_n) \geq -\Delta_\infty^{\varepsilon_n} T_{\varepsilon_n} \varphi(y_n).$$

Note also that $y_n \to y$ as $n \to \infty$.

Stringing our inequalities together, we obtain

$$T_{2\varepsilon}(-\Delta_\infty^+ \varphi)(y_n) \leq f(y_n),$$

for all large $n > 0$. Since $y_n \to y$ and $-\Delta_\infty^+ \varphi$ is lower semicontinuous, we may send $n \to \infty$ and obtain $-\Delta_\infty^+ \varphi(y) \leq f(y)$. □

2.6 Graph-theoretic results

A graph-theoretic abstraction of the finite difference infinity Laplacian (2.2.3) is useful for the purposes of numerical approximation. It permits a certain uniformity of presentation in the sequel. We remark that this section is an analytic translation of the game-theoretic ideas of Peres, Schramm, Sheffield, and Wilson [34]. In particular, none of these results are new. It is the presentation and language that is different. Most interesting is Lemma 2.6.3 which makes clear the fact that the patching theorem of Crandall, Gunnarsson and Wang [11] and the backtracking strategy of [34] are actually the same idea.

Let $G := (X, E, Y)$ denote a finite diameter graph with vertex set X, edge set E, and a distinguished non-empty set of *boundary vertices* $Y \subseteq X$. Recall that a *path of length* m in G is a tuple of vertices $(z_0, ..., z_m) \in X^{m+1}$ such that $z_i \sim_E z_{i+1}$ for $i = 0, ..., m-1$. Our assumption that G has *finite diameter* means that there is an $M < \infty$ such that for every pair of vertices $x, y \in X$ there a path $(x, z_1, ..., z_{m-1}, y)$ in G of length $m \leq M$.

Given a bounded function $u : X \to \mathbb{R}$, we define the functions $S^+_G u, S^-_G u, \Delta^G_\infty u : X \setminus Y \to \mathbb{R}$ by

$$S^+_G u(x) = \sup_{y \sim_E x} (u(y) - u(x)), \tag{2.6.1}$$

$$S^-_G u(x) = \sup_{y \sim_E x} (u(x) - u(y)), \tag{2.6.2}$$

and

$$-\Delta^G_\infty u(x) = S^-_G u(x) - S^+_G u(x). \tag{2.6.3}$$

We call Δ^G_∞ the *discrete infinity Laplacian on* G.

Remark 2.6.1. The finite difference infinity Laplacian $\Delta^\varepsilon_\infty$ for $U \subseteq \mathbb{R}^n$ is a rescaling of the discrete infinity Laplacian Δ^G_∞ for the graph

$$G := (U, E, U \setminus U_\varepsilon),$$

where

$$E := \{\{x, y\} \subseteq U : x \in U_\varepsilon \text{ and } 0 < |x - y| \leq \varepsilon\}.$$

Indeed, if $u : U \to \mathbb{R}$ is bounded, then

$$\varepsilon^2 \Delta^\varepsilon_\infty u = \Delta^G_\infty u.$$

We need the following gradient estimate for our numerical results in Chapter 4. Its proof uses a "marching" argument.

Lemma 2.6.2. *If $u : X \to \mathbb{R}$ is bounded and satisfies*

$$-\Delta_\infty^G u = 0 \quad \text{on } X \setminus Y,$$

then

$$\sup_{X\setminus Y} S_G^+ u \le \sup_{x,y\in Y} \frac{u(x)-u(y)}{d(x,y)}. \tag{2.6.4}$$

Proof. Suppose $\{x_0, y_0\} \in E$ and $u(x_0) - u(y_0) = k > 0$. Using $-\Delta_\infty^G u = 0$ on $X \setminus Y$, we may iteratively select $x_1, x_2, ..., x_m$ such that $u(x_{i+1}) - u(x_i) \ge k$ and $x_m \in Y$. Similarly, we may select $y_1, y_2, ..., y_n$ such that $u(y_i) - u(y_{i+1}) \ge k$ and $y_n \in Y$. Thus

$$\frac{u(x_m)-u(y_n)}{d(x_m,y_n)} \ge \frac{u(x_m)-u(y_n)}{n+m+1} \ge k,$$

and (2.6.4). □

The next lemma is a patching lemma for infinity subharmonic functions on graphs. It shows that we can always perturb to the positive gradient case.

Lemma 2.6.3. *If $u : X \to \mathbb{R}$ is bounded from above and*

$$-\Delta_\infty^G u \le 0 \quad \text{on } X \setminus Y,$$

and $k > 0$, there is a function $v : X \to \mathbb{R}$ that satisfies

$$u \ge v \ge u - 2k\,\mathrm{dist}(\cdot, Y), \tag{2.6.5}$$

$$S_G^+ v \ge k, \tag{2.6.6}$$

and

$$-\Delta_\infty^G v \le 0, \tag{2.6.7}$$

on $X \setminus Y$.

Proof. 1. Consider the set

$$Z := \{S^+ u < k\} \subseteq X \setminus Y,$$

and let P denote the set of paths $(x_0, ..., x_m)$ such that $m > 0$, $x_0, ..., x_{m-1} \in Z$ and $x_m \in X \setminus Z$. Define $w : Z \to \mathbb{R}$ by

$$w(x) = \sup\{u(x_m) - km : (x, x_1, ..., x_m) \in P\},$$

and then define $v : X \to \mathbb{R}$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in X \setminus Z, \\ w(x) & \text{if } x \in Z. \end{cases}$$

We claim that v satisfies (2.6.5), (2.6.6), and (2.6.7).

2. Given $(x_0, x_1, ..., x_m) \in P$, compute

$$\begin{aligned} u(x_m) - km &\leq u(x_m) - \sum_{i=1}^{m} S^+u(x_{i-1}) \\ &\leq u(x_m) - \sum_{i=1}^{m} (u(x_i) - u(x_{i-1})) \\ &= u(x_0). \end{aligned}$$

Thus $w \leq u$ on Z. For the other half of (2.6.5), fix and arbitrary $x_0 \in Z$ and select a path $(x_0, ..., x_m) \in P$ such that $m \leq \operatorname{dist}(x_0, Y)$. Compute

$$\begin{aligned} w(x_0) &\geq u(x_m) - km \\ &= u(x_0) + \sum_{i=1}^{m} (u(x_i) - u(x_{i-1})) - km \\ &\geq u(x_0) - \sum_{i=1}^{m} S_G^- u(x_{i-1}) - km \\ &\geq u(x_0) - \sum_{i=1}^{m} S_G^+ u(x_{i-1}) - km \\ &\geq u(x_0) - 2km \\ &\geq u(x_0) - 2k \operatorname{dist}(x_0, Y). \end{aligned}$$

3. To prove (2.6.6), suppose first that $x_0 \in Z$. Given $\varepsilon > 0$, select $(x_0, ..., x_m) \in P$ such that

$$v(x_0) \leq u(x_m) - km + \varepsilon.$$

Observe that

$$S^+v(x_0) \geq v(x_1) - v(x_0) \geq [u(x_m) - k(m-1)] - [u(x_m) - km + \varepsilon] = k - \varepsilon.$$

Sending $\varepsilon \to 0$, we see that $S^+v \geq k$ in Z.

Next, suppose $x_0 \in X \setminus (Y \cup Z)$. Suppose $\varepsilon \in (0, k/4)$ and $(x_0, ..., x_m)$ is a path such that

$$u(x_{i+1}) - u(x_i) \geq S_G^+ u(x_i) - \frac{\varepsilon}{2^i},$$

and $x_i \in X \setminus Y$ for $i = 0, ..., m-1$. Since

$$S_G^+ u(x_{i+1}) \geq S_G^- u(x_{i+1}) \geq u(x_{i+1}) - u(x_i) \geq S_G^+ u(x_i) - \frac{\varepsilon}{2^i},$$

we see that

$$S_G^+ u(x_i) \geq u(x_{i+1}) - u(x_i) \geq S_G^+ u(x_0) - 2\varepsilon.$$

Since u is bounded from above and $\varepsilon < S_G^+ u(x_0)/4$, we have $m \leq M$ for some constant $M > 0$ independent of ε. Selecting a maximal path, we obtain $x_m \in Y$. If $x_1 \notin Z$, then

$$S_G^+ v(x_0) \geq u(x_1) - u(x_0) \geq S_G^+ u(x_0) - 2\varepsilon.$$

Otherwise, since $x_m \in Y$, there is an $l \leq m$ such that $(x_1, ..., x_l) \in P$ and we have

$$\begin{aligned} S_G^+ v(x_0) &\geq u(x_1) - u(x_0) \\ &\geq u(x_l) - u(x_0) - (l-1) S_G^+ u(x_0) \\ &\geq l(S_G^+ u(x_0) - 2\varepsilon) - (l-1) S_G^+ u(x_0) \\ &= S_G^+ u(x_0) - 2\varepsilon M. \end{aligned}$$

Sending $\varepsilon \to 0$, we obtain

$$S^+ v(x_0) \geq S^+ u(x_0), \tag{2.6.8}$$

and therefore (2.6.6).

4. To prove (2.6.7), suppose first that $x \in X \setminus (Z \cup Y)$. The definition of w guarantees that

$$S_G^- v(x) \leq \max\{k, S_G^- u(x)\}.$$

Since $S_G^+ v(x) \geq S_G^- u(x) \geq k$ by (2.6.8), we see that (2.6.7) holds at x.

Next, suppose that $x \in Z$. We claim that $S_G^- v(x) \leq k$. For contradiction, suppose $u(x) - u(y) > k$ and $y \sim_E x$. If $y \in Z$, then $v(y) \geq v(x) - k$ by the definition of w. Thus $y \in X \setminus Z$, and we may compute

$$k < v(x) - v(y) \leq u(x) - u(y) \leq S_G^- u(x) \leq S_G^+ u(x),$$

contradicting the definition of Z. Thus $S_G^- u \leq k$ on Z and (2.6.6) implies that (2.6.7) holds at x. □

The following lemma is a "strictness" transformation for the discrete infinity Laplacian. It shows that, when the gradient is positive, subsolutions perturb to strict subsolutions.

Lemma 2.6.4. *Suppose $u : X \to \mathbb{R}$ is bounded and satisfies $u \geq 0$ and*

$$-\Delta_\infty^G u \leq 0 \quad \text{on } X \setminus Y.$$

For every $k > 0$, the function $v := u + ku^2$ satisfies

$$-\Delta_\infty^G v \leq -\Delta_\infty^G u - k(S_G^+ u)^2 \quad \text{on } X \setminus Y. \tag{2.6.9}$$

Proof. Fix $x \in X \setminus Y$ and suppose there are $y, z \sim_E x$ such that

$$S^+u(x) = u(y) - u(x) \quad \text{and} \quad S^-u(x) = u(x) - u(z).$$

Since the map $t \mapsto t + kt^2$ is monotone on the range of u, we compute

$$\begin{aligned} S^+v(x) &= v(y) - v(x) \\ &= S^+u(x) + kv(y)^2 - kv(x)^2 \\ &= S^+u(x) + kS^+u(x)(v(y) + v(x))) \\ &= S^+u(x) + kS^+u(x)(2v(x) + S^+u(x)), \end{aligned}$$

and

$$\begin{aligned} S^-v(x) &= v(x) - v(z) \\ &= S^-u(x) + kv(x)^2 - kv(z)^2 \\ &= S^-u(x) + kS^-u(x)(v(x) + v(z)) \\ &\leq S^-u(x) + kS^+u(x)(2v(x)). \end{aligned}$$

Combining these inequalities gives (2.6.9).

In general, there are no $y, z \sim_E x$ that achieve $S^+_G u(x)$ and $S^-_G u(x)$. Instead, we fix $\varepsilon > 0$, and choose y and z such that

$$S^+u(x) \leq u(y) - u(x) + \varepsilon \quad \text{and} \quad S^-u(x) \leq u(x) - u(z) + \varepsilon.$$

Going through the above calculation again, we obtain

$$-\Delta^G_\infty v(x) \leq -\Delta^G_\infty u(x) - k(S^+_G u)^2(x) + O(\varepsilon).$$

Now, sending $\varepsilon \to 0$ gives (2.6.9). □

Putting the patching and strictness lemmas together, we obtain a general comparison result on graphs. Note that the Theorem below is strictly weaker than what the Le Gruyer argument yielded in Lemma 2.3.1. This is because we no longer have the topology of $\mathbb{R}^n$ at our disposal.

Theorem 2.6.5. *Suppose $u, v : X \to \mathbb{R}$ are bounded and satisfy*

$$-\Delta^G_\infty u \leq f \leq g \leq -\Delta^G_\infty v \quad \text{on } X \setminus Y,$$

for some $f, g : X \setminus Y \to \mathbb{R}$. If $\sup_{X\setminus Y}(f - g) < 0$, $f \equiv 0$, $\sup_{X\setminus Y} f < 0$, or $\inf_{X\setminus Y} f > 0$, then

$$\sup_X (u - v) = \sup_Y (u - v). \tag{2.6.10}$$

Proof. 1. We first consider the case $\sup_{X\setminus Y}(f-g)<0$. Assume that

$$\sup_X(u-v)>\sup_Y(u-v).$$

Thus, given $\varepsilon>0$, we may select a vertex $x\in X\setminus Y$ such that

$$(u-v)(x)\geq \sup_X(u-v)(x)-\varepsilon/2.$$

Observe that

$$S_G^+u(x)=\sup_{y\sim_E x}(u(y)-u(x))\leq \sup_{y\sim_E x}(u(y)-v(x)+\varepsilon/2)=S_G^+v(x)+\varepsilon/2,$$

and similarly

$$S_G^-u(x)\geq S_G^-v(x)-\varepsilon/2,$$

Thus

$$f(x)\geq -\Delta_\infty^G u(x)\geq -\Delta_\infty^G v(x)-\varepsilon/2\geq g(x)-\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, we obtain

$$\sup_{X\setminus Y}(f-g)\geq 0.$$

2. Next we suppse that $\sup_{X\setminus Y} f<0$. Since $-\Delta_\infty^G u=S_G^-u-S_G^+u$, we obtain $\inf_{X\setminus Y} S^+u>0$. Thus Lemma 2.6.4 gives

$$-\Delta_\infty^G(u+ku^2)\leq f\leq g-\delta\quad\text{on } X\setminus Y,$$

for some $\delta>0$ and all $k>0$. Now part one of the proof gives

$$\sup_X(u+ku^2-v)=\sup_Y(u+ku^2-v),$$

for all $k>0$. Sending $k\to 0$ gives (2.6.10).

3. The case $\inf_{X\setminus Y} g>0$ is symmetric to $\sup_{X\setminus Y} f<0$, so we may assume $f\equiv 0$. In this case, Lemma 2.6.3 gives a family of functions $u_k : X\to Y$ such that

$$\inf_{X\setminus Y} S^+u_k\geq k,$$

$$-\Delta_\infty u_k\leq 0\quad\text{on } X\setminus Y,$$

and

$$\sup_X|u-u_k|\leq O(k),$$

for every $k>0$. Since $\inf_{X\setminus Y} S^+u_k>0$, the argument in part two of the proof gives

$$\sup_X(u_k-v)=\sup_Y(u_k-v).$$

Sending $k\to 0$, we obtain (2.6.10). □

Finally, we prove existence of solution for the graph-theoretic boundary value problem.

Theorem 2.6.6. *If $g : Y \to \mathbb{R}$ and $f : X \setminus Y \to \mathbb{R}$ are bounded, then there is a unique bounded function $u : X \to \mathbb{R}$ such that*

$$\begin{cases} -\Delta_\infty^G u = f & \text{on } X \setminus Y, \\ u = g & \text{on } Y. \end{cases} \tag{2.6.11}$$

Proof. Let $d := \operatorname{diam}(G)$ and $c := 2\sup_Y |g| + \sup_{X\setminus Y} |f|$. Given $y \in Y$, consider the function

$$v(x) := g(y) - c(1+d^2)\operatorname{dist}(y,x) + c\operatorname{dist}(y,x)^2. \tag{2.6.12}$$

We claim that v satisfies

$$\begin{cases} -\Delta_\infty^G v \le c & \text{on } X \setminus Y, \\ u \le g & \text{on } Y. \end{cases}$$

Indeed, if $k := \operatorname{dist}(x,y) \ge 1$, then

$$v(x) \le v(y) - c \le v(y) - 2\sup_Y |g| \le g(x).$$

If, in addition $x \in X \setminus Y$, then

$$\begin{aligned} S_G^- u(x) &\le [g(y) - c(1+d^2)k + ck^2] - [g(y) - c(1+d^2)(k+1) + c(k+1)^2] \\ &\le c(1+d^2) - c(2k+1). \end{aligned}$$

Moreover, since there is a $z \in X$ such that $z \sim_E x$ and $\operatorname{dist}(z,y) = k-1$, we have

$$\begin{aligned} S_G^+ u(x) &\ge u(z) - u(x) \\ &= [g(y) - c(1+d^2)k + ck^2] - [g(y) - c(1+d^2)(k-1) + c(k-1)^2] \\ &= c(1+d^2) + c(2k-1). \end{aligned}$$

Thus

$$-\Delta_\infty^G u(x) = S_G^- u(x) - S_G^+ u(x) \le -2c \le c.$$

Similarly, the function

$$w(x) := g(y) + c(1+d^2)\operatorname{dist}(y,x) - c\operatorname{dist}(y,x)^2.$$

satisfies

$$\begin{cases} -\Delta_\infty^G w \ge c & \text{on } X \setminus Y, \\ w \ge g & \text{on } Y. \end{cases} \tag{2.6.13}$$

Now, let $u : X \to \mathbb{R}$ be the supremum of all functions $u : X \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta_\infty^G u \leq f & \text{on } X \setminus Y, \\ u \leq g & \text{on } Y. \end{cases} \tag{2.6.14}$$

Using the function v constructed above, we see that the supremum is non-empty. Using the function w and Theorem 2.6.5, we see that $u < \infty$. By varying the vertex y used to define v, we see that $u = g$ on Y. Thus we need only show $-\Delta_\infty^G u = f$ on $X \setminus Y$.

That $-\Delta_\infty^G u \leq f$ on $X \setminus Y$ is trivial from the observation that

$$-\Delta_\infty^G \max\{u_1, u_2\} \leq \max\{-\Delta_\infty^G u_1, -\Delta_\infty^G u_2\},$$

for any bounded functions $u_1, u_2 : X \to \mathbb{R}$.

Suppose for contradiction that $-\Delta_\infty^G u(x_0) = f(x_0) + \delta$ for some $\delta > 0$ and $x_0 \in X \setminus Y$. Consider $\tilde{u} : X \to \mathbb{R}$ given by

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \neq x_0, \\ u(x) + \delta/2 & \text{if } x = x_0. \end{cases}$$

Since $S_G^- \tilde{u} \leq S_G^- u$ and $S_G^+ \tilde{u} \geq S_G^+ u$ on $X \setminus (Y \cup \{x_0\})$ and $-\Delta_\infty \tilde{u}(x_0) = f(x_0)$, we see that $\tilde{u}$ satisfies (2.6.14). As $\tilde{u}(x_0) > u(x_0)$, this contradicts the definition of u. In particular, u solves (2.6.11). $\square$

Remark 2.6.7. Suppose each edge $\{x, y\} \in E$ has a weight $d(x, y) \in (0, \infty)$. If we have $d^- := \inf_{E \cap [X \setminus Y]^2} d > 0$ and $d^+ := \sup_{E \cap [X \setminus Y]^2} d < 0$, then the above results easily generalize when we incorporate the weights. That is, when we define

$$S_G^+ u(x) := \sup_{y \sim_E x} \frac{u(y) - u(x)}{d(y, x)},$$

and

$$S_G^- u(x) := \sup_{y \sim_E x} \frac{u(x) - u(y)}{d(y, x)},$$

and define the length of a path $(x_0, .., x_m)$ to be $\sum_i d(x_i, x_{i+1})$. The only difference in the results is that the constants in the estimates (2.6.5) and (2.6.9) now depend on the ratio d^+/d^- and that $\operatorname{diam}(G)$ must be measured using the weights.

2.7 Continuous dependence

For the purposes of building numerical approximations, it is useful to know how the solution of (2.0.1) varies as one changes the right-hand side. In this section we prove two continuous dependence estimates. The first works for arbitrary boundary data while the second only works in some special cases. We suspect that the second estimate is in fact true for arbitrary boundary data.

Theorem 2.7.1. *Suppose $u_k \in C(\bar{U})$ solves*

$$\begin{cases} -\Delta_\infty u_k = k & \text{in } U, \\ u_k = g & \text{on } \partial U, \end{cases}$$

for every $k \in \mathbb{R}$. There is a constant $C > 0$ depending only on $\operatorname{diam}(U)$ *and* $\|g\|_{L^\infty(\partial U))}$ *such that*

$$\|u_0 - u_k\|_{L^\infty(U)} \leq C|k|^{1/3},$$

for all sufficiently small $k \in \mathbb{R}$.

Proof. We may assume that $k \in (-1, 0)$ and $2\operatorname{diam}(U) \leq u \leq 2\operatorname{diam}(U) + 1$. Fix $\varepsilon > 0$. Theorem 2.2.3 implies that

$$-\Delta_\infty^\varepsilon T^\varepsilon u_0 \leq 0 \quad \text{in } U_{2\varepsilon}.$$

Using Lemma 2.6.3, select a $v : U_\varepsilon \to \mathbb{R}$ such that

$$-\Delta_\infty^\varepsilon v \leq 0, \quad S_\varepsilon^+ v \geq |k|^{1/3}, \quad \text{and} \quad T^\varepsilon u_0 \geq v \geq T^\varepsilon u_0 - 2|k|^{1/3} \operatorname{dist}(\cdot, U_\varepsilon \setminus U_{2\varepsilon}),$$

in $U_{2\varepsilon}$. Since $v \geq 0$, we may set

$$w := v - k^{1/3} v^2,$$

and conclude by Lemma 2.6.4 that

$$\Delta_\infty^\varepsilon w \leq k \quad \text{in } U_{2\varepsilon}.$$

and

$$\|w - T^\varepsilon u_0\|_{L^\infty(U_\varepsilon)} \leq C k^{1/3}.$$

we compute

$$\begin{aligned} \sup_{U_\varepsilon}(T^\varepsilon u_0 - T_\varepsilon u_k) &\leq \sup_{U_\varepsilon}(w - T_\varepsilon u_k) + C|k|^{1/3} \\ &= \sup_{U_\varepsilon \setminus U_{2\varepsilon}} (w - T_\varepsilon u_k) + C|k|^{1/3} \\ &\leq \sup_{U_\varepsilon \setminus U_{2\varepsilon}} (T^\varepsilon u_0 - T_\varepsilon u_k) + 2C|k|^{1/3}. \end{aligned}$$

Since $u_k \leq u_0$ by Theorem 2.4.1, sending $\varepsilon \to 0$ yields $\|u_0 - u_k\|_{L^\infty(U)} \leq C|k|^{1/3}$. □

We can improve the power in the above estimate from 1/3 to 1 in some special cases. That it can be improved when the magnitude of the gradient is bounded away from 0 is trivial. However, it is new and unexpected for the Aronsson function. Moreover, this strongly suggests that the improvement is possible for arbitrary boundary data. Indeed, the Aronsson function has historically served as a "universal" counterexample for conjectures about infinity harmonic functions.

We remark that this improvement is also possible whenever $u \in C^2(\bar{U})$, as a result of Yu [37] implies that the magnitude of the gradient is bounded away from zero in this case.

Proposition 2.7.2. *Suppose the u_k are as in the previous theorem. If* $\inf_U |Du_0| > 0$ *or* $U \subseteq \mathbb{R}^2$ *and* $u_0(x,y) = x^{4/3} - y^{4/3}$*, then*

$$\|u_0 - u_k\|_{L^\infty(U)} \leq C|k|, \tag{2.7.1}$$

for some constant $C > 0$*.*

Proof. If $\inf_U |Du_0| = \alpha > 0$, then we have $S_\varepsilon^+ u_0 \geq \alpha$ for all $\varepsilon > 0$ by (2.2.7). Thus the proof of Theorem 2.7.1 yields (2.7.1). Indeed, in this case we can avoid the application of Lemma 2.6.3 and apply 2.6.4 with the parameter $2\alpha^{-2}|k|$ instead of $2|k|^{1/3}$.

Now suppose $U \subseteq \mathbb{R}^2$ and $u_0(x,y) = |x|^{4/3} - |y|^{4/3}$. Consider

$$w := u_0 - \frac{4}{3}k|u_0|^{3/2},$$

for $k < 0$. Assume temporarily that u_0 and w are smooth. Compute

$$Dw = (1 - 2k|u_0|^{1/2})Du_0,$$

$$D^2 w = (1 - 2k{u_0}^{1/2})D^2 u_0 - k{u_0}^{-1/2} Du_0 \otimes Du_0,$$

and thus

$$-\Delta_\infty w = -(1 - 2k|u_0|^{1/2})\Delta_\infty u_0 + k{u_0}^{-1/2}|Du_0|^2.$$

Since $|Du_0| \geq |u|^{1/4}$ in $\mathbb{R}^2$, we have

$$-\Delta_\infty w \leq k, \tag{2.7.2}$$

where u_0 and w are smooth.

In particular, the inequality (2.7.2) holds in the viscosity sense in $\mathbb{R}^n \setminus \{u = 0\}$. That it holds on all of $\{u = 0\}$ follows because w can not be touched from above by a smooth function on the set $\{u = 0\} \setminus \{0\}$ and that $w \geq |x|^2$ on the set $\{y = 0\}$.

Thus, it follows from Theorem 2.4.1 that $u_0 \geq u_k \geq u_0 + Ck$ for some constant $C > 0$ independent of $k < 0$. $\square$

Chapter 3

Numerical methods for the infinity Laplacian

This chapter concerns the numerical approximation of the unique solution of

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{3.0.1}$$

where $U \subseteq \mathbb{R}^n$ is bounded and open and $g \in C(\partial U)$ is Lipschitz.

Using the Max-Ball Theorem and the results of Section 2.6, we give an error analysis for the finite difference scheme of Oberman [32]. We prove that Oberman's scheme converges at a rate of $O(h^{1/5})$ in general and $O(h^{1/3})$ in some interesting special cases.

These rates are slow, but this is not terribly unusual for schemes approximating viscosity solutions. They reflect that fact that large stencil sizes are required for consistency when the solutions are not smooth. Indeed, it appears to be difficult to construct fast numerical methods that are capable of resolving non-smooth viscosity solutions of fully nonlinear operators [13, 31, 33].

To address the problem of large stencils, we introduce a new adaptive grid method. The Max-Ball Thoerem and continuous dependence estimates from Chapter 3 provide an easily computed a posteriori error estimate for approximate solutions of (3.0.1). We use this estimate to automatically concentrate grid points near the non-smooth parts of solutions.

We point out two examples of related work. The first is the master's thesis of Hansson [19], who used FEMLAB to approximate p-harmonic extensions for large p. Hansson used this analysis to investigate the concentration of gradient flow-lines as $p \to \infty$. The second is the vanishing moment method of Feng and Neilan [18], who used a finite element method together with a fourth-order regularlization term. The results of these two papers are in a different direction from what we present here. Indeed, we are interested in methods with explicit rates of convergence and error estimates. It is still unknown how quickly the p-harmonic and vanishing-moment approximations converge to infinity harmonic extensions.

3.1 Oberman's scheme

While discussing Oberman's scheme, we assume that

$$U := \{\max\{|x_1|, |x_2|\} < 1\} \subseteq \mathbb{R}^2,$$

This is not much of a restriction, since the generalization to arbitrary bounded and open sets $U \subseteq \mathbb{R}^n$ is trivial. However, when $n > 2$ the scheme is computationally intractable. Indeed, the stencils we define below have D^n points in them, where n is the dimension of the ambient space and D an integer. To obtain accurate solutions, we need to choose fairly large D. When $n > 2$, the stencils are too large for reasonable study on a laptop (in 2010).

3.1.1 Definition of the scheme

Select integers $N > D > 0$ and define the grid points

$$X := \{(i/N, j/N) : i, j \in \mathbb{Z} \text{ and } -N \leq i, j \leq N\},$$

and the boundary points

$$Y := \{(i/N, j/N) : i, j \in \mathbb{Z} \text{ and } \max\{|i|, |j|\} = N\}.$$

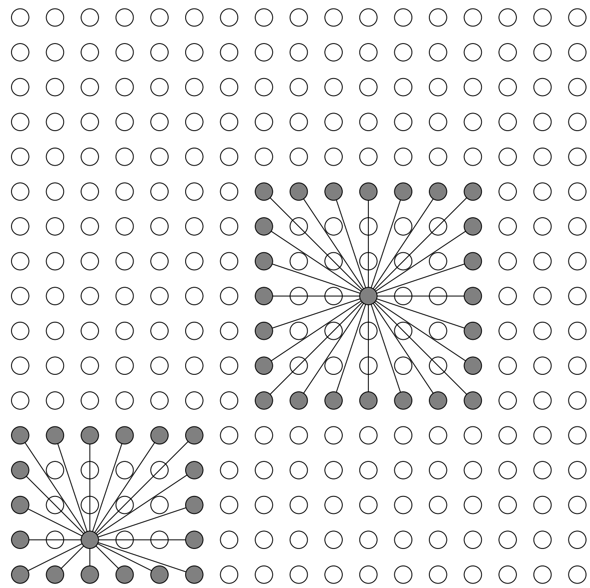
Put a graph structure on X by letting the edge set $E \subseteq [X]^2$ be such that $\{x, y\} \in E$ if and only if $x \in X$ and either

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} = \frac{D}{N},$$

or

$$\max\{|x_1 - y_1|, |x_2 - y_2|\} < \frac{D}{N} \quad \text{and} \quad y \in Y.$$

The following picture shows two neighbor sets in the case $N = 8$ and $D = 3$.



Note that when a point is near the boundary its stencil has a different shape. The purpose of this is to make path distance in the graph between any two points on the boundary close to the Euclidean distance between the two points. This has the effect of making affine functions close to being solutions of the finite difference scheme. This seems to improve the accuracy of the scheme by a large constant factor.

Given $g \in C(\partial U)$, there is a unique function $u : X \to \mathbb{R}$ such that

$$\begin{cases} -\Delta_\infty^{N,D} u = 0 & \text{on } X \setminus Y, \\ u = g & \text{on } Y, \end{cases} \tag{3.1.1}$$

where

$$-\Delta_\infty^{N,D} u(x) := \max_{y \sim_E x} \frac{u(x) - u(y)}{|y - x|} - \max_{y \sim_E x} \frac{u(y) - u(x)}{|y - x|}, \tag{3.1.2}$$

for $x \in X \setminus Y$. Observe that $\Delta_\infty^{N,D}$ is exactly Δ_∞^G for the graph

$$G := (X, E, Y),$$

with edge weights $d(x, y) = |x - y|$ by Remark 2.6.7.

Oberman [32] proved the following convergence result.

Theorem 3.1.1 (Oberman). *If $D_k \to \infty$ and $N_k/D_k \to \infty$ as $k \to \infty$ and the u_k solve* (3.1.1) *for N_k and D_k, then $u_k \to u$ the unique solution of* (3.0.1) *as $k \to \infty$.*

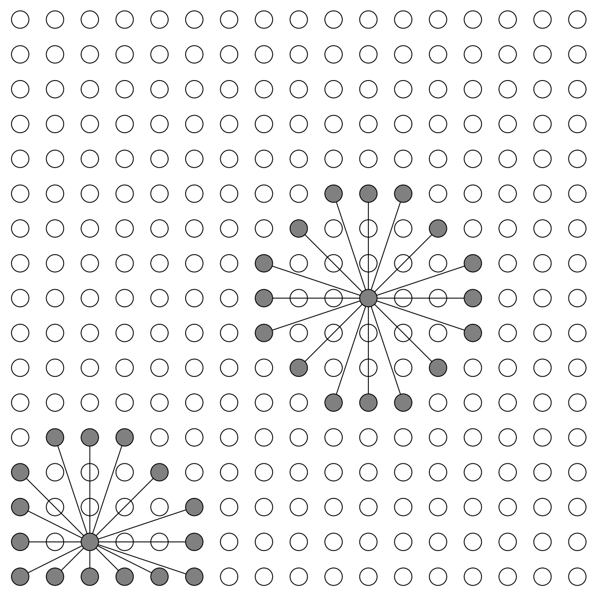
This follows from Barles and Souganidis [8], using the fact that (3.1.1) is monotone and consistent. This result leaves open two important questions. It says nothing about the rate of convergence nor how to choose the ratio N/D.

3.1.2 Circular stencils

Using the max-ball theorem to analyze the scheme (3.1.1) is complicated by the fact that the stencils are square-shaped. While it is possible to carry this out, the extra effort does not yield anything interesting. Instead, we redefine the edge set E to be

$$E := \left\{ \{x, y\} \in [X]^2 : x \in X \text{ and } \frac{D}{N} - \frac{1}{2N} < |x - y| < \frac{D}{N} + \frac{1}{2N} \right\}.$$

The following picture shows the new stencils in the case $N = 8$ and $D = 3$.



The advantage of this modification is made clear in the rate of convergence proof below. For now, we simply observe that as $D \to \infty$ and $N/D \to \infty$, the stencils converge to circles. Since the max-ball theorem operates on disks, this is a good sign.

We remark that Oberman's convergence theorem [32] still applies in this case.

3.1.3 Rate of convergence

The first step in our convergence analysis is to estimate the error from the discretization of $\Delta_\infty^\varepsilon$ by Δ_∞^G. The reader may find it strange that we only compute the discretization error for subsolutions. This assumption allows guarantees the $\max_{\bar{B}(x,\varepsilon)} u$ is attained on $\partial B(x, \varepsilon)$ for all $x \in U_\varepsilon$. We need this because our stencils approximate the boundary of a ball and do not contain interior points.

Lemma 3.1.2. *If $u \in C(U)$ satisfies*

$$-\Delta_\infty u \leq 0 \quad \textit{in } U,$$

and $\varepsilon = D/N$*, then*

$$-\Delta_\infty^G u \le -\varepsilon^2 \Delta_\infty^\varepsilon u + C \operatorname{Lip}(u,U) N^{-1} \quad \text{on } X \cap U_\varepsilon, \tag{3.1.3}$$

where $C > 0$ *is a universal constant.*

Proof. Since $-\Delta_\infty \ge 0$, Lemma 2.2.1 implies that

$$\max_{\bar{B}(x,\varepsilon)} u = \max_{\partial B(x,\varepsilon)} u \quad \text{for every } x \in U_{2\varepsilon}.$$

Observe that if $x \in X \cap U_{2\varepsilon}$ and $y \in \partial B(x,\varepsilon)$, then there is a $z \in X$ such that $z \sim_E x$ and $|y - z| \le CN^{-1}$. Thus, if $x \in X \cap U_{2\varepsilon}$, we compute

$$\begin{aligned} -\Delta_\infty^G u(x) &\le \left[2u(x) - \min_{\partial B(x,\varepsilon)} u - \max_{\partial B(x,\varepsilon)} u \right] + C \operatorname{Lip}(u,U) N^{-1} \\ &\le \left[2u(x) - \min_{\bar{B}(x,\varepsilon)} u - \max_{\bar{B}(x,\varepsilon)} u \right] + C \operatorname{Lip}(u,U) N^{-1} \\ &= -\varepsilon^2 \Delta_\infty^\varepsilon u(x) + C \operatorname{Lip}(u,U) N^{-1}. \end{aligned}$$

□

Using Theorem 2.7.1, it is now fairly easy to obtain an $O(h^{1/5})$ rate of convergence for arbitrary boundary data.

Theorem 3.1.3. *If* $D = \lceil N^{4/5} \rceil$*,* u *solves* (3.0.1)*, and* $\tilde{u}$ *solves* (3.1.1)*, then*

$$\max_X |u - \tilde{u}| \le C N^{-1/5} \operatorname{Lip}(g, \partial U), \tag{3.1.4}$$

for some universal constant $C > 0$*. Here* $\lceil z \rceil$ *denotes the least integer larger than* z*.*

Proof. Define $\varepsilon := D/N \approx N^{-1/5}$ and observe that for any $x \in X \cap U_{2\varepsilon}$ and $y \in \partial B(x,\varepsilon)$, there is a $z \in X$ such that $|y - z| \le C\varepsilon^5$. For each $k > 0$, Theorem 2.7.1 provides a $u_k \in C(\bar{U})$ such that

$$-\Delta_\infty u_k \ge k \quad \text{in } U,$$

$$\operatorname{Lip}(u_k, U) \le C(\operatorname{Lip}(g, \partial U) + k),$$

and

$$\sup_{\bar{U}} |u - u_k| \le C k^{1/3}.$$

Since $-\Delta_\infty T_\varepsilon u_k \ge k$ in $U_{2\varepsilon}$, the inequality (3.1.3) gives

$$-\Delta_\infty^G T_\varepsilon u_k \ge k\varepsilon^2 + C \operatorname{Lip}(u_k, U) \varepsilon^5 \quad \text{on } X \cap U_{2\varepsilon}.$$

Thus if we set $k := C \operatorname{Lip}(u_k, U) \varepsilon^3$, we obtain

$$-\Delta_\infty^G T_\varepsilon u_k \ge 0 \quad \text{on } X \cap U_{2\varepsilon},$$

and

$$\sup_U |u - T_\varepsilon u_k| \le C\varepsilon.$$

Now, Lemma 2.6.10 implies that

$$\sup_X (\tilde{u} - T_\varepsilon u_k) = \sup_{X \setminus U_{2\varepsilon}} (\tilde{u} - T_\varepsilon u_k),$$

and Lemma 2.6.2 implies

$$\sup_{X \setminus U_{2\varepsilon}} |u - \tilde{u}| \le C \operatorname{Lip}(g, \partial U)\varepsilon.$$

The last three inequalities together imply that

$$\tilde{u} \le u + C \operatorname{Lip}(g, \partial U)\varepsilon \quad \text{on } X.$$

The other half of (3.1.4) is symmetric. □

Using Proposition 2.7.2 in place of Theorem 2.7.1, we obtain an $O(h^{1/3})$ rate of convergence for certain examples. As is the case for Proposition 2.7.2, we suspect that this rate is attained for all boundary data.

Proposition 3.1.4. *Suppose u solves (3.0.1) and either $\inf_U Lu > 0$ or $u(x, y) = x^{4/3} - y^{4/3}$. If $D = \lceil N^{2/3} \rceil$ and $\tilde{u}$ solves (3.1.1), then*

$$\max_X |u - \tilde{u}| \le C N^{-1/3}, \tag{3.1.5}$$

for some constant $C > 0$ depending on u.

Proof. Using $\varepsilon := D/N \approx N^{-1/3}$ and Proposition 2.7.2 in place of Theorem 2.7.1 in the proof of the above theorem, we obtain the estimates

$$\sup_U |u - u_k| \le Ck$$

and

$$-\Delta^G_\infty T_\varepsilon u_k \ge k\varepsilon^2 - C\varepsilon^3,$$

instead of

$$\sup_U |u - u_k| \le Ck^{1/3}.$$

and

$$-\Delta^G_\infty T_\varepsilon u_k \ge k\varepsilon^2 - C\varepsilon^5.$$

Thus we can set $k := C\varepsilon$ and the rest of the proof goes through as before. □

Remark 3.1.5. We suspect that even the faster rate (3.1.5) is pessimistic on account of the following heuristic calculation. Suppose $T_\varepsilon u$ and $T^\varepsilon u$ happen to be C^2. In this case, the discretization error (3.1.3) would be

$$-\Delta^G_\infty u \le -\varepsilon^2 \Delta^\varepsilon_\infty u + C \operatorname{Lip}(u, U) N^{-2}.$$

If we also assume linear continuous dependence (2.7.2), then we could set $D := \lceil N^{1/3} \rceil$ and obtain an $O(h^{2/3})$ rate of convergence.

3.1.4 Implementation notes

To solve the scheme (3.1.1), one typically computes the fixed point of the operator $\mathcal{F}$, where if $u : X \to \mathbb{R}$ then $\mathcal{F}u : X \to \mathbb{R}$ is the unique function satisfying

$$\begin{cases} \mathcal{F}u(x) = u(x) & \text{if } x \in Y, \\ \max_{y \sim_E x} \frac{\mathcal{F}u(x) - u(y)}{|y-x|} = \max_{y \sim_E x} \frac{u(y) - \mathcal{F}u(x)}{|y-x|} & \text{if } x \in X \setminus Y. \end{cases}$$

One must use a relaxation parameter $\alpha \in (0,1)$ and iterate

$$u \mapsto \alpha u + (1-\alpha)\mathcal{F}u,$$

in order to achieve convergence. Any parameter $\alpha > 0$ will do, although the optimal choice of α seems to be problem-dependent.

Whether there is a faster solution method is an interesting open problem, as (3.1.1) is highly non-linear. The other standard algorithm is to iteratively fill in the steepest path. That is, to iterate the following process.

> Select a path $(x_0, ..., x_m)$ in X such that $x_0, x_m \in Y$, $x_1, .., x_{m-1} \in X \setminus Y$, and $s := (u(x_m) - u(x_0)) / \sum_i d(x_i, x_{i+1})$ is as large as possible. Set $u(x_k) = u(x_0) + s \sum_{i=0}^{k-1} d(x_i, x_{i+1})$ for $k = 1, ..., m-1$ and add $x_1, .., x_{m-1}$ to Y.

The naive implementation of this has worst-case time complexity $O(N^4 D^2 \log(N)^2)$, and is much slower than the iterative process described above.

We remark that while increasing D increases the cost of computing $\mathcal{F}$, it reduces the number of iterations required to converge. In practice, increasing D actually reduces the total computation time. This is due to the fact that a large D means information travels farther during each iteration. Thus, when considering how to choose the optimal D for a particular N, we can safely focus on accuracy alone.

3.2 Adapting the grid

The large stencil sizes in Oberman's scheme are required for consistency. Indeed, large stencils appear to be principal obstacle in developing fast numerical methods capable of resolving of non-smooth viscosity solutions of fully nonlinear equations [13, 31, 33]. To get around this, we design a scheme that resorts to large stencil sizes only when necessary.

3.2.1 An a posteriori error estimate

Using the Max-Ball Theorem and the continuous dependence estimates from Chapter 2, we obtain the following a posteriori error estimate.

Theorem 3.2.1. *If u solves* (3.0.1) *and $v \in C(\bar{U})$ is Lipschitz and satisfies $v = g$ on ∂U, then*

$$\sup_U |u - v| \le C(\varepsilon \operatorname{Lip}(v, U) + \sup_{U_{2\varepsilon}} |\Delta_\infty^\varepsilon v|^{1/3}), \tag{3.2.1}$$

for any $\varepsilon \in (0, 1)$ and a constant $C > 0$ that depends only on $\operatorname{diam}(U)$ *and $Lip(g, \partial U)$. If in addition* $\inf_U Lu > 0$ *or $U \subseteq \mathbb{R}^2$ and $u(x, y) = x^{4/3} - y^{4/3}$, then*

$$\sup_U |u - v| \le C(\varepsilon \operatorname{Lip}(v, U) + \sup_{U_{2\varepsilon}} |\Delta_\infty^\varepsilon v|). \tag{3.2.2}$$

Proof. Let $k := \sup_{U_{2\varepsilon}} |\Delta_\infty^\varepsilon v|$. Theorem 2.7.1 provides a function $w \in C(\bar{U})$ such that

$$-\Delta_\infty w \ge k \quad \text{in } U,$$

$$\operatorname{Lip}(w, U) \le C(1 + k),$$

and

$$\sup_U |w - u| \le Ck^{1/3}.$$

The Max-Ball Theorem implies that

$$-\Delta_\infty^\varepsilon T_\varepsilon w \ge k \quad \text{in } U_{2\varepsilon},$$

and thus Lemma 2.3.1 implies that

$$\sup_{U_\varepsilon}(v - w) = \sup_{U_\varepsilon \setminus U_{2\varepsilon}} (v - w).$$

On the other hand,

$$\sup_{\bar{U} \setminus U_{2\varepsilon}} |u - v| \le 2(\operatorname{Lip}(v, U) + \operatorname{Lip}(g, \partial U))\varepsilon.$$

Stringing these inequalities together, we obtain

$$v \le u + C(\varepsilon \operatorname{Lip}(v, U) + k^{1/3}) \quad \text{in } U.$$

A symmetric argument yields

$$v \ge u - C(\varepsilon \operatorname{Lip}(v, U) + k^{1/3}) \quad \text{in } U,$$

and thus (3.2.1).

In the special case that $\inf_U Lu > 0$ or $u(x, y) = x^{4/3} - y^{4/3}$, Proposition (2.7.2) gives the better estimate

$$\sup_U |w - u| \le Ck.$$

This gives (3.2.2). □

3.2.2 Boundary modification

To construct our scheme, we extending the definition of $\Delta_\infty^\varepsilon u$ to all of U. This is analogous the stencil modifications near the boundary in Oberman's scheme. Given a bounded function $u : U \to \mathbb{R}$ and $x \in U$, we define

$$S_\varepsilon^- u(x) := \sup_{|y-x|\le\varepsilon} \frac{u(x)-u(y)}{\rho_\varepsilon(x,y)},$$

$$S_\varepsilon^+ u(x) := \sup_{|y-x|\le\varepsilon} \frac{u(y)-u(x)}{\rho_\varepsilon(x,y)},$$

and

$$-\Delta_\infty^\varepsilon u(x) := \frac{1}{\varepsilon}(S_\varepsilon^- u(x) - S_\varepsilon^+ u(x)),$$

where

$$\rho_\varepsilon(x,y) = \begin{cases} |x-y| & \text{if } x \in \partial U \text{ or } y \in \partial U, \\ \max\{|x-y|,\varepsilon\} & \text{if } x,y \in U. \end{cases}$$

Observe that these new definitions coincide with the old definitions on U_ε.

The corresponding boundary value problem is

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{3.2.3}$$

Existence and comparison of solutions for (3.2.3) follows by Remark 2.6.7.

3.2.3 A linearly interpolating finite difference scheme

Of course, the computer can not directly approximate (3.2.3). Instead, we suppose that $(\mathcal{H},\mathcal{V})$ is a triangulation of U given by a finite set of vertices $\mathcal{V} \subseteq U$ and triangles $\mathcal{H} \subseteq [\mathcal{V}]^3$. Given a function $u : \mathcal{V} \to \mathbb{R}$, we define $\mathcal{H}u : \bar{U} \to \mathbb{R}$ to be the piecewise linear interpolation of u on $\bar{U}$.

Theorem 3.2.2. *Given $\varepsilon > 0$ and $g : \mathcal{V} \cap \partial U \to \mathbb{R}$, there is a unique function $u : \mathcal{V} \to \mathbb{R}$ satisfying*

$$\begin{cases} u = g & \text{on } \mathcal{V} \cap \partial U, \\ -\Delta_\infty^\varepsilon \mathcal{H}u = 0 & \text{on } \mathcal{V} \cap U, \end{cases} \tag{3.2.4}$$

Proof. For uniqueness, we follow Le Gruyer's argument and patch it to work for linear interpolation on triangulations. Suppose $u, v : \mathcal{V} \to \mathbb{R}$ and

$$-\Delta_\infty^\varepsilon \mathcal{H}u \le 0 \le -\Delta_\infty^\varepsilon \mathcal{H}v \quad \text{on } \mathcal{V} \cap U.$$

Suppose, for contradiction, that

$$k := \max_{\mathcal{V}}(u - v) > \max_{\mathcal{V}\cap\partial U}(u - v).$$

Define

$$E := \{x \in \mathcal{V} : (u - v)(x) = k\},$$

and

$$F := \{x \in E : u(x) = \max_{E} u\}.$$

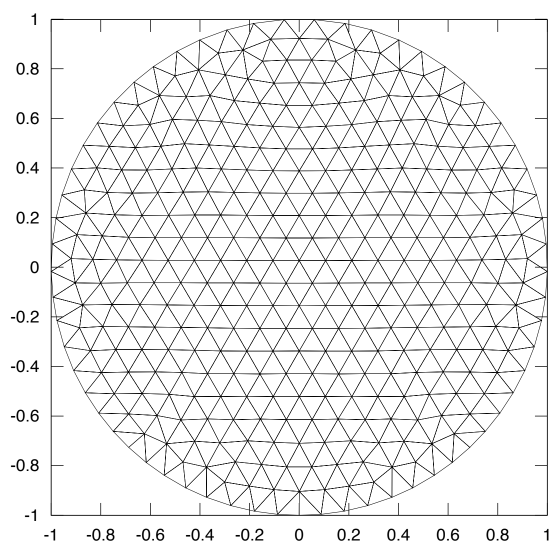
Since $\max_{\bar{U}}(\mathcal{H}u - \mathcal{H}v) = k$, we conclude as in the proof of Lemma 2.3.1 that

$$S_\varepsilon^+\mathcal{H}u = S_\varepsilon^+\mathcal{H}v \quad \text{and} \quad S_\varepsilon^-\mathcal{H}u = S_\varepsilon^-\mathcal{H}v \quad \text{on } U.$$

Now, suppose $x \in F$ and $S_\varepsilon^+\mathcal{H}u(x) > 0$ is realized at some point $y \in t \cap \bar{B}(x, \varepsilon)$ with $t \in \mathcal{H}$. Since $u(z) - v(z) \leq k$ for each vertex z of t and necessarily $\mathcal{H}u(y) - \mathcal{H}v(y) = k$ for some $y \in t$, we must have $u(z) - v(z) = k$ for each vertex z of t. Thus, there is a vertex $z \in E$ with $u(z) > u(x)$, contradicting the definition of F.

Thus $S_\varepsilon^+\mathcal{H}u(x) = 0$ for every $x \in F$. Since $S_\varepsilon^-\mathcal{H}u(x) \leq S_\varepsilon^+\mathcal{H}u(x)$, we conclude that $\mathcal{H}u$ is constant on $\{x \in \bar{U} : \operatorname{dist}(x, F) \leq \varepsilon\}$. However, as $\mathcal{H}u$ is the linear interpolation of u on a triangulation, this implies u is constant on $\mathcal{V}$. Similarly, v is constant on $\mathcal{V}$. □

The boundary value problem (3.2.4) comprises one half of our new numerical scheme. Missing is a good method for choosing the triangulation $(\mathcal{H}, \mathcal{V})$. If we apply this method to regular triangulations like the one shown here,



this scheme has performance roughly equivalent to that of Oberman's scheme (3.1.1). While the scheme incurs are large per-vertex penalty for linear interpolation, some additional accuracy is obtained by making affine functions exact solutions. These two effects seem to offset one another.

3.2.4 Minimizing the residual

Using Theorem 3.2.1, we can estimate how close a solution of (3.2.4) is to the solution of (3.0.1). In fact, Theorem 3.2.1 suggests that we should look for triangulations that minimize the residual.

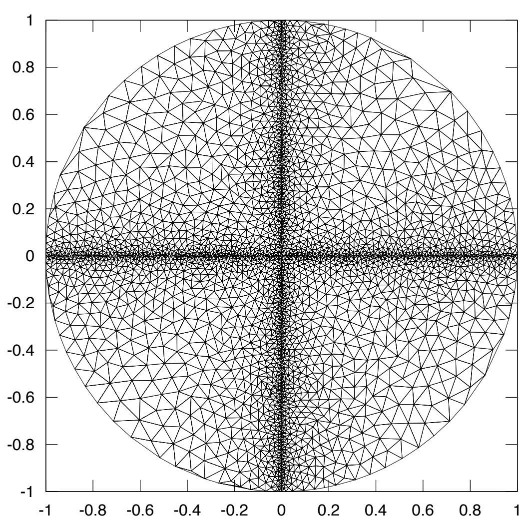
If one knows in advance the shock structure of the solutions, then one can easily find such triangulations. For example, the Aronsson function

$$u(x, y) = x^{4/3} - y^{4/3},$$

fails to be twice differentiable on the coordinate axes. Thus, we want more triangles near the coordinate axes. If we fix in advance the total number of triangles and try to minimize

$$\max \left\{ \sup_{U_\varepsilon} |\Delta_\infty^\varepsilon T^\varepsilon u|, \ \sup_{U_\varepsilon} |\Delta_\infty^\varepsilon T_\varepsilon u| \right\},$$

we obtain a triangulations like the following.



The scheme (3.2.4) performs well on such triangulations. Of course, we do not usually know in advance the shock structure of the solutions.

3.2.5 Automatic refinement

Theorem 3.2.1 suggest a natural way to generate good triangulations automatically. We select $\varepsilon > 0$, a residual threshold $\eta > 0$, and an initial triangulation $(\mathcal{H}_0, \mathcal{V}_0)$ of U with approximate spacing ε. At stage k, we compute the unique $u_k : \mathcal{V}_k \to \mathbb{R}$ that solves

$$\begin{cases} u_k = g & \text{on } \mathcal{V}_k \cap \partial U, \\ -\Delta_\infty^\varepsilon \mathcal{H}_k u_k = 0 & \text{on } \mathcal{V}_k \cap U. \end{cases}$$

If $\sup_U |\Delta_\infty^\varepsilon \mathcal{H}_k u_k| < \eta$, then we stop. Otherwise, we construct $\mathcal{V}_{k+1}$ from $\mathcal{V}_k$ by including the circumcenter of every triangle $t \in \mathcal{H}_k$ such that $\sup_t |\Delta_\infty^\varepsilon \mathcal{H}_k u_k| \geq \eta$. Then we let $\mathcal{H}_{k+1}$ be the Delaunay triangulation of $\mathcal{V}_k$.

Using Theorem 3.2.1, this algorithm can guarantee any desired accuracy. Indeed, the constant in the estimate (3.2.1) can be computed explicitly, and this will tell us how small $\varepsilon, \eta > 0$ need to be in order to meet any accuracy requirement.

Below we give five examples of generated triangulations. In each case, we use the domain $U = B(0,1)$ and the parameters $\varepsilon = \eta = 0.1$. The automatically generated triangulations are significantly rougher than the one we hand-made for the Aronsson function above. This is intentional. The scheme (3.2.4) does not care about element quality, so we sacrificed quality for speed in our refinement algorithm.

Observe that the mesh refinement algorithm appears to uncover the "hidden" shock structure of the solutions. The third and fourth examples make this particularly clear.

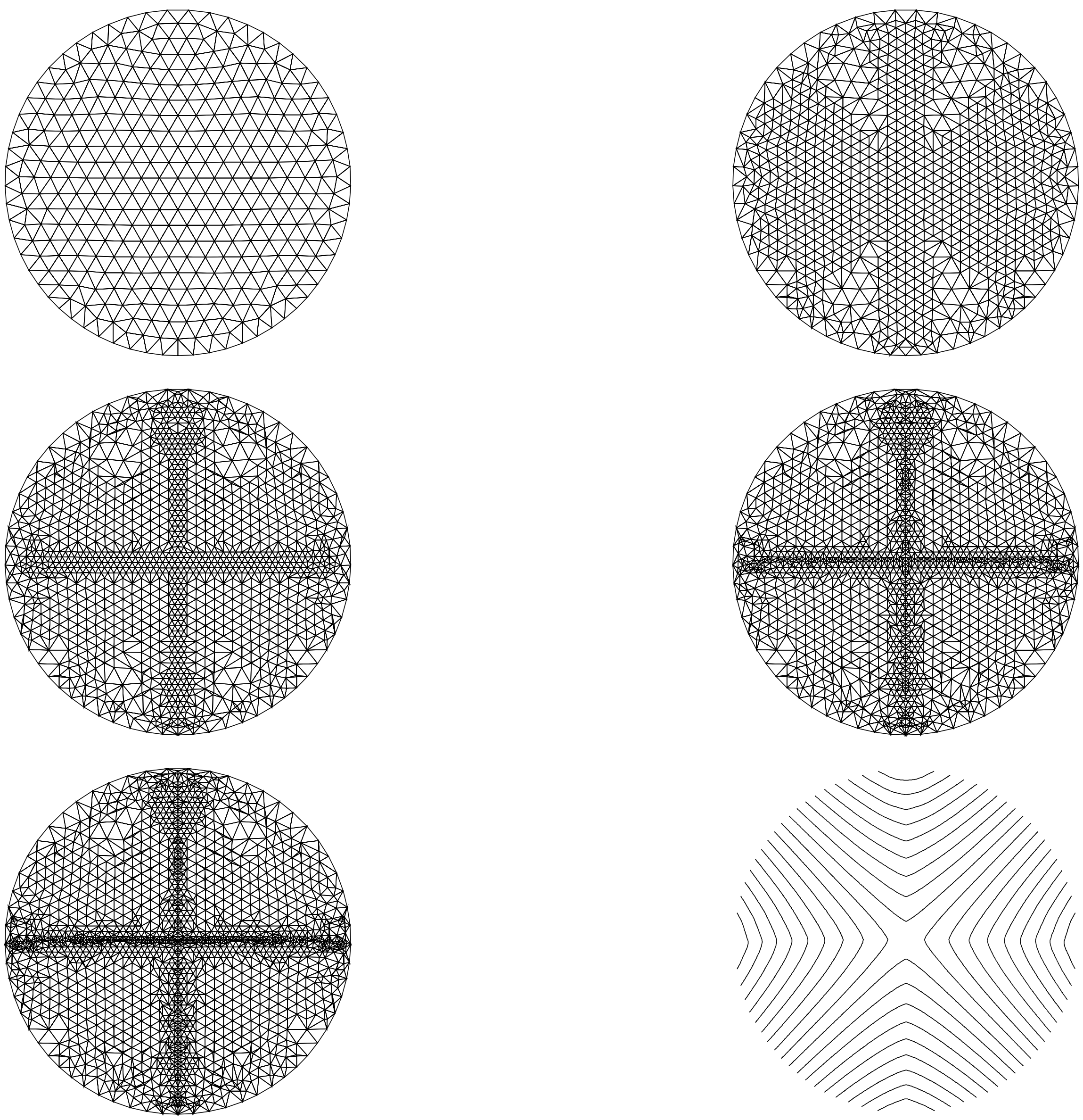
A careful implementation of our adaptive method seems to outperform Oberman's scheme in tests. However, neither method is particularly fast. The principal advantage of Oberman's scheme is its relatively simple formulation. It is easily implemented in an afternoon. Our adaptive method is significantly more complicated. However, it succeeds in avoiding large stencils in regions where the solutions are smooth.

Example 1

When the boundary data is the Aronsson function,

$$g(x,y) = x^{4/3} - y^{4/3},$$

we obtain the following sequence of triangulations and computed solution contour lines.

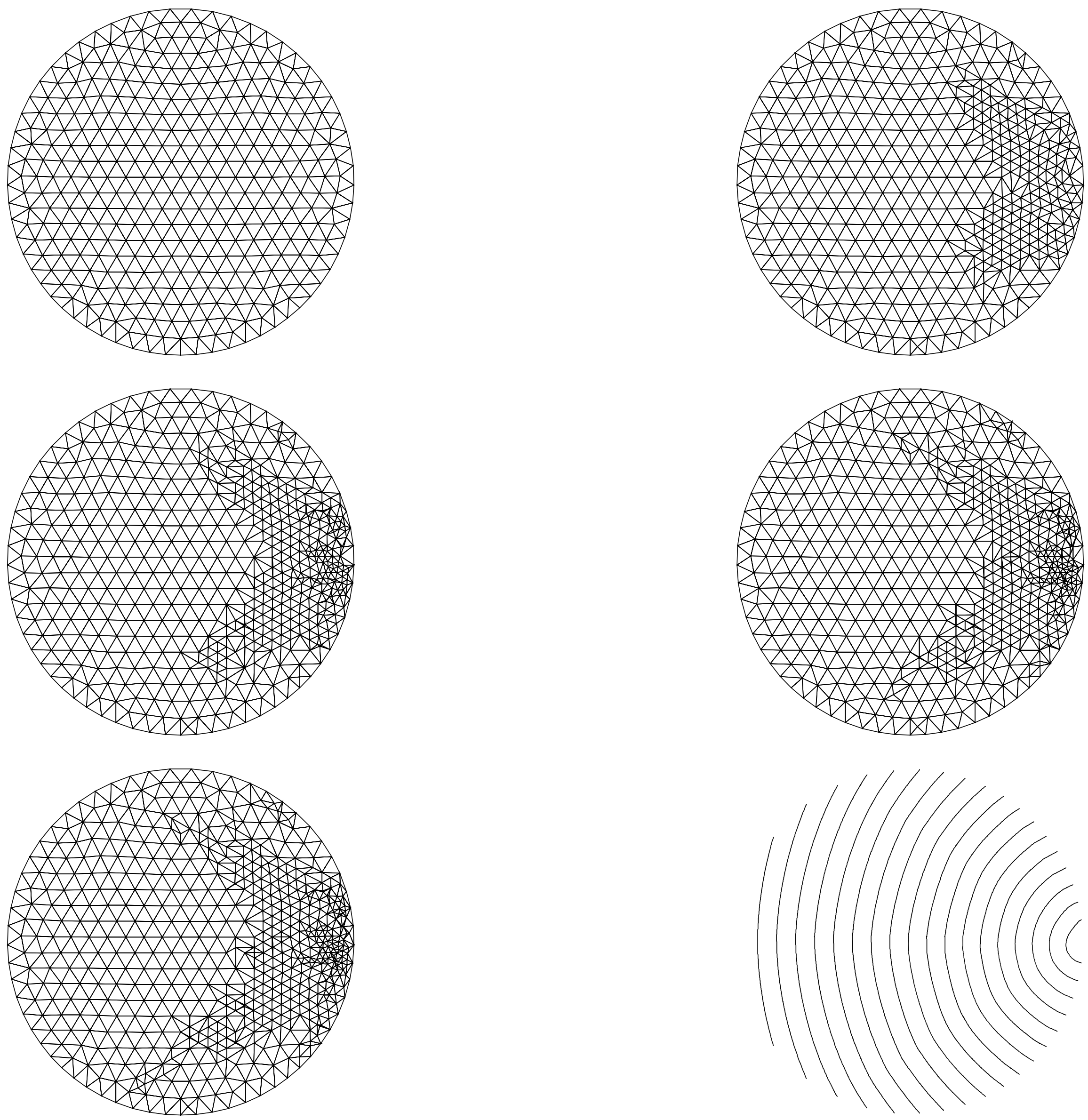


Example 2

When the boundary data is a cone,

$$g(x, y) = |(x, y) - (1, 0)|,$$

we obtain the following sequence of triangulations and computed solution contour lines.

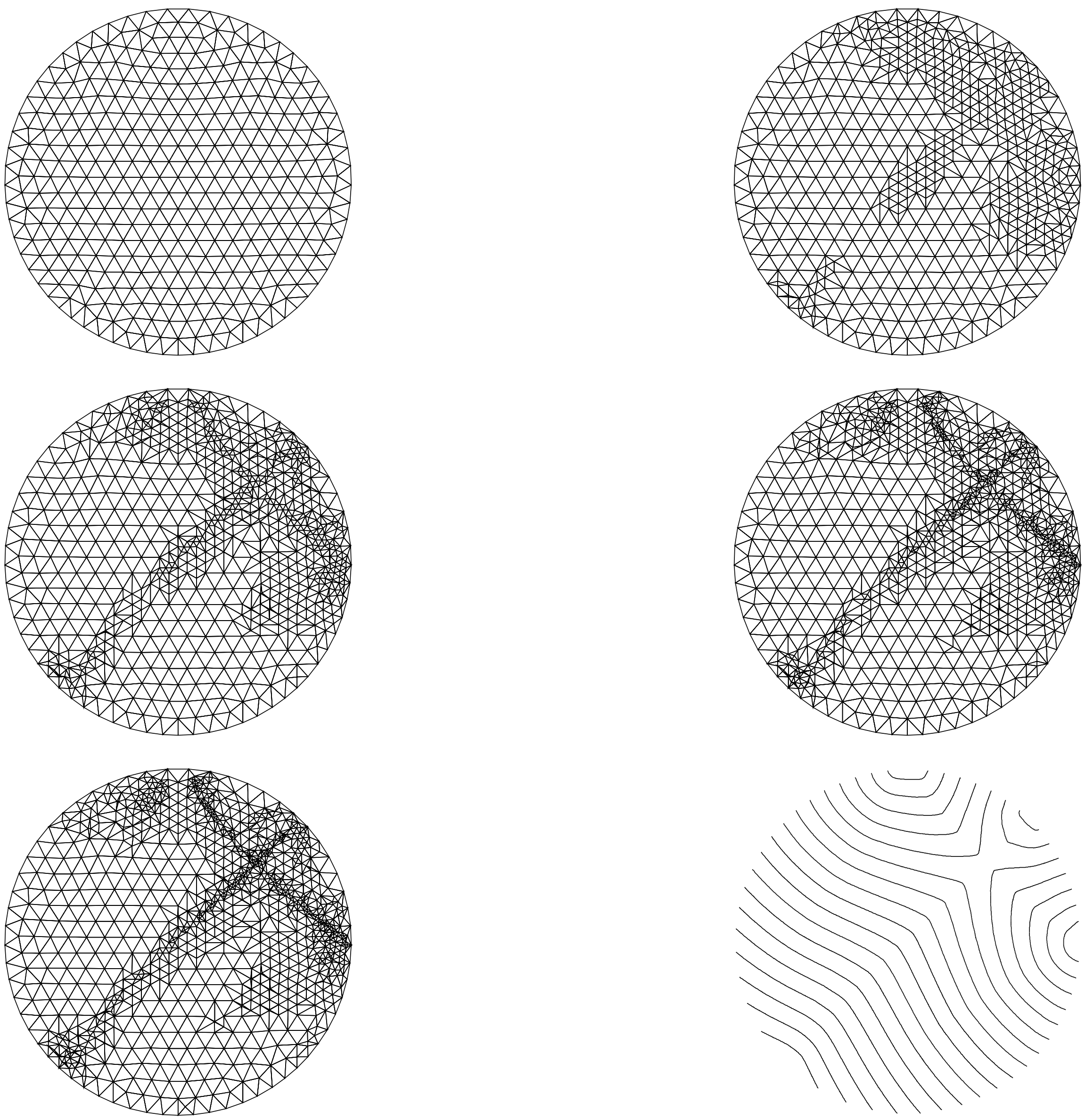


Example 3

When the boundary data is the infimum of two cones,

$$g(x, y) = \min\{|(x, y) - (1, 0)|, |(x, y) - (0, 1)|\},$$

we obtain the following sequence of triangulations and computed solution contour lines.

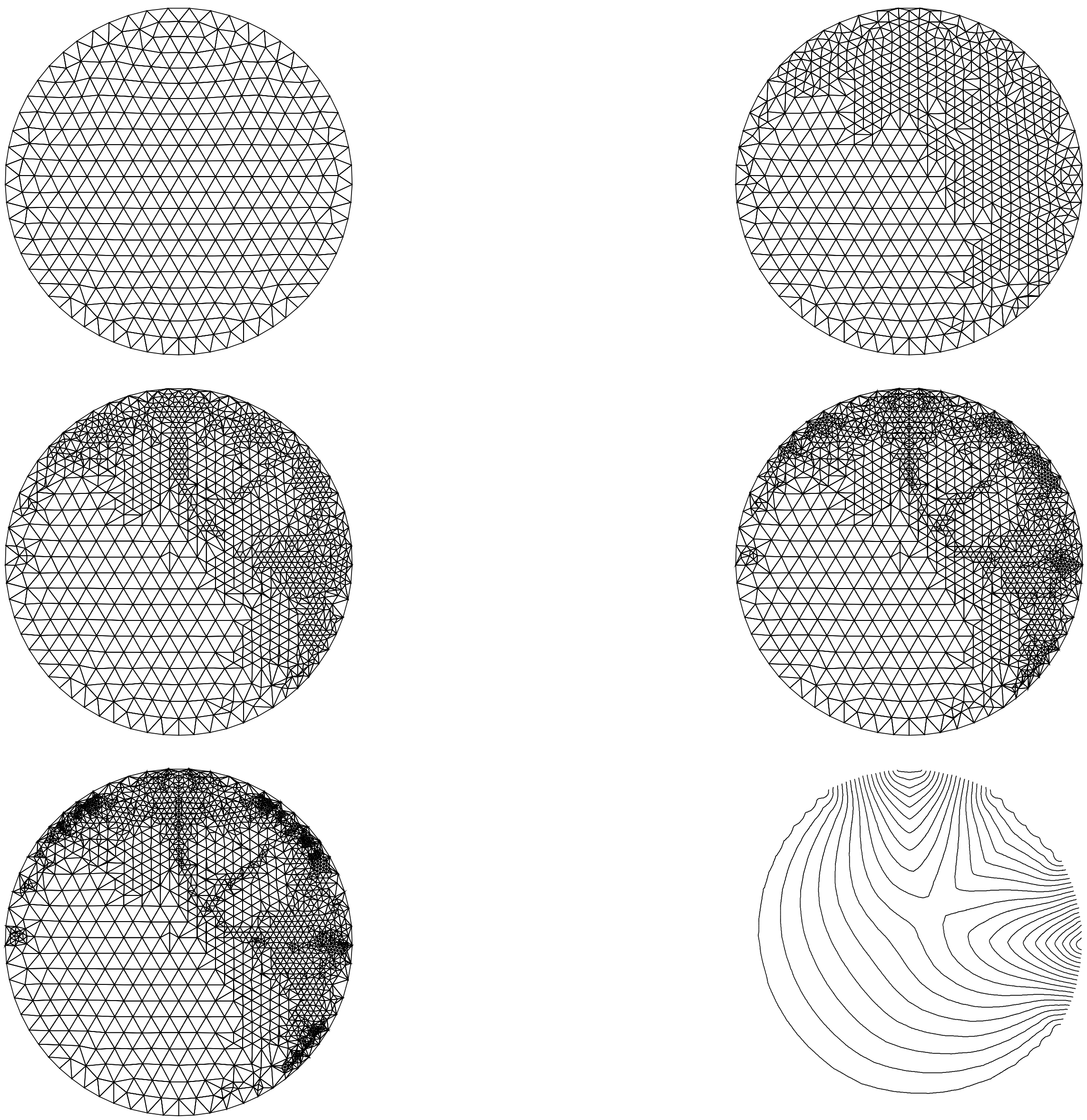


Example 4

When the boundary data is given by

$$g(x, y) = \min\{1/2, |(x, y) - (1, 0)|, |(x, y) - (0, 1)|\},$$

we obtain the following sequence of triangulations and computed solution contour lines.



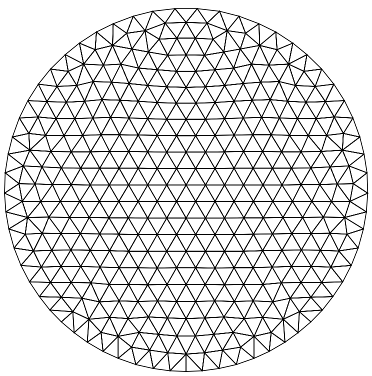
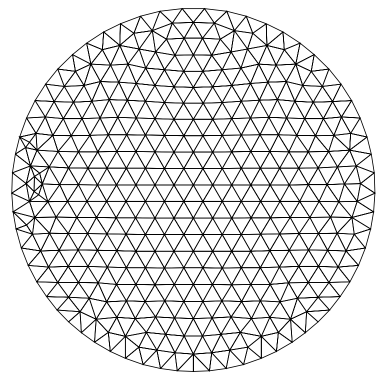
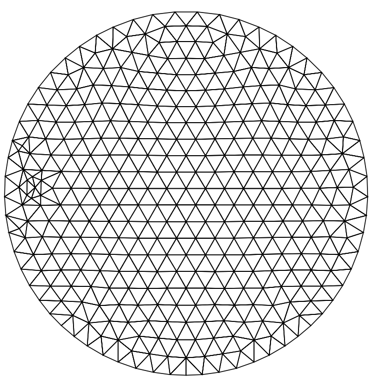
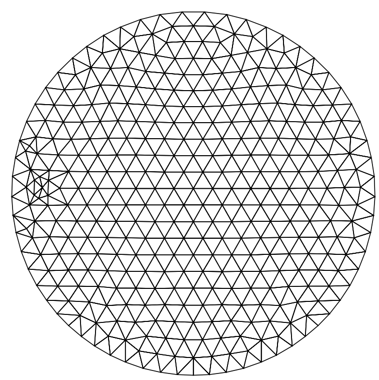
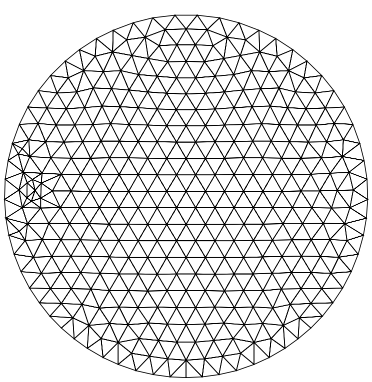
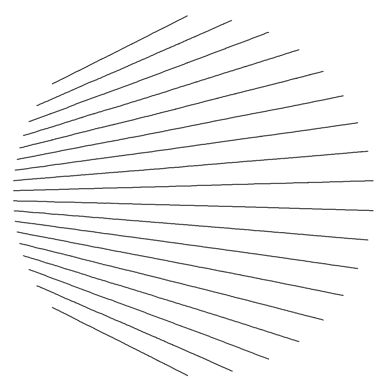
Note the complicated shock structure being revealed as the mesh is refined. The contour wiggles near the boundary are artifacts of the coarse boundary discretization.

Example 5

When the boundary data is the argument function,

$$g(x, y) = \tan^{-1}(y/(x+2)),$$

we obtain the following sequence of triangulations and computed solution contour lines.

Note in this case that the mesh converges after one iteration.

Chapter 4

Interpreting Hasson's example

4.1 Introduction

We assume familiarity with basic model theory [24] and stability theory [35]. In particular, we assume the reader is familiar with Morley rank, forking dependence, imaginaries, and canonical bases. Unless otherwise specified, we assume that theories T are complete and eliminate quantifiers in a countable and relational language $L(T)$. We drop the qualifier Morley from Morley rank and Morley degree.

4.1.1 Definable rank and degree

Recall that a theory T has *definable* rank if for every $\phi(\mathbf{x}, \mathbf{y}) \in L(T)$ and $r \in \mathbb{N}$, there is a $\theta(\mathbf{y}) \in L(T)$ such that

$$\mathrm{RM}(\phi(\mathbf{x}, \mathbf{a})) = r \quad \text{if and only if} \quad M \models \theta(\mathbf{a}),$$

whenever $M \models T$ and $\mathbf{a} \in M$. Similarly, T has *definable* degree if for $\phi(\mathbf{x}, \mathbf{y}) \in L(T)$ and $d \in \mathbb{N}$, there is a $\theta(\mathbf{y}) \in L(T)$ such that

$$\mathrm{dM}(\phi(\mathbf{x}, \mathbf{a})) = d \quad \text{if and only if} \quad M \models \theta(\mathbf{a}),$$

whenever $M \models T$ and $\mathbf{a} \in M$.

A theory T with definable rank has *definably bounded* degree if for every $\phi(\mathbf{x}, \mathbf{y}) \in L(T)$ there is a $d \in \mathbb{N}$ such that

$$\mathrm{dM}(\phi(\mathbf{x}, \mathbf{a})) \leq d.$$

whenever $M \models T$ and $\mathbf{a} \in M$. By compactness, any theory with definable rank and degree has definably bounded degree.

In the literature, definable rank and degree is usually called the *definable multiplicity property (DMP)*, and Hrushovski and Hasson [23] call definable rank and definably bounded degree the *weak definable multiplicity property (wDMP)*. We use definable rank and definable and definably bounded degree here, as we believe it to be more clear.

4.1.2 Fusion

Suppose T_1 and T_2 are theories of finite rank in disjoint languages. A *fusion* of T_1 and T_2 is a complete theory $T \models T_1 \cup T_2$ in a language $L(T) \supseteq L(T_1) \cup L(T_2)$ such that rank in T satisfies the following condition.

(Generic intersections) Whenever $M \models T$, $\phi_i(\mathbf{x}, \mathbf{y}) \in L(T_i)$ for $i = 1, 2$, and $\mathbf{a} \in M$, we have

$$\mathrm{RM}_T(\phi_1(\mathbf{x}, \mathbf{a}) \wedge \phi_2(\mathbf{x}, \mathbf{a})) = v_1 \,\mathrm{RM}_{T_1}(\phi_1(\mathbf{x}, \mathbf{a})) + v_2 \,\mathrm{RM}_{T_2}(\phi_2(\mathbf{x}, \mathbf{a})) - N|\mathbf{x}|,$$

where $N = \mathrm{lcm}(N_1, N_2)$ and $v_i := N/N_i$.

A theorem of Ziegler [38] states that any two theories T_1 and T_2 in disjoint languages with finite definable rank and degree such that $\mathrm{dM}(T_1) = \mathrm{dM}(T_2)$ admit a fusion. This is an extension of Hrushovski [25], who fused strongly minimal sets with definable rank and degree.

4.1.3 Interpretation

Recall that a theory T_1 is *interpretable* in a theory T_2 if there are structures $M_1 \models T_1$ and $M_2 \models T_2$ and an injective map $\tau : M_1 \to M_2^k$, such that the image of every definable subset in M_1^l for $l > 0$ is a definable subset of M_2^{kl}. If M_1 and M_2 are countably saturated and the map τ preserves the Morley rank of definable sets, we say that the interpretation is *rank preserving*. The following result allows us to focus on fusion constructions instead of rank-preserving itnerpretations.

Theorem 4.1.1 (Ziegler [38])**.** *If T has finite rank and admits a fusion with any theory T_2 with definable rank and degree such that $\mathrm{dM}(T) = \mathrm{dM}(T_2)$, then T has a rank-preserving interpretation in a strongly minimal set.*

4.1.4 Hasson's example

Hasson [20] proved that any theory with finite definable rank and degree admits a rank-preserving interpretation in a strongly minimal theory. As a test case for the necessity of definable degree, he proposed the following example. Let

$$M := (M, E, A, B_i, C_i, +_A, +_i, S_i, \pi)_{i \in \mathbb{N}},$$

be a structure with the following properties.

1. E is an equivalence relation on M with infinitely many infinite classes.

2. A, B_i, C_i are 1-ary and pick out distinct classes of E.

3. $+_A$ and $+_i$ are 3-ary and satisfy $(A, +_A) \equiv (B_i, +_i) \equiv (\mathbb{Q}, +)$.

4. S_i is 1-ary and divides C_i into two infinite sets.

5. π is 2-ary and defines a bijection $\pi : M/E \to A$ that maps $\{A\} \cup \{B_i\} \cup \{C_i\}$ to an indiscernible set in $(A, +_A)$.

It is routine to check that $Th(M)$ has finite definable rank and definably bounded degree. What makes M interesting is that it has no rank-preserving expansion with definable degree. Indeed, recall that a rank preserving expansion of $(\mathbb{Q}, +)$ is necessarily degree 1. In particular, if $N \supseteq M$ is a rank-preserving expansion, then $\mathrm{dM}^N(A) = \mathrm{dM}^N(B_i) = 1$ and $\mathrm{dM}^N(C_i) \geq 2$. If N had definable degree, then there would be a definable set $D \subseteq A$ such that $\pi(B_i) \in D$ and $\pi(C_i) \in A \setminus D$, contradicting our observation that $\mathrm{dM}^N(A) = 1$.

Thus, if $\tau : M \to S^k$ is an interpretation of M in a strongly minimal set S, then S can not have definable degree.

4.2 A new fusion construction

In this section, we prove the following theorem.

Theorem 4.2.1. *If T_1 and T_2 have finite definable Morley rank, the same degree, and nice codes, then T_1 and T_2 admit a fusion.*

The definition of *nice codes* appears in Section 4.2.5. For now, we remark that Theorem 4.2.1 applies to Hasson's example.

Our proof follows the standard outline of any Hrushovski construction. We first compute the Fraisse limit of a large class of finite structures and obtain a theory T_∞ of infinite rank. By carefully analyzing the finite-rank types in T_∞, we are able to collapse them to algebraic types by restricting the finite structures in our Fraisse limit. This yields a new theory T_μ with the desired properties.

The principal difficulty lies in keeping the restricted class of finite structures definable. This was handled elegantly in [38], when definable degree was available. In our case, we need some additional machinery.

4.2.1 Free fusion

In this section, we recall the free fusion construction described in [38, 22]. We stop short of building T_∞, describing only the amalgamation class $(\mathcal{K}_\infty, \leq_s)$ that T_∞ is the Fraisse limit of. We assume throughout that T_1 and T_2 have degree 1 and finite definable rank and that $L(T_1) \cap L(T_2) = \emptyset$.

We consider $L(T_1) \cup L(T_2)$-structures $A \models T_1^\forall \cup T_2^\forall$. Recall that for any such structure we can find an ω-saturated model $M \models T_1 \cup T_2$ such that $A \subseteq M$. Given such an M, we can compute $\mathrm{RM}^M_{T_1}(A)$ and $\mathrm{RM}^M_{T_2}(A)$ in the reducts $M|L(T_1)$ and $M|L(T_2)$. However, by quantifier elimination, the ranks we compute do *not* depend on the choice of M. Indeed, they depend only on $\mathrm{qftp}(A)$. Thus we can safely talk about $\mathrm{RM}_{T_i}(A)$ without selecting an ambient model M. Similarly we can make sense of $\mathrm{acl}^{eq}_{T_i}(A)$, although we must be careful about the automorphisms over $\mathrm{dcl}^{eq}_{T_i}(A)$. Alternatively, we could assume everything we do takes place inside some λ-saturated and λ-homogeneous $M \models T_1 \cup T_2$ for some huge $\lambda > 0$.

The amalgamation class $(\mathcal{K}_\infty, \leq_s)$ is given by the following definition.

Definition 4.2.2. Let K, v_1, v_2 be integers so that

$$K = v_1\,\mathrm{RM}(T_1) = v_2\,\mathrm{RM}(T_2)$$

For $A \subseteq B \models T_1^\forall \cup T_2^\forall$ with $B \setminus A$ finite, we define the *prerank of B over A* to be

$$\delta(B/A) := v_1\,\mathrm{RM}_{T_1}(B/A) + v_2\,\mathrm{RM}_{T_2}(B/A) - K|B \setminus A|.$$

Using δ, we define the class of structures

$$\mathcal{K}_\infty := \{A \models T_1^\forall \cup T_2^\forall : \delta(B) \geq 0 \text{ for all finite } B \subseteq A\}.$$

If $A \subseteq B \in \mathcal{K}_\infty$ and

$$\delta(A \cup C/A) \geq 0 \quad \text{ for all finite } C \subseteq B,$$

then we say that A is a *strong substructure of B* and write $A \leq_s B$.

The notions of prerank and strong substructure in $\mathcal{K}_\infty$ enjoy the following nice properties. All of these are easy consequences of the fact that rank is additive and submodular in T_1 and T_2.

Lemma 4.2.3 ([38, 22])**.** *The following properties hold for all $A, B, C \in \mathcal{K}_\infty$.*

1. *If $A \subseteq B \subseteq C$, then $\delta(C/A) = \delta(C/B) + \delta(B/A)$.*
2. *If $A, B \subseteq C$, then $\delta(A/A \cap B) \geq \delta(A \cup B/B)$.*
3. *If $A \leq_s B \leq_s C$, then $A \leq_s C$.*
4. *If $A, B \leq_s C$, then $A \cap B \leq_s C$.*
5. *If $A \subseteq B$, then*
$$\mathrm{cl}_B(A) := \bigcap\{A' \leq_s B : A' \supseteq A\} \leq_s B.$$
We call $\mathrm{cl}_B(A)$ the strong closure of A in B.

In the sequel we need an approximation of strong substructure that is first-order definable.

Definition 4.2.4. If $A \subseteq B \in \mathcal{K}_\infty$, $m > 0$, and $\delta(A \cup C/A) \geq 0$ for all $C \subseteq B$ with $|C| < m$, then we write $A \leq_{s,m} B$.

Lemma 4.2.5. *If $A \subseteq B \in \mathcal{K}_\infty$, then there is a $\mathrm{cl}_{B,m}(A) \leq_{s,m} B$ such that $A \subseteq \mathrm{cl}_{B,m}(A)$ and $\mathrm{cl}_{B,m} \subseteq C$ whenever $A \subseteq C \leq_{s,m} B$.*

Proof. Call $A' \subseteq A''$ an *m-step* if $|A'' \setminus A| < m$, $\delta(A''/A') < 0$, and $\delta(A^*/A') \geq 0$ whenever $A' \subseteq A^* \subsetneq A''$. Choose some maximal chain $A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n$ of m-steps. Set $\mathrm{cl}_{B,m}(A) := A_n$ and note that $\mathrm{cl}_{B,m} \leq_{s,m} B$.

Now, suppose $A \subseteq C \leq_{s,m} B$ and $\mathrm{cl}_{B,m} \not\subseteq C$. Let $i < n$ be least so that $A_{i+1} \not\subseteq C$. Then $0 > \delta(A_{i+1}/C \cap A_i) \geq \delta(A_{i+1} \cup C/C)$, which contradicts our assumption that $C \leq_{s,m} B$. □

We extend δ and $\leq_{s,m}$ to imaginary elements as follows.

Definition 4.2.6. If $A \in \mathcal{K}_\infty$, we define

$$\mathrm{acl}^{eq}_\infty(A) := \mathrm{acl}^{eq}_{T_1}(A) \times \mathrm{acl}^{eq}_{T_2}(A)$$

and include $A \subseteq \mathrm{acl}^{eq}_\infty(A)$ via $a \mapsto (a, a)$. If Σ is the home sort shared by T_1 and T_2 then for $X \subseteq Y \subseteq \mathrm{acl}^{eq}_\infty(C)$ define

$$\delta(Y/X) := v_1 \,\mathrm{RM}_{T_1}(\pi_1(Y)/\pi_1(X)) + v_2 \,\mathrm{RM}_{T_2}(\pi_2(Y)/\pi_2(X)) - N|(Y \setminus X) \cap \Sigma|.$$

For $A \subseteq B$ and $X \subseteq \mathrm{acl}^{eq}_\infty(B)$, write $X \leq_{s,m} A$ if $X \cap \Sigma \subseteq A$ and $\delta(X \cup C/X) \geq 0$ whenever $C \subseteq X$ and $|C| < m$.

Lemma 4.2.7. *If $A \subseteq B \in \mathcal{K}_\infty$ and $X \subseteq \mathrm{acl}^{eq}_\infty(B)$, then there is a $\mathrm{cl}_{A,m}(X) \subseteq A$ such that $X \cup \mathrm{cl}_{A,m}(X) \leq_{s,m} A$ and $\mathrm{cl}_{A,m}(X) \subseteq C$ whenever $C \subseteq A$ and $X \cup C \leq_{s,m} A$.*

Proof. Same as the proof of Lemma 4.2.5. □

The first step in our analysis of finite rank types in T_∞ is given by the following lemma. The ideas is that any extension $A \leq_s B \in \mathcal{K}_\infty$ where $B \setminus A$ is finite can be decomposed as a sequence of *minimal* extensions $A \leq_s C_1 \leq_s \cdots \leq_s C_k \leq_s B$, whose types are easy to analyze.

Definition 4.2.8. An extension $A \leq_s B \in \mathcal{K}_\infty$ is *minimal* if there is no C with $A \leq_s C \leq_s B$, $A \neq C$, and $C \neq B$.

Lemma 4.2.9 ([38, 22])**.** *If the extension $A \leq_s B \in \mathcal{K}_\infty$ is minimal, then $B \setminus A$ is finite and one of the following holds.*

1. *The extension is algebraic, that is, $\delta(B/A) = 0$, $B = A \cup \{b\}$, and for some $i = 1, 2$, $\mathrm{tp}_{T_i}(b/A)$ is algebraic and $\mathrm{tp}_{T_{2-i}}(b/A)$ generic.*

2. *The extension is prealgebraic, that is, $\delta(B/A) = 0$ and $\mathrm{tp}_{T_i}(b/A)$ is not algebraic for any $b \in B \setminus A$ and $i = 1, 2$.*

3. *The extension is transcendental, that is, $N \geq \delta(B/A) > 0$ and $\mathrm{tp}_{T_i}(b/A)$ is not algebraic for any $b \in B \setminus A$ and $i = 1, 2$.*

4.2.2 Codes

In order to definably analyze types in T_∞, we need a special notion of normal formula, called a *code*. In this section we will repeat the code construction of [38] and make a few minor adjustments. We fix a theory T with finite rank and the definably bounded degree for the rest of this section.

Definition 4.2.10. A *code* c is a parameter-free formula $\phi_c(\mathbf{x}; y)$ with the following properties.

1. $\mathbf{x}$ is a tuple of real variables, $|\mathbf{x}| = n_c$, and $y \in T^{eq}$.
2. Consistent $\phi_c(\mathbf{x}; a)$ have rank k_c and degree at most D_c. If $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ then the elements of $\mathbf{b}$ are distinct and for each $S \subsetneq \{1, ..., n_c\}$
$$\mathrm{RM}(\mathbf{b}/a\mathbf{b}_S) \le k_{c,S}$$
with equality for generic $\mathbf{b}$. If a is generic in $\exists \mathbf{x} \phi_c(\mathbf{x}; y)$ then $\phi_c(\mathbf{x}; a)$ has degree 1. Finally, $k_{c,\{i\}} < k_c$ for all i.
3. If $\mathrm{RM}(\phi_c(\mathbf{x}; a) \wedge \phi_c(\mathbf{x}; a')) = k_c$ then $a = a'$.
4. There is a $G_c \le Sym(n_c)$ such that for each consistent $\phi_c(\mathbf{x}; a)$ and $\sigma \in Sym(n_c)$,
   (a) $\sigma \in G_c$ implies $\phi_c(\mathbf{x}; a) \equiv \phi_c(\mathbf{x}^\sigma; a)$.
   (b) $\sigma \notin G_c$ implies $\mathrm{RM}(\phi_c(\mathbf{x}; a) \wedge \phi_c(\mathbf{x}^\sigma; a')) < k_c$ for all a'.

This definition of codes differs from the definable rank case in one critical way. The degree of consistent instances $\phi_c(\mathbf{x}; a)$ is not always 1. In fact, if $D_c = 1$, then the two definitions coincide.

A formula $\psi(\mathbf{x}; d)$ is *simple* if it has degree 1, the components of its realizations are distinct, and the components of any generic realization lie outside $\mathrm{acl}(d)$. For any two formulas $\psi_1(\mathbf{x}; d_1)$ and $\psi_2(\mathbf{x}; d_2)$ with the same free variables, we write
$$\psi_1(\mathbf{x}; d_1) \sim \psi_2(\mathbf{x}; d_2)$$
when
$$\mathrm{RM}(\psi_1(\mathbf{x}; d_1) \triangle \psi_2(\mathbf{x}; d_2)) < \mathrm{RM}(\psi_1(\mathbf{x}; d_1)) = \mathrm{RM}(\psi_2(\mathbf{x}; d_2)).$$
If $\psi(\mathbf{x}; d)$ is simple and $\phi_c(\mathbf{x}; a) \sim \psi(\mathbf{x}; d)$, then we say that c *encodes* $\psi(\mathbf{x}; d)$. If $\psi(\mathbf{x}; d)$ is simple and $\mathrm{RM}(\phi_c(\mathbf{x}; a) \wedge \psi(\mathbf{x}; d)) = k_c = \mathrm{RM}(\psi(\mathbf{x}; d))$, then we say that c *covers* $\psi(\mathbf{x}; d)$.

Lemma 4.2.11. *Every simple $\psi(\mathbf{x}; d)$ is encoded by some code c.*

Proof. Let a be the canonical base of the global type isolated by $\psi(\mathbf{x}; d)$ and let $\phi_c(\mathbf{x}; y)$ be parameter-free so that $\phi_c(\mathbf{x}; a) \sim \psi(\mathbf{x}; d)$. We will strengthen $\phi_c(\mathbf{x}; y)$ to meet the requirements above.

Let $\mathbf{b}$ be a generic realization of $\phi_c(\mathbf{x}; a)$. Let $k_{c,S} = \mathrm{RM}(\mathbf{b}/a\mathbf{b}_S)$ for $S \subsetneq \{1, ..., n_c\}$. Strengthening $\phi_c(\mathbf{x}; y)$, we may assume

$$\mathrm{RM}(\phi_c(\mathbf{x}; a) \wedge \mathbf{x}_S = \mathbf{b}_S) = k_{c,S}$$

for all S. Let $\theta(y)$ isolate $\mathrm{tp}(a)$ in its rank. Replace $\phi_c(\mathbf{x}; y)$ with

$$\phi_c(\mathbf{x}; y) \wedge \theta(y) \wedge \bigwedge_S \mathrm{RM}_{\mathbf{z}}(\phi_c(\mathbf{z}; y) \wedge \mathbf{z}_S = \mathbf{x}_S) = k_{c,S}.$$

Now, the wDMP implies the existence of D_c, the choice of $\theta(y)$ forces $\phi_c(\mathbf{x}; a')$ to have degree 1 for any a' generic in $\exists \mathbf{x} \phi_c(\mathbf{x}; y)$, and $k_{c,\{i\}} < k_c$ follows from the simplicity of $\psi(\mathbf{x}; d)$. Thus we have (2).

Let $p(y) = \mathrm{tp}(a)$ and note that since a is a canonical base,

$$p(y) \wedge p(y') \wedge \mathrm{RM}_{\mathbf{x}}(\phi_c(\mathbf{x}; y) \wedge \phi_c(\mathbf{x}; y')) = k_c \to y = y'.$$

By compactness there is some $\theta(y) \in p(y)$ which works in place of $p(y)$ above. If we replace $\phi_c(\mathbf{x}; y)$ with $\phi_c(\mathbf{x}; y) \wedge \theta(y)$ we get (3).

To achieve (4), first note that the collection of all $\sigma \in Sym(n_c)$ such that $\phi_c(\mathbf{x}; a) \sim \phi_c(\mathbf{x}^\sigma; a^\sigma)$ for some $a^\sigma \equiv a$ forms a subgroup $G_c \leq Sym(n_c)$. Replacing $\phi(\mathbf{x}; y)$ with

$$\bigwedge_{\sigma \in G_c} \phi_c(\mathbf{x}^\sigma; y) \wedge \mathrm{RM}_{\mathbf{x}} \left(\bigwedge_{\sigma \in G_c} \phi_c(\mathbf{x}^\sigma; y) \right) = k_c,$$

we have (4a). Since, for $\sigma \in Sym(n_c) \setminus G_c$,

$$p(y) \wedge p(y') \to \mathrm{RM}_{\mathbf{x}}(\phi(\mathbf{x}; y) \wedge \phi_c(\mathbf{x}^\sigma; y')) < k_c,$$

there is (by compactness) a $\theta(y) \in p(y)$ such that

$$\phi_c(\mathbf{x}; y) \wedge \theta(y)$$

satisfies (4b) as well. □

Lemma 4.2.12. *There exists a set of codes $\mathcal{C}$ such that*

1. *Every simple formula is covered by a unique $c \in \mathcal{C}$.*

2. *If $c \in \mathcal{C}$ and $\sigma \in Sym(n_c)$ there is a unique $c^\sigma \in \mathcal{C}$ with $\phi_c(\mathbf{x}^\sigma; y) \equiv \phi_{c^\sigma}(\mathbf{x}; y)$.*

Proof. We will build $\mathcal{C}$ as a limit of finite sets, starting with $\mathcal{C} = \emptyset$ and inductively maintaining (1)' and (2), where

(1)' Every simple formula is covered by at most one $c \in \mathcal{C}$.

Suppose $\psi(\mathbf{x}; d)$ is a simple formula not covered by some code in $\mathcal{C}$. Choose c which encodes $\psi(\mathbf{x}; d)$. Replace $\phi_c(\mathbf{x}; y)$ with

$$\phi_c(\mathbf{x}; y) \wedge \bigwedge_{c' \in \mathcal{C}'} \forall y' \ \mathrm{RM}_{\mathbf{x}}(\phi_{c'}(\mathbf{x}; y') \wedge \phi_c(\mathbf{x}; y)) < k_c,$$

where $\mathcal{C}' := \{c' \in \mathcal{C} : n_c = n_{c'} \text{ and } k_c = k_{c'}\}$, and note that this is still a code.

Choose representatives $\sigma_1, ..., \sigma_m$ of the right cosets of G_c and define, for $\sigma \in Sym(n_c)$, c^σ to be the code with $\phi_{c^\sigma}(\mathbf{x}; y) := \phi_c(\mathbf{x}^\sigma; y)$. Now $\mathcal{C} \cup \{c^{\sigma_1}, ..., c^{\sigma_m}\}$ satisfies (1)' and (2) and covers $\psi(\mathbf{x}; d)$. □

We call a collection of codes $\mathcal{C}$ satisfying the conclusion of the lemma above a *system of codes* for T.

Lemma 4.2.13. *For every code c there is a constant m_c and a $\emptyset$-definable partial function f_c so that if $\mathbf{b}_1, ..., \mathbf{b}_{m_c}$ are independent realizations of $\phi_c(\mathbf{x}; a)$, then $a = f_c(\mathbf{b}_1, ..., \mathbf{b}_{m_c})$.*

Proof. This is a standard stability fact. See Remark 2.26 of [35]. □

4.2.3 Prealgebraic Codes

We are now ready to definably analyze types in T_∞. We once again assume that T_1 and T_2 have degree 1 and finite definable rank and that $L(T_1) \cap L(T_2) = \emptyset$. We fix a system of codes $\mathcal{C}_i$ for each T_i.

Definition 4.2.14. A *prealgebraic code* is a pair $c = (c_1, c_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ so that

1. $n_c := n_{c_1} = n_{c_2}$,
2. $v_1 k_{c_1} + v_2 k_{c_2} - K n_c = 0$,
3. $v_1 k_{c_1,S} + v_2 k_{c_2,S} - K(n_c - |S|) < 0$ for $\emptyset \subsetneq S \subsetneq \{1, ..., n_c\}$,
4. $\phi_c(\mathbf{x}; y) := \phi_{c_1}(\mathbf{x}; y_1) \wedge \phi_{c_2}(\mathbf{x}; y_2)$,
5. $D_c := D_{c_1} \cdot D_{c_2}$,
6. $G_c := G_{c_1} \cap G_{c_2}$.

We say a prealgebraic code instance $\phi_c(\mathbf{x}; a)$ is *over* $A \in \mathcal{K}_\infty$ if $a \in \mathrm{acl}^{eq}_\infty(A)$; i.e., if $a = (a_1, a_2) \in \mathrm{acl}^{eq}_{T_1}(A) \times \mathrm{acl}^{eq}_{T_2}(A)$.

Definition 4.2.15. Suppose $\phi_c(\mathbf{x}; a)$ is over $A \in cK_\infty$ and $B, \mathbf{b} \subseteq A$. We say that $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ is *B-generic* if $\mathrm{RM}_{T_i}(\mathbf{b}/Ba_i) = k_{c_i}$ for $i = 1, 2$. We say that a sequence of realizations $\mathbf{b}_1, ..., \mathbf{b}_N$ of $\phi_c(\mathbf{x}; a)$ is independent if and only if it is independent over a_i in each T_i.

The following lemma is proved in [38], but we include a proof here because it helps explain the purpose of prealgebraic codes.

Lemma 4.2.16 (Ziegler [38])**.** *If $A \leq_s A \cup \{\mathbf{b}\} \in \mathcal{K}_\infty$ is prealgebraic there is a unique prealgebraic code c and parameter $a \in \mathrm{acl}^{eq}(A)$ such that $\mathbf{b}$ is an A-generic realization of $\phi_c(\mathbf{x}; a)$.*

On the other hand, if $\mathbf{b} \nsubseteq A$, $a \in \mathrm{acl}^{eq}(A)$, and $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ then $\delta(\mathbf{b}/A) \leq 0$. Moreover $\delta(\mathbf{b}/A) = 0$ if and only if $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic if and only if $\mathbf{b}$ is an A-generic realization of $\phi_c(\mathbf{x}; a)$.

Proof. Suppose $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic. Since $\mathrm{tp}_{T_i}(\mathbf{b}/A)$ is not algebraic, there is a simple $\psi_i(\mathbf{x}; d_i) \in L_i$ such that $d_i \in \mathrm{acl}^{eq}_{T_i}(A)$ and $\mathbf{b}$ is an A generic realization of $\psi_i(\mathbf{x}; d_i)$. Now choose $c_i \in \mathcal{C}_i$ and $a_i \in \mathrm{acl}^{eq}_{T_i}(A)$ such that

$$\mathrm{RM}_{T_i}(\psi_i(\mathbf{x}; d_i) \wedge \phi_{c_i}(\mathbf{x}; a)) = \mathrm{RM}_{T_i}(\psi_i(\mathbf{x}; d_i)) = k_c.$$

Because $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic, $\delta(\mathbf{b}/A) = 0$ and $\delta(\mathbf{b}/A\mathbf{b}_s) < 0$ whenever $\emptyset \subsetneq S \subsetneq \{1, ..., n_c\}$. It follows that $v_1 k_{c_1} + v_2 k_{c_2} - K n_c = 0$ and $v_1 k_{c_1,S} + v_2 k_{c_2,S} - K(n_c - |S|) < 0$ whenever $\emptyset \subsetneq S \subsetneq \{1, ..., n_c\}$. Thus $c = (c_1, c_2)$ is a prealgebraic code and $\mathbf{b}$ is an A-generic realization of $\phi_c(\mathbf{x}; a)$ where $a = (a_1, a_2) \in \mathrm{acl}^{eq}(A)$.

For the second part, note that if $A \cap \{\mathbf{b}\} \neq \emptyset$, then $\delta(\mathbf{b}/A) \leq v_1 k_{c_1,S} + v_2 k_{c_2,S} - K(n_c - |S|) < 0$, where $S = \{i \mid b_i \in A\}$. Furthermore, if $A \cap \{\mathbf{b}\} = \emptyset$, then $\delta(\mathbf{b}/A) \leq v_1 k_{c_1} + v_2 k_{c_2} - K n_c = 0$. □

Lemma 4.2.17. *For each prealgebraic code c we can find an integer $m_c \geq n_c$ so that if $A \leq_{s,m_c} B$, $a \in \mathrm{acl}^{eq}(B)$, and $a \notin \mathrm{dcl}^{eq}(A)$, then fewer than m_c distinct realizations of $\phi_c(\mathbf{x}; a)$ intersect A. Moreover, for any distinct $\mathbf{b}_1, .., \mathbf{b}_{m_c}$ there is at most one parameter a such that $\mathbf{b}_i \models \phi_c(\mathbf{x}; a)$ for all $i \leq m_c$.*

Proof. It suffices to prove the lemma for set-wise distinct realizations.

Suppose $\mathbf{b}_1, ..., \mathbf{b}_m \models \phi_c(\mathbf{x}; a)$ and $\mathbf{b}_i \nsubseteq \bigcup_{j<i} \mathbf{b}_j$ for all $i < m$. By the additivity of δ,

$$\delta(\mathbf{b}_1 ... \mathbf{b}_m) \leq \delta(a) + \sum_{i \leq m} \delta(\mathbf{b}_i / a\mathbf{b}_1 ... \mathbf{b}_{i-1}).$$

By Lemma 4.2.16, $\mathbf{b}_i$ is a non-generic realization of $\phi_c(\mathbf{x}; a)$ over $a\mathbf{b}_1 ... \mathbf{b}_{i-1}$ if and only if $\delta(\mathbf{b}_i / a\mathbf{b}_1 ... \mathbf{b}_{i-1}) < 0$. Since $\delta(\mathbf{b}_1 ... \mathbf{b}_N) \geq 0$, $\mathbf{b}_i$ must be $a\mathbf{b}_1 ... \mathbf{b}_{i-1}$-generic for all but at most $\delta(a)$ of the $i < m$. Moreover, $\delta(a)$ is bounded uniformly in c.

The above paragraph shows that given a sufficiently long sequences $\mathbf{b}_1, ..., \mathbf{b}_m$ of set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$, more than half of the length m_{c_i} $(i = 1, 2)$ subsequences are independent. Thus given a sufficiently long sequence, a_i is the consensus value of f_{c_i} on the length m_{c_i} subsequences. Hence a is uniquely determined.

Suppose $A \leq_{s,m_c} B$, $a \in \mathrm{acl}^{eq}(B)$, and $a \notin \mathrm{dcl}(A)$. Since $|\,\mathrm{cl}_{B,2n_c}(a)| < 2n_c\delta(a)$ there is a finite bound M_c on the number of $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ with $\mathbf{b} \subseteq A$ or $\mathbf{b} \subseteq \mathrm{cl}_{B,2n_c}(a)$. By Lemma 4.2.16, any two set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$ which are not contained in $\mathrm{cl}_{B,2n_c}(a)$ are disjoint. Thus if $\mathbf{b}_1, ..., \mathbf{b}_m$ are set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$ with $\mathbf{b}_i \cap A \neq \emptyset$, then

$$0 \leq \delta(\mathbf{b}_1...\mathbf{b}_k a/A) \leq \delta(a/A) - (m - M_c).$$

Thus we can increase m to the desired m_c. □

Definition 4.2.18. We say that a prealgebraic code instance $\phi_c(\mathbf{x}; a)$ is *strongly based* on a set A if A contains at least m_c distinct realizations of $\phi_c(\mathbf{x}; a)$.

Choose an injective function $c \mapsto s_c$ on the prealgebraic codes such that

$$s_c > (m_c n_c + 1)! + 2m_c\delta(a)$$

for all consistent $\phi_c(\mathbf{x}; a)$.

Definition 4.2.19. We say a prealgebraic code instance $\phi_c(\mathbf{x}; a)$ over A is *long in A* if and there are more than s_c distinct realizations of $\phi_c(\mathbf{x}; a)$ in A. If $\mathbf{b}_1, ..., \mathbf{b}_N$ are distinct realizations of some $\phi_c(\mathbf{x}; a)$ and $N > s_c$, then we say that $\{\mathbf{b}_i\}$ is a *long sequence in $\phi_c(\mathbf{x}; a)$*.

We now give the main combinatorial argument in our construction. We call this the *Decomposition Lemma*. This lemma allows us to definably analyze almost orthogonality of prealgebraic codes in T_∞.

Lemma 4.2.20. *Suppose $A \leq_s B \in \mathcal{K}_\infty$ and $B \setminus A$ is finite. We can find*

$$A \leq_s X \subsetneq B$$

such that if

$$Z := \{\mathbf{b} \subseteq B \mid \mathbf{b} \nsubseteq X \text{ is an element of a long sequence strongly based on } X\},$$

then

1. *$\delta(\mathbf{b}\mathbf{b}'/X) = 0$ for all $\mathbf{b}, \mathbf{b}' \in Z$.*
2. *For every long $\phi_c(\mathbf{x}; a)$ either*

   (a) $\phi_c(\mathbf{x}; a)$ is strongly based on X and $\mathrm{cl}_{B,m_c}(a) \subseteq X$,

   or (b) there is a $\mathbf{b} \in Z$ such that $X \cup \{\mathbf{b}\}$ contains every realization of $\phi_c(\mathbf{x}; a)$.

Proof. We will build X in stages starting with $X = A$ and inductively maintaining the following conditions.

- $\delta(\mathbf{b}\mathbf{b}'/X) = 0$ for all $\mathbf{b}, \mathbf{b}' \in Z$.
- If (2) fails for $\phi_c(\mathbf{x}; a)$, then $X \leq_{s,m_c} B$, $X \cup \{\mathbf{b}\} \leq_{s,m_c} B$ for all $\mathbf{b} \in Z$, and $||Z|| > 2m_c\delta(X/A)$ where $||Z||$ is the number of set-wise distinct elements in Z.

Choose a $\phi_c(\mathbf{x}; a)$ that witnesses the failure of (2). Since $X \leq_{s,m_c} B$, it can not be the case that $\phi_c(\mathbf{x}; a)$ is strongly based on X. In fact, fewer than m_c realizations of $\phi_c(\mathbf{x}; a)$ intersect X by Lemma 4.2.17. Since $c \mapsto s_c$ is injective, we may choose $\phi_c(\mathbf{x}; a)$ which maximizes m_c.

If there is a $\mathbf{b} \in Z$ with $\phi_c(\mathbf{x}; a)$ is strongly based on $X \cup \{\mathbf{b}\}$, then set $\tilde{X} := X \cup \{\mathbf{b}\}$. Otherwise, choose $\mathbf{b}_1, ..., \mathbf{b}_{m_c} \models \phi_c(\mathbf{x}; a)$ and set $\tilde{X} := X \cup \bigcup_i \{\mathbf{b}_i\}$. By the proof of Lemma 4.2.17, we can select the $\mathbf{b}_i$ which include all the realizations of $\phi_c(\mathbf{x}; a)$ which intersect X. Moreover, we can select the $\mathbf{b}_i$ such that set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$ not contained in $\tilde{X}$ are pairwise disjoint.

Define
$$\tilde{Y} := \{\mathbf{b} \in \tilde{Z} \mid \mathbf{b} \in Z \text{ or } \mathbf{b} \models \phi_c(\mathbf{x}; a)\}$$
and note that $||\tilde{Y}|| > 2m_c\delta(\tilde{X}/A)$, because $s_c > (m_c n_c + 1)! + 2m_c\delta(a)$.

Now, close $\tilde{X}$ under the following three operations.

- If $\tilde{X} \not\leq_{s,m_c} B$ then set $\tilde{X} := \mathrm{cl}_{B,m_c}(\tilde{X})$.
- If $\tilde{X} \cup \{\mathbf{b}\} \not\leq_{s,m_c} B$ for some $\mathbf{b} \in \tilde{Z}$ then set $\tilde{X} := \mathrm{cl}_{B,m_c}(\tilde{X} \cup \{\mathbf{b}\})$.
- If there are $\mathbf{b}, \mathbf{b}' \in \tilde{Z}$ with $\delta(\mathbf{b}\mathbf{b}'/X) < 0$ then set $\tilde{X} := \tilde{X} \cup \{\mathbf{b}, \mathbf{b}'\}$.

By the maximality of m_c and induction, each closure step reduces $||\tilde{Y}||$ by at most $2m_c$ and reduces $\delta(\tilde{X}/A)$ by at least 1. It follows that after closing, we have

$$||\tilde{Z}|| \geq ||\tilde{Y}|| > 2m_c\delta(\tilde{X}/A)$$

and the rest of the induction hypothesis. Moreover, $\phi_c(\mathbf{x}; a)$ no longer witnesses the failure of (2).

Iteration of this process must stop because $B \setminus A$ is finite. Once finished, (1) and (2) must hold and $||Z|| > 0$ implies $X \subsetneq B$. □

4.2.4 Weak Closure

We need one final ingredient to definably analyze prealgebraic codes in T_∞. Given prealgebraic code instance $\phi_c(\mathbf{x}; a)$ over some $A \in \mathcal{K}_\infty$, we need a first-order approximation or $\mathrm{cl}_A(a)$.

For each prealgebraic code c, define

$$\Phi_c(\mathbf{x}_1, ..., \mathbf{x}_{m_c+1}) := \bigwedge_{i<j} \mathbf{x}_i \neq \mathbf{x}_j \wedge \bigwedge_i \phi_c(\mathbf{x}_i; y),$$

and

$$\Gamma_c := \{\Phi_{c'} : s_c > s_{c'}\}.$$

Lemma 4.2.21. *We may assume that if* $\phi_c(\mathbf{x}; a)$ *is over* A *and* $\mathbf{b}, \mathbf{b}' \models \phi_c(\mathbf{x}; a)$ *are* A*-generic, then* $\mathrm{qftp}_{\Gamma_c}(\mathbf{b}/A) = \mathrm{qftp}_{\Gamma_c}(\mathbf{b}'/A)$.

Proof. The easiest way to obtain this is to redo the code constructions in each T_i. Make sure that the lemma is true in T_i for $\Gamma_{c_i} := \{\Phi_{c'_i} : n_{c_i} > m_{c'_i} \cdot n_{c'_i}\}$. Now, since $s_c > s_{c'}$ implies $n_{c_i} > m_{c'_i} \cdot n_{c'_i}$ for $i = 1, 2$, the lemma follows. □

Lemma 4.2.22. *For any prealgebraic code instance* $\phi_c(\mathbf{x}; a)$ *over* A*, there is a unique minimal subset* $W \subseteq A$ *with the following properties.*

1. *Suppose for some* A*-generic* $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ *there is a* $\phi_{c'}(\mathbf{x}'; a')$ *with a long sequence in* $\mathbf{b}$*. If*

   $$Y := \{\mathbf{b}' \subseteq A \cup \{\mathbf{b}\} \mid \mathbf{b}' \models \phi_{c'}(\mathbf{x}'; a')\},$$

   then $A \cap \bigcup Y \subseteq W$*.*

2. *If* $\mathbf{b} \subseteq A$*,* $\mathbf{b} \models \phi_c(\mathbf{x}; a)$*, and* $\mathrm{qftp}_{\Gamma_c}(\mathbf{b}/W)$ *is not generic, then* $\mathbf{b} \subseteq W$*.*

Moreover, W *is contained in* $\mathrm{cl}_{A,n_c}(a)$*, and first-order definable.*

Proof. First we show $\mathrm{cl}_{A,n_c}(a)$ satisfies (1) and (2).

Condition (2) is easy, because if $\mathrm{qftp}_{\Gamma_c}(\mathbf{b}/\mathrm{cl}_{A,n_c}(a))$ fails to be generic, then $\delta(\mathbf{b}/\mathrm{cl}_{A,n_c}(a)) < 0$. This contradicts the assumption $\mathrm{cl}_{A,n_c}(a) \leq_{s,n_c} A$.

For condition (1), suppose $\mathbf{b} \models \phi_c(\mathbf{x}; a)$, $\phi_{c'}(\mathbf{x}'; a')$ is long in $\mathbf{b}$, $\mathbf{b}' \subseteq A \cup \{\mathbf{b}\}$, $\mathbf{b}' \nsubseteq \mathbf{b}$, and $\mathbf{b}' \models \phi_{c'}(\mathbf{x}'; a')$. Since $A \cap \{\mathbf{b}'\} \downarrow_a^{T_i} a'$ and $a' \notin \mathrm{acl}^{eq}(a)$, we have $\mathbf{b}' \subseteq \mathrm{cl}_{A,n_c}(a)$ by Lemma 4.2.16.

The class of sets satisfying (1) and (2) is closed under intersection. Thus uniqueness and containment in $\mathrm{cl}_{A,n_c}(a)$ follows from the fact that $\mathrm{cl}_{A,n_c}(a)$ is finite (recall $|\mathrm{cl}_{A,n_c}(a)| < n_c\delta(a)$).

Since checking condition (1) and (2) is first-order for a set of fixed size and we have a bound on the size of W, W is first-order definable. □

Definition 4.2.23. With W as in the lemma above, we define

$$\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) := W,$$

and call it the *weak closure of* $\phi_c(\mathbf{x}; a)$ *in* A.

Lemma 4.2.24. *If $\phi_c(\mathbf{x}; a)$ is over A, $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ is A-generic and $\phi_{c'}(\mathbf{x}'; a')$ is long in $\mathbf{b}$, then* $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a')) \subseteq \mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$.

Proof. Note that by Lemma 4.2.21, we can restrict condition (1) above to a single generic realization.

Because $\phi_{c'}(\mathbf{x}'; a')$ is long in $\mathbf{b}$, there is a $\mathbf{b}' \subseteq \mathbf{b}$ such that $\mathbf{b}' \models \phi_{c'}(\mathbf{x}'; a')$ is $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a'))$-generic. Since $\Gamma_{c'} \subseteq \Gamma_c$, $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$ satisfies conditions (1) and (2) for $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a'))$. □

4.2.5 Nice Codes

In this subsection, we temporarily move back to the context of a single theory T with finite definable rank and definably bounded degree. We need to make additional assumptions about the codes in T in order to progress further. We find these assumptions by looking more closely at our intended application.

Hasson's example is rank and degree preserving biinterpretable with a theory T that has an equivalence relation E such that:

1. T/E is strongly minimal with definable rank and degree.
2. The structure of each E-class has rank 1, degree $\leq D$, and definable rank and degree,
3. Distinct E-classes are orthogonal.
4. Generic E-classes are pure sets.

For the rest of this section, fix such a theory T. We write $[a]$ for the equivalence class coded by an imaginary $a \in T/E$. Thus, we write $Th([a])$ for the induced structure on the equivalence class a represents. We assume $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$.

Let $\{a_n\}$ enumerate $\mathrm{dcl}^{eq}(\emptyset) \cap (T/E)$. For each n let $d_n := \mathrm{dM}([a_n])$ and add predicates $\{P_{n,k} : k \leq d_n\}$ which partition $[a_n]$ into strongly minimal sets.

Lemma 4.2.25. *There is a system of codes $\mathcal{C}$ with the following two properties.*

1. *If $\psi(\mathbf{x}; d)$ is simple and covered by $c \in \mathcal{C}$, there is a parameter a and a conjuction $\theta(\mathbf{x})$ of atoms $P_{n,k}(x_i)$ such that $\psi(\mathbf{x}; d) \sim \phi_c(\mathbf{x}; a) \wedge \theta(\mathbf{x})$.*
2. *If $\phi_c(\mathbf{x}; a)$ is over A, $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ is A-generic, $b_i \in P_{n,k}$, and $\phi_c(\mathbf{x}; a) \not\models P_{n,k}(x_i)$, then $\phi_c(\mathbf{x}; a) \models \bigvee_{j\leq d_n} P_{n,j}(x_i)$ and for any $j \leq d_n$ we can change b_i so that $b_i \in P_{n,j}$ while maintaining that $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ is A-generic.*

Proof. Suppose we are building a code for the simple formula $\psi(\mathbf{x}; d)$. Since $\psi(\mathbf{x}; d)$ is simple, we may assume it implies a complete atomic E-type $\xi(\mathbf{x})$. Let $S_1 \cup \cdots \cup S_m = \{1, ..., |\mathbf{x}|\}$

be a partition such that $\xi(\mathbf{x})$ implies $x_i E x_j$ if and only if $i, j \in S_k$ for some k. By the orthogonality condition (3),

$$\psi(\mathbf{x}; d) \sim \bigwedge_k \exists \mathbf{x}_{\{1,...,|x|\} \setminus S_k} \psi(\mathbf{x}; d).$$

If we choose codes c_k which encode $\exists \mathbf{x}_{\{1,...,|x|\} \setminus S_k} \psi(\mathbf{x}; d)$, then

$$\phi_c(\mathbf{x}; y) := \xi(\mathbf{x}) \wedge \bigwedge_k \phi_{c_k}(\mathbf{x}_{S_k}; y_k)$$

is a code which encodes $\psi(\mathbf{x}; d)$. Thus we may assume $\psi(\mathbf{x}; d) \to \bigwedge_{i<j} x_i E x_j$.

Case 1: If b_1/E is generic over d for generic $\mathbf{b} \models \psi(\mathbf{x}; d)$, then, since generic E-classes are pure sets, we must have $\psi(\mathbf{x}; d) \sim \bigwedge_{i<j} x_i E x_j$. In this case, $\phi_c(\mathbf{x}) := \bigwedge_{i<j} x_i E x_j \wedge x_i \neq x_j$ is a code which encodes $\psi(\mathbf{x}; d)$. Since $\phi_c(\mathbf{x})$ has degree 1, properties (1) and (2) are trivial.

Case 2: If $b_1/E \in \operatorname{acl}(d)$ for generic $\mathbf{b} \models \psi(\mathbf{x}; d)$, then we can strengthen $\psi(\mathbf{x}; d)$ such that $\psi(\mathbf{x}; d) \to \mathbf{x} \subseteq [a]$ for some $a \in (T/E) \cap \operatorname{acl}(d)$.

Case 2a: If $\mathrm{RM}(a) = 0$, then we may assume $a \in \operatorname{dcl}(\emptyset)$ and choose a $Th([a])$-code $\phi_c(\mathbf{x}; y)$ which encodes $\psi(\mathbf{x}; d)$. Since $Th([a])$ has definable rank and degree, all instances of ϕ_c have degree 1. Thus (1) and (2) are again trivial.

Case 2b: If $\mathrm{RM}(a) = 1$, then $[a]$ is a pure set and $\psi(\mathbf{x}; d) \sim \mathbf{x} \subseteq [a]$. Thus the code $\phi_c(\mathbf{x}; y) \equiv \mathbf{x} \subseteq [y] \wedge \bigwedge_{i<j} x_i \neq x_j$ works. Note that $\mathrm{dM}(\phi_c(\mathbf{x}; a)) = \mathrm{dM}([a])^{n_c}$. In particular, $\phi_c(\mathbf{x}; a_n)$ is partitioned into $(d_n)^{n_c}$ degree 1 sets by the formulas

$$\{\phi_c(\mathbf{x}; a_n) \wedge \bigwedge_{i \leq n_c} P_{n,k_i}(x_i) : \mathbf{k} \in \{1, ..., d_n\}^{n_c}\}.$$

From this (1) and (2) follow. □

Definition 4.2.26. If $\mathcal{C}$ is a system of codes and there are disjoint predicates $\{P_{n,k} \mid k \leq d_n\}$ which make the above lemma true, we say that $\mathcal{C}$ is a *nice system of codes*. Note that any system of codes for a theory with definable rank and degree is nice via $d_n = 1$ and $P_{n,1} = \emptyset$.

Suppose $\mathcal{C}$ is a nice system of codes. Write Σ_n for the set of complete $\{P_{m,k} : m < n, k \leq d_n\}$-formulas. Given a code $c \in \mathcal{C}$ and $\theta(\mathbf{x}) \in \Sigma_n$ with $|\mathbf{x}| = n_c$, let $c \wedge \theta$ be the code with

$$\phi_{c \wedge \theta}(\mathbf{x}; y) \equiv \phi_c(\mathbf{x}; y) \wedge \theta(\mathbf{x}) \wedge \mathrm{RM}_{\mathbf{x}}(\phi_c(\mathbf{x}; y) \wedge \theta(\mathbf{x})) = k_c.$$

We will call $c \wedge \theta$ a Σ_n*-specialization* of c. Note that by Lemma 4.2.25, $c \wedge \theta \in \mathcal{C}$ if and only if $\phi_c(\mathbf{x}; y) \models \theta(\mathbf{x})$ already.

4.2.6 The Class $\mathcal{K}_\mu$

We now have everything we need to describe the restricted amalgamation class $\mathcal{K}_\mu$. We assume that each theory T_i has a nice system of code $\mathcal{C}_i$ via the predicates $\{P^i_{n,k} : n \in \mathbb{N} \text{ and } k \leq d^i_n\}$.

We write $\Sigma_n := \Sigma^1_n \times \Sigma^2_n$. For a prealgebraic code c and a $\theta \in \Sigma_n$, write $c \wedge \theta$ for the *Σ_n-specialized prealgebraic code* $(c_1 \wedge \theta_1, c_2 \wedge \theta_2)$. Note that specializations $c \wedge \theta$ still code prealgebraic extensions in the sense of Lemma 4.2.16.

We define a class $\mathcal{K}_\mu \subseteq \mathcal{K}_\infty$ by saying that $A \in \mathcal{K}_\mu$ when

$$\dim_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) \leq \mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$$

for all specialized prealgebraic codes $c \wedge \theta$ and $a \in \mathrm{acl}^{eq}(A)$. Of course, we have yet to define $\dim_A$ and μ_A.

If $\phi_{c\wedge\theta}(\mathbf{x}; a))$ a specialized prealgebraic instance over A, then let $\dim_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ be the cardinality of the set

$$\{\mathbf{b} \subseteq A : \mathbf{b} \not\subseteq \mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \text{ and } \mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)\};$$

that is, the number of realizations outside of the weak closure.

For unspecialized prealgebraic codes c, let

$$\mu_A(\phi_c(\mathbf{x}; a)) = (D_c!)^{D_c} \cdot (s_c + m_c + 1).$$

For Σ_n-specializations $c \wedge \theta$, we will simultaneously define $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ and first-order approximations $\mathcal{K}_{c,n} \subseteq \mathcal{K}_\infty$ to the final $\mathcal{K}_\mu$.

Suppose $c \wedge \theta$ is a Σ_n-specialization of c. We inductively assume μ_A has been defined for instances of specialized prealgebraic codes $c' \wedge \theta'$ whenever $s_{c'} < s_c$ or $\theta' \in \Sigma_{n-1}$. Using the induction hypothesis, let $\mathcal{K}_{c,n}$ be the class of all $A \in \mathcal{K}_\infty$ such that

$$\dim_A(\phi_{c'\wedge\theta'}(\mathbf{x}'; a')) \leq \mu_A(\phi_{c'\wedge\theta'}(\mathbf{x}'; a'))$$

for $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ over A with $s_{c'} < s_c$ and $\theta' \in \Sigma_n$. If $A \in \mathcal{K}_{c,n}$ and $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is over A, we say that $\phi_{c\wedge\theta}(\mathbf{x}; a)$ *extendible over* A when there is an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ so that $A \cup \{\mathbf{b}\} \in \mathcal{K}_{c,n}$. For A-extendible $\phi_{c\wedge\theta}(\mathbf{x}; a)$ define

$$\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) := \mu_A(\phi_{c\wedge\theta^-}(\mathbf{x}; a))/D,$$

where $\theta^- \in \Sigma_{n-1}$, $\theta \to \theta^-$, and D is the number of $\theta' \in \Sigma_n$ with $\theta' \to \theta^-$ and $\phi_{c\wedge\theta'}(\mathbf{x}; a)$ extendible over A. For non-A-extendible $\phi_{c\wedge\theta}(\mathbf{x}; a)$ define

$$\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) := 0.$$

Lemma 4.2.27. *If $A \in \mathcal{K}_{c,n}$ and $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-extendible, then $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) > s_c + m_c$.*

Proof. The degree of any prealgebraic code instance $\phi_c(\mathbf{x}; a)$ is bounded by D_c. Thus each time we divide by D in the definition of μ_A, we have $D \leq D_c$. Moreover, we divide by a number greater than 1 at most D_c times. □

Lemma 4.2.28. *If $A \in \mathcal{K}_{c,n}$, $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is over A, and $\theta \in \Sigma_n$ then $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ depends only on $\mathrm{qftp}_{\Sigma_n \cup \Gamma_c}(\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\})$ for A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$.*

Proof. The quantifier-free type above is uniquely determined by Lemma 4.2.21.

Suppose $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-generic and $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ witnesses $A \cup \{\mathbf{b}\} \notin \mathcal{K}_{c,n}$. Note that all of the realizations of $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ are contained in $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$. By induction, we know that $\mu_{A\cup\{\mathbf{b}\}}(\phi_{c'\wedge\theta'}(\mathbf{x}'; a'))$ is completely determined by $\mathrm{qftp}_{\Sigma_n \cup \Gamma_c}(\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a')) \cup \{\mathbf{b}'\})$ for some (any) $A \cup \{\mathbf{b}\}$-generic $\mathbf{b}' \models \phi_{c'\wedge\theta'}(\mathbf{x}'; a')$.

Note that $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a')) \subseteq \mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$, every realization of $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ is contained in $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$, and $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$ computes the same value for $\mu_{c'\wedge\theta'}(\mathbf{x}'; a'))$ as $A \cup \{\mathbf{b}\}$. It follows that the failure $A \cup \{\mathbf{b}\} \notin \mathcal{K}_{c,n}$ is encoded in $\mathrm{qftp}_{\Sigma_n \cup \Gamma_c}(\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\})$ and that $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is not A-extendible.

Thus the A-extendibility of $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is encoded in $\mathrm{qftp}_{\Sigma_n \cup \Gamma_c}(\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\})$. Unrolling the definition of $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ we see that it too is encoded. □

Lemma 4.2.29. *If $A \in \mathcal{K}_{c,n}$, $\theta \in \Sigma_n$, $\mathbf{b} \subseteq A$, $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$, and $\mathbf{b} \nsubseteq \mathrm{wcl}_A(\phi_c(\mathbf{x}; a))$ then $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is extendible over A.*

Proof. Note that $\mathbf{b}$ has the same quantifier-free $\Sigma_n \cup \Gamma_c$ type over $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a))$ as any A-generic $\mathbf{b}' \models \phi_{c\wedge\theta}(\mathbf{x}; a)$. Since $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\} \subseteq A \in \mathcal{K}_{c,n}$ we can apply the proof of the previous lemma to get $A \cup \{\mathbf{b}'\} \in \mathcal{K}_{c,n}$. □

Lemma 4.2.30. *For all prealgebraic codes c and $n \in \mathbb{N}$, $\mathcal{K}_{c,n+1} \subseteq \mathcal{K}_{c,n}$.*

Proof. This an easy consequence of the previous lemma and the definition of μ_A. □

In the following lemma we use the Decomposition Lemma and nice code assumption to show that our first order approximations $\mathcal{K}_{c,n} \supseteq \mathcal{K}_\mu$ are well-behaved.

Lemma 4.2.31. *Suppose $A \in \mathcal{K}_{c,n+1}$, $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-extendible, and $\theta \in \Sigma_n$. There is a $\theta^* \in \Sigma_{n+1}$ such that $\theta^* \to \theta$ and $\phi_{c\wedge\theta^*}(\mathbf{x}; a)$ is A-extendible.*

Proof. We induct on $S \subseteq \{1, ..., n_c\}$ to prove the following claim.

Claim. *There exists an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ such that $A \cup \{\mathbf{b}_S\} \in \mathcal{K}_{c,n+1}$.*

Suppose $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-generic and $S \subseteq \{1, ..., n_c\}$. Applying the Decomposition Lemma to $A \leq_s B = A \cup \{\mathbf{b}_S\}$, we get $A \leq_s X \subsetneq B$ and Z at stated there. Since $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ being A-generic completely determines $\mathrm{qftp}_{\Gamma_c}(\mathbf{b}/A)$ and the values of δ on subsets of $A \cup \{\mathbf{b}\}$, the decomposition is the same for all A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$. Thus we may assume that $X \in \mathcal{K}_{c,n+1}$ by induction.

If $\mathbf{b}' \in Z$, then $\mathbf{b}'$ is an X-generic realization of some Σ_n-specialized prealgebraic code instance $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ strongly based on X. Since $X \in \mathcal{K}_{c,n+1}$ and $X \cup \{\mathbf{b}'\} \in \mathcal{K}_{c,n}$, we know that $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ is extendible over X. Because $s_{c'} < s_c$ we can use this lemma to find a $\theta'' \in \Sigma_{n+1}$ so that $\theta'' \to \theta$ and $\phi_{c'\wedge\theta''}(\mathbf{x}'; a')$ is X-extendible. By Lemma 4.2.25, we may assume that $\mathbf{b}' \models \phi_{c'\wedge\theta''}(\mathbf{x}'; a')$. Because $\mathrm{wcl}_B(\phi_{c'}(\mathbf{x}; a')) \subseteq X$ and $\mathbf{b}'$ is X-generic we have $X \cup \{\mathbf{b}'\} \in \mathcal{K}_{c,n+1}$.

Since the set-wise distinct elements of Z are pairwise disjoint, we can do this for all $\mathbf{b}' \in Z$ simultaneously.

Now, if $B \notin \mathcal{K}_{c,n+1}$ it must be because some Σ_n-specialized prealgebraic code instance $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ has a further Σ_{n+1}-specialization with too many realizations. By the above, we must have $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ strongly based on X.

Let $c' \wedge \theta_1, ..., c' \wedge \theta_D$ enumerate the X-extendible Σ_{n+1}-specializations of c' which further specialize $c' \wedge \theta'$. We may assume

$$\dim_B(\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')) > \mu_B(\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')) = \mu_X(\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')).$$

Since $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ doesn't have too many realizations in B, we may assume that

$$\dim_B(\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')) < \mu_B(\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')) = \mu_X(\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')).$$

Since $X \in \mathcal{K}_{c,n+1}$, there is a $\mathbf{b}' \in Z$ realizing $\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')$. Using Lemma 4.2.25 we can change $\mathbf{b}'$ into a realization of $\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')$.

If $\phi_{c''\wedge\theta''}(\mathbf{x}''; a'')$ is any other Σ_{n+1}-specialized prealgebraic code instance over X, then its dimension is unchanged by this operation unless

$$\phi_{c''\wedge\theta''}(\mathbf{x}'; a'') \equiv \phi_{c'\wedge\theta_i}((\mathbf{x}')^\sigma; a')$$

for some $\sigma \in Sym(n_c)$ and $i = 1, 2$. If this latter condition holds, then $|x''| = |x'|$ and

$$\mu_X(\phi_{c''\wedge\theta''}(\mathbf{x}'; a'')) = \mu_X(\phi_{c'\wedge\theta_i}(\mathbf{x}'; a')).$$

Thus the net effect of changing $\mathbf{b}'$ is to reduce the total number of violations to the multiplicity rules. Iterating this process, we eventually get $B \in \mathcal{K}_{c,n+1}$. □

Lemma 4.2.32. *Suppose $A \in \mathcal{K}_\mu$, $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-extendible, and* $\dim_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) < \mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$. *There is an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ such that $A \cup \{\mathbf{b}\} \in \mathcal{K}_\mu$.*

Proof. Suppose $\theta \in \Sigma_n$. By the previous lemma, there is at least one $\theta^* \in \Sigma_{n+1}$ so that $\theta^* \to \theta$ and $\phi_{c\wedge\theta^*}(\mathbf{x}; a)$ is A-extendible. Since $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ is divided evenly amongst these θ^*, we can choose θ^* such that $\dim_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) < \mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$. Iterating this process, we can find an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ so that $A \cup \{\mathbf{b}\} \in \mathcal{K}_{c,n'}$ for all $n' > n$.

If $A \cup \{\mathbf{b}\} \notin \mathcal{K}_\mu$, then it must be the case that

$$\dim_{A\cup\{\mathbf{b}\}}(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) > \mu_{A\cup\{\mathbf{b}\}}(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$$

for some $\theta^* \in \Sigma_{n'}$ with $n' > n$ and $\mathbf{b} \models \phi_{c\wedge\theta^*}(\mathbf{x}; a)$. But $\mu_{A\cup\{\mathbf{b}\}}(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) = \mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$ and we constructed $\mathbf{b}$ so that $\mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) > \dim_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$. Thus $\dim_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) = \mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$, a contradiction. □

4.2.7 The Theory T_μ

We now argue that the Fraisse limit of $(\mathcal{K}_\mu, \leq_s)$ is the fustion we are looking for.

Lemma 4.2.33. *If $A \leq_s A \cup \{b\}$ is algebraic or transcendental, then $A \in \mathcal{K}_\mu$ implies $A \cup \{b\} \in \mathcal{K}_\mu$.*

Proof. Suppose $\mathbf{b}_1, ..., \mathbf{b}_N \models \phi_{c \wedge \theta}(\mathbf{x}; a)$ witnesses $A \cup \{\mathbf{b}\} \notin \mathcal{K}_\mu$. Since $a \in \mathrm{acl}^{eq}(A)$, A and $A \cup \{b\}$ compute the same value for $\mu(\phi_{c\wedge\theta}(\mathbf{x}; a))$. Thus it can not be the case that $\mathbf{b}_i \subseteq A$ for all $i \leq N$, so we may assume $b \in \mathbf{b}_1$. This contradicts Lemma 4.2.16 and the assumption that $A \leq_s A \cup \{b\}$ is not prealgebraic. □

Lemma 4.2.34. *The class $\mathcal{K}_\mu$ has the amalgamation property with respect to $\leq_s$.*

Proof. Suppose $A \leq_s B, C \in \mathcal{K}_\mu$. We need to find a $D \in \mathcal{K}_\mu$ with $A \leq_s C \leq_s D$ and a $B' \leq_s D$ such that $B' \equiv_A B$. By induction, we may assume that both $A \leq_s B$ and $A \leq_s C$ are minimal.

Suppose $A \leq_s B$ is algebraic, say because $B = A \cup \{b\}$ and $\mathrm{tp}_{T_1}(b/A)$ is algebraic. If $\mathrm{tp}_{T_1}(b/A)$ is realized by $c \in C \setminus A$, then $B \equiv_A C$. Otherwise, we may assume $\mathrm{tp}_{T_1}(b/C)$ is some extension of $\mathrm{tp}_{T_1}(b/C)$ which implies $b \notin C$ and $\mathrm{tp}_{T_2}(b/C)$ is generic. It is then easy to check $C \leq_s C \cup \{b\}$, so $D = C \cup \{b\}$ works by the previous lemma.

Thus we may assume neither $A \leq_s B$ nor $A \leq_s C$ are algebraic. We compute the free fusion of B and C over A by assuming $\mathrm{tp}_{T_i}(B/C)$ is some non-forking extension of $\mathrm{tp}_{T_i}(B/A)$ and letting $D = B \cup C$. By the submodularity of δ, we have $B, C \leq_s D$.

Suppose $D \notin \mathcal{K}_\mu$ is witnessed by distinct $\mathbf{b}_1, ..., \mathbf{b}_N \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ with N too large. We may assume $\phi_{c\wedge\theta}(\mathbf{x}; a)$ has degree 1.

By Lemma 4.2.17, we may assume that $a \in \mathrm{acl}^{eq}(B)$ and thus $\mathrm{cl}_D(a) \subseteq B$. It follows that B and D compute the same value for $\mu(\phi_{c\wedge\theta}(\mathbf{x}; a))$. Since $B \in \mathcal{K}_\mu$, we may assume $\mathbf{b}_1 \nsubseteq B$. By Lemma 4.2.16, $C = A \cup \{\mathbf{b}_1\}$. Since $B \downarrow_A^{T_i} C$, we must have $a \in \mathrm{acl}^{eq}(A)$ and thus $\mathrm{cl}_D(a) \subseteq A$. By repeating the argument just given, we may assume $B = A \cup \{\mathbf{b}_2\}$.

Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are both A-generic realizations of a degree 1 prealgebraic code instance over A, we must have $\mathbf{b}_1 \equiv_A \mathbf{b}_2$. Thus $B \equiv_A C$. □

We call an $M \in \mathcal{K}_\mu$ *rich* if for all finite $A \leq_s M$ and finite $A \leq_s B \in \mathcal{K}_\mu$ there is a $C \leq_s M$ with $B \equiv_A C$. The amalgamation property shows that for every $A \in \mathcal{K}_\mu$ we can find a rich $M \in \mathcal{K}_\mu$ with $A \leq_s M$.

Assumption 4.2.35. If $K > 1$, then $\mathrm{RM}(T_1) \leq \mathrm{RM}(T_2)$, in T_1 every element is interalgebraic with infinitely many elements, and in T_2 there are infinitely many disjoint unary predicates of rank $\mathrm{RM}(T_2) - 1$.

Let T_μ be the theory which says, for $M \models T_\mu$, that

1. $M \in \mathcal{K}_\mu$,

2. $M \upharpoonright L_i \models T_i$ for $i = 1, 2$,

3. there is no prealgebraic extension $M \leq_s N \in \mathcal{K}_\mu$.

Note that axiom (3) is first order by Lemma 4.2.32.

Theorem 4.2.36. *The theory T_μ is consistent, complete, and the ω-saturated models of T_μ are exactly the rich structures on $\mathcal{K}_\mu$. Moreover, T_μ has rank K, nice codes, and*

$$\mathrm{RM}_T(\phi(x; a)) = v_i \,\mathrm{RM}_{T_i}(\phi(x; a)) \text{ and } \mathrm{dM}_T(\phi(x; a)) = \mathrm{dM}_{T_i}(\phi(x; a))$$

for all $\phi(x; y) \in L(T_i^{eq})$ and $i = 1, 2$.

Proof. We have set up the machinery required to run the proof of the corresponding theorem in [38]. The only thing that needs mention is that the pairs of predicates $P^1_{n,k} \wedge P^2_{n',k'}$ provide nice codes for T_μ. □

Proof of Theorem 4.2.1. This has the same proof as the corresponding theorem in [38]. The main point is that if we are willing to expand the language, i.e., $L(T) \supsetneq L(T_1) \cup L(T_2)$, then we can obtain assumption 4.2.35 and apply Theorem 4.2.36. □

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