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Abstract

A financial institution's counterparty credit exposures may be correlated with the credit quality of a counterparty; wrong way risk refers to the case where this correlation is negative. Hull and White [9] are the first to model wrong way risk in Credit Value Adjustment (CVA) calculations by expressing the counterparty's default intensity in terms of the financial institution's credit exposure to the counterparty. We derive a formula for CVA for a class of models that includes the formulation of Hull and White [9], and we show that wrong way risk does not affect the credit quality of the counterparty. We provide numerical examples based on the Hull and White [9] formulation to estimate CVA for forward contracts and European options. These examples demonstrate that independent CVA can exceed wrong way CVA. This is inconsistent with the scalar multiples of independent CVA that have been adopted by regulators as a proxy for wrong way CVA.

1 Introduction

Consider a portfolio of OTC derivative contracts that a financial institution, such as a a derivatives dealer, holds with a counterparty. The Credit Value Adjustment[1] (CVA) is the difference between the portfolio value before and after adjustment for the risk that the counterparty might default. In other words, CVA is the market value of counterparty credit risk. Since the 2007 credit crisis, the emphasis on counterparty credit risk by both global and US regulators has increased dramatically. CVA is one of the most important counterparty credit risk measures;

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[1]Throughout this paper, we consider the unilateral CVA. See Chapter 7 of Gregory [7] for discussions on Bilateral CVA. In this paper we only consider Unilateral CVA; hereafter, referred to as CVA.

according to Basel III,[2] banks are required to hold regulatory capital based on CVA *charges* against each of their counterparties, (see, for instance, [1]).

The Credit Value Adjustment is expressed in terms of the dealer's counterparty exposure, V, which is the maximum of zero and the future (collateralized or uncollateralized) value of the portfolio; counterparty exposure is always non-negative. It also depends on the maturity, T, of the longest transaction in the portfolio and the default time, τ, of the counterparty. The Credit Value Adjustment can be expressed as a risk-neutral expected discounted loss:

$$\mathrm{CVA} = E[(1-R)D_\tau V_\tau \mathbf{1}\{\tau \le T\}], \tag{1}$$

where D_t is the stochastic discount factor at time t, $1\{\cdot\}$ is an indicator function, and R is the financial institution's recovery rate.[3] Hereafter we suppress the dependence of the CVA to the recovery rate, R. A widely adopted assumption is that credit exposure, V, and the counterparty's default time, τ, are independent. The leads to independent CVA, denoted CVA_I, and it is expressed in terms of the density, f, of τ:

$$\mathrm{CVA}_I = \int_0^T E[D_\tau V_\tau | \tau = t] f(t)\, dt = \int_0^T E[D_t V_t] f(t)\, dt, \tag{2}$$

where the last equality follows from the independence of τ and V. In practice, the counterparty's default time distribution is approximated from counterparty credit spreads observed in the market. Monte Carlo simulation is then used to estimate independent CVA by estimating $E[D_t V_t]$ based on a discrete time grid.

The efficacy of independent CVA is limited since there are important practical cases where credit exposure, V, and the counterparty's default time, τ, are correlated, (see Chapter 8 of Gregory [7]).[4] When credit exposure is negatively correlated with a counterparty's credit quality, the exposure and its associated risk measures are said to be *wrong way.* Wrong way CVA, denoted CVA_W, refers to CVA in the presence of wrong way risk. When the correlation is positive, the exposure and its associated risk measures are said to be *right way.* To simplify the exposition, we concentrate on wrong way CVA. However, there are analogous results for right way CVA.

In Section 2, we summarize the model developed by Hull and White in [9]. In that model, the logarithm of the counterparty's default intensity is an affine function of the dealer's exposure to the counterparty. In Section 3, we consider a class of intensity models of CVA that includes the formulation of Hull and White [9]. We derive a formula for CVA that applies to this class of stochastic intensity models. We also show that neither wrong way risk nor right way risk affects the counterparty's credit quality. In Section 4, we provide numerical examples of independent and wrong way CVA in the Hull and White model. For simple instruments such as

[2] Basel III is a global regulatory standard on bank capital adequacy, stress testing and market liquidity risk agreed upon by the members of the Basel Committee on Banking Supervision in 2010-11, and scheduled to be introduced from 2013 until 2018.

[3] A derivation of Formula (1) is in Chapter 7 of Gregory [7].

[4] A basic example of wrong way risk occurs when a counterparty issues a put option on his own company. Or, more practically, when the derivatives dealer takes a long position in a put option on a stock of a company which has fortunes that are highly positively correlated to those of the counterparty.

forwards and European put options,[5] we find that the percentage difference between wrong way and independent CVA can depend materially on maturity, and that wrong way CVA need not exceed independent CVA. Our findings are inconsistent with the scalar multiples of independent CVA that have been adopted by regulators as a proxy for wrong way CVA; they provide an occasion to re-examine the assumptions underlying those proxies.

2 The Hull and White Stochastic Intensity Model of CVA

Hull and White [9] are the first to include wrong way risk in CVA calculations by formulating a counterparty's default intensity in terms of a dealer's credit exposure to a counterparty. They assume the intensity of counterparty's default time, τ, denoted by λ, is given by:

$$\lambda_t = e^{bV_t + a_t} \tag{3}$$

where b is a constant and a_t is a deterministic function of time. To calibrate the their model, Hull and White first specify the parameter b using "subjective judgement."[6] Let s_t denotes the counterparty's maturity-t credit spread and let R denotes the recovery rate.[7] Given b, the piecewise constant a_t are sequentially chosen to satisfy:

$$e^{-\frac{ts_t}{1-R}} = E\left[e^{-\int_0^t \lambda_u du}\right], \tag{4}$$

as closely as possible. Hull and White [9] use the left side of (4) as an approximation of the counterparty's probability of survival until time $t > 0$, i.e., $P(\tau > t)$. The Appendix of [9] details how a_t is sequentially specified using Formula (4); the expectation on the right side is estimated by Monte Carlo simulation after time discretization of the integral of the intensity process, λ.

3 Intensity-Based Models of CVA

Inspired by the model in Hull and White [9], we consider similar intensity models of CVA in which a counterparty's default intensity, λ, is driven by a single risk factor, V.[8] The real-valued process $\{V_t\}_{t\geq 0}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \{G_t\}_{t\geq 0}, P)$ where $\{G_t\}_{t\geq 0}$ denotes the filtration generated by V. To incorporate wrong way risk, the intensity, λ, is defined as an increasing function of exposure, V. In this setting the default time, τ, admits a stochastic

[5] In practice dealer portfolios are complex, and there are almost always collateral and netting agreements associated with positions. However, in order to effectively communicate our main results, we consider uncollateralized contract-level exposure in our numerical examples.

[6] Hull and White [9] discuss, but do not implement, an estimation scheme based on historical observations of the exposure V and credit spread of the counterparty.

[7] This is the recovery rate associated with the credit default swap contract "on" the counterparty, and it may or may not be equal to the recovery rate that appears in the CVA formula. The recovery rate in the CVA formula refers to the fraction of loss that is recovered by the financial institution (a derivatives dealer) if the counterparty defaults.

[8] We address the case where λ depends on additional sources of randomness in a subsequent paper.

intensity λ, as shown in the Appendix. A consequence of this is an expression for the survival probability:

$$P(\tau > t) = E\left[e^{-\int_0^t \lambda_u \, du}\right], \tag{5}$$

which implies that the density of the default time τ is given by:

$$f_\tau(t) = E\left[\lambda_t e^{-\int_0^t \lambda_u \, du}\right]. \tag{6}$$

3.1 Calibration

The calibration scheme for the intensity models of CVA that are considered in this paper uses (5) by assuming that survival probabilities on the left side of (5) are approximated from a counterparty's market observable credit default swap spreads or bond spreads.

Many of the credit default swap contract valuation models assume that the default time is the first event of a non-homogenous Poisson process. Using this assumption, the calibration scheme (5) can be viewed as follows. Let $\tilde{\tau}$ denote the time to the first event of a non-homogenous Poisson process with deterministic time dependent intensity $\tilde{\lambda}$. The survival probability by time $t > 0$ is given by:[9]

$$P(\tilde{\tau} > t) = e^{-\int_0^t \tilde{\lambda}_u \, du}.$$

Suppose that a set of maturity-t credit default swap spreads, s_t, are observed in the market. A well-known approximation used in the valuation of credit default swaps is as follows:

$$\frac{1}{t}\int_0^t \tilde{\lambda}_u \, du \approx \frac{s_t}{1-R}.$$

The calibration scheme (5) chooses the parameters of λ such that:

$$P(\tau > t) = P(\tilde{\tau} > t) \approx e^{-\frac{ts_t}{1-R}}, \tag{7}$$

for $t \in [0, T]$. That is, this calibration strategy implies that the counterparty's default time, τ, with stochastic intensity, λ, is equal in distribution to $\tilde{\tau}$ for $t \in [0, T]$. The distribution of $\tilde{\tau}$ for $t \in [0, T]$ is approximated based on market observed credit spreads and independent of the dealer's credit exposure.

3.2 Counterparty Survival Probabilities

Wrong way exposures are first defined by Canabarro and Duffie [3] as credit exposures that are negatively correlated with the credit quality of the counterparty, (see page 123 of [3]). In what follows, we show that stochastic intensity models of CVA capture this basic definition. However, we emphasize that wrong way risk does not affect the credit quality of the counterparty. The

[9] See Chapter 5 of [12].

counterparty's survival probabilities $P(\tau > T)$, for $T > 0$, are considered as the measure of its credit quality.

The default time, τ, admits a stochastic intensity, λ, which in the presence of wrong way risk is defined as an increasing function of the exposure V. Then, for a given $T > 0$, we can write:

$$P(\tau > T \mid G_T) = e^{-\int_0^T \lambda_u \, du}. \tag{8}$$

That is, conditional on a given sample path of the exposure process in $[0, T]$, the counterparty default time can be viewed as the first event of a non-homogeneous Poisson process with intensity λ_t.

Consider two given sample paths of the exposure process, $\{V_t^k \ ; \ t \leq T\}$, $k = 1, 2$, for which:

$$\int_0^T \lambda(V_t^1) dt < \int_0^T \lambda(V_t^2) dt.$$

This implies that the counterparty's credit quality is lower along the second sample path. In other words, wrong way risk affects a counterparty's credit quality on a path-wise basis. However, the calibration strategy that uses (5), or equivalently (7), forces the average of conditional survival probabilities to be equal to the market observed survival probabilities:

$$P(\tau > T) = E[P(\tau > T \mid G_T)] = P(\tilde{\tau} > T).$$

An analogous argument shows that the calibration strategy of this class of stochastic intensity models implies that right way risk does not affect the credit quality of the counterparty.

3.3 Independent and Wrong Way CVA

In an intensity-based CVA model, Formula (6) implies that independent CVA can be written as:

$$\text{CVA}_I = \int_0^T E[D_t V_t] f_\tau(t) \, dt = \int_0^T E[D_t V_t] E\left[\lambda_t e^{-\int_0^t \lambda_u \, du}\right] \, dt. \tag{9}$$

In Lemma 1 of the Appendix, we derive the following formula for wrong way intensity-based CVA, which assumes that the stochastic intensity of counterparty's default time, τ, is a function of dealer's credit exposure, V:

$$\text{CVA}_W = E[D_\tau V_\tau \mathbf{1}\{\tau \leq T\}] = \int_0^T E\left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u \, du}\right] \, dt. \tag{10}$$

A comparison of independent CVA (right hand side of Formula (9)) and wrong way CVA (right hand side of Formula (10)) suggests that wrong way CVA need not exceed independent CVA. While a positive correlation between λ and V implies $E[D_t V_t \lambda_t] \geq E[D_t V_t] E[\lambda_t]$, it has no consistent implication for the pair of terms:

$$E[D_t V_t] E\left[\lambda_t e^{-\int_0^t \lambda_u \, du}\right] \text{ and } E\left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u \, du}\right],$$

or for the time integrals of those terms. In Section 4, we give examples of forward contracts and put options for which $\text{CVA}_I > \text{CVA}_W$ in the model of Hull and White [9].

4 Numerical Examples

This section is a summary of our numerical examples based on the Hull and White model [9]. They demonstrate that independent CVA can exceed wrong way CVA. There are many practical instances where Monte Carlo estimates of CVA_I and CVA_W are close but the former exceeds the latter. We consider contract level exposures for forward contracts and put options.

In what follows we assume that the risk free rate, r, is constant. That is, the discount factor is $D_t = e^{-rt}$ and independent and wrong way CVA are:

$$\text{CVA}_I = \int_0^T D_t E[V_t] f(t)\, dt \quad \text{and} \quad \text{CVA}_W = \int_0^T D_t E\left[V_t \lambda_t e^{-\int_0^t \lambda_u\, du}\right] dt,$$

where V_t denotes the time $t \geq 0$ value of the derivative contract and T is the maturity of the contract. Also, λ is the stochastic intensity proposed by Hull and White, i.e., $\lambda_t = \exp(bV_t + a_t)$. Assuming that b is given, the piece-wise constant deterministic function a_t is approximated based on counterparty's t-maturity credit spreads, s_t, and (4), (see the details in the Appendix of [9]).

The expected exposures, $E[V_t]$, are with respect to the physical measure in our numerical examples. There is no consensus in counterparty credit risk around choices of measure for CVA calculations, (see [8] and Chapters 7 and 9 of [7] for discussions on the use of risk-neutral and physical measure in CVA calculations). [10]

Monte Carlo CVA Estimation Monte Carlo estimators of CVA_I and CVA_W, denoted $\hat{\theta}_I$ and $\hat{\theta}_W$, are defined as follows. Consider the time grid, $0 \equiv t_0 < t_1 < ... < t_n \equiv T$,

$$\hat{\theta}_I = \sum_{i=1}^{n} D_i \bar{V}_i f(t_i) \Delta_i, \quad \text{and} \quad \hat{\theta}_W = \sum_{i=1}^{n} D_i \xi_i \Delta_i,$$

where $\Delta_i \equiv t_i - t_{i-1}$, and, $\bar{V}_i = \frac{1}{m}\left(\sum_{j=1}^{m} V_{ij}\right)$, with V_{ij} being the jth Monte Carlo realization of $V_i \equiv V_{t_i}$. Similarly, ξ_i is the m-simulation-run average of $V_i \lambda_i e^{-\sum_{k=1}^{i} \lambda_k \tilde{\Delta}_k}$, with $\tilde{\Delta}_k = \tilde{t}_k - \tilde{t}_{k-1}$ being defined based on a finer time grid, $0 \equiv \tilde{t}_0 < \tilde{t}_1 < ... < \tilde{t}_l \equiv T$, $l > n$.

Let $\{S_t\}_{t\geq 0}$ denote a geometric Brownian motion, $S_t = S_0 e^{X_t}$, where $\{X_t\}_{t\geq 0}$ is a Brownian motion with drift μ and volatility σ. We sample from the risk factor S_t based on the physical measure. Then, given the Monte Carlo realization of S, the valuation is based on the risk-neutral measure.[11] This implies $V_t = e^{-r(T-t)} E[S_T \mid S_t] = S_t$ for the forward contract. For the put options, we simply set $V_t = e^{-r(T-t)} E[(K - S_T)^+ \mid S_t]$.

The credit curve is assumed to be flat at s. So, in the independent case, the default time, τ, is an exponential random variable with mean $1/s$. This leads to the following closed form

[10] Note that in the above setting $\{D_t V_t\}_{t\geq 0}$ is a martingale under the risk-neutral measure. We have chosen the physical measure to avoid the trivial case, $V_0 = E[D_t V_t]$ and so $\text{CVA}_I = V_0 P(\tau \leq T)$, resulting from $\{D_t V_t\}_{t\geq 0}$ being a martingale.

[11] This is the common and well known practice in risk management: sampling from the risk factors based on the physical measure and then risk neutral valuation, (see, for instance, Chapter 9 of [6]).

formula for independent CVA in the forward contract case, $\text{CVA}_I = \frac{sS_0}{\alpha}(\exp(\alpha T) - 1)$ with $\alpha = \mu + \frac{\sigma^2}{2} - r - s$.

Numerical Results CVA estimates in the following numerical examples are based on $m = 10^5$ simulation runs.[12] We assume a recovery rate of $R = 0$, a constant risk free rate of $r = .01$, and an annualized volatility of 25%. The credit quality of the counterparty is investment grade with a flat spread curve at 100 basis points. The family of forward contracts presented in Table 1 and the family of in-the-money put options analyzed in Table 2 are both examples where independent CVA and wrong way CVA are close, but CVA_I exceeds CVA_W at each maturity. The coefficient $b = .02$ in both Tables 1 and 2 indicates a relatively low dependence of stochastic intensity on exposure. Table 3 presents another 20% in-the-money put option example where CVA_W exceeds CVA_I at each maturity; note that the difference is most pronounced for $T = 1$. The coefficient $b = 1$ in Table 3 indicates a relatively higher dependence of intensity on exposure.

T	.1	.2	.4	.6	.8	1
CVA_I	2	4	8	12	16.1	20.1
$\hat{\theta}_W$	1.9	3.7	7.5	10.5	15.5	19.6

Table 1: Forward contract: CVA numbers and estimates are of order 10^{-3}, $m = 10^5$, $b = .02$, $\mu = 0$, $\sigma = .25$, $S_0 = 2$, spread $= .01$, $\Delta = 5\tilde{\Delta}$, $\tilde{\Delta} = .01$ for $T = 1, .8, .6, .4$, and $\tilde{\Delta} = .001$ for $T = .1, .2$.

T	.1	.2	.4	.6	.8	1
$\hat{\theta}_I$	2	4	8.1	11.5	17.1	21.9
$\hat{\theta}_W$	1.9	3.7	7.6	11.1	16.7	21.6

Table 2: Put option: CVA estimates are of order 10^{-3}, $m = 10^5$, $b = .02$, $\mu = 0$, $\sigma = .25$, $S_0 = 10$, $K = 12$, spread $= .01$, $\Delta = 5\tilde{\Delta}$, $\tilde{\Delta} = .01$ for $T = 1, .8, .6, .4$, and $\tilde{\Delta} = .001$ for $T = .1, .2$.

We also came across unrealistic cases of put options where CVA_I exceeds CVA_W in a more pronounced way. For instance, consider the case where credit spread is flat at 10^6 basis points, i.e., $s = 100$. This gives $\text{CVA}_I = .0169$ and $\text{CVA}_W = .0057$ for $T = 1$. That is, independent

[12] We use MATLAB to produce the results.

T	.1	.2	.4	.6	.8	1
$\hat{\theta}_I$	2	4	8.1	11.5	17.1	21.9
$\hat{\theta}_W$	2.2	4.8	11.27	17.6	29.4	37.9

Table 3: Put option: CVA estimates are of order 10^{-3}, $m = 10^5$, $b = 1$, $\mu = 0$, $\sigma = .25$, $S_0 = 10$, $K = 12$, spread = .01, $\Delta = 5\tilde{\Delta}$, $\tilde{\Delta} = .01$ for $T = 1, .8, .6, .4$, and $\tilde{\Delta} = .001$ for $T = .1, .2$.

CVA is roughly 3 times larger than wrong way CVA.[13]

Note that $\hat{\theta}_I$ and $\hat{\theta}_W$ are biased estimators of CVA_I and CVA_W due to the time-discretization. Ideally, the mean square error of these estimators should be estimated. This is computationally extremely expensive in our setting. To get a feel for the statistical efficiency of our estimators, we note that for the forward contract example presented in Table 1, CVA_I is analytically calculated, and Monte Carlo estimates of CVA_I coincide with the exact values. Since Monte Carlo estimation of CVA is computationally intensive, a valuable line of research is to improve the calibration strategies of intensity models and to develop efficient Monte Carlo estimators of wrong way CVA.[14]

5 Conclusion

A mathematical model is required to compare independent CVA and worng way or right way (dependent) CVA. In this note, we focus on a class of stochastic intensity CVA models that includes the formulation of Hull and White [9]. We derive a formula for CVA for models in this class and show that neither wrong way risk nor right way risk affects a counterparty's credit quality. Using the model of Hull and White [9], we generate numerical examples that demonstrate wrong way CVA need not exceed independent CVA. Analogous arguments and numerical examples would show that right way CVA could exceed independent CVA. Our findings are inconsistent with the scalar multiples of independent CVA that have been adopted by Basel II and III regulators as a proxy for wrong way CVA; they provide an occasion to re-examine the assumptions underlying those proxies.

[13] The remaining parameters for this unrealistic example are $\sigma = .3$, $b = 2$, $\mu = 0$, $S_0 = 1$, $K = 1.5$. Also, $\tilde{\Delta} = .01$ and $\Delta = .05$

[14] See [5] for efficient Monte Carlo independent CVA estimation.

A Default Times with Stochastic Intensity and the Proof of the Counterparty Risk Pricing Formula

It is well known that a default time, τ, defined on a filtered probability space, $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, P)$, admits a stochastic intensity, λ, when the process,

$$1\{\tau \leq t\} - \int_0^{t\wedge\tau} \lambda_u du,$$

is a martingale, (where $t\wedge\tau \equiv \min\{t,\tau\}$). To make the martingale property precise the filtration is to be specified. For the general case see Chapter 2 of [2]. In what follows, we do this for our setting. A consequence of the existence of an intensity is the identity:

$$P(\tau > t) = E\left[e^{-\int_0^t \lambda_u\, du}\right],$$

which is used throughout this paper and in the proof Lemma 1.

Doubly Stochastic Random Times Let τ be a default time on a filtered probability space $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, P)$. Let $\{H_t\}_{t\geq 0}$ denote the filtration generated by the default indicator process $1\{\tau \leq t\}$. Suppose that the distribution of τ depends on additional information denoted by $\{G_t\}_{t\geq 0}$. Set $F_t \equiv G_t \vee H_t$ where F_t is the smallest σ-algebra that contains G_t and H_t.[15] The default time, τ, is called *doubly stochastic* when for all $t > 0$,[16]

$$P(\tau \leq t \,|\, G_\infty) = P(\tau \leq t \,|\, G_t),$$

and when conditional on G_t, $\int_0^t \lambda_u du$ is strictly increasing.[17]

In our setting $\{G_t\}_{t\geq 0}$ is the filtration generated by the exposure process V. The first condition implies that given the past values, $u \leq t$, of V, the future, $s > t$ does not contain any extra information for predicting the probability that τ occurs before t.[18]

The credit exposure process, V, could have jumps due to the the expiration of trades before prior to the maturity of the longest instrument in the portfolio. In this case, where V has points of discontinuity, τ may not be doubly stochastic. But, it can be shown that τ still admits a stochastic intensity λ, (see Definition D7 and Theorem D8 of [2]).

Lemma 1. *Consider a real-valued process V defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{G_t\}_{t\geq 0}$, denote the filtration generated by V, i.e., $G_t = \sigma\{V_s;\ 0 \leq s \leq t\}$, the smallest σ-field with respect to which V_s is measurable for every $s \in [0,t]$, and let $G \equiv G_\infty \subset \mathcal{F}$. Let D denote a real-valued process that is adapted to $\{G_t\}_{t\geq 0}$. Let τ denote a counterparty's default time, which admits the stochastic intensity λ that is adapted to $\{G_t\}_{t\geq 0}$. For $t \geq 0$,*

[15] By definition, τ is an H_t-stopping time. Note that τ is also a $F_t \equiv G_t \vee H_t$-stopping time for any $\{G_t\}_{t\geq 0}$.

[16] it represents the first event time of a conditional or doubly stochastic Poisson process.

[17] see, for instance, Chapter 9 of [10].

[18] Many of the stochastic intensity models in the credit literature work under this doubly stochastic framework, (see, for instance, [4]).

$$P(\tau > t \,|\, G) = e^{-\int_0^t \lambda_u \, du} \quad and \quad P(\tau > t) = E\left[e^{-\int_0^t \lambda_u \, du}\right]. \tag{11}$$

Then, the following holds for any given $T \geq 0$:

$$E[D_\tau V_\tau \boldsymbol{1}\{\tau \leq T\}] = \int_0^T E\left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u \, du}\right] dt.$$

Proof. *Conditional on G we can write,*

$$\begin{aligned} E[D_\tau V_\tau \boldsymbol{1}\{\tau \leq T\} \,|\, G] &= \int_0^T E[D_\tau V_\tau \,|\, G, \tau = t] f_{\tau|G}(t) \, dt \\ &= \int_0^T D_t V_t f_{\tau|G}(t) \, dt = \int_0^T D_t V_t \lambda_t e^{-\int_0^t \lambda_u \, du} \, dt, \end{aligned}$$

where $f_{\tau|G}$ is the conditional density of τ and is derived based on the left side of (11). Then, the Lemma follows by noting that:

$$E\left[D_\tau V_\tau \boldsymbol{1}\{\tau \leq T\}\right] = E\left[E[D_\tau V_\tau \boldsymbol{1}\{\tau \leq T\}|G]\right],$$

and

$$E\left[\int_0^T D_t V_t \lambda_t e^{-\int_0^t \lambda_u \, du} \, dt\right] = \int_0^T E\left[D_t V_t \lambda_t e^{-\int_0^t \lambda_u \, du}\right] dt.$$

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