Topological Distance Between Nonplanar Transportation Networks

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1 Introduction

In recent years, street network analysis has received substantial attention in scholarly and professional urban planning and transportation engineering [7]. Street networks are typically modeled as a graph with intersections as nodes connected to one another by street segments as edges. This graph model enables the computation of various metric, connectivity, and topological measures by efficient algorithms, including accessibility [36, 40], connectivity [21, 25, 28], centrality [15, 30, 39] eccentricity [34], betweenness [3], clustering [27], block lengths [4, 9], and average circuity [6, 8, 20, 23].

One drawback of most existing research in street network analysis is the common assumption of planarity [11] or approximate planarity [15] of the network. In a planar model, the street network is represented in two dimensions such that grade-separated edge crossings such as bridges and tunnels create artificial nodes in the graph. While planar simplifications can be useful for computational tractability, they misrepresent many real-world urban street networks. The failure of this assumption can lead to substantial errors in analytical results [10].

For the present study, we focus in particular on *map comparison*, which can serve the field of urban morphology [5, 26, 38] and the evaluation of map reconstruction algorithms [2]. A recent line of work showed the advantage of topology-based measures for map comparison [1]. Using the common representation of streets by their centerlines [11, 14, 18, 25, 29, 30, 33], streets are split into a sequence of line segments or *street segments* in 2D. Leveraging insights from the nascent field of topological data analysis, the shape of the network can be described by tracing how the connectivity of street segments evolves as segments are gradually thickened [16]. These methods of topological map comparison can be applied in transportation engineering and urban planning research/practice to quantify the difference between two graphs—accounting for non-planarity—to measure street network evolution over time, to compare the structural and functional differences among proposed urban design alternatives, or to match trip trajectories to pre-existing infrastructure models.

In this abstract, we extend and enhance the topology-based measure presented in [1] to accommodate grade-separated nonplanar street networks. Our work is similar in spirit to the study of *multi-layered environments* [35] in motion planning. With the aim of extending known algorithms and data structures for 2D spaces to accommodate surfaces embedded in 3D, the surface is partitioned into *layers*. Each layer is required to project onto the ground plane without self-intersection. The connections between layers can be thought of as staircases and ramps. Ignoring heights, paths across layers are measured by the projected distance in $\mathbb{R}^2$ [32]. Similar ideas have been explored in architecture and urban planning to model circulation paths through multi-story buildings [31, 37].

2 Preliminaries

Topological data analysis is fueled by the concept of persistent homology, which exposes the shape of data by tracing the evolution of topological features [17]. By varying the scale, a sequence of nested complexes derived from the data yields a *filtration*. Computing the persistent homology on the filtration reveals all the topological features in the data, where each feature is described by its dimension, and the scales at which it appears and disappears [41]. This is often encoded as a *barcode*, which consists of an interval representing the lifespan of each feature from its birth time to its death time [19]. In that way, persistent homology enables us to compare two data sets by comparing their respective barcodes. One way to carry out this comparison is to find the minimum-cost matching between the intervals in each barcode as used to define the *bottleneck distance*. The validity of this distance measure stems from its *stability* against small changes in the data [12]. For further information, the reader is referred to [16, 22].

Our work is an enhancement and extension of the topology-based distance for street networks proposed in [1]. Under the assumption of planarity, a point is fixed in the plane and the goal is to define a *topological signature* of the local neighborhood of this point. To define the neighborhood, a *window* is placed with the point at its center and the street networks are restricted within this window by clipping any segments that extend beyond. A filtration is defined by tracing the topology of the thickened clipped graph, relative to the boundary of the clipping region. The persistence diagram summarizing this filtration is the *local persistent homology* (LPH) diagram. Given two street networks, the signatures can be compared by computing their bottleneck distance. This distance can be integrated by varying the scale and center point used to define the window to obtain a pseudo-metric; see [1] for the details.

3 Defining and Computing Layered LPH Distance

In this abstract, we propose a new distance between street networks which relaxes the planarity assumption. We also describe the computation of this distance by explaining how to compute the Layered Čech Filtration. If all street segments lie in the same layer, the desired distance can be found by examining the segment Voronoi diagram [24], and using the proposed distance for planar LPH distance in [1]. In what follows, we define the (local) Layered Čech Filtration by 'thickening' the graph G while allowing travel through portals. Similar to the one-layer case, our zero-simplices of the Layered Čech Filtration correspond to straight-line embedded edges and appear at radius zero, our one-simplices appear at the radius when two thickened edges overlap. We note that the edges may thicken *through* a portal, which is where the interesting part of the algorithm lies. Finally, the two-simplices correspond to a three-way intersection of thickened edges (again, possibly thickening through portals). Since our domain is a street network, the most interesting topology lies in the connected components and looping behavior, so we do not compute higher-dimensional simplices.

Assumptions and Notation Let $G = (V, E)$ be our piecewise-linear graph immersed in $\mathbb{R}^2$. In what follows, we assume that an oracle $\ell \colon E \to \mathbb{N}$ assigns an edge $e \in E$ to layer $\ell(e)$. We may further assume that $\ell(e) \leq m$ for all edges $e \in E$, for some $m \in \mathbb{N}$. We denote a layer $L_i := \ell^{-1}(i)$. Whenever $\ell(e) = i$, with $e = (u, v)$, we say say that the vertices u, v are also in L_i. We define a *portal* as a vertex shared between at least two layers, and assume any layer has at most k portals, for some $k \in \mathbb{N}$. Notice that while edges map to a single layer, a portal may map to multiple layers (which happens when adjacent edges map to different layers).

3.1 The Layered Čech Filtration

Since we assume that the graph has at most m layers, we think of this graph as m graphs, each of which resides in its own copy of $\mathbb{R}^2$, with portals defining identifications between the layers. We denote this embedding space (m copies of $\mathbb{R}^2$

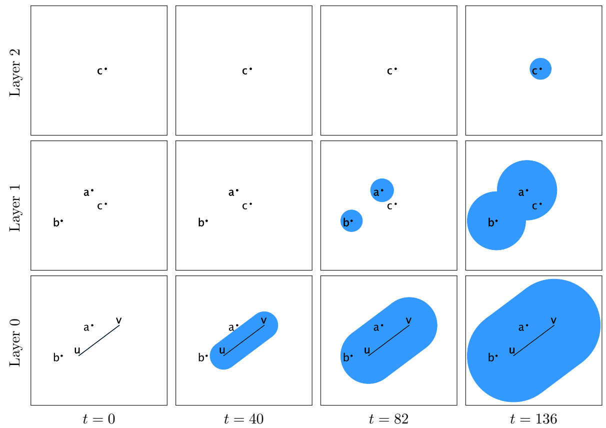

Figure 1: Offset of a segment $\overline{uv}$ across three layers. Notice that the thickening in the third layer starts when the portal c is reached in the second layer.

with an equivalence relation induced from portals) by $\mathbb{L}$. For $\epsilon > 0$, we define the thickened graph G^ϵ to be the tubular neighborhood around G of radius ϵ. We can also think of this as the union of thickened edges. A thickened edge looks like a tubular neighborhood in one layer and a union of disks centered at portals in other layers. The disks need not have the same radius.

One way to approximate the union of thickened sets is through computing the *nerve*. In our setting, we call this nerve filtration the *Layered Čech Filtration*. Specifically, the zero-simplices correspond to the set of edges in the input graph G, the one-simplices correspond to pairwise intersections between edges (and appears at half the distance needed for that intersection to be nontrivial), and the two-simplices correspond to the three-way intersections between thickened edges. When intersections between sets are collapsible, the *nerve lemma* states that the nerve of the set is homotopic to the union of sets. In our setting, however, the portals allow for nontrivial intersections between thickened edges (in fact, a thickened edge might not be contractible). Nonetheless, we use the nerve, as the local connectivity information is important to capture.

3.2 Computation

To compute the Layered Čech Filtration, we must be able to determine values for the thickening radius such that pairs and triples of thickened edges begin to intersect. Since the thickening radius between two edges is half the distance between the edges, first, we describe how to compute distances between edges in $\mathbb{L}$. Then, we consider three-way intersections of thickened edges in $\mathbb{L}$.

Given two edges $e_1, e_2 \in E$, as the triangle inequality still holds between layers, if e_1 and e_2 are on the same layer L, there exists a shortest path between e_1 and e_2 that remains within layer L. If, however, the edges are in different layers, the shortest path passes through multiple layers, perhaps even some layers that do not contain either e_1 or e_2. To compute the length of the path, we introduce an auxiliary data structure called the *Portal Graph*.

The Portal Graph The only way to move between layers is through portals. To help with subsequent distance computations, we precompute a complete weighted graph $\mathcal{P}$ on the $O(km)$ portals. We create a graph $\widetilde{\mathcal{P}}$ with the vertex set corresponding to the portals, and the initial weight of the edge between two portals p and q is:

$$w_0(p,q) = \begin{cases} 0, & \text{if } p = q, \\ d_2(p,q), & \text{if } \exists i \text{ } s.t. \text{ } p,q \in L_i, \\ \infty, & \text{otherwise.} \end{cases}$$

Using this initial weight assignment, we run an all-pairs shortest paths algorithm on this weighted graph to obtain the shortest-path distance $d_P \colon P \times P \to \mathbb{R}$ between all pairs of portals. The portal graph $\mathcal{P}$ is the graph where the edge-weights are the lengths of the shortest paths in $\widetilde{\mathcal{P}}$. Precomputation costs $O(k^3m^3)$ by standard algorithms [13].

Pairwise Distances (Two-way Intersections) Next, we describe how to compute distances between edges in $\mathbb{L}$. Let edges $e_i \in L_i$

and $e_j \in L_j$. If $i = j$, the distance between e_i and e_j in $\mathbb{L}$ is the Euclidean distance between e_i and e_j. If $i \neq j$, we compute the distance between e_i and e_j in $\mathbb{L}$ by augmenting $\mathcal{P}$ as follows. First, let $\mathcal{P}^*$ be the subgraph of $\mathcal{P}$ consisting of portals in L_i and L_j and the edges between the portals in the two layers. Add a vertex u to $\mathcal{P}^*$ where u represents e_i. For each portal $p \in L_i$, add edge (u, p) to $\mathcal{P}^*$ weighted as the Euclidean distance between e_i and p. Similarly, add a vertex v representing e_j and edges to all portals in L_j. The length of the shortest path from u to v in $\mathcal{P}^*$ is the distance from e_i to e_j in $\mathbb{L}$. Since $\mathcal{P}^*$ has at most $2k + 2$ vertices and at most $k^2 + 2k$ edges, the cost of computing a shorted path between u and v in $\mathcal{P}^*$ is $O(k^2)$. Once we have the distance d between edges e_i and e_j in $\mathbb{L}$, the thickening radius where e_i and e_j intersect is $d/2$.

Two-Simplices (Three-way Intersections) Given a triplet of edges, we are interested in finding the thickening radius where the edges intersect. We begin with a few observations. First, observe that a three-way intersection in $\mathbb{L}$ may occur in any layer. Second, observe that for a thickened edge $e \in L_i$, when e thickens enough to reach a portal $p_i \in L_i$, edge e continues to thicken on every layer in which p_i connects. In particular, let $p_j \in L_j$ be a portal on layer L_j in which p_i connects. As we continue to thicken e, we observe that a disk centered at p_j starts growing.

From the observations, we can enumerate the four configurations of how thickened objects can intersect on a layer. In particular the configurationss are: (C1) three thickened edges, (C2) two thickened edges and a disk, (C3) one thickened edge and two disks, (C4) three disks.

We can further use the observations to identify the thickening radius with brute force. In particular, for each layer L_i, we enumerate all configurations and determine r_i the least thickening radius yielding a non-trivial three-way overlap over all configurations on L_i. Finally, the thickening radius r is $\min(\{r_i, \ldots, r_m\})$.

4 Discussion

In this paper, we extend the results of [1] to the more realistic setting that allows for bridges and tunnels to be represented. In particular, we break the graph into layers. We assume the layering structure is part of the input and focus on the problem of computing a topology-based signature for the purposes of map comparison. Future work includes improving the algorithm by exploring the order-k Voronoi diagram of additively-weighted line segments, extending the definition of the Layered Čech Filtration to the *Local* Layered Čech Filtration, and implementing the algorithm.

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