ISIMA Project Report

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Following the work on ray orbits in spatially hyperbolic systems by Mass and Lam (1995) and Rieutord and Valdettaro (1997) we seek to examine the behaviour of the shear layer emitted at the critical latitude in 3D in a spherical shell filled with rotating fluid. We compare the (previously known) 3D and the 2D solutions for a sphere in an infinite domain to find the major difference being a logarithmic singularity on the rotation axis formed by a cone of shear converging to an apex. We then consider the "split disc" arrangement first considered by Walton to examine this singularity in more detail. We also consider the behaviour of the Moore and Saffman shear layers under the influence of a large-scale forcing; our motivation is primarily the dissipation of tidal energies in astrophysical binary systems.

1 Motivation

It is well known that tidal forces in binary systems leads to a large scale deformation of both fluid masses, and also to unsteady flows dissipative flows. Since the Ekman number $= \nu/(\Omega R^2)$, the balance between viscous and Coriolis forces, is so small in most astrophysical domains, this dissipation leads to an effect on the evolution of binary systems that acts on secular timescales (e.g. Zahn, 1976 §3, Rieutord, 2003) to affect the angular momentum distribution of the system and convert gravitational potential energy into thermal energy by way of dissipation.

This tides will also drive shear layers in the interior of each star (see e.g. Rieutord and Valdettaro, 2010) which can produce high degrees of transport in very localised regions. It is a topic which requires further investigation, and seems likely to influence the strength and location (and therefore, for example, the distribution of metallicity) in the star proper.

2 Mathematical Framework

We consider incompressible and inviscid fluid with temporal dependence $\propto \exp(i\omega t)$, acting under the Coriolis force. The governing equations

$$i\omega\vec{u} + \vec{e_z} \times u = -\nabla p$$

and

$$\nabla.\vec{u} = 0$$

*Advised during by Michel Rieutord.

facillitate the introduction of a streamfunction ψ, with

$$u_x = \partial_z \psi,\, u_z = -\partial_x \psi$$

which leads quickly in 2D cylindrical co-ordinates to the governing Poincarè's equation

$$\partial_{xx}\psi - \frac{\alpha}{\omega 2}\partial_{zz}\psi = 0$$

with $\alpha^2 = 1 - \omega^2 > 0$; this is spatially hyperbolic; in a 3D fluid domain we again have a hyperbolic structure and additional curvature terms; the equation for the pressure field can be shown to be (e.g. Rieutord and Noui, 1998)

$$\Delta P - \frac{1}{\omega^2}\partial_{zz}P = 0$$

and there exists an extensive literature on the expected wave structures with such a governing equation. In particular we wish to consider the inertial waves in a spherical shell with no-penetration boundary conditions on the inner and outer radius.

3 2D and 3D solution

Since the critical latitude shear layer is emitted by the inner sphere, it is reasonable as to examine its behaviour in an infinite domain; following the work of Rieutord and Noui (1998) we make use of a singular transformation

$$\alpha x = \sqrt{(1-\eta^2)(1-\mu^2)}$$

$$\omega z = \mu\eta$$

with

$$1 \geq \eta \geq \omega,\, \eta \geq \mu \geq \omega$$

to convert our governing equation into Laplace's equation; our no-penetration boundary condition that $\psi = const$ on the circular boundary must now be applied on a hyperbola. After several more co-ordinate transformations we may write the solutions for 2D (3D) in elliptic (oblate spheroid) co-ordinates as follows. With reference to a book by Claycomb, we write the 2D solution as

$$\psi_p(\eta,\mu) = \frac{P_s^p(\mu) + kP_c^p(\mu)}{\left(i\sqrt{1-\eta 2} + i\eta\right)^p}$$

and we refer to RN1 for the 3D solution for the pressure field

$$P_l^m = Q_l^m(\eta)P_l^m(\mu)e^{im\theta}$$

with notation to be explained. For the 2D case we may think of p as a separation constant governing the rate at which our solution decays in the far field, and P_s^p, P_c^p trigonometric polynomials such that $P_s^p(\sin\theta) = \sin(p\theta)$ and similarly for P_c^p. For the 3D case we may think of (m, l) as the usual spherical wavenumbers, and P_l^m, Q_l^m Legendre polynomials of the first and second kind respectively. Note particularly that although the mapping is singular on the lines of critical latitude $\eta^2 = \mu^2$, the 2D streamfunction is entire and the 3D pressure field is singular at a finite number of points; we discuss these below.

A full discussion of elliptic or oblate spheroidal co-ordinates is rather involved and not relevant here, expect to state that it is η which may tend to the point at ∞; μ may become arbitrarly large but must remain finite. We shall note any points of interest and give their physical location in the more usual cylindrical co-ordinates.

We note that in these co-ordinates the solution (or an obvious conjugate) given by Maas and Lamb (1995) for the streamfunction inside a semi-elliptic basin

$$\psi = x^3 + z^2 x - x$$

becomes in these co-ordinates

$$\psi = \cosh 3\xi \cos 3\chi$$

where we have made the natural identification

$$\tau^2 \equiv \frac{N^2}{\omega^2} \leftrightarrow \frac{\alpha^2}{\omega^2}.$$

3.1 Critical Latitude Singularity

Since the transformation between 2D cylindrical coordinates and 2D elliptical coordinates is extremely closely related to the transformation between 3D spherical coordinates and 3D oblate spheroidal coordinates, we may note that the 2D velocity field suffers from a similar divergence to that found in 3D; we may write for the velocity parallel to one of the critical latitude shear layers that

$$\begin{gathered} DV_{||} = D\left(\omega\partial_z\psi + \alpha\partial_x\psi\right) \\ = \left(\omega^2\mu(\eta^2-1) - \alpha 2\eta(\eta^2-1)^{1/2}(\mu^2-1)^{1/2}\right)\partial_\eta\psi \\ -\mu \leftrightarrow \eta \end{gathered}$$

and for the velocity perpendicular that

$$\begin{gathered} V_\perp = \alpha\partial_z\psi - \omega\partial_x\psi \\ = \frac{\omega}{a\left(\mu\sqrt{1-\eta^2} + \eta\sqrt{1-\mu^2}\right)}\left((\sqrt{\eta^2-1})\partial_\eta\psi - (\sqrt{\mu^2-1})\partial_\mu\psi\right) \end{gathered}$$

where $D \equiv \eta^2 - \mu^2 = 0$ on the critical latitude shear layers.

4 Comparison of 2D with 3D

It is plain that in 2D there is a symmetry along the line $r = z$ with $\alpha \leftrightarrow \omega$, since we may simply rewrite Poincaré's equation to have the same functional form. It is equally plain that in 3D no such simple symmetry exists due to the curvature terms in the Poincaré operator; we see a global geometrical distinction between the solutions. If we write again the solutions for 2D

$$\psi_p(\eta,\mu) = \frac{P_s^p(\mu) + kP_c^p(\mu)}{\left(i\sqrt{1-\eta^2} + i\eta\right)^p}$$

and 3D

$$P_l^m = Q_l^m(\eta)P_l^m(\mu)e^{im\theta}$$

we may note that for all complex values of η the denominator in the 2D case remains finite; the solution is regular everywhere in the domain since μ is not allowed to diverge. However if, in the 3D case, we write out the form of Q_l^m

$$Q_l^m(\eta) = f_l^m(\eta)\log\left(\frac{1+\eta}{1-\eta}\right) + g_l^m(\eta)$$

we see that we must consider the location of the point $\eta = 1$; it lies on the rotation axis at the apex of the cone, where a cone of shear emitted from the critical latitude converge; $(r, z, \theta) = (0, \pm 1/\omega, 0)$.

We briefly consider now Riemann's method; if we consider an $m = 0$ axisymmetric mode and focus on the topologies of the attractors which localise the dissipation of energy in the domain, we realise that may have not only an irreducible path which encloses the "stellar core", but also irreducible paths which do or do not enclose this inviscid logarithmic singularity. A conjecture emerges, therefore, that for fixed forcing frequency ω, a star might dissipate preferentially on one attractor over another based on its path around the domain. This remains to be investigated numerically; to make analytic progress we consider a domain in which a full analytic solution is available.

5 The Split-Disc and the Viscous Crossing Singularity

We refer to Walton (1974) and Kerswell (1995) to consider a "split-disc" arrangement; we consider two semi-infinite plates at a distance $2h$ apart, separated by viscous incompressible fluid. The entire system is rotating with angular velocity Ω and there is an inner disc of radius a on the upper plate which, in the rest frame of the plates, is rotating with constant angular velocity ω. This discontinuity in the boundary conditions emits a family of shear layers which converge to a point on the axis of rotation which, by moving the lower plate, we are free to place at the origin. We assume that the local regularisation in this split disc case will hold true for the same singularity in the spherical case.

We transcribe from Walton (1974) the entire analytic form of the azimuthal velocity in terms of Bessel functions.

$$\frac{r}{\epsilon a \Omega} u_\psi e^{-i\omega\Omega t} = \chi = r\sum_{s=1}^{6}\int_0^\infty A_s(k)J_1(kr)exp(\alpha_s z)\mathrm{dk}$$

a sum of six terms; two representing the large scale flow, two representing the Ekman layers near the upper and lower discs, and two representing the internal shear layers in which we are interested. We may therefore neglect the majority of these terms, as in Walton (1974) to write

$$\chi = \frac{rc}{(4-\omega^2)^{1/2}}\int_{-\infty}^{\infty}\frac{kJ_0(ka)J_1(ka)\cosh((h+z)\alpha)}{\sinh 2h\alpha}\mathrm{dk}$$

c being some constant. Here we take α, the vertical attenuation coefficient being a solution to a known sextic which at small k gives $\alpha \sim ik\omega/(4-\omega^2)^{1/2} +$

$16k^3E/(4-\omega^2)^{5/2}+...$ with the intention of taking $R\to\infty$; then, $i\alpha$ represents the vertical wavenumber, $\text{Re}(k)$ represents the radial wavenumber, and $\text{Im}(k)$ represents the radial attenuation. As may be seen from the form of the integrand, there are poles whenever $\alpha = n\pi i/2h$, representing the vertical Fourier modes; these poles may be used to obtain an asymptotic expansion for the integral.

There exists here a somewhat subtle point, since the integral does not converge with this expansion of α unless $R=\infty$; more seriously, the series is not asymptotic. Walton then noted the poles in the upper half complex plane whenever $\alpha = n\pi/2h$, and so applied the method of residues in summing the far-field expansion of Hankel functions. After taking $R=\infty$ and an Euler sum, we find that that, for $r = ord(1)$, we have four terms, each corresponding to a family of characteristics travelling either inward and upward, outward and downward, or otherwise in the natural senses; the azimuthal flow reads

$$\chi \propto \Re\left(\frac{1}{1-exp(\frac{n\pi}{2h}(h+z+\frac{r-a}{\alpha_0})}+...+...+...\right)$$

which diverges precisely on those shear layers. We must modify this solution when $r = O(E^{1/3})$ which is the expected scaling of the inner region; the far-field expansion of Hankel or Bessel functions is no longer appropriate for those poles which do not have $|k_n| \gg E^{-1/3}$. After repeating the above method, including the adopting of $R=\infty$, we may write that, near the axis of rotation,

$$\chi = \frac{i\alpha_{10}rc}{\omega}\Re\Bigg\{2\pi i\sum_{n=1}^{N-1}k_n\left(\frac{2}{\pi k_n a}\right)^{1/2}\left(\frac{k_n r}{2}-\frac{(k_n r)^3}{16}\right)$$
$$\times\cos(k_n a-\frac{\pi}{4})\exp(\frac{h+z}{2h}n\pi i)$$
$$+\frac{2\pi i}{\sqrt{\pi^2 ar}}\left[\frac{i\exp(N\pi i(\frac{r-a}{\alpha_0}+h+z)/2h)}{1-\exp(\pi i(\frac{r-a}{\alpha_0}+h+z)/2h))}-...+...+...\right]\Bigg\}.$$

where our polynomial terms have come from approximating $J_1(kr)$ by its inner (outer) asymptotic limit for $kr<1$ ($kr>1$); this introduces some small error on the order of 3% into the solution. $N = \left\lceil\frac{2h\alpha_{10}}{\pi r}\right\rceil$ is the index of the first pole for which $|k_N r|>1$; we see that its existence has introduced an ultraviolet catastrophe into the solution: As we allow r to approach 0 the radial wavenumber of the solution diverges and we are dominated by arbitrarily small scales. We would like to successfully re-introduce viscosity to properly analyse this point.

6 The Re-introduction of Viscosity

As previously noted, those series used to obtain the inviscid flow are not asymptotic and do not converge when $E\neq 0$. We consider briefly the inverse problem - given that $\alpha = n\pi i/2h$, some fixed vertical harmonic, what associated radial wavenumber k_n may contribute to the series? We find that we must examine the suitable solution branch for k_n, given that $\alpha = n\pi/2h$ leads to

$$\left(((n\pi/2h)^2-k_n^2)E-i\omega\right)^2\left((n\pi/2h)^2-k_n^2\right)+4(n\pi/2h)^2=0$$

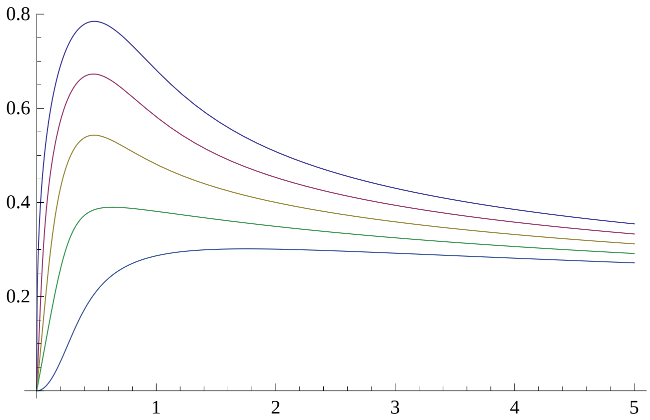

Figure 1: A plot of $\mathrm{Im}(k_n/\sqrt{E\omega})$, the rescaled real component of the radial wavenumber, as a function of the rescaled index $\sigma\sqrt{\omega h/E}n$. Plotted from top to bottom are $\omega \in 0, 0.5, 1, 1.5, 2$

and this may be readily calculated (see figure).

By allowing n to vary continuously between 0 and ∞, we see that there is a well-defined single turning point which corresponds to the lengthscale at which viscosity becomes dominant; this is to be expected, since we have a well defined vertical lengthscale. If we rescale the (now continuous) variable n such that $\sigma = \sqrt{E^{-1}\omega^3 h}n$, we find the turning point occuring at the maximum of

$$\mathrm{Im}\ \bar{k}_n(\sigma,\omega) = \frac{\sqrt{E^{-1}\omega}}{2\sqrt{3}}\mathrm{Im}\Big\{3\pi^2\sigma^2\omega^2 - 8i + \Big(-2i + 3\pi\sigma(9\pi\sigma + \sqrt{-12i + 81\pi^2\sigma^2})\Big)^{(1/3)} + \Big(-2i + 3\pi\sigma(9\pi\sigma + \sqrt{-12i + 81\pi^2\sigma^2})\Big)^{(-1/3)}\Big\}^{1/2}$$

and we see that - if our integrand has terms of the form $J_1(kr)$ - we expect a single exponentially dominant term $\propto exp(k_n r)$. This is somewhat alarming; we see our solution growing rather than decaying with r[1] and diverging as $r \to \infty$. This is due to the fact that we have not yet had to consider the Sommerfeld radiation condition; we currently have incoming waves at $r = \infty$. If we then attempt to chose a radial basis of decaying Hankel functions representing only outgoing waves $\sim e^{-kr}$ in the far field, we must contend with the fact that Hankel functions are singular at the origin.

We conjecture a resolution: The appropriate radial basis is a mix of $J_1(kr)$ when $r < a$ and $H_1^1(kr)$ when $r > a$. Physically this is demanded; inside the upper disc of radius a we will have waves both incoming and outgoing with respect to the origin; outside we will have only outgoing. We expect, then, the solution behaviour to be of the amplitude $|k_n|\cosh(\mathrm{Im}(k_n)(a \pm r))$ for $r < a$ and $|k_n|\exp(\mathrm{Im}(k_n)(a - r))$ for $r > a$. Due to the geometrical nature of the singularity - that of a cone of rays converging - it remains to be investigated as to how many terms in the $k \sim E^{1/3}$ region are necessary to resolves the inner singularity.

[1]Thanks to G.I.Ogilvie for a helpful discussion on the satisfaction of boundary conditions in wave-propagation problems and this problem in particular.

7 Internal Structure of Forced Shear Layers

We examine now the structure of the internal shear layers which are known to scale like $E^{1/3}$ (see e.g. Moore and Saffman 1967, hereafter M&S). It is derived in M&S that in the far-field absence of curvature we may write the spreading of a shear layer in a self-similar way; here, as usual, the zeroth order azimuthal component of velocity is given, and the corresponding relations for other components may be quickly found in the literature.

$$\partial_{YYY} u^0_\psi = -i\partial_q u^0_\psi$$

and hence

$$u^0_\phi = q^m H_m \left(\eta \equiv \frac{Y}{q^{1/3}}\right)$$

where q is an order unity rescaling of the coordinate along the length of the ray, and $Y = E^{1/3}y$ is the small scale co-ordinate across the width of the ray. For H_m, we follow the convention of Ogilvie 2005 to write

$$H_m(t) = i\int_0^\infty e^{i-pt-p3} p^{-3ik_m}\, dp$$

whose asymptotic behaviour for large t may be found easily by the method of stationary phase. We consider the axisymmetric set of equations labelled (3.6) given in RVG 2009 with the addition of a smooth, large scale forcing term $\vec{f}$

$$\lambda u_{||} - \omega u_\psi = -\partial_x p + \frac{\omega p}{2s} + E\left(\nabla 2 - \frac{\omega 2}{s2}\right) u_{||} + \frac{\alpha\omega E}{s2} u_\perp + f_x$$

$$\lambda u_\psi + \omega u_{||} - \alpha u_\perp = E\nabla 2 u_\psi + f_\psi$$

$$\lambda u_\perp + \alpha u_\psi = -\partial_y p - \frac{\alpha p}{2s} + E\left(\nabla 2 + \frac{\alpha 2}{s2}\right) u_\perp - \frac{\alpha\omega E}{s2} u_{||} + f_y$$

and mass conservation through

$$\partial_x u_{||} + \partial_y u_\perp + \frac{\omega u_{||} - \alpha u_\perp}{2s} = 0$$

with the Laplacian modified to read $\nabla 2 = \partial_{xx} + \partial_{yy} + \frac{1}{4(\omega x - \alpha y)2}$. If we neglect now curvature and pose the natural expansion as demanded by mass conservation we may follow the Moore and Saffman derivation to find the governing equation including forcing

$$\partial_{YYY} u^0_\psi = -i\partial_q u^0_\psi + F_q$$

where the forcing term is given by

$$F = \frac{1}{2}\left(\frac{i}{\alpha} f^0_y + \frac{1}{\omega} f^0_\psi\right).$$

Now, since we have stipulated that $\vec{f}$ be large scale and have no boundary layers, we may write that $\partial_Y \vec{f} = O(E^{1/3})$ and consider the forcing to be constant at zeroth order across the boundary layer. Therefore we may rewrite our governing equation as

$$\partial_{YYY}(u^0_\psi + iF) = -i\partial_q(u^0_\psi + iF) + O(E)$$

Figure 2: A schematic of the upper-right hand plane showing the inner circle or sphere. Points of interest include $A = (0, \frac{1}{\omega})$, $P = (\omega, \alpha)$ and $B = (\frac{1}{\alpha}, 0)$. Lined in pink is the position of the critical latitude singularity.

and argue that our solution is essentially unchanged and reads

$$u_\phi^0 = q^m H_m \left(\eta \equiv \frac{Y}{q^{1/3}} \right) + iF.$$

giving the simple but slightly counterintuitive result that the azimuthal velocity can 'see' the forcing both along and *perpendicular* to the ray.

7.1 Neglect of Curvature

This derivation involved the neglect of curvature terms to reproduce the Cartesian argument that appears in Moore & Saffman. It is plain that this approximation is likely to be at best marginally valid when terms such as $\frac{\omega p}{2s}$ are the same order of magnitude as terms such as $\partial_x p$, which is here the case since we are considering an exterior bounding surface of radius 1. It is also plain that this approximation will become catastrophically poor when we allow $s \to 0$ if the ray approaches the axis of rotation; we therefore cannot expect these Moore and Saffman solutions to more than qualitatively describe equatorial attractors (which always have $s > \eta$, η the inner rbounding radius, and cannot be expected at all to explain polar attractors, nor the ray cone emitted from the critical latitude which has $s < \eta$ and directly approaches the axis.

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