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ON THE PENETRATION EFFECT IN ELECTRIC DIPOLE INTERNAL CONVERSION

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ABSTRACT

Nuclear-structure-dependent contributions to the internal-conversion process are considered in conjunction with highly retarded electric dipole transitions. Formulas for a theory of anomalous internal conversion for electric transitions are given. These formulas are applied to explain the strong  $E1$  anomalies for  $L$  conversion found experimentally by Asaro, Stephens, Hollander, and Perlman.<sup>4</sup>

Two different types of anomalous matrix elements occur, one associated with the nuclear charge, the other with the nuclear current. It turns out that the contribution associated with charge is negligible. In the current terms we distinguish two parts, the convection current and the spin current, which are associated with different selection rules and generally both important. From the empirical data of the  $L_I$  and  $L_{II}$  anomaly the sign of the ratio of nuclear matrix elements which gives the dominant contribution to the anomalous conversion amplitude is deduced.

The anomalous nuclear matrix elements are evaluated for the single-particle model. Effects of pair correlation are discussed. The agreement with experiments is satisfactory.

ON THE PENETRATION EFFECT  
IN ELECTRIC DIPOLE INTERNAL CONVERSION\*

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Introduction

The occurrence of so-called anomalous internal conversion can now be used as a means of exploring details of nuclear structure. Deviations from the point-nucleus internal-conversion coefficients (ICC)<sup>1</sup> can occur because of two effects. The first effect we might label as static. There the finite radial extension of the central-charge distribution changes the electron wave functions outside the nucleus relative to the point-charge case. A satisfactory account of this effect has changed the assumed values of the theoretical ICC by appreciable amounts.<sup>2,3</sup>

The other effect is sometimes referred to as dynamical and is connected with the penetration of the electron wave function inside the nuclear surface, and is also thus present for all finite nuclei. However, it is generally a small effect because of the small probability of the electrons being inside the small nuclear volume. This penetration usually gives

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rise to additional nuclear matrix elements not present in the  $\gamma$ -decay. More specifically, the anomaly caused by the penetration depends on the ratio of the nuclear matrix element due to penetration to the normal  $\gamma$ -ray matrix element. It is clear that the effect should be noticeable only if the normal nuclear matrix element is small, i.e. the corresponding  $\gamma$ -transition hindered. Obviously the probability for the electron to be inside the nucleus is strongly increased with increasing Z and A. The strongest anomalies are found for E1 transitions, and — as expected in line with what is said above — they occur in the actinide region<sup>4</sup> where the volume factor for penetration is most favorable and where the  $\gamma$ -hindrance factors for E1 transitions are particularly large. We limit ourselves from section 2 on to considering only anomalous ICC of E1 transitions.

An increase in the values of the ICC by a factor of up to 20 is encountered by Asaro, Stephens, Hollander and Perlman.<sup>4</sup> The data have been analyzed and discussed with some qualitative success (as to selection rules) in papers by Nilsson and Rasmussen.<sup>5,6</sup>

A reinvestigation of these anomalies appears to be called for as Greene and Rose<sup>7</sup> and Church and Weneser<sup>8</sup> have pointed out the possible or even probable importance of a penetration term neglected in refs. 5 and 6 in view of an occurring cancellation in the anomaly terms that would otherwise be dominant. Such extra terms were also derived earlier by Kramer<sup>9</sup> but not further explored for specific cases.

We give here a new derivation of all the penetration terms, employing a formulation close to that of refs. 5 and 6. The result is the occurrence of anomalous matrix elements of two different types, one associated with charge, the other with current.

We evaluate the electron matrix elements inside the nucleus for both these contributions. It then turns out that usually the contribution associated with charge is negligible, owing to the mentioned cancellation effects occurring for the combination  $(ff' + gg')$  of the radial electron wave functions inside the nucleus. In such a case it is possible to relate the  $L_I$  anomaly to the  $L_{II}$  anomaly independently of any knowledge of the nuclear matrix elements. From an analysis based on the electron wave functions it also becomes clear why any possible anomaly of the  $L_{III}$  coefficient must be smaller than that of  $L_I$  and  $L_{II}$  by several orders of magnitude.

The next step in any quantitative analysis must be an evaluation of the ratio of the nuclear matrix elements involved. The matrix element occurring in the denominator of the anomaly term is a conspicuously small number in the cases considered, i.e., the hindrance factors are large. It is obviously very difficult to calculate this quantity with enough reliability (in view of its smallness relative to its unhindered value) on the basis of any presently developed nuclear model. Thus even the sign of the calculated quantity appears uncertain. On the other hand, the absolute value may be considered as given empirically from the  $\gamma$ -ray lifetime of the transitions.

We have nevertheless attempted to calculate the  $E1$  matrix elements theoretically. Although the order of magnitude of hindrance is in rough agreement with the experimental values, there is generally no quantitative agreement.

We believe, however, that the anomalous matrix elements, which are much less or in one case not at all inhibited, can be calculated with somewhat better reliability.

Because the major part of the penetration effect comes from the contribution of the nuclear current, the form of which is not completely known

theoretically, we are forced to limit ourselves to applying a specific nuclear model not only for the calculation of the nuclear wave functions but thus also for the specification of the electromagnetic current.\*

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\* It is well known that for the gamma-transition matrix elements this ambiguity can be removed. (Refs. 10 and 11.)



1. Derivation of the Internal Conversion Anomaly Terms

In order to facilitate a comparison with ref. 5 we adopt the formulation of the first paper of ref. 9 based on an expansion of the electromagnetic field into multipole components.

Using perturbation theory and working in the Coulomb gauge for the photons, one can derive the following expression for the electric conversion amplitude (see ref. 9):

$$U_{fi} = \sum_{LM} \sum_{n=1}^A U_{fi}(LM,n) + U_{fi}(C), \quad (1)$$

where

$$U_{fi}(L,M,n) = \lim_{\sigma \rightarrow 0} \int_0^{\infty} \frac{dk}{(W+i\sigma)^2 - k^2} \times \left\{ \int \psi_f^* \vec{j}_n \cdot \vec{A}_{LM}^*(\vec{r}_n) \psi_i d\tau_n \cdot \int \phi_f^* \vec{j}_e \cdot \vec{A}_{LM}(\vec{r}_e) \phi_i d\tau_e \right\} \quad (2)$$

and

$$U_{fi}(C) = \int \psi_f^* \phi_f^* \frac{e_n e_e}{|\vec{r}_n - \vec{r}_e|} \psi_i \phi_i d\tau_n d\tau_e. \quad (3)$$

The contribution from the transversal photons is represented by eq. (2). The direct Coulomb interaction (scalar and longitudinal photons in the Lorentz gauge) is responsible for the amplitude  $U_{fi}(C)$  given by eq. (3).

In the Coulomb gauge the electric multipoles are defined as <sup>12</sup>

$$\vec{A}_{LM} = N_L \left[ \vec{\nabla} \left( \frac{d}{dr} r j_L(kr) \right) + k^2 \vec{r} j_L(kr) \right] Y_{LM}. \quad (4)$$

A more familiar expression for  $\vec{A}_{LM}$  is

$$\vec{A}_{LM} = -N_L \frac{1}{k} \text{curl} (\vec{r} \times \vec{\nabla}) k j_L(kr) Y_{LM}. \quad (5)$$

The normalization constant is given as

$$N_L = [(\pi/2) L (L+1)]^{-1/2}, \quad (6)$$

and  $j_L$  is the spherical Bessel function. Furthermore,  $W$  denotes the nuclear excitation energy,  $\vec{j}_n$  and  $\vec{j}_e$  the nucleon and electron current operators. Finally  $\psi_f$ ,  $\psi_i$ , and  $\phi_f$ , and  $\phi_i$  are the wave functions of the nucleus and the electrons in the final and initial states, respectively. With the specific purpose to avoid nonexistent integrals in the  $k$  integration, it is convenient first to exploit the continuity equation for the nucleon and electron currents, respectively. This amounts to replacing  $\vec{j} \cdot \vec{\nabla} \cdot S(\vec{r})$ , where  $S(\vec{r})$  is an arbitrary function of  $\vec{r}$  and where  $\vec{j}$  stands for alternatively the nucleon or electron current, by the commutator<sup>11</sup>

$$\vec{j} \cdot \vec{\nabla} S(\vec{r}) = ei[H_0, S(\vec{r})]. \quad (7)$$

The quantity  $H_0$  is the nuclear and electronic Hamiltonian, respectively, in the absence of the electromagnetic field.

Solely by the use of eq. (7) we arrive at the following expression for  $U(LM)$ . (In the following we omit the indices  $f$ ,  $i$ , and  $n$  in  $U$ .)

$$U(LM) = N_L^2 \int \frac{dk}{w^2 - k^2} \int \psi_f^* \phi_f^* \left\{ (-iW \frac{d}{dr_n} r_n j_n Y_{LM}^*) (iW \frac{d}{dr_e} r_e j_e + k^2 \vec{j}_e \cdot \vec{r}_e j_e) Y_{LM} \right. \\ \left. + \left( k^2 \vec{j}_n \cdot \vec{r}_n Y_{LM}^* \right) \vec{j}_e \cdot \left( \vec{\Delta} \frac{d}{dr_e} r_e + k^2 r_e \right) j_e Y_{LM} \right\} \psi_i \phi_i d\tau_e d\tau_n. \quad (8)$$

The next step, the integration over  $k$  (the momentum of the photon field), is discussed in some detail in the Appendix, and gives as a result

$$\begin{aligned}
U(LM) = & N_L^2 \frac{\pi}{2} \frac{1}{iW} \left\{ \int_0^\infty (-iW\rho_n \frac{d}{dr_n} r_n + W^2 \vec{j}_n \cdot \vec{r}_n) j_L Y_{LM}^* d\tau_n \right. \\
& \times \int_0^\infty (iW\rho_e \frac{d}{dr_e} r_e + W^2 \vec{j}_e \cdot \vec{r}_e) h_L Y_{LM} d\tau_e \quad (9a) \\
& + \int_0^\infty (-iW\rho_n Y_{LM}^* F(r_n)) d\tau_n \\
& \left. + \int_0^\infty W^2 \vec{j}_n \cdot \vec{r}_n Y_{LM}^* G(r_n) d\tau_n \right\}
\end{aligned}$$

The functions  $F(r_n)$  and  $G(r_n)$  contain integrals over the charge and current of the electron. They are

$$\begin{aligned}
F(r_n) = & \left( \frac{d}{dr_n} r_n h_L(Wr_n) \right) \int_0^{r_n} (iW\rho_e \frac{d}{dr_e} r_e + W^2 \vec{j}_e \cdot \vec{r}_e) j_L(Wr_e) Y_{LM} d\tau_e \\
& - \left( \frac{d}{dr_n} r_n j_L(Wr_n) \right) \int_0^{r_n} (iW\rho_e \frac{d}{dr_e} r_e + W^2 \vec{j}_e \cdot \vec{r}_e) h_L(Wr_e) Y_{LM} d\tau_e \quad (9b)
\end{aligned}$$

and

$$\begin{aligned}
G(r_n) = & h_L(Wr_n) \int_0^{r_n} \vec{j}_e \cdot (\vec{\nabla} \frac{d}{dr_e} r_e + W^2 \vec{r}_e) j_L(Wr_e) Y_{LM} d\tau_e \\
& - j_L(Wr_n) \int_0^{r_n} \vec{j}_e \cdot (\vec{\nabla} \frac{d}{dr_e} r_e + W^2 \vec{r}_e) h_L(Wr_e) Y_{LM} d\tau_e \quad (9c)
\end{aligned}$$

Here  $h_L$  is the spherical Hankel function of the first kind. For convenience we have used the same symbols for the transition charge and transition currents of the nucleon and electron as for the corresponding operators used earlier.

It can be shown quite easily that formula (9a) together with (9b) and (9c) agrees with the formula for the electric conversion derived in ref. 9.

The expression for  $G(r_n)$  above can be transformed into a more convenient form by a partial integration, the use of the continuity equation, and Gauss's theorem. One obtains, e.g., for the first part of  $G(r_n)$  containing the current,

$$\begin{aligned} & \int_0^{r_n} \vec{j}_e \cdot \vec{\nabla} \frac{d}{dr_e} r_e j_L Y_{LM} d\tau_e \\ &= iW \int_0^{r_n} \rho_e \frac{d}{dr_e} r_e j_L Y_{LM} d\tau_e \\ & \quad + \int [r_e^2 \vec{j}_e \cdot \frac{\vec{r}_e}{r_e} \frac{d}{dr_e} r_e j_L Y_{LM}]_{r_e=r_n} d\Omega_e. \end{aligned} \quad (10)$$

By such a procedure (9) can be brought into the more compact form

$$\begin{aligned} U(LM) = N_L^2 \frac{\pi}{2} \frac{1}{iW} \left\{ \int_0^\infty Q_n^*(j_L) d\tau_n \cdot \int_0^\infty Q_e(h_L) d\tau_e \right. \\ + \int_0^\infty Q_n^*(h_L) \int_0^{r_n} Q_e(j_L) d\tau_e d\tau_n - \int_0^\infty Q_n^*(j_L) \int_0^{r_n} Q_e(h_L) d\tau_e d\tau_n \\ \left. - \int_0^\infty W^2 \vec{j}_n \cdot \vec{r}_n Y_{LM}^* \cdot \frac{i}{W} \left[ \int \vec{j}_e \cdot \vec{r}_e Y_{LM} d\Omega_e \right]_{r_e=r_n} d\tau_n \right\}, \end{aligned} \quad (11)$$

after some substitution based on the identity

$$h_{L-1} j_L - j_{L-1} h_L = \frac{i}{(Wr)^2}$$

and the adoption of the following definition for  $Q_\lambda(j_L)$  with  $\lambda=n$  or  $e$ :

$$Q_\lambda(j_L) = (iW\rho_\lambda \frac{d}{dr_\lambda} r_\lambda + W^2 \vec{j}_\lambda \cdot \vec{r}_\lambda) j_L(Wr_\lambda) Y_{LM}. \quad (12)$$

The same definition is used for  $Q_\lambda(h_L)$ , with  $j_L$  replaced by  $h_L$  in (12).

In eq. (11) we distinguish penetration terms of basically different character, some associated with the nuclear current, others with the nuclear charge. In ref. 5 only the terms of the latter type have been considered. This was because in the expression for the electric multipole defined by eq. (4) only the gradient term was retained for the nucleon-photon interaction, whereas both terms were kept for the electron-photon interaction. After the  $k$  integration was performed, the larger part of the terms originating from the second term of eq. (4) are small, of order  $W_r$ , compared with terms stemming from the gradient part of  $\vec{A}_{LM}$ . They can thus be neglected, as anticipated. However, because of the complicated nature of the  $k$  integration, an additional surface term originating from the second part of eq. (4) emerges, as can be seen from eq. (11). This — the last term in eq. (11) — is the current term mentioned. In addition, because a particular cancellation occurs for the penetration terms associated with charge, the current term is not only nonnegligible but in fact dominant (see the discussion below in sections 3 and 4).

2. Further Reduction of the Electron Integrals

The angular integration over the electron wave function is identical to that occurring in the normal internal conversion problem, hence we leave out all details of this. After this <sup>angular</sup> integration the amplitude takes the form

$$\begin{aligned}
 U(LM) = & \frac{\pi}{2} N_L^2 O_{LM} \int_0^\infty \int_0^\infty (-iW \rho_n \frac{d}{dr_n} + W^2 \vec{j}_n \cdot \vec{r}_n) d\tau_n \\
 & \times \int_0^\infty [(f_{K'} f_{K'} + g_K g_{K'}) \frac{d}{dr} r + (f_K g_{K'} - g_K f_{K'}) Wr] h_L(Wr) r^2 dr \\
 & + \int_0^\infty (-iW \rho_n) Y_{LM}^* \alpha(r_n) \\
 & + \int_0^\infty W^2 \vec{j}_n \cdot \vec{r}_n Y_{LM}^* \beta(r_n),
 \end{aligned} \tag{13}$$

where the functions  $\alpha(r)$  and  $\beta(r)$  are defined in terms of the radial wave functions of the electron in the initial and final state. We write

$$\alpha(r) = f_1(r) + g_1(r), \tag{13a}$$

$$\beta(r) = f_2(r) + g_2(r) + h(r). \tag{13b}$$

Then the functions  $f_1, g_1, f_2, g_2,$  and  $h$  are defined as

$$f_1(r) = \frac{d}{dr}(r h_L) I_j(r) - \frac{d}{dr}(r j_L) I_h(r), \tag{13c}$$

$$g_1(r) = \frac{d}{dr}(r h_L) J_j(r) - \frac{d}{dr}(r j_L) J_h(r), \tag{13d}$$

$$f_2(r) = h_L I_j(r) - j_L I_h(r), \tag{13e}$$

$$g_2(r) = h_L J_j(r) - j_L J_h(r), \tag{13f}$$

$$h(r) = (1/i(Wr)^2)(f_K g_{K'} - g_K f_{K'}) r^3 \tag{13g}$$

where  $I_j$  and  $J_j$  stand for radial integrals of the electron radial wave functions,

$$I_j(r) = \int_0^r dr' r'^2 (f_\kappa f_{\kappa'} + g_\kappa g_{\kappa'}) \frac{d}{dr'} r' j_L \quad (13h)$$

$$J_j(r) = \int_0^r dr' r'^2 (f_\kappa g_{\kappa'} - g_\kappa f_{\kappa'}) W r j_L \quad (13j)$$

and  $I_h(r)$ ,  $J_h(r)$  are defined in the same way as  $I_j(r)$ ,  $J_j(r)$ , provided the Bessel function  $j_L$  is replaced by the Hankel function in (13h) and (13j).

The first term in eq. (13) can easily be transformed by a partial integration into the form originally employed for calculation of the normal conversion coefficients (see refs. 1, 2 and 9).

In eq. (13)  $O_{LM}$  is a factor common to both normal and anomalous terms and contains all the geometrical factors resulting from the angular integration.

The functions  $f_\kappa$  and  $g_\kappa$  are the "small" and "large" components of the electron wave functions as defined, e.g., in ref. 12. The quantity  $\kappa$ , conveniently representing both the  $l$  and  $j$  quantum numbers of the electrons, is used in the conventional definition (see, e.g., ref. 12).

We determine the quantities  $\alpha(r)$  and  $\beta(r)$  in the form of an expansion in powers of  $(r/R)$  from the Dirac equation for a homogeneous isotropic charge distribution inside the nucleus. Those solutions are fitted to the solutions outside the nuclear surface which have been determined by numerical calculations. In the bound-electron states we have used the recent calculations by Cohen<sup>13</sup> (Hartree-Fock), and for the continuum wave functions those tabulated by Reitz<sup>14</sup> (Thomas-Fermi).

Retaining the three lowest powers in  $r/R$  in  $\beta(r)$ , we may write

$$\beta(r) = (r/R)^2 [d_0 + d_2 (r/R)^2 + d_4 (r/R)^4 + \dots] \quad (14)$$

The coefficients  $d_0$ ,  $d_2$ ,  $d_4$  are given in table 1 for the specific empirical cases considered. A similar expansion for  $\alpha(r)$  can be found. However,

it turns out that the expansion coefficients in  $\alpha(r)$  are smaller than those of  $\beta(r)$  by nearly two orders of magnitude. This point is discussed in more detail in section 4.

### 3. Analysis of the Empirical Data. Conclusions Independent of Nuclear Models

The experimental cases considered are all taken from the article by Asaro, Stephens, Hollander, and Perlman (ref. 4). We have selected such cases for which the empirical data are most complete. We have furthermore confined ourselves to nuclei for which the deformed coupling scheme appears to be well established.

Of the cases exhibited in table 2 the most conspicuous ones are Pa<sup>231</sup> and Pa<sup>233</sup>. For Am<sup>243</sup> only the L<sub>I</sub> conversion appears anomalous compared with the value given by Sliv and Band, which lies just barely outside the experimental limits of error, whereas the L<sub>I</sub> ICC given by Rose falls within these limits. This is therefore probably not an anomalous case at all. It is also associated with a relatively small hindrance factor (see the last column of Table 2, where the hindrance factor is defined in accordance with ref. 4). The Am<sup>243</sup> case is included in this discussion to verify that the theoretical calculations can verify the absence of anomaly in this case. One may observe that as a general trend the degree of anomaly increases with the hindrance factor.

A feature common to all the anomalous cases listed is that the L<sub>III</sub> value is never anomalous.

Instead of trying to compare directly the measured anomalous ICC's with those calculated from theory (of course such a calculation involves the evaluation of nuclear matrix elements), we found it advantageous to



extract the anomaly of the ICC's in terms of an anomaly amplitude  $\Delta$ . In the next paragraph we shall attempt to calculate these  $\Delta$ 's on the basis of a specific nuclear model. For electric dipole  $L_I$  and  $L_{II}$  conversion the following electron transitions occur:

$$\left. \begin{array}{l}
 L_I : \left\{ \begin{array}{l}
 1) s_{1/2} \rightarrow p_{1/2} \\
 2) s_{1/2} \rightarrow p_{3/2}
 \end{array} \right\} \\
 \\
 L_{II} : \left\{ \begin{array}{l}
 3) p_{1/2} \rightarrow s_{1/2} \\
 4) p_{1/2} \rightarrow d_{3/2}
 \end{array} \right\}
 \end{array} \right\} \quad (15)$$

The contribution to the anomaly from the second and fourth transitions is entirely negligible, as the appropriate expansion coefficients  $d_i$  of eq. (14) are smaller by several orders of magnitude than those of the first and third transition. This is connected with the large centrifugal barrier of the larger total angular momenta in the final electron states in cases 2 and 4.

We write the  $L_I$  and  $L_{II}$  ICC's as

$$\alpha_{L_I} = \alpha_1 + \alpha_2, \quad (16)$$

$$\alpha_{L_{II}} = \alpha_3 + \alpha_4, \quad (17)$$

where the index notation is in accordance with (15). Then, in line with the discussion above,  $\alpha_1$  and  $\alpha_3$  are assumed to contain all the anomaly. We then define the quantities  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  by the relations

$$\alpha_1 = \frac{1}{6} | R_1 + \Delta_{L_I} |^2, \quad (18)$$

$$\alpha_3 = \frac{1}{6} | R_3 + \Delta_{L_{II}} |^2. \quad (19)$$

Here  $R_1$  and  $R_3$  are the normal conversion amplitudes which have been calculated earlier by Sliv and Band.<sup>15</sup> \* The factor  $1/6$  is a geometrical factor.

The quantities  $\alpha_{L_I}$  and  $\alpha_{L_{II}}$  are then identified with the measured ICC's as given in table 2. From these  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  are calculated on the basis of eqs. (16) through (19). Inside the approximations in section 2 the quantities  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  are purely real. As  $R_1$  and  $R_3$  are also almost purely real - the imaginary part is smaller than the real part by an order of magnitude or more - the normal and anomalous components interfere strongly. This is one reason why the anomalous conversion is seen even for moderately hindered transitions. In this connection we should also point out that  $R_1$  and  $R_3$  have the same sign and are of comparable magnitude ( $R_1/R_3 \approx 3$ ). According to the theory given in sections 1 and 2 the anomaly amplitudes  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  are approximately equal to the product of the expansion coefficient  $d_0$ , as defined in eq. (14), and a certain ratio of nuclear matrix elements, which latter enters in both the  $L_I$  and  $L_{II}$  anomaly amplitude. As the coefficients  $d_0$  have opposite signs in the two cases, it is obvious that the interference of the normal and anomalous amplitudes (see (18) and (19)) is constructive in one case and destructive in another. If  $\alpha_{L_I}$  and  $\alpha_{L_{II}}$  are added, a considerable fraction of the interference terms cancels. The occurrence of such an effect is suggested by fig. 7 in ref. 4.

The results for  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  are collected in table 3 (the indices 1, 2 refer to the first and second solution of eqs (18) and (19)). Under the condition that the lowest-order term in (14) is dominant, which means that only one nuclear matrix element enters the anomaly amplitudes, then

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\* A table of the quantities  $R_1$ ,  $R_3$ ,  $\alpha_2$ , and  $\alpha_4$  was kindly supplied to us by Professor L. A. Sliv and Professor I. M. Band.

because  $d_0$  has opposite signs in the  $L_I$  and  $L_{II}$  cases (see table 1) also  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  must have opposite signs. On the basis of this fact we have been able to pair off two sets of solutions from the four values of  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$ . As is assumed above, we have the approximate relations

$$\Delta_{L_I} \approx d_0 (L_I) \lambda_0, \quad (20)$$

$$\Delta_{L_{II}} \approx d_0 (L_{II}) \lambda_0, \quad (21)$$

where  $\lambda_0$  is a ratio of nuclear matrix elements. From (20) and (21) we obtain

$$\Delta_{L_I} / \Delta_{L_{II}} \approx d_0 (L_I) / d_0 (L_{II}). \quad (22)$$

As the ratio on the right side of (22) is known and tabulated in table 3, we can rule out one of the two sets of solutions for the pair  $(\Delta_{L_I}, \Delta_{L_{II}})$  in most of the cases. The result of such a comparison is presented in the last four columns of table 3. By application of the relation (20) or (21) the sign of  $\lambda_0$  is then determined. The sign of  $\lambda_0$  could also be tested experimentally in a direct way by angular correlation measurements.

Here the question arises whether eqs. (20) and (21) are sufficiently accurate to validate such a determination of the sign of  $\lambda_0$ . In the case that the nuclear matrix element of the higher powers in  $r/R$  are nonnegligible, eqs. (20) and (21) have to be replaced by the equations

$$\Delta_{L_I} \approx d_0 (L_I) \lambda_0 + d_2 (L_I) \lambda_2 + d_4 (L_I) \lambda_4, \quad (23)$$

$$\Delta_{L_{II}} \approx d_0 (L_{II}) \lambda_0 + d_2 (L_{II}) \lambda_2 + d_4 (L_{II}) \lambda_4, \quad (24)$$

where  $\lambda_2$  and  $\lambda_4$  contain matrix elements with two and four additional powers in  $r/R$ . Even if  $\lambda_2$  and  $\lambda_4$  are of the same order of magnitude as  $\lambda_0$  eq. (22) still holds with very good accuracy, as  $d_2/d_0$  and  $d_4/d_0$  are approximately equal for  $L_I$  and  $L_{II}$ . It is clear that the analysis may be more complicated if  $\lambda_2$  or  $\lambda_4$  is considerably larger than  $\lambda_0$ .

The evaluation of  $\beta(r)$  as defined in section 2 for  $L_{III}$  conversion leads to an expansion, of the following form, similar to eqs. (23) and (24):

$$\Delta_{L_{III}} \approx d_2 (L_{III}) \lambda_2 + d_4 (L_{III}) \lambda_4. \tag{25}$$

One may note that  $\lambda_0$  does not occur in (25). From the fact that  $\Delta_{L_{III}}$  depends on the same ratios of nuclear matrix elements as  $\Delta(L_I)$  and  $\Delta(L_{II})$  and from the smallness of  $d_2(L_{III})$  and  $d_4(L_{III})$  - see table 1 - it follows that  $\Delta_{L_{III}}$  is very much smaller than  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$ .

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\* The detailed calculation of these matrix elements for a particular nuclear model bears out this contention in all the cases considered in this paper.

#### 4. Calculation of Nuclear Matrix Elements

In this section we discuss the calculation of the ratio of nuclear matrix elements  $\lambda_s$  entering into eqs. (23) and (24).

We may write

$$\lambda_s = \frac{3}{2} (1/M\omega R^2)^{\frac{s+2}{2}} \omega R A_s(E1)/N(E1) \quad (26)$$

with only  $s$  values equal to 0, 2, and 4 considered.

The quantity  $N(E1)$  is the normal  $\gamma$ -transition matrix element, more specifically the matrix element of  $\rho(4\pi/3)^{1/2} Y_{lm}$ . There and in the following the radial operator in the nuclear matrix elements have been expressed in terms of the dimensionless quantity  $\rho = (M\omega)^{1/2} r$ . The magnitude of this matrix element may be determined empirically from the known E1 life time  $1/T$  through the formula

$$T(E1) = (8\pi/9) \cdot (\Delta E)^3 \cdot M\omega \cdot (4\pi/3) N(E1)^2 \cdot e_{\text{eff}}^2 \\ \approx 2.37 \times 10^{15} (E/\text{Mev})^3 N(E1)^2 \cdot e_{\text{eff}}^2, \quad (27)$$

where the second line in eq. (27) is accurate only for elements with  $A \approx 240$  and  $e_{\text{eff}}$  is the effective charge equal to  $\frac{N}{A}e$  for a proton and  $-\frac{Z}{A}e$  for a neutron.

The  $A_s(E1)$  are the matrix elements of the operator  $i\vec{j}_n \cdot \hat{r}_n (4\pi/3)^{1/2} Y_{lm} \rho^{s+2}$  ( $s = 0, 2, 4$ ) in units of  $e_{\text{eff}}$ .

In this calculation we assume for the nucleon current the following expression

$$\vec{j}_n = \vec{j}_n^{\text{cc}} + \vec{j}_n^{\text{sc}} \quad (28)$$

where  $\vec{j}_n^{\text{cc}}$  is the convection part of the current:

$$\vec{j}_n^{\text{cc}} = (e_n/2Mi) \left( \psi_f^* \vec{\nabla}_n \psi_i - (\vec{\nabla}_n \psi_f^*) \psi_i \right). \quad (28a)$$

and  $j_n^{sc}$  is the so-called spin current:

$$\vec{j}_n^{sc} = \frac{e}{2M} \mu_n \vec{\nabla}_n \times \psi_f^* \vec{\sigma}_n \psi_i. \quad (28b)$$

In (28b)  $\mu_n$  is the magnetic moment of the nucleon. The matrix elements of the current were evaluated using for  $e_n$  in the convection current the effective charge and for  $\mu_n$  in the spin current effective values of  $\mu_n = 2$  for protons and  $\mu_n = -1/2$  for neutrons. These values of  $\mu_n$  account in an approximate way for the effects of the spin polarization.<sup>16</sup>

Of course, the expression in eq. (28) for the nucleon current is not complete even for a single-particle model if a spin-orbit force is included.<sup>17</sup> However, the contribution to the nucleon current resulting from the spin-orbit force is perpendicular to  $\vec{r}_n$  and thus does not contribute to  $\vec{j}_n \cdot \hat{r}_n$ , which is the quantity of interest here.\*

In a single-particle oscillator model the matrix element of  $\vec{j}^{cc} \cdot \hat{r} \cdot r^{s+2}$  can be expressed in terms of matrix elements involving only powers of  $r$ . In this model, therefore, one has

$$\begin{aligned} &\langle N' \ell' | i \vec{j}^{cc} \cdot \hat{r} (r/R)^{s+2} | N \ell \rangle \\ &= \frac{\omega R}{s+3} \cdot \left\{ -(N'-N) \langle N' \ell' | (r/R)^{s+3} | N \ell \rangle \right. \\ &\quad \left. + \left( 1/(2M\omega) \right) \left[ \ell' (\ell'+1) - \ell(\ell+1) \right] \langle N' \ell' | (r/R)^{s+1} | N \ell \rangle \right\}, \quad (29) \end{aligned}$$

for  $s = 0, 2, 4$ .

In formula (29)  $N$  and  $\ell$  refer to the total number of nodes in the oscillator wave function and angular momentum  $\ell$  respectively, and  $\omega$  is the characteristic oscillator energy.

From this formula we may estimate the order of magnitude of the contribution of the convection current terms compared with the charge terms. The

\* The same holds true for the contribution from an  $\vec{l} \cdot \vec{l}$  force.

structure of the terms connected with the charge is  $\langle N'l' | (r/R)^{s+3} Nl \rangle$ . The ratio of expression (29) to this matrix element is then of the order of  $\omega R$ , which is approximately 0.3. It should be emphasized that it is the oscillator shell spacing energy  $\omega$  that enters, and not the spacing between the initial and final nuclear states  $W$ ,<sup>8</sup> which is usually much smaller than  $\omega$ . Although the nuclear matrix elements due to the current are smaller than those due to the charge by the factor  $\omega R$ , the contribution of the latter can be neglected owing to the smallness of the corresponding coefficients  $d_i$ . (For a definition of  $d_i$  see section 2.) As can be seen from eqs. (13), (13a), (13c), and (13h) the combination  $(f_K f_{K'} + g_K g_{K'})$  of radial electron wave function enters into these  $d_i$  coefficients. The corresponding factor for the current term is  $(f_K g_{K'} - g_K f_{K'})$  according to eqs. (13), (13b), and (13g). The first combination of electron amplitudes is small inside the nucleus compared with the second by a magnitude of the order of  $mR$  (where  $m$  is the electron mass)<sup>6</sup> which is small of the order of  $10^{-2}$ . Additional geometrical factors also favor the current terms by a factor of 10 or more.

One may note that the asymptotic selection rules of the convection current and charge matrix elements are identical. Thus, provided the spin current contribution is minor, the qualitative analysis of ref. 5 is valid also for the current terms.

It can be shown that the spin current contribution can be written in the form

$$\int i \vec{j}_n^{sc} \cdot \hat{r} \left( \frac{r}{R} \right)^{s+2} Y_{lm} d\tau_n = \frac{e}{2MR} \mu_n \int \psi_f^* \vec{\sigma}_n \psi_i \cdot \vec{L} \left( \frac{r}{R} \right)^{s+1} Y_{lm} d\tau_n \quad (30)$$

The  $m=0$  and  $m=1$  components of the operator involved in this term are proportional to  $(\sigma_+ Y_{1-1} + \sigma_- Y_{11})$  and  $(\sqrt{2} \sigma_Z Y_{11} + \sigma_+ Y_{10})$ , respectively. It is apparent that these terms are associated with different selection rules — for instance spin flip is allowed — than the terms contributed by the convection current. In Table 7 the selection rules for the "asymptotic" single-particle wave functions are given corresponding to the two cases  $s=0$  and  $s=2$  in eq. (30).

Basically the spin current term is smaller than the leading convection current term by a factor  $\frac{\mu_n(s+3)}{2MR \cdot \omega R}$  which is of the order  $\frac{1}{3}$ . However, because the spin current term has different selection rules, it may be very important in cases where the spin current contribution is unhindered but the convection term hindered. In the cases treated here the spin current matrix elements for  $s=0$  are always classified as hindered, why its effect in the cases presently treated are less important. However for  $s=2$  most of the spin current matrix elements are unhindered. All the six experimental examples considered refer to nuclei for which the deformed coupling scheme is well established. The wave functions can be written as<sup>18</sup>

$$\begin{aligned} \psi_{MK}^I &= \left( (2I+1)/16\pi^2 \right)^{1/2} \left[ \chi_K \mathcal{D}_{MK}^I + R_1 (\chi_K \mathcal{D}_{MK}^I) \right] \\ &= \left( (2I+1)/16\pi^2 \right)^{1/2} \left[ \chi_K \mathcal{D}_{MK}^I + (-)^{I+l-1/2} \chi_{-K} \mathcal{D}_{M-K}^I \right], \end{aligned} \quad (31)$$

where  $\chi_K$  is the intrinsic nuclear wave function and  $\mathcal{D}_{MK}^I$  describes the rotation of the nucleus as a whole. The nuclear matrix elements of interest can be given in terms of the intrinsic wave function,  $\chi_K$ . For instance, for  $N(E1)$  we have

$$\begin{aligned} N(E1, IK \rightarrow I'K') &= \left\{ \langle I \ 1 \ K \ K' \ -K | I' \ K' \rangle \langle \chi_K | (4\pi/3)^{1/2} \rho Y_{1K'-K} | \chi_K \rangle + \right. \\ &\quad \left. (-)^{I'-l'+1/2} \langle I \ 1 \ K \ -K' \ -K | I' \ -K' \rangle \langle \chi_{-K} | (4\pi/3)^{1/2} \rho Y_{1-K'-K} | \chi_K \rangle \right\} \end{aligned} \quad (32)$$

In eq. (31) the effect of the Coriolis interaction is neglected. However, the inclusion of this effect gives rise to a coupling between bands with angular momentum components  $K$  and  $K \pm 1$  to lowest order in perturbation theory. The generalization of (31) and (32) to include such Coriolis admixtures is obvious.



For the examples of anomalous ICC considered, single-particle states involved are listed in table 4. The states are identified in that table by their asymptotic quantum numbers  $[Nn_ZAK]$ . The table also lists the coefficients of Coriolis admixtures that we have employed in our evaluation of the nuclear matrix elements.

The detailed wave functions of ref. 19 were employed in the calculations of  $N(E1)$  and  $A_s(E1)$  for  $s = 0, 2$ . The results are exhibited in table 5.

The separate contributions of the convection current and the spin current are denoted  $A_s^{cc}$  and  $A_s^{sc}$ , respectively. The contribution of the amplitude  $A_s$  to  $\lambda_s$  for  $s > 0$  are very much reduced by the proportionality factor  $(1/M\omega R)^{2s/2}$ .

We see from table 5 that the experimental  $\gamma$ -transition matrix elements are reduced relative to the unhindered single particle value, which is normally of the order one, by a factor between  $10^{-2}$  and  $10^{-3}$ . This general feature of a large inhibition is borne out by the detailed calculations of  $N(E1)$ ; the results are listed as  $N_{eff}^{theo}$  in table 5. (In the asymptotic limit all these matrix elements vanish.) Nevertheless in individual cases there are discrepancies of the order of 10 in both directions. This indicates that the model employed is not capable of predicting these very small matrix elements with any reliability, not even as to their signs. The smallness of these matrix elements makes it plausible that any neglected effects such as higher configuration mixing could decisively influence their value and might tend to increase them generally. On the other hand the effects of pair correlation<sup>20</sup> give rise to an inhibition factor  $(uu'-vv')$  which becomes particularly small and may even vanish when the initial and final single-particle states lie on opposite sides of the Fermi surface. This effect might in some case be responsible for a change of sign in  $N(E1)$ .

Regarding the anomalous matrix elements  $A_0^{cc}$  and  $A_2^{cc}$ , we notice that the three first cases listed in table 5 are always smaller by a factor of almost 10 than the fourth one, which latter is classified as unhindered by the asymptotic selection rules.<sup>5</sup> The first three are hindered by the same rules. The two last ones are forbidden by the K- selection rule. They show, however, very large impurities in K, as may be found by inspection of table 4.\* The dominant K- admixture is also associated with an unhindered transition. This explains the intermediate values of  $A_0^{cc}$  and  $A_2^{cc}$  in these last two cases. The nuclear matrix elements of  $\rho \cdot \vec{\sigma} \cdot \vec{L}(Y_{IM})$  corresponding to  $A_0^{sc}$  are all small because of the intrinsic hindrance already discussed. Their relative importance is much increased by the additional factor  $\frac{e}{e_{\text{eff}}} \mu_n$ .

In order to obtain the anomaly amplitudes  $\Delta$ , we calculate  $\lambda_0$  and  $\lambda_2$  in accordance with eq. (26). For comparison we use both the empirical and the theoretical values for  $N(\text{El})$ . In the former case we list  $\lambda_0$  and  $\lambda_2$  in columns 2 and 4 of table 6 (denoted  $\lambda_0^s$  and  $\lambda_2^s$ ). The signs of these two quantities are arbitrarily chosen so that  $N^{\text{exp}}$  is negative for all the six cases. The completely theoretical values  $\lambda_0^t$  and  $\lambda_2^t$  are exhibited in columns 3 and 5 of table 6. We have not calculated  $\lambda_4$  because it would make a negligible contribution to  $\Delta$ , even if it were of the same order of magnitude as  $\lambda_0$  or  $\lambda_2$ , owing to the smallness of  $d_4$  (see table 1). The final results for  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  can be found in columns 8 and 12. These results have to be compared with the values of  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  (columns 9 and 13) obtained directly from an analysis of the experimental data by the procedure described in section 3.

The semitheoretical values of  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$ , i.e.  $\lambda_0^s d_0 + \lambda_2^s d_2$  are in fair agreement with the experimental  $\Delta$ 's as to their magnitudes — the signs are arbitrary. — In all cases, except  $\text{Pu}^{239}$  the theoretical values are below

\* Some of these amplitudes of the K- impurities have been calculated by Dr. F. S. Stephens, Jr. We are grateful for his permission to use his values in advance of publication.

the experimental ones. The exceptional Pu<sup>239</sup> case is discussed separately below. The last two transitions are K-forbidden, and all the other transitions are hindered in the asymptotic quantum numbers, hence small admixtures of configurations neglected here that give unhindered matrix elements may increase the theoretical values and improve the agreement. We further note that a less strong  $\vec{I}^2$ -force in the proton potential ( $\mu = 0.45$  vs.  $\mu = 0.70$  for the protons) generally increases the nuclear matrix element by approximately 30%.

The Pu<sup>239</sup> transition corresponds to an anomalous matrix element which is unhindered. For this case, therefore, the theoretical estimate should be most reliable. The observed deviation with experiments being of the order of 40% can largely be accounted for in terms of pair correlation effects. As pointed out earlier pair correlations modify the usual E1 transition matrix element  $N(E1)$  in the quasi-particle approximation by the factor  $(uu' - vv')$ . However the anomalous matrix elements  $A_s(E1)$  are instead modified by a factor  $(uu' + vv')$ . The value of this latter factor is usually very close to 1 as the single-particle states involved lie close to the Fermi surface. We have an exception in Pu<sup>239</sup>, where the transition takes place between two rather highly excited states separated from the Fermi surface by single-particle energies of the order of half the gap energy. The initial state corresponds to a hole state in a pure single-particle description, and the final state is a single-particle excitation in this picture. Therefore in such a pure single-particle model the transition is forbidden due to the fact that a change of two particles would be required for the transition. In the pair correlation model such a transition is no longer forbidden because of the smearing of the pairs near the Fermi surface which is quantitatively reflected in the factors  $(uu' - vv')$  for  $N(E1)$  and  $(uu' + vv')$  for  $A_s(E1)$ . This latter factor is estimated to be of the order of 0.7 for Pu<sup>239</sup> in the quasi-particle model.

This accounts for a reduction by 30% of the semitheoretical  $\lambda$ -value of Pu<sup>239</sup> because the factor  $(uu' - vv')$ , associated with  $N(E_1)$ , is taken care of by the use of an experimental value for  $N(E_1)$ . One may expect a further, though not very significant, reduction factor from blocking effects.<sup>21,22</sup>

We observe that the  $\Delta_{L_I}$  and  $\Delta_{L_{II}}$  as obtained from the analysis of the experimental data contain some additional information so far not exploited. The experimental  $\Delta$ 's can be used to determine the relative sign between the anomalous matrix elements and  $N(E_1)$ . In our analysis this sign is given by the sign of  $\lambda_0^t$  exhibited in column 3 of table 6. This sign should be equal to the sign of  $\Delta_{L_{II}}$  (column 13 of table 6). However, as mentioned earlier we cannot expect to predict the sign of  $N(E_1)$  reliably due to the large hindrance of the single-particle matrix elements and furthermore due to the occurrence of the factor  $(uu' - vv')$  in the quasi-particle description.

In summary we may conclude that the over-all order of discrepancy as to the general magnitude of the effect encountered in an attempted quantitative discussion in ref. 5 is now removed owing to the larger magnitude of the current contribution.

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APPENDIX

Before we integrate over  $k$  in eq. (8) we notice that the following identities hold:

$$\frac{d}{dr} r j_L = kr j_{L-1} - L j_L, \quad (A1)$$

$$\left( \vec{\nabla} \frac{d}{dr} r + k^2 \vec{r} \right) j_L Y_{LM} = \left( kr j_{L-1} - L j_L \right) \vec{\nabla} Y_{LM} + \frac{L(L+1)}{r^2} \vec{r} j_L Y_{LM}. \quad (A2)$$

The following three integrals have to be evaluated:

$$I_1 = \int_0^\infty dk \frac{1}{W^2 - k^2} \left( k^2 r_n r_e j_{L-1}(kr_n) j_{L-1}(kr_e) + L^2 j_L(kr_n) j_L(kr_e) - L kr_n j_{L-1}(kr_n) j_L(kr_e) - L j_L(kr_n) kr_e j_{L-1}(kr_e) \right), \quad (A3)$$

$$I_2 = \int_0^\infty dk \frac{1}{W^2 - k^2} \left( kr_n j_{L-1}(kr_n) - L j_L(kr_n) \right) k^2 j_L(kr_e), \quad (A4)$$

$$\vec{I}_3 = \int_0^\infty dk \frac{1}{W^2 - k^2} k^2 j_L(kr_n) \left[ \left( kr_e j_{L-1}(kr_e) - L j_L(kr_e) \right) \vec{\nabla} Y_{LM} + \frac{L(L+1)}{r_e^2} \vec{r}_e j_L(kr_e) Y_{LM} \right]. \quad (A5)$$

Using the identity  $\frac{1}{W^2 - k^2} = \frac{1}{W^2} \left( \frac{k^2}{W^2 - k^2} + 1 \right)$  (A6)

for the three last terms in (A3), we can write, for the integral  $I_1$ ,

$$\begin{aligned}
 I_1 = \int_0^\infty dk \frac{1}{W^2 - k^2} \frac{k^2}{W^2} & \left( W^2 r_n r_e j_{L-1}(kr_n) j_{L-1}(kr_e) + L^2 \cdot j_L(kr_n) j_L(kr_e) \right. \\
 & \left. - Lkr_n j_{L-1}(kr_n) j_L(kr_e) - Lj_L(kr_n) kr_e j_{L-1}(kr_e) \right) \\
 & + \frac{1}{W^2} \int_0^\infty dk \left( L^2 j_L(kr_n) j_L(kr_e) - Lkr_n j_{L-1}(kr_n) j_L(kr_e) - Lj_L(kr_n) kr_e j_{L-1}(kr_e) \right)
 \end{aligned}
 \tag{A7}$$

One observes that the integrals as defined by (A3), (A4), and (A5) exist.<sup>25</sup> With the help of the identity (A6) all three integrals can be evaluated easily if we apply the formulas given in ref. 25:

$$\int_0^\infty \frac{j_L(kr) j_L(kr')}{W^2 - k^2} k^2 dk = -\frac{i\pi}{2} W \begin{cases} j_L(Wr) h_L(Wr') & \text{for } r' \geq r \\ h_L(Wr) j_L(Wr') & \text{for } r > r' \end{cases}, \tag{A8}$$

$$\int_0^\infty \frac{j_{L-1}(kr) j_L(kr')}{W^2 - k^2} k^3 dk = -\frac{i\pi}{2} W^2 \begin{cases} j_{L-1}(Wr) h_L(Wr') & \text{for } r' \geq r \\ h_{L-1}(Wr) j_L(Wr') & \text{for } r > r' \end{cases}, \tag{A9}$$

$$\int_0^\infty j_L(kr) j_L(kr') dk = \frac{\pi}{2} \frac{1}{L+1} \begin{cases} \frac{r^L}{r'^{L+1}} & \text{for } r' \geq r \\ \frac{r'^L}{r^{L+1}} & \text{for } r > r' \end{cases}, \tag{A10}$$

and

$$\int_0^{\infty} k j_{L-1}(kr) j_L(kr') = \frac{\pi}{2} \begin{cases} \frac{r^{L-1}}{r'^{L+1}} & r' > r \\ \frac{1}{2r^2} & r = r' \\ 0 & r > r' \end{cases} \quad (A11)$$

Then the results for  $I_1$ ,  $I_2$ , and  $I_3$  are

$$I_1 = \frac{\pi}{2} \frac{1}{iW} \left\{ \begin{array}{l} \frac{d}{dr_n} r_n j_L(Wr_n) \frac{d}{dr_e} r_e h_L(Wr_e) \\ \frac{d}{dr_n} r_n h_L(Wr_n) \frac{d}{dr_e} r_e j_L(Wr_e) \end{array} \right\} \quad (A12)$$

$$= \frac{\pi}{2} \frac{1}{W} \frac{L(L+1)}{2L+1} \begin{cases} \frac{r_n^L}{r_e^{L+1}} & \text{for } r_e \geq r_n \\ \frac{r_e^L}{r_n^{L+1}} & \text{for } r_n > r_e \end{cases}$$

$$I_2 = \frac{\pi}{2} \frac{1}{iW} W^2 \begin{cases} \frac{d}{dr_n} W r_n j_L(Wr_n) h_L(Wr_e) & \text{for } r_e \geq r_n \\ \frac{d}{dr_n} W r_n h_L(Wr_n) j_L(Wr_e) & \text{for } r_n > r_e \end{cases} \quad (A13)$$

$$\vec{I}_3 = \frac{\pi}{2} \frac{1}{iW} W^2 \begin{cases} j_L(Wr_n) \left( \vec{\nabla} \frac{d}{dr_e} r_e + W^2 \vec{r}_e \right) h_L(Wr_e) Y_{LM} & \text{for } r_e \geq r_n \\ h_L(Wr_n) \left( \vec{\nabla} \frac{d}{dr_e} r_e + W^2 \vec{r}_e \right) j_L(Wr_e) Y_{LM} & \text{for } r_n > r_e \end{cases}$$

(A14)

We remember now that the direct Coulomb interaction can be expanded in multipoles as follows:

$$\frac{1}{|\vec{r}_n - \vec{r}_e|} = \sum_{L,M} \frac{4\pi}{2L+1} \left\{ \begin{array}{l} \frac{r_n^L}{r_e^{L+1}} \\ \frac{r_e^L}{r_n^{L+1}} \end{array} \right\} \cdot Y_{LM}^*(\hat{r}_n) Y_{LM}(\hat{r}_e) \quad \begin{array}{l} \text{for } r_e \geq r_n \\ \text{for } r_e < r_n \end{array} \quad (A15)$$

It is then apparent that the contribution of the second term in (A12) is equal to the negative of the direct Coulomb interaction term defined by eq. (3).

If we insert the results for the integrals  $I_1$ ,  $I_2$ ,  $I_3$  into (8) we obtain eq. (9a), and the direct Coulomb term of eq. (1) is exactly canceled.



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## LEGENDS TO TABLES

Table 1. Expansion coefficients of  $\beta(r)$ . The numbers 1, 2, ..., 6 in the first column refer to the specific E1 transitions considered in this paper, and they may be identified with the help of table 2. The numbers in parenthesis in the last two columns correspond to powers of 10. Thus, the  $d_s$  coefficients of  $L_{III}$  are on the average smaller than those of  $L_I$  and  $L_{II}$  by a factor of almost  $10^{-3}$ .

Table 2. List of some empirically known anomalous E1 transitions. The E1 transitions considered in this paper are identified in the first five columns. Columns 4 and 5 give the single-particle state assignments as  $I K \pi [N n_Z \Lambda]$ . Columns 6 through 8 are taken from ref. 4. The denominators contain the theoretical conversion coefficients of ref. 2, with those of ref. 3 given in parenthesis. The hindrance factor  $h$  expressed in units of  $10^6$  is given in the last column; it is taken from ref. 4 (see also ref. 6).

Table 3. Analysis of the experimental data in terms of the anomaly amplitudes  $\Delta$ . The two sets of pairs of  $\Delta(L_I)$  and  $\Delta(L_{II})$ , of opposite relative sign, are designated as  $\Delta^{(1)}$  and  $\Delta^{(2)}$ . The two numbers given in each column correspond to the limits of error of the empirical data as quoted by ref. 4. The empirical  $\Delta$  ratios for the two solutions are compared to the theoretical values  $d_o(L_I)/d_o(L_{II})$ . The sign of  $\lambda_o$ , as concluded from this comparison, is exhibited in the last column.

Table 4. The assumed K impurities of the initial and final states of the E1 transitions considered. (For a discussion of state assignments see ref. 4 and ref. 23.) The table lists the amplitudes of the different components of single-particle states, labeled  $[N n_Z \Lambda K]$ . The cases denoted by a have been obtained by F. Stephens<sup>24)</sup> from a detailed analysis of the empirical spectra. The other numbers, which are probably somewhat more uncertain, have been calculated by a simple lowest-order perturbation treatment of the Coriolis interaction.

Table 5. Gamma-transition amplitudes  $N \cdot e_{\text{eff}}$  as defined by eq. (27) and anomalous conversion amplitudes  $A_0^{\text{cc}}$ ,  $A_2^{\text{cc}}$ , and  $A_0^{\text{sc}}$  and  $A_2^{\text{sc}}$ , the first two corresponding to the convection current and the last two ones to the spin current. Note that the numbers in parenthesis in columns 4 and 5 denote powers of 10.

Table 6. Comparison of the results of the calculations with the anomaly amplitudes  $\Delta^{\text{exp}}$ , obtained from the analysis of the experimental data in terms of the normal conversion amplitudes, as given by ref. 15. The quantities  $\lambda_0$  and  $\lambda_2$  are ratios of nuclear conversion matrix elements as defined in the main text. The superscript t denotes that both N and  $A_1$  have been taken from theory. In the semitheoretical case, labeled by s, the experimental value of  $N \cdot e_{\text{eff}}$  given by the E1  $\gamma$ -ray lifetime, has been employed instead of the calculated one. The quantities  $d_0$  and  $d_2$  refer to  $L_{\text{I}}$  conversion,  $d_0'$  and  $d_2'$  to  $L_{\text{II}}$  conversion; they are listed in table 1.

Table 7. Selection rules for the spin current matrix elements for  $s = 0$  and  $s = 2$  in terms of the "asymptotic" quantum numbers.

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Table 1

Transition	$L_I$ conversion $s_{1/2} \rightarrow p_{1/2}$		$L_{II}$ conversion $p_{1/2} \rightarrow s_{1/2}$		$L_{III}$ conversion $p_{3/2} \rightarrow s_{1/2}$	
	$d_0$	$d_2$	$d_0$	$d_2$	$d_2$	$d_4$
1	-1.231	0.2776	1.077	-0.2430	-1.584(-3)	0.7601(-3)
2	-0.3493	0.07887	0.2756	-0.06225	-0.4069(-3)	0.1956(-3)
3	-0.2398	0.05418	0.1773	-0.04008	-0.2625(-3)	0.1260(-3)
4	-0.1808	0.04089	0.1279	-0.02894	-0.1896(-3)	0.09098(-3)
5	-0.2293	0.05183	0.1685	-0.03809	-0.2494(-3)	0.1196(-3)
6	-0.2247	0.05080	0.1641	-0.03710	-0.2434(-3)	0.1167(-3)

Table 2

Case No.	Nuclide	$\Delta E$ (keV)	Initial state	Final state	$\alpha_{L_I}$ (exp) $\alpha_{L_I}$ (theor)	$\alpha_{L_{II}}$ (exp) $\alpha_{L_{II}}$ (theor)	$\alpha_{L_{III}}$ (exp) $\alpha_{L_{III}}$ (theor)	N(E1) exper.	h
1	$Np^{237}$	26.4	5/2 5/2- [523]	7/2 5/2+ [642]	$\frac{2.0 \pm 0.4}{0.55(0.22)}$	$\frac{3.9 \pm 0.8}{1.1(0.55)}$	$\frac{1.2 \pm 0.2}{1.4(1.3)}$	2.52 (-3)	0.38 (6)
2	$Np^{237}$	59.6	5/2 5/2- [523]	5/2 5/2+ [642]	$\frac{0.22 \pm 0.02}{0.13(0.11)}$	$\frac{0.46 \pm 0.05}{0.12(0.10)}$	$\frac{0.12 \pm 0.03}{0.13(0.13)}$	2.81 (-3)	0.31 (6)
3	$Am^{243}$	83.9	5/2 5/2+ [642]	5/2 5/2- [523]	$\frac{0.047 \pm 0.011}{0.068(0.054)}$	$\frac{0.057 \pm 0.013}{0.052(0.042)}$	$\frac{0.041 \pm 0.009}{0.046(0.045)}$	13.8 (-3)	0.013 (6)
4	$Pu^{239}$	106.1	7/2 7/2- [743]	5/2 5/2+ [622]	$\frac{0.062 \pm 0.007}{0.041(0.035)}$	$\frac{0.071 \pm 0.007}{0.026(0.021)}$	$\frac{0.021}{0.021(0.021)}$	1.01 (-3)	2.4 (6)
5	$Pa^{231}$	84.2	5/2 5/2+ [642]	3/2 1/2- [530]	$\frac{1.3 \pm 0.2}{0.064(0.054)}$	$\frac{0.65 \pm 0.15}{0.042(0.037)}$	$\frac{0.046 \pm 0.014}{0.039(0.039)}$	0.928 (-3)	2.8 (6)
6	$Pa^{233}$	86.3	5/2 5/2+ [642]	3/2 1/2- [530]	$\frac{0.35 \pm 0.15}{0.060(0.052)}$	$\frac{0.57 \pm 0.26}{0.039(0.034)}$	$\frac{0.08 \pm 0.08}{0.036(0.036)}$	1.32 (-3)	1.4 (6)

Table 3

	$L_I$		$L_{II}$		$-\Delta_{L_I}^{(1)}/\Delta_{L_{II}}^{(1)}$	$-\Delta_{L_I}^{(2)}/\Delta_{L_{II}}^{(2)}$	$-d_o(L_I)/d_o(L_{II})$	$\lambda_o$
	$\Delta^{(1)}$	$\Delta^{(2)}$	$\Delta^{(1)}$	$\Delta^{(2)}$				
1	3.95	-1.75	-3.09	3.89	0.93	0.35	1.143	< 0
2	1.26	-0.30	-1.18	1.46	0.91	0.18	1.267	< 0
3	0.30	0.30	0.09	0.09	—	—	1.353	undet.
4	0.57	-0.13	-0.41	0.56	1.16	-0.20	1.414	< 0
5	2.81	-2.23	-1.56	1.74	1.38	1.00	1.361	undet.
6	1.24	-0.67	-1.18	1.36	0.59	0.30	1.369	< 0 ?





Table 5.

number	Transition nuclide	energy	$ N^{\text{exp}} \cdot e_{\text{eff}} $	$N^{\text{theo}} \cdot e_{\text{eff}}$	$A_0^{\text{cc}}$	$A_2^{\text{cc}}$	$A_0^{\text{sc}}$	$A_2^{\text{sc}}$
1	Np <sup>237</sup>	26	2.52 (-3)	0.778 (-3)	0.0844	0.562	0.155	0.118
2	Np <sup>237</sup>	60	2.81 (-3)	-0.911 (-3)	0.138	0.964	0.0255	-1.86
3	Am <sup>243</sup>	84	13.8 (-3)	-0.914 (-3)	-0.148	1.04	0.0054	2.22
4	Pu <sup>239</sup>	106	1.01 (-3)	-32.7 (-3)	0.703	5.66	-0.226	-3.21
5	Pa <sup>231</sup>	84	0.928 (-3)	22.7 (-3)	0.404	1.56	-0.157	-2.16
6	Pa <sup>233</sup>	86	1.32 (-3)	22.8 (-3)	0.404	1.56	-0.156	-2.15

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