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Remarks on the McKay Conjecture

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Abstract The McKay Conjecture (MC) asserts the existence of a bijection between the (inequivalent) complex irreducible representations of degree coprime to p (p a prime) of a finite group G and those of the subgroup N, the normalizer of Sylow p-subgroup. In this paper we observe that MC implies the existence of analogous bijections involving various pairs of algebras, including certain crossed products, and that MC is *equivalent* to the analogous statement for (twisted) quantum doubles. Using standard conjectures in orbifold conformal field theory, MC is *equivalent* to parallel statements about holomorphic orbifolds V^G, V^N. There is a uniform formulation of MC covering these different situations which involves quantum dimensions of objects in pairs of ribbon fusion categories.

Keywords McKay correspondence · Quantum double

Mathematics Subject Classification (2000) 20C05

1 Introduction

The following notation will be used throughout the paper: G is a finite group, p a prime, P a Sylow p-subgroup of G, $N = N(P)$ the *normalizer* of P in G, G' the *commutator subgroup*, X a (finite, non-empty, left-) G-set, ϕ the Euler phi-function.

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All algebras and modules are finite-dimensional and defined over $\mathbb{C}$. $\mathbb{C}[G]$ is the group algebra of G and $\mathbb{C}[G]^*$ the *dual* group algebra.

For an algebra A, let

$$\mu(A) = \#\text{ inequivalent simple } A\text{-modules of dimension coprime to } p. \tag{1}$$

We say that a pair of algebras (A, B) is an *M-pair* in case $\mu(A) = \mu(B)$. The McKay Conjecture (MC) is the assertion that $(\mathbb{C}[G], \mathbb{C}[N])$ is an M-pair. The reader may consult the paper [7] of Isaacs, Malle and Navarro for the current status of this conjecture. The idea of the present paper is to *extend* MC beyond its original formulation for groups. First we show how it may be extended to large classes of algebras that are not group algebras. Examples include *crossed product* algebras, where we show that $(\mathbb{C}[H]^*\#_\sigma\mathbb{C}[G], \mathbb{C}[H]^*\#_\sigma\mathbb{C}[N])$ is an M-pair. Here, G acts on the group H and σ is a certain 2-cocycle. (See [8, 11] for background.) A particularly interesting case is that of quantum doubles $D(G)$ (see [4, 9] and below for more details). In this case we establish

$$\text{MC is true if, and only if, } (D(G), D(N)) \text{ is an M-pair for all } G \text{ and } N. \tag{2}$$

Note that quantum doubles $D(G)$ are generally not group algebras (unless G is abelian).

For a multiplicative 3-cocycle $\omega \in Z^3(G, \mathbb{C}^*)$, we show that MC implies the same result for *twisted quantum doubles*. That is, $(D^\omega(G), D^\omega(N))$ is an M-pair. Now there is a standard Ansatz in orbifold conformal field theory (CFT) due to Dijkgraaf et al. [6] which, when interpreted appropriately, says that the tensor category $D^\omega(G)$-Mod is equivalent to the module category V^G-Mod of a so-called holomorphic G-orbifold for a suitable vertex operator algebra V admitting G as automorphisms. Therefore, granted the DPR conjecture, MC is *equivalent* to a CFT-formulation involving a bijection between certain sets of simple modules for V^G and V^N. It is not necessary for the reader to be familiar with this language; the point is simply that modules for V^G are infinite-dimensional and the idea of an M-pair based on definition (1) makes no sense. In fact, all three types of M-pairs that we have discussed (i.e. for groups, (quasi-)Hopf algebras and orbifolds) may be uniformly described in the following setting: a pair of ribbon categories admitting a bijection between objects whose quantum dimension is integral and coprime to p.

All of the proofs in this paper are elementary and involve nothing beyond a few facts about finite groups, their representations, and their cohomology. In Section 2 we discuss some algebras $D_X(G)$ constructed from G and a G-set X and show that MC implies that $(D_X(G), D_X(N)$ is an M-pair. We also establish Eq. 2. In Section 3 we carry out the twisted analog of this construction. Together, these results cover several of the connections with crossed products and twisted quantum doubles mentioned above. In Section 4 we discuss the connections with CFT and ribbon categories. We assure the reader that no knowledge of CFT is required to understand the contents of this paper.

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2 The Algebras $D_X(G)$

We use the following additional notation: for $H \leq G$, $g \in G$, $H^g = \{g^{-1}hg \mid h \in H\}$. For $x \in X$, $\mathrm{Stab}_G(x) = \{g \in G | \; g.x = x\}$.

We now introduce the algebras $D_X(G)$, which were mentioned briefly in [9]. Let $\mathbb{C}[X]^*$ be the space of complex-valued functions on X. One sees that it is a G-module algebra, as follows. The algebra structure is pointwise multiplication, with basis the Dirac delta functions

$$e(x) : y \mapsto \delta_{x,y} \quad x, y \in X.$$

Thus

$$e(x)e(y) = \delta_{x,y} e(x).$$

G acts on the left of $\mathbb{C}[X]^*$ as algebra automorphisms via

$$g : e(x) \mapsto e(g.x).$$

Consider the linear space

$$D_X(G) = \mathbb{C}[X]^* \otimes_{\mathbb{C}} \mathbb{C}[G]. \tag{3}$$

It becomes an algebra via the product

$$\begin{aligned}(e(x) \otimes g)(e(y) \otimes h) &= e(x)e(g.y) \otimes gh \\ &= \delta_{x,g.y} e(x) \otimes gh \end{aligned} \tag{4}$$

for $x, y \in X$ and $g, h \in G$. One readily checks that this is *associative*. There is a decomposition into 2-sided ideals

$$D_X(G) = \bigoplus_Y D_Y(G) \tag{5}$$

where Y ranges over the (transitive) G-orbits of X. For the most part, this reduces questions about $D_X(G)$ for general X to the transitive case.

A special example of this construction is the quantum double of G. Here, we take $X = G_{\mathrm{conj}}$, i.e. $X = G$ and the left action of G is left conjugation $g : x \mapsto gxg^{-1}$. In this case we write $D(G)$ in place of $D_{G_{\mathrm{conj}}}(G)$. $D(G)$ is in fact a Hopf algebra, but at the moment we only require the algebra structure.

Next we describe the category of (left-) $D_X(G)$-modules (cf. [6, 8, 9]). For $x \in X$ and a left $\mathrm{Stab}_G(x)$-module V, set $V_x = e(x) \otimes V$. This is a left $\mathbb{C}[X]^* \otimes \mathrm{Stab}_G(x)$-module via

$$(e(y) \otimes g).(e(x) \otimes v) = e(y)e(x) \otimes g.v = \delta_{x,y} e(x) \otimes g.v, \;\; g \in \mathrm{Stab}_G(x). \tag{6}$$

From Eq. 4 it follows that $\mathbb{C}[X]^* \otimes \mathrm{Stab}_G(x)$ is a subalgebra of $D_X(G)$, so that $D_X(G) \otimes_{\mathbb{C}[X]^* \otimes \mathrm{Stab}_G(x)} V_x$ is a left $D_X(G)$-module.

Proposition 2.1 *Suppose that X is a* transitive *G-set and $x \in X$. Then the map*

$$\begin{aligned}\mathbb{C}[Stab_G(x)]\text{-}Mod &\to D_X(G)\text{-}Mod, \\ V &\mapsto D_X(G) \otimes_{\mathbb{C}[X]^* \otimes Stab_G(x)} V_x, \end{aligned} \tag{7}$$

is a Morita equivalence.

Proof For this and more, see [9] and Section 3 of [8]. In these references X is a group, but this is not necessary and the proofs go through without change. Via the natural identification $\mathbb{C}[\mathrm{Stab}_G(x)] \stackrel{\cong}{\to} e(x) \otimes \mathbb{C}[\mathrm{Stab}_G(x)]$, $g \mapsto e(x) \otimes g$, the object map inverse to (7) is $W \mapsto (e(x) \otimes 1)W$. □

Proposition 2.2 *Suppose that X is a transitive G-set and $x \in X$. The following hold:*

(a) *If p does not divide $|X|$ then $(D_X(G), Stab_G(x))$ is an M-pair.*
(b) *If p divides $|X|$ then $\mu(D_X(G)) = 0$.*

Proof Let T be a set of right coset representatives in G for $\mathrm{Stab}_G(x)$, so that there is a disjoint union $G = \cup_{t \in T} t\mathrm{Stab}_G(x)$. Because X is transitive then $X = \{t.x | t \in T\}$. From Eq. 4, observe that $D_X(G)$ is a *free* right $\mathbb{C}[X]^* \otimes \mathrm{Stab}_G(x)$-module with free basis $\{e(t.x) \otimes t | t \in T\}$. Using this observation, it follows that for a left $\mathrm{Stab}_G(x)$-module V,

$$\begin{aligned}\dim(D_X(G) \otimes_{\mathbb{C}[X]^* \otimes \mathrm{Stab}_G(x)} V_x) &= |T| \dim V \\ &= |X| \dim V.\end{aligned}$$

Therefore, in the Morita equivalence (7) modules of dimension d are mapped to modules of dimension $d|X|$. Parts (a) and (b) both follow immediately from this. □

Lemma 2.3 $\mu(D_X(G)) = \sum_Y \mu(D_Y(G))$ *where Y ranges over the G-orbits of X.*

Proof This follows from the decomposition (5) into 2-sided ideals. □

We will also need the following standard result.

Lemma 2.4 *The number of G-orbits of X of cardinality coprime to p is equal to the number of N-orbits of X of cardinality coprime to p.*

Proof By considering the decomposition of X into G-orbits, we see that we must prove the following assertion:

If X is a *transitive* G-set, then either (a) p divides $|X|$ and N has *no* orbits of cardinality coprime to p, or (b) p does *not* divide $|X|$ and there is a *unique* N-orbit of cardinality coprime to p. (8)

Assume, then, that X is a transitive G-set, and that $x, y \in X$ lie in N-orbits of cardinality coprime to p. In such an N-orbit, P must fix at least one, and therefore all, elements in the N-orbit. In particular, P lies in the stabilizers of both x and y. By transitivity there is $g \in G$ with $g.x = y$. Then P and P^g are both Sylow p-subgroups of $\mathrm{Stab}_G(x)$ and by Sylow's theorem there is $t \in \mathrm{Stab}_G(x)$ such that $P^{gt} = P$. Then $gt \in N$ and $(gt).x = y$. This shows that x, y lie in the *same* N-orbit, so that there is at most one N-orbit of cardinality coprime to p. Equation (8) is easily deduced from this, and the lemma is proved. □

Consider the following statements:

MC**1** : $(\mathbb{C}[G], \mathbb{C}[N])$ is an M-pair for all G,

MCD : $(D(G), D(N))$ is an M-pair for all G,

MCX : $(D_X(G), D_X(N))$ is an M-pair for all G and all X,

MCT : $(D_X(G), D_X(N))$ is an M-pair for all G and all *transitive* X.

The McKay Conjecture is of course the assertion that MC**1** is true.

Theorem 2.5 *MC***1**, *MCD*, *MCX and MCT are* equivalent *statements.*

Proof

$MCX \Leftrightarrow MCT$ This follows from Lemma 2.4.

$MCT \Rightarrow MC\mathbf{1}$ This holds because if $X = \mathbf{1}$ is the one-element set, then $D_X(G) = \mathbb{C}[G]$.

$MC\mathbf{1} \Rightarrow MCT$ Let X be a transitive G-set. If p divides $|X|$ then $\mu(D_X(G)) = 0$ by Proposition 2.2(b). Similarly, $\mu(D_X(N)) = 0$ by Lemma 2.3, (8)(a), and Proposition 2.2(b) (applied to N). Now suppose that p does *not* divide $|X|$. Then $\mu(D_X(G)) = \mu(\mathbb{C}[\mathrm{Stab}_G(x)])$ for any $x \in X$ by Proposition 2.2(a). Moreover by Lemma 2.4 there is a *unique* N-orbit of cardinality coprime to p, call it $Y \subseteq X$. By Lemmas 2.3, (8) and Proposition 2.2 once more we find that $\mu(D_X(N)) = \mu(D_Y(N)) = \mu(\mathbb{C}[\mathrm{Stab}_N(y)])$ for $y \in Y$. Note that because $|Y|$ is coprime to p then $P \leq \mathrm{Stab}_G(y)$ and $N_{\mathrm{Stab}_G(y)}(P) = \mathrm{Stab}_N(y)$. The assumption that $MC\mathbf{1}$ holds (applied to $\mathrm{Stab}_G(y)$) tells us that $\mu(\mathbb{C}[\mathrm{Stab}_G(y)]) = \mu(\mathbb{C}[\mathrm{Stab}_N(y)])$, whence $\mu(D_X(G)) = \mu(D_X(N))$.

$MCX \Rightarrow MCD$ Let $Y_1, \dots, Y_h$ be the N-orbits of G_{conj} of cardinality coprime to p. We have $Y_i \subseteq N$ for each index i, so that they are also the N-orbits of N_{conj} of cardinality coprime to p. Taking $X = G_{\mathrm{conj}}$, MCX together with Lemma 2.3 and Proposition 2.2, we conclude that $\mu(D(G)) = \mu(D_{G_{\mathrm{conj}}}(N)) = \sum_{i=1}^{h} \mu(D_{Y_i}(N)) = \mu(D_{N_{\mathrm{conj}}}(N)) = \mu(D(N))$, as required.

$MCD \Rightarrow MC\mathbf{1}$ We prove this using induction on $|G|$. Retain the notation of the last paragraph, and choose $y_i \in Y_i$. By Lemma 2.4, $y_1, \dots, y_h$ are representatives for the G-orbits of G_{conj} (i.e., conjugacy classes of G) of cardinality coprime to p. By Proposition 2.2, $\mu(D(G)) = \sum_{i=1}^{h} \mu(\mathbb{C}[C_G(y_i)])$. Since each $y_i \in N$, we similarly have $\mu(D(N)) = \sum_{i=1}^{h} \mu(\mathbb{C}[C_N(y_i)])$. If $C_G(y_i)$ is a *proper* subgroup of G then by induction $\mu(\mathbb{C}[C_G(y_i)]) = \mu(\mathbb{C}[C_N(y_i)])$. Then the assumption MCD tells us that $\sum_{i'} \mu(\mathbb{C}[G]) = \sum_{i'} \mu(\mathbb{C}[N])$ where i' ranges over those indices for which $y_{i'}$ lies in the *center* $Z(G)$ of G. We conclude that $|Z(G)|\mu(\mathbb{C}[G]) = |Z(G)|\mu(\mathbb{C}[N])$, whence $\mu(\mathbb{C}[G]) = \mu(\mathbb{C}[N])$. This completes the proof of the theorem. □

3 Twisted Algebras

In this section we explain how to extend the results of the previous section to the *twisted case*, i.e. the incorporation of a cocycle. Let $\theta \in Z^2(G, \mathbb{C}^*)$ be a (normalized) multiplicative 2-cocycle. Thus $\theta : G^2 \to \mathbb{C}^*$ satisfies the identities

$$\theta(h,k)\theta(g,hk) = \theta(gh,k)\theta(g,h), \quad g,h,k \in G,$$
$$\theta(1,g) = \theta(g,1) = 1.$$

The corresponding twisted group algebra is $\mathbb{C}^\theta[G]$. It has the same underlying linear space as $\mathbb{C}[G]$ with multiplication $g \circ h = \theta(g,h)gh$ for $g,h \in G$. The cocycle identities ensure that this is an associative algebra with identity element 1. For a subgroup $H \leq G$ we identify θ with its *restriction* $\mathrm{Res}^G_H\theta$ to H. Then $\mathbb{C}^\theta[H]$ is a subalgebra of $\mathbb{C}^\theta[G]$. For more information on this subject, including results that we use below, see for example [3].

The cohomological analog of Proposition 2.2(b) is the following

Lemma 3.1 *Suppose that the* cohomology class $[\theta] \in H^2(G, \mathbb{C}^*)$ *determined by* θ *has order k. If k is divisible by p then* $\mu(\mathbb{C}^\theta[G]) = \mu(\mathbb{C}^\theta[N]) = 0$.

Proof One knows (loc.cit.) that there is a central extension

$$1 \to Z \to L \xrightarrow{\pi} G \to 1$$

such that $\mathbb{Z}_k \cong Z \leq L'$, and $\mathbb{C}^\theta[G]$ is the algebra summand of $\mathbb{C}[L]$ corresponding to the irreducible representations of L in which a generator z of Z acts as multiplication by a prescribed primitive kth root of unity, say λ. If V is a simple $\mathbb{C}^\theta[G]$-module of dimension d then the determinant of z considered as operator on V is clearly λ^d. On the other hand $z \in L'$, so that this determinant is necessarily 1. So $\lambda^d = 1$, whence $k|d$. In particular, if $p|k$ then $\mu(\mathbb{C}^\theta[G]) = 0$.

Now it is well-known that the restriction map $\mathrm{Res}^G_N : H^2(G, \mathbb{C}^*) \to H^2(N, \mathbb{C}^*)$ is an *injection* on the p-part of $H^2(G, \mathbb{C}^*)$. In particular, if $p|k$ then $\mathrm{Res}^G_N[\theta]$ is divisible by p. Then the result of the last paragraph also applies to $\mathbb{C}^\theta[N]$, and we obtain $\mu(\mathbb{C}^\theta[N]) = 0$. This completes the proof of the lemma. □

The McKay Conjecture implies that the twisted analog is also true. This is the content of

Proposition 3.2 *Suppose that MC***1** *holds. Then* $(\mathbb{C}^\theta[G]), \mathbb{C}^\theta[N])$ *is an M-pair for all G and all* θ.

Proof Let the notation be as in Lemma 3.1. Although it is not really necessary to do so, because of Lemma 3.1 we may, and shall, assume that k is not divisible by p. Let P_1 be a Sylow p-subgroup of L with $\pi : P_1 \xrightarrow{\cong} P$. Applying MC**1** to pairs $(L/Z_0, N_L(P_1)/Z_0)$ with $Z_0 \leq Z$, we see that the number l of irreducible representations of both $\mathbb{C}[L]$ and $\mathbb{C}[N_L(P_1)]$ which have degree coprime to p and in which z acts as *some* primitive kth root of unity are equal. Since $\mathbb{C}^\theta[G]$ is the algebra summand of $\mathbb{C}[L]$ corresponding to λ, then $\mu(\mathbb{C}^\theta[G]) = l/\phi(k)$. On the other hand, $N_L(P_1) = ZK$ where $K \le L$ is such that $\mathbb{C}^\theta[N]$ is the algebra summand of $\mathbb{C}[K]$ corresponding to λ^t, where $t|k$ and k/t is the order of $\mathrm{Res}^G_N[\theta]$. By slightly modifying the previous argument, we also find that $\mu(\mathbb{C}^\theta[N]) = l/\phi(k)$, and the Proposition is proved. □

We can now treat the twisted version of $D_X(G)$. Let $U = U(\mathbb{C}[X]^*)$ be the group of units in $\mathbb{C}[X]^*$. Then

$$U = \left\{ \sum \lambda_x e(x) \mid \lambda_x \neq 0 \right\}$$

is a *multiplicative* left G-module. Let $\alpha \in Z^2(G, U)$ be a normalized 2-cocycle with coefficients in U, and set $\alpha(g, h) = \sum_{x\in X} \alpha_x(g, h)e(x)$. Here, the cocycle property amounts to the identity

$$\alpha_x(g, h)\alpha_x(gh, k) = \alpha_x(g, hk)\alpha_{g^{-1}.x}(h, k). \tag{9}$$

Define $D^\alpha_X(G)$ to be the linear space $D_X(G)$ with multiplication being the twisted version of Eq. 4. That is,

$$(e(x) \otimes g)(e(y) \otimes h) = \alpha_x(g, h)\delta_{x,g.y}e(x) \otimes gh. \tag{10}$$

Equation 9 is exactly what is needed to show that Eq. 10 is *associative*. Note also from Eq. 9 that for fixed $x \in X$, α_x defines an element in $Z^2(\mathrm{Stab}_G(x), \mathbb{C}^*)$ and that as a subalgebra of $D^\alpha_X(G)$, $e(x) \otimes \mathbb{C}[\mathrm{Stab}_G(x)] \cong \mathbb{C}^{\alpha_x}[\mathrm{Stab}_G(x)]$. The proof of Proposition 2.1 still applies in this situation (cf. [8]). It provides a Morita equivalence of categories

$$\mathbb{C}^{\alpha_x}[\mathrm{Stab}_G(x)]\text{-Mod} \xrightarrow{\sim} D^\alpha_X(G)\text{-Mod}.$$

The proof of the twisted version of Theorem 2.5 then goes through too. We just state a part of this as

Theorem 3.3 *Suppose that MC***1** *holds. Then* $(D^\alpha_X(G), (D^\alpha_X(N))$ *is an M-pair for all* G, X *and* α.

Special cases of $D^\alpha_X(G)$ include certain kinds of crossed products and abelian extensions of Hopf algebras. See, for example, [8] for further details.

Once again the case of the quantum double, when $X = G_{\mathrm{conj}}$, is of special interest (cf. [2, 5, 6, 9] for more details and further background.) Here, one twists $D(G)$ by a normalized *three cocycle* $\omega \in Z^3(G, \mathbb{C}^*)$. The resulting object is denoted by $D^\omega(G)$. It is a quasi-Hopf algebra, but not a Hopf algebra in general. To connect with previous paragraphs, we observe that there is a map [6]

$$Z^3(G, \mathbb{C}^*) \to Z^2(G, G_{\mathrm{conj}})$$

for which

$$\alpha_x(g, h) = \frac{\omega(x, g, h)\omega(g, h, (gh)^{-1}x(gh))}{\omega(g, g^{-1}xg, h)}. \tag{11}$$

There is a natural interpretation of this map in terms of the loop space LBG, but we will not need it. The twisted product in $D^{\omega}(G)$ is as in Eq. 10 using Eq. 11. This gives the algebra structure, and as before leads to

Theorem 3.4 *Suppose that MC***1** *holds. Then* $(D^{\omega}(G), D^{\omega}(N))$ *is an M-pair for all G and* ω.

The statement and proof of Theorem 3.4 only require the algebra structure of $D^{\omega}(G)$. However, we will make use of other structural features of $D^{\omega}(G)$ in the next section.

4 Orbifolds and Ribbon Categories

We refer the interested reader to [5] for background concerning vertex operator algebras. Let V be a holomorphic vertex operator algebra admitting G as a group of automorphisms, with V^G the subalgebra of G-invariants. One expects that the module category V^G-Mod is a (braided, ribbon) tensor category and that it is equivalent to the tensor category $D^{\omega}(G)$-Mod for a 3-cocycle ω which describes the associativity constraint in V^G-Mod. If this is so, we deduce from Theorem 3.4 that there are bijections between the simple objects of V^G-Mod and V^N-Mod which themselves correspond to the simple modules of $D^{\omega}(G)$-Mod and $D^{\omega}(N)$-Mod respectively which have dimension coprime to p.

We seek a direct definition of an M-pair for modules over orbifolds such as V^G and V^N. We cannot use Eq. 1 as it stands because it makes no sense for infinite-dimensional spaces such as a module over a vertex operator algebra. Instead, we can make use of the expected structure of V^G-Mod as a ribbon tensor category, whereby the objects have a *quantum dimension*. Indeed, $D^{\omega}(G)$-Mod has a *canonical* ribbon structure (cf. [1, 10]), and the quantum dimensions of simple objects are the usual dimensions. Granted the equivalence of V^G-Mod and $D^{\omega}(G)$-Mod, it follows that the quantum dimension of simple objects in V^G-Mod are also integers. Then the definition of an M-pair makes sense if we use quantum dimension in place of dimension.

Thus we arrive at the following situation: a pair of ribbon fusion categories $\mathcal{G}, \mathcal{N}$ whose simple objects have quantum dimensions that are rational integers. We say that $(\mathcal{G}, \mathcal{N})$ is an M-pair if $\mu(\mathcal{G}) = \mu(\mathcal{N})$, where we use (1) with quantum dimension in place of dimension in order to define μ. As we have seen, taking $\mathcal{G}$ to be $\mathbb{C}[G]$-Mod, $D^{\omega}(G)$-Mod or V^G-Mod and $\mathcal{N}$ to be $\mathbb{C}[N]$-Mod, $D^{\omega}(N)$-Mod or V^N-Mod respectively results (conjecturally) in an M-pair. Furthermore, the three versions of MC for groups, quantum doubles of groups, and holomorphic orbifolds, are *equivalent*.

References

1. Altschuler, D., Coste, A.: Quasi-quantum groups, knots, three-manifolds, and topological field theory. Comm. Math. Phys. **150**(1), 83–107 (1992)
2. Kassel, C.: Quantum Groups, Graduate Texts in Mathematics, vol. 155. Springer, New York (1995)
3. Curtis, C., Reiner, I.: Methods of Representation Theory, vol. 1. Wiley-Interscience, New York (1981)
4. Drinfeld, V.: Quasi-Hopf algebras. Leningr. Math. J. **1**, 1419–1457 (1990)
5. Dong, C., Mason, G.: Vertex operator algebras and moonshine: a survey. In: Adv. Studies in Pure Math. vol. 24, pp. 101–136. Progress in Algebraic Combinatorics, Math. Soc. of Japan, Tokyo (1996)
6. Dijkgraaf, R., Pasquier, V., Roche, P.: Quasi-quantum groups related to orbifold models. In: Modern Quantum Field Theory (Bombay, 1990), pp. 375–383. World Scientific, Singapore (1991)
7. Isaacs, M., Malle, G., Navarro, G.: A reduction theorem for the McKay conjecture. Invent. Math. **170**, 33–101 (2007)
8. Kashina, Y., Mason, G., Montgomery, S.: Computing the Frobenius-Schur indicator for Abelian extensions of Hopf algebras. J. Algebra **251**, 888–913 (2002)
9. Mason, G.: The quantum double of a finite group and its role in conformal field theory. In: Groups '93 Galway/St. Andrews, Lond. Math. Soc. Lect. Notes Ser. 212, vol. 2. CUP (1995)
10. Mason, G., Ng, S.-H.: Central invariants and Frobenius-Schur indicators for semi-simple quasi-Hopf algebras. Adv. Math. **190**, 161–195 (2005)
11. Montgomery, S.: Hopf Algebras and Their Actions on Rings, CBMS Number 82. Amer. Math. Soc. (1993)