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A SIMPLE 3-DIMENSIONAL VISCOELASTIC MODEL ACCOUNTING FOR DAMAGE EFFECTS

by

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Part I: Infinitesimal Theory

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ABSTRACT

A simple three- dimensional viscoelastic constitutive model is presented which is capable of accounting for damage effects characteristic of some highly filled elastomeric polymers. The model predicts degradation of the *loss* and *stored* moduli with increasing maximum strain amplitude in a cyclic test, in agreement with experimental results. The Finite Element implementation of the proposed 3-dimensional, which is discussed in some detail, shows that this model is particularly suited for computational applications. In addition, its simplicity allows the extension to the fully non-linear theory. Numerical results are presented which indicate that steady-state solutions in cyclic tests are achieve in very few cycles when the level of strains is held constant.

CONTENTS

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1. INTRODUCTION

Material damage in polymers is a complex process involving chain and multichain damage, and microstructural damage such as microvoid formation. See e.g., Wool [1980] and references therein. In this paper we shall adopt a phenomenological point of view and attempt to describe some of the experimentally observed facts by means of a three dimensional viscoelastic model, suitable for a realistic numerical implementation within the framework of the finite element method.

In the cyclic test of highly filled polymers, it is experimentally observed that the *stored* and *loss moduli* of the composite *decrease* with *increasing* maximum strain. Furthermore, it is also observed that a steady state response of the composite is attained in very few cycles, provided the maximum strain amplitude remains constant. The damage mechanism associated with this degradation of the viscoelastic properties of the composite is characteristic of *highly filled polymers*, and appears to be related to the re-arrangement of the structure of the filler occurring as a result of the increasing strain.

From a phenomenological standpoint and under the assumption that the material is isotropic and remains isotropic in its damaged states, this experimental evidence suggests the use of a measure of the maximum strain to which the specimen is subjected as an *internal variable* characterizing the damage process. In this paper we proposed a simple *three-dimensional* viscoelastic model capable of accounting for such damage effects. The essential idea is to introduce a damage variable defined as the maximum value of the Euclidean norm of the deviatoric strain tensor during the loading history. In accord with the described experimental evidence, our basic assumption is that no further damage in the material occurs if the norm of the deviatoric strains does not surpass the previously attained maximum.

We then considers a representation for the deviatoric strain tensor in the form of a convolution integral, analogous to that of classical viscoelasticity, but now involving a non-linear function of the history of the rate of deviatoric strains and the damage variable. The idea of introducing representations of this type is certainly not a new one, and goes back at least to Leaderman [1943], who considered a power of the strain rate in the convolution integral.

The structure of the non-linear function of the strain rate in the convolution representation adopted here, is inspired in a one dimensional model proposed by Browning, Gurtin & Williams [1983]. In our 3-dimensional model, we assume this non-linear function to be the product of two functions. The first one depends only on the damage variable, and the second function involves both the damage variable and the strain deviators.

Clearly, the model is far too general for any practical implementation, and further specialization is needed. We propose a simple choice of the non-linear functions appearing in the convolution integral, which captures the essential features of some experimentally observed behavior of highly filled polymers.

The resulting model is not only particularly well suited for computational purposes, specially in the context of a finite element formulation, but its simplicity permits an extension to the fully non-linear theory.

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2. BASIC ASSUMPTIONS

The proposed three dimensional viscoelastic damage model is based upon the introduction of the following physical assumptions:

(i) The material is assumed to be isotropic in its virgin state and in any of its damaged states.

(ii) The volumetric response of the material is assumed to be elastic; i.e., viscoelastic effects in bulk response are not considered.

(iii) The amount of damage is independent of hydrostatic pressures, i.e., no damage in the material takes place as a result of hydrostatic loading.

(iv) The amount of damage depends on the maximum value attained by a norm of the deviatoric strain during the loading history.

For the purpose of the present discussion, it will prove convenient to introduce the following inner product and associated norm:

$$<\mathbf{e},\bar{\mathbf{e}}> = e_{ij}\,\bar{e}_{ij}\ ; \qquad ||\mathbf{e}|| \equiv <\mathbf{e},\mathbf{e}>^{1/2} = \sqrt{2\,J_2(\mathbf{e})} \tag{1}$$

where $J_2(\mathbf{e})$ stands for the second invariant of the *deviatoric* part of the strain tensor $\boldsymbol{\epsilon}$. Denoting by $\mathbf{s}(t)$ the deviatoric part of the stress tensor $\boldsymbol{\sigma}(t)$, we have the usual definitions:

$$\mathbf{e}(t) = \boldsymbol{\epsilon}(t) - \frac{1}{3}\mathrm{tr}\,\boldsymbol{\epsilon}(t)\ \mathbf{1}\ , \qquad \mathbf{s}(t) = \boldsymbol{\sigma}(t) - \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}(t)\ \mathbf{1} \tag{2}$$

In accordance with assumption (iv) we introduce a *scalar damage variable* ψ_t defined in terms of the Euclidean norm $(1)_2$ as follows. For a given *deviatoric* strain history $H_D \equiv \{s \rightarrow \mathbf{e}(s)\ \mid s\in(-\infty, t]\ \}$, the *damage variable* ψ_t is the function of the strain history H_D defined as

$$\psi_t = \max_{s\in(-\infty,t]} ||\mathbf{e}(s)|| \tag{3.a}$$

We shall denote by $D_{\psi_t}\subset H_D$ the set of elements in the deviatoric strain history H_D for which this maximum is attained; that is,

$$D_{\psi_t} \equiv \{\ \mathbf{e}_M = \mathbf{e}(t_M)\ \mid\ ||\mathbf{e}_M|| = \psi_t\ ,\ \ t_M\in(-\infty, t]\ \} \tag{3.b}$$

By assumption (ii) the response function for the stress tensor $\boldsymbol{\sigma}(t)$ may be decomposed according to

$$\boldsymbol{\sigma}(t) = 3\,K\,\mathrm{tr}\,\boldsymbol{\epsilon}\,\mathbf{1} + \mathbf{s}(t) \tag{4}$$

where viscoelastic effects and, as a result of assumption (iii), damage effects are contained in the response function associated with the stress deviator $\mathbf{s}(t)$. To account for these effects, we introduce a convolution representation for the deviatoric stress tensor $\mathbf{s}(t)$ involving a non-linear function of the strain deviator and the damage variable ψ_t of the form

$$\mathbf{s}(t) = 2\,G(t-s) * \dot{\boldsymbol{\pi}}(\mathbf{e}(s), \psi_s) \tag{5}$$

where "*" stands for convolution in $L_2(-\infty,\infty)$, $G(t)$ is the relaxation function of the classical linear theory, and $\boldsymbol{\pi}(\mathbf{e}(t), \psi_t)$ is a (non-linear) function of the strain history and the damage variable ψ_t. In particular, we consider the following special representation for the function $\boldsymbol{\pi}(\mathbf{e}(t), \psi_t)$:

$$\boldsymbol{\pi}(\mathbf{e}(t), \psi_t) = g(\psi_t)\cdot\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t) \tag{6.a}$$

where the function $\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t)$ satisfies the conditions

$$||\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t)|| = 1 \qquad iff \qquad ||\mathbf{e}(t)|| \equiv \psi_t \tag{6.b}$$

and

$$\mathrm{tr}\,[\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t)] \equiv 0 \tag{6.c}$$

Remarks

1. The idea of characterizing the response function of the stress deviator through the convolution representation (5) typical of classical linear viscoelasticity, but with the strain rate replaced by the rate of a non-linear function of the strain, goes back at least to Leaderman [1943] who used as non-linear function a power of the strain deviator. Schapery [1966], among others, considered also a related non-linear representation obtained through thermodynamic arguments. Pipkin and Rogers [1968] arrived at analogous representation as the first term of a formal series expansion. The representation (6.a)-(6.c) is inspired in (and generalizes) a one dimensional form proposed by Browning, Gurtin & Williams [1983].

2. One often speaks of the function $\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t)$ as the "damage function" and, motivated by the one-dimensional case, we often refer to $g(\psi_t)$ as the "loading function". Since $\psi_t = \sqrt{2\,J_2(\mathbf{e}_M)}$, for $\mathbf{e}_M \in D_{\psi_t}$, the choice of ψ_t as "damage variable" is consistent with the isotropy assumption (i). As a result of this assumption, the characterization of damage in the function $\boldsymbol{\pi}$ should involve only the invariants J_2 and J_3 of the deviatoric strain history. The reasons for condition (6.b) will become apparent in what follows.

For a given choice of functions $g(\psi_t)$ and $\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t)$ the rate $\dot{\pi}(\mathbf{e}(s), \psi_s)$ appearing in the convolution representation (5) is obtained, for $s \in (-\infty, t]$, as follows. First, we note that the damage variable ψ_t defined by (3.a) satisfies the following *rate equation*:

$$\frac{d\psi_s}{ds} = \begin{cases} \frac{1}{||\mathbf{e}(s)||} < \mathbf{e}(s), \dot{\mathbf{e}}(s) > , & iff \quad ||\mathbf{e}(s)|| = \psi_s \\ 0 & iff \quad ||\mathbf{e}(s)|| < \psi_s \end{cases} \qquad s \in (-\infty, t] \tag{7.a}$$

Conversely, by integrating (7.a) subjected to the initial condition $\psi_s \Big|_{s \to -\infty} = 0$, one recovers definition (3.a). The initial condition $\psi_s \Big|_{s \to -\infty} = 0$ expresses the fact that the material is assumed to be initially *undamaged*. (Clearly, one may consider an initial time $t_0 \neq -\infty$). Thus, by making use of the chain rule and (7.a) we are led to the following expression for the rate $\dot{\pi}(\mathbf{e}(s), \psi_s)$

$$\dot{\pi} = \begin{cases} \left\{ \frac{g'(||\mathbf{e}||)}{||\mathbf{e}||} \boldsymbol{\gamma} \otimes \mathbf{e} + \frac{g(||\mathbf{e}||)}{||\mathbf{e}||} \left[\frac{\partial \boldsymbol{\gamma}}{\partial \psi_t} \otimes \mathbf{e} + \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{e}} \right] \right\} : \dot{\mathbf{e}} & iff \quad ||\mathbf{e}|| \equiv \psi_s \text{ and } < \mathbf{e}, \dot{\mathbf{e}} > \, > 0 \\ \left\{ g(\psi_t) \frac{\partial \boldsymbol{\gamma}}{\partial \mathbf{e}} \right\} : \dot{\mathbf{e}} \; ; & \textit{otherwise} \end{cases} \tag{7.b}$$

Obviously, the key to the success of the model lies in a simple and at the same time rational choice of the functions $g(\psi_t)$ and $\boldsymbol{\gamma}(\mathbf{e}, \psi_t)$. We consider this issue in some detail next.

3. THE CHOICE OF FUNCTIONS $g(\psi_t)$ and $\boldsymbol{\gamma}(\mathbf{e}, \psi_t)$

Choice of $\boldsymbol{\gamma}(\mathbf{e}, \psi_t)$. It is generally accepted that most polymers exhibit little, if any at all, *permament set* upon removal of the loading. Motivated by this fact, we introduce the following additional assumption:

(v) The "damage function " $\boldsymbol{\gamma}(\mathbf{e}, \psi_t)$ is continuous for $\psi_t \neq 0$ and such that:

$$\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t) \Big|_{\mathbf{e}=0} = 0 \tag{8}$$

We consider perhaps the simplest choice of $\boldsymbol{\gamma}(\mathbf{e}, \psi_t)$ satisfying conditions (6.b), (6.c) and (8); namely:

$$\boldsymbol{\gamma}(\mathbf{e}(t), \psi_t) = \frac{\mathbf{e}(t)}{\psi_t} \tag{9}$$

Choice of the function $g(\psi_t)$. To motivate the choice of the function $g(\psi_t)$, consider a specimen subjected to a pure shear test. Then, according to (5) and (6.a)-(6.c) we must have

$$s_{12}(t) = 2\,G(t-s) * \dot{\pi}_{12}(e_{12}, \psi_t)$$

$$\pi_{12} = \frac{g(\psi_t)}{|e_{12}|} e_{12} \tag{10}$$

$$\psi_t = \max_{s \in (-\infty, t]} |e_{12}(s)|$$

Further, consider a very fast relaxation test; i.e., if τ is a characteristic relaxation time of the material, we have the strain history:

$$t << \tau \ \ (i.e., \ \frac{t}{\tau} \to 0) \tag{11}$$

$$0 < e_{12}(s) = e_{12}(0) = \psi_t \, ; \qquad \text{for all} \quad s \in [0, t]$$

The final stress then is a function of the initial amplitude, given by:

$$\sigma_{12} \approx 2\,G(0)\,\frac{g(\psi_t)}{\psi_t} e_{12} \equiv 2\,G(0)\,g(\psi_t)$$

Experimentally it is observed that the stress depends on the amplitude in a monotonically decreasing fashion; i.e., as a result of damage the material softens. Moreover, it is often observed in the experimental testing of highly filled polymers that the shear modulus of the composite material tends to that of the pure gum with increasing values of the maximum shear strain, (Ferry [1970]). We consider the idealized behavior represented in Fig. 1a which essentially captures the basic experimental facts and leads to the following simplified expression for the derivative $g'(\psi_t)$:

$$g'(\psi_t) = \beta + (1-\beta)\, e^{-\psi_t/\alpha} \qquad \beta \in [0,1] \, , \quad \alpha \in [0, \infty) \tag{12}$$

On integrating (12) we obtain the following expression the behavior of which is illustrated in Fig. 1b,

$$g(\psi_t) = \left[\beta + (1-\beta)\,\frac{1 - e^{-\psi_t/\alpha}}{\psi_t/\alpha} \right] \psi_t \quad \beta \in [0,1] \, , \quad \alpha \in [0, \infty) \tag{13}$$

Remark. Clearly, the function $g(\psi_t)$ could also be determined by performing the same test as described above but very slowly (i.e., a quasi-static test for which $t >> \tau$). We then would have:

$$\sigma_{12} = 2\,G(\infty)\,g(|e_{12}(\infty)|) \tag{14}$$

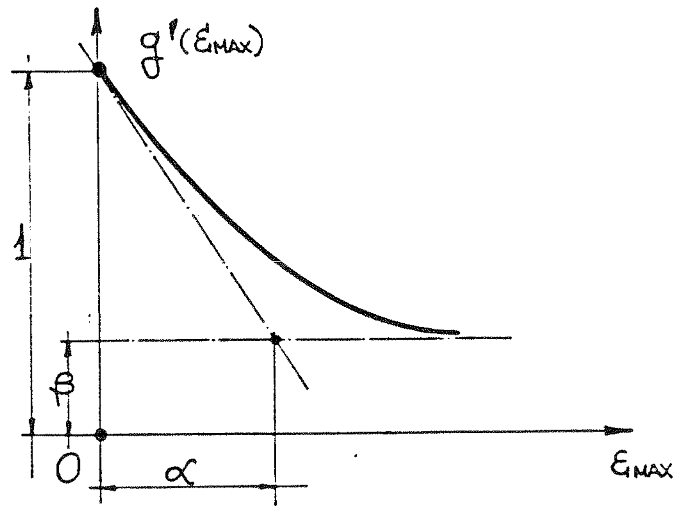

Figure 1a. Behavior of the derivative $g'(\psi_t)$ of the "loading function".

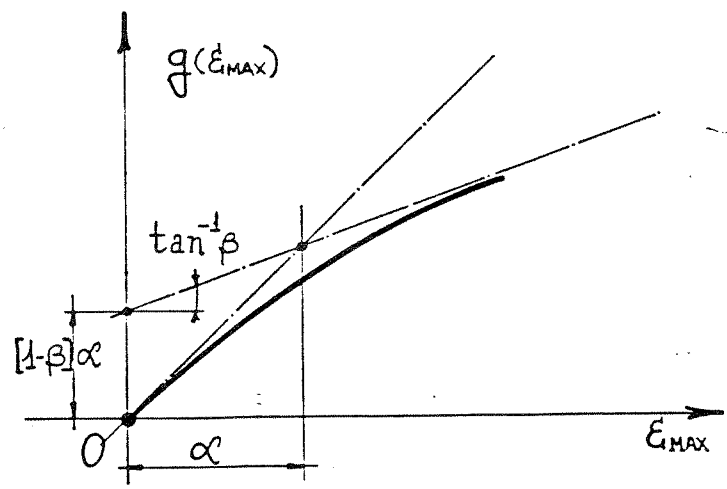

Figure 1b. Behavior of the "loading function" $g(\psi_t)$.

4. SUMMARY OF THE MODEL

For convenience, let us first define

$$g(\psi_t) = \bar{g}(\psi_t)\cdot\psi_t \tag{15}$$

where $\bar{g}(x)$ is given by:

$$\bar{g}(x) = \beta + (1-\beta)\frac{1-e^{-x/\alpha}}{x/\alpha}; \qquad \beta\in[0,1]\ , \quad \alpha\in[0,\infty) \tag{16}$$

Thus, $\pi(\mathbf{e}(s), \psi_s)$ becomes:

$$\pi(\mathbf{e}(s), \psi_t) = \bar{g}(\psi_s)\, \mathbf{e}(s)\ , \qquad s \in (-\infty, t] \tag{17}$$

And

$$\sigma(t) = 3\,K\, \mathrm{tr}\,\epsilon\, \mathbf{1} + 2\,G(t-s) * \dot{\pi}(\mathbf{e}(s), \psi_s) \tag{18}$$

For definiteness we shall assume that the relaxation function $G(t)$ is that corresponding to a standard solid and thus given by

$$G(t) \equiv G_\infty + (G_o - G_\infty)\ e^{-t/\tau} \tag{19}$$

More realistic forms could be obtained simply by considering the usual series expansion in terms of exponentials associated with a spectrum of relaxation times.

Remarks

1. Notice that if $\beta \rightarrow 1$, then $\bar{g}(x) \rightarrow 1$ and the model reduces to classical linear viscoelasticity.
2. All that is needed for computational purposes is an expression for $\dot{\pi}$. Particularizing (7.b) we obtain:

$$\dot{\pi}(s) = \mathbf{C}(\mathbf{e}(s), \psi_s) : \dot{\mathbf{e}}$$
$$\equiv \begin{cases} \left[\bar{g}(\|\mathbf{e}\|)\,\mathbf{1} + (\bar{g}'(\|\mathbf{e}\|)\,\|\mathbf{e}\|)\ \mathbf{n}\otimes\mathbf{n}\right] : \dot{\mathbf{e}}\ , & iff \quad \|\mathbf{e}\| \equiv \psi_s \text{ and } \langle \mathbf{e}, \dot{\mathbf{e}} \rangle > 0 \\ \left[\bar{g}(\psi_t)\,\mathbf{1}\right] : \dot{\mathbf{e}}\ , & otherwise \end{cases} \tag{20}$$

where $\mathbf{n} \equiv \dfrac{\mathbf{e}}{\|\mathbf{e}\|}$. □

5. FINITE ELEMENT IMPLEMENTATION.

In this Section we examine the basic aspects involved in the numerical solution by the Finite Element method of the boundary value problem for a solid exhibiting viscoelastic response governed by the nonlinear constitutive model summarized in Section 4, which accounts for damage effects. Details regarding the spatial discretization of the weak form of momentum balance (virtual work) may be found in standard references on the subject (see, e.g., Zienkiewicz [1977]) and thus will be omitted. Our main concern shall be a summary account of the algorithm employed in the numerical integration of the constitutive model, and the appropriate expression for the tangent elasticities.

Consider a viscoelastic body initially occupying a compact region $\Omega \subset \mathbb{R}^3$ at time $t=0$, the response of which is seek throughout the time interval $[0, T] \subset \mathbb{R}$. Let $\mathbf{b}(\mathbf{x})$, $\mathbf{x} \in \Omega$, be the body force field, $\bar{\mathbf{t}}(\mathbf{x})$ the prescribed traction field on $\partial_\sigma \Omega$, and $\bar{\mathbf{u}}(\mathbf{x})$ the prescribed displacement field on $\partial_u \Omega$; where $\overline{\partial_u \Omega} \cup \overline{\partial_\sigma \Omega} = \overline{\partial \Omega}$ and $\partial_u \Omega \cap \partial_\sigma \Omega = \varnothing$. We shall denote by V the space of kinematically admissible variations defined by

$$V \equiv \{ \boldsymbol{\eta} : \Omega \rightarrow \mathbb{R}^3 \mid \boldsymbol{\eta}\big|_{\partial_u \Omega} = \mathbf{0} \} \subset H^1(\Omega) \ . \tag{21}$$

If for simplicity we confine our attention to the static case, the *weak form* of momentum balance is given, at any time $t \in [0, T]$, by[†]

$$G(\mathbf{u}(t), \boldsymbol{\eta}) \equiv \int_\Omega tr[\boldsymbol{\sigma}(t) \nabla \boldsymbol{\eta}] \, d\Omega - \int_\Omega \mathbf{b} \cdot \boldsymbol{\eta} \, d\Omega - \int_{\partial_\sigma \Omega} \bar{\mathbf{t}} \cdot \boldsymbol{\eta} \, d\Omega = 0 \ , \tag{22}$$

for any kinematically admissible variation $\boldsymbol{\eta} \in V$. The response of the body is governed by equation (21) in which the stress tensor $\boldsymbol{\sigma}(t)$ is given by the *nonlinear* viscoelastic model (Eq. (18)) summarized in Section 4. The numerical solution of (18) and (22) requires first an algorithm for the numerical integration of (18) at a discrete set of N points $\{ t_n \mid t_n \in [0, T] \ , \text{ and } t_n < t_{n+1} \ , \ n = 1, 2, \ldots, N \}$. The resulting equation is *nonlinear* due to the presence of damage, and relates the current stress at $t_{n+1} \in [0, T]$ to the history of stress and the *incremental strain* as shown below. The simultaneous solution of (22) and (18) then leads to a classical iterative solution procedure based on Newton's method. We shall denote by $(.)_{n+1}^i$ the value of the variable $(.)$ at iteration i within the time step $[t_n, t_{n+1}]$. Omission of the iteration superscript will indicate converged value.

Recursive Algorithm for Stress Recovery. In order to numerically integrate the constitutive equation (18), the essential idea is to evaluate the convolution integral in (18) through a recurrence relation. A related procedure was employed by Key & Krieg [1982]. Considering the time interval $[0, t_n]$, $t_n < T$, the stress $\boldsymbol{\sigma}_{n+1}$ at time $t_{n+1} \in [0, T]$ may be computed from (17) and (18) according to

$$\boldsymbol{\sigma}_{n+1} = 3\, K \operatorname{tr} \Delta \boldsymbol{\epsilon}_{n+1} \mathbf{1} + \boldsymbol{\sigma}_n + [e^{-\Delta t_n / \tau} - 1] \int_0^{t_n} 2\,(G_o - G_\infty)\, e^{-(t_n - s)/\tau} \dot{\boldsymbol{\pi}}(s)\, ds$$

$$+ \int_{t_n}^{t_{n+1}} 2\,[G_\infty + (G_o - G_\infty)\, e^{-(t_{n+1} - s)/\tau}]\, \dot{\boldsymbol{\pi}}(s)\, ds \tag{23}$$

[†]For notational convenience, the spatial dependence of the variables appearing (22) on the point $\mathbf{x} \in \Omega$ is not explicitly indicated.

To evaluate (23) we introduce, in addition to the stress tensor $\boldsymbol{\sigma}(t_n)$ given by (18), the history variable $\mathbf{h}(t_n)$ defined at any time $t_n \in [0, T]$ by

$$\mathbf{h}_n = \mathbf{h}(t_n) \equiv \int_0^{t_n} 2\,(G_o - G_\infty)\, e^{-(t_n - s)/\tau}\, \dot{\boldsymbol{\pi}}(s)\, ds \tag{24}$$

Given converged values $\boldsymbol{\sigma}_n$ and $\mathbf{h}_n$ of the stress and auxiliary history variable at the left point of the interval $[t_n, t_{n+1}]$, and the incremental strain $\Delta\boldsymbol{\epsilon}^i_{n+1}$ at iteration i, use of a generalized mid-point rule and the mean value theorem results in following algorithm for calculating the stress $\boldsymbol{\sigma}^i_{n+1}$:

$$\boldsymbol{\epsilon}^i_{n+1} = \boldsymbol{\epsilon}_n + \Delta\boldsymbol{\epsilon}^i_{n+1}\,, \qquad \mathbf{e}^i_{n+1} = \boldsymbol{\epsilon}^i_{n+1} - \operatorname{tr}\Delta\boldsymbol{\epsilon}^i_{n+1}\,\mathbf{1} \tag{25}$$

$$\Delta\boldsymbol{\pi}^i_{n+1} = \boldsymbol{\pi}(\mathbf{e}^i_{n+1}, \psi^i_{n+1}) - \boldsymbol{\pi}(\mathbf{e}_n, \psi_n)\,, \quad \text{where:} \quad \psi^i_{n+1} \equiv \max\{\,||\mathbf{e}^i_{n+1}||, ||\mathbf{e}_n||\,\} \tag{26}$$

$$\Delta\mathbf{h}^i_{n+1} = [e^{-\Delta t_n/\tau} - 1]\,\mathbf{h}_n + 2\,(G_o - G_\infty)\,\frac{1 - e^{-\Delta t_n/\tau}}{\Delta t_n/\tau}\,\Delta\boldsymbol{\pi}^i_{n+1} \tag{27}$$

$$\boldsymbol{\sigma}^i_{n+1} = \boldsymbol{\sigma}_n + 3\,K \operatorname{tr}\Delta\boldsymbol{\epsilon}^i_{n+1}\,\mathbf{1} + 2\,G_\infty\,\Delta\boldsymbol{\pi}^i_{n+1} \tag{28}$$

Once convergence is achieved the stress $\boldsymbol{\pi}_n$ and history variable $\mathbf{h}_n$ are updated at the end of the step in the obvious manner as

$$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n + \Delta\boldsymbol{\pi}_{n+1}\,, \qquad \mathbf{h}_{n+1} = \mathbf{h}_n + \Delta\mathbf{h}_{n+1} \tag{29}$$

Linearization and Tangent Elasticities. The linearization of the weak form of momentum balance given by (22) leads, for a typical iteration i within the time step $[t_n, t_{n+1}]$, to the following linear problem for the incremental strain $\Delta\boldsymbol{\epsilon}^{i+1}_{n+1}$

$$\int_\Omega \operatorname{tr}\Big[\nabla\boldsymbol{\eta}\cdot(\mathbf{D}\boldsymbol{\sigma}^i_{n+1} : \Delta\boldsymbol{\epsilon}^{i+1}_{n+1})\Big]\, d\Omega = -G(\mathbf{u}^i_{n+1}, \boldsymbol{\eta})\,, \quad \text{for all} \quad \boldsymbol{\eta} \in V\,, \tag{30}$$

where the matrix of "tangent elasticities" $\mathbf{D}\boldsymbol{\sigma}^i_{n+1}$ is obtained according to

$$\mathbf{D}\boldsymbol{\sigma}^i_{n+1} : \Delta\boldsymbol{\epsilon}^{i+1}_{n+1} = 3\,K\,[\mathbf{1}\otimes\mathbf{1}] : \Delta\boldsymbol{\epsilon}^{i+1}_{n+1}$$
$$+\, 2\left[G_\infty + (G_o - G_\infty)\,\frac{1 - e^{-\Delta t_n/\tau}}{\Delta t_n/\tau}\right]\mathbf{C}(\mathbf{e}^i_{n+1}, \psi^i_{n+1}) : \Delta\mathbf{e}^{i+1}_{n+1}\,, \tag{31}$$

with the tangent matrix $\mathbf{C}(\mathbf{e}^i_{n+1}, \psi^i_{n+1})$ defined by (20).

Once the linearized problem (30) is discretized by standard Finite Element procedures, the classical Newton solution procedure simply involves, for each time step $[t_n, t_{n+1}]$, the iterative solution of (30) with the stress in the right hand side evaluated according to (28) and the tangent stiffness computed according to (31).

6. NUMERICAL EXAMPLES

The finite element formulation discussed in Section 4 of the the proposed viscoelastic constitutive model accounting for damage effects, is currently implemented in the general purpose finite element computer program FEAP discussed in Zienkiewicz [1977], Chap.24. It is emphasised that the current version of FEAP at U.C. Berkeley has been modified in order to achieve *full compatibility* with the treatment of the constitutive equations employed in the computer program NIKE 2D (Hallquist [1983]), presently used at the L.L.N.L.

Two simple examples that illustrate the proposed viscoelastic model have been considered. In the first example a specimen is subjected to a *pure shear cyclic* test, whereas in the second example the specimen is subjected to a *simple tension cyclic* test. A "saw tooth" strain history and a sinusoidal strain history were the strain input considered in both examples. The stress-strain diagrams together with the corresponding strain histories are collected in Fig. 2 to Fig. 5. From these figures we note the following

(i) The *stored* and *loss* moduli *decrease* with increasing maximum strain amplitude, as expected.

(ii) If the maximum strain amplitude of the cycles is held fixed, then the steady state solution is achieved in *about two cycles*.

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DAMAGE MODEL: PURE SHEAR CYCLIC TEST

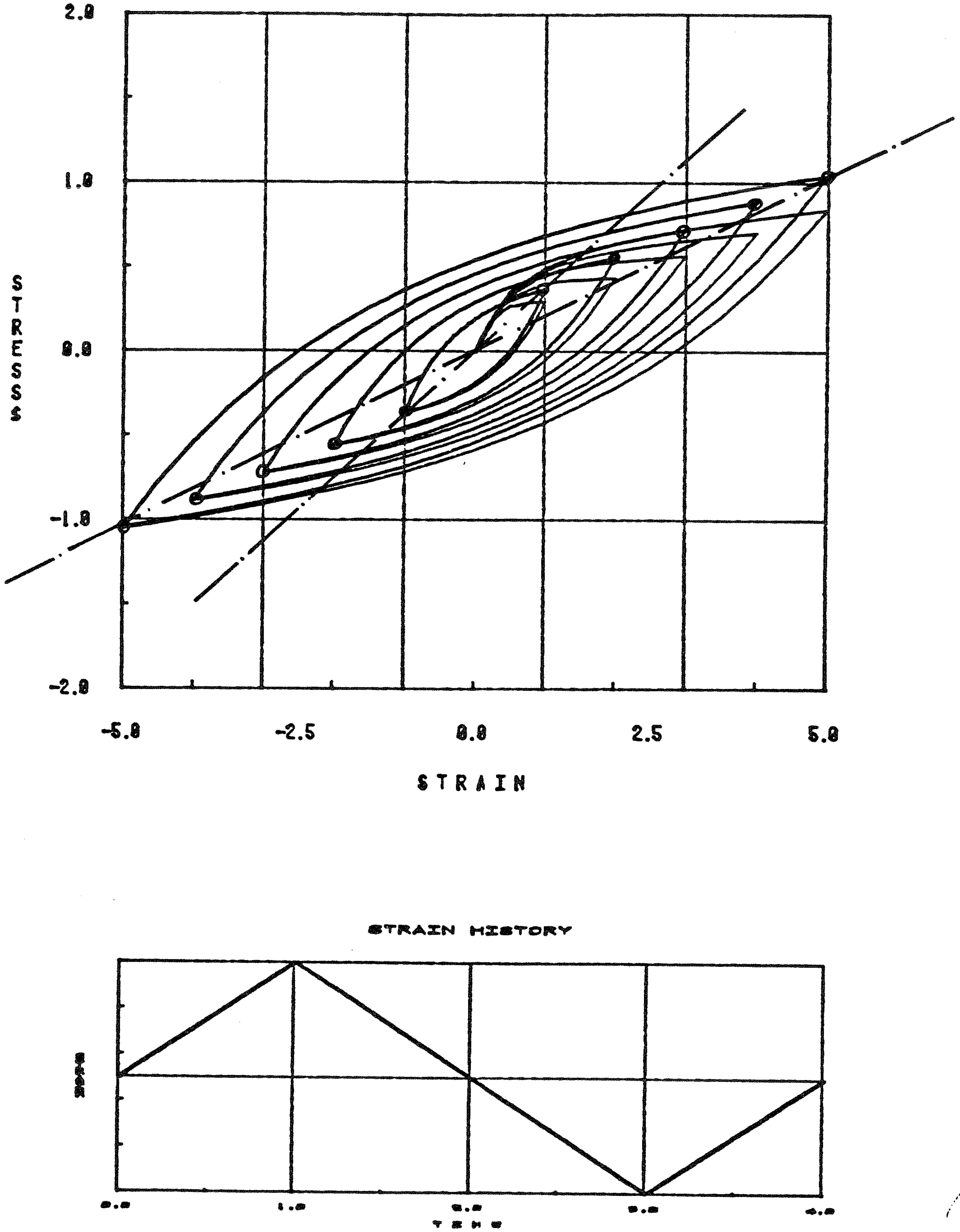


Figure 2

DAMAGE MODEL: PURE SHEAR CYCLIC TEST

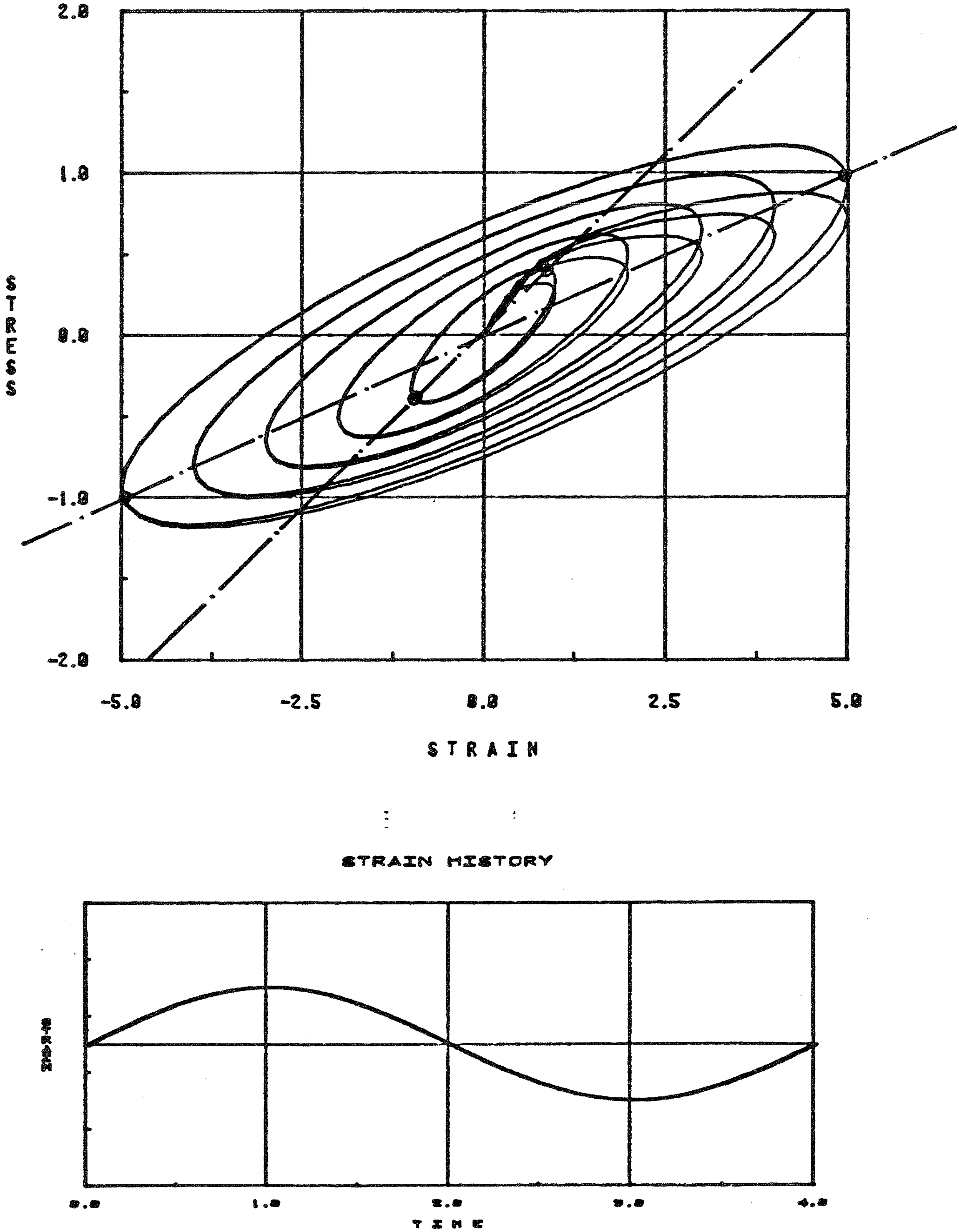
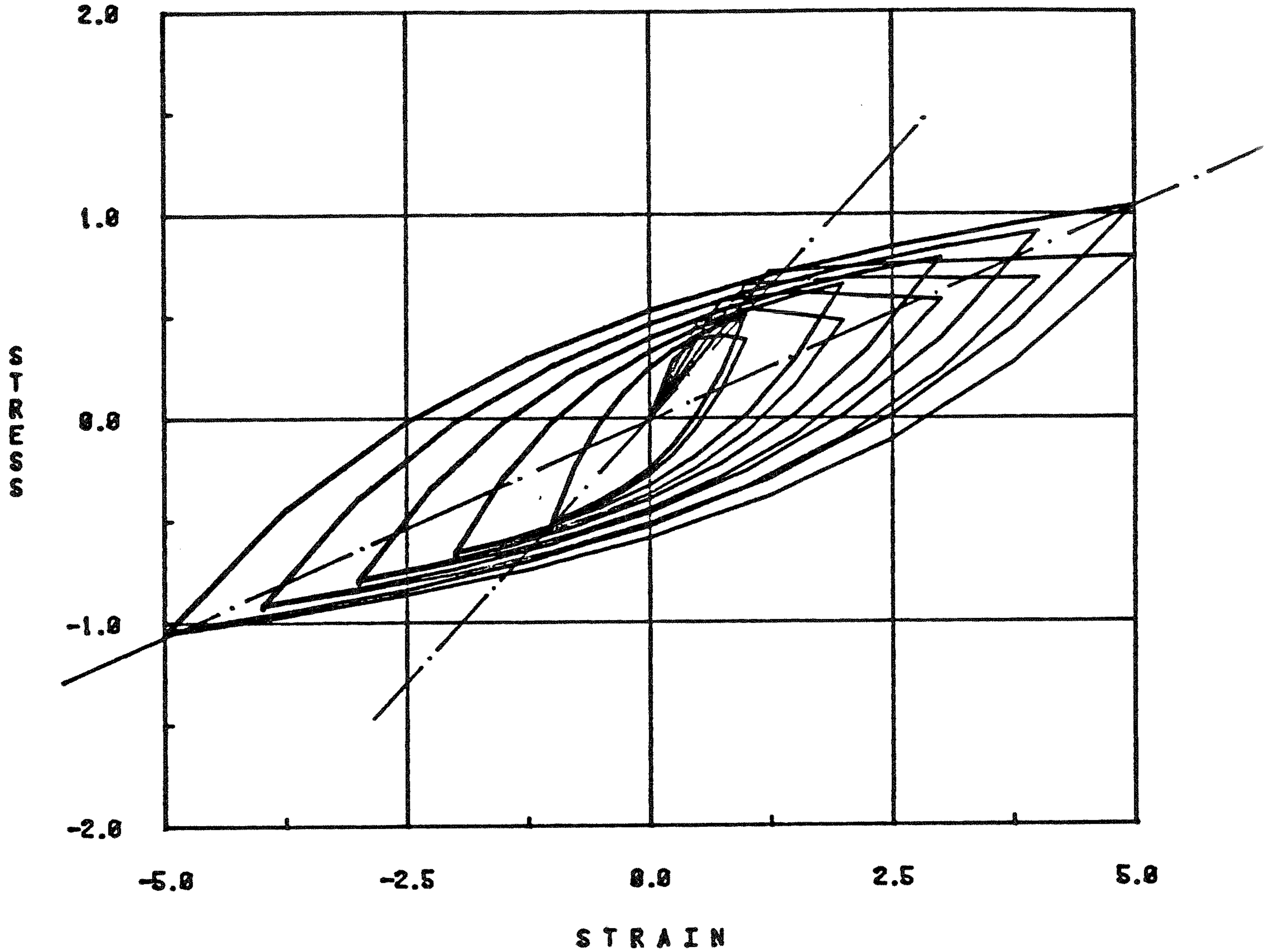


Figure 3

DAMAGE MODEL: SIMPLE EXTENSION CYCLIC TEST



STRAIN HISTORY

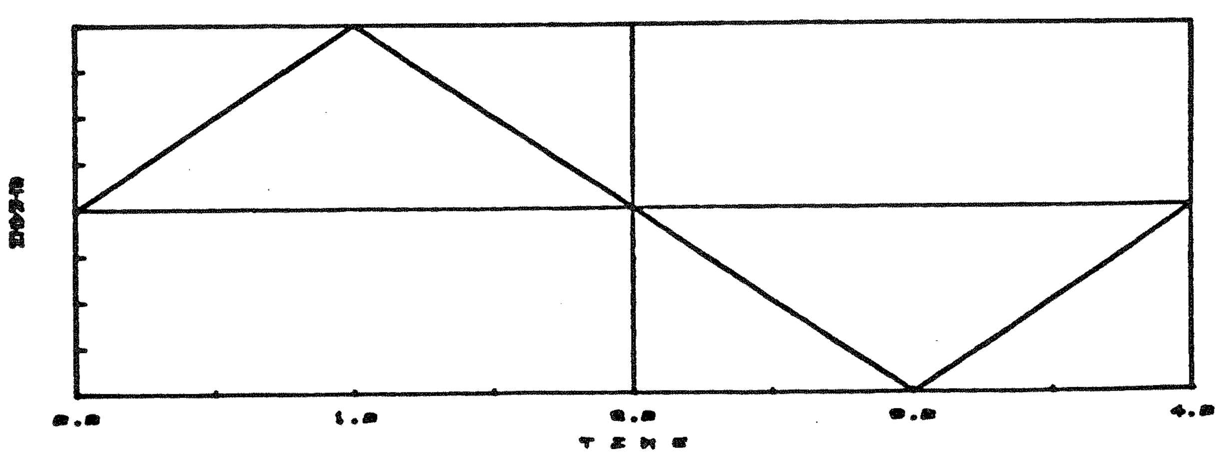


Figure 4

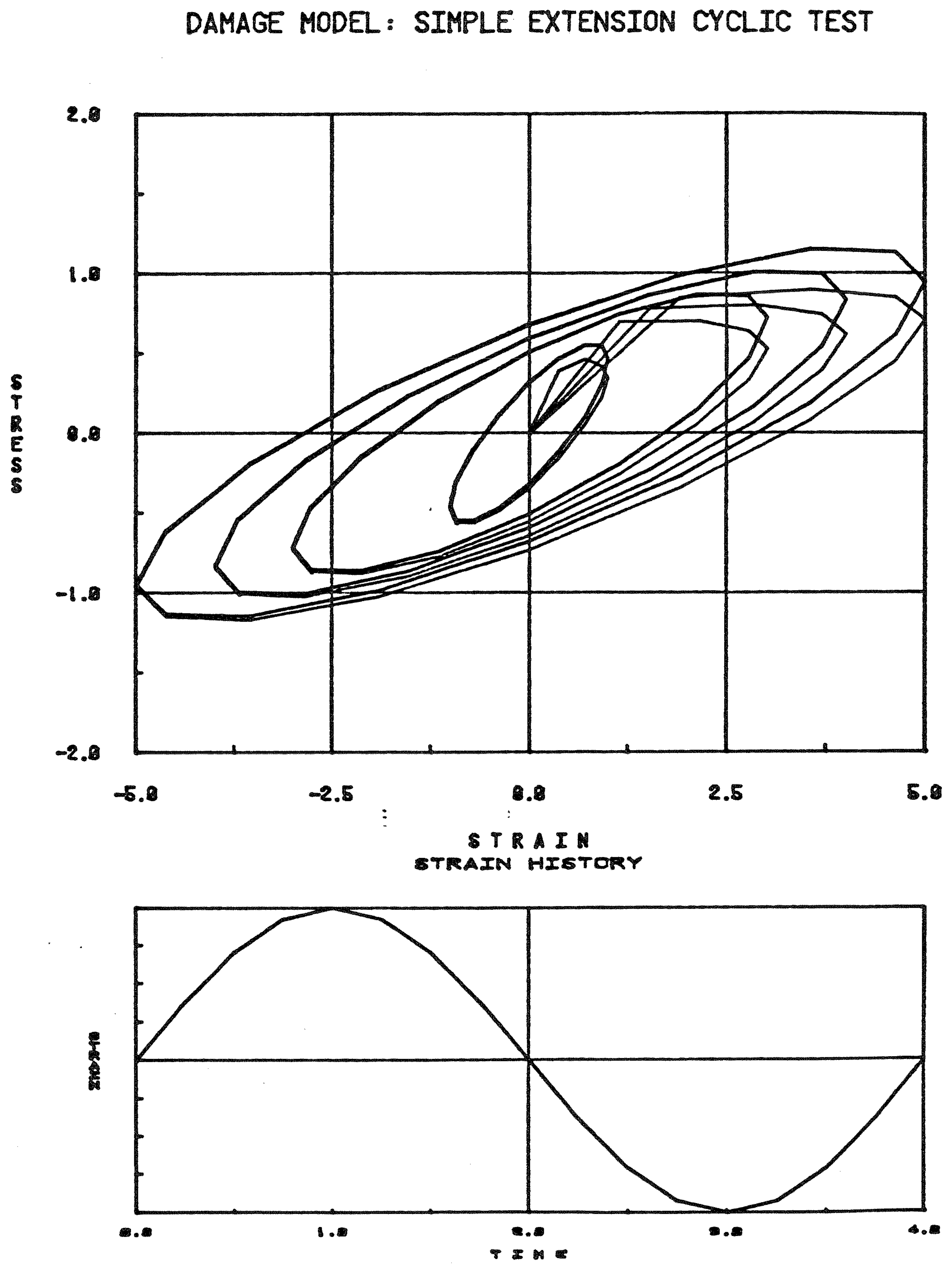

DAMAGE MODEL: SIMPLE EXTENSION CYCLIC TEST
2.0
1.0
0.0
-1.0
-2.0
STRESS
-5.0
-2.5
0.0
2.5
5.0
STRAIN
STRAIN HISTORY
TIME

Figure 5