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Manifolds with Integral and Intermediate Ricci Curvature Bounds

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Yousef K. Chahine

Committee in charge:

Professor Guofang Wei, Chair
Professor Xianzhe Dai
Professor Xin Zhou

September 2019

The Dissertation of Yousef K. Chahine is approved.

Professor Xianzhe Dai

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September 2019

Manifolds with Integral and Intermediate Ricci Curvature Bounds

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Yousef K. Chahine

For my parents

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Math Connections 2018, UC Riverside, May 2018

Heintze-Karcher type inequalities and their applications
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Graduate Geometry Seminar, UC Santa Barbara, 2018

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Abstract

Manifolds with Integral and Intermediate Ricci Curvature Bounds

by

Yousef K. Chahine

In this work, we study interactions between the curvature of a Riemannian manifold and the geometry of its submanifolds. In particular, we consider manifolds with intermediate Ricci curvature bounded below and manifolds with integral curvature bounds.

First, we develop the tools for studying manifolds with intermediate Ricci curvature bounds. In particular, we prove a comparison theorem for the Hessian of the distance function to a submanifold based on a lower bound for the k-Ricci curvature.

The main result is a generalization of the inequality of E. Heintze and H. Karcher [18] for the volume of tubes around minimal submanifolds to an inequality based on integral bounds for k-Ricci curvature. Even in the case of a pointwise bound, this generalizes the classical inequality by replacing a sectional curvature bound with a k-Ricci bound. This theorem is motivated by the estimates of Petersen-Shteingold-Wei for the volume of tubes around a geodesic [27] and generalizes their result.

Finally, we give several applications of these comparison theorems to the geometry and topology of submanifolds in spaces with curvature bounded below. The first is a uniform lower bound on the volume of minimal submanifolds in spaces with integral curvature bounds. We then bound the relative growth of the fundamental group of a closed minimal submanifold in terms of the growth of the fundamental group of the embedding space when the latter has nonnegative intermediate Ricci curvature. We conclude with an application of the comparison theory for intermediate Ricci curvature to certain geometric inequalities which are motivated by questions in general relativity.

Contents

Chapter 1

Introduction

Intermediate Ricci curvature and integral curvature bounds

The curvature of a Riemannian manifold (M, g) is the fundamental local invariant of the metric g which characterizes the geometry of the manifold. In its full form, it is a smooth 4-tensor on M which assigns to three vector fields X, Y, Z on M the vector field

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where ∇ is the covariant derivative induced by the metric g and $[X, Y]$ is the commutator of X and Y. Since its discovery by Riemann as the coefficient of the second order term in a series expansion of the metric, the curvature tensor has been the central object of study in our attempts to understand the relationship between the local geometry of a space and its global geometry and topology. As it has turned out, the curvature tensor is a rather subtle object and this relationship is deep and complicated; far from completely understood.

One of the natural assumptions to make on the curvature with important global consequences is some type of positivity condition. The importance of this type of condition has been evident since the foundations of the subject in the form of the Gauss-Bonnet theorem together with the classification of surfaces. Generalizing this type of theorem to higher dimensions is one of the main problems in Riemannian geometry. However, in higher dimensions the curvature tensor is a much more complicated object (reducing to a simple scalar quantity for surfaces) and so the nature of these generalizations is far more complex. Even the notion of positivity becomes more complicated to formulate.

Early in the study of these spaces it was understood that the curvature tensor could be completely described by the *sectional curvature*, a function on the space of 2-planes Π in the tangent bundle TM defined by

$$\sec(\Pi) = \langle R(e_1, e_2)e_2, e_1 \rangle$$

where e_1, e_2 is any orthonormal basis of Π. Assuming a uniform lower bound on the sectional curvature of the form $\sec \geq K$ for some constant K is one of the strongest senses in which one can consider the curvature to be bounded below.

Another positivity condition on the curvature tensor comes from its contraction, the Ricci tensor defined by

$$Ric(X, Y) = \sum_{i=1}^{n} \langle R(e_i, X)Y, e_i \rangle$$

where $e_1, ..., e_n$ is any orthonormal basis of the tangent space. We say that the Ricci curvature is bounded below if for every unit vector $u \in TM$ we have $Ric(u, u) \geq (n-1)K$ for some constant K. Aside from being the only non-trivial contraction of the full curvature tensor, the study of the Ricci curvature is also motivated by the general theory of relativity as it appears prominently in the Einstein equation relating the stress-energy of matter to the curvature of spacetime. In this context, positivity of the Ricci curvature is closely related to positive energy conditions for the stress-energy of matter.

A lower bound on the Ricci curvature is equivalent to a lower bound on the average of the sectional curvatures taken over all 2-planes containing a preferred vector u, since

$$Ric(u,u) = \sum_{i=1}^{n-1} \langle R(e_i, u)u, e_i \rangle$$

where $e_1, ..., e_{n-1}$ is any orthonormal basis for the subspace orthogonal to u. This has led some to consider certain partial averages of the sectional curvatures known as the *intermediate Ricci curvature* or *k-Ricci curvature*. The k-Ricci curvature interpolates between sectional curvature and Ricci curvature by taking an average of sectional curvatures over a k-dimensional subspace of the tangent space. Specifically, given a unit vector u tangent to M and k-dimensional subspace $\mathcal{V}$ of the tangent space orthogonal to u the k-Ricci curvature of $(u, \mathcal{V})$ is defined by

$$Ric_k(u, \mathcal{V}) = \frac{1}{k} \sum_{i=1}^{k} \langle R(e_i, u)u, e_i \rangle$$

where $e_1, ..., e_k$ form an orthonormal basis of $\mathcal{V}$. Notice that Ric_{n-1} is equivalent to the Ricci curvature and Ric_1 is equivalent to sectional curvature. We say that a manifold has k-Ricci curvature bounded below by K if $Ric_k(u, \mathcal{V}) \geq K$ for all unit vectors $u \in TM$ and k-dimensional subspaces $\mathcal{V} \perp u$.

Some of the earliest global results using k-Ricci lower bounds as a partial positivity condition for curvature were obtained by G. Galloway [11] and H. Wu [41], though the notion had been introduced much earlier by Bishop and Crittenden [4, p. 253]. A significant literature has since developed which bridges a gap between the global results based on sectional curvature bounds and those based on Ricci curvature bounds [34, 35, 39, 31, 42, 17, 21].

In a different direction, it has been shown that some global results still hold even without pointwise curvature bounds, provided the part of the curvature which violates a pointwise bound is small in an L^p sense [10, 43, 28, 27, 30, 2]. To make this precise, for a real-valued function f let $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ denote the positive and negative parts of f, respectively. Given a manifold (M, g), let $\rho_k(x)$ denote the minimum of $Ric_k(u, \mathcal{V})$ where $u \in T_xM$ is a unit tangent vector at x and $\mathcal{V}$ is a k-dimensional subspace orthogonal to u. For a fixed constant K we may then consider the integral norms

$$\|(\rho_k - K)_-\|_p = \left(\int_M (\rho_k - K)_-^p \; dvol_g \right)^{1/p} \tag{1.1}$$

which measure the amount of k-Ricci curvature below K.

The study of manifolds with bounds on the integral norms (1.1) appears to have originated in the work of S. Gallot who obtained an isoperimetric inequality on n-manifolds with bounds on $\|(\rho_{n-1})_-\|_p$ for $p > n/2$ which has a number of well-known consequences [10]. One of the key tools in this work was an extension of an estimate of E. Heintze and H. Karcher for the volume of the tube of a given radius around a compact hypersurface based on Ricci curvature bounds [18]. The original estimate of Heintze-Karcher assumed a pointwise ($p = \infty$) bound on the Ricci curvature, which was weakened to an L^p bound by Gallot. A few years later, D. Yang [43] and then Petersen-Wei [28] obtained volume estimates for geodesic balls based on integral norms of the Ricci curvature. The latter established a Laplacian comparison using integral curvature bounds and led to the generalization of many results based on pointwise curvature bounds to the setting of L^p curvature bounds. At the same time, volume estimates for tubes around geodesics using integral bounds on the sectional curvature were also obtained by Petersen-Shteingold-Wei [27] and used to generalize the Grove-Petersen finiteness theorem to manifolds with integral curvature bounds.

In this dissertation, we study the geometry of manifolds with pointwise and integral bounds on the intermediate Ricci curvature. In particular, we study the relationship between the k-Ricci curvature of a manifold and the geometry and topology of its submanifolds. In Chapter 3, we develop some basic comparison theory for manifolds with k-Ricci curvature bounds. Our main result is to complete the program of generalizing the volume estimate of Heintze-Karcher to the setting of integral curvature bounds by obtaining a volume estimate for the tubes around a submanifold of arbitrary codimension (Chapter 4). In Chapter 5, we develop several applications of the comparison theory. Before proceeding to the main work, we give a brief outline of the results.

Volume inequalities based on lower curvature bounds

The geodesic tube of radius r around a closed submanifold Σ^m of a Riemannian manifold M^n, denoted $T(\Sigma, r)$, is the set of all points whose distance to Σ is at most r. In Chapter 4, we give upper bounds for the volume of $T(\Sigma, r)$ based on L^p norms of the negative part of the k-Ricci curvature of M. For $p = \infty$, we prove that the well-known estimate of E. Heintze and H. Karcher based on pointwise sectional curvature bounds requires only k-Ricci bounds (Theorem 3.18). The main result is the case $p < \infty$, where we give the first estimates for the volume of tubes around submanifolds of general codimension using integral curvature bounds.

Theorem 1.1. *Let M^n be a complete Riemannian manifold and let $\Sigma^m \subset M$ be a closed minimal submanifold with $0 < m < n-1$.*

Put $k = \min\{m, n-m-1\}$. If $K \leq 0$ and $p > n-k$ then

$$\operatorname{vol}(T(\Sigma, r)) \leq \left(f(r)^{n-m-1} + \|(\rho_k - K)_-\|_p^{\beta p} f(r)^p \right) e^{\kappa r^{2\alpha}} \tag{1.2}$$

where $0 < \alpha, \beta < 1$ *are constants,* $\kappa = C_3|K|^\alpha$*,*

$$f(r) = C_1 \left(\mathrm{vol}(\mathbb{S}^{n-m-1})\,\mathrm{vol}(\Sigma) r^{n-m}\right)^{\frac{1}{n-m-1}} + C_2 \left\|(\rho_k - K)_-\right\|_p^{1-\beta} r^2,$$

and $C_1(n,m)$*,* $C_2(n,m,p)$ *and* $C_3(n,m,p)$ *are constants.*

Remark 1.2. As mentioned above, estimates when Σ is a point or a hypersurface have already been obtained in [10, 28] so we do not repeat this case.

Remark 1.3. In the case of a pointwise lower bound $Ric_m \geq 0$ (i.e. $\|(\rho_m)_-\|_p = 0$) and $k = m$ (see also Remark 4.4) the estimate above reduces to

$$\mathrm{vol}(T(\Sigma, r)) \leq \frac{1}{n-m}\,\mathrm{vol}(\Sigma)\,\mathrm{vol}(\mathbb{S}^{n-m-1}) r^{n-m}.$$

In particular, this shows that the Heintze-Karcher estimate [18, Corollary 3.3.1] holds for tubes around minimal submanifolds assuming only a k-Ricci lower bound in place of a sectional curvature lower bound. In fact, in Theorem 3.18 below we show that the Heintze-Karcher estimate holds for tubes around *any* closed submanifold assuming only a pointwise k-Ricci lower bound (see also [16] for a related volume comparison using pointwise k-Ricci bounds).

Loosely speaking, the estimate of Theorem 1.1 shows that it does not matter how the negative part of the curvature concentrates around the submanifold, a uniform estimate holds for all manifolds with $\|(\rho_k - K)_-\|_p$ bounded above by a constant as long as p is chosen sufficiently large. The estimates of Gallot and Petersen-Wei for tubes around hypersurfaces and geodesic balls require $p > n/2$, whereas the estimates of Petersen-Shteingold-Wei for the tubes around a geodesic require $p > n-1$. Notice that our requirement $p > n-k$ is a natural generalization of both of these conditions as $n-k$ is bounded below by $n/2$.

The work of Petersen-Shteingold-Wei for the volume of the tube around a geodesic illustrates the increase in difficulty in the case that the central submanifold has arbitrary codimension. In particular, estimates were needed for certain quadratic invariants of the Hessian of the distance function which were completely new to comparison geometry [27, Lemma 3.1]. Our methods are based on the ideas of [27]; however, in that work a number of simplifications were employed specific to the 1-dimensional case which make modification to general codimension nontrivial. Indeed, if one naively adapts the arguments of [27] the resulting volume estimates require stronger assumptions on the curvature of M and on the second fundamental form of Σ than are necessary. As it turns out, an understanding of the interaction between k-Ricci curvature and volume greatly facilitates the generalization to tubes around submanifolds of all dimensions.

Thus, before proving Theorem 1.1 we introduce our main ideas by proving a new Hessian comparison for distance functions based on k-Ricci curvature bounds which is of independent interest. Specifically, if $r(x) = d(x, \Sigma)$ is the distance to a closed submanifold Σ we prove an upper bound for certain partial traces of the Hessian $\nabla^2 r$ given pointwise lower bounds on Ric_k. This Hessian comparison unifies and generalizes a number of distinct Hessian and Laplacian comparisons for the distance function to a point. Recall that in a space of constant curvature K the eigenvalues of the Hessian of the distance function to a point are given by $\mathrm{cs}_K(r)/\,\mathrm{sn}_K(r)$ where sn_K and cs_K are the generalized trigonometric functions defined in Section 2.1.1.

Theorem 1.4 (Hessian Comparison)**.** *Let Σ^m be an m-dimensional submanifold of a complete Riemannian manifold M^n and let $r(x) = d(x, \Sigma)$ be the distance function to Σ. Let γ be any geodesic segment satisfying $r(\gamma(t)) = t$.*

If $Ric_k(\dot\gamma, \cdot) \geq K$ then for any orthonormal k-frame $\{e_1(t), ..., e_k(t)\} \subset \dot\gamma(t)^\perp$ which *is parallel along* γ *we have*

$$\sum_{i=1}^{k} \nabla^2 r(e_i, e_i) \leq \begin{cases} k \log(\mathrm{cs}_K + h_0 \,\mathrm{sn}_K)'(r) & \text{if } \{e_1(0), ..., e_k(0)\} \subset T\Sigma \\ k \log(\mathrm{sn}_K)'(r) & \text{otherwise} \end{cases}$$

where $h_0 = \frac{1}{k}\sum_{i=1}^{k}\langle S_{\dot\gamma}(e_i), e_i\rangle$ *and* $S_{\dot\gamma}$ *is the shape operator of* Σ *for the normal* $\dot\gamma(0)$*.*

Remark 1.5. Notice that taking Σ to be a point, the usual Hessian and Laplacian comparisons follow from this theorem by taking $k = 1$ and $k = n - 1$, respectively. When Σ is a point, the result was proved by Shen [34, Lemma 11] and Li-Wang [23, Theorem 1.2].

Remark 1.6. This result implies the mean curvature comparison of [16] when Σ is totally geodesic.

In Section 3.3 we give a slightly more general version of this theorem which also treats the question of rigidity when equality holds. This comparison theorem should be compared with that of Guijarro-Wilhelm which gives comparison along a family of k-dimensional subspaces determined by Jacobi fields rather than parallel subspaces [17, Lemma 2.23]. That comparison is based on Wilking's transverse Jacobi equation; in Section 3.2.1 we give another proof of that comparison and explain its connection to the Hessian comparison above.

Applications to the geometry and topology of submanifolds

In Chapter 5, we develop a number of applications of the comparison theorems above to the global geometry and topology of manifolds with intermediate Ricci curvature bounds. The first is a uniform lower bound for the volume of closed minimal submanifolds in spaces with integral curvature bounds.

Corollary 1.7. *Given integers n and m with $n \geq 3$ and $0 < m < n-1$, and real numbers $K \leq 0$, $v_0, D > 0$ and $p > n-k$ where $k = \min\{m, n-m-1\}$, there exist constants $\epsilon(n,m,p,K,v_0,D) > 0$ and $\delta(n,m,p,K,v_0,D) > 0$ such that every closed n-dimensional Riemannian manifold M satisfying*

$$\mathrm{vol}(M) \geq v_0, \quad \mathrm{diam}(M) \leq D, \quad \|(\rho_k - K)_-\|_p \leq \epsilon$$

has the property that all closed m-dimensional minimal submanifolds have volume bounded below by δ.

Remark 1.8. This should be thought of as a generalization of Cheeger's lemma. For the case of 1-dimensional minimal submanifolds (closed geodesics) this result was obtained already in [27, Theorem 1.2]. The proof follows easily from the observation that our uniform upper bound (1.2) for the tube around a minimal submanifold approaches 0 as $\mathrm{vol}(\Sigma), \|(\rho_k - K)_-\|_p \to 0$.

Continuing our study of minimal submanifolds, we give a Frankel-type theorem for the image of the fundamental group of an immersed minimal submanifold Σ induced by the immersion $\iota : \Sigma \to M$. Recall that Frankel's theorem states that if Σ is a closed minimal hypersurface in a complete manifold M with $Ric > 0$ then the induced map $\iota_* : \pi_1(\Sigma) \to \pi_1(M)$ is a surjection [9]. Of course, if one reduces the assumption to $Ric \geq 0$ the map need not be surjective (consider e.g. the flat n-torus). G. Galloway classified the image of the fundamental group $\iota_*(\Sigma)$ in the case that Σ is a minimal hypersurface and M has nonnegative Ricci curvature and, in particular, showed that the image is a "relatively large" subgroup of $\pi_1(M)$ (see Theorem 5.5). For minimal submanifolds of higher codimension the situation is much less rigid; however, adapting some ideas of J. Milnor and M. Anderson and using the volume comparison theorem for minimal submanifolds above we show that if the growth of the fundamental group of M

is sufficiently large, then the relative growth of the image $\iota_*(\Sigma)$ is large provided $Ric_k \geq 0$ for suitably chosen k (refer to Section 5.2 for the relevant definitions).

Theorem 1.9. *Let M^n be a complete Riemannian manifold with $Ric_k \geq 0$. If the fundamental group $\pi_1(M)$ has polynomial growth of order p and M has asymptotic volume growth of order q then for any immersed compact minimal submanifold $\iota : \Sigma^m \to M$ with $m \geq k$ and either $m = n-1$ or $m \leq n-k-1$ the image of the fundamental group $\iota_*(\pi_1(\Sigma)) \subset \pi_1(M)$ has relative growth of order at least $(p+q)-(n-m)$.*

In Section 5.4 we give another application of the Hessian comparison above to some geometric inequalities which arise in connection to certain problems in general relativity. Specifically, we generalize an inequality of G. Qiu and C. Xia relating the volume of a compact manifold with boundary to the total inverse mean curvature of the boundary [32, Theorem 1.3]. They originally proved this inequality assuming a lower bound on sectional curvature and ask whether the inequality holds assuming only a Ricci lower bound. In Theorem 5.40 we give a broader context for this inequality and show that an $(n-2)$-Ricci lower bound suffices.

Betti number bounds for Intermediate Ricci curvature

In 1981, Gromov showed that there exist universal constants $C(n)$ depending only on dimension such that all complete Riemannian n-manifolds with $\sec \geq 0$ have total Betti number at most $C(n)$ [15]. On the other hand, it is well-known that all complete n-manifolds with $Ric \geq 0$ have $b_1(M;\mathbb{R})$ and $b_{n-1}(M;\mathbb{R})$ bounded above by n (see Section 5.3). This begs the following very natural question.

Question 1.10. *Do there exist constants $C(n)$ depending on n such that all complete*

Riemannian n-manifolds with $Ric_k \geq 0$ satisfy

$$b_k(M, \mathbb{R}) \leq C(n)?$$

In Section 5.3 we give some limited evidence for this conjecture by showing that such a bound exists for compact manifolds with a simple curvature operator. We say that the curvature operator of a Riemannian manifold M is *simple* if its eigenvectors are simple bivectors, i.e. bivectors of the form $u \wedge v$ where $u, v \in TM$ (see Section 5.3 for a more thorough treatment of this condition).

Theorem 1.11. *If (M^n, g) is a compact Riemannian manifold with simple curvature operator and $Ric_k \geq 0$ then*

$$b_k(M, \mathbb{R}) \leq \binom{n}{k}$$

and $b_k = 0$ if $Ric_k > 0$ at some point.

The proof of this statement is based on a careful treatment of the curvature terms in the Bochner formula for k-forms and this technique is not expected to work without the assumption on the curvature operator. On the other hand, the result of Gromov suggests that another approach may still yield an affirmative answer to Question 1.10.

Notation and conventions

In the following development all manifolds are assumed to be connected unless otherwise noted. We refer to an embedded submanifold simply as a submanifold, explicitly stating when a submanifold is immersed. As defined in Chapter 2, the normalized mean curvature vector of a submanifold is denoted η, and more generally greek letters ν, ξ are used for vectors normal to a submanifold with capital latin letters X, Y, etc. used for general vector fields.

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Chapter 2

Background

In this chapter, we review some of the basic tools for studying manifolds with curvature lower bounds. The fundamental observation on which many results are based is that curvature controls how families of neighboring geodesics spread out. In order to make this notion precise, we review in the first section some basic tools involving geodesic vector fields, Jacobi fields, and the Riccati equation. We then review the notion of distance functions which can be thought of as potentials for an important class of geodesic vector fields. Finally, we review some notions important in the geometry of submanifolds, in particular those involving the normal bundle and the normal exponential map.

This chapter should serve as a reference for the material presented later, but also fixes some notation and terminology. We have attempted to set off all notation and terminology as a formal definition so the reader familiar with this material may skim this section and proceed to the next chapter referring back as needed.

2.1 The Riccati equation and the Jacobi equation

Let (M, g) be a complete Riemannian manifold.

Definition 2.1. A *geodesic vector field* on an open subset $\Omega \subset M$ is a smooth vector field V on Ω such that $\nabla_V V = 0$.

Observe that the integral curves of a geodesic vector field V are geodesics with the exception of constant curves where $V = 0$. This terminology should not be confused with what is often termed *the* geodesic vector field, which is a unique global vector field on the tangent bundle TM. To understand the local behavior of the congruence of geodesics correspoding to a geodesic vector field V, we study the total covariant derivative ∇V which we shall denote by H_V.

Proposition 2.2 (Riccati equation)**.** *The total covariant derivative $H_V = \nabla V$ of a geodesic vector field V satisfies*

$$H_V' + H_V^2 = -R_V$$

where H_V' denotes the covariant derivative $\nabla_V H_V$, H_V^2 denotes the composition $H_V \circ H_V$, and $R_V = R(\cdot, V)V$ is the directional curvature operator in the direction V.

Proof. The proof is a simple computation using the identity $\nabla_V V = 0$,

$$(\nabla_V H_V)(X) = \nabla_V(\nabla_X V) - \nabla_{\nabla_V X} V = \nabla_V \nabla_X V - \nabla_X \nabla_V V - \nabla_{[V,X]} V - \nabla_{\nabla_X V} V$$

where we recognize the expression on the right as $R(V, X)V - H_V^2(X)$. □

Note that at this point we do not assume H_V is self-adjoint, as is often the case (e.g. if V is the gradient of a distance function). In particular, we observe that for any smooth vector field X the tensor ∇X decomposes into symmetric and anti-symmetric parts via the equation

$$g(\nabla X, \cdot) = \frac{1}{2}\mathcal{L}_X g + \frac{1}{2} d\theta_X$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric and $\theta_X = g(X, \cdot)$ is the one-form dual to X.

Definition 2.3. A vector field X is *irrotational* if $d\theta_X = 0$, i.e., if $H_X = \nabla X$ is symmetric.

An irrotational geodesic vector field on a connected open set which vanishes at a point is identically zero. By the Frobenius theorem a smooth non-vanishing geodesic vector field V is irrotational if and only if the orthogonal distribution $V^\perp$ is integrable, i.e. the vector field V is hypersurface orthogonal. In this case, V is locally the gradient of a smooth function and the covariant derivative H_V is equivalent to the Hessian operator. The irrotational geodesic vector fields are of particular interest and we will return to this special case in Section 2.2.

In addition to the total covariant derivative H_V, another perspective for studying the local behavior of geodesics tangent to V is via the vector fields which commute with V, that is, the vector fields which are invariant under the flow of V.

Definition 2.4. If V is a geodesic vector field then a vector field J is a *V-Jacobi field* if $\mathcal{L}_V J = 0$.

Note that the condition defining a V-Jacobi field only requires J to be defined along an integral curve γ of V. In this case, if we wish to be specific we may say that J is a V-Jacobi field *along* γ. The relationship to the Riemannian structure of M is obtained by computing the covariant derivative of the commuting vector fields along V.

Proposition 2.5 (Jacobi equation)**.** *If V is a geodesic vector field and J is a V-Jacobi field then*

$$J'' = -R_V(J)$$

where $J'' = \nabla_V \nabla_V J$ denotes the second covariant derivative along V.

Proof. Since $\nabla_V V = 0$ and $[J, V] = 0$ implies $\nabla_J V = \nabla_V J$ we have

$$R_V(J) = \nabla_J \nabla_V V - \nabla_V \nabla_J V - \nabla_{[J,V]} V = -\nabla_V \nabla_V J.$$

□

Although exceedingly simple in their derivation, a great deal of the geometry of M can be understood by application of the two ordinary differential equations above. The Riccati equation for H_V and the Jacobi equation for V-Jacobi fields J are, in fact, equivalent. The connection between the two is established by the fact that V-Jacobi fields are precisely those vector fields which satisfy

$$H_V(J) = J'. \tag{2.1}$$

Covariantly differentiating this equation along V we obtain

$$J'' = H_V'(J) + H_V(J') = H_V'(J) + H_V^2(J).$$

Since any vector u can be locally extended to a V-Jacobi field J_u this shows how the Riccati equation is obtained from the Jacobi equation and vice versa.

Finally, we note the following relationship between the total covariant derivative and the volume element spanned by a collection of commuting vector fields which will be of central importance in the sequel.

Proposition 2.6. *Let X be any smooth vector field. Given any linearly independent vector fields $J_1, ..., J_k$ which commute with X we have*

$$\log(|J_1 \wedge ... \wedge J_k|)' = \mathrm{tr}_{\mathcal{V}}(H_X)$$

where $H_X = \nabla X$, $\mathcal{V} = \mathrm{span}\{J_1, ..., J_k\}$, *and the derivative is along* X.

Remark 2.7. Here, we introduce the notation $\mathrm{tr}_{\mathcal{V}}$ to denote the *projected* or *partial trace* of a linear operator A on a subspace $\mathcal{V}$ of an inner product space defined by

$$\mathrm{tr}_{\mathcal{V}}(A) = \mathrm{tr}(A \circ P)$$

where P is the orthogonal projection onto $\mathcal{V}$.

Proof. Recall that

$$|J_1 \wedge ... \wedge J_k| = \sqrt{\det(J_{ij})}$$

where J_{ij} is the matrix of inner products $J_{ij} = \langle J_i, J_j \rangle$. We may compute the logarithmic derivative of the determinant using Jacobi's formula

$$\log(|J_1 \wedge ... \wedge J_k|)' = \frac{1}{2}\,\mathrm{tr}(J_{ij}^{-1} J_{ij}').$$

Note that at any fixed point $x \in M$, one can apply a constant change of basis to the vector fields J_i so that $J_i\big|_x$ are orthonormal without changing the logarithmic derivative above. Thus we may assume the vector fields J_i are orthonormal at the point we take the derivative to get

$$\log(|J_1 \wedge ... \wedge J_k|)' = \frac{1}{2}\,\mathrm{tr}(J_{ij}') = \frac{1}{2}\sum_{i=1}^{k} X\langle J_i, J_i \rangle = \sum_{i=1}^{k} \langle H_X(J_i), J_i \rangle = \mathrm{tr}_{\mathcal{V}}(H_X).$$

□

The proposition above refines the following basic result of Riemannian geometry.

Corollary 2.8. *Let* $(x^1, ..., x^n)$ *be any local coordinates on a Riemannian manifold* M.

The density $\sqrt{\det g}$ *of the volume element in these coordinates satsifies*

$$\partial_i \log(\sqrt{\det g}) = \mathrm{tr}(\nabla \partial_i) = \mathrm{div}\, \partial_i.$$

Our main interest, of course, will be in the case when X is a geodesic vector field. In this case, we can use the Riccati equation to relate the curvature of M to the logarithmic derivative of the k-dimensional volume element spanned by Jacobi fields via the previous proposition.

Before moving to geometric applications, we briefly review the comparison theory for these ordinary differential equations.

2.1.1 Comparison theory for Riccati ODEs

The perspective we take in this dissertation is primarily that obtained through the analysis of the Riccati differential equation. There is a classical observation which corresponds to the equivalence between the Jacobi and Riccati equations seen above, namely, if a real-valued function $h : (a, b) \to \mathbb{R}$ satisfies the scalar Riccati equation

$$h' + h^2 + \rho = 0$$

for some real-valued function ρ then for any anti-derivative H of h the function $f = e^H$ satisfies the scalar Jacobi equation

$$f'' + \rho f = 0.$$

Conversely, if a *non-vanishing* function f satisfies the latter equation then $h = f'/f$ satisfies the scalar Riccati equation.

Although the Riccati equation introduced in the previous section is a matrix equation, all of the analysis we require can be obtained through the comparison theory for the scalar Riccati equation. The proof is standard but included for completeness.

Proposition 2.9 (Riccati Comparison Principle). *Let $f, h : (0, t_0] \to \mathbb{R}$ be differentiable functions on the interval $(0, t_0]$ satisfying*

$$f' + f^2 \leq h' + h^2.$$

If $\liminf_{t\to 0} h$ is bounded below and $\limsup_{t\to 0}[h(t) - f(t)] \geq 0$ then $f \leq h$ on $(0, t_0]$ with equality at t_0 if and only if $f \equiv h$ on $(0, t_0]$.

Proof. First, rewrite the inequality in the form

$$(h - f)' + (h + f)(h - f) \geq 0.$$

Multiply by an integrating factor $\mu(t) = e^{F(t)}$ where F is any anti-derivative of $h+f$ and integrate from $\delta > 0$ to $t \leq t_0$ to obtain

$$h(t) - f(t) \geq e^{-F(t)} e^{F(\delta)} [h(\delta) - f(\delta)].$$

By our assumptions $F' = h + f$ is bounded below on $(0, t_0]$ and hence e^F is bounded above and so taking any sequence $\delta \to 0$ such that $h(\delta) - f(\delta) \to \epsilon \geq 0$ shows that the left hand side is nonnegative for all $t > 0$.

Moreover, if equality holds at t_0 then the inequality above shows that $h(\delta) \leq f(\delta)$ for all $\delta < t_0$, which together with the previous observation implies $f \equiv h$ on $(0, t_0]$. □

When applying this principle, we shall frequently make reference to specific model solutions to dominate the geometric quantity of interest.

Definition 2.10. For a given real constant K define the *generalized sine* $\operatorname{sn}_K : \mathbb{R} \to \mathbb{R}$ as the solution to the equation

$$f'' + Kf = 0$$

with $f(0) = 0$ and $f'(0) = 1$. Similarly, the *generalized cosine* $\operatorname{cs}_K = \operatorname{sn}_K'$ is the solution with $f(0) = 1$ and $f'(0) = 0$.

Explicitly, we have

$$\operatorname{sn}_K(t) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) & K > 0 \\ t & K = 0 \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}t) & K < 0 \end{cases} \qquad \operatorname{cs}_K(t) = \begin{cases} \cos(\sqrt{K}t) & K > 0 \\ 1 & K = 0 \\ \cosh(\sqrt{-K}t) & K < 0. \end{cases}$$

Since these form a complete basis of solutions to the scalar Jacobi equation $f'' + Kf = 0$ one can see from our discussion above that every solution to the Riccati differential equation

$$h' + h^2 + K = 0$$

on an interval $(0, t_0)$ with $h(0+) = h_0 \in [-\infty, \infty]$ is of the form

$$h(t) = \log(\alpha \operatorname{cs}_K(t) + \beta \operatorname{sn}_K(t))' = \begin{cases} \log(\operatorname{sn}_K(t+\delta))' & h_0^2 > -K \\ h_0 & h_0^2 = -K \\ \log(\operatorname{cs}_K(t+\delta))' & h_0^2 < -K \end{cases}$$

where $h_0 = \beta/\alpha$ and the shift δ in the second form of the solution is defined by $\operatorname{cs}_K(\delta)/\operatorname{sn}_K(\delta) = h_0$ or $-K \operatorname{sn}_K(\delta)/\operatorname{cs}_K(\delta) = h_0$ depending on whether $h_0^2 > -K$ or $h_0^2 < -K$, respectively.

2.2 Distance functions, mean curvature, and volume

In this section we review the theory of distance functions on a Riemannian manifold and the connection between the mean curvature and volume of its level sets. This connection is a geometric realization of the connection between the Riccati and Jacobi equations described above.

2.2.1 Distance functions and shape operators

We now turn to the special case of irrotational geodesic vector fields. Up to a constant rescaling, these are completely captured by the following notion of a distance function.

Definition 2.11. A *distance function* on a Riemannian manifold M is a real-valued function $r : M \to \mathbb{R}$ which is smooth on a dense open subset Ω with $|\nabla r| \equiv 1$.

Examples are provided by the functions of the form $r(x) = d(x, \Sigma) = \inf_{y\in\Sigma} d(x, y)$ where Σ is a submanifold of M and $d(x, y)$ denotes the Riemannian distance in M. We shall call the maximal open subset on which r is differentiable the *regular set* of r.

We now fix some notation for distance functions. For convenience we denote the gradient of a distance function ∇r by ∂_r. The *Hessian operator* of r is the $(1, 1)$-tensor field $H_{\partial_r} = \nabla\partial_r$ corresponding to the Hessian $\nabla^2 r$.

Proposition 2.12. *The gradient of a distance function is an irrotational geodesic vector field on its regular set, and hence satisfies the Riccati equation*

$$H'_{\partial_r} + H^2_{\partial_r} = -R_{\partial_r}. \tag{2.2}$$

Proof. For any smooth vector field X on the regular set we have

$$\langle \nabla_X \partial_r, \partial_r\rangle = X\langle \partial_r, \partial_r\rangle - \langle \partial_r, \nabla_X \partial_r\rangle = -\langle \partial_r, \nabla_X \partial_r\rangle$$

which shows that the above expression vanishes. Since the Hessian operator is self-adjoint we have $\langle \nabla_{\partial_r}\partial_r, X\rangle = \langle \nabla_X \partial_r, \partial_r\rangle = 0$. □

Since a distance function r has a non-vanishing gradient on its regular set Ω, the level set $\Sigma_t = r^{-1}(t) \cap \Omega$ of a distance function restricted to its regular set is a smooth embedded hypersurface with unit normal ∂_r. Moreover, since ∂_r is a unit normal along Σ_t the Hessian operator $H_{\partial_r} = \nabla \partial_r$ is related directly to the extrinsic geometry of the level sets.

Definition 2.13. If Σ^m is a submanifold of a Riemannian manifold M^n and ξ is a unit normal vector to Σ at a point $x \in \Sigma$ the *shape operator* of Σ with respect to ξ is the linear map $S_\xi : T_x\Sigma \to T_x\Sigma$ defined by

$$S_\xi(X) = (\nabla_X \xi)^\top$$

where $^\top$ denotes the orthogonal projection $T_xM \to T_x\Sigma$ and ξ is extended arbitrarily to a unit normal vector field along Σ. The *mean curvature* of Σ with respect to the unit normal ξ is given by the trace $h(\xi) = \mathrm{tr}(S_\xi)$. The (normalized) *mean curvature vector* of Σ is the normal vector field η along Σ defined by $\langle \eta, \xi\rangle = -\mathrm{tr}(S_\xi)/m$ for all unit vectors $\xi \perp \Sigma$.

Note that in the case that Σ is a hypersurface with unit normal ξ, the orthogonal projection in the definition of the shape operator is redundant. As a result, we have

Proposition 2.14. *Let r be a distance function with regular set Ω and let $\Sigma_t = r^{-1}(t) \cap \Omega$ denote the regular part of the level set. The gradient ∂_r is a unit normal along Σ_t and so the shape operator $S_{\partial_r} : T_x\Sigma_t \to T_x\Sigma_t$ is given by*

$$S_{\partial_r}(X) = H_{\partial_r}(X).$$

Note that since the Hessian operator H_{∂_r} vanishes on the orthogonal complement of $T\Sigma_t$, the trace of the Hessian H_{∂_r} is the same as the trace of S_{∂_r} which is just the mean curvature $h(\partial_r)$ of Σ_t. We thus have the equivalence of the Laplacian of a distance function r, the divergence of the geodesic vector field ∂_r, and the mean curvature of the level hypersurfaces

$$\Delta r = \mathrm{div}(\partial_r) = \mathrm{tr}(H_{\partial_r}) = \mathrm{tr}(S_{\partial_r}) = h(\partial_r). \tag{2.3}$$

In order to make use of the Riccati equation for the Hessian operator of a distance function, we naturally need initial conditions. In the next section we give the initial conditions for the case of main interest to us, that corresponding to distance functions from submanifolds.

2.2.2 Series expansion for the Hessian

In this section we show that the Hessian of the distance function to a submanifold is the same as that in Euclidean space to leading order in the distance r from the submanifold. This result is well-known but we give a proof for completeness since it is not readily available in the literature.

Proposition 2.15. *Let Σ be a submanifold of a Riemannian manifold M and let $r(x) = d(x, \Sigma)$ denote the distance to Σ. Let $\gamma(t)$ be any geodesic with $r(\gamma(t)) = t$.*

Identifying the subspaces $T_{\gamma(t)}M$ via parallel transport along γ, the Hessian operator satisfies

$$H_{\partial_r} = \frac{1}{r}P + S_{\dot\gamma} + O(r) \tag{2.4}$$

where $P : T_{\gamma(0)}M \to T_{\gamma(0)}M$ is the orthogonal projection onto $(T_{\gamma(0)}\Sigma)^\perp \cap \dot\gamma^\perp$ and $S_{\dot\gamma}$ is the shape operator of Σ with respect to $\dot\gamma(0)$ extended to $T_{\gamma(0)}M$ so as to vanish on the *orthogonal complement of* $T_{\gamma(0)}\Sigma$.

Proof. Since the function $\frac{1}{2}r^2$ is smooth on a neighborhood of Σ, the Hessian $\nabla^2(\frac{1}{2}r^2) = r\nabla^2 r + dr^2$ is a smooth tensor field in a neighborhood of Σ. Let $B(t) : T_{\gamma(t)}M \to T_{\gamma(t)}M$ denote the Hessian operator of $\frac{1}{2}r^2$ at $\gamma(t)$ and note that for $t > 0$ we have $B(t) = tH_{\partial_r} + P_{\dot\gamma}$ where $P_{\dot\gamma}$ is the orthogonal projection onto span$\{\dot\gamma\}$. Identifying the tangent spaces $T_{\gamma(t)}M$ via parallel translation along γ, we will now show that

$$B(t) = P_{\boldsymbol{\nu}} + tS_{\dot\gamma} + O(t^2)$$

where $P_{\boldsymbol{\nu}}$ is the orthogonal projection onto the space $\boldsymbol{\nu}_{\gamma(0)}(\Sigma)$ of normal vectors to Σ at $\gamma(0)$, from which the result follows.

To compute $B(0)$, first note that since $\frac{1}{2}r^2$ identically achieves its minimum on Σ it follows that $B(0)$ vanishes on $T_{\gamma(0)}\Sigma$. Moreover, for any vector ξ normal to Σ at $\gamma(0)$ we can extend it to a smooth local vector field X such that $\nabla_X X = 0$, and hence

$$\langle B(0)\xi, \xi\rangle = \xi(X(\frac{1}{2}r^2)) - \nabla_\xi X(\frac{1}{2}r^2) = \frac{1}{2}\xi(X(r^2)) = \langle \xi, \xi\rangle.$$

As this holds for all ξ normal to Σ it follows that $B(0) = P_{\boldsymbol{\nu}}$, as required.

To show that $B'(0) = S_{\dot\gamma}$, we use the Riccati equation (2.2) for $t > 0$ to get

$$B'(t) = H_{\partial_r} + tH'_{\partial_r} = H_{\partial_r} - tH^2_{\partial_r} - tR_{\dot\gamma} = H_{\partial_r}[I - tH_{\partial_r}] + O(t),$$

where I is the identity on $T_{\gamma(t)}M$. To first order, $tH_{\partial_r} = P_{\boldsymbol{\nu}} - P_{\dot\gamma} + tB'(0) + O(t^2)$, so

$$B'(t) = H_{\partial_r}(P_\Sigma + P_{\dot\gamma}) - tH_{\partial_r}B'(0) + O(t) = H_{\partial_r}P_\Sigma + (P_{\dot\gamma} - P_{\boldsymbol{\nu}})B'(0) + O(t). \quad (2.5)$$

where P_Σ is the orthogonal projection onto $T_{\gamma(0)}\Sigma$. We thus have

$$B'(t) + (P_{\boldsymbol{\nu}} - P_{\dot\gamma})B'(0) = H_{\partial_r}P_\Sigma + O(t).$$

Letting $t \to 0$ we see that $H_{\partial_r}P_\Sigma$ extends smoothly to $t = 0$ and that $B'(0) = B'(0)P_\Sigma$. Since $B'(0)$ is self-adjoint, it follows that $B'(0) = P_\Sigma B'(0)P_\Sigma$ and substituting this in the equation above we see that $B'(t) = H_{\partial_r}P_\Sigma + O(t)$. It is now easily checked that $H_{\partial_r}P_\Sigma \to S_{\dot\gamma}$ as $t \to 0$ and hence $B'(0) = S_{\dot\gamma}$, completing the proof of (2.4).

□

2.2.3 Mean curvature and volume

We have now seen that the gradient of a distance function is a geodesic vector field and since it has unit length, the gradient of a distance function has the special property that its total covariant derivative also describes the extrinsic geometry of its level sets.

In this section we take the dual point of view and study the vector fields which commute with the gradient of a distance function. Distance functions are distinguished in this regard by the fact that the volume element of its level sets has a very simple relationship to the mean curvature of the level sets.

Proposition 2.16. *Let r be a distance function and let $(x^1, ..., x^{n-1}, r)$ be any local coordinates on a subset Ω of its regular set such that $(x^1, ..., x^{n-1})$ restrict to coordinates on the level hypersurfaces $\Sigma_t = r^{-1}(t)$. The density of the volume element $\mathcal{J} = \sqrt{\det g}$ of Σ_t in these coordinates and the mean curvature h of Σ_t define functions on Ω satisfying*

$$\partial_r \log(\mathcal{J}) = h(\partial_r).$$

Proof. The coordinate vector fields $\partial_1, ..., \partial_{n-1}, \partial_r$ are all ∂_r-Jacobi fields and since ∂_r is a unit vector orthogonal to ∂_i we have

$$\mathcal{J} = |\partial_1 \wedge ... \wedge \partial_{n-1}| = |\partial_1 \wedge ... \wedge \partial_{n-1} \wedge \partial_r|.$$

Now apply Corollary 2.8 together with Equation (2.3). □

Having developed some of the basic properties of general distance functions, we now develop further the theory of distance functions of the form $r(x) = d(x, \Sigma)$ where Σ is a submanifold of M. A useful tool for studying these distance functions is given by the exponential map of M restricted to the normal bundle of Σ.

2.3 The normal exponential map of a submanifold

Let Σ be an m-dimensional submanifold of a complete Riemannian manifold M. Let $\boldsymbol{\nu} = \boldsymbol{\nu}(\Sigma)$ denote the normal bundle of Σ in M with projection $\pi : \boldsymbol{\nu} \to \Sigma$. Denote the fiber over $x \in \Sigma$ by $\boldsymbol{\nu}_x$ and let $\hat{\boldsymbol{\nu}}$ denote the unit normal bundle. In this section, $r(x) = d(x, \Sigma)$ shall denote the distance to Σ.

The *normal exponential map* of Σ is the map $\exp_{\boldsymbol{\nu}} : \boldsymbol{\nu} \to M$ defined as the restriction of the exponential map of M to $\boldsymbol{\nu}$.

Definition 2.17. The *segment domain* of Σ is the subset $\mathrm{seg}(\Sigma) \subset \boldsymbol{\nu}$ defined by

$$\mathrm{seg}(\Sigma) = \{\nu \in \boldsymbol{\nu} : r(\exp_{\boldsymbol{\nu}}(\nu)) = |\nu|\}.$$

Proposition 2.18. *The interior of the segment domain, denoted* $\mathrm{seg}^0(\Sigma)$*, is given by*

$$\mathrm{seg}^0(\Sigma) = \{\nu \in \boldsymbol{\nu} : r(\exp_{\boldsymbol{\nu}}((1+\epsilon)\nu)) = (1+\epsilon)|\nu| \text{ for some } \epsilon > 0\}.$$

The normal exponential map restricts to a diffeomorphism from $\mathrm{seg}^0(\Sigma)$ *onto its image and the complement* $M \setminus \exp_{\boldsymbol{\nu}}(\mathrm{seg}^0(\Sigma))$ *has measure zero.*

The proof in the case that Σ is a point is standard (see e.g. [29]). The submanifold case can be found in [24].

The importance of this proposition is that it says that we can realize M, up to a set of measure zero, as the diffeomorphic image of an open neighborhood $\mathrm{seg}^0(\Sigma)$ of the normal bundle $\boldsymbol{\nu}$ which is star-shaped with respect to the zero-section of $\boldsymbol{\nu}$. This can be useful, for example, by giving us a way to put local coordinates on M of the type described in Proposition 2.2.4 with respect to the distance function r.

However, one can go further. In particular, we would like to use the exponential map to relate the geometry of M to the geometry of Σ even far away from Σ using the radial curvature R_{∂_r}. In order to do this, we introduce a canonical metric on $\boldsymbol{\nu}$ for which Σ is isometric to the zero section and the projection π is a Riemannian submersion.

2.3.1 The canonical metric on the normal bundle

The Riemannian connection on M induces a connection $\nabla^{\boldsymbol{\nu}}$ on the normal bundle defined for all smooth vector fields X tangent to Σ and sections ν of $\boldsymbol{\nu}$ by

$$(\nabla^{\boldsymbol{\nu}}_X \nu) = (\nabla_X \nu)^{\perp}$$

where $^{\perp}$ denotes orthogonal projection of T_xM onto $\boldsymbol{\nu}_x$. Viewing $\boldsymbol{\nu}$ as a smooth manifold, we can define a metric on $\boldsymbol{\nu}$ determined by

$$|\dot{\nu}|^2 = |\dot{x}|^2 + |D^{\boldsymbol{\nu}}_t \nu|^2$$

where $\nu(t)$ is any smooth curve in $\boldsymbol{\nu}$, $x(t) = \pi(\nu(t))$, and $D_t^{\boldsymbol{\nu}}$ is the covariant derivative along the base curve $x(t)$ induced by the normal connection.

This metric can be understood as follows. For any point x in Σ and any small star-shaped neighborhood U of x in Σ one can identify all of the fibers $\boldsymbol{\nu}_u$ over $u \in U$ with $\boldsymbol{\nu}_x$ by parallel translation along radial geodesics in Σ using the normal connection. One can thus obtain a map $\Phi_x : \pi^{-1}(U) \to \mathbb{R}^{n-m}$ by choosing any orthonormal basis of $\boldsymbol{\nu}_x$. The metric described above has the property that $\pi : \boldsymbol{\nu} \to \Sigma$ is a Riemannian submersion and all such maps $\Phi_x : \boldsymbol{\nu} \to \mathbb{R}^{n-m}$ restict to an isometry on each fiber $\boldsymbol{\nu}_u$ of π and the fibers of Φ_x are orthogonal to $\boldsymbol{\nu}_x$. In particular, since π is a Riemannian submersion one can apply Fubini's theorem to integrate over $\boldsymbol{\nu}$ by first integrating over each fiber and then integrating over the base Σ.

Using this metric we can understand the geometry of M in terms of the much simpler geometry of $\boldsymbol{\nu}$ together with the geometric properties of the normal exponential map.

2.3.2 The Jacobian determinant of the normal exponential map and the polar volume density

In order to relate the volume element in M to that in $\boldsymbol{\nu}$, we can compute the Jacobian determinant of the exponential map $|(d\exp_{\boldsymbol{\nu}})_\nu|$. In fact, the differential of the normal exponential map is related in a very natural way to the ∂_r-Jacobi fields where r is the distance function to Σ.

Fix a unit-speed geodesic $\gamma(t) = \exp_{\boldsymbol{\nu}}(t\xi)$ with $\gamma(0) = x$ in Σ and $\dot{\gamma}(0) = \xi$ in $\hat{\boldsymbol{\nu}}$. Let $x^1, ..., x^m$ denote normal coordinates for the zero-section σ of $\boldsymbol{\nu}$ centered at $\sigma(x)$ and put spherical coordinates $\theta^1, ..., \theta^{n-m-1}, t$ on the fibers of $\boldsymbol{\nu}$ near $\boldsymbol{\nu}_x$ via the map Φ_x described in the previous section, with the angular coordinates oriented so that the coordinate singularities in $\boldsymbol{\nu}_x$ are at vectors orthogonal to ξ. Here, the coordinate t is the radius with respect to the metric of M on the fibers of $\boldsymbol{\nu}$. On the interior of the segment domain $\text{seg}^0(\Sigma)$ these can be pushed forward to give local coordinates $x_*^1, ..., x_*^m, \theta_*^1, ..., \theta_*^{n-m-1}, t_*$ on a neighborhood of γ in $M \cap \exp_{\boldsymbol{\nu}}(\text{seg}^0(\Sigma))$ where the t-coordinate becomes the distance $t_*(p) = t(\exp^{-1}(p)) = r(p)$.

Let $Y_1(t), ..., Y_m(t)$ denote the coordinate vector fields for the coordinates $x_*^1, ..., x_*^m$ at $\gamma(t)$, and let $J_1(t), ..., J_{n-m-1}(t)$ denote the coordinate vector fields corresponding to the coordinates $\theta_*^1, ..., \theta_*^{n-m-1}$ at $\gamma(t)$. Since all of these coordinate vector fields commute with the coordinate vector field ∂_r, they are all ∂_r-Jacobi fields. Moreover, they are the push forward of the corresponding coordinate vector fields on $\boldsymbol{\nu}$ via the exponential map

$$Y_i(t) = (d\exp_{\boldsymbol{\nu}})_{t\xi}\left(\frac{\partial}{\partial x^i}\right), \qquad J_i(t) = (d\exp_{\boldsymbol{\nu}})_{t\xi}\left(\frac{\partial}{\partial \theta^i}\right), \qquad \partial_r = (d\exp_{\boldsymbol{\nu}})\left(\frac{\partial}{\partial t}\right).$$

In particular, we can compute the Jacobian determinant of the exponential map as

$$|(d\exp_{\boldsymbol{\nu}})_{t\xi}| = \frac{|Y_1(t) \wedge ... \wedge Y_m(t) \wedge J_1(t) \wedge ... \wedge J_{n-m-1}(t) \wedge \partial_r|}{|\frac{\partial}{\partial x^1} \wedge ... \wedge \frac{\partial}{\partial x^m} \wedge \frac{\partial}{\partial \theta^1} \wedge ... \wedge \frac{\partial}{\partial \theta^{n-m-1}} \wedge \partial_t|}.$$

By our choice of spherical coordinates the denominator is just t^{n-m-1}, and since ∂_r is a unit orthogonal vector to the surfaces of constant r we have

$$|(d\exp_{\boldsymbol{\nu}})_{t\xi}| = \frac{|Y_1(t) \wedge ... \wedge Y_m(t) \wedge J_1(t) \wedge ... \wedge J_{n-m-1}(t)|}{t^{n-m-1}}.$$

Recalling Proposition 2.16, we are motivated to introduce the following function on $\boldsymbol{\nu}$.

Definition 2.19. The *polar volume density* of M with respect to Σ is the function $\mathcal{A} : [0, \infty) \times \hat{\boldsymbol{\nu}} \to \mathbb{R}$ defined by

$$\mathcal{A}(t, \xi) = |Y_1(t) \wedge ... \wedge Y_m(t) \wedge J_1(t) \wedge ... \wedge J_{n-m-1}(t)|$$

where $Y_i(t), J_i(t)$ are as defined as above for $t\xi$ in $\mathrm{seg}^0(\Sigma)$, with $\mathcal{A} \equiv 0$ otherwise.

The polar volume density is thus related to the Jacobian determinant of the exponential map via

$$\mathcal{A}(t,\xi) = |(d\exp_{\boldsymbol{\nu}})_{t\xi}|t^{n-m-1}. \tag{2.6}$$

In particular, the volume of the tube of radius R can be written

$$\mathrm{vol}(T(\Sigma,R)) = \int_{\{\nu\in\mathrm{seg}^0(\Sigma):|\nu|<R\}} |(d\exp_{\boldsymbol{\nu}})_\nu| d\,\mathrm{vol}_{\boldsymbol{\nu}} = \int_0^R \int_{\hat{\boldsymbol{\nu}}} \mathcal{A}(t,\xi) d\xi dt \tag{2.7}$$

where $d\xi = d\,\mathrm{vol}_{\hat{\boldsymbol{\nu}}}$ is the volume element of $\hat{\boldsymbol{\nu}}$.

Remark 2.20. A note on the terminology "polar volume density." Observe that the polar volume density $\mathcal{A}(t,\xi)$ is indeed the density of the volume element of M written in polar coordinates around Σ where the spherical coordinates are oriented according to ξ so that all of the factors involving sines of the angles θ^i become unity. Notice, however, that the coordinate system which defines the density $\mathcal{A}$ depends on the direction of ξ; there is not one single choice of coordinates for which $\mathcal{A}(t,\xi)$ is the density of the volume element of M at all points $\exp_{\boldsymbol{\nu}}(t\xi)$. By defining the polar volume density this way, we hide all of the factors involving the Jacobian determinant for spherical coordinates in the volume element $d\,\mathrm{vol}_{\hat{\boldsymbol{\nu}}}$ of the unit normal bundle.

Proposition 2.21. *Let Σ^m be an m-dimensional submanifold of a complete Riemannian manifold M^n and let $r(x) = d(x,\Sigma)$ be the distance function to Σ. The polar volume density of Σ satisfies*

$$\lim_{t\to 0} \frac{\mathcal{A}(t,\xi)}{t^{n-m-1}} = 1$$

and

$$\mathcal{A}'(t,\xi) = h(t,\xi)\mathcal{A}(t,\xi) \tag{2.8}$$

for $t\xi$ *in* $\mathrm{seg}^0(\Sigma)$ *where* $h(t,\xi) = \mathrm{tr}(H_{\partial_r})$ *evaluated at* $\exp(t\xi)$.

Proof. Since the normal exponential map is an isometry on the zero-section of $\boldsymbol{\nu}$, the Jacobian determinant of the exponential map satisfies $|(d\exp_{\boldsymbol{\nu}})_{0\xi}| = 1$, from which the first equation follows immediately. The second equation is a direct application of Proposition 2.16. □

2.4 The symplectic vector space of Jacobi fields

In this section we briefly review another useful formalism for treating families of geodesics. Sometimes it is useful to consider families of geodesics which may intersect, and so cannot be fully described by a geodesic vector field as above. Although most comparison theorems only hold on domains free of such intersections which can be described by the notions above, it is still useful to have a formalism which can fully describe such families of geodesics, which we now review. The main references for this section are [14] and [17].

Definition 2.22. Given a geodesic $\gamma : \mathbb{R} \to M$ a *Jacobi field* along γ is a vector field $J(t)$ along γ such that $J''(t) = -R_{\dot\gamma}(J)$.

The set of Jacobi fields along γ forms a real $2n$-dimensional symplectic vector space $\mathfrak{J}(\gamma)$ with symplectic form ω given by

$$\omega(J_1, J_2) = \langle J_1'(t), J_2(t)\rangle - \langle J_1(t), J_2'(t)\rangle$$

where the value on the right is independent of the choice of t by the Jacobi equation and the symmetries of the curvature R. The set $\mathfrak{J}^\perp(\gamma)$ of normal Jacobi fields $J(t)$ along γ satisfying $\langle J(t), \dot\gamma\rangle \equiv 0$ forms a $(2n-2)$-dimensional symplectic subspace of $\mathfrak{J}(\gamma)$.

As a first step in relating this to the notions introduced above, we have

Definition 2.23. A k-parameter family of geodesics is a smooth map $\Gamma : (-\epsilon, \epsilon)^k \times \mathbb{R} \to M$ such that the curves $\Gamma(s^1, ..., s^k, t) = \gamma_{s^1,...,s^k}(t)$ are geodesics. A k-parameter variation of γ through geodesics is a k-parameter family Γ such that $\Gamma(\mathbf{0}, t) = \gamma(t)$.

Proposition 2.24. *Given any k-dimensional subspace $\mathcal{V}$ of $\mathfrak{J}(\gamma)$ there exists a k-parameter variation of γ through geodesics such that the vector fields*

$$J_i(t) = \frac{\partial \Gamma}{\partial s^i}(\mathbf{0}, t)$$

span $\mathcal{V}$.

Proof. Let $J_1, ..., J_k$ be a basis of $\mathcal{V}$. Any solution $J_i(t)$ of the Jacobi equation is uniquely determined by $q_i = J_i(0)$ and $p_i = J_i'(0)$. First define a smooth map $\Phi : (-\epsilon, \epsilon)^k \to M$ by

$$\Phi(s^1, ..., s^k) = \exp_{\gamma(0)}(s^i q_i)$$

where we use the Einstein summation convention over the indices $i = 1, ..., k$. Next, let σ denote the radial line segment in $(-\epsilon, \epsilon)^k$ from $(0, ..., 0)$ to $(s^1, ..., s^k)$ and define vector fields $u(s^1, ..., s^k)$ and $v_i(s^1, ..., s^k)$ to be the parallel transports of $\dot{\gamma}$ and p_i along the curve $\Phi(\sigma)$, respectively.

Finally, we construct the variation

$$\Gamma(s^1, ..., s^k, t) = \exp_{\Phi(s^1,...,s^k)}(t[u(s^1, ..., s^k) + s^i v_i(s^1, ..., s^k)]).$$

□

Corresponding to the irrotational geodesic vector fields (distance functions) are the Lagrangian subspaces of $\mathfrak{J}(\gamma)$ defined by $\omega \equiv 0$. More generally, we will consider the isotropic subspaces. Given any isotropic subspace $\mathcal{V} \subset \mathfrak{J}^{\perp}(\gamma)$ of dimension $\dim \mathcal{V} = k$

define a family of subspaces $\mathcal{V}_t \subset T_{\gamma(t)}M$ by

$$\mathcal{V}_t = \{J(t) : J \in \mathcal{V}\} \oplus \{J'(t) : J \in \mathcal{V}, J(t) = 0\}. \tag{2.9}$$

It is easily shown that $\dim \mathcal{V}_t = \dim \mathcal{V} = k$ and that the sum is indeed direct with the second summand vanishing for almost all t (see e.g. [14]). We shall call any point $\gamma(t)$ at which the second summand does not vanish a *focal point* of $\mathcal{V}$. Define $\mathcal{V}_t^\perp$ as the orthogonal complement of $\mathcal{V}_t$ in $\dot{\gamma}(t)^\perp$ so that

$$\dot{\gamma}(t)^\perp = \mathcal{V}_t \oplus \mathcal{V}_t^\perp.$$

Given any interval $(0, t_0)$ for which $\mathcal{V}$ has no focal points there is a well-defined *tangential Riccati operator* $\bar{S}_t : \mathcal{V}_t \to \mathcal{V}_t$ associated to $\mathcal{V}$ defined by

$$\bar{S}_t(u) = P_t(J_u'(t))$$

where $J_u \in \mathcal{V}$ is the (unique) Jacobi field such that $J_u(t) = u$ and $P_t : \dot{\gamma}(t)^\perp \to \mathcal{V}_t$ is the orthogonal projection (if $\mathcal{V}$ is Lagrangian the projection may be omitted). Similarly, we also have a one parameter family of linear maps $A_t : \mathcal{V}_t \to \mathcal{V}_t^\perp$ defined by

$$A_t(u) = Q_t(J_u'(t)) \tag{2.10}$$

where Q_t is the orthogonal projection $\dot{\gamma}(t)^\perp \to \mathcal{V}_t^\perp$. We shall refer to the map A_t as the A-tensor of $\mathcal{V}$; it is also sometimes known as *Wilking's A-tensor.*

Furthermore, if Λ is any Lagrangian extension of the isotropic subspace $\mathcal{V}$ then on any interval $(0, t_0)$ on which Λ has no focal points there exists a well-defined *transversal*

Riccati operator $\hat{S}_t : \mathcal{V}_t^\perp \to \mathcal{V}_t^\perp$ associated to the pair $(\mathcal{V}, \Lambda)$ on $(0, t_0)$, defined by

$$\hat{S}_t(u) = Q_t(J_u'(t))$$

where $J_u(t) \in \Lambda$ is the unique Jacobi field such that $J_u(t) = u$.

In the case that $\dim \mathcal{V} = n - 1$, i.e. $\mathcal{V} = \Lambda$ is itself a Lagrangian subspace, only the tangential Riccati operator $\bar{S}$ is non-trivial, and is given on any interval without focal points simply by

$$S(v) = J_v'.$$

In this case we shall omit the overbar notation and simply refer to S as the *Riccati operator* of Λ associated to an interval $(0, t_0)$. Note that since the Riccati operator of a Lagrangian subspace is defined on the parallel subspace $\dot{\gamma}^\perp$, the covariant derivative of S along γ is well-defined.

Proposition 2.25. *The Riccati operator of a Lagrangian subspace of Jacobi fields satisfies the Riccati equation*

$$S' + S^2 = -R_{\dot{\gamma}} \tag{2.11}$$

on any interval $(0, t_0)$ *on which* Λ *has no focal points.*

Proof. Since Λ has no focal points, for each vector $v \in \dot{\gamma}(t)^\perp$ there exists a unique Jacobi field $J_v \in \Lambda$ such that $J_v(t) = v$. We thus have

$$S'(v) = S'(J_v) = S(J_v)' - S(J_v') = J_v'' - S^2(J_v) = -R_{\dot{\gamma}}(v) - S^2(v).$$

□

We now establish the connection to the previous sections. First, we note that the existence and uniqueness theorem for ordinary differential equations applied to the Jacobi equation implies that the vector space of Jacobi fields along a geodesic on some interval $(0, t_0)$ is naturally isomorphic to the vector space on any larger interval.

Proposition 2.26. *Given a geodesic vector field V on an open subset $\Omega \subset M$ and a geodesic $\gamma : (a, b) \to \Omega$ with $\dot{\gamma} = V$ the set of V-Jacobi fields along γ forms an n-dimensional subspace $\mathcal{V} \subset \mathfrak{J}(\gamma)$. Moreover, V is irrotational along γ if and only if $\mathcal{V}$ is a Lagrangian subspace of $\mathfrak{J}(\gamma)$.*

Similarly, the space of normal V-Jacobi fields along γ forms an $(n-1)$-dimensional subspace of $\mathfrak{J}^{\perp}(\gamma)$ which is Lagrangian if and only if V is irrotational along γ.

Example 2.27. Let $\gamma : [0, t_0) \to M$ be a geodesic and let $r(x) = d(x, \gamma(0))$ be the distance function from $\gamma(0)$ with gradient ∂_r. The Lagrangian subspace Λ of $\mathfrak{J}^{\perp}(\gamma)$ consisting of normal Jacobi fields with $J(0) = 0$ is precisely the vector space of normal ∂_r-Jacobi fields along γ.

2.5 Comparison theorems for sectional and Ricci curvature

In this section, we recall some of the basic comparison theorems in Riemannian geometry. First, we recall the Hessian comparison theorem for sectional curvature bounds and the Laplacian comparison for Ricci curvature bounds, which are comparison theorems for the Hessian of the distance function to a point. We then discuss the comparison theorem of Heintze-Karcher which is a generalization of these familiar results to include distance functions to a submanifold. One of our main goals in Chapter 4 is a generalization of the latter to the setting of intermediate and integral Ricci curvature bounds.

2.5.1 Sectional curvature comparison

One of the earliest comparison theorems in Riemannian geometry is a result of Rauch bounding the length of a ∂_r-Jacobi field in terms of a lower sectional curvature bound where r is the distance function from a point. There is a more modern formulation of this theorem, known as the Hessian comparison theorem, which is stated explicitly in terms of the distance function.

Theorem 2.28 (Hessian comparison). *Let $M^{n+m}, \bar{M}^n$ be complete Riemannian manifolds and let $r(x)$ and $\bar{r}(x)$ denote the distance functions from points $p \in M$ and $\bar{p} \in \bar{M}$, respectively. Let γ, $\bar{\gamma} : [0, t_0) \to M, \bar{M}$ be any geodesics with $r(\gamma(t)) = \bar{r}(\bar{\gamma}(t)) = t$.*

If $\sec(\dot{\gamma}(t), \cdot) \geq \sec(\dot{\bar{\gamma}}(t), \cdot)$ then for all unit vectors $u, \bar{u}$ orthogonal to $\dot{\gamma}(t), \dot{\bar{\gamma}}(t)$ we have

$$\nabla^2 r(u, u) \leq \bar{\nabla}^2 \bar{r}(\bar{u}, \bar{u}).$$

A proof can be found in [29], but we will prove a generalized version of the case where $\bar{M}$ is a space of constant curvature in Chapter 3. The dual point of view gives the following theorem of Rauch bounding the length of ∂_r-Jacobi fields for the distance function to a point.

Theorem 2.29 (Rauch). *Let $\gamma : [0, t_0] \to M^{n+m}$ and $\bar{\gamma} : [0, t_0] \to \bar{M}^n$ be unit speed geodesics. Let J and $\bar{J}$ be normal Jacobi fields along γ and $\bar{\gamma}$, respectively, such that*

$$J(0) = \bar{J}(0) = 0$$
$$|J'(0)| = |\bar{J}'(0)|$$

If γ has no conjugate points on $(0, t_0]$ and $\sec(\dot{\gamma}(t), \cdot) \geq \sec(\dot{\bar{\gamma}}(t), \cdot)$ then

$$|J(t)| \leq |\bar{J}(t)|.$$

We note that both of these comparisons are of local differential quantities, Jacobi fields and the Hessian of a function. However, one can, in a sense, integrate these inequalities to obtain global comparison theorems. The primary example of this is Toponogov's triangle comparison theorem. This global comparison theorem can be obtained either through a rather technical argument using the Rauch comparison or via a much simplified proof using a weak version of the Hessian comparison for modified distance functions (see e.g. [29]).

2.5.2 Ricci curvature comparison

There is an analagous comparison theorem which comes from averaging the sectional curvatures over all planes containing the radial vector field ∂_r. In this case, one obtains a comparison for the $(n-1)$-dimensional volume element spanned by ∂_r-Jacobi fields. The modern formulation is given in terms of the Laplacian of the distance function. In contrast to the sectional curvature case, the averaging of the sectional curvatures requires that the comparison be made against a model space of constant sectional curvature K.

Theorem 2.30 (Laplacian comparison). *Let M be a complete Riemannian manifold and let $r(x) = d(x,p)$ denote the distance from a point $p \in M$. Let $\gamma : (0, t_0) \to M$ be any geodesic with $r(\gamma(t)) = t$.*

If $Ric(\dot\gamma, \dot\gamma) \geq (n-1)K$ for some constant K then

$$(\Delta r)(\gamma(t)) \leq (n-1)\log(\mathrm{sn}_K)'(t).$$

In [4], Bishop-Crittenden proved the following related Jacobi field comparison for ∂_r-Jacobi fields for the distance function to a point based on a k-Ricci curvature bound.

Theorem 2.31 (Jacobi field comparison I). *Let $\gamma : [0, t_0] \to M^n$ be a unit speed geodesic* *without conjugate points on* $[0, t_0)$*. Let* $J_1, ..., J_k$ *be normal Jacobi fields along* γ *with initial conditions* $J_i(0) = 0$ *and* $\langle J_i'(0), J_j'(0)\rangle = \delta_{ij}$*.*

If $Ric_k(\dot{\gamma}, \cdot) \geq K$ *for any constant* K*, then*

$$|J_1(t) \wedge ... \wedge J_k(t)| \leq \mathrm{sn}_K^k(t).$$

Again, these comparisons are of local differential quantities. Just as the Hessian comparison theorem can be integrated to obtain the global Toponogov triangle comparison, the Laplacian comparison theorem can be integrated to obtain the global Bishop-Gromov volume comparison (see e.g. [29]).

2.5.3 The Heintze-Karcher comparison theorem

All of the comparison theorems above are comparison theorems for V-Jacobi fields where the geodesic vector field V is taken to be the gradient of the distance function to a point. The Heintze-Karcher comparison theorem generalizes these theorems to a comparison theorem for the distance functions $r(x) = d(x, \Sigma)$ to any submanifold Σ.

The first part of the Heintze-Karcher comparison was a Jacobi field comparison for families of ∂_r-Jacobi fields for distance functions to a hypersurface Σ based on a sectional curvature bound.

Theorem 2.32 (Jacobi field comparison II)**.** *Let* Σ^{n-1} *be an immersed hypersurface in* M^n*, let* $\xi \in \hat{\boldsymbol{\nu}}_x(\Sigma)$ *be a unit normal vector, and put* $\gamma(t) = \exp_x(t\xi)$*. Let* $J_1, ..., J_k$ *be any normal Jacobi fields along* γ *such that* $J_i'(0) = S_\xi(J_i(0))$ *and* $|J_1(0) \wedge ... \wedge J_k(0)| = 1$*.*

If $\sec(\dot{\gamma}, \cdot) \geq K$ *and* Σ *has no focal points on* $[0, t_0)$ *then for* $0 \leq t \leq t_0$

$$|J_1(t) \wedge ... \wedge J_k(t)| \leq (\mathrm{cs}_K(t) + \lambda\, \mathrm{sn}_K(t))^k$$

where λ is the largest eigenvalue of S_ξ.

The second part of the Heintze-Karcher comparison can be stated in modern form as a Laplacian comparison for distance functions from a submanifold of any dimension. A modern treatment of the following theorem is given by W. Ballmann in [3].

Theorem 2.33 (Laplacian comparison for submanifolds)**.** *Let Σ^m be a submanifold of a complete Riemannian manifold M^n and let $r(x) = d(x, \Sigma)$ denote the distance from Σ. Let $\gamma : (0, t_0) \to M$ be any geodesic with $r(\gamma(t)) = t$.*

If $\sec(\dot\gamma, \cdot) \geq K$ or $m = 0, n-1$ and $Ric(\dot\gamma, \dot\gamma) \geq (n-1)K$ then

$$(\Delta r)(\gamma(t)) \leq (n - m - 1)\log(\mathrm{sn}_K)'(t) + m\log(\mathrm{cs}_K - \langle \eta, \dot\gamma(0)\rangle \,\mathrm{sn}_K)'(t),$$

where η is the mean curvature vector of Σ.

Notice that when the dimension m of the submanifold Σ is between $0 < m < n-1$ a sectional curvature bound is used to obtain the Laplacian comparison. The fundamental reason for this is the initial conditions for the shape operator along geodesics which leave Σ orthogonally split into an m-dimensional subspace with finite eigenvalues (principle curvatures of Σ) and an $(n-m-1)$-dimensional subspace where the eigenvalues are asymptotic to $1/r$ (recall Proposition 2.15). Thus, averaging the sectional curvatures over an $(n-1)$-dimensional subspace one cannot take advantage of the finite initial conditions. This observation suggests, however, that a bound on certain partial averages of sectional curvatures might be sufficient in place of a sectional curvature bound. In Chapter 3 it is shown that this is indeed the case.

Before proceeding to this generalization, we continue the pattern established in the previous sections by noting that the local version of the Heintze-Karcher comparison can be integrated to obtain a global volume comparison [18].

Theorem 2.34 (Heintze-Karcher volume comparison)**.** *Let M^n be a complete Riemannian manifold and let Σ^m be a closed m-dimensional submanifold. If* $\sec \geq K$*, or* $m = 0, n-1$ *and* $Ric \geq (n-1)K$*, then*

$$\mathrm{vol}(T(\Sigma, R)) \leq \int_{\hat{\boldsymbol{\nu}}} \int_0^{f(R,\xi)} (\mathrm{cs}_K(t) - \langle \eta, \xi \rangle \, \mathrm{sn}_K(t))^m \, \mathrm{sn}_K(t)^{n-m-1} dt d\xi.$$

where $f(R, \xi)$ denotes the minimum of R and the first zero of the integrand.

Chapter 3

Comparison theorems based on intermediate Ricci curvature

In this chapter, we develop the comparison geometry of k-Ricci curvature bounds. The motivating problem concerns the volume comparison for tubes around submanifolds given in Section 3.4. The ideas developed in this Chapter will also be useful in understanding the proof of the volume estimate for tubes around submanifolds using integral curvature bounds given in Chapter 4.

In Section 3.1 we prove a Hessian comparison theorem in the context of abstract Jacobi fields as introduced in Section 2.4. We show how this comparison immediately yields a generalization of Jacobi field comparisons I and II as stated above (Theorems 2.31 and 2.32). In Section 3.2 we then study in more detail the evolution of a k-dimensional volume element under the flow of a geodesic congruence and give a unified context for some recent work of Ketterer-Mondino and Guijarro-Wilhelm on k-Ricci curvature bounds.

In Section 3.3 we combine the abstract Hessian comparison with the Taylor series expansion of Section 2.2.2 to obtain a Hessian comparison theorem for the distance function to a submanifold based on k-Ricci curvature bounds. This yields a direct generalization of the Heintze-Karcher volume comparison based on k-Ricci curvature bounds.

3.1 Jacobi field comparison theorems for k-Ricci curvature

In this section, we prove a general Jacobi field comparison theorem for k-dimensional families of Jacobi fields based on k-Ricci curvature bounds. The theorem is a consequence of the following comparison theorem for the Riccati operator of a Lagrangian subspace of Jacobi fields.

Proposition 3.1 (Abstract Hessian comparison)**.** *Let $\gamma : [0, t_0] \to M^n$ be a geodesic and let $\Lambda \subset \mathfrak{J}^{\perp}(\gamma)$ be a Lagrangian subspace of normal Jacobi fields with no focal points on $(0, t_0)$.*

If $\mathcal{W}_t \subset \dot{\gamma}(t)^{\perp}$ is any family of k-dimensional subspaces which are parallel along γ and $Ric_k(\dot{\gamma}, \mathcal{W}_t) \geq K$ then the Riccati operator S_t of Λ on $(0, t_0)$ satisfies

$$\operatorname{tr}_{\mathcal{W}_t}(S_t) \leq \begin{cases} k \log(\operatorname{cs}_K + (w_0/k) \operatorname{sn}_K)'(t) & \text{if } \mathcal{W}_0 \subset \{J(0) : J \in \Lambda\} \\ k \log(\operatorname{sn}_K)'(t) & \text{otherwise} \end{cases} \tag{3.1}$$

where $w_0 = \sum_{i=1}^{k} \langle J_i'(0), J_i(0) \rangle$ and $J_i(0)$ is any orthonormal basis of $\mathcal{W}_0$ with $J_i \in \Lambda$. If equality holds at t then equality holds on $(0, t]$ and

1. *K is an eigenvalue of $R_{\dot{\gamma}}$ with $\mathcal{W}_t$ contained in the corresponding eigenspace,*
2. *$\mathcal{W}_t^{\perp}$ is an invariant subspace of $R_{\dot{\gamma}}$.*

Proof. Using the fact that $\mathcal{W}_t$ is parallel along γ, we may choose a parallel orthonormal frame $\{e_1, ..., e_{n-1}, \dot{\gamma}\}$ along γ such that $\{e_1, ..., e_k\}$ form a parallel orthonormal basis of

$\mathcal{W}_t$. The Riccati operator S_t of Λ satisfies the Riccati equation (2.11), and since $\mathcal{W}_t$ is parallel along γ the projected trace $\mathrm{tr}_{\mathcal{W}_t}$ commutes with the covariant derivative along γ and hence

$$\mathrm{tr}_{\mathcal{W}_t}(S_t)' + \mathrm{tr}_{\mathcal{W}_t}(S_t^2) = -Ric_k(\dot{\gamma}, \mathcal{W}_t).$$

Putting $s_{ij} = \langle S_t e_i, e_j \rangle$ and noting that $s_{ij} = s_{ji}$ since S_t is self-adjoint we have

$$\mathrm{tr}_{\mathcal{W}_t}(S_t^2) = \sum_{i=1}^{k} \sum_{j=1}^{n-1} s_{ij}^2 \geq \sum_{i,j=1}^{k} s_{ij}^2 \geq \sum_{i=1}^{k} s_{ii}^2 \geq \frac{1}{k} \left(\sum_{i=1}^{k} s_{ii} \right)^2 = \frac{1}{k} \mathrm{tr}_{\mathcal{W}_t}(S_t)^2$$

where the last inequality follows from the Cauchy-Schwarz inequality. Putting $w(t) = \mathrm{tr}_{\mathcal{W}_t}(S_t)/k$ we have

$$w'(t) + w(t)^2 \leq -Ric_k(\dot{\gamma}, \mathcal{W}_t) \leq -K. \tag{3.2}$$

Noting that the model functions on the right hand side of Equation (3.1) satisfy the scalar Riccati equation, we may apply the comparison theory for this equation developed in Section 2.1.1 provided we match the initial conditions. If $\mathcal{W}_0 \subset \{J(0) : J \in \Lambda\}$ then $w(t) \to w_0/k$ as $t \to 0$. In any case, since every Jacobi field J has the form $J(t) = J(0) + tJ'(0) + O(t^2)$ it follows that $\lim_{t\to 0} t\langle J'(t), J(t)\rangle / \langle J(t), J(t)\rangle \leq 1$ and hence $\limsup_{t\to 0} tw(t) \leq 1$. The inequality then follows from the comparison theory for the scalar Riccati equation given in Proposition 2.9.

If equality holds at $t_1 < t_0$, then the aforementioned Riccati comparison principle implies that equality holds on $(0, t_1]$. From the inequalities above, it follows that with respect to the parallel basis $\{e_i\}$ the matrix representation of S_t on $(0, t_1]$ is block diagonal of the form

$$\begin{pmatrix} w(t) I_k & 0 \\ 0 & * \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix. Since this decomposition holds on $(0, t_1]$, it follows that $R_{\dot\gamma}$ is also block diagonal of the same form, replacing $w(t)$ with $-w' - w^2 = K$. $\square$

This Hessian comparison now leads immediately to the following Jacobi field comparisons. First, we give the case of Jacobi fields with initial conditions $J_i(0) = 0$.

Proposition 3.2 (Jacobi field comparison I). *Let $\gamma : [0, t_0] \to M$ be a geodesic without conjugate points on $[0, t_0)$ and let $J_1, ..., J_k$ be Jacobi fields along γ with initial conditions $J_i(0) = 0$ and $\langle J_i'(0), J_j'(0)\rangle = \delta_{ij}$.*

If $Ric_k(\dot\gamma, \cdot) \geq K$ then for $0 \leq t \leq t_0$,

$$\frac{|J_1(t) \wedge ... \wedge J_k(t)|}{\mathrm{sn}_K^k(t)} \leq 1$$

and the ratio is non-increasing.

Remark 3.3. Note that if $\bar J_1, .., \bar J_k$ are Jacobi fields along a geodesic $\bar\gamma$ in a space form of constant sectional curvature K satisfying $\bar J_i(0) = 0$ and $\langle \bar J_i'(0), \bar J_j'(0)\rangle = \delta_{ij}$ then

$$|\bar J_1(t) \wedge ... \wedge \bar J_k(t)| = \mathrm{sn}_K^k(t).$$

This theorem strengthens Theorem 2.31 by showing that the ratio is non-increasing (this monotonicity is already well-known in the cases $k = 1$ and $k = n - 1$).

Proof. Let Λ be the Lagrangian extension of $\mathcal{V}$ consisting of Jacobi fields which vanish at $\gamma(0)$ and let S_t denote the Riccati operator of Λ on $(0, t_0)$. Using the abstract Hessian comparison of Proposition 3.1 we have

$$\log(|J_1 \wedge ... \wedge J_k|)'(t) = \mathrm{tr}_{\mathcal{V}_t}(S_t) \leq \log(\mathrm{sn}_K^k)'(t).$$

Noting that any positive differentiable functions $f_1, f_2 : (0, t_0) \to \mathbb{R}$ satisfy

$$\left(\frac{f_1}{f_2}\right)' = \frac{f_1}{f_2}[\log(f_1)' - \log(f_2)'],$$

the result follows since $|J_1 \wedge ... \wedge J_k| \sim t^k |J_1'(0) \wedge ... \wedge J_k'(0)|$ as $t \to 0$. □

Note that the comparison does not necessarily hold all the way to the first focal point of the subspace $\mathcal{V} = \text{span}\{J_1, ..., J_k\}$ but only holds up to the first conjugate point, i.e. to the first focal point of the Lagrangian extension of $\mathcal{V}$ consisting of all normal Jacobi fields which vanish at $\gamma(0)$. A similar restriction is required for non-zero initial conditions, where in place of the notion of conjugate points we use the following definition.

Definition 3.4. Let $\gamma : [0, t_0] \to M$ be a geodesic and let $\mathcal{V} \subset \mathfrak{J}^\perp(\gamma)$ be an isotropic subspace of normal Jacobi fields. We shall say that $\mathcal{V}$ is *supported by a Lagrangian* on $[0, t_0)$ if either $\mathcal{V}$ is a Lagrangian subspace without focal points on $(0, t_0)$ or the A-tensor of $\mathcal{V}$ vanishes at $\gamma(0)$ and there exists a Lagrangian extension Λ of $\mathcal{V}$ without focal points on $(0, t_0)$ such that

$$\limsup_{t \to 0} \lambda_{\max}(\hat{S}_t) \leq \liminf_{t \to 0} \lambda_{min}(\bar{S}_t)$$

where $\lambda_{max}, \lambda_{min}$ denote the largest and smallest eigenvalues, respectively.

First we note that an isotropic subspace $\mathcal{V}^k$ of normal Jacobi fields along a geodesic γ consisting of Jacobi fields which vanish at $\gamma(0)$ is supported by a Lagrangian on $[0, t_0)$ if and only if γ has no conjugate points on $[0, t_0)$.

On the other hand, if $\mathcal{V}^k$ is an isotropic subspace and $\gamma(0)$ is not a focal point then $\mathcal{V}$ is supported by a Lagrangian on some interval if and only if there exists a hypersurface orthogonal to $\gamma(0)$ whose shape operator restricts to $\bar{S}_0$ on $\mathcal{V}_0$ such that the sum of the largest k principle curvatures is no larger than the trace of $\bar{S}_0$, in which case $\mathcal{V}$ is supported up to the first focal point $\gamma(t_0)$ of the hypersurface.

Example 3.5. Suppose $J_1, ..., J_k$ are normal Jacobi fields along a geodesic γ such that $J_i(0)$ are linearly independent and $J_i'(0) = 0$. Let B_ϵ denote a ball in $\dot\gamma(0)^\perp$ of radius ϵ. Up to the first focal point $\gamma(t_0)$ of the hypersurface $\exp_{\gamma(0)}(B_\epsilon)$ in the direction $\dot\gamma$, the isotropic subspace $\mathcal{V}$ spanned by $J_1, ..., J_k$ is supported by the Lagrangian $\Lambda = \{J \in \mathfrak{J}^\perp(\gamma) : J_i'(0) = 0\}$ on $[0, t_0)$.

Proposition 3.6 (Jacobi field comparison II)**.** *Let $\gamma : [0, t_0] \to M$ be a geodesic and let $\mathcal{V}$ be an isotropic k-dimensional subspace of normal Jacobi fields which is supported by a Lagrangian on $[0, t_0)$ and assume that $\gamma(0)$ is not a focal point of $\mathcal{V}$.*

If $Ric_k(\dot\gamma, \cdot) \geq K$ then for any basis $J_1, ..., J_k$ of $\mathcal{V}$ with $\langle J_i(0), J_j(0)\rangle = \delta_{ij}$ and $v_0 = \sum_{i=1}^k \langle J_i'(0), J_i(0)\rangle$ we have for $0 \leq t \leq t_0$ that

$$\frac{|J_1(t) \wedge ... \wedge J_k(t)|}{(\mathrm{cs}_K(t) + (v_0/k)\,\mathrm{sn}_K(t))^k} \leq 1$$

and the ratio is non-increasing.

Remark 3.7. Note that if $\bar J_1, ..., \bar J_k$ are parallel Jacobi fields along a geodesic in a space form of constant curvature K with $\langle \bar J_i(0), \bar J_j(0)\rangle = \delta_{ij}$ and $|\bar J_i'(0)| = v_0/k$ then

$$|\bar J_1(t) \wedge ... \wedge \bar J_k(t)| = (\mathrm{cs}_K(t) + (v_0/k)\,\mathrm{sn}_K(t))^k.$$

Proof. Let Λ be a Lagrangian support for $\mathcal{V}$ on $[0, t_0)$, and let S_t denote the Riccati operator of Λ on $[0, t_0)$. Using the abstract Hessian comparison of Proposition 3.1 one has

$$\mathrm{tr}_{\mathcal{V}_t}(S_t) \leq k \log(\mathrm{cs}_K + (w_0/k)\,\mathrm{sn}_K)'(t).$$

where $w_0 = \mathrm{tr}_{\mathcal{W}_0}(S_0)$ and $\mathcal{W}_0$ is the subspace of $\dot\gamma(0)^\perp$ parallel to $\mathcal{V}_t$. By the definition of a Lagrangian support we have

$$w_0 = \mathrm{tr}_{\mathcal{W}_0}(S_0) \leq \mathrm{tr}_{\mathcal{V}_0}(S_0) = v_0$$

and hence

$$\log(|J_1 \wedge ... \wedge J_k|)'(t) = \mathrm{tr}_{\mathcal{V}_t}(S) \leq k \log(\mathrm{cs}_K + (v_0/k)\,\mathrm{sn}_K)'(t).$$

Noting that $|J_1(0) \wedge ... \wedge J_k(0)| = (\mathrm{cs}_K(0) + (v_0/k)\,\mathrm{sn}_K(0))^k$ the result follows as in the proof of Proposition 3.2. □

We conclude this section with a few remarks concerning the notion of a Lagrangian support. First, we note that the vanishing of the A-tensor required by the definition of a Lagrangian support is a necessary condition for the Jacobi field comparison above to hold on any interval $[0, \epsilon)$. For example, one may consider the Jacobi field $J(t) = \cos(t)E_1(t) + \sin(t)E_2(t)$ along any geodesic γ in $\mathbb{S}^4$ where $E_1, E_2, E_3, \dot{\gamma}$ form a parallel orthonormal frame along γ. The isotropic subspace $\mathcal{V} = \mathrm{span}\{J\}$ admits the Lagrangian extension

$$\Lambda = \mathrm{span}\{J(t), \sin(t)E_1 + \cos(t)E_2, \cos(t)E_3\}$$

which satisfies $\lambda_{max}(\hat{S}_0) = 0 = \lambda_{min}(\bar{S}_0)$.

Second, we note that if the A-tensor of an isotropic subspace $\mathcal{V}^k \subset \mathfrak{J}^{\perp}(\gamma)$ vanishes at $\gamma(0)$ then there exists an $\epsilon > 0$ such that $\mathcal{V}$ is supported by a Lagrangian on the interval $[0, \epsilon)$. Specifically, one can construct a Lagrangian support by adjoining to $\mathcal{V}$ the Jacobi fields $J_1, ..., J_{n-k-1}$ defined by $J_i(0) = e_i$ and $J_i'(0) = \lambda e_i$ where $e_1, ..., e_{n-k-1}$ is any orthonormal basis of $\mathcal{V}_0^{\perp}$ and $\lambda = \liminf_{t\to 0} \lambda_{min}(\bar{S}_t)$ (if $\lambda = \infty$ then $\mathcal{V}$ is supported up to the first conjugate point of $\gamma(0)$ by the Lagrangian subspace of Jacobi fields which vanish at $\gamma(0)$).

3.2 The projected Riccati equations

We now consider in more detail the evolution of a k-dimensional volume element under the flow of a congruence of geodesics. Let $\gamma : [0, t_0] \to M$ be a geodesic and let $\mathcal{V} \subset \mathfrak{J}^{\perp}(\gamma)$ be a k-dimensional isotropic subspace of normal Jacobi fields along γ with no focal points on $(0, t_0)$.

Let $P_t, Q_t : \dot\gamma(t)^{\perp} \to \dot\gamma(t)^{\perp}$ denote the orthogonal projections onto $\mathcal{V}_t$ and $\mathcal{V}_t^{\perp}$, respectively. Let Λ be any Lagrangian extension of $\mathcal{V}$ and decompose the Riccati operator S_t with respect to these subspaces as

$$S_t = \bar{S}_t + \hat{S}_t + A_t + A_t^*$$

where $\bar{S} = PSP, \hat{S} = QSQ, A = QSP$, and $A^* = PSQ$ is the adjoint of A. Note that $\bar{S}$, $\hat{S}$, and A are just the tangential/transverse Riccati operators and A-tensor discussed in Section 2.4, except that their domain of definition is extended to all of $\dot\gamma(t)^{\perp}$ so their covariant derivatives along γ are well-defined. Note also that only $\hat{S}$ depends on the extension Λ.

We now derive the evolution equations for the tangential and transverse operators $\bar{S}$ and $\hat{S}$. To state them, we decompose the curvature $R_{\dot\gamma}$ in terms of the same projected subspaces as

$$R_{\dot\gamma} = \bar{R} + \hat{R} + \widetilde{R} + \widetilde{R}^*$$

where $\bar{R} = PR_{\dot\gamma}P$, $\hat{R} = QR_{\dot\gamma}Q$, and $\widetilde{R} = QR_{\dot\gamma}P$.

Proposition 3.8 (Projected Riccati Equations)**.** *The operators S and $R_{\dot\gamma}$ satisfy*

$$\bar{S}' + \bar{S}^2 - A^*A - [A\bar{S} + (A\bar{S})^*] = -\bar{R} \tag{3.3}$$

$$\hat{S}' + \hat{S}^2 + 3AA^* + [\hat{S}A + (\hat{S}A)^*] = -\hat{R} \tag{3.4}$$

where the terms in brackets are traceless. The A-tensor satisfies

$$A' + 2A\bar{S} + [A^*A - AA^*] = -\widetilde{R}. \tag{3.5}$$

Before proving these identities, we observe that taking the trace yields the following inequalities for the trace of the Riccati operator along the subspaces $\mathcal{V}_t$ and $\mathcal{V}_t^\perp$.

Proposition 3.9. *Let $\mathcal{V}$ be a k-dimensional isotropic subspace of normal Jacobi fields along a geodesic. On any interval for which $\mathcal{V}$ has no focal points the tangential Riccati operator $\bar{S}$ and A-tensor satisfy*

$$\operatorname{tr}(\bar{S})' + \frac{1}{k}\operatorname{tr}(\bar{S})^2 - \|A\|^2 \leq -k \cdot Ric_k(\dot{\gamma}, \mathcal{V}_t) \tag{3.6}$$

$$\frac{1}{2}(\|A\|^2)' - 2\|\bar{S}\|\,\|A\|^2 \leq \|A\|\,\|\widetilde{R}\|. \tag{3.7}$$

Furthermore, for any Lagrangian extension Λ the transverse Riccati operator satisfies

$$\operatorname{tr}(\hat{S})' + \frac{1}{n-k-1}\operatorname{tr}(\hat{S})^2 + 3\,\|A\|^2 \leq -(n-k-1)Ric_{n-k-1}(\dot{\gamma}, \mathcal{V}_t^\perp). \tag{3.8}$$

Remark 3.10. The inequality for $\operatorname{tr}(\bar{S})$ was given by Ketterer-Mondino [21] in the context of characterizing k-Ricci curvature bounds in terms of optimal transport. The differential inequality for $\operatorname{tr}(\hat{S})$ can also be obtained from Wilking's transverse Jacobi equation which was first used to study k-Ricci curvature bounds by Guijarro-Wilhelm [17] (as mentioned in Section 2.4 the off-diagonal term A is sometimes referred to as Wilking's A-tensor in this context).

The projected Riccati equations as formulated above are obtained as a consequence of a simple but remarkable lemma showing that the covariant derivative of the projection operators P_t along γ is just the sum of the A-tensor and its adjoint.

Lemma 3.11. *The covariant derivative of the projection P_t along γ at any point which is not a focal point of $\mathcal{V}$ is given by*

$$P' = A + A^*$$

The complementary projection Q satisfies $Q' = -A - A^*$.

Proof. First we compute P' restricted to $\mathcal{V}_t$. For any vector v in $\mathcal{V}_t$ let J_v be the Jacobi field in $\mathcal{V}$ with $J_v(t) = v$. Since $P(J_v) \equiv J_v$ and $Q(J_v) \equiv 0$, we have

$$P'(v) = P'(J_v) = P(J_v)' - P(J_v') = J_v' - P(J_v') = S(J_v) - PS(J_v) = QS(v).$$

It follows that $P' = P'(P + Q) = QSP + P'Q = A + P'Q$.

To compute the second term, first observe that $P'Q = (PQ)' - PQ' = -PQ'$. Noting that since Q' is self-adjoint (since Q is self-adjoint) we can compute the adjoint of PQ'

$$\langle PQ'(u), v\rangle = \langle u, Q'(Pv)\rangle = \langle u, Q'(J_{Pv})\rangle = \langle u, Q(J_{Pv})' - Q(J_{Pv}')\rangle = \langle u, -QS(J_{Pv})\rangle$$

where J_{Pv} is defined as above. We thus have

$$\langle PQ'(u), v\rangle = \langle u, -QS(J_{Pv})\rangle = \langle u, -QS(Pv)\rangle = \langle u, -Av\rangle$$

and thus $PQ' = (-A)^*$. It follows that $P'Q = A^*$ and so $P' = A + A^*$. The identity for Q' follows since $(P + Q)' = 0$ implies $Q' = -P'$. □

Using this observation, we now derive the projected Riccati equations.

Proof of Proposition 3.8. The proof is now a direct consequence of Lemma 3.11 obtained by conjugating the Riccati equation (2.11) by the projection operators

$$PS'P + PS^2P = -PR_{\dot{\gamma}}P.$$

Rewriting the first term as $PS'P = (PSP)' - P'SP - PSP'$, the second term as $PS(P + Q)SP$, and substituting $P' = A + A^*$ yields the first identity. The second is obtained similarly by conjugating equation (2.11) by Q. To see that the bracketed terms are traceless observe that

$$\operatorname{tr}(A\bar{S}) = \operatorname{tr}(QSPSP) = \operatorname{tr}([PQ]SPS) = 0$$

and a similar computation shows that the traces of the other bracketed terms also vanish.

□

Finally, we obtain the inequalities of Proposition 3.9 from the projected Riccati equations.

Proof of Proposition 3.9. The inequalities for $\bar{S}$ and $\hat{S}$ follow immediately from taking the trace of the projected Riccati equations, and applying the Cauchy-Schwarz inequality to the trace of the square of the projected Riccati operators. For example, since $\bar{S}$ has at most k non-zero eigenvalues $\lambda_1, ..., \lambda_k$ we have

$$\operatorname{tr}(\bar{S}^2) = \sum_{i=1}^{k} \lambda_i^2 \geq \frac{1}{k}\left(\sum_{i=1}^{k} \lambda_i\right)^2 = \frac{1}{k}\operatorname{tr}(\bar{S})^2.$$

For the A-tensor inequality, we first multiply Equation (3.5) by A^* and take the adjoint to obtain the identities

$$A^*A' + 2A^*A\bar{S} + A^*[A^*A - AA^*] = -A^*\widetilde{R}$$
$$(A^*)'A + 2(A^*A\bar{S})^* + [A^*A - AA^*]A = -\widetilde{R}^*A$$

Summing these equations and taking the trace yields

$$\operatorname{tr}(A^*A)' + 4\operatorname{tr}(A^*A\bar{S}) = -2\operatorname{tr}(A^*\widetilde{R}).$$

Applying the Cauchy-Schwarz inequality to the second term and the right hand side we obtain the inequality

$$(\|A\|^2)' - 4\|\bar{S}\|\,\|A\|^2 \leq 2\,\|A\|\,\|\widetilde{R}\|.$$

□

In interpreting the projected Riccati equations of Proposition 3.8, it may be helpful to consider the case of a single Jacobi field J (i.e. $k = 1$). In this case, the Cauchy-Schwarz inequality is not needed on the term $A^*A\bar{S}$ as used above since the operators A^*A and $\bar{S}$ each have at most one non-zero eigenvalue α^2 and h, respectively, with common eigenvector J, and so in place of the inequalities of Proposition 3.9 we obtain a system of coupled ordinary differential equations:

Corollary 3.12. *Let γ be a geodesic and let J be a non-vanishing normal Jacobi field on an interval (a, b). Putting $h = \log(|J|)'$ and $\alpha = |\hat{J}'|$ where $\hat{J} = J/|J|$ we have*

$$h' + h^2 - \alpha^2 = \sec(\dot{\gamma}, \hat{J}) \tag{3.9}$$
$$\frac{1}{2}(\alpha^2)' + 2h\alpha^2 = -\alpha Rm(\hat{J}, \dot{\gamma}, \dot{\gamma}, E_{\hat{J}'}) \tag{3.10}$$

where $E_{\hat{J}'}$ is the unit vector orthogonal to $\hat{J}$ in the direction of $\hat{J}'$.

We now discuss the applications of Proposition 3.9.

3.2.1 Tangential Riccati comparison

Observe that the sign of the off-diagonal term $\|A\|^2$ in the inequality (3.6) for the tangential Riccati operator is such that the term cannot be omitted. Moreover, the inequality for the A-tensor given by (3.7) involves the off-diagonal term $\widetilde{R}$ of the curvature operator, which illustrates the fact that one cannot control the logarithmic growth of a single k-dimensional family of Jacobi fields up to the first focal point purely in terms of the k-Ricci curvature. This is precisely the reason that the notion of a Lagrangian support was needed for the Jacobi field comparisons of Section 3.1.

Nonetheless, it was observed by Ketterer and Mondino [21] that the single inequality (3.6), including the off-diagonal term $\|A\|^2$, is sufficient to completely characterize a k-Ricci curvature lower bound (we note however, that the definition of k-Ricci curvature used in that work is slightly different than the traditional definition that we have used due to the fact that they do not restrict to normal Jacobi fields).

On the other hand, the sign of the off-diagonal term in the inequality (3.8) for the transverse Riccati operator is such that the term can be omitted. This was observed by Guijarro and Wilhelm [17] and immediately yields a direct analogue of Proposition 3.1 where the parallel subspaces $\mathcal{W}_t$ are replaced by any family of subspaces $\mathcal{W}_t = \mathcal{V}_t^\perp$ where $\mathcal{V}^{n-k-1} \subset \Lambda$ is any subspace of the Lagrangian subspace Λ.

Remarkably, even though the tangential Riccati equation (3.6) does not directly yield a comparison, by recalling our notion of a Lagrangian support, the *transverse* inequality (3.8) yields a comparison for the *tangential* Riccati operator which also gives a Jacobi field comparison for Jacobi fields of mixed type.

Proposition 3.13 (Tangential Riccati comparison)**.** *Let $\gamma : [0, t_0] \to M$ be a geodesic and let $\mathcal{V}^m$ be an isotropic m-dimensional subspace of normal Jacobi fields which is supported by a Lagrangian on $[0, t_0)$. Put $k = \dim\{J(0) : J \in \mathcal{V}\}$.*

If $Ric_k(\dot\gamma, \cdot) \geq K$ and $Ric_{m-k}(\dot\gamma, \cdot) \geq K$ then for $0 < t < t_0$

$$\operatorname{tr}(\bar{S}_t) \leq (m-k)\log(\operatorname{sn}_K)'(t) + k\log(\operatorname{cs}_K + (v_0/k)\operatorname{sn}_K)'(t)$$

where $v_0 = \sum\langle J_i'(0), J_i(0)\rangle$ and J_i is any basis of $\mathcal{V}$ such that $J_1(0), ..., J_k(0), J_{k+1}'(0), ..., J_m'(0)$ are orthonormal (cf. (2.9)). In particular, given any such basis we have for $0 \leq t \leq t_0$

$$\frac{|J_1(t) \wedge ... \wedge J_m(t)|}{(\operatorname{cs}_K(t) + (v_0/k)\operatorname{sn}_K(t))^k \operatorname{sn}_K^{m-k}(t)} \leq 1$$

and the ratio is non-increasing.

Remark 3.14. If $k = 0$ or $m-k = 0$ we omit the corresponding condition on the curvature. These cases were already proved in Propositions 3.2 and 3.6.

Remark 3.15. If $\mathcal{V}$ is Lagrangian the comparison holds up to the first focal point of $\mathcal{V}$, otherwise the comparison only holds up to the first focal point of a Lagrangian support for $\mathcal{V}$.

Proof. Let Λ be a Lagrangian support for $\mathcal{V}$ on $[0, t_0)$ and let S_t denote the Riccati operator of Λ on $(0, t_0)$. Let $\mathcal{F} \subset \mathcal{V}$ denote the subspace of Jacobi fields in $\mathcal{V}$ which vanish at $\gamma(0)$ and define $\mathcal{H}_t$ via the orthogonal decomposition $\mathcal{V}_t = \mathcal{F}_t \oplus \mathcal{H}_t$. Fix $\tau \in (0, t_0)$ and let $\mathcal{W} \subset \Lambda$ be the subspace defined by

$$\mathcal{W} = \{J \in \Lambda : \langle J(\tau), \mathcal{H}_\tau\rangle = 0\}.$$

In particular, note that $\mathcal{W}$ contains $\mathcal{F}$ and thus $\mathcal{W}_0^\perp \subset \mathcal{F}_0^\perp$ and so by the definition of a

Lagrangian support we have

$$\limsup_{t\to 0} \operatorname{tr}_{\mathcal{W}_t^\perp}(S_t) \le v_0.$$

Applying the inequality (3.8) for the trace $\operatorname{tr}_{\mathcal{W}_t^\perp}(S_t)$ of the transverse Riccati operator of $\mathcal{W} \subset \Lambda$ and using the Riccati comparison principle (2.9) we obtain

$$\operatorname{tr}_{\mathcal{H}_\tau}(S_\tau) = \operatorname{tr}_{\mathcal{W}_\tau^\perp}(S_\tau) \le k \log(\operatorname{cs}_K + (v_0/k)\operatorname{sn}_K)'(\tau).$$

On the other hand, it follows immediately from Proposition 3.1 that

$$\operatorname{tr}_{\mathcal{F}_\tau}(S_\tau) \le (m-k)\log(\operatorname{sn}_K)'(\tau)$$

The result follows from the observation that $\operatorname{tr}(\bar{S}_\tau) = \operatorname{tr}_{\mathcal{H}_\tau}(S_\tau) + \operatorname{tr}_{\mathcal{F}_\tau}(S_\tau)$. The second part of the proposition follows as in the proof of Proposition 3.2 from the observation that $|J_1(t) \wedge ... \wedge J_m(t)| \sim t^{m-k}|J_1(0) \wedge ... \wedge J_k(0) \wedge J'_{k+1}(0) \wedge ... \wedge J'_m(0)|$ as $t \to 0$. □

Note that in the key step of the proof we apply the transverse Riccati equation (3.8) not to the subspace $\mathcal{V}$, but to the subspace $\mathcal{W}$ which is constructed in such a way as to contain the Jacobi fields in $\mathcal{V}$ which vanish at $\gamma(0)$.

3.3 Hessian comparison theorem for the distance to a submanifold

Combining the abstract Hessian comparison given in Proposition 3.1 with the series expansion for the Hessian of the distance function from a submanifold (Proposition 2.15) we obtain the following comparison for distance functions to a submanifold.

Theorem 3.16 (Hessian Comparison). *Let Σ^m be an m-dimensional submanifold of a* *complete Riemannian manifold M^n. Let $r(x) = d(x, \Sigma)$ denote the distance to Σ and let H_{∂_r} denote the Hessian operator. Let $\gamma : [0, t_0] \to M$ be any geodesic with $r(\gamma(t)) = t$.*

If $\mathcal{W}_t \subset \dot\gamma(t)^\perp$ is any family of k-dimensional subspaces which are parallel along γ and $Ric_k(\dot\gamma, \mathcal{W}_t) \geq K$ then for $0 < t < t_0$

$$\operatorname{tr}_{\mathcal{W}_t}(H_{\partial_r}) \leq \begin{cases} k \log(\operatorname{cs}_K + (w_0/k)\operatorname{sn}_K)'(t) & \text{if } \mathcal{W}_0 \subset T\Sigma \\ k \log(\operatorname{sn}_K)'(t) & \text{otherwise} \end{cases} \tag{3.11}$$

where $w_0 = \operatorname{tr}_{\mathcal{W}_0}(S_{\dot\gamma(0)})$.

If equality holds at some t then equality holds on $(0, t]$ and either $\mathcal{W}_0 \subset \boldsymbol{\nu}(\Sigma)$ or $\mathcal{W}_0$ is contained in an eigenspace of the shape operator $S_{\dot\gamma(0)}$. Moreover, $\mathcal{W}_t$, $\mathcal{W}_t^\perp$ are invariant subspaces of $R_{\dot\gamma}$ with $\mathcal{W}_t$ contained in an eigenspace with eigenvalue K.

Proof. The Hessian operator H_{∂_r} can be identified with the Riccati operator S for the set of ∂_r-Jacobi fields along γ, which has no focal points on $(0, t_0)$ since γ is distance minimizing on this interval. Now apply Proposition 3.1 together with the series expansion (2.4). □

From the proof of Proposition 3.1 it is easily seen that equality is realized if M is a space of constant curvature K and Σ is a submanifold such that either $\mathcal{W}_0 \subset \boldsymbol{\nu}(\Sigma)$ or $\mathcal{W}_0$ is contained in an eigenspace of $S_{\dot\gamma(0)}$.

We will develop a number of applications of this lemma in the following sections; however, we conclude this section by noting that it also recovers an upper bound on the focal radius which was recently obtained by Guijarro and Wilhelm in [17].

Corollary 3.17. *Let Σ be a submanifold of a complete Riemannian manifold M^n with $\dim(\Sigma) \geq k$. If $Ric_k \geq K > 0$ then the focal radius of Σ is at most $\frac{\pi}{2\sqrt{K}}$ and this focal radius is achieved if and only if Σ is totally geodesic.*

Proof. Given $x \in \Sigma$ let $\xi \in \hat{\boldsymbol{\nu}}(\Sigma)$ be any unit normal based at x, put $\gamma(t) = \exp_{\boldsymbol{\nu}}(t\xi)$. By replacing ξ with $-\xi$ if necessary, we may assume $\langle \eta, \xi \rangle \geq 0$ where η is the mean curvature vector of Σ. Putting $\mathcal{W}_0 = T_x\Sigma$ and applying the lemma, we find that $\mathrm{tr}_{\mathcal{W}_t}(H_{\partial_r})$ diverges to $-\infty$ for some $t \leq \pi/(2\sqrt{K})$. Moreover, if equality holds for all $\xi \in \hat{\boldsymbol{\nu}}(\Sigma)$ then $\eta \equiv 0$ and it follows from the equality case of the lemma that Σ is totally umbilic, and hence totally geodesic. □

3.4 Volume comparison theorem for k-Ricci curvature

Our first application of the Hessian comparison above is to prove that the Heintze-Karcher volume comparison holds using k-Ricci lower bounds.

Theorem 3.18. *Let M^n be a complete Riemannian manifold and let Σ^m be a closed m-dimensional submanifold. Put $k = \min\{m, n-m-1\}$. If $Ric_k \geq K$ then*

$$\mathrm{vol}(T(\Sigma, R)) \leq \int_{\hat{\boldsymbol{\nu}}} \int_0^{f(R,\xi)} (\mathrm{cs}_K(t) - \langle \eta, \xi \rangle \, \mathrm{sn}_K(t))^m \, \mathrm{sn}_K(t)^{n-m-1} dt d\xi.$$

where $f(R, \xi)$ denotes the minimum of R and the first zero of the integrand.

Proof. Let r denote the distance function to Σ and let $\gamma : [0, t_0] \to M$ be a maximally extended geodesic such that $r(\gamma(t)) = t$. Put $\xi = \dot{\gamma}(0)$.

Let $\mathcal{W}_t$ and $\mathcal{V}_t$ denote the subspaces of $T_{\gamma(t)}M$ parallel along γ to $T_x\Sigma$ and $\boldsymbol{\nu}_x \cap \xi^\perp$, respectively. Since $\mathcal{W}_t$, $\mathcal{V}_t$, and ∂_r are orthogonal, the trace $h(t, \xi) = \mathrm{tr}(H_{\partial_r})$ at $\gamma(t)$ is given by

$$h = \varphi + \psi$$

where $\varphi = \mathrm{tr}_{\mathcal{W}_t}(H_{\partial_r})$ and $\psi = \mathrm{tr}_{\mathcal{V}_t}(H_{\partial_r})$. Using the assumption $Ric_k \geq K$, Theorem 3.16

gives

$$h(t, \xi) \leq m \log(\mathrm{cs}_K - \langle \eta, \xi \rangle \, \mathrm{sn}_K)'(t) + (n - m - 1) \log(\mathrm{sn}_K)'(t)$$

for $0 < t < t_0$. From Equation (2.8) we have for $0 < t < t_0$ the inequality

$$\log(\mathcal{A})' \leq \log[(\mathrm{cs}_K - \langle \eta, \xi \rangle \, \mathrm{sn}_K)^m \, \mathrm{sn}_K^{n-m-1}]'$$

where $\mathcal{A}(t, \xi)$ is the polar volume density (2.6). Now, since $\mathcal{A}(t, \xi) \sim t^{n-m-1}$ as $t \to 0$ (Proposition 2.21) we can integrate from 0 to $t \leq t_0$ to obtain

$$\mathcal{A}(t, \xi) \leq (\mathrm{cs}_K - \langle \eta, \xi \rangle \, \mathrm{sn}_K)^m \, \mathrm{sn}_K^{n-m-1} .$$

The result follows from Equation (2.7).

□

Notice that we could also have proved this Theorem as an immediate consequence of Proposition 3.13. Instead, we have chosen to give a proof based directly on the Hessian comparison of Theorem 3.16 as it facilitates a more natural generalization to the volume estimates for manifolds with integral curvature bounds developed in Chapter 4.

Chapter 4

Volume estimates using integral curvature bounds

One of the primary motivations for developing the comparison theory of Chapter 3 is to develop the background needed to generalize the inequality of E. Heintze and H. Karcher [18] for the volume of tubes around submanifolds based on sectional curvature bounds to the setting of integral curvature lower bounds.

As in the introduction, for a given Riemannian manifold (M, g) we define the function $\rho_k(x)$ on M to be the minimum of $Ric_k(u, \mathcal{V})$ where $u \in T_xM$ is a unit tangent vector at x and $\mathcal{V}$ is a k-dimensional subspace orthogonal to u. For a fixed constant K we may then consider the norms

$$\|(\rho_k - K)_-\|_p = \left(\int_M (\rho_k - K)_-^p \; dvol_g\right)^{1/p}$$

which measure the amount of k-Ricci curvature below K.

For convenience, we now recall the statement of the volume estimate given in Theorem 1.1 including the values of all constants.

Theorem 4.1. *Let M^n be a complete Riemannian manifold and let $\Sigma^m \subset M$ be a closed minimal submanifold with $0 < m < n-1$.*

Put $k = \min\{m, n-m-1\}$. If $K \le 0$ and $p > n-k$ then

$$\operatorname{vol}(T(\Sigma, r)) \le \left(w(r)^{n-m-1} + 2^{p/\alpha} \left\| (\rho_k - K)_- \right\|_p^{\beta p} w(r)^p \right) e^{\kappa r^{2\alpha}}$$

where $\alpha = \frac{n-k-1}{n-k}$, $\beta = \frac{1}{n-m-1} - \frac{1}{p}$,

$$w(r) = \left(\tfrac{\alpha}{n-m-1}\right)^{\frac{1}{n-k-1}} \left(\operatorname{vol}(\mathbb{S}^{n-m-1}) \operatorname{vol}(\Sigma) r^{n-m} \right)^{\frac{1}{n-m-1}} + \delta \left\| (\rho_k - K)_- \right\|_p^{1-\beta} r^2,$$

and $\kappa = (\delta |K|)^\alpha / (2\alpha)$ with

$$\delta = 4(n-k-1) + \frac{4}{k} \left(\frac{2p-1}{p-n+k} \right).$$

The proof given in this section is entirely independent of the theory developed in Chapter 3; however, an understanding of the proof of the pointwise volume comparison given in Theorem 3.18 will be helpful in understanding the general approach taken in the proof below. The only necessary prerequisite is a review of the notation and preliminaries given in Sections 2.2 and 2.3.

4.1 Proof of the volume estimate

Throughout this section we assume $0 < m < n-1$. In the proof of Theorem 3.18 we assumed a pointwise lower curvature bound and used the projected traces of the Riccati equation (2.2) to bound the mean curvature h of the distance level sets explicitly and thus bound the logarithmic growth of the polar volume density $\mathcal{A}$. Unfortunately, in order to obtain a bound depending only on integrals of the curvature this approach fails since one cannot use the comparison theory for the Riccati differential equation.

As in the proof of Theorem 3.18, we fix $\xi \in \hat{\boldsymbol{\nu}}(\Sigma)$ and put $\gamma(t) = \exp_{\boldsymbol{\nu}}(t\xi)$. Let H_{∂_r} denote the Hessian operator of the distance function to Σ and let $\mathcal{A}(t,\xi)$ and $h(t,\xi)$ be the polar volume density and mean curvature of the distance level sets as in Proposition 2.21. Differentiating Equation (2.8) gives

$$\mathcal{A}'' = (h' + h^2)\mathcal{A}. \tag{4.1}$$

Taking the trace of the Riccati equation (2.2) gives $h' + \operatorname{tr}(H_{\partial_r}^2) = -Ric(\dot\gamma,\dot\gamma)$ which leaves us to control the second order invariant $\operatorname{tr}(H_{\partial_r})^2 - \operatorname{tr}(H_{\partial_r}^2)$ in (4.1) in terms of curvature. This motivates us [10, 43] to consider in place of $\mathcal{A}$ the function J with $\mathcal{A} = J^{n-1}$ which satisfies

$$J'' = \frac{1}{n-1}\left(h' + \frac{h^2}{n-1}\right)J \le -\frac{Ric(\dot\gamma,\dot\gamma)}{n-1}J$$

allowing us to control the second order invariant of H_{∂_r} using the Cauchy-Schwarz inequality. However, if $0 < m < n-1$ then the initial conditions for the function J are unusable, namely $J(0) = 0$ and $J'(0) = \infty$.

The remarkable observation of [27] was that one can control certain products of eigenvalues of H_{∂_r} directly in terms of integrals of sectional curvature without relying on the Cauchy-Schwarz inequality, provided one of the eigenvalues vanishes at Σ. For $m = 1$, putting $\mathcal{A} = J^{n-2}$ (so that $J'(0) = 1$) and assuming Σ is a geodesic then allows control of the second order invariant of H_{∂_r}. However, generalizing this directly to higher dimensional submanifolds by setting $\mathcal{A} = J^{n-m-1}$ then yields an estimate only when Σ is totally geodesic.

Instead, motivated by the pointwise comparison of the previous section, we decompose the mean curvature as $h = \varphi + \psi$ as in the proof of Theorem 3.18, and then decompose the polar volume density $\mathcal{A}$ into two functions $\mathcal{A}(t) = \mathcal{J}(t)\mathcal{Y}(t)$ where $\mathcal{J}(t)$ is defined by the equation

$$\begin{cases} \mathcal{J}' = \varphi\mathcal{J}, \\ \mathcal{J}(0) = 1 \end{cases}$$

from which it follows that $\mathcal{Y}$ satisfies $\mathcal{Y}' = \psi\mathcal{Y}$. Putting $\mathcal{J} = J^m$ and $\mathcal{Y} = Y^{n-m-1}$ we have $\mathcal{A} = J^m Y^{n-m-1}$ with

$$J'' = \frac{1}{m}\left(\varphi' + \frac{\varphi^2}{m}\right) J \leq -Ric_m(\dot{\gamma}, \mathcal{W}_t)J \tag{4.2}$$

and

$$Y'' = \frac{1}{n-m-1}\left(\psi' + \frac{\psi^2}{n-m-1}\right) Y \leq -Ric_{n-m-1}(\dot{\gamma}, \mathcal{V}_t)Y. \tag{4.3}$$

The initial conditions for J and Y are easily found to be

$$\begin{cases} J(0) = 1, \\ J'(0) = -\langle \eta, \xi \rangle, \end{cases} \qquad \begin{cases} Y(0) = 0, \\ Y'(0) = 1. \end{cases}$$

The main challenge introduced with this decomposition is that we only want to consider expressions involving curvature multiplied by the full volume density $\mathcal{A}$, rather than curvature multiplied by just the function J or Y as in (4.2) and (4.3). With this in mind, rather than integrating the two inequalities directly these considerations motivate the following lemma.

Lemma 4.2. *If $0 < m < n-1$ the functions J and Y defined above satisfy*

$$J'(t)Y^{\frac{n-m-1}{m}}(t) \leq \int_0^t (\rho_m)_- \mathcal{A}^{1/m} ds + \frac{1}{m^2}\int_0^t (\varphi_+\psi_+)\mathcal{A}^{1/m} ds \tag{4.4}$$

and

$$Y'(t)J^{\frac{m}{n-m-1}}(t) \leq 1 + \int_0^t (\rho_{n-m-1})_- \mathcal{A}^{\frac{1}{n-m-1}} ds + \frac{1}{(n-m-1)^2}\int_0^t (\varphi_+\psi_+)\mathcal{A}^{\frac{1}{n-m-1}} ds.$$

Proof. For the first inequality, for any $\delta > 0$ we have

$$(J'Y^\delta)' = J''Y^\delta + \delta Y^{\delta-1}J'Y'.$$

Using the identities $J' = (\varphi/m)J$ and $Y' = [\psi/(n-m-1)]Y$ together with eqs. (4.2) and (4.3) we get

$$(J'Y^\delta)' \leq \left(-\rho_m + \delta\frac{\varphi\psi}{m(n-m-1)}\right) JY^\delta.$$

Now, if $J'(t) \leq 0$ the inequality (4.4) holds automatically so we only need to show the inequality for all values of t such that $J'(t) > 0$. Moreover, since $J'(0)Y^\delta(0) = 0$ it follows that all such values of t are contained in an interval $[t_0, t]$ such that $J'(t_0)Y^\delta(t_0) = 0$ and $J' \geq 0$ on $[t_0, t]$. On such an interval, $J' \geq 0$ implies $\varphi \geq 0$ and hence $\varphi\psi \leq \varphi_+\psi_+$ on $[t_0, t]$. Using also that $-\rho_m \leq (\rho_m)_-$ and integrating over the interval $[t_0, t]$ gives

$$J'(t)Y^\delta(t) \leq \int_{t_0}^t (\rho_m)_- JY^\delta + \delta\int_{t_0}^t \frac{\varphi_+\psi_+}{m(n-m-1)} JY^\delta.$$

Since the integrands are nonnegative, we can replace the lower bound t_0 with 0 and preserve the inequality. Finally, taking $\delta = (n-m-1)/m$ gives the result (4.4).

Analogous reasoning leads to the second inequality, except that the initial condition $Y'(0)J^\delta(0) = 1$ leads to the extra term on the right hand side of the inequality. □

Based on this lemma, one now only needs to control the product $\varphi_+\psi_+$ in terms of curvature. We prove that this is possible provided φ_+ vanishes at Σ, generalizing the eigenvalue estimate in [27].

Lemma 4.3. *Let φ, ψ, and $\mathcal{A}$ be as above. Put $k = \min\{m, n-m-1\}$. If $\varphi_+(t)\psi_+(t)$ is bounded as $t \to 0$ then for any $p > n-k$,*

$$\left(\int_0^t (\varphi_+\psi_+)^p \mathcal{A} ds\right)^{\frac{1}{p}} \leq \frac{2p-1}{p-(n-k)} \left(\int_0^t (\rho_k)_-^p \mathcal{A} ds\right)^{\frac{1}{p}}.$$

The proof is given at the end of this section and is a straightforward modification of the proof in [27]. Note that if the submanifold Σ is minimal then it follows from equation (2.4) that $\varphi_+\psi_+$ is bounded as $t \to 0$.

Using these inequalities, we now consider the area of the equidistant hypersurfaces $v(t) = \mathrm{vol}(\Sigma_t)$ given by the integral

$$v(t) = \int_{\hat{\boldsymbol{\nu}}} J^m(t,\xi) Y^{n-m-1}(t,\xi) d\xi. \tag{4.5}$$

The volume $V(r) = \mathrm{vol}(T(\Sigma, r))$ can then be written

$$V(r) = \int_0^r v(t) dt.$$

We wish to differentiate $v(t)$ via the expression (4.5). Note that the integrand is smooth and nonnegative on the interior of the segment domain $\mathrm{seg}^0(\Sigma)$ defined in Section 2.3 and vanishes on $\boldsymbol{\nu} \setminus \mathrm{seg}^0(\Sigma)$, but may be discontinuous on the boundary of $\mathrm{seg}^0(\Sigma)$. However, since $\mathrm{seg}^0(\Sigma)$ is star-shaped with respect to the zero section of $\boldsymbol{\nu}$ (i.e. $\nu \in \mathrm{seg}^0(\Sigma)$ implies $\lambda\nu \in \mathrm{seg}^0(\Sigma)$ for $0 \leq \lambda \leq 1$) it follows that v is an almost everywhere differentiable lower semi-continuous function (see [2]) and

$$v'(t) \leq \int_{\hat{\boldsymbol{\nu}}} mJ^{m-1}Y^{n-m-1}J' + (n-m-1)J^m Y^{n-m-2} Y' d\xi.$$

We now substitute the two inequalities of Lemma 4.2 and use two applications of Hölder's inequality. For example, the first term satisfies

$$\begin{aligned} m\int_{\hat{\boldsymbol{\nu}}} J^{m-1}Y^{n-m-1}J'd\xi &\leq m\int_{\hat{\boldsymbol{\nu}}} \mathcal{A}^{\frac{m-1}{m}}\left(\int_0^t (\rho_m)_-\mathcal{A}^{\frac{1}{m}}ds + \frac{1}{m^2}\int_0^t (\varphi_+\psi_+)\mathcal{A}^{\frac{1}{m}}ds\right)d\xi \\ &\leq mv(t)^{\frac{m-1}{m}}\left[\left(\int_{\hat{\boldsymbol{\nu}}}\Big(\int_0^t (\rho_m)_-\mathcal{A}^{\frac{1}{m}}ds\Big)^m d\xi\right)^{\frac{1}{m}} + \frac{1}{m^2}\left(\int_{\hat{\boldsymbol{\nu}}}\Big(\int_0^t (\varphi_+\psi_+)\mathcal{A}^{\frac{1}{m}}ds\Big)^m d\xi\right)^{\frac{1}{m}}\right] \\ &\leq mv(t)^{\frac{m-1}{m}}t^{\frac{m-1}{m}}\left[\left(\int_{\hat{\boldsymbol{\nu}}}\int_0^t (\rho_m)_-^m\mathcal{A}dsd\xi\right)^{\frac{1}{m}} + \frac{1}{m^2}\left(\int_{\hat{\boldsymbol{\nu}}}\int_0^t (\varphi_+\psi_+)^m\mathcal{A}dsd\xi\right)^{\frac{1}{m}}\right]. \end{aligned}$$

Handling the second term of the integral in a similar fashion one easily checks that

$$\begin{aligned} v'(t) &\leq (n-m-1)\,\mathrm{vol}(\hat{\boldsymbol{\nu}})^{\frac{1}{n-m-1}}v(t)^{\frac{n-m-2}{n-m-1}} \\ &+ \left[(n-m-1)\left\|(\rho_{n-m-1})_-\right\|_{n-m-1,t} + \tfrac{1}{n-m-1}\left\|\varphi_+\psi_+\right\|_{n-m-1,t}\right](tv(t))^{\frac{n-m-2}{n-m-1}} \\ &+ \left[m\left\|(\rho_m)_-\right\|_{m,t} + \tfrac{1}{m}\left\|\varphi_+\psi_+\right\|_{m,t}\right](tv(t))^{\frac{m-1}{m}} \end{aligned} \tag{4.6}$$

where $\|\cdot\|_{p,t}$ is the usual L^p norm on the tube $T(\Sigma,t)$. In order to make use of the estimate in Lemma 4.3 it is necessary to raise the exponents in the expression above at the cost of a volume term via the inequality

$$\|f\|_{q,t} \leq \|f\|_{p,t}\,V(t)^{\frac{1}{q}-\frac{1}{p}} \tag{4.7}$$

provided $p \geq q \geq 1$.

Henceforth, set $k = \min\{m, n-m-1\}$ and note that $\rho_k \leq \rho_{n-k-1}$. Using the inequality (4.7) together with Lemma 4.3, we have for any $p > n-k \geq q$

$$\|\varphi_+\psi_+\|_{q,t} \leq \frac{2p-1}{p-(n-k)}V(t)^{\frac{1}{q}-\frac{1}{p}}\left\|(\rho_k)_-\right\|_{p,t}.$$

Applying these observations to (4.6) we obtain

$$
\begin{aligned}
V''(t) \leq{} & (n-m-1)\operatorname{vol}(\hat{\boldsymbol{\nu}})^{\frac{1}{n-m-1}} V'(t)^{\frac{n-m-2}{n-m-1}} \\
& + \|(\rho_k)_-\|_{p,t}\left(C_1 V(t)^{\frac{1}{m}-\frac{1}{p}}(tV'(t))^{\frac{m-1}{m}} + C_2 V(t)^{\frac{1}{n-m-1}-\frac{1}{p}}(tV'(t))^{\frac{n-m-2}{n-m-1}}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= m + \frac{2p-1}{m(p-n+k)}, \\
C_2 &= (n-m-1) + \frac{2p-1}{(n-m-1)(p-n+k)}.
\end{aligned}
$$

In order to obtain an inequality which depends more generally on $\|(\rho_k - K)_-\|$ we observe that for $K \leq 0$,

$$
(\rho_k)_- \leq (\rho_k - K)_- + |K|
$$

and hence $(\rho_k)_-^p \leq 2^{p-1}\left((\rho_k - K)_-^p + |K|^p\right)$. It then follows that

$$
\|(\rho_k)_-\|_{p,t} \leq 2^{\frac{p-1}{p}}\left(\|(\rho_k - K)_-\|_{p,t} + |K| V(t)^{\frac{1}{p}}\right).
$$

Substituting this back into the inequality above and estimating $2^{\frac{p-1}{p}} < 2$ we obtain

$$
\begin{aligned}
V''(t) \leq{} & (n-m-1)\operatorname{vol}(\hat{\boldsymbol{\nu}})^{\frac{1}{n-m-1}} V'(t)^{\frac{n-m-2}{n-m-1}} \\
& + 2\|(\rho_k - K)_-\|_p\left(C_1 V(t)^{\frac{1}{m}-\frac{1}{p}}(tV'(t))^{\frac{m-1}{m}} + C_2 V(t)^{\frac{1}{n-m-1}-\frac{1}{p}}(tV'(t))^{\frac{n-m-2}{n-m-1}}\right) \\
& + 2|K|\left(C_1 V(t)^{\frac{1}{m}}(tV'(t))^{\frac{m-1}{m}} + C_2 V(t)^{\frac{1}{n-m-1}}(tV'(t))^{\frac{n-m-2}{n-m-1}}\right).
\end{aligned}
$$

It remains to use this differential inequality to obtain an estimate for $V(t)$. To simplify notation we introduce the constants

$$\begin{aligned}
a &= (n-m-1)(\operatorname{vol}(\mathbb{S}^{n-m-1})\operatorname{vol}(\Sigma))^{\frac{1}{n-m-1}},\\
b &= \|(\rho_k - K)_-\|_p\,,\\
c &= 2(n-k-1) + \frac{2}{k}\left(\frac{2p-1}{p-n+k}\right).
\end{aligned}$$

Noting that $\operatorname{vol}(\hat{\boldsymbol{\nu}}) = \operatorname{vol}(\mathbb{S}^{n-m-1})\operatorname{vol}(\Sigma)$ the previous inequality then implies

$$\begin{aligned}
V''(t) \le a(V')^{1-\frac{1}{n-m-1}} &+ cb\left(V^{\frac{1}{m}-\frac{1}{p}}(tV')^{1-\frac{1}{m}} + V^{\frac{1}{n-m-1}-\frac{1}{p}}(tV')^{1-\frac{1}{n-m-1}}\right)\\
&+ c|K|\left(V(t)^{\frac{1}{m}}(tV'(t))^{1-\frac{1}{m}} + V(t)^{\frac{1}{n-m-1}}(tV'(t))^{1-\frac{1}{n-m-1}}\right).
\end{aligned} \tag{4.8}$$

Multiplying through by the nonnegative quantity $(V')^{\frac{1}{n-k-1}}$ and putting

$$\delta_1 = \frac{1}{n-m-1} - \frac{1}{n-k-1}, \qquad \delta_2 = \frac{1}{k} - \frac{1}{n-k-1}, \qquad \delta_3 = \frac{1}{n-k-1} - \frac{1}{p},$$

and $\alpha = (n-k-1)/(n-k)$ gives

$$\begin{aligned}
V''(t)(V')^{\frac{1}{n-k-1}} \le a(V')^{1-\delta_1} &+ cbt^{\frac{k-1}{k}}V^{\delta_2+\delta_3}(V')^{1-\delta_2} + \tfrac{cb}{1+\delta_3}t^{\frac{n-k-2}{n-k-1}}(V^{1+\delta_3})'\\
&+ c|K|t^{\frac{k-1}{k}}V^{\frac{1}{k}}(V')^{1-\delta_2} + \alpha c|K|t^{\frac{n-k-2}{n-k-1}}(V^{1/\alpha})'
\end{aligned} \tag{4.9}$$

We now integrate both sides from 0 to t as follows. Noting that $0 \le \delta_1, \delta_2, \delta_3 < 1$ and using Hölder's inequality we get the inequalities

$$\int_0^t (V')^{1-\delta_1} ds \le t^{\delta_1}\left(\int_0^t V' ds\right)^{1-\delta_1} = t^{\delta_1}V(t)^{1-\delta_1}$$

and

$$\begin{aligned}\int_0^t s^{\frac{k-1}{k}} V^{\delta_2+\delta_3}(V')^{1-\delta_2} ds &\leq t^{\frac{k-1}{k}} \int_0^t V^{\delta_2+\delta_3}(V')^{1-\delta_2} ds \\ &\leq t^{\frac{k-1}{k}} t^{\delta_2} \left(\int_0^t V^{(\delta_2+\delta_3)/(1-\delta_2)}(V') ds \right)^{1-\delta_2} \\ &= t^{\frac{n-k-2}{n-k-1}} \left(\frac{1-\delta_2}{1+\delta_3} \right)^{1-\delta_2} V^{1+\delta_3}.\end{aligned}$$

Handling the integral of the fourth term on the right hand side of equation (4.9) in the same manner, we integrate equation (4.9) from 0 to t to obtain

$$\begin{aligned}(V')^{1/\alpha} \leq \frac{1}{\alpha} &\left(a t^{\delta_1} V^{1-\delta_1} + cb \left[\left(\tfrac{1-\delta_2}{1+\delta_3} \right)^{1-\delta_2} + \tfrac{1}{1+\delta_3} \right] t^{\frac{n-k-2}{n-k-1}} V^{1+\delta_3} \right) \\ &+ \frac{c|K|}{\alpha} \left(\left[\left(1 - \tfrac{\alpha}{k}\right)^{1-\delta_2} + \alpha \right] t^{\frac{n-k-2}{n-k-1}} V^{1/\alpha} \right).\end{aligned}$$

Noting that both quantities in brackets are bounded above by 2 and since $0 < \alpha < 1$ the inequality $(x+y)^\alpha \leq x^\alpha + y^\alpha$ for $x, y \geq 0$ implies

$$V' \leq \alpha^{-\alpha} \left(a t^{\delta_1} V^{1-\delta_1} + 2cb t^{\frac{n-k-2}{n-k-1}} V^{1+\delta_3} \right)^\alpha + (2c|K|/\alpha)^\alpha t^{2\alpha-1} V$$

Multiplying by the integrating factor $\mu(t) = e^{-\kappa t^{2\alpha}}$ where $\kappa = (2c|K|/\alpha)^\alpha/(2\alpha)$ transforms this inequality into

$$(\mu V)' \leq \alpha^{-\alpha} \left(a t^{\delta_1} V^{1-\delta_1} + 2cb t^{\frac{n-k-2}{n-k-1}} V^{1+\delta_3} \right)^\alpha \mu$$

and using the fact that $0 < \mu \leq 1$ for $t \geq 0$ we can write

$$(\mu V)' \leq \alpha^{-\alpha} \left(a t^{\delta_1} (\mu V)^{1-\delta_1} + 2cb t^{\frac{n-k-2}{n-k-1}} (\mu V)^{1+\delta_3} \right)^\alpha. \tag{4.10}$$

Put $r_0 = \inf\{r : b\mu(r)V(r) \geq 1\}$ with $r_0 = \infty$ if $b\mu(r)V(r) < 1$ for all $r > 0$. Define the function $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = \begin{cases} \mu(t)V(t) & \text{if } t \leq r_0 \\ \max\{\mu(t)V(t), 1/b\} & \text{if } t > r_0 \end{cases}$$

and observe that f is absolutely continuous and satisfies the differential inequality (4.10) with f in place of μV. We now use this inequality to derive an upper bound for the function $f(t)$. To integrate this inequality, first notice that the exponents satisfy $1-\delta_2 < 1+\delta_3$. On the interval $[0, r_0]$, since $bf \leq 1$ we thus have

$$f' \leq \alpha^{-\alpha}\left(at^{\delta_1} + 2cb^{1-\delta_1-\delta_3}t^{\frac{n-k-2}{n-k-1}}\right)^{\alpha} f^{1-\frac{\alpha}{n-m-1}}, \qquad (t \leq r_0).$$

Putting $\beta = \delta_1 + \delta_3$ it follows that for $0 \leq t \leq r_0$ there holds

$$\left(f^{\frac{\alpha}{n-m-1}}\right)' \leq \tfrac{1}{n-m-1}\alpha^{1-\alpha}\left(at^{\delta_1} + 2cb^{1-\beta}t^{\frac{n-k-2}{n-k-1}}\right)^{\alpha}, \qquad (t \leq r_0).$$

Integrating the right hand side from 0 to $r \leq r_0$ using Hölder's inequality we find

$$\int_0^r \left(at^{\delta_1} + 2cb^{1-\beta}t^{\frac{n-k-2}{n-k-1}}\right)^{\alpha} dt \leq r^{1-\alpha}\left(\int_0^r at^{\delta_1} + 2cb^{1-\beta}t^{\frac{n-k-2}{n-k-1}}\, dt\right)^{\alpha}$$

and so carrying out the integration yields

$$f(r) \leq \left(\tfrac{\alpha^{1-\alpha}}{n-m-1}\right)^{\frac{n-m-1}{\alpha}}\left(\frac{a}{1+\delta_1}r^{\frac{n-m}{n-m-1}} + \frac{2cb^{1-\beta}}{2-\frac{1}{n-k-1}}r^2\right)^{n-m-1}, \qquad (r \leq r_0).$$

Simplifying the expression, noting that $1/(1+\delta_1) \leq 1$, $2 - 1/(n-k-1) \geq 1$, and

$\alpha^{1-\alpha}/(n-m-1) \leq 1$ we finally obtain

$$f(r) \leq w(r)^{n-m-1}, \qquad (r \leq r_0) \quad (4.11)$$

where

$$w(r) = \left(\tfrac{\alpha}{n-m-1}\right)^{\frac{1}{n-k-1}} \mathrm{vol}(\hat{\boldsymbol{\nu}})^{\frac{1}{n-m-1}} r^{\frac{n-m}{n-m-1}} + 2cb^{1-\beta} r^2.$$

For $t \geq r_0$ we may assume $b \neq 0$ and since $bf \geq 1$ we have

$$f' \leq \alpha^{-\alpha} \left(ab^{\delta_1+\delta_3} t^{\delta_1} + 2cbt^{\frac{n-k-2}{n-k-1}} \right)^{\alpha} f^{1-\frac{\alpha}{p}}, \qquad (t \geq r_0).$$

Proceeding as above except that we integrate from r_0 to $r > r_0$ it is easy to check that for $r \geq r_0$,

$$f(r) \leq \frac{1}{b} \left[1 + b^{\frac{\alpha}{n-m-1}} w(r)^{\alpha} \right]^{p/\alpha}.$$

Moreover, for $r \geq r_0$ we have $bw(r)^{n-m-1} \geq bw(r_0)^{n-m-1} \geq bf(r_0) = 1$ and hence

$$b^{-1}(1 + b^{\frac{\alpha}{n-m-1}} w(r)^{\alpha})^{p/\alpha} \leq b^{-1}(2b^{\frac{\alpha}{n-m-1}} w(r)^{\alpha})^{p/\alpha} = 2^{p/\alpha} b^{\beta p} w(r)^p$$

and thus for $r \geq r_0$ we have

$$f(r) \leq 2^{p/\alpha} b^{\beta p} w(r)^p, \qquad (r \geq r_0). \quad (4.12)$$

Combining equations (4.11) and (4.12) it then follows that for all $r \geq 0$ the function f satisfies

$$f(r) \leq w(r)^{n-m-1} + 2^{p/\alpha} b^{\beta p} w(r)^p.$$

Noting that $f(r) \geq \mu(r)V(r)$ it follows that the volume of the tube $T(\Sigma, r)$ satisfies

$$V(r) \leq \left(w(r)^{n-m-1} + 2^{p/\alpha} b^{\beta p} w(r)^p\right) e^{\kappa r^{2\alpha}}$$

This completes the proof of Theorem 1.1 contingent on our proof of Lemma 4.3.

Remark 4.4. As mentioned in the introduction, in the case of a pointwise lower curvature bound $\|(\rho_k)_-\|_p = 0$ with $k = m$, the estimate reduces to

$$V(r) \leq \frac{1}{n-m} \operatorname{vol}(\Sigma) \operatorname{vol}(\mathbb{S}^{n-m-1}) r^{n-m} \tag{4.13}$$

which is precisely the volume of a tube around a piece of an m-plane in $\mathbb{R}^n$. The loss of sharpness in the pointwise case when $k \neq m$ comes from the use of Hölder's inequality above, and could be removed by setting $\|(\rho_k)_-\|_p = 0$ in (4.8) earlier in the computation.

Proof of Lemma 4.3. Define

$$\sigma = \min\{\varphi_+/m,\ \psi_+/(n-m-1)\}$$
$$\tau = \max\{\varphi_+/m,\ \psi_+/(n-m-1)\}$$

and observe that both σ and τ are absolutely continuous and from equation (3.2), using the fact that $0 \leq (\rho_{n-k-1})_- \leq (\rho_k)_-$ they satisfy

$$\sigma' + \sigma^2 \leq (\rho_k)_-$$
$$\tau' + \tau^2 \leq (\rho_k)_-.$$

Multiplying the first equation by $(\sigma\tau)^{p-1}\mathcal{A}$ and integrating, we have

$$\int_0^r \sigma'(\sigma\tau)^{p-1}\mathcal{A}dt + \int_0^r \sigma^{p+1}\tau^{p-1}\mathcal{A} \leq \int_0^r (\rho_k)_-(\sigma\tau)^{p-1}\mathcal{A}dt. \tag{4.14}$$

Integrating the first term by parts we find that

$$\begin{aligned}\frac{1}{p}\int_0^r (\sigma^p)'\tau^{p-1}\mathcal{A}dt &= \frac{1}{p}\sigma^p\tau^{p-1}\mathcal{A}\Big|_0^r - \frac{p-1}{p}\int_0^r \sigma^p\tau^{p-2}\tau'\mathcal{A}dt - \frac{1}{p}\int_0^r \sigma^p\tau^{p-1}h\mathcal{A}dt \\ &\geq 0 - \frac{p-1}{p}\int_0^r \sigma^p\tau^{p-2}((\rho_k)_- - \tau^2)\mathcal{A}dt - \frac{n-k-1}{p}\int_0^r \sigma^p\tau^{p-1}(\sigma+\tau)\mathcal{A}dt\end{aligned}$$

where we have used the fact that $\sigma\tau$ is bounded as $t \to 0$ and $\mathcal{A}(0) = 0$ for $m < n-1$. The last term uses the observation $h = \varphi + \psi \leq (n-k-1)(\sigma+\tau)$. Substituting back into (4.14), we now have

$$\begin{aligned}\frac{p-(n-k)}{p}\int_0^r (\sigma\tau)^p\mathcal{A}dt + \left(1 - \frac{n-k-1}{p}\right)\int_0^r \sigma^{p+1}\tau^{p-1}\mathcal{A}dt& \\ \leq \frac{p-1}{p}\int_0^r (\rho_k)_-\sigma^p\tau^{p-2}\mathcal{A}dt + \int_0^r (\rho_k)_-(\sigma\tau)^{p-1}\mathcal{A}dt.&\end{aligned}$$

Assuming $p > n-k$ the first term is positive, the second term is non-negative and can be dropped, and since $0 \leq \sigma \leq \tau$ we can use $\sigma^p\tau^{p-2} \leq (\sigma\tau)^{p-1}$ to obtain

$$\int_0^r (\sigma\tau)^p\mathcal{A}dt \leq \frac{2p-1}{p-(n-k)}\int_0^r (\rho_k)_-(\sigma\tau)^{p-1}\mathcal{A}dt.$$

Finally, using Hölder's inequality on the right hand side we have

$$\int_0^r (\rho_k)_-(\sigma\tau)^{p-1}\mathcal{A}dt \leq \left(\int_0^r (\rho_k)_-^p\mathcal{A}dt\right)^{1/p}\left(\int_0^r (\sigma\tau)^p\mathcal{A}dt\right)^{1-\frac{1}{p}}$$

and the lemma follows immediately. □

Chapter 5

Applications

5.1 Volume estimates for minimal submanifolds

As discussed in the introduction, Theorem 1.1 implies the following uniform lower bound for the volume of closed minimal submanifolds in spaces with integral curvature bounds, which can be thought of as a generalization of Cheeger's lemma concerning the length of the shortest closed geodesic. As in Chapter 4, we denote by $\rho_k(x)$ the minimum of the k-Ricci curvatures in the tangent space at x.

Corollary 5.1. *Given integers n and m with $n \geq 3$ and $0 < m < n-1$, and real numbers $K \leq 0$, $v_0, D > 0$ and $p > n-k$ where $k = \min\{m, n-m-1\}$, there exist constants $\epsilon(n, m, p, K, v_0, D) > 0$ and $\delta(n, m, p, K, v_0, D) > 0$ such that every closed n-dimensional Riemannian manifold M satisfying*

$$\mathrm{vol}(M) \geq v_0, \quad \mathrm{diam}(M) \leq D, \quad \|(\rho_k - K)_-\|_p \leq \epsilon$$

has the property that all closed m-dimensional minimal submanifolds have volume bounded below by δ.

Proof of Corollary 5.1. Fix n, m, p, K, v_0, D as in the statement of the Corollary. By Theorem 1.1, there exists a function $F(a, b, r)$ with the property that $F \to 0$ as $a, b \to 0$ such that for any closed m-dimensional minimal submanifold Σ of a complete n-dimensional Riemannian manifold M the volume of the tube around Σ satisfies $\mathrm{vol}(T(\Sigma, r)) \leq F(\mathrm{vol}(\Sigma), \|(\rho_k - K)_-\|_p, r)$.

Given a closed minimal submanifold Σ^m of an n-dimensional closed Riemannian manifold satisfying $\mathrm{vol}(M) \geq v_0$ and $\mathrm{diam}(M) \leq D$, since $M \subset T(\Sigma, D)$ we have

$$v_0 \leq \mathrm{vol}(M) = \mathrm{vol}(T(\Sigma, D)) \leq F(\mathrm{vol}(\Sigma), \|(\rho_k - K)_-\|_p, D).$$

Since v_0 is fixed, for sufficiently small ϵ there exists a number $\delta > 0$ such that if $\|(\rho_k - K)_-\|_p \leq \epsilon$ then $\mathrm{vol}(\Sigma) \geq \delta$. □

The case of compact 1-dimensional minimal submanifolds (closed geodesics) was proved by Petersen-Shteingold-Wei as a fundamental tool in the generalization of the Grove-Petersen finiteness theorem to the setting of integral curvature bounds.

5.2 The fundamental group of a minimal submanifold

In this section, we continue our study of minimal submanifolds by proving a Frankel-type theorem for the image of the fundamental group of an immersed minimal submanifold induced by the immersion for spaces with nonnegative k-Ricci curvature.

Given an immersed submanifold $\iota : \Sigma^m \to M^n$ the immersion naturally induces a homomorphism of the fundamental group $\iota_* : \pi_1(\Sigma) \to \pi_1(M)$ for which it is natural to study the image $\iota_*(\pi_1(\Sigma)) \subset \pi_1(M)$. For minimal hypersurfaces in spaces of positive

Ricci curvature, Frankel proved [9]

Theorem 5.2 (Frankel, 1966). *Let M^n be a complete Riemannian manifold with positive Ricci curvature. Let Σ^{n-1} be a compact immersed minimal hypersurface. Then the natural homomorphism of fundamental groups $\iota_* : \pi_1(\Sigma) \to \pi_1(M)$ is surjective.*

The fundamental observation on which this result is based is that two minimal hypersurfaces in a complete manifold of positive Ricci curvature must intersect (a consequence of the Laplacian comparison theorem).

At the same time, Frankel also proved

Theorem 5.3 (Frankel, 1966). *Let M^n be a complete Riemannian manifold with positive sectional curvature. Let Σ^m be a compact totally geodesic submanifold with $2m \geq n$. Then the natural homomorphism of fundamental groups $\iota_* : \pi_1(\Sigma) \to \pi_1(M)$ is surjective.*

This theorem is based on a similar idea, namely that any two totally geodesic submanifolds, whose dimension sum is at least n must intersect. It was observed by K. Kenmotsu and C.Y. Xia [20] that one can weaken the assumption of positivity of the sectional curvature to positivity of the k-Ricci curvature at the cost of requiring the dimension sum to be at least $n + k - 1$.

Theorem 5.4 (Kenmotsu-Xia, 1995). *Let M^n be a complete Riemannian manifold with positive k-Ricci curvature. Let Σ^m be a compact totally geodesic submanifold with $2m \geq n + k - 1$. Then the homomorphism of fundamental groups $\iota_* : \pi_1(\Sigma) \to \pi_1(M)$ is surjective.*

The basic idea of all of these theorems is the same, namely, to use the Laplacian comparison in strictly positive curvature to show that two submanifolds must intersect.

Naturally, one might consider what can be said about the image of the fundamental group of a minimal submanifold in the case of nonnegative curvature. Of course, in nonnegative curvature two minimal submanifolds need not intersect. For minimal hypersurfaces, a classification of the image of the fundamental group in manifolds of nonnegative Ricci curvature was given by Galloway [12].

Theorem 5.5 (Galloway, 1987)**.** *Let M^n be a complete Riemannian manifold with nonnegative Ricci curvature. Suppose $\iota : \Sigma^{n-1} \to M$ is a minimal immersion of a compact manifold Σ.*

If the homomorphism $\iota_* : \pi_1(\Sigma) \to \pi_1(M)$ is not surjective, then $\iota(\Sigma)$ is a totally geodesic embedded submanifold and either $\pi_1(M)/\iota_*(\pi_1(\Sigma)) = \mathbb{Z}/2\mathbb{Z}$ or M (or its double cover) is isometric to a possibly twisted product of $\mathbb{S}^1$ and Σ (or a double covering of Σ).

On the other hand, much of the work on the case of minimal submanifolds of higher codimension in nonnegative sectional curvature typically introduces additional assumptions [13, 31].

In this section, we study what can be said about the image $\iota_*(\pi_1(\Sigma)) \subset \pi_1(M)$ for general minimal submanifolds $\iota : \Sigma \to M$ assuming only nonnegative k-Ricci curvature. Loosely speaking, we show that if the fundamental group of M is sufficiently large, then the image of the fundamental group of the minimal submanifold is correspondingly large. As a preliminary, the following examples should serve to illustrate how Frankel's theorem fails in nonnegative curvature.

Example 5.6. The n-dimensional torus $\mathbb{T}^n = \mathbb{T}^m \times \mathbb{T}^{n-m}$ has many totally geodesic submanifolds of the form $\mathbb{T}^m \times \{p\}$ for which the fundamental group injects as a free abelian subgroup $\mathbb{Z}^m$ of rank m in the fundamental group $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.

Example 5.7. The Riemannian product manifold $M^n = \mathbb{S}^m \times \mathbb{T}^{n-m}$ has many totally geodesic submanifolds of the form $\mathbb{S}^m \times \{p\}$, and the image of the fundamental group $\iota_*(\pi_1(\mathbb{S}^m \times \{p\})) \subset \pi_1(M)$ is trivial.

These examples show that surjectivity is not expected, and moreover the image can be trivial. However, one can rule out the latter possibility by assuming the fundamental group of M is sufficiently large, as in the case of the n-torus of the first example. This can be made precise in the sense of the asymptotic growth of the fundamental group.

Definition 5.8. Let Γ be a finitely generated group and let $\{\gamma_1, ..., \gamma_r\}$ be a set of generators. A *word* in the generators is any ordered sequence $\gamma_{i_1}^{\pm 1}\gamma_{i_2}^{\pm 1}...\gamma_{i_\ell}^{\pm 1}$; the *length* of a word is the length ℓ of the sequence. The *length* $|g|$ of an element $g \in \Gamma$ with respect to the generating set is the minimal length of all words which represent g. For any $s > 0$ define

$$\Gamma(s) = \{g \in \Gamma : |g| \leq s\}.$$

and define the *growth function* of Γ with respect to the generating set as the number of elements in the set

$$|\Gamma(s)| = \#\{g \in \Gamma : |g| \leq s\}.$$

Definition 5.9. A finitely generated group Γ has *polynomial growth of order at most p* if there exists a finite generating set such that $|\Gamma(s)| = O(s^p)$ as $s \to \infty$. Similarly, Γ has *at least polynomial growth of order p* if $\liminf_{s\to\infty} |\Gamma(s)|/s^p > 0$.

A subgroup $\Gamma' \subset \Gamma$ has *relative growth* of order at least p if $\liminf_{s\to\infty} |\Gamma'(s)|/s^p > 0$ where $\Gamma'(s) = \Gamma(s) \cap \Gamma'$, i.e. it is the set of all elements of Γ' which can be represented as a word of length less than s in the chosen generating set of Γ.

It can be readily shown that the order of growth of a group and the relative growth of its subgroups is independent of the choice of generating set. However, we note that if the subgroup is itself finitely generated, the relative growth of a subgroup may be different from the absolute growth with respect to its own generating set.

Example 5.10. Let Γ denote the Heisenberg group

$$\Gamma = \langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

The cyclic subgroup $\Gamma' = \langle c \rangle$ has quadratic relative growth in Γ. To see that Γ' has at least quadratic relative growth, note that $a^{-n}b^{-n}a^{n}b^{n} = c^{n^2}$. Multiplying by c^m for $0 \leq m \leq 2n + 1$ shows that $|\Gamma'(4n + 2n + 1)| \geq (n + 1)^2$ for all n. On the other hand, the absolute growth of any infinite cyclic group is linear.

An easy exercise shows that the relative growth of a subgroup is bounded below by the absolute growth, and in the case of a finitely generated abelian group Γ, the relative growth of its subgroups is the same as the absolute growth.

The classical theorem of Milnor [25] demonstrated the connection between the growth of the fundamental group of a complete Riemannian manifold and the asymptotic volume growth of its universal cover.

Definition 5.11. A complete Riemannian manifold M has asymptotic volume growth of order at least q if

$$\liminf_{r \to \infty} \mathrm{vol}(B(x, r))/r^q > 0$$

for some (and hence any) point $x \in M$.

M. Anderson [1] later refined these methods with further applications of volume comparison in the universal cover. Inspired by this work, we obtain the following Frankel-type theorem for the image of the fundamental group of a compact minimal submanifold under inclusion.

Theorem 5.12. *Let M^n be a complete Riemannian manifold with $Ric_k \geq 0$. If the fundamental group $\pi_1(M)$ has polynomial growth of order at least p and M has asymptotic* *volume growth of order at least q then for any immersed compact minimal submanifold $\iota : \Sigma^m \to M$ with $m \geq k$ and either $m = n-1$ or $m \leq n-k-1$ the image of the fundamental group $\iota_*(\pi_1(\Sigma)) \subset \pi_1(M)$ has relative growth of order at least $(p+q)-(n-m)$.*

Remark 5.13. In the assumption of polynomial growth we implicitly assume that $\pi_1(M)$ is finitely generated. By Milnor's theorem, the assumption $Ric \geq 0$ then implies the growth of $\pi_1(M)$ is polynomial of order at most n.

Remark 5.14. Note that if $q > n-m$ then the theorem implies that M has no compact minimal submanifolds of dimension m. This is also a direct consequence of Theorem 3.18.

Remark 5.15. If M is compact, the splitting theorem of Cheeger and Gromoll implies that the fundamental group $\pi_1(M)$ has a finite index subgroup $\mathbb{Z}^p$ for some $p \geq 0$. Thus, in this case the assumption on the growth of $\pi_1(M)$ amounts to an assumption on the size of this finite index subgroup.

Remark 5.16. In the case of nonnegative Ricci curvature, the theorem states that for any immersed compact minimal hypersurface $\iota : \Sigma^{n-1} \to M$ the image of the fundamental group has relative growth of order at least $p+q-1$ where $\pi_1(M)$ has growth of order p and M has asymptotic volume growth of order q. This is of course already clear from the theorem of Galloway above.

Proof. Let $\iota : \Sigma^m \to M$ be a minimal immersion of a compact manifold Σ and fix a point x_0 in $\iota(\Sigma)$. Put $\Gamma = \pi_1(M, x_0)$ and let Γ_Σ denote the image of $\pi_1(\Sigma)$ in $\pi_1(M, x_0)$ induced by the immersion $\iota : \Sigma \hookrightarrow M$.

Fix any generating set $\{\gamma_1, ..., \gamma_r\}$ for Γ and let $\Gamma(s)$ and $\Gamma_\Sigma(s)$ be defined as above. For each $s > 0$ consider the collection of cosets

$$\bar{\Gamma}(s) = \{g\Gamma_\Sigma : \text{ there exists } \sigma \in \Gamma_\Sigma \text{ with } |g\sigma| \leq s\}.$$

Observe that

$$|\bar{\Gamma}(s)| \geq |\Gamma(s)|/|\Gamma_\Sigma(2s)| \tag{5.1}$$

since the mapping $\Gamma(s) \to \bar{\Gamma}(s)$ defined by $g \mapsto g\Gamma_\Sigma$ has at most $|\Gamma_\Sigma(2s)|$ elements in each fiber, for if $g_1\Gamma_\Sigma = g_2\Gamma_\Sigma$ then $g_1^{-1}g_2 \in \Gamma_\Sigma$ has length less than $2s$ and hence all elements in the fiber are of the form $g_1\sigma$ for some $\sigma \in \Gamma_\Sigma(2s)$.

The quotient $\bar{M} = \widetilde{M}/\Gamma_\Sigma$ gives a sequence of covering spaces

$$\widetilde{M} \xrightarrow{\varphi} \bar{M} \xrightarrow{\pi} M$$

such that the covering map $\pi : \bar{M} \to M$ has the property that $\iota_*(\pi_1(\Sigma)) = \pi_*(\pi_1(\bar{M}))$; hence, the immersion $\iota : \Sigma \to M$ lifts to an immersion $\bar{\iota} : \Sigma \to \bar{M}$ whose image $\bar{\Sigma}$ is thus an immersed compact minimal submanifold of $\bar{M}$.

Let F be the Dirichlet fundamental domain for the action of $\pi_1(M, x_0)$ on the universal cover $\widetilde{M}$ defined by

$$F = \bigcap_{g \in \pi_1(M)} \{\widetilde{x} \in \widetilde{M} : d(\widetilde{x}, \widetilde{x}_0) \leq d(g \cdot \widetilde{x}, \widetilde{x}_0)\}$$

where $\widetilde{x}_0$ in the fiber over x_0 is chosen so that $\varphi(\widetilde{x}_0) = \bar{x}_0 \in \bar{\Sigma}$. Put $\ell = \max_i d(\gamma_i \cdot \widetilde{x}_0, \widetilde{x}_0)$. Note that since φ is distance non-increasing, for each $g\Gamma_\Sigma \in \bar{\Gamma}(s)$ taking a representative with $|g| < s$ we have

$$\varphi\left(g \cdot (B(\widetilde{x}_0, s) \cap F)\right) \subset \varphi(B(\widetilde{x}_0, (\ell+1)s)) \subset B(\bar{x}_0, (\ell+1)s).$$

Note that $\varphi(g_1F) = \varphi(g_2F)$ if $g_1\Gamma_\Sigma = g_2\Gamma_\Sigma$ and otherwise the two sets are disjoint up to a set of measure zero. Since $\operatorname{vol}(B(\widetilde{x}_0, s) \cap F) = \operatorname{vol}(B(x_0, s))$ it follows that

$$|\bar{\Gamma}(s)| \operatorname{vol}(B(x_0, s)) \leq \operatorname{vol}(B(\bar{x}_0, (\ell+1)s) \leq \operatorname{vol}(T(\bar{\Sigma}, (\ell+1)s))$$

where $T(\bar{\Sigma}, s)$ denotes the tube around $\bar{\Sigma}$ of radius s. Using the volume comparison theorem (Theorem 3.18) we have

$$|\bar{\Gamma}(s)| \operatorname{vol}(B(x_0, s)) \leq \frac{\omega_{n-m-1}}{n-m} \operatorname{vol}(\bar{\Sigma})(\ell+1)^{n-m} s^{n-m}$$

where ω_{n-m-1} is the volume of the unit $(n-m-1)$-sphere. By equation (5.1) we have

$$|\Gamma(s)| \frac{\operatorname{vol}(B(x_0, s))}{s^{n-m}} \leq \frac{\omega_{n-m-1}}{n-m} \operatorname{vol}(\bar{\Sigma})(\ell+1)^{n-m} |\Gamma_\Sigma(2s)|$$

and thus

$$|\Gamma_\Sigma(2s)| \geq C s^{(p+q)-(n-m)}$$

from which the theorem follows immediately. □

One can see that the lower bound $(p+q)-(n-m)$ is sharp by considering the examples $\mathbb{T}^m \times \mathbb{T}^{n-m-q} \times \mathbb{R}^q$ and $\mathbb{S}^m \times \mathbb{T}^{n-m-q} \times \mathbb{R}^q$. As a final remark, we emphasize that it is the *relative* growth of $\iota_*(\pi_1(\Sigma))$ in $\pi_1(M)$ that determines the asymptotic volume growth of the quotient $\widetilde{M}/\iota_*(\pi_1(\Sigma))$. A recent study of the relationship between the relative growth and absolute growth of subgroups by can be found in [8], while earlier work can be found in the references therein.

5.3 Betti number bounds for k-Ricci curvature

In manifolds of strictly positive k-Ricci curvature, the following theorem of Z. Shen showed that under certain conditions the higher Betti numbers vanish. To state the theorem, we say that a complete open n-manifold M is *proper* if the Busemann function at some point p is proper.

Theorem 5.17 (Shen [34]). *Let M be a proper open n-manifold with $Ric_k > 0$. Then M has the homotopy type of a CW complex with cells each of dimension at most $k-1$. In particular, $H_i(M, \mathbb{R}) = 0$ for $i \geq k$.*

We now investigate what might be said about the Betti numbers of a manifold with nonnegative k-Ricci curvature. Of course, a remarkable theorem of Gromov showed that for $k = 1$, there exists a universal constant bound for the total Betti number [15].

Theorem 5.18 (Gromov, 1981). *If M^n is a complete Riemannian manifold with* $\sec \geq 0$. *There exists a constant $C(n)$ such that*

$$\sum_{i=1}^{n} b_i(M, \mathbb{R}) \leq C(n).$$

On the other hand, in the Ricci curvature case $k = n-1$ one obtains a finiteness theorem for b_{n-1}. The result follows from the Bochner technique for compact manifolds and the noncompact case was proved by Yau [44] (see also [36]).

Theorem 5.19 (Yau, 1976). *If M is a complete Riemannian manifold with $Ric \geq 0$ then*

$$b_{n-1}(M, \mathbb{R}) \leq n$$

and $b_{n-1}(M, \mathbb{R}) = 0$ if $Ric > 0$ at some point.

Considering the previous result of Shen on noncompact manifolds, this motivates the following very natural question.

Question 5.20. *Does there exist a constant $C(n)$ depending on n such that all complete Riemannian n-manifolds M with $Ric_k \geq 0$ satisfy*

$$b_k(M, \mathbb{R}) \leq C(n)?$$

From the discussion above this question has an affirmative answer in the cases of nonnegative sectional and Ricci curvature, corresponding to $k = 1$ and $k = n - 1$, respectively. The proofs in the two cases are very different; the latter uses some analysis of the Laplacian operator and subharmonic functions on a Riemannian manifold [44], while the proof of the former uses an ingenious geometric argument of Gromov.

In the compact case, a much simpler approach is available through the Bochner technique. For nonnegative Ricci curvature, the Weitzenböck formula establishes the finiteness of b_1 and b_{n-1}; however, for nonnegative sectional curvature, in order to make use of the Weitzenböck formula one has to make the stronger assumption of a nonnegative curvature operator in place of nonnegative sectional curvature.

In this section we provide some evidence for the conjecture above by showing that it holds in the compact case under the additional requirement that the eigenvectors of the curvature operator are simple bivectors.

Definition 5.21. We shall say that the curvature operator $\hat{R}$ of a Riemannian manifold is *simple* if at each point $p \in M$ it can be diagonalized by a basis of simple bivectors, i.e. vectors of the form $x \wedge y$ where $x, y \in T_pM$.

Example 5.22. A Riemannian manifold (M, g) with nonnegative sectional curvature has a simple curvature operator only if it has a nonnegative curvature operator.

Example 5.23. Any Riemannian manifold (M^n, g) which can be locally isometrically immersed as a hypersurface in a space of constant curvature K has a simple curvature operator given by the Gauss equation

$$\hat{R} = S \wedge S - K(I \wedge I)$$

where S is the shape operator of the immersed hypersurface, I is the identity map of T_pM, and $\wedge$ denotes the product defined for linear operators $A, B : T_pM \to T_pM$ by

$$(A \wedge B)(x \wedge y) = \frac{1}{2}(Ax \wedge By + Bx \wedge Ay).$$

If $\{e_1, ..., e_n\}$ is an orthonormal eigenbasis for the self-adjoint operator S then $e_i \wedge e_j$ form an orthonormal eigenbasis for the curvature operator.

Example 5.24. If the Weyl curvature W of (M, g) vanishes then the curvature operator is simple since the curvature operator decomposes as

$$\hat{R} = 2\hat{Ric} \wedge I - \frac{(2n-3)s}{n(n-1)} I \wedge I + \hat{W}$$

where s is the scalar curvature and $\hat{Ric}$ and $\hat{W}$ are the endomorphisms corresponding to the Ricci and Weyl tensors, respectively. If $\hat{W} = 0$ then taking an orthonormal eigenbasis $\{e_1, ..., e_n\}$ for the self-adjoint operator $\hat{Ric}$ gives a simple eigenbasis $e_i \wedge e_j$ of $\hat{R}$. Thus, all 3-dimensional Riemannian manifolds and locally conformally flat manifolds have a simple curvature operator.

Example 5.25. Any product of Riemannian manifolds with simple curvature operators has a simple curvature operator. More generally, a curvature operator is simple if it can be written as a sum

$$\hat{R} = \sum_i B_i \wedge B_i$$

where $B_i : T_pM \to T_pM$ are commuting self-adjoint linear operators.

Example 5.26. There are smooth manifolds which do not admit a metric with simple curvature operator. For example, if a Riemannian manifold has a simple curvature operator then the Pontryagin forms must vanish [5].

Theorem 5.27. *If (M^n, g) is a compact Riemannian manifold with simple curvature operator and $Ric_k \geq 0$ then all harmonic k-forms and $(n-k)$-forms are parallel and vanish if $Ric_k > 0$ at some point.*

Corollary 5.28. *If (M^n, g) is a compact Riemannian manifold with simple curvature operator and $Ric_k \geq 0$ then*

$$b_k(M, \mathbb{R}) \leq \binom{n}{k}$$

and $b_k = 0$ if $Ric_k > 0$ at some point.

Remark 5.29. For the case of strictly positive curvature one cannot remove the assumption on the curvature operator. For example, $\mathbb{C}P^2$ with the Fubini-Study metric has $Ric_2 > 0$ but $b_2(\mathbb{C}P^2) = 1$.

Remark 5.30. As mentioned above, a sufficient condition which implies the curvature operator is simple is local conformal flatness. However, a classification of compact locally conformally flat manifolds with nonnegative Ricci curvature is already known [46]: the universal cover is either conformally equivalent to $\mathbb{S}^n$ or isometric to $\mathbb{R}^n$ or $\mathbb{R} \times \mathbb{S}^{n-1}$.

Proof. Recall the Weitzenböck formula for k-forms gives

$$\triangle\omega = \nabla^*\nabla\omega + \frac{1}{4}\sum_{i,j=1}^{n} [\theta^i \cdot \theta^j, R(e_i, e_j)\omega]$$

where $e_1, ..., e_n$ is any orthonormal frame, $\theta^1, ..., \theta^n$ the dual coframe, and ω is any smooth k-form. Here, $\theta^i \cdot \theta^j$ denotes Clifford multiplication on the space of forms. A simple computation [29, p. 220] yields

$$\sum_{i,j=1}^{n} g([\theta^i \cdot \theta^j, R(e_i, e_j)\omega], \omega) = \sum_{\alpha} \lambda_\alpha |[\Theta_\alpha, \omega]|^2$$

where Θ_α are the duals of any eigenvector basis for the curvature operator with corresponding eigenvalues λ_α.

Now, suppose the curvature operator can be diagonalized by simple bivectors $e_i \wedge e_j$ in $\Lambda^2 T_p M$. In this case, we have

$$\sum_{i,j=1}^{n} g([\theta^i \cdot \theta^j, R(e_i, e_j)\omega], \omega) = \sum_{i<j} \sec(e_i, e_j) |[\theta^i \cdot \theta^j, \omega]|^2. \tag{5.2}$$

Writing the k-form ω in terms of the basis $\theta^{i_1} \cdot ... \cdot \theta^{i_k}$ $(i_1 < i_2 < ... < i_k)$ of $\Lambda^k T_p^* M$ we have for any $i < j$

$$\begin{aligned} [\theta^i \cdot \theta^j, \omega] &= \sum_{i_1 < ... < i_k} \omega_{i_1 i_2 ... i_k} [\theta^i \cdot \theta^j, \theta^{i_1} \cdot ... \cdot \theta^{i_k}] \\ &= \sum_{i_1 < ... < i_k} \omega_{i_1 ... i_k} (1 - \sigma_{i i_1} ... \sigma_{i i_k} \sigma_{j i_1} ... \sigma_{j i_k}) \theta^i \cdot \theta^j \cdot \theta^{i_1} \cdot ... \cdot \theta^{i_k} \end{aligned}$$

where $\sigma_{ab} = 1$ if $a = b$ and $\sigma_{ab} = -1$ if $a \neq b$. This mixed form decomposes into three components, a $(k+2)$-form, a k-form, and a $(k-2)$-form. However, by pairing up the terms $\sigma_{i i_l} \sigma_{j i_l}$ for each $l = 1, ..., k$, we see that both the $(k+2)$ and $(k-2)$ components vanish. Indeed, if neither i nor j is in $\{i_1, ..., i_k\}$ then each of these pairs satisfies $\sigma_{i i_l} \sigma_{j i_l} = 1$ and so all of the coefficients of the $(k+2)$-forms vanish. Similarly, if both i, j are contained in $\{i_1, ..., i_k\}$ then since $i \neq j$ precisely two of the pairs satisfy $\sigma_{i i_l} \sigma_{j i_l} = -1$ and so once again all of these coefficients vanish. We are thus left only with the k-form

$$[\theta^i \cdot \theta^j, \omega] = \sum_{\substack{i\in\{i_1,\dots,i_k\}\\ j\notin\{i_1,\dots,i_k\}}} \pm 2\omega_{i_1\dots i_k}\theta^j \cdot \theta^{i_1} \cdot \dots \cdot \hat{\theta}^i \dots \cdot \theta^{i_k}$$
$$+ \sum_{\substack{i\notin\{i_1,\dots,i_k\}\\ j\in\{i_1,\dots,i_k\}}} \pm 2\omega_{i_1\dots i_k}\theta^i \cdot \theta^{i_1} \cdot \dots \cdot \hat{\theta}^j \dots \cdot \theta^{i_k}$$

where the sign of the coefficients is not important for us so we have left them indeterminate. Note that the first summation is orthogonal to the second, and hence

$$\left|[\theta^i \cdot \theta^j, \omega]\right|^2 = \sum_{\substack{i\in\{i_1,\dots,i_k\}\\ j\notin\{i_1,\dots,i_k\}}} 4\omega^2_{i_1\dots i_k} + \sum_{\substack{i\notin\{i_1,\dots,i_k\}\\ j\in\{i_1,\dots,i_k\}}} 4\omega^2_{i_1\dots i_k}.$$

Referring back to our equation (5.2) and rearranging terms we can obtain

$$\frac{1}{4}\sum_{i<j}\sec(e_i,e_j)\left|[\theta^i \cdot \theta^j, \omega]\right|^2 = \sum_{i_1<\dots<i_k} \omega^2_{i_1\dots i_k} \sum_{m\notin\{i_1,\dots,i_k\}}\sum_{l=1}^{k}\sec(e_m, e_{i_l}). \tag{5.3}$$

Thus, if $Ric_k \geq 0$ then the last summation is nonnegative, and hence the entire expression is nonnegative. Similarly, switching the order of the double summation we see that $Ric_{n-k} \geq 0$ also implies this expression is nonnegative. In either case, applying the Weitzenböck formula we have

$$\begin{aligned} 0 = \int_M g(\triangle\omega,\omega) &= \int_M g(\nabla^*\nabla\omega,\omega)dV_g + \int_M \frac{1}{4}\sum_{i,j=1}^{n} g([\theta^i\cdot\theta^j, R(e_i,e_j)\omega],\omega)dV_g \\ &\geq \int_M |\nabla\omega|^2 dV_g \\ &\geq 0. \end{aligned}$$

Thus, if either $Ric_k \geq 0$ or $Ric_{n-k} \geq 0$ then $|\nabla\omega| = 0$ and so ω is parallel. It follows that if $Ric_k \geq 0$ then all harmonic k-forms and $(n-k)$-forms are parallel and if $Ric_k > 0$ at some point p then any such form must vanish at p by (5.3). □

From the example in Remark 1, we cannot expect this argument to work without the assumption on the eigenvectors of the curvature operator. However, Gromov's bound on the total Betti number using nonnegative sectional curvature gives some suggestion that another approach might still yield an affirmative answer to Question 5.20.

5.4 Geometric inequalities for manifolds with mean-convex boundary

As a further application of the Hessian comparison (Theorem 3.16) we give a generalization of another Heintze-Karcher type inequality given by G. Qiu and C. Xia in [32]. The inequality of Qiu-Xia is inspired by similar inequalities of A. Ros and S. Brendle which have been used to prove Alexandrov's Theorem in various contexts (see [33, 6, 32]). Recall that Alexandrov's theorem states that every compact embedded hypersurface of constant mean curvature in Euclidean space is a sphere.

These geometric inequalities and their application to generalizations of Alexandrov's theorem are motivated by certain problems that arise from considerations in general relativity. The notions of mass and energy in the context of general relativity are still not completely understood; specifically, there are difficulties in characterizing the energy associated to the gravitational field itself. It is now a widely accepted consequence of the equivalence principle that there is no purely local description of the energy-momentum of the gravitational field; i.e. an expression for a local energy density as a covariant quantity constucted from derivatives of the metric tensor. The Hamiltonian formulation of the Einstein equations introduced by Arnowitt, Deser, and Misner led to an accepted notion of the total mass of an isolated system; however, it is still an active area of research to determine if there is a good "quasi-local" notion of mass which may be used to refine the expression for the total ADM mass (see e.g. the survey [37]).

In addition to the total mass, several suggestions have been made to introduce a notion of the center of mass of an isolated system, i.e. a conserved quantity associated to the boost symmetry of Minkowski space. Motivated by this latter problem, it was shown by Huisken and Yau that any asymptotically flat 3-manifold with positive ADM mass can be foliated by constant mean curvature surfaces outside a compact set, which is unique under certain additional assumptions [19, 45]. The uniqueness of the foliation is an important step in justifying an interpretation of the constant mean curvature foliation as being associated to the center of mass of the initial data set represented by the asymptotically flat Riemannian 3-manifold. A further justification was provided by Brendle, who classified the constant mean curvature surfaces in the Schwarzshild manifold [6] (see Example 5.34).

Theorem 5.31 (Brendle, 2013)**.** *Every closed, embedded hypersurface of constant mean curvature in the Schwarzschild manifold is a sphere of symmetry.*

More generally, Brendle classified the constant mean curvature surfaces in a broad class of warped product manifolds which also includes the de Sitter-Schwarzschild and Reissner-Nordstrom manifolds (initial data sets for the associated black hole spacetimes).

The proof of these generalized versions of Alexandrov's theorem are based on a Heintze-Karcher type inequality which was first used by Montiel and Ros [26] to greatly simplify Alexandrov's original proof.

Theorem 5.32 (Heintze-Karcher inequality)**.** *Let (M^n, g) be a compact Riemannian manifold with mean-convex boundary Σ, i.e. the mean curvature vector η is an inward-*

pointing normal. If Ric ≥ 0 then

$$\int_\Sigma \frac{1}{|\eta|} \geq n\,\mathrm{vol}(M).$$

This inequality is a direct consequence of the Heintze-Karcher volume comparison applied to the distance function from the boundary (Theorem 3.18), and together with the Minkowski formula for compact hypersurfaces Σ in Euclidean space

$$\mathrm{vol}(\Sigma) = -\int_\Sigma \langle \eta(x), x\rangle dx$$

yields a simple proof of Alexandrov's theorem for constant mean curvatures hypersurfaces in Euclidean space [33].

Brendle (together with an observation of Eichmair [6, p. 257]) generalized the Heintze-Karcher inequality by replacing the nonnegative Ricci curvature condition with the notion of a substatic potential.

Definition 5.33. Given a Riemannian manifold (M, g), a positive function $u \in C^\infty(M)$ is a *static potential* on M if

$$\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u} + Ric = 0.$$

A positive function u is a *substatic potential* if

$$\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u} + Ric \geq 0.$$

The notion of a static potential is very natural in the context of general relativity. Given a Riemannian 3-manifold (M, g), one can naturally attempt to construct a static spacetime (i.e. Lorentzian 4-manifold with hypersurface orthogonal timelike Killing field) of the form $M \times_u \mathbb{R}$ with metric

$$g - u^2 dt^2$$

where u is any positive smooth function on M. The Einstein equations for the spacetime $M \times_u \mathbb{R}$ with a general stress-energy tensor $T = \bar{T} + \rho u^2 dt^2$ where $\bar{T}$ is a symmetric 2-tensor on M representing the stress-tensor of the static frame can then be written

$$\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u} + \left(Ric - \frac{1}{2} s \right) g = \bar{T} \tag{5.4}$$

$$\frac{1}{2} s = \rho \tag{5.5}$$

where Ric and s are the Ricci and scalar curvatures of M, respectively. In other words, given a Riemannian 3-manifold (M, g) and a positive function $u \in C^\infty(M)$ the Lorentzian warped product $M \times_u \mathbb{R}$ satisfies the Einstein equations for the stress energy tensor

$$T = -\frac{1}{2} s \left(g - u^2 dt^2 \right) + \left(\frac{\Delta u}{u} \right) g - \frac{\nabla^2 u}{u} + Ric. \tag{5.6}$$

Thus, the function u is static precisely if the stress-energy tensor above is either vacuum ($T = 0$) or a perfect fluid of the form $T = -\rho(g - u^2 dt^2)$. Moreover, the notion of a substatic potential corresponds directly to common energy conditions in general relativity. Specifically, the null energy condition holds if and only if u is substatic, the weak energy condition holds if and only if u is substatic and the scalar curvature s is nonnegative, and the strong energy condition holds if and only if u is substatic and subharmonic ($\Delta u \geq 0$). The definitions of these energy conditions can be found in any standard reference on general relativity (e.g. [38]) and the proof of these statements is a straightforward consequence of the expression (5.6) for the stress-energy tensor.

Example 5.34 (Schwarzschild manifold)**.** The Schwarzschild manifold is defined by

$M^n = \mathbb{S}^{n-1} \times (2m, \infty)$ with metric

$$g = \frac{1}{1 - 2m/r^{n-2}} dr^2 + r^2 g_{\mathbb{S}^{n-1}}.$$

The function $u = \sqrt{1 - 2m/r^{n-2}}$ is a static potential on M. The Lorentzian warped product $M \times_u \mathbb{R}$ is the well-known exterior Schwarzschild black hole spacetime.

Example 5.35. If M has constant sectional curvature K then for any fixed point $p \in M$ putting $r(x) = d(x, p)$ the function $\mathrm{cs}_K(r)$ is a substatic potential on $M \setminus \mathrm{cut}\{p\}$.

Example 5.36. If M is any Riemannian manifold with $Ric \geq 0$ then the constant function $u \equiv 1$ is a substatic potential on M.

With this background, we now state Brendle's inequality.

Theorem 5.37 (Brendle, 2013)**.** *Let M^n be a compact Riemannian manifold with connected mean-convex boundary Σ and substatic potential $u \in C^\infty(M)$. Then*

$$\int_\Sigma \frac{u}{|\eta|} dvol_\Sigma \geq n \int_M u \, dvol_M$$

with equality only if Σ is umbilic.

A more general version with multiple boundary components was proved by J. Li and C. Xia [22]. In Example 5.36 we observed that for any manifold M with $Ric \geq 0$ the constant function $u \equiv 1$ is a substatic potential, which recovers the previous Heintze-Karcher inequality.

In light of the fact that every manifold M with $Ric \geq 0$ admits a Brendle-type inequality, one may ask if such an inequality holds for manifolds with other curvature bounds. In [32], G. Qiu and C. Xia obtained a version of Brendle's inequality which holds for manifolds with a sectional curvature bound $\sec \geq -1$.

Theorem 5.38 (Qiu-Xia, 2014). *Let (M^n, g) be a compact Riemannian manifold with mean-convex boundary Σ and $\sec \geq -1$. If M is star-shaped with respect to $p \in M$ and $u(x) = \cosh(r(x))$ where $r(x) = d(x, p)$ then*

$$\int_\Sigma \frac{u}{|\eta|} dvol_\Sigma \geq \int_M (\Delta u) dvol_M. \tag{5.7}$$

Equality holds only if M is a geodesic ball in a space form of constant sectional curvature -1.

In that paper, the authors ask if the inequality holds assuming only a Ricci lower bound $Ric \geq -(n-1)g$. In this section, we show that the inequality holds under a lower bound for the $(n-2)$-Ricci curvature.

Before proving this, we remark that this inequality, despite its apparent similarity with Brendle's inequality, is of quite a different nature. In particular, no claim is made that the function u is a substatic potential, in fact, in the course of the proof one finds that

$$\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u} - (n-1)g \leq 0 \tag{5.8}$$

which appears to be in the opposite direction from the substatic potentials used in Brendle's inequality. At the same time, the resulting inequality (5.7) is weaker since it follows from (5.8) that $\Delta u \leq nu$.

To better understand this situation, we will show that the inequality of Qiu-Xia can be understood more generally in terms of static solutions to the Einstein equations with cosmological constant. To emphasize this distinction we introduce the following definition.

Definition 5.39. Given a Riemannian manifold (M, g) we shall call a positive continuous function u on M which is twice differentiable almost everywhere a *static Einstein potential* with Einstein constant λ if

$$\frac{\Delta u}{u}g - \frac{\nabla^2 u}{u} + \lambda g = 0.$$

for some constant λ. We say that u is a *substatic Einstein potential* if

$$\frac{1}{\lambda}\left(\frac{\Delta u}{u}g - \frac{\nabla^2 u}{u}\right) + g \geq 0.$$

Note that if (M, g) is an Einstein manifold with $Ric = \lambda g$ then a static Einstein potential is equivalent to a static potential. Note also that when $\lambda < 0$, the substatic condition implies an upper bound on the tensor in brackets as in (5.8). These notions can again be interpreted in terms of the Lorentzian warped product $M \times_u \mathbb{R}$ above. Specifically, adding a non-zero cosmological constant λ to the Einstein equation (5.4) the stress tensor of the static rest frame becomes

$$\bar{T} = \frac{\Delta u}{u}g - \frac{\nabla^2 u}{u} + \lambda g + \left(Ric - \frac{1}{2}sg\right).$$

Thus, a positive function u is a static Einstein potential precisely when the stress tensor $\bar{T}$ of the static frame reduces to the Einstein tensor $G = Ric - (1/2)sg$ of M. For non-zero cosmological constant λ, the substatic condition is equivalent to $\bar{T}/\lambda \geq G/\lambda$.

We now show that the inequality of Qiu-Xia can be understood in terms of a general inequality for substatic Einstein potentials in spaces with Ricci curvature bounded below.

Theorem 5.40. *Let (M^n, g) be a compact Riemannian manifold with mean-convex boundary Σ and $Ric \geq (n-1)Kg$ for some non-zero constant K. If u is a substatic Einstein potential for the Einstein constant $\lambda = (n-1)K$, then*

$$\int_\Sigma \frac{u}{|\eta|} dvol_\Sigma \geq -\frac{1}{K}\int_M (\Delta u) dvol_M. \tag{5.9}$$

Equality holds only if M is a geodesic ball in a space form of constant curvature K and $u(x) = \mathrm{cs}_K(r(x))$ where $r(x) = d(x,p)$ for $p \in M$.

Before giving the proof, we prove the following proposition showing that spaces with a lower bound on the $(n-2)$-Ricci curvature admit many substatic Einstein potentials.

Proposition 5.41. *Let (M^n, g) be a Riemannian manifold (possibly with boundary) which is star-shaped with respect to $p \in M$ and put $r(x) = d(x,p)$. If $Ric_{n-2}(\nabla r, \cdot) \geq K$ for some non-zero constant K and $f \in C^2([0,\infty))$ is any positive function satisfying*

(i) $0 \leq -\frac{f'}{Kf} \leq \frac{\mathrm{sn}_K}{\mathrm{cs}_K}$

(ii) $-\frac{f''}{Kf} \leq 1$

then the radial function $u(x) = f(r(x))$ is a substatic Einstein potential on M for the constant $\lambda = (n-1)K$.

Remark 5.42. This proposition, together with Theorem 5.40 implies that the Qiu-Xia inequality of Theorem 5.38 holds with the sectional curvature lower bound replaced by an $(n-2)$-Ricci lower bound.

Proof. Let f be as in the statement of the proposition and put $u(x) = f(r(x))$. A direct computation gives

$$\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u} = \frac{f''(r)}{f(r)}[g - dr^2] + \frac{f'(r)}{f(r)}[(\Delta r)g - \nabla^2 r].$$

Multiplying both sides by $1/K$ and using the first condition on f, together with the k-Ricci Hessian comparison theorem (Theorem 3.16) we obtain

$$\frac{1}{K}\left(\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u}\right) \geq (n-1)\frac{f'}{Kf}\frac{\mathrm{cs}_K}{\mathrm{sn}_K} dr^2 + \left(\frac{f''}{Kf} + (n-2)\frac{f'}{Kf}\frac{\mathrm{cs}_K}{\mathrm{sn}_K}\right)[g - dr^2].$$

Again applying both conditions we find

$$\frac{1}{K}\left(\frac{\Delta u}{u}g-\frac{\nabla^2 u}{u}\right)\geq -(n-1)g.$$

□

Finally, we conclude with the proof of Theorem 5.40. In the case of a negative Einstein constant $\lambda<0$ the proof is essentially the same as that given by Qiu-Xia and is based on the following weighted version of Reilly's formula [32].

Lemma 5.43 (Qiu-Xia, 2014)**.** *Let (M^n,g) be a compact Riemannian manifold with smooth boundary Σ and outward unit normal ξ and let $u:M\to\mathbb{R}$ be a.e. twice differentiable. For any real constant K and smooth function $f\in C^\infty(M)$ such that $f|_\Sigma=f_0$ is constant*

$$\begin{aligned}(n-1)&\int_\Sigma 2u\langle\nabla f,\xi\rangle Kf_0-\langle\nabla u,\xi\rangle Kf_0^2-u\langle\eta,\xi\rangle\langle\nabla f,\xi\rangle^2 dvol_\Sigma\\ &=\int_M u\big((\Delta f+nKf)^2-|\nabla^2 f+Kfg|^2-[Ric-(n-1)Kg](\nabla f,\nabla f)\big)\,dvol_M\\ &+\int_M\big(\Delta ug-\nabla^2 u+u(n-1)Kg\big)(\nabla f,\nabla f)-(n-1)K(\Delta u+nKu)f^2 dvol_M\end{aligned}$$

where η is the mean curvature vector of Σ.

In the case of a positive Einstein constant $\lambda>0$, the integral formula above does not appear to yield the inequality (5.9) due to the sign of the last term on the right hand side. However, in this case, we show that the inequality (5.9) is actually a consequence of Brendle's inequality.

Proof of Theorem 5.40. First we consider the case of substatic Einstein potential u for a negative constant $\lambda=(n-1)K<0$. Let f be the solution to the Dirichlet boundary value problem

$$\begin{cases} \Delta f = -nKf \\ f|_\Sigma = c > 0 \end{cases}$$

From the substatic inequality it follows that

$$\left(\Delta u g - \nabla^2 u + u(n-1)Kg\right)(\nabla f, \nabla f) \leq 0$$

and

$$-(n-1)K(\Delta u + nKu)f^2 \leq 0$$

and hence the integral formula of Theorem 5.43 gives

$$\int_\Sigma u|\eta|\langle \nabla f, \xi\rangle^2 \leq -K \int_\Sigma \left(2u\langle \nabla f, \xi\rangle c - \langle \nabla u, \xi\rangle c^2\right). \tag{5.10}$$

Now, using Hölder's inequality followed by the previous inequality we get

$$\begin{aligned} \left(\int_\Sigma u\langle \nabla f, \xi\rangle\right)^2 &\leq \int_\Sigma \frac{u}{|\eta|} \int_\Sigma u|\eta|\langle \nabla f, \xi\rangle^2 \\ &\leq -K \int_\Sigma \frac{u}{|\eta|} \int_\Sigma \left(2u\langle \nabla f, \xi\rangle c - \langle \nabla u, \xi\rangle c^2\right). \end{aligned}$$

It follows that

$$\left(\int_\Sigma u\langle \nabla f, \xi\rangle + cK\int_\Sigma \frac{u}{|\eta|}\right)^2 - c^2K^2\left(\int_\Sigma \frac{u}{|\eta|}\right)^2 \leq c^2 K \int_\Sigma \frac{u}{|\eta|}\int_\Sigma \langle \nabla u, \xi\rangle.$$

Dropping the first term (since it's nonnegative) and dividing out non-zero factors we obtain

$$\int_\Sigma \frac{u}{|\eta|} \geq -\frac{1}{K}\int_\Sigma \langle \nabla u, \xi\rangle = -\frac{1}{K}\int_M \Delta u$$

as desired. If equality holds than equality must hold in (5.10) and hence $(\Delta f + nKf)^2 = |\nabla^2 f + Kfg|^2$. By construction $\Delta f + nKf = 0$ and hence in the equality case we have $\nabla^2 f = -Kfg$ in M. Since $f|_\Sigma = c$ it follows from an Obata type rigidity result that M must be a geodesic ball in a space form of constant curvature -1 (see e.g. [40, Theorem 5.1]).

The case of a substatic Einstein potential u with positive constant $\lambda = (n-1)K > 0$ follows from Brendle's inequality. Specifically, since $Ric \geq (n-1)Kg$ it follows that u is, in particular, a substatic potential

$$\frac{\Delta u}{u} g - \frac{\nabla^2 u}{u} + Ric \geq 0.$$

Applying Brendle's inequality, we find

$$\int_\Sigma \frac{u}{|\eta|} dvol_M \geq \int_M nu \, dvol_M.$$

Moreover, taking the trace of the defining inequality for a substatic Einstein potential with $\lambda = (n-1)K$ gives

$$\frac{1}{K}\frac{\Delta u}{u} + n \geq 0 \tag{5.11}$$

and hence

$$\int_\Sigma \frac{u}{|\eta|} dvol_M \geq \int_M nu \, dvol_M \geq -\frac{1}{K}\int_M \Delta u \, dvol_M.$$

If equality holds, then equality holds in (5.11). Since u is a substatic Einstein potential we thus have $\nabla^2 u \leq (\Delta u)g + u(n-1)Kg = -uKg$ which combined with $\Delta u = -nKu$ implies $\nabla^2 u = -uKg$. As above, it follows from Obata rigidity that M has constant sectional curvature K and $u = \mathrm{cs}_K(d(x,p))$. By rigidity in Brendle's inequality Σ is totally umbilic and hence is a geodesic sphere. □

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