Strong Macdonald Theory and the Brylinski Filtration for Affine Lie Algebras

by

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Abstract

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The strong Macdonald theorems state that, for L reductive and s an odd variable, the cohomology algebras $H^*(L[z]/z^N)$ and $H^*(L[z,s])$ are freely generated, and describe the cohomological, s-, and z-degrees of the generators. The resulting identity for the z-weighted Euler characteristic is equivalent to Macdonald's constant term identity for a finite root system. The proof of the strong Macdonald theorems, due to Fishel, Grojnowski, and Teleman, uses a Laplacian calculation for the (continuous) cohomology of $L[[z]]$ with coefficients in the symmetric algebra of the (continuous) dual of $L[[z]]$.

Our main result is a generalization of this Laplacian calculation to the setting of a general parahoric $\mathfrak{p}$ of a (possibly twisted) loop algebra $\mathfrak{g}$. As part of this result, we give a detailed exposition of one of the key ingredients in Fishel, Grojnowski, and Teleman's proof, a version of Nakano's identity for infinite-dimensional Lie algebras.

We apply this Laplacian result to prove new strong Macdonald theorems for $H^*(\mathfrak{p}/z^N\mathfrak{p})$ and $H^*(\mathfrak{p}[s])$, where $\mathfrak{p}$ is a standard parahoric in a twisted loop algebra. We show that $H^*(\mathfrak{p}/z^N\mathfrak{p})$ contains a parabolic subalgebra of the coinvariant algebra of the fixed-point subgroup of the Weyl group of L, and thus is no longer free. We also prove a strong Macdonald theorem for $H^*(\mathfrak{b};S^*\mathfrak{n}^*)$ and $H^*(\mathfrak{b}\,/z^N\mathfrak{n})$ when $\mathfrak{b}$ and $\mathfrak{n}$ are Iwahori and nilpotent subalgebras respectively of a twisted loop algebra. For each strong Macdonald theorem proved, taking z-weighted Euler characteristics gives an identity equivalent to Macdonald's constant term identity for the corresponding affine root system. As part of the proof, we study the regular adjoint orbits for the adjoint action of the twisted arc group associated to L, proving an analogue of the Kostant slice theorem.

Our Laplacian calculation can also be adapted to the case when $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. In this case, the Laplacian calculation leads to a generalization of the Brylinski identity for affine Kac-Moody algebras. In the semisimple case, the Brylinski identity states that, at dominant weights, the q-analog of weight multiplicity is equal to the Poincare series of the principal nilpotent filtration of the weight space. This filtration is known as the Brylinski filtration. We show that this identity holds in the affine case, as long as the principal nilpotent filtration is replaced by the principal Heisenberg. We also give an example to show that the Poincare series of the principal nilpotent filtration is not always equal to the q-analog of weight multiplicity, and give some partial results for indefinite Kac-Moody algebras.

To Anyu

For your constant love and support.

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Chapter 1

Introduction

The purpose of this dissertation is to study the Lie algebra cohomology of affine Kac-Moody algebras and loop algebras, with two goals: extending the Brylinski filtration for semisimple Lie algebras to affine Kac-Moody algebras, and proving new strong Macdonald theorems for parahoric subalgebras of (possibly twisted) affine Kac-Moody algebras. This chapter contains an overview of these results, and explains how they fit into the rich theory of semisimple Lie algebras and Kac-Moody algebras. The reader is assumed to be familiar with the basic representation theory and structure theory of semisimple Lie algebras. Section 1.1 gives a short introduction, with references, to Kac-Moody algebras and Lie algebra cohomology. Section 1.2 explains the development of the Brylinski filtration, starting with Kostant's generalized exponents, and ending with the Brylinski filtration for a Kac-Moody algebra. Finally, Section 1.3 covers the Macdonald constant term identity, the strong Macdonald theorems of Fishel, Grojnowski, and Teleman, and strong Macdonald theorems for parahoric subalgebras. New material is covered in Subsections 1.2.1 and 1.3.1; most of what is discussed has previously been presented in [Sl11a] and [Sl11b].

1.1 Background

1.1.1 Kac-Moody algebras

Let L be a complex semisimple Lie algebra. It is well-known that L has a presentation as the free Lie algebra generated by elements $\{h_1, \ldots, h_l\}$, $\{e_1, \ldots, e_l\}$, and $\{f_1, \ldots, f_l\}$, satisfying the Serre relations

$$\begin{aligned}
&[h_i, e_j] = A_{ij} e_j, \quad 1 \leq i, j \leq l, \\
&[e_i, f_j] = \delta_{ij} h_i, \quad 1 \leq i, j \leq l, \\
&\mathrm{ad}(e_i)^{1-A_{ij}}(e_j) = 0, \quad 1 \leq i \neq j \leq l, \text{ and} \\
&\mathrm{ad}(f_i)^{1-A_{ij}}(f_j) = 0, \quad 1 \leq i \neq j \leq l,
\end{aligned}$$

where δ_{ij} is the Kronecker delta, and A is an $l \times l$ matrix with integer coefficients. The matrix A satisfies the following conditions:

- $A_{ii} = 2$,
- $A_{ij} \leq 0$ if $i \neq j$,
- $A_{ij} = 0$ if $A_{ji} = 0$, and
- A is positive definite.

Any matrix satisfying these conditions is called a Cartan matrix, and gives rise to a presentation of a semisimple Lie algebra. A matrix satisfying the first three conditions, but which is not necessarily positive-definite, is called a generalized Cartan matrix. Just as Cartan matrices correspond to semisimple Lie algebras, the generalized Cartan matrices correspond to the larger family of Kac-Moody algebras. Specifically, if A is a generalized Cartan matrix, let $\mathfrak{h}$ be a vector space of dimension $l + \operatorname{corank} A$, such that $\mathfrak{h}$ contains the free vector space spanned by symbols $\{h_1, \ldots, h_l\}$. Choose $\alpha_1, \ldots, \alpha_l$ in $\mathfrak{h}^*$ such that $\alpha_j(h_i) = A_{ij}$. The Kac-Moody algebra associated to a generalized Cartan matrix is the free Lie algebra generated by $\mathfrak{h}$, $\{e_1, \ldots, e_l\}$, and $\{f_1, \ldots, f_l\}$, satisfying the relations

$$
\begin{aligned}
&[h, e_j] = \alpha_j(h) e_j, \quad 1 j \leq l, \\
&[e_i, f_j] = \delta_{ij} h_i, \quad 1 \leq i, j \leq l, \\
&\operatorname{ad}(e_i)^{1-A_{ij}}(e_j) = 0, \quad 1 \leq i \neq j \leq l, \text{ and} \\
&\operatorname{ad}(f_i)^{1-A_{ij}}(f_j) = 0, \quad 1 \leq i \neq j \leq l.
\end{aligned}
$$

Kac-Moody algebras were discovered by Kac [Ka67] and Moody [Mo67], and have proven to be very important and useful objects in mathematics and physics. For example, they appear in string theory and conformal field theory. One of the most surprising things about Kac-Moody algebras is the extent to which they are analogous to finite-dimensional Lie algebras. Kac-Moody algebras have a Weyl group, a root system, flag varieties, and irreducible highest weight representations $\mathcal{L}(\lambda)$ parametrized by dominant weights. A general overview of the structure and representation theory of Kac-Moody algebras can be found in [Ka83]. For flag varieties and representation theory, more background can be found in [Ku02]. We follow [Ku02] in our presentation except where noted.

A generalized Cartan matrix is said to be indecomposable if no conjugate of A by a permutation matrix has a non-trivial block decomposition. The classification theorem for Kac-Moody algebras divides indecomposable Kac-Moody algebras into three types: if the generalized Cartan matrix is positive definite, then the Kac-Moody algebra is a simple finite-dimensional Lie algebra, and is said to be of finite type; if the generalized Cartan matrix positive semi-definite, then the Kac-Moody algebra is said to be of affine type; otherwise the generalized Cartan matrix is indefinite, and the Kac-Moody algebra is said to be of indefinite type. Affine Kac-Moody algebras are of particular interest, perhaps because they can be constructed using loop algebras, as we will see in the next section. In contrast, no nice construction for indefinite Kac-Moody algebras is known.

A Kac-Moody algebra $\mathfrak{g}$ can be given a $\mathbb{Z}$-grading by assigning every generator e_i a non-negative degree $d_i \geq 0$. In the Kac terminology such a grading is called a grading of type d [Ka83]. Let $\mathfrak{g}_n$ denote the degree n component of $\mathfrak{g}$ with respect to some grading of type d. Then $\mathfrak{g}_0$ is again a Kac-Moody algebra, and $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{g}_0 \oplus \mathfrak{u}$, where $\mathfrak{u} = \bigoplus_{n>0} \mathfrak{g}_n$ and $\overline{\mathfrak{u}} = \bigoplus_{n<0} \mathfrak{g}_n$. A subalgebra $\mathfrak{p}$ is said to be a parahoric of $\mathfrak{g}$ if it is of the form $\mathfrak{p} = \bigoplus_{n\geq 0} \mathfrak{g}_n$ for some grading of type d. When every d_i is strictly positive, $\mathfrak{g}_0 = \mathfrak{h}$, and the decomposition $\mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{h} \oplus \mathfrak{u}$ is an analogue of the triangular decomposition for semisimple Lie algebras. If $\mathfrak{g}$ is an affine Kac-Moody algebra constructed from a (possibly twisted) loop algebra $L[z^{\pm 1}]^{\tilde{\sigma}}$ (this construction will be explained in the next section), let $\mathfrak{p}_0$ be a fixed parabolic subalgebra of L^{σ}, and let $\mathfrak{p}$ denote the subalgebra $\{f \in L[z] : f(0) \in \mathfrak{p}_0\} \oplus \mathbb{C}c \oplus \mathbb{C}d$. The algebra $\mathfrak{p}$ is a parahoric, and we call subalgebras of this form standard parahorics.

A Kac-Moody algebra is said to be symmetrizable if there is a diagonal matrix D with positive integral entries such that $D^{-1}A$ is symmetric. A symmetrizable Kac-Moody algebra has an invariant bilinear form. All finite and affine Kac-Moody algebras are symmetrizable.

1.1.2 Loop algebras and affine Kac-Moody algebras

Let L be a reductive Lie algebra with diagram automorphism σ of finite order k. By definition L has a triangular decomposition $L = \overline{\mathfrak{u}_0} \oplus \mathfrak{h} \oplus \mathfrak{u}_0$ where the Cartan algebra $\mathfrak{h}$ and nilpotent radicals $\overline{\mathfrak{u}_0}$ and $\mathfrak{u}_0$ are σ-invariant, and such that σ permutes the simple roots corresponding to the Borel $\mathfrak{h} \oplus \mathfrak{u}_0$. We say that a Cartan, Borel, or nilpotent radical is compatible with σ if it appears in such a decomposition. The twisted loop algebra is the Lie algebra $\mathfrak{g} = L[z^{\pm 1}]^{\tilde{\sigma}}$, where $\tilde{\sigma}$ is the automorphism sending $f(z) \mapsto \sigma(f(q^{-1}z))$ for q a fixed kth root of unity. $\mathfrak{g}$ can be written as

$$\mathfrak{g} = \bigoplus_{i=0}^{k-1} L_a \otimes z^a \mathbb{C}[z^{\pm k}],$$

where L_a is the q^ath eigenspace of σ. If L is simple then each L_a is an irreducible L_0-module. In particular if L is simple then L_0 is also simple; in general L_0 will be reductive. A reductive Lie algebra L has an anti-linear Cartan involution $\bar{\cdot}$ and a contragradient positive-definite Hermitian form $\{,\}$. These two structures extend to the twisted loop algebra $\mathfrak{g}$ so that for any grading of type d, $\overline{\mathfrak{g}_n} = \mathfrak{g}_{-n}$ and $\mathfrak{g}_m \perp \mathfrak{g}_n$ when $m \neq n$.

The root system of $\mathfrak{g}$ can be described as follows. Let $\mathfrak{h}$ be a Cartan subalgebra of L compatible with the diagram automorphism. Then $\mathfrak{h}_0 := \mathfrak{h}^{\sigma}$ is a Cartan in L_0, and L_0 has a set $\alpha_1, \ldots, \alpha_l$ of simple roots which are projections of simple roots of L. The roots of $\mathfrak{g}$ can be described as $\alpha + n\delta \in \mathfrak{h}_0^* \times \mathbb{Z}$ where either α is a weight of L_a with $n \equiv a \mod k$, or $\alpha = 0$ and $n \neq 0$, and δ comes from the rotation action of $\mathbb{C}^*$ on $\mathfrak{g}$. Assume that L is simple, and let ψ be either the highest weight of L_1 (an irreducible L_0-module) if $k > 1$, or the highest root of L, if $k = 1$. Then the set $\alpha_0 = \delta - \psi, \alpha_1, \ldots, \alpha_l$ is a complete set of simple roots for $\mathfrak{g}$. If L is reductive then we can choose a set of simple roots by decomposing L as a direct sum of σ-invariant simple subalgebras plus centre, and taking the simple root sets from each corresponding factor of $\mathfrak{g}$.

As with Kac-Moody algebras, the twisted loop algebra $\mathfrak{g}$ can be given a $\mathbb{Z}$-grading by assigning degree $d_i \geq 0$ to the positive root vector associated to α_i. A parahoric subalgebra of $\mathfrak{g}$ is a subalgebra of the form $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$, for some $\mathbb{Z}$-grading of $\mathfrak{g}$ of type d. A parahoric subalgebra contains a nilpotent subalgebra defined by $\mathfrak{u} = \bigoplus_{n>0} \mathfrak{g}_n$. We will say that a parahoric is standard with respect to the choice of simple roots if it comes from a grading of type d such that $d_i > 0$ whenever α_i is of the form $\delta - \psi$ for $\psi \in \mathfrak{h}_0^*$. Suppose $\mathfrak{p}$ is a standard parahoric. Let $S = \{\alpha_i : d_i = 0\}$, and $\mathfrak{p}_0$ be the standard parabolic subalgebra of L_0 defined by

$$\mathfrak{p}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta^+} (L_0)_\alpha \oplus \bigoplus_{\alpha \in \Delta^- \cap \mathbb{Z}[S]} (L_0)_\alpha,$$

where $\Delta^\pm$ are the positive and negative roots of L_0 with respect to the chosen simple roots. Then $\mathfrak{p} = \{f \in \mathfrak{g} : f(0) \in \mathfrak{p}_0\}$, while $\mathfrak{u} = \{f \in \mathfrak{g} : f(0) \in \mathfrak{u}_0\}$, where $\mathfrak{u}_0$ is the nilpotent radical of $\mathfrak{p}_0$. Note that in this context the nilpotent radical of an algebra $\mathfrak{k}$ is defined to be the largest nilpotent ideal in $[\mathfrak{k}, \mathfrak{k}]$ (or equivalently the intersection of the kernels of all irreducible representations), so that $\mathfrak{u}_0$ does not intersect the centre of L. If $\mathfrak{p}_0$ is a Borel, then $\mathfrak{p}$ is called a standard Iwahori subalgebra.

If L is simple, define $\tilde{\mathfrak{g}}$ to be the Lie algebra $L[z^{\pm 1}]^{\tilde{\sigma}} \oplus \mathbb{C}c \oplus \mathbb{C}d$, where the bracket is defined, for $x, y \in L$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \mathbb{C}$, by

$$[xz^m + \gamma_1 c + \beta_1 d, yz^n + \gamma_2 c + \beta_2 d] = \\ [x,y]z^{m+n} + \beta_1 n y z^n - \beta_2 m x z^m + \delta_{m,-n} m \langle x, y \rangle c,$$

for $\langle , \rangle$ a symmetric invariant bilinear form on L. Then $\tilde{\mathfrak{g}}$ is an indecomposable affine Kac-Moody algebra, and every indecomposable affine Kac-Moody algebra arises in this fashion. When σ is the identity, the affine Kac-Moody algebra is said to be untwisted; otherwise the Kac-Moody algebra is said to be twisted. The root system of $\tilde{\mathfrak{g}}$ as a Kac-Moody algebra is the same as the root system of the loop algebra $\mathfrak{g}$ described above. If L is semisimple then it is still possible to turn $\mathfrak{g}$ into a (possibly decomposable) Kac-Moody algebra, but this construction needs to be repeated for every simple component of L.

1.1.3 Lie algebra cohomology

Suppose that $\mathfrak{g}$ is a Lie algebra, $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, and V is an L-module. The Koszul homology complex (also called the Chevalley, Chevalley-Eilenberg, or Koszul-Chevalley complex) C_* for the pair $(\mathfrak{g}, \mathfrak{k})$ with coefficients in V is defined as the space of coinvariants

$$C_p = \left(\bigwedge^p \mathfrak{g}/\mathfrak{k} \otimes V \right) \Big/ \mathfrak{k} \cdot \left(\bigwedge^p \mathfrak{g}/\mathfrak{k} \otimes V \right),$$

with differential d sending

$$x_1 \wedge \cdots x_k \otimes v \mapsto \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \hat{x_j} \cdots \wedge x_k \\ + \sum_i (-1)^i x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_k \otimes x_i v,$$

where $\hat{x_i}$ denotes that the symbol is omitted. The Lie algebra homology $H_*(\mathfrak{g}, \mathfrak{k}; V)$ is the homology of the complex (C_*, d). Cohomology spaces are defined similarly: the Koszul complex C^* is defined to be the space of invariants,

$$C^p = \left(\left(\bigwedge^p \mathfrak{g}/\mathfrak{k} \right)^* \otimes V \right)^{\mathfrak{k}} \cong \mathrm{Hom}_{\mathfrak{k}} \left(\bigwedge^p \mathfrak{g}/\mathfrak{k}, V \right)$$

with differential $\bar{\partial}$ defined, for $f \in \mathrm{Hom}_{\mathfrak{k}}(\bigwedge^k \mathfrak{g}/\mathfrak{k}, V)$, by

$$(\bar{\partial} f)(x_1, \ldots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \hat{x_i}, \cdots, \hat{x_j}, \cdots, x_{k+1}) \\ + \sum_i (-1)^{i+1} x_i f(x_1, \ldots, \hat{x_i}, \ldots, x_{k+1}).$$

The cohomology spaces $H^*(\mathfrak{g}, \mathfrak{k}; V)$ are then defined to be the homology spaces of the complex $(C^*, \bar{\partial})$. Lie algebra cohomology was first defined by Chevalley and Eilenberg [CE48]; the treatment here is based on [Ku02].

Cohomology and homology groups are dual, in the sense that $H^*(\mathfrak{g}, \mathfrak{k}; V^*) \cong H_*(\mathfrak{g}, \mathfrak{k}; V)^*$. If $\mathfrak{g}$ is infinite-dimensional, then $H_*(\mathfrak{g}, \mathfrak{k}; V)$ can be infinite-dimensional, and hence the cohomology spaces can be very large: the full dual of an infinite-dimensional vector space. If $\mathfrak{g}$ is a topological Lie algebra, and V is a continuous representation, then we can take, in the Koszul complex, $\mathrm{Hom}_{\mathfrak{k}}(\bigwedge^* \mathfrak{g}/\mathfrak{k}, V)$ to be the space of continuous $\mathfrak{k}$-invariant maps. The cohomology of the resulting complex is called the continuous cohomology of $(\mathfrak{g}, \mathfrak{k})$ with coefficients in V, and the resulting cohomology spaces will be denoted by $H^*_{cts}(\mathfrak{g}, \mathfrak{k}; V)$. For more background on continuous cohomology, see [Fu86]. Continuous cohomology is a very valuable tool for working with infinite-dimensional algebras, and we use it without exception. Furthermore, the dual $(\mathfrak{g}/\mathfrak{k})^*$ will always denote the continuous dual. However, for our purposes it is enough to consider only one special case of this construction. Suppose $\mathfrak{p}$ is a $\mathbb{Z}_{\geq 0}$-graded Lie algebra with finite-dimensional homogeneous components, so $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{p}_n$ with $[\mathfrak{p}_n, \mathfrak{p}_m] \subset \mathfrak{p}_{m+n}$ and $\dim \mathfrak{p}_n < +\infty$. Take $\mathfrak{k} = \mathfrak{p}_0$. The algebra $\mathfrak{p}$ has a completion

$$\hat{\mathfrak{p}} = \varprojlim_k \mathfrak{p} \Big/ \bigoplus_{n \geq k} \mathfrak{p}_n,$$

where lim denotes the inverse limit. The completion $\hat{\mathfrak{p}}$ can given the inverse limit topology, and $(\bigwedge^* \hat{\mathfrak{p}}/\mathfrak{p}_0)^* = \bigwedge^*(\hat{\mathfrak{p}}/\mathfrak{p}_0)^*$, where the continuous dual $(\hat{\mathfrak{p}}/\mathfrak{p}_0)^*$ is linearly isomorphic to $\bigoplus_{n>0} \mathfrak{p}_n^*$, a direct sum of duals of finite-dimensional spaces. Assuming that V has a compatible $\mathbb{Z}_{\geq 0}$-grading with finite-dimensional components, we can define the completion $\hat{V}$ in a similar manner. The homology spaces $H_*(\mathfrak{p}, \mathfrak{p}_0; V)$ and cohomology spaces $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{p}_0; \hat{V})$ then inherit $\mathbb{Z}_{>0}$-gradings, and the homogeneous components are finite-dimensional and dual to each other.

Lie algebra cohomology has been an important object of study throughout the development of the theory of semisimple Lie algebras and groups. We highlight two historically significant results. The first is the calculation of the cohomology of a reductive Lie algebra. Recall that if G is a Lie group, a well-known theorem of Hopf states that the cohomology $H^*(G)$ is a free super-commutative algebra generated in odd degrees. Accordingly, the Poincare polynomial of G can be written as

$$\sum q^i \dim H^i(G) = \prod_j \left(1 + q^{2m_i+1}\right)$$

for some non-negative collection of integers $m_1, \ldots, m_l$. This theorem has a long history, a contemporaneous summary of which can be found in [Sa52]. In particular, the theorem was first proved for the classical Lie groups by Pontryagin [Po39], and also by Cartan and Brauer [Car36] [Bra35]. If G is a connected reductive complex Lie group then the cohomology algebra of G is isomorphic to the cohomology algebra of the associated Lie algebra L [CE48]. So if L is reductive, $H^*(L)$ is a free super-commutative algebra, with generators occurring in homogeneous degrees $2m_1 + 1, \ldots, 2m_l + 1$. In this case, l is equal to the rank of L and the numbers $m_1, \ldots, m_l$ have become known as the exponents of L. As will become apparent in the following sections, the exponents, along with subsequent extensions, have an important place in Lie theory, with deep connections to geometry, combinatorics, and representation theory.

Second, we point out Kostant's theorem, which describes the cohomology $H^*(\mathfrak{n}; \mathbb{C}_\lambda)$ of a nilpotent radical $\mathfrak{n}$ of a semisimple Lie algebra L, with coefficients in a weight module $\mathbb{C}_\lambda$ [Ko59]. Kostant originally proved this theorem by calculating the kernel of the Laplacian $\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ of the Lie algebra cohomology differential $\bar{\partial}$, where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ in the metric on the Koszul complex C^* induced from the invariant bilinear form on L. Computing the kernel of the Laplacian has become a standard tool for Lie algebra cohomology calculations. Kostant's theorem can be regarded as a Lie algebra cohomology analogue of the Borel-Weil-Bott theorem, which describes the cohomology of equivariant line bundles on the full flag variety. In fact, Kostant used this theorem to give a proof of the Borel-Weil-Bott theorem. The Lie algebra perspective turns out to be very useful for working with Kac-Moody algebras. The full flag variety of a Kac-Moody algebra can be infinite-dimensional, so it is difficult to work with the cohomology. In contrast, Kostant's theorem on Lie algebra cohomology generalizes to the case of Kac-Moody algebras in relatively straightforward fashion. This generalisation is due to Garland-Lepowsky [GL76], while Kostant's

Laplacian calculation has been extended to the case of Kac-Moody algebras by Kumar [Ku84]. Kumar has also proven a Borel-Weil-Bott theorem for the flag variety of a Kac-Moody algebra [Ku87].

1.2 Generalized exponents and the Brylinski filtration

As mentioned in the previous section, the exponents of a reductive Lie algebra L arise as the degrees of the generators of the cohomology algebra $H^*(L)$. If $m_1, \ldots, m_l$ is the list of exponents, then the generators occur in degrees $2m_1 + 1, \ldots, 2m_l + 1$. One of the first questions asked about the exponents was how to compute them from the root system of L. A simple procedure was given by Shapiro: if b_k is the number of positive roots of order k, where the order of a root is the sum of the coefficients with respect to a simple basis, then the multiplicity of m as an exponent is $b_m - b_{m+1}$ (to handle reductive Lie algebras, set b_0 to the rank of L) [Ko59]. Another procedure was given by Coxeter: the exponents of L can be calculated from the eigenvalues of a certain element of the Weyl group, known as the Coxeter-Killing transformation [Cox51]. In the case of $\mathfrak{sl}_n$, the Weyl group is the permutation group S_n, and a Coxeter-Killing transformation is a long cycle, for example $(12 \cdots n)$ in disjoint cycle notation. Since the exponents were known for the simple Lie algebras, it was clear from the empirical evidence that the procedures of Shapiro and Coxeter gave the exponents. A proof of the correctness of Coxeter's procedure relying on a small amount of empirical evidence was given by Coleman [Col58]. Kostant, in a seminal paper, used the concept of a principal nilpotent of L to prove the equivalence of the different procedures and definitions without reference to any empirical data [Ko59]. An element e of L is nilpotent if e belongs to $[L, L]$ and $\mathrm{ad}(e)$ is nilpotent on L, and regular if the centralizer L^e has dimension equal to the rank of L. An element which is both nilpotent and regular is called a principal nilpotent. Every principal nilpotent belongs to an $\mathfrak{sl}_2$-triple $\{x, e, f\}$, i.e. a 3-tuple satisfying the determining relations $[x, e] = 2e$, $[x, f] = -2f$, and $[e, f] = x$ for $\mathfrak{sl}_2$. Kostant showed that the exponents of L are the highest weights of L as a module for the subalgebra determined by a principal $\mathfrak{sl}_2$-triple, or in other words the eigenvalues of $\mathrm{ad}(x)$ on the centralizer L^e. In the case of $\mathfrak{sl}_n$, the principal nilpotents are those matrices having a single Jordan block, or in other words the matrices which are conjugate to the matrix with ones down the first off-diagonal, and zeroes elsewhere. The exponents for $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ are $1, \ldots, n-1$ and $0, \ldots, n-1$ respectively.

Kostant followed his study of the exponents of L with another seminal paper, this time describing the L-module structure of the ring S^*L^* of polynomials on the Lie algebra [Ko63b]. Central to this paper is the geometry of the adjoint orbits in L, and the multiplicity of the highest-weight L-module $\mathcal{L}(\lambda)$ in S^*L^* can be determined from this geometry. Specifically, Kostant observed that S^*L^* is free as a $J = (S^*L^*)^L$-module, and furthermore that there is a homogeneous L-submodule H of S^*L^* such that the multiplication map $J \otimes H \to S^*L^*$ is an L-module isomorphism. J is constant on any G-orbit $\mathcal{O}$ in L, so there is an L-module homomorphism $H \to \mathbb{C}[\mathcal{O}]$. Kostant showed that this morphism is an isomorphism if $\mathcal{O}$ is the orbit of a regular element x. Thus H is isomorphic to the L-module $\mathbb{C}[G]^{G^x}$, and the multiplicity of $\mathcal{L}(\lambda)$ in H can be determined from the Peter-Weyl theorem. Determining the multiplicity of $\mathcal{L}(\lambda)$ in the submodule $S^k L^*$ of polynomials of some fixed degree is more difficult, and leads to a notion of generalized exponents. Specifically, the generalized exponents for a highest weight representation $\mathcal{L}(\lambda)$ are defined by stating that the multiplicity m as a generalized exponent is equal to the multiplicity of $\mathcal{L}(\lambda)$ in the degree m component of H, where H has a grading inherited from $S^* L^*$. The generalized exponents can also be computed using a principal $\mathfrak{sl}_2$-triple $\{x, e, f\}$: they are equal to the eigenvalues of x on the subspace $\mathcal{L}(\lambda)^{G^e}$. To prove this last fact, Kostant turns again to the orbit geometry of L, showing that the GIT quotient map $L \to L//G = \operatorname{Spec} J$ restricts to a submersion over the open subset L^{reg} of regular elements in L, and that there is an affine subset $\nu \subset L^{reg}$, called the Kostant slice, such that $\nu \hookrightarrow L \to L//G$ is an isomorphism. Incidentally, the Kostant slice can be used to show that J is a free commutative algebra, with homogeneous generators in degrees $m_1 + 1, \ldots, m_l + 1$. There is one further perspective on the L-module H. The ideal of $S^* L^*$ generated by the positive-degree elements of J corresponds to an algebraic variety $\mathcal{N} \subset L$. This variety consists of the nilpotent elements, and is known as the nilpotent cone. Hesselink showed that $\mathcal{N}$ has only rational singularities, and in fact has a resolution $T^*X \to \mathcal{N}$ where X is the full flag variety $X = G/B$ of G [He76]. It follows from Hesselink's work that $\mathbb{C}[T^*X]$ and $\mathbb{C}[\mathcal{N}]$ are isomorphic to H as L-modules.

Computing the generalized exponents using Kostant's definition requires working with representations of L. The generalized exponents can also be computed directly from the root system and Weyl group using a formula independently discovered by Hesselink and Peterson. Let $m_0^\lambda(q) = \sum d_i q^i$ where d_m is the multiplicity of m as an exponent. The Hesselink-Peterson formula states that

$$m_0^\lambda(q) = \sum_{w \in W} \epsilon(w) K(w(\lambda + \rho) - \rho; q),$$

where W is the Weyl group of L, ϵ is the usual sign representation, ρ is the half-sum of positive roots, and $K(\beta)$ is the Kostant partition function giving the coefficient of $[e^\beta]$ in $\prod_{\alpha \in \Delta^+} (1 - q e^\alpha)^{-1}$. The formula was discovered by Hesselink using cohomological methods, using ideas similar to those appearing in [He76]. Both Hesselink and Peterson later gave non-cohomological proofs [He80] [Pe78]. Hesselink and Peterson's formula connects the generalized exponents to other areas of representation theory and algebraic combinatorics. The polynomials $m_0^\lambda(q)$ are a special case of Lusztig's q-analog of weight multiplicity, defined for a weight μ of $\mathcal{L}(\lambda)$ by

$$m_\mu^\lambda(q) = \sum_{w \in W} \epsilon(w) K(w(\lambda + \rho) - \mu - \rho; q)$$

These polynomials are analogs of weight multiplicity because $m_\mu^\lambda(1) = \dim \mathcal{L}(\lambda)_\mu$, the multiplicity of the μth weight-space of $\mathcal{L}(\lambda)$. In combinatorics, these polynomials are equal to

Kostka-Foulkes polynomials, which express the characters of the highest-weight representations in terms of Hall-Littlewood polynomials [Kat82] (see [St05] for an expository reference). Lusztig observed that the $m^\lambda_\mu(q)$ are Kazhdan-Lusztig polynomials for the affine Weyl group [Lus83].

An interesting feature of Lusztig's q-analogs is that the coefficients of $m^\lambda_\mu(q)$ are non-negative when μ is a dominant weight of $\mathcal{L}(\lambda)$. There is an explanation for this phenomenon, first conjectured by Lusztig [Lus83]: the weight space $\mathcal{L}(\lambda)_\mu$ has an increasing filtration ${}^eF^*$ such that $m^\lambda_\mu(q)$ is equal to the Poincare polynomial

$$ {}^eP^\lambda_\mu(q) = \sum_{i \geq 0} q^i \dim {}^eF^i\mathcal{L}(\lambda)_\mu \ / \ {}^eF^{i-1}\mathcal{L}(\lambda)_\mu \tag{1.1} $$

of the associated graded space. This identity was first proved by Brylinski for μ regular or $\mathfrak{g}$ of classical type; the filtration ${}^eF^*$ is known as the Brylinski or Brylinski-Kostant filtration, and is defined by

$$ {}^eF^i(\mathcal{L}(\lambda)_\mu) = \{v \in \mathcal{L}(\lambda)_\mu : e^{i+1}v = 0\}, $$

where e is a principal nilpotent. Brylinski's proof states that the coefficient of q^m in $m^\lambda_\mu(q)$ is the multiplicity of $\mathcal{L}(\lambda)$ in the mth component of the graded space of $\Gamma(\mathcal{O}, F^{-\mu})$, where $\mathcal{O}$ is the orbit of a regular semisimple element in L, $F^{-\mu}$ is a line bundle defined over X by the character μ and pulled back to μ, and $\Gamma(\mathcal{O}, F^{-\mu})$ is the space of sections of $F^{-\mu}$, filtered by polynomial degree of S^*L^*. Hence the polynomials $m^\lambda_\mu(q)$ can be regarded as μ-twisted generalized exponents of $\mathcal{L}(\lambda)$. Brylinski's proof was extended to all dominant weights by Broer [Bro93]. More recently Joseph, Letzter, and Zelikson gave a purely algebraic proof of the identity $m^\lambda_\mu = {}^eP^\lambda_\mu$, and determined ${}^eP^\lambda_\mu$ for μ non-dominant [JLZ00].

The q-analogs of weight multiplicity are also connected to one of the best-known theorems of geometric representation theory, the geometric Satake isomorphism. This theorem states that there is an equivalence between the representation category of G, and the category of equivariant perverse sheaves on the loop Grassmannian $\mathrm{Gr} = G^\vee((z))/G^\vee[[z]]$ of the Langlands dual group $G^\vee$. The loop Grassmannian is an ind-variety, realized as an increasing disjoint union of Schubert varieties Gr^λ parametrized by weights of G. Under the equivalence, a highest-weight representation $\mathcal{L}(\lambda)$ is sent to the intersection cohomology complex IC^λ of $\overline{\mathrm{Gr}^\lambda}$. In addition to conjecturing the equality $m^\lambda_\mu(q) = {}^eP^\lambda_\mu$, Lusztig showed in [Lus83] that $m^\lambda_\mu(q)$ is equal (after a degree shift) to the generating function $\mathrm{IC}^\lambda_\mu(q)$ for the dimensions of the stalk of the complex IC^λ_μ at a point in $\mathrm{Gr}^\mu \subset \overline{\mathrm{Gr}^\lambda}$. A direct isomorphism between the stalks IC^λ_μ and the graded spaces $\mathrm{gr}\,\mathcal{L}(\lambda)_\mu$ appears in the geometric Satake isomorphism [Gi95] [MV07], leading to another proof that $m^\lambda_\mu = {}^eP^\lambda_\mu$ (see [Gi95] in particular).

1.2.1 The Brylinski filtration for affine Kac-Moody algebras

The q-analogs of weight multiplicity $m^\lambda_\mu(q)$ can also be defined for an arbitrary symmetrizable Kac-Moody algebra.In contrast to the case of semisimple Lie algebras, the root spaces of a

Kac-Moody algebra are not necessarily one-dimensional; the dimension of a root space is called the multiplicity of the root. The multiplicities have to be inserted into the Kostant partition function, so that $K(\beta; q)$ is defined as the coefficient of $[e^\beta]$ in $\prod_{\alpha\in\Delta^+}(1-qe^\alpha)^{-\operatorname{mult}\alpha}$. Viswanath has shown that the q-analogs of weight multiplicity of an arbitrary symmetrizable Kac-Moody are Kostka-Foulkes polynomials for generalized Hall-Littlewood polynomials, and determined $m_\mu^\lambda(q)$ at some simple μ for an untwisted affine Kac-Moody [Vi08]. A principal nilpotent of a Kac-Moody algebra is defined to be a linear combination $e = \sum c_i e_i$ of the positive generators, where all the coefficients c_i are non-zero. With this definition, the principal nilpotent filtration ${}^eF^*$ of the weight space of a highest-weight representation can be defined as in the finite case. Braverman and Finkelberg have proposed a conjectural analog of the geometric Satake isomorphism for affine Kac-Moody groups [BF10]. Their conjecture relates representations of $\mathfrak{g}$ to perverse sheaves on an analog of the loop Grassmannian $\mathfrak{g}^\vee$, where $\mathfrak{g}^\vee$ is an untwisted affine Kac-Moody algebra and $\mathfrak{g}$ is the Langlands dual to $\mathfrak{g}$. Their model leads them to conjecture that $m_\mu^\lambda(q) = {}^eP_\mu^\lambda$ in the affine case, with both related to the intersection cohomology stalks as in the finite case.

Brylinski's original proof of the identity $m_\mu^\lambda = {}^eP_\mu^\lambda$ uses at a crucial point the fact that the cohomology groups $H^q(X, F^{-\mu} \otimes S^*TX)$ are zero if $q > 0$ and μ is a dominant and regular weight. Cohomology vanishing theorems are a standard tool in complex algebraic geometry, and Brylinski deduces this cohomology vanishing theorem from a standard result on positive vector bundles. In general, it is difficult to adapt these standard tools to the infinite-dimensional setting. Teleman observes in [Te95] that one standard tool, Nakano's identity, can be adapted for the relative Lie algebra cohomology of the pair $(L[z], L)$ (this Lie algebra pair corresponds to the homogeneous space Gr of the loop group). Nakano's identity is used in [FGT08] to calculate the Laplacian for the cohomology of the pair $(L[z], L)$ with coefficients in $S^*L[[z]]^*$ with respect to the unique Kahler metric for the loop Grassmannian. The Laplacian calculation implies that the cohomology groups $H^q(L[z], L; \mathcal{L} \otimes S^*L[[z]]^*)$ vanish for $q > 0$ when $\mathcal{L}$ is a positive-level representation of the loop group $L[z^{\pm 1}]$. From this, Fishel, Grojnowski, and Teleman deduce a Brylinski-like theorem: there is a filtration of the G-invariant subspace $\mathcal{L}^G$ of $\mathcal{L}$ such that the graded space $\operatorname{gr}^* \mathcal{L}^G$ is isomorphic to the space $(\mathcal{L} \otimes S^*L[[z]]^*)^{L[z]}$.

One of the main results of this dissertation is an extension of Brylinski's result to affine (i.e. indecomposable of affine type) Kac-Moody algebras. This extension follows from the vanishing of the cohomology groups $H^q(\mathfrak{b}, \mathfrak{h}; S^*\hat{\mathfrak{u}}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu})$ for $q > 0$, where $\mathfrak{b}$ is the analog of the Borel in a Kac-Moody algebra $\mathfrak{g}$, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{u}$ is the analog of the nilpotent subalgebra of $\mathfrak{b}$, $\mathcal{L}(\lambda)$ is a highest-weight integrable representation of $\mathfrak{g}$, and μ is a dominant weight of $\mathcal{L}(\lambda)$. To prove this vanishing theorem, we use Nakano's identity as in [FGT08]. Teleman's version of Nakano's identity is given only for $L[z]$, but the proof applies in a very general setting; this is explained in Chapter 2. While the loop Grassmannian has a unique Kahler metric, the corresponding homogeneous space for $(\mathfrak{b}, \mathfrak{h})$ has many Kahler metrics. We show that there is a particular Kahler metric that allows us to imitate the Laplacian calculation of [FGT08]. Again, this can be done in a very general setting; this is explained in Chapter 3.

From the cohomology vanishing theorem we conclude that, as in the finite-dimensional case, there is a filtration on $\mathcal{L}(\lambda)_\mu$ such that when μ is dominant, $m_\mu^\lambda(q)$ is equal to the Poincare series of the associated graded space. Unlike the finite-dimensional case, the principal nilpotent is not sufficient to define the filtration in the affine case; instead we use the positive part of the principal Heisenberg (this form of Brylinski's identity was first conjectured by Teleman). We give examples to show that $m_\mu^\lambda(q)$ is not necessarily equal to ${}^eP_\mu^\lambda$, so our result gives a correction of Braverman and Finkelberg's conjecture. There are two difficulties in extending this result to indefinite symmetrizable Kac-Moody algebras: there does not seem to be a simple analogue of the Brylinski filtration, and the cohomology vanishing result does not extend for all dominant weights μ. We can overcome these difficulties by replacing the Brylinski filtration with an intermediate filtration, and by requiring that the root $\lambda - \mu$ has affine support. Thus we get some partial non-negativity results for the coefficients of $m_\mu^\lambda(q)$ even when $\mathfrak{g}$ is of indefinite type. The extension of Brylinski's filtration to Kac-Moody algebras is explained in Chapter 6.

1.3 Strong Macdonald theory

One of the first significant achievements in the theory of Kac-Moody algebras was the proof, by Kac [Ka74], of a generalization of the Weyl character formula for highest weight representations to symmetrizable Kac-Moody algebras. This character formula synthesized a number of previously discovered identities in algebraic combinatorics; in particular, the Weyl denominator identity for affine Kac-Moody algebras is equivalent to certain Dedekind's η-function identities discovered by Macdonald [Ma72a].

The strong Macdonald theorems give a connection between the Lie algebra cohomology of loop algebras, and Macdonald's constant term identity. The constant term identity states that if Δ is a reduced root system then

$$[e^0] \prod_{\alpha\in\Delta^+} \prod_{i=1}^{N} (1-q^{i-1}e^{-\alpha})(1-q^i e^{\alpha}) = \prod_{i=1}^{l} \binom{N(m_i+1)}{N}_q, \tag{1.2}$$

where $m_1, \ldots, m_l$ is the list of exponents of L and $\binom{a}{b}_q$ is the q-binomial coefficient, defined by

$$\binom{a}{b}_q = \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-b+1})}{(1-q^b)(1-q^{b-1})\cdots(1-q)}.$$

Equation (1.2) makes sense when $N = +\infty$, in which case the identity is just the Weyl denominator formula. Thus the constant term identity can be regarded as a truncation of the denominator formula. Macdonald presented the identity as a conjecture in [Ma82], and observed that it constitutes the untwisted case of a constant term identity for affine root systems. Further extensions (including a (q,t)-version) and proofs for individual affine root systems followed (see for instance [BZ85] [Hab86] [Ze87] [St88] [Ze88] [Ma88] [Gu90] [GG91] [Kad94]) until Cherednik gave a uniform proof of the most general version using double affine Hecke algebras [Ch95].

Suppose Δ is the root system of a semisimple Lie algebra L with exponents $m_1, \ldots, m_l$. Prior to Cherednik's proof, Hanlon observed in [Ha86] that the constant term identity would follow from a stronger conjecture:

> The cohomology $H^*\left(L[z]/z^N\right)$ is a free super-commutative algebra with N generators of cohomological degree $2m_i + 1$ for each $i = 1, \ldots, l$, of which, for fixed i, one has z-degree 0 and the others have z-degree $Nm_i + j$ for $j = 1, \ldots, N-1$. (1.3)

Hanlon termed this the strong Macdonald conjecture, and gave a proof for $L = \mathfrak{sl}_n$. Feigin observed in [Fe91] that the identity of (1.2) and the theorem of (1.3) follow from:

> The (continuous) cohomology $H^*(L[z,s])$ for s an odd variable is a free super-commutative algebra with generators of tensor degree $2m_i + 1$ and $2m_i + 2$, z-degree n, for $i = 1, \ldots, l$ and $n \geq 0$, where tensor degree refers to combined cohomological and s-degree. (1.4)

This version of the strong Macdonald conjecture corresponds to the (q,t) version of the Macdonald constant term conjecture. However, an error was discovered in Feigin's proof of (1.4). A complete proof of (1.3) and (1.4) was given by Fishel, Grojnowski, and Teleman [FGT08], using an explicit description of the relative cocycles combined with Feigin's idea (a spectral sequence argument) to prove (1.3) from (1.4). The free algebra $H^*(L[s])$ (which can easily be calculated from the Hochschild-Serre spectral sequence) appears as a subalgebra of $H^*(L[z,s])$, and Fishel, Grojnowski, and Teleman also prove that if $\mathfrak{b}$ is the Iwahori subalgebra $\{f \in L[z] : f(0) \in \mathfrak{b}_0\}$ then $H^*(\mathfrak{b}[s])$ is the free algebra $H^*(\mathfrak{b}_0[s]) \otimes_{H^*(L[s])} H^*(L[z,s])$. In this case their proof does not yield explicit generating cocycles.

The strong Macdonald theorem is connected with the version of the Brylinski filtration given by Fishel, Grojnowski, and Teleman. Recall that the proof of the Brylinski filtration for Kac-Moody algebras was based on the vanishing of the cohomology groups $H^*_{cts}(\mathfrak{b}, \mathfrak{h}; S^*\mathfrak{u}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu})$, where $\mathfrak{b}$ is an Iwahori in a Kac-Moody algebra, and λ is a dominant weight. This vanishing theorem holds even when $\mathcal{L}(\lambda)$ is the trivial representation. Fishel, Grojnowski, and Teleman's version of the Brylinski identity depended on the vanishing of $H^*_{cts}(L[z], L; \mathcal{L} \otimes S^*L[[z]]^*)$, where $\mathcal{L}$ is a positive-level representation of the loop algebra. In contrast to the Kac-Moody case, this vanishing theorem does not hold when $\mathcal{L}$ is trivial; instead $H^*_{cts}(L[z], L; S^*L[[z]]^*)$ is isomorphic to $H^*_{cts}(L[z,s], L)$, which in turn is a free subalgebra of $H^*_{cts}(L[z,s])$ with countably many generators. So the strong Macdonald theorem can be thought of as the failure of this cohomology vanishing theorem at level zero.

1.3.1 Strong Macdonald conjectures for the parahoric

The second main result of this thesis is a strong Macdonald theorem for $H^*(\mathfrak{p}[s])$ when $\mathfrak{p}$ is a standard parahoric in the twisted loop algebra $L[z^{\pm 1}]^{\tilde{\sigma}}$, for σ a (possibly trivial) diagram automorphism of L. Our proof is along the same lines as [FGT08]; in particular, we are able to give an explicit description of cocycles for the relative cohomology, and hence apply Feigin's spectral sequence to determine the cohomology of the truncations $\mathfrak{p}/z^N\mathfrak{p}$ when N is a multiple of the order of σ. Combined, our results for $L[z]^{\tilde{\sigma}}$ give an extension of the strong Macdonald theorems to match the affine version of Macdonald's constant term identity. For a general parahoric, our calculation reveals that $H^*(\mathfrak{p}[s])$ is isomorphic to $H(\mathfrak{p}_0[s]) \otimes_{H^*(L_0[s])} H(L[z,s]^{\tilde{\sigma}})$, and hence can be viewed as providing an interpolation between the two extremal results of Fishel, Grojnowski, and Teleman.

The algebras $H^*(\mathfrak{p}/z^N\mathfrak{p})$ also have an interesting description. As in (1.3), the algebras $H^*(L[z]^{\tilde{\sigma}}/z^N)$ are free, but this is no longer the case with a non-trivial parabolic component. The algebra $H^*(\mathfrak{p}/z^N\mathfrak{p})$ is isomorphic to $H^*(\mathfrak{g}_0)\otimes \text{Coinv}(L^\sigma, \mathfrak{g}_0)\otimes_{H^*(L^\sigma)} H^*(L[z]^{\tilde{\sigma}}/z^N)$, where $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$ is the reductive component of the parabolic $\mathfrak{p}_0$, and $\text{Coinv}(L^\sigma, \mathfrak{g}_0)$ is the (parabolic subalgebra of the) coinvariant algebra of the Weyl group of L^σ. A classic theorem of Borel states that $\text{Coinv}(L^\sigma, \mathfrak{g}_0)$ is isomorphic to the cohomology algebra of the generalized flag variety X corresponding to the Lie algebra pair $(L^\sigma, \mathfrak{p}_0)$ [Bo53] [BGG73]. The cohomology of X is in turn isomorphic to the Lie algebra cohomology algebra $H^*(L^\sigma, \mathfrak{g}_0)$. If $\mathfrak{p}$ is a parahoric in an untwisted loop algebra, then it is not hard to show that $H^*(\mathfrak{p}/z\mathfrak{p}, \mathfrak{g}_0)$ is isomorphic to $H^*(L^\sigma, \mathfrak{g}_0)$, and hence in the simplest case our result gives a Lie algebraic proof of Borel's theorem.

One intriguing consequence of Hanlon's conjecture is that $H^*(L[z]/z^N)$ is isomorphic as a vector space to $H^*(L)^{\otimes N}$. Since $L[z]/(z^N - t) \cong L^{\oplus N}$ for $t \neq 0$, this means that while the structure of $L[z]/(z^N - t)$ changes dramatically as t degenerates to zero, the cohomology is unchanged. Hanlon termed this "property M", and conjectured that it holds not only for semisimple Lie algebras, but also for the nilpotent radical of a parabolic in a semisimple Lie algebra and the Heisenberg Lie algebras [Ha90]. Kumar gave counterexamples to property M for the nilpotent radical of a parabolic [Ku99]. The conjecture for Heisenberg Lie algebras remains open, along with a number of other questions [Ha94] [HW03]. In the case of a parahoric in a twisted loop algebra $L[z^{\pm 1}]^{\tilde{\sigma}}$, if $t \neq 0$ then the truncation $\mathfrak{p}/(z^N - t)\mathfrak{p}$ is isomorphic to $L^{\bigoplus N/k}$, irregardless of the parahoric component. Our calculation shows that the cohomology is unchanged for $L[z]^{\tilde{\sigma}}/(z^N - t)$ as t degenerates to zero, but degenerates from $H^*(L)^{\otimes N/k}$ to $H^*(\mathfrak{g}_0)\otimes H^*(L^\sigma, \mathfrak{g}_0)\otimes_{H^*(L^\sigma)} H^*(L)^{\otimes N/k}$ for a general parahoric truncation $\mathfrak{p}/(z^N - t)\mathfrak{p}$.

The proof of the strong Macdonald theorem in [FGT08] is based on a Laplacian calculation for $H^*(L[z,s])$ using the unique Kahler metric on the loop Grassmannian. The Laplacian calculation shows that the ring of harmonic forms is isomorphic to a ring of basic and invariant forms on the arc space $L[[z]]$. The well-known facts about adjoint orbits in a reductive Lie algebra extend immediately from L to $L[[z]]$, and can be used to determine the ring of basic and invariant forms on $L[[z]]$. As occurs with the cohomology vanishing theorem used in the proof of the Brylinski identity, the homogeneous space corresponding to a parahoric can have many Kahler metrics; we show that there is one particular choice of Kahler metric that makes an analogous Laplacian calculation work. The ring of harmonic forms is isomorphic to (a ring similar to) the ring of basic and invariant forms on $\hat{\mathfrak{p}}$. To calculate this ring, we study the adjoint orbits on the twisted arc space $L[[z]]^{\tilde{\sigma}}$ (the significant facts about adjoint orbits no longer extend immediately). As part of our calculation of the basic and invariant forms, we show that the GIT quotient of $L[[z]]^{\tilde{\sigma}}$ by $G[[z]]^{\tilde{\sigma}}$ is $Q[[z]]^{\tilde{\sigma}}$, where $Q = L//G$. We also prove a slice theorem for twisted arcs in the regular semisimple locus, and an analogue of the Kostant slice theorem. This is done in Chapter 4.

Removing the super-notation, the cohomology ring of $\mathfrak{p}[s]$ is isomorphic to the cohomology ring of $\mathfrak{p}$ with coefficients in the symmetric algebra $S^*\mathfrak{p}^*$ of the restricted dual of $\mathfrak{p}$. Frenkel and Teleman have shown that $H^*(\mathfrak{b}; S^*\hat{\mathfrak{n}}^*)$ is a free algebra (and determined the degrees of the generators) when $\mathfrak{b}$ and $\mathfrak{n}$ are Iwahori and nilpotent subalgebras respectively of an untwisted loop algebra [FT06]. We prove Frenkel and Teleman's result in the twisted case and calculate the cohomology of the corresponding truncation $\mathfrak{b}/z^N\mathfrak{n}$. More generally, strong Macdonald theorems for different choices of coefficients might allow us to determine the cohomology of other truncations, such as $L[z]^{\tilde{\sigma}}/z^N$ when N is not divisible by k. At the moment, this question appears to be open.

1.4 Organization

Chapters 2-4 establish the necessary ingredients for proving strong Macdonald theorems and developing the Brylinski filtration. Chapter 2 gives a proof of the Lie algebraic version of Nakano's identity using semi-infinite cohomology. Chapter 3 uses Nakano's identity to make the essential Laplacian calculations. Finally, Chapter 4 is concerned with proving analogues of the Kostant slice theorems for twisted arc and jet spaces.

Chapter 5 concerns strong Macdonald theorems for parahoric subalgebras, and the corresponding truncations. This chapter uses the material from Chapters 2-4.

Chapter 6 concerns the Brylinski filtration for affine Kac-Moody algebras. This chapter uses material from Chapters 2, and gives a new Laplacian calculation, analogous to one of the Laplacian calculations from Chapter 3, but more applicable to symmetrizable Kac-Moody algebras.

Chapter 2

Semi-infinite cohomology and Nakano's identity

In this section we will construct the semi-infinite chain complex and use it to prove a Lie algebraic version of Nakano's identity. This version of Nakano's identity applies to any $\mathbb{Z}$-graded Lie algebra with finite-dimensional homogeneous components and the additional data of a grading-reversing anti-linear automorphism. As such, it generalizes the Lie algebraic version of Nakano's identity first given by Teleman for the loop algebra [Te95]. However, the material in this chapter should not be regarded as new, as it follows straight-forwardly from placing Teleman's proof in the standard framework for semi-infinite cohomology. The treatment of semi-infinite cohomology is based on [FGZ86] and [Vo93]. One novel detail is the explicit formula given for the semi-infinite cocycle, which makes it easy to compute the cocycle for specific examples.

We use the following terminology through this section:

- $\mathfrak{g} = \bigoplus \mathfrak{g}_n$ will be a $\mathbb{Z}$-graded Lie algebra such that $\dim \mathfrak{g}_n < +\infty$ for all n.

- The grading of $\mathfrak{g}$ induces a triangular decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{u}^+$, where $\mathfrak{u}^+ = \bigoplus_{k>0} \mathfrak{g}_k$, and $\mathfrak{u}^- = \bigoplus_{k<0} \mathfrak{g}_k$. The direct sum $\mathfrak{g}_0 \oplus \mathfrak{u}^+$ will be denoted by $\mathfrak{p}$.

- If $V = \bigoplus_n V_n$ is a $\mathbb{Z}$-graded vector space, the restricted dual is defined to be $\bigoplus_n V_n^*$, and will be denoted by V^t. In the case of $\mathfrak{u}^+$, the restricted dual $(\mathfrak{u}^+)^t$ is the same as the continuous dual $\widehat{\mathfrak{u}^+}^*$ of the completion of $\mathfrak{u}^+$ defined in the introduction. However, we use the restricted dual in this section so that we can also take a "small dual" of $\mathbb{Z}$-graded spaces such as $\mathfrak{g}$.

- The Clifford algebra $\mathcal{C}$ of the vector space $\mathfrak{g} \oplus \mathfrak{g}^t$ with respect to the dual pairing $\langle , \rangle$ is the universal associative algebra containing $\mathfrak{g} \oplus \mathfrak{g}^t$ such that $xy + yx = 2\langle x, y \rangle \mathbb{1}$ for all $x, y \in \mathfrak{g} \oplus \mathfrak{g}^t$.

2.1 Semi-infinite forms

Pick a homogeneous basis $\{z_i\}_{i\in\mathbb{Z}}$ of $\mathfrak{g}$ by running through bases of the graded components

$$\ldots,\ \mathfrak{g}_{k-1},\ \mathfrak{g}_k,\ \mathfrak{g}_{k+1},\ \ldots$$

in order, with indexing chosen so that $z_i \in \mathfrak{u}^-$ if and only if $i < 0$. Let $\{z^i\}$ denote the corresponding dual basis of $\mathfrak{g}^t$. The space of semi-infinite forms Λ^*_∞ is defined to be the span of forms

$$z^{i_1} \wedge z^{i_2} \wedge \ldots \wedge z^{i_k} \wedge \ldots$$

where the sequence $i_1, i_2, \ldots$ satisfies a stability condition: there is some N such that $i_{k+1} = i_k - 1$ for any $k > N$. Otherwise the forms behave as in the finite-dimensional case. Suppose that T is a graded operator on $\mathfrak{g}$ of degree zero. Then T will act on Λ^*_∞ via the diagonal action as long as T satisfies the following stability condition: there is some N such that for $k < N$, $T|_{\mathfrak{g}_k}$ is an element of $\mathrm{SL}(\mathfrak{g}_k)$. Thus our construction of semi-infinite forms really depends on a choice of "semi-infinite volume" for $\mathfrak{g}$, rather than a choice of basis.

Note that if T had non-zero degree then any reasonable extension of T by the diagonal action will be zero. On the other hand T can be extended to Λ^*_∞ as a derivation. For example, if $n \neq 0$ then $\mathfrak{g}_n$ acts on $\bigwedge^*_\infty$ by the adjoint action. Denote this action by ρ. On the other hand, degree zero maps can't necessarily be extended by derivations. The rest of this section is about how to define a coadjoint action of $\mathfrak{g}_0$ on Λ^*_∞.

Semi-infinite forms are by definition constructed from the restricted dual, and consequently interior and exterior multiplication are still well-defined. If $x \in \mathfrak{g}$ then $\iota(x)$ will denote interior multiplication by x. Similarly if $f \in \mathfrak{g}^t$ then $\epsilon(f)$ will denote exterior multiplication by f. As in the finite-dimensional case, the anti-commutator $[\epsilon(f), \iota(x)]$ acts as scalar multiplication by $f(x)$, so Λ^*_∞ is a $\mathcal{C}$-module. It is not hard to see that it is irreducible. Let $\langle , \rangle$ denote the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^t$. The algebra $\mathfrak{g}$ acts on itself by the adjoint action, and the coadjoint action on $\mathfrak{g}^t$ is defined so that $\langle x \cdot y, f\rangle = -\langle y, x \cdot f\rangle$. It follows that $\mathfrak{g}$ acts on the Clifford algebra $\mathcal{C}$. Note that if $x \in \mathfrak{g}_n, n \neq 0$, then

$$\rho(x)c\omega = (x \cdot c)\omega + c\rho(x)\omega.$$

Another way to think of this is that $[\rho(x), \iota(z)] = \iota([x, z])$ and $[\rho(x), \epsilon(f)] = \epsilon(\mathrm{ad}^t(x)f)$. This idea is the starting point for defining the action of $\mathfrak{g}_0$. Let ω_0 denote the homogeneous form

$$\omega_0 = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge \ldots,$$

and define $\rho(x)\omega_0 = \beta(x)\omega_0$ for all $x \in \mathfrak{g}_0$, where β is chosen from $\mathfrak{g}_0^t$. The annihilator of ω_0 in $\mathcal{C}$ is generated by the elements of $\mathfrak{p}$ and $(\mathfrak{u}^-)^t$, and is thus preserved by $\mathfrak{g}_0$. Thus $\rho(x)$ can be extended to all semi-infinite forms by the formula $\rho(x)c\omega_0 = (x \cdot c)\omega_0 + \beta(x)c\omega_0$. Because $\mathfrak{g}$ acts by algebra derivations on $\mathcal{C}$, it follows immediately that $\rho(x)c\omega = (x \cdot c)\omega + c\rho(x)\omega$ for all $x \in \mathfrak{g}$ and forms ω. In particular, $\rho(x)$ is completely determined by $\rho(x)\omega_0$. For instance, $[\rho(x), \rho(y)]c\omega_0 = ([x, y] \cdot c)\omega_0 + c[\rho(x), \rho(y)]\omega_0$, so ρ is a Lie algebra action if and only if $[\rho(x), \rho(x)]\omega_0 = \rho([x, y])\omega_0$. Since $[\rho(x), \rho(y)] = 0$ for $x, y \in \mathfrak{g}_0$, ρ is only projective in general. This can be fixed for $\mathfrak{g}_0$ by choosing β vanishing on $[\mathfrak{g}_0, \mathfrak{g}_0]$, but cannot be fixed over $\mathfrak{g}$.

Proposition 2.1.1. *Let ρ denote the action of $\mathfrak{g}$ on Λ^*_∞ defined above. Then ρ can be expressed in terms of the $\mathcal{C}$ action as*

$$\rho(x) = \sum_{i \geq 0} \epsilon(x \cdot z^i)\iota(z_i) - \sum_{i<0} \iota(z_i)\epsilon(x \cdot z^i) + \beta(x),$$

where the formal sums are finite on any element of Λ^*_∞.

From an intuitive perspective, given ω and $x \in \mathfrak{g}$ it is possible to find $c \in \mathcal{C}$ such that $\rho(x)\omega = c\omega$, and the formal sum allows us to find c in a nice way. More formally, the formal sum represents the action of an element of a completed Clifford algebra $\hat{\mathcal{C}}$ constructed as follows: take the obvious linear isomorphism

$$\mathcal{C} \cong \bigwedge^* \left(\mathfrak{u}^- \oplus \mathfrak{p}^t\right) \otimes \bigwedge^* \left(\left(\mathfrak{u}^-\right)^t \oplus \mathfrak{p}\right),$$

and grade the second exterior factor by giving $(\mathfrak{u}^-)^t$ the natural $\mathbb{Z}_{>0}$-grading which reverses the grading on $\mathfrak{u}^-$. Define $\hat{\mathcal{C}}$ to be the completion of $\mathcal{C}$ with respect to this grading, i.e. elements of $\hat{\mathcal{C}}$ are sums $\sum_{n \geq 0} a_n b_n$ where $a_n \in \bigwedge^* (\mathfrak{u}^- \oplus \mathfrak{p}^t)$ and b_n is an element of total degree n in $\bigwedge^* \left((\mathfrak{u}^-)^t \oplus \mathfrak{p}\right)$. The action of $\mathcal{C}$ on $\bigwedge^*_\infty$ extends to an action of $\hat{\mathcal{C}}$.

Proof of Proposition 2.1.1. Suppose $x \in \mathfrak{g}_n, n \neq 0$, and $\omega \in \Lambda^*_\infty$. Then there is some N such that

$$\rho(x)\omega = \sum_{i=-N}^{N} \epsilon(x \cdot z^i)\iota(z_i)\omega.$$

Since $x \cdot z^i$ is zero on z_i, $\epsilon(x \cdot z^i)$ anti-commutes with $\iota(z_i)$ in $\mathcal{C}$, so the formal sum is valid in this case. If $x \in \mathfrak{g}_0$ then all the terms except $\beta(x)$ vanish on ω_0, and only finitely many terms are non-zero on any element of Λ^*_∞.

Since only finitely many terms are relevant at any given time, the commutator calculations

$$[\epsilon(x \cdot z^i)\iota(z_i), \iota(z_k)] = z^i([x, z_k])\iota(z_i) = -[\iota(z_i)\epsilon(x \cdot z^i), \iota(z_k)]$$

and

$$[\epsilon(x \cdot z^i)\iota(z_i), \epsilon(z^k)] = \delta_{ik}\epsilon(x \cdot z^k) = -[\iota(z_i)\epsilon(x \cdot z^i), \epsilon(z^k)]$$

prove that the commutator of the formal sum with $c \in \mathcal{C}$ gives $x \cdot c$. Thus $\rho(x)$ is given by the formal sum for $x \in \mathfrak{g}_0$ as well. $\square$

One of the keys to working with the formal sum is the following lemma:

$$\sum_{z_i \in \mathfrak{g}_n} x \cdot z^i \otimes z_i + \sum_{z_j \in \mathfrak{g}_{n-k}} z^j \otimes [x, z_j] = 0$$

for fixed $n \in \mathbb{Z}$ and $x \in \mathfrak{g}_k$.

Proposition 2.1.2. *Let ρ be the action of $\mathfrak{g}$ constructed above from $\beta \in \mathfrak{g}_0^t$. Then:*

- *ρ is a projective action, in the sense that*

$$[\rho(x), \rho(y)] = \rho([x, y]) + \gamma(x, y),$$

where $\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a 2-cocycle. In addition $\gamma|_{\mathfrak{g}_m \times \mathfrak{g}_n} = 0$ unless $m + n = 0$, and $\gamma|_{\mathfrak{g}_0 \times \mathfrak{g}_0} = d\beta$, where d is the Lie algebra cohomology operator.

- *Denote the dependence of γ on β by $\gamma \equiv \gamma_\beta$. Then*

$$\gamma_{\beta_0} - \gamma_{\beta_1} = d(\beta_0 - \beta_1),$$

where d is the Lie algebra boundary map for $\mathfrak{g}$.

- *If $[\gamma]$ is a trivial cohomology class in $H^2(\mathfrak{g}, \mathbb{C})$ then there is a choice of $\beta \in \mathfrak{g}_0^t$ such that $\gamma = 0$.*

Proof. Suppose $x \in \mathfrak{g}_k$, $y \in \mathfrak{g}_l$. Using the action of $\mathfrak{g}$ on $\mathcal{C}$ combined with Proposition 2.1.1, we get that

$$[\rho(x), \rho(y)] = \sum_{i \geq 0} \epsilon(x \cdot y \cdot z^i)\iota(z_i) + \epsilon(y \cdot z^i)\iota(x \cdot z_i) - \sum_{i<0} \iota(x \cdot z_i)\epsilon(y \cdot z^i) + \iota(z_i)\epsilon(x \cdot y \cdot z^i).$$

Now for $n \geq 0$, the term

$$\sum_{z_i \in \mathfrak{g}_n} \epsilon(y \cdot z^i)\iota(x \cdot z_i)$$

can be replaced with

$$-\sum_{z_i \in \mathfrak{g}_{n+k}} \epsilon(y \cdot x \cdot z^i)\iota(z_i),$$

and similarly if $n < 0$. Thus it immediately follows that

$$[\rho(x), \rho(y)] = \rho([x, y]) - \beta([x, y]) + c_{xy},$$

where

$$\begin{aligned} c_{xy} &= -\sum_{k \leq n < 0} \sum_{z_i \in \mathfrak{g}_n} [\epsilon(y \cdot x \cdot z^i), \iota(z_i)] + \sum_{0 \leq n < k} \sum_{z_i \in \mathfrak{g}_n} [\iota(z_i), \epsilon(y \cdot x \cdot z^i)] \\ &= -\sum_{k \leq n < 0} \sum_{z_i \in \mathfrak{g}_n} z^i([x, [y, z_i]]) + \sum_{0 \leq n < k} \sum_{z_i \in \mathfrak{g}_n} z^i([x, [y, z_i]]). \end{aligned}$$

If $k + l \neq 0$ then $[x, [y, z_i]] \notin \mathfrak{g}_{n+k}$ in each summand, so c_{xy} will be zero. Let $\gamma(x, y) = c_{xy} + d\beta(x, y)$. That γ is a cocycle follows from the Jacobi identities for $\mathfrak{g}$ and $\mathrm{End}(\Lambda_\infty^*)$, and the fact that $[\rho(x), \rho(y)]$ is skew-symmetric in x and y.

Finally, suppose $\gamma = d\alpha$ for some $\alpha \in \mathfrak{g}^t$. Since $\gamma|_{\mathfrak{g}_m \times \mathfrak{g}_n} = 0$, $m + n \neq 0$, we can assume that $\alpha \in \mathfrak{g}_0^t$. By replacing β with $\beta - \alpha$, we get the cocycle $\gamma - d\alpha = 0$. □

If $x \in \mathfrak{g}_k$, $y \in \mathfrak{g}_{-k}$, and $k > 0$ then the operator $\mathrm{ad}(x)\,\mathrm{ad}(y)$ acts on each $\mathfrak{g}_n$. The quantity c_{xy} appearing in the proof is easily seen to be $\sum_{0 \leq n < k} \mathrm{tr}_{\mathfrak{g}_n}(\mathrm{ad}(x)\,\mathrm{ad}(y))$, and consequently

$$\gamma(x, y) = \sum_{0 \leq n < k} \mathrm{tr}_{\mathfrak{g}_n}(\mathrm{ad}(x)\,\mathrm{ad}(y)) + d\beta(x, y).$$

This formula can easily be verified by calculating $([\rho(x), \rho(y)] - \rho([x, y]))\,\omega_0$, and observing that

$$\sum_{0 \leq n < k} \mathrm{tr}_{\mathfrak{g}_n}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = \sum_{-k \leq n < 0} \mathrm{tr}_{\mathfrak{g}_n}(\mathrm{ad}(y)\,\mathrm{ad}(x)).$$

We will make heavy use of this formula for γ.

So far we have only assumed that $\mathfrak{g}$ is $\mathbb{Z}$-graded with finite-dimensional components. However, our construction of semi-infinite forms that $\mathfrak{g}$ is infinite-dimensional, and indeed the existence of semi-infinite forms stems from the fact that infinite-dimensional Clifford algebras have more irreducible representations than their finite-dimensional analogues. Nonetheless, as long as our arguments can be phrased in terms of the Clifford action, they hold for finite-dimensional Lie algebras just as well; hence the results of this and the following sections do hold for finite-dimensional Lie algebras where Λ_∞ is replaced with the ordinary exterior algebra $\bigwedge^* \mathfrak{g}^t$. If we place $\mathfrak{g}$ in degree zero, then the semi-infinite cohomology is the same as the ordinary cohomology, but if we choose a different grading we can get something new.

2.2 The semi-infinite Chevalley complex

The degree of a semi-infinite monomial $\omega = z^{i_1} \wedge z^{i_2} \wedge \ldots$ is defined to be

$$|\{i_1, i_2, \ldots\} \cap \mathbb{Z}_{\geq 0}| - |\mathbb{Z}_{<0} \setminus \{i_1, i_2, \ldots\}|.$$

Another way to define the degree is to declare that ω_0 has degree 0, that $\mathfrak{g}^t \subset \mathcal{C}$ acts by degree $+1$, and that $\mathfrak{g} \subset \mathcal{C}$ acts by degree -1. Let Λ_∞^k be the span of the monomials of degree k, and define a boundary operator $d : \Lambda_\infty^k \to \Lambda_\infty^{k+1}$ by

$$d = \frac{1}{2} \sum_{i \geq 0} \epsilon(z^i)\rho(z_i) + \frac{1}{2} \sum_{i < 0} \rho(z_i)\epsilon(z^i) + \frac{1}{2}\epsilon(\beta),$$

where ρ and β are as in the previous section. Note that changing β to $\beta + \beta'$ changes d to $d + \epsilon(\beta')$. Also d is an operator of degree 1.

The Clifford algebra $\mathcal{C}$ contains the exterior algebra $\bigwedge^* \mathfrak{g}^t$, which acts on Λ^*_∞ by exterior multiplication. Similarly the completed Clifford algebra $\hat{\mathcal{C}}$ contains a completed exterior algebra $\hat{\bigwedge}^* \mathfrak{g}^t$ which extends the action of $\bigwedge^* \mathfrak{g}^t$. Specifically, $\hat{\bigwedge}^* \mathfrak{g}^t$ is the completion of $\bigwedge^* \mathfrak{p}^t \otimes \bigwedge^* (\mathfrak{u}^-)^t$ with respect to the grading on the second factor, or equivalently the tensor product $\bigwedge^* \mathfrak{p}^t \otimes (\bigwedge^* \mathfrak{u}^-)^*$, where we take the full dual in the second factor. Intuitively, elements of $\hat{\bigwedge}^* \mathfrak{g}^t$ are elements α of the full dual of $\bigwedge^* \mathfrak{g}$ such that for any k the restriction of α to $\bigoplus_{n>k} \mathfrak{g}_n$ is contained in the restricted dual $\bigwedge^* \mathfrak{g}^t$. The semi-infinite cocycle is a particular example of $\hat{\bigwedge}^* \mathfrak{g}^t$. The completed exterior algebra $\hat{\bigwedge}^* \mathfrak{g}^t$ is closed under the ordinary Lie algebra cohomology differential.

Proposition 2.2.1. *Let d be the boundary operator defined above. Then:*

- *The anticommutator $[d, \iota(x)] = \rho(x)$ for all $x \in \mathfrak{g}$.*
- *The commutator $[\rho(x), d] = [\iota(x), \epsilon(\gamma)]$ for all $x \in \mathfrak{g}$.*
- *$d^2 = \epsilon(\gamma)$, where γ is the projective cocycle corresponding to the projective action ρ.*
- *Suppose $\alpha \in \hat{\bigwedge}^* \mathfrak{g}^t$. Then $[d, \epsilon(\alpha)] = \epsilon(d\alpha)$, where $d\alpha$ refers to the ordinary boundary map in Lie algebra cohomology.*

Proof. It is useful to set $\tilde{d} = d - \epsilon(\beta)/2$. Using the graded commutator, we get

$$\begin{aligned}\left[\tilde{d}, \iota(x)\right] &= \frac{1}{2}\sum_i z^i(x)\rho(z_i) + \frac{1}{2}\sum_{i\geq 0}\epsilon(z^i)\iota([z_i,x]) - \frac{1}{2}\sum_{i<0}\iota([z_i,x])\epsilon(z^i) \\ &= \frac{\rho(x)}{2} + \frac{1}{2}\sum_{n\geq 0}\sum_{z_i\in\mathfrak{g}_{n+k}}\epsilon(x\cdot z^i)\iota(z_i) - \frac{1}{2}\sum_{n<0}\sum_{z_i\in\mathfrak{g}_{n+k}}\iota(z_i)\epsilon(x\cdot z^i),\end{aligned}$$

where $x \in \mathfrak{g}_k$. If $k \neq 0$ then $\epsilon(x \cdot z^i)$ anticommutes with $\iota(z_i)$, so the last two summands are equal to $(\rho(x) - \beta(x))/2$. But if $x \in \mathfrak{g}_0$ there is no need to take anticommutators to show that these summands equal to $(\rho(x) - \beta(x))/2$, and hence the whole expression is equal to $\rho(x) - \beta(x)/2$. On the other hand $[\epsilon(\beta), \iota(x)] = \beta(x)$, so it follows immediately that $[d, \iota(x)] = \rho(x)$.

Next, $\left[\rho(x), \tilde{d}\right] = A + B$, where

$$A = \frac{1}{2}\sum_{i\geq 0}\epsilon(z_i)[\rho(x),\rho(z_i)] + \frac{1}{2}\sum_{i<0}[\rho(x),\rho(z_i)]\epsilon(z^i),$$

$$\begin{aligned}B &= \frac{1}{2}\sum_{i\geq 0}\epsilon(x\cdot z^i)\rho(z_i) + \frac{1}{2}\sum_{i<0}\rho(z_i)\epsilon(x\cdot z^i) \\ &= -\frac{1}{2}\sum_{i\geq 0}\epsilon(z^i)\rho([x,z_i]) - \frac{1}{2}\sum_{i<0}\rho([x,z_i])\epsilon(z^i) - \frac{c_x}{2},\end{aligned}$$

for $x \in \mathfrak{g}_k$ and

$$\begin{aligned} c_x &= \sum_{-k \le n < 0} \sum_{z_i \in \mathfrak{g}_n} [\epsilon(z^i), \rho([x, z_i])] + \sum_{0 \le n < -k} \sum_{z_i \in \mathfrak{g}_n} [\rho([x, z_i]), \epsilon(z^i)] \\ &= - \sum_{-k \le n < 0} \sum_{z_i \in \mathfrak{g}_n} \epsilon([x, z_i] \cdot z^i) + \sum_{0 \le n < -k} \sum_{z_i \in \mathfrak{g}_n} \epsilon([x, z_i] \cdot z^i). \end{aligned}$$

So

$$\left[\rho(x), \tilde{d}\right] = \frac{1}{2} \sum_i \gamma(x, z_i) \epsilon(z^i) - \frac{c_x}{2}$$

is multiplication by a 1-form. Applying this 1-form to $y \in \mathfrak{g}_l$ we get

$$\frac{\gamma(x,y)}{2} + \frac{1}{2} \sum_{-k \le n < 0} \sum_{z_i \in \mathfrak{g}_n} z^i([y, [x, z_i]]) - \frac{1}{2} \sum_{0 \le n < -k} \sum_{z_i \in \mathfrak{g}_n} z^i([y, [x, z_i]]).$$

Now this is zero unless $k + l = 0$, in which case using the expression for $\gamma(x, y)$ from the previous section this is equal to

$$\frac{\gamma(x,y)}{2} - \frac{\gamma(y,x) + \beta([y,x])}{2} = \gamma(x,y) + \frac{\beta([x,y])}{2}.$$

Multiplication by the 1-form sending y to $\gamma(x, y)$ agrees with the operator $[\iota(x), \epsilon(\gamma)]$, so

$$[\rho(x), d] = \left[\rho(x), \tilde{d}\right] + \frac{\epsilon(x \cdot \beta)}{2} = [\iota(x), \epsilon(\gamma)].$$

To get d^2, let ω be any form $z^{N-1} \wedge z^{N-2} \wedge z^{N-3} \wedge \cdots$, where there is some $n > 0$ such that $z_{N-1}, z_{N-2}, \ldots$ runs through bases for $\mathfrak{g}_k$, $k < n$, in descending order. All pure semi-infinite forms can be constructed by successively applying the operators $\iota(x)$ to forms ω of this type. Now $d\omega = 0$, so $d^2\omega = 0 = \epsilon(\gamma)\omega$. Also

$$\begin{aligned} [\iota(x), d^2 - \epsilon(\gamma)] &= [\iota(x), d]d - d[\iota(x), d] - [\iota(x), \epsilon(\gamma)] \\ &= \rho(x)d - d\rho(x) - [\iota(x), \epsilon(\gamma)] = 0. \end{aligned}$$

So $\ker(d^2 - \epsilon(\gamma))$ is closed under successive interior multiplications. Thus $d^2 = \epsilon(\gamma)$ on Λ^*_∞.

To prove the last part of the Proposition for $\alpha \in \bigwedge^* \mathfrak{g}^t$, we only need to show that $[d, \epsilon(\alpha)] = \epsilon(d\alpha)$ for α a 1-form, since then we will have $d\alpha \wedge \omega = (d\alpha) \wedge \omega + \alpha \wedge d\omega$. If α is a 1-form, then

$$\begin{aligned} [d, \epsilon(\alpha)] &= \frac{1}{2} \sum_{i \ge 0} [\epsilon(\alpha), \rho(z_i)] \epsilon(z^i) + \frac{1}{2} \sum_{i < 0} \epsilon(z^i) [\rho(z_i), \epsilon(\alpha)] \\ &= \frac{1}{2} \sum_i \epsilon(z^i) \epsilon(z_i \cdot \alpha) = \epsilon(d\alpha). \end{aligned}$$

If α is a completed k-form, let α_N be the restriction of α to $\bigoplus_{n\geq N}\mathfrak{g}_n$. Then $(d\alpha)_N$ depends only on α_{2N}, and in fact $(d\alpha)_N = (d\alpha_{2N})_N$. If ω is any semi-infinite form then there is N such that $\alpha\wedge\omega = \alpha_K\wedge\omega$ for all $K < N$ and forms $\alpha \in \hat{\bigwedge}^*\mathfrak{g}^t$. Choosing $N < 0$ small enough so that this property holds simultaneously for ω and $d\omega$ simultaneously, we get that

$$\begin{aligned}[d,\epsilon(\alpha)]\omega &= [d,\epsilon(\alpha_{2N}]\omega = \epsilon(d\alpha_{2N})\omega \\ &= (d\alpha_{2N})_N\omega = \epsilon(d\alpha)\omega.\end{aligned}$$

□

One consequence of this proposition is that if ω is a completed k-form on $\mathfrak{g}$, then $[d,\epsilon(\omega)] = 0$ if and only if ω is a cocycle.

Definition 2.2.2. *Let V be a $\mathbb{Z}$-graded vector space. A $\mathfrak{g}$-action on V is simply a graded linear map $\pi : \mathfrak{g} \to \mathrm{End}(V)$ (the idea is to allow projective representations and other such things). We say that the action is* negative-energy *if there is $N \in \mathbb{Z}$ such that $V_n = 0$ for $n \geq N0$.*

The space of semi-infinite forms with coefficients in V is the graded space $\Lambda^*_\infty \otimes V$, and is denoted by $\Lambda^*_\infty(V)$. If V has a $\mathfrak{p}$-finite action, then the boundary map d_V is defined to be

$$d_V = d\otimes \mathbb{1}_V + \sum_i \epsilon(z^i)\pi(z_i).$$

The action $\rho+\pi$ of $\mathfrak{g}$ on $\Lambda^*_\infty(V)$ will be denoted by θ.

We use negative energy actions for simplicity, but it is likely that the class of actions we consider can be considerably loosened; for example, locally negative energy (i.e. for all $v \in V$ there is N such that $\pi(\mathfrak{g}_n)v = 0$ for $n \geq N$) is fine.

Corollary 2.2.3. *Let V be a graded vector space with an negative-energy action. Let Θ be the $\mathrm{End}(V)$-valued 2-form $\sum_{i<j} z^i\wedge z^j\left([\pi(z_i),\pi(z_j)] - \pi([z_i,z_j])\right)$. Then:*

- $d_V^2 = \epsilon(\gamma+\Theta)$.
- $[\iota(x), d_V] = \theta(x)$.
- $[\theta(x), d_V] = [\iota(x), \epsilon(\gamma+\Theta)]$.

Θ should be thought of as a curvature term.

Proof. Recall that the anticommutator $[\epsilon(f), d] = \epsilon(df) = \frac{1}{2}\sum_i \epsilon(z^i)\epsilon(z_i f)$. So

$$\begin{aligned}
d_V^2 &= d^2 \otimes \mathbb{1} + \sum_i [\epsilon(z^i), d]\pi(z_i) + \sum_{i,j} \epsilon(z^i)\epsilon(z^j)\pi(z_i)\pi(z_j) \\
&= \epsilon(\gamma) + \frac{1}{2}\sum_{i,j} \epsilon(z^i)\epsilon(z_i z^j)\pi(z_j) + \sum_{i,j} \epsilon(z^i)\epsilon(z^j)\pi(z_i)\pi(z_j) \\
&= \epsilon(\gamma) - \frac{1}{2}\sum_{i,j} \epsilon(z^i)\epsilon(z^j)\pi([z_i, z_j]) + \sum_{i<j} \epsilon(z^i)\epsilon(z^j)[\pi(z_i), \pi(z_j)] \\
&= \epsilon(\gamma) + \sum_{i<j} \epsilon(z^i)\epsilon(z^j)\left([\pi(z_i), \pi(z_j)] - \pi([z_i, z_j])\right).
\end{aligned}$$

Next

$$[\iota(x), d_V] = [\iota(x), d] + \sum_i [\iota(x), \epsilon(z^i)]\pi(z_i) = \rho(x) + \pi(x) = \theta(x).$$

Finally

$$[\rho(x), d_V] = [\iota(x), \epsilon(\gamma)] + \sum_i \epsilon(x \cdot z^i)\pi(z_i) = [\iota(x), \epsilon(\gamma)] - \sum_i \epsilon(z^i)\pi([x, z_i]).$$

while

$$[\pi(x), d_V] = \sum_i \epsilon(z^i)[\pi(x), \pi(z_i)]$$

□

So far we have defined $\Lambda^*_\infty(V) \equiv \Lambda^*_\infty(\mathfrak{g}; V)$. Now we define the relative Chevalley complex.

Corollary 2.2.4. *Let $\mathfrak{h}$ be a subgroup of $\mathfrak{g}$. The space of relative forms $\Lambda^*_\infty(\mathfrak{g}, \mathfrak{h}; V)$ is defined to be the intersection of the kernels of the operators $\theta(x), x \in \mathfrak{h}$ and $\iota(x), x \in \mathfrak{h}$.*

- *d_V restricts to an operator on $\Lambda^*_\infty(\mathfrak{g}, \mathfrak{h}; V)$ if and only if $\epsilon(\gamma+\Theta)$ restricts to an operator on $\Lambda^*_\infty(\mathfrak{g}, \mathfrak{h}; V)$.*
- *$[\iota(x), \epsilon(\gamma+\Theta)] = 0$ for all $x \in \mathfrak{h}$ is a sufficient condition for d_V to restrict to $\Lambda^*_\infty(\mathfrak{g}, \mathfrak{h}; V)$.*
- *$[\iota(x), \epsilon(\gamma)] = 0$ for all $x \in \mathfrak{h}$ if and only if $\gamma|_{\mathfrak{h}\times\mathfrak{g}} = 0$, while $[\iota(x), \epsilon(\Theta)] = 0$ if and only if $\pi([x, y]) = [\pi(x), \pi(y)]$ for all $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$.*

Proof. If d_V restricts, then so does $d_V^2 = \epsilon(\gamma + \Theta)$. Going in the other direction, if $\epsilon(\gamma + \Theta)$ restricts to an operator on $\Lambda^*_\infty(\mathfrak{g}, h; V)$ and ω is in this latter space then $\iota(x)d_V\omega = -d_V\iota(x)\omega + \theta(x)\omega = 0$, while $\theta(x)d_V\omega = d_V\theta(x)\omega + [\iota(x), \epsilon(\gamma + \Theta)]\omega = 0$. This also gives the second part of the proposition.

The last part of the proposition is obvious, since $[\iota(x), \epsilon(\gamma)] = \epsilon(\alpha)$, where α is the 1-chain sending $y \mapsto \gamma(x, y)$, and a similar argument works for Θ. □

Example 2.2.5. *If $\mathfrak{h} = \mathfrak{g}_0$ then $\gamma|_{\mathfrak{g}_0 \times \mathfrak{g}} = 0$ if and only if $\beta|_{[\mathfrak{g}_0, \mathfrak{g}_0]} = 0$.*

Example 2.2.6. *The hypotheses of this corollary hold very naturally if $\mathfrak{h} = \mathfrak{g}_0$, $\beta = 0$, and π is a representation such that $\pi|_{\mathfrak{p}}$ and $\pi|_{\mathfrak{p}^-}$ both preserve the Lie bracket, where $\mathfrak{p}^-$ is the algebra $\mathfrak{u}^- \rtimes \mathfrak{g}_0$. For example this happens when V is one of the truncated spaces $\mathfrak{u}^+ = \mathfrak{g}/\mathfrak{p}^-$ or $\mathfrak{p} = \mathfrak{g}/\mathfrak{u}^-$.*

2.3 A bigrading on semi-infinite forms

The basic idea of this section is to give Λ^*_∞ a bigrading as follows: a form $z^{i_1} \wedge z^{i_2} \wedge z^{i_3} \wedge \cdots$ has bidegree (a, b), where

$$a = |\{i_1, i_2, \ldots\} \cap \mathbb{Z}_{\geq 0}| \text{ and } b = -|\{i_1, i_2, \ldots\} \cap \mathbb{Z}_{<0}|.$$

This grading could also be constructed by giving ω_0 bidegree $(0, 0)$, declaring $\epsilon(f)$ to have bidegree $(0, 1)$ when $f \in \mathfrak{p}^t$, and declaring $\iota(x)$ to have bidegree $(-1, 0)$ when $x \in \mathfrak{u}^-$. The total degree of ω is $a + b$ as before.

Another way to define the bigrading is to look at the space

$$C^{a,b} \equiv C_\beta^{(a,b)} = \bigwedge^b \mathfrak{p}^t \otimes \bigwedge^{-a} \mathfrak{u}^- \otimes \mathbb{C}_\beta.$$

There is an obvious bidegree-preserving bijection between $C^{*,*}$ and Λ^*_∞— namely, that which sends $\alpha \otimes b$ to $\epsilon(\alpha)\iota(b_1)\ldots\iota(b_k)\omega_0$. The term $\mathbb{C}_\beta$ is not relevant to the vector space structure, but does affect the $\mathfrak{g}_0$-module structure on $C^{a,b}$. There is actually an action of $\mathfrak{p} \oplus \mathfrak{u}^-$ (direct sum of Lie algebras, so $[\mathfrak{p}, \mathfrak{u}^-]$ is defined to be zero) on $C^{a,b}$, given by considering $\mathfrak{u}^-$ as the $\mathfrak{p}$-module $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}^t$ as the dual of the $\mathfrak{u}^-$-module $\mathfrak{g}/\mathfrak{u}^-$. Denote the action of $\mathfrak{p} \oplus \mathfrak{u}^-$ on itself by $\widetilde{\mathrm{ad}}$, and the dual action by $\widetilde{\mathrm{ad}}^t$. The map $C^{a,b} \cong \Lambda^{a,b}_\infty$ is $\mathfrak{g}_0$-linear, but neither $\mathfrak{u}^+$ nor $\mathfrak{u}^-$-linear. The map $C^{a,b} \cong \Lambda^{a,b}_\infty$ is also $\mathcal{C}$-linear, where $\mathfrak{p}$ and $\mathfrak{p}^t$ act as usual on $\bigwedge^b \mathfrak{p}^t$, and $\mathfrak{u}^-$ and $(\mathfrak{u}^-)^t$ act on $\bigwedge^{-a} \mathfrak{u}^-$ by exterior and interior multiplication respectively. Note that the action of $\mathfrak{u}^-$ and $(\mathfrak{u}^-)^t$ is defined with a sign, so that for example $x(\alpha \otimes b) = (-1)^{|\alpha|}\alpha \otimes x \wedge b$ for any $x \in \mathfrak{u}^-$.

Proposition 2.3.1. *The boundary operator $d = D + \bar\partial$, where D has bidegree $(1, 0)$ and $\bar\partial$ has bidegree $(0, 1)$. On $C^{a,b}$, $\bar\partial$ is the Lie algebra cohomology boundary operator*

$$\bar\partial = \sum_{i \geq 0} \epsilon(z^i) \left(\frac{1}{2} \mathrm{ad}^t_{\mathfrak{p}}(z_i) + \widetilde{\mathrm{ad}}(z_i) + \beta(z_i) \right)$$

for $\mathfrak{p}$ with coefficients in $\bigwedge^* \mathfrak{u}^- \otimes \mathbb{C}_\beta$, while D is the Lie algebra homology boundary operator

$$D = \sum_{i<0} \left(\frac{1}{2} \mathrm{ad}_{\mathfrak{u}^-}(z_i) + \widetilde{\mathrm{ad}}^t(z_i) \right) \iota(z^i).$$

for $\mathfrak{u}^-$ with coefficients in $\bigwedge^* \mathfrak{p}^t$.

Proof. Let $x \in \mathfrak{g}_k$, $k \neq 0$. If $k \geq 0$ then

$$\rho(x) = \sum_{n \geq k} \sum_{z_i \in \mathfrak{g}_n} \epsilon(x \cdot z^i)\iota(z_i) - \sum_{i<0} \iota(z_i)\epsilon(x \cdot z^i) + \sum_{0 \leq n < k} \sum_{z_i \in \mathfrak{g}_n} \epsilon(x \cdot z^i)\iota(z_i) + \beta(x),$$

where the first two summands have degree $(0,0)$ and the last summand has degree $(1,-1)$. If $k < 0$ then

$$\rho(x) = \sum_{i \geq 0} \epsilon(x \cdot z^i)\iota(z_i) - \sum_{n<k} \sum_{z_i \in \mathfrak{g}_n} \iota(z_i)\epsilon(x \cdot z^i) - \sum_{k \leq n < 0} \sum_{z_i \in \mathfrak{g}_n} \iota(z_i)\epsilon(x \cdot z^i),$$

where again the last summand has degree $(-1,1)$ and the first two have degree $(0,0)$.

From this it is clear that $d = D + \bar{\partial}$ where

$$2\bar{\partial} = \sum_{n \geq k \geq 0} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \epsilon(z^i)\epsilon(z_i \cdot z^j)\iota(z_j) - \sum_{\substack{i \geq 0 \\ j<0}} \epsilon(z^i)\iota(z_j)\epsilon(z_i \cdot z^j)$$
$$- \sum_{k \leq n < 0} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \iota(z_j)\epsilon(z_i \cdot z^j)\epsilon(z^i) + 2\epsilon(\beta),$$

and

$$2D = \sum_{\substack{j \geq 0 \\ i<0}} \epsilon(z_i \cdot z^j)\iota(z_j)\epsilon(z^i) - \sum_{n<k<0} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \iota(z_j)\epsilon(z_i \cdot z^j)\epsilon(z^i)$$
$$+ \sum_{0 \leq n < k} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \epsilon(z^i)\epsilon(z_i \cdot z^j)\iota(z_j).$$

Start with $2\bar{\partial}$. The first sum is equal to

$$\sum_{i,j \geq 0} \epsilon(z^i)\epsilon(\mathrm{ad}^t_{\mathfrak{p}}(z_i)z^j)\iota(z_j) = \sum_{i \geq 0} \epsilon(z^i)\,\mathrm{ad}^t_{\mathfrak{p}}(z_i),$$

which translates straightforwardly on $C^{a,b}$ to the cohomology boundary operator for $\mathfrak{p}$. The last two sums

$$-\sum_{\substack{i \geq 0 \\ j<0}} \epsilon(z^i)\iota(z_j)\epsilon(z_i \cdot z^j) = \sum_{n<-k \leq 0} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \epsilon(z^i)\iota([z_i, z_j])\epsilon(z^j)$$

and

$$-\sum_{k \leq n < 0} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \iota(z_j)\epsilon(z_i \cdot z^j)\epsilon(z^i) = \sum_{k<-n \leq 0} \sum_{\substack{z_i \in \mathfrak{g}_k \\ z_j \in \mathfrak{g}_n}} \iota([z_i, z_j])\epsilon(z^j)\epsilon(z^i)$$

are equal, since in the second sum $\iota([z_i, z_j])$ and $\epsilon(z^j)$ anti-commute. Added together, these two sums translate to

$$2\sum_{i\geq 0, j<0} \epsilon(z^i)\epsilon(\widetilde{\mathrm{ad}}(z_i)z_j)\iota(z^j) = 2\sum_{i\geq 0}\epsilon(z^i)\widetilde{ad}(z_i)$$

on $C^{a,b}$.

Next, we look at $2D$. We have

$$\sum_{j\geq 0>i} \epsilon(z_i \cdot z^j)\iota(z_j)\epsilon(z^i) = \sum_{i<0} \widetilde{\mathrm{ad}}^t(z_i)\epsilon(z^i),$$

which translates to the Lie algebra differential for $\mathfrak{u}^-$, while

$$\begin{aligned}\sum_{0\leq n<k}\sum_{\substack{z_i\in\mathfrak{g}_k\\ z_j\in\mathfrak{g}_n}} \epsilon(z^i)\epsilon(z_i\cdot z^j)\iota(z_j) &= -\sum_{k\geq -l>0}\sum_{\substack{z_i\in\mathfrak{g}_k\\ z_j\in\mathfrak{g}_l}} \epsilon(z^i)\epsilon(z^j)\iota([z_i, z_j]) \\ &= -\sum_{\substack{s\geq 0\\ l<0}}\sum_{\substack{z_i\in\mathfrak{g}_s\\ z_j\in\mathfrak{g}_l}} \epsilon(z_j\cdot z^i)\epsilon(z^j)\iota(z_i) \\ &= \sum_{i\geq 0>j} \epsilon(z_j\cdot z^i)\iota(z_i)\epsilon(z^j)\end{aligned}$$

is equal to the above equation. When written on $C^{a,b}$, the sum of the two becomes

$$2\sum_{i<0}\widetilde{\mathrm{ad}}^t(z_i)\iota(z^i).$$

Finally

$$-\sum_{n<k<0}\sum_{\substack{z_i\in\mathfrak{g}_k\\ z_j\in\mathfrak{g}_n}} \iota(z_j)\epsilon(z_i\cdot z^j)\epsilon(z^i) = \sum_{i,j<0}\iota([z_i, z_j])\epsilon(z^j)\epsilon(z^i),$$

which on $C^{a,b}$ becomes

$$\sum_{i,j<0}\epsilon([z_i, z_j])\iota(z^j)\iota(z^i) = \sum_{i<0}\mathrm{ad}_{\mathfrak{u}^-}(z_i)\iota(z^i).$$

□

If V has a negative energy action, define $C^{a,b}(V) = C^{a,b}\otimes V$. Then $d_V = \bar{\partial}_V + D_V$, where

$$\bar{\partial}_V = \bar{\partial}\otimes \mathbb{1} + \sum_{i\geq 0}\epsilon(z^i)\pi(z_i)$$

and

$$D_V = D\otimes \mathbb{1} + \sum_{i<0}\iota(z_i)\pi(z_i)$$

on $C^{a,b}(V)$.

2.4 Hodge star and fundamental form

Let $\mathfrak{g}$ be a graded Lie algebra as before, but with the additional structure of an anti-isomorphism $x \mapsto \bar{x}$ such that $\overline{\mathfrak{g}_n} = \mathfrak{g}_{-n}$. In addition, suppose that a (positive-definite) Hermitian metric is given on $\mathfrak{u}^+$. As usual in Hodge theory, this metric can be used to define a Hodge star operator and an action of $\mathfrak{sl}(2,\mathbb{C})$ on the subspace $\hat{C}^{a,b}(V) := \bigwedge^b(\mathfrak{u}^+)^t \otimes \bigwedge^{-a}\mathfrak{u}^- \otimes \mathbb{C}_\beta \otimes V$ of $C^{a,b}_\beta(V)$. In this section we define this action, and prove a rudimentary Kahler identity.

Extend the metric on $\mathfrak{u}^+$ to $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ by making $\mathfrak{u}^+ \perp \mathfrak{u}^-$ and setting $(\bar{x},\bar{y}) = \overline{(x,y)} = (y,x)$ if $x, y \in \mathfrak{u}^-$. Let $\phi : \mathfrak{u}^- \to (\mathfrak{u}^+)^t$ be the isomorphism induced by the metric, i.e. $\phi(a)(b) = (b, \bar{a})$. The dual metric on $(\mathfrak{u}^+)^t$ is the metric which makes ϕ an isometry. Give $\hat{C}^{a,b}(V)$ a metric by multilinear extension. Explicitly the metric on an exterior power is

$$\begin{aligned}(a_1 \wedge \cdots \wedge a_n, b_1 \wedge \cdots \wedge b_n) &= \frac{1}{n!}\sum_{\sigma,\rho\in S_n} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho)\prod_{i=1}^n (a_{\sigma(i)}, b_{\rho(i)})\\ &= \sum_{\rho\in S_n}\operatorname{sgn}(\rho)\prod_{i=1}^n (a_i, b_{\rho(i)}) \quad \text{alternatively}\\ &= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^n (a_{\sigma(i)}, b_i) \quad \text{alternatively.}\end{aligned}$$

For the purposes of the next two sections, pick a basis $\{z_i\}_{i\neq 0}$ for $\mathfrak{u}^+ \oplus \mathfrak{u}^-$ as before but omitting $\mathfrak{g}_0$, and obeying the convention that $\overline{z_i} = z_{-i}$ and $z_i \in \mathfrak{u}^+$ if and only. if $i > 0$. In some situations it will be helpful to assume that $\{z_i\}$ is orthonormal.

Definition 2.4.1. *The Hodge star $*$ is the operator*

$$\hat{C}^{-p,q} \to \hat{C}^{-q,p} : \alpha \otimes a \mapsto (-1)^{pq}\phi(a) \otimes \phi^{-1}(\alpha).$$

Clearly $*^2 = \mathbb{1}$.

Lemma 2.4.2. *If $\alpha, \beta \in \bigwedge^*(\mathfrak{u}^+)^t$, $a, b \in \bigwedge^*\mathfrak{u}^-$, then $(\alpha \otimes a, *\beta \otimes b) = (-1)^{|\beta||b|}\alpha(\bar{b}) \cdot \overline{\beta(\bar{a})}$.*

One way to think about this lemma is that if $f \in \hat{C}^{-p,q}$ is regarded as a map $\bigwedge^q \mathfrak{u}^+ \to \bigwedge^p \mathfrak{u}^-$ then $(f, *\beta \otimes b) = \overline{\beta\left(\overline{f(\bar{b})}\right)}$. A consequence of this lemma is that $*$ is self-adjoint.

If S is an endomorphism of $\mathfrak{u}^+$, let S^t denote the operator on $\bigwedge^*\mathfrak{u}^+$ induced by dualizing and then extending the dual as a derivation. Let $\bar{S}$ denote the extension of the conjugate on $\mathfrak{u}^-$ to $\bigwedge^*\mathfrak{u}^-$.

Lemma 2.4.3. *Let $\{z_i\}$ be an orthonormal basis following the above convention.*

- *If $i > 0$ then $\iota(z_i)^* = \epsilon(z^i)$. If $i < 0$ then $\iota(z^i)^* = \epsilon(z_i)$.*

- *If* $f \in (\mathfrak{u}^+)^t$ *and* $z \in \mathfrak{u}^+$ *then* $*\epsilon(f)* = \epsilon(\phi^{-1}(f))$, *and* $*\iota(z)* = \iota\left(\overline{\phi(\bar{z})}\right)$.
- *If* $S, T \in \mathrm{End}(\mathfrak{u}^+)$ *then* $(S^t \otimes \bar{T})^* = *(T^t \otimes \bar{S})*$.

Proof. The adjoints are obvious after taking an ordered orthonormal basis. If $\alpha \otimes \beta \in \hat{C}^{p,q}$ then

$$*\epsilon(f) * \alpha \otimes b = (-1)^{|\alpha||b|} * f \wedge \phi(b) \otimes \phi^{-1}(\alpha) = (-1)^{|\alpha|}\alpha \otimes \phi^{-1}(f) \wedge b = \epsilon(\phi^{-1}(f))\alpha \otimes b.$$

while

$$*\iota(z)*\alpha\otimes b = *(-1)^{|\alpha||b|}\sum_i (-1)^i \phi(b_i)(z)\phi(b_0)\wedge\cdots\wedge\check{\phi(b_i)}\wedge\cdots\otimes\phi^{-1}(\alpha) = (-1)^{|\alpha|}\alpha\otimes\iota\left(\overline{\phi(\bar{z})}\right)b,$$

since $\phi(b_i)(z) = \overline{\phi(\bar{z})\left(\overline{b_i}\right)}$, and this latter expression is $\iota(\phi(z))\alpha \otimes b$.

Finally,

$$\big((S^t \otimes \bar{T})(\alpha \otimes a), *\beta \otimes b\big) = \alpha(S\bar{b})\overline{\beta\left(T(\bar{a})\right)} = \alpha\left(\overline{\bar{S}(b)}\right)\overline{T^t(\beta)(\bar{a})} = \big(\alpha \otimes a, *(T^t \otimes \bar{S})(\beta \otimes b)\big)\,.$$

Thus $\big((S^t \otimes \bar{T})A, B\big) = \big((S^t \otimes \bar{T})A, **B\big) = \big(A, *(T^t \otimes \bar{S}) * B\big)$. □

The first part of the proposition can be expressed in coordinate free fashion using the fact that $\phi(z_{-i}) = z^i$.

By definition, the fundamental form of the metric is $\omega = -i\sum_{i,j\geq 1}(z_i, z_j)z^i \wedge z^{-j}$. The operator $\epsilon(\omega)$ can be written on $\hat{C}^{a,b}(V)$ as $L = -i\sum(z_i, z_j)\epsilon(z^i)\iota(z^{-j})$. For convenience, assume that $\{z^i\}$ is an orthonormal basis so that $L = -i\sum_{i\geq 1}\epsilon(z^i)\iota(z^{-i})$. Then L is an operator of bidegree $(1,1)$, and

$$\Lambda = L^* = i\sum_{i\geq 1}\epsilon(z_{-i})\iota(z_i)$$

is an operator of type $(-1,-1)$. Now

$$H := [\Lambda, L] = \sum_{i,j\geq 1}[\epsilon(z_{-i})\iota(z_i), \epsilon(z^j)\iota(z^{-j})] = \sum_{i\geq 1}\epsilon(z_{-i})\iota(z^{-i}) - \epsilon(z^j)\iota(z_j)$$

acts on $\hat{C}^{p,q}$ by $-p-q$. Thus $[H, L] = -2L$ and $[H, \Lambda] = 2\Lambda$. This proves that the operators $\{H, \Lambda, L\}$ give an action of $\mathfrak{sl}(2,\mathbb{C})$ on $\hat{C}^{*,*}(V)$.

The proof of the following proposition (for $\mathfrak{g}$ a loop algebra) can be found in [Te95].

Proposition 2.4.4. *Let* V *be a* $\mathbb{Z}$*-graded vector space with a negative-energy* $\mathfrak{g}$*-action such that* V_n *is finite-dimensional for all* n*. Then the action of* $\mathfrak{sl}(2,\mathbb{C})$ *on* $\hat{C}^{*,*}(V)$ *lifts to an action of* $\mathrm{SL}(2,\mathbb{C})$*. Consequently*

$$\exp\left(\frac{\pi}{2}(\Lambda - L)\right) = i^{-p+q} * .$$

on $\hat{C}^{p,q}(V)$.

Finally, if T *is a bigraded operator on* $\hat{C}^{*,*}(V)$ *of total degree one and* $[L,T]=0$*, then*

$$*T* = \begin{cases} i[\Lambda, T] & T \text{ has degree } (1,0) \\ -i[\Lambda, T] & T \text{ has degree } (0,1). \end{cases}$$

The last part of this proposition is a version of what are called Kahler identities.

Proof. Note that Λ annihilates $\hat{C}^{-1,0} = \mathfrak{u}^-$ and sends $\hat{C}^{0,1}$ to $\hat{C}^{-1,0}$, while L annihilates $\hat{C}^{0,1}$ and sends $\hat{C}^{-1,0}$ to $\hat{C}^{0,1}$. Thus $M = \hat{C}^{-1,0} \oplus \hat{C}^{0,1} = (\mathfrak{u}^+)^t \oplus \mathfrak{u}^-$ is an $\mathfrak{sl}(2,\mathbb{C})$-submodule of $\hat{C}^{*,*}$. Because of the grading assumption, M is a direct sum of irreducible representations of highest weight 1, so M is integrable. It is helpful to write the action of Λ, L, and H on M in block matrix form with respect to the direct sum decomposition $M = \mathfrak{u}^- \oplus (\mathfrak{u}^+)^t$:

$$\Lambda = \begin{pmatrix} 0 & i* \\ 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ -i* & 0 \end{pmatrix}, \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On M the operator $\Lambda - L$ is equal to $i*$, so

$$\exp\left(\theta(\Lambda - L)\right) = \cos(\theta)\mathbb{1} + \sin(\theta) i * .$$

Now $\hat{C}^{*,*}$ is the exterior algebra of M, albeit with a modified grading. Extending the $\mathfrak{sl}(2,\mathbb{C})$-action on M by derivations gives an $\mathfrak{sl}(2,\mathbb{C})$-action on $\hat{C}^{*,*}$. But H,Λ, and L act by derivations on $\hat{C}^{*,*}$, so this new action agrees with the one we started with. Thus the action of $\mathfrak{sl}(2,\mathbb{C})$ on $\hat{C}^{*,*}$ can be integrated by taking the diagonal action of $\mathrm{SL}(2,\mathbb{C})$ on $\bigwedge^* M$. Thanks to the term $(-1)^{pq}$ in the definition of the Hodge star, the Hodge star on $\hat{C}^{*,*}$ agrees with its diagonal extension to $\bigwedge^* M$. Consequently

$$\exp\left(\frac{\pi}{2}(\Lambda - L)\right) = i^{-p+q} * .$$

on $\hat{C}^{p,q}$.

Finally, $\mathrm{SL}(2)$ acts on $\mathrm{End}(\hat{C}^{*,*})$ by conjugation. If T is a graded operator of degree $(1,0)$ or $(0,1)$ on $\hat{C}^{*,*}$ then $[H,T] = -T$. If $[L,T] = 0$ then T is a lowest weight vector of weight -1, so $\{T, \Lambda T\}$ is an irreducible subrepresentation of highest weight 1. If v is a lowest weight vector in this irreducible representation then $L\Lambda v = v$, so

$$\exp\left(\theta(\Lambda - L)\right) = \cos\theta v + \sin\theta \Lambda(v).$$

The conjugate of T by $i^{-p+q}*$ is $-i * T*$ if T has degree $(1,0)$, and $i * T*$ if T has degree $(0,1)$. Comparing these two expressions for $\exp(\pi(\Lambda - L)/2)T$ finishes the proposition. □

2.5 The Kahler identities

In this section we continue the conventions of the previous section, including the choice of basis for $\mathfrak{u}^+ \oplus \mathfrak{u}^-$. Additionally, suppose that d_V preserves the space $\Lambda^*_\infty(\mathfrak{g}, \mathfrak{g}_0; V)$. The map $C^{a,b}_\beta(V) \cong \Lambda^*_\infty(V)$ is $\mathfrak{g}_0$-linear, and preserves the action of $\iota(x)$ for all $x \in \mathfrak{g}_0$. The space of $\iota(x)$-invariants, $x \in \mathfrak{g}_0$, can thus be identified with the space $\hat{C}^{a,b}(V)$ defined in the previous section. The space $\Lambda^*_\infty(\mathfrak{g}, \mathfrak{g}_0; V)$ can be identified with the space $\hat{C}^{a,b}(V)^{\mathfrak{g}_0}$ of $\mathfrak{g}_0$-invariants in $\hat{C}^{a,b}(V)$. While the operator d_V, and consequently the operators $\bar{\partial}$ and D, act on the latter space of $\mathfrak{g}_0$-invariants, it is not clear that $\bar{\partial}$ preserves the space $\hat{C}^{a,b}(V)$. However, on the space of $\mathfrak{g}_0$-invariants $\bar{\partial}$ agrees with the Lie algebra cohomology differential

$$\bar{\partial}_{res} := \sum_{i \geq 1} \epsilon(z^i) \left(\frac{1}{2} \operatorname{ad}^t_{\mathfrak{u}^+}(z_i) + \widetilde{\operatorname{ad}}(z_i) + \pi(z_i) \right).$$

for $\mathfrak{u}^+$ with coefficients in $\bigwedge^* \mathfrak{u}^- \otimes V$. Thus we can use this as the definition of $\bar{\partial}_V$ on $\hat{C}^{a,b}(V)$.

Proposition 2.5.1. *Let V be a vector space with a negative-energy $\mathfrak{g}$-action π, and a metric $(,)$ such that $\pi(x)^* = -\pi(\bar{x})$. If $\bar{\partial}_V \equiv \bar{\partial}_{res}$ and D_V are the relative boundary operators on $\hat{C}^{a,b}(V)$ then $\bar{\partial}^*_V = - * D_V *$, and $D^*_V = - * \bar{\partial}_V *$.*

Proof. Suppose $\{z_i\}$ is an orthonormal basis. If $z \in \mathfrak{u}^+$, then $\operatorname{ad}^t_{\mathfrak{u}^+}(z)$ is the dual of $-\operatorname{ad}_{\mathfrak{u}^+}(z)$, while $\overline{\operatorname{ad}_{\mathfrak{u}^+}(z)} = \operatorname{ad}_{\mathfrak{u}^-}(\bar{z})$. So $\operatorname{ad}^t_{\mathfrak{u}^+}(z)^* = - * \operatorname{ad}_{\mathfrak{u}^-}(\bar{z}) *$. Similarly if $z \in \mathfrak{u}^-$ then $\widetilde{\operatorname{ad}}^t(z)$ on $(\mathfrak{u}^+)^t$ is the dual of $-\widetilde{\operatorname{ad}}(z)$ on $\mathfrak{u}^+$, while $\overline{\widetilde{\operatorname{ad}}(z)} = \widetilde{\operatorname{ad}}(\bar{z})$, so if $z \in \mathfrak{u}^+$ then $\widetilde{\operatorname{ad}}(z)^* = - * \widetilde{\operatorname{ad}}^t(\bar{z}) *$. Finally π commutes with $*$, so $\pi(z)^* = - * \pi(\bar{z}) *$. Thus

$$\begin{aligned} \bar{\partial}^* &= \sum_{i \geq 1} \left(\frac{1}{2} \operatorname{ad}^t_{\mathfrak{u}^+}(z_i)^* + \widetilde{\operatorname{ad}}(z_i)^* + \pi(z_i)^* \right) \iota(z_i) \\ &= - \sum_{i \geq 1} * \left(\frac{1}{2} \operatorname{ad}_{\mathfrak{u}^-}(z_{-i}) + \widetilde{\operatorname{ad}}^t(z_{-i}) + \pi(z_{-i}) \right) * (* \iota(z^{-i}) *) \\ &= - * D * . \end{aligned}$$

Since $*$ is self-adjoint, this also proves that $D^* = - * \bar{\partial} *$. □

Definition 2.5.2. *Say that a metric on $\mathfrak{g}$ is* graded *if $\mathfrak{g}_n \perp \mathfrak{g}_m$ for $m \neq n$. A* Kahler metric *for $(\mathfrak{g}, \mathfrak{g}_0)$ is a graded $\mathfrak{g}_0$-contragradient positive-definite Hermitian metric $(,)$ on $\mathfrak{u}^+$ such that the corresponding fundamental form $\omega = -i \sum_{i,j \geq 1} (z_i, z_j) z^i \wedge z^{-j}$ is a cocycle.*

A metric is $\mathfrak{g}_0$-contragradient if $\operatorname{ad}(x)^* = -\operatorname{ad}(\overline{x})$ for all $x \in \mathfrak{g}_0$. A Kahler metric can be regarded as a metric on $\mathfrak{g}$ by extending by conjugation on $\mathfrak{u}^-$ (as done in the previous section), and by zero on $\mathfrak{g}_0$. The point of $\mathfrak{g}_0$-contragradience is that the adjoint of a $\mathfrak{g}_0$-linear map with respect to a $\mathfrak{g}_0$-contragradient metric will be $\mathfrak{g}_0$-linear. The fundamental form ω

of a metric on $\mathfrak{u}^+$ can be written explicitly as the 2-form on $\mathfrak{g}$ sending $x \in \mathfrak{u}^+$, $y \in \mathfrak{u}^-$ to $-i\omega(x, y)$. The scalar multiple of $-i$ is chosen so that ω is conjugation equivariant. The cocycle condition states that

$$\omega([x, y], z) - \omega([x, z], y) + \omega([y, z], x) = 0$$

for all $x, y, z \in \mathfrak{g}$. Since ω is zero on $\mathfrak{g}_0$ by convention, the requirement that $(,)$ be $\mathfrak{g}_0$-contragradient is implied by the cocycle condition when one of $x, y, z \in \mathfrak{g}_0$. Since ω is conjugation invariant and of type $(1, 1)$, if $\mathfrak{g}_0$-contragradience is assumed then the cocycle condition only needs to be checked for $x, y \in \mathfrak{u}^+$, $z \in \mathfrak{u}^-$. Hence the cocycle condition can be written in terms of the metric as

$$([x, y], \overline{z}) + (y, \overline{[x, z]}) - (x, \overline{[y, z]}) = 0$$

for all $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_m$, $z \in \mathfrak{g}_{-m-n}$, $m, n > 0$.

Example 2.5.3. *Consider the loop algebra $\mathfrak{g} = L[z^{\pm 1}]$ graded by z-degree, where L is a simple finite-dimensional Lie algebra. Then $(\mathfrak{g}, \mathfrak{g}_0)$ has a unique (up to positive scalar multiple) Kahler metric given by $(xz^n, yz^m) = \delta_{mn} m(x, y)_{inv}$, where $(,)_{inv}$ is a contragradient metric on L.*

To show that this is unique, observe that $\mathfrak{g}_0 = L$, and that $\mathfrak{g}_n$ is isomorphic to the adjoint representation of L as a $\mathfrak{g}_0$-module. The graded and $\mathfrak{g}_0$-contragradience assumptions imply that the restriction of a Kahler metric $(,)$ to $\mathfrak{g}_n$ is a scalar multiple of the unique contragradient metric on L, i.e. $(xz^n, yz^m) = \delta_{mn} c_m (x, y)_{inv}$. For the cocycle condition, we then have to check that $c_{m+n} - c_m - c_n = 0$, so $c_n = nc_1$.

If ω is the fundamental form of a Kahler metric then $[\epsilon(\omega), d_V] = 0$. Comparing gradings it follows that $[\epsilon(\omega), \bar{\partial}_V] = [\epsilon(\omega), D_V] = 0$. Now $L = \epsilon(\omega)$ preserves $\hat{C}^{*,*}(V)$, annihilates $\mathfrak{g}_0^t$, and acts as an algebra derivation. So L also preserves the obvious complementary subspace to $\hat{C}^{*,*}(V)$ in $C^{*,*}(V)$, namely the ideal of the exterior algebra generated by $\mathfrak{g}_0^t$ (for contrast, D does not preserve this complementary subspace). If $\alpha \in \hat{C}^{a,b}(V)$, then $\bar{\partial}_V \alpha = \bar{\partial}_{res} \alpha + (*)$, where $(*)$ is in this complementary subspace. So $[\bar{\partial}_{res}, L] = 0 = [D, L]$ on $\hat{C}^{a,b}(V)$. If V has a metric such that $\pi(x)^* = -\pi(\bar{x})$ then Propositions 2.4.4 and 2.5.1 combine to give

$$D^* = i[\Lambda, \bar{\partial}] \text{ and } \bar{\partial}^* = -i[\Lambda, D]$$

on $\hat{C}^{a,b}(V)$. These are the canonical Kahler identities.

Since the operators D, D^*, $\bar{\partial}$, and $\bar{\partial}^*$ are odd, the graded commutators $\Box = [D_V, D_V^*]$ and $\overline{\Box} = [\bar{\partial}_V, \bar{\partial}_V^*]$ can be interpreted as Laplacians. That leads to the following proposition, whose proof once again is taken from [Te95].

Proposition 2.5.4 (Nakano's identity)**.** *Suppose V is a $\mathbb{Z}$-graded vector space with* $\dim V_n < +\infty$ *and a negative-energy $\mathfrak{g}$-action. Let γ be the semi-infinite cocycle and let Θ be the curvature of V. If V has a metric such that $\pi(x)^* = -\pi(\bar{x})$ then*

$$\overline{\Box} = \Box + i[\epsilon(\gamma + \Theta), \Lambda]$$

on $\hat{C}^{a,b}(V)^{\mathfrak{g}_0}$.

Proof. Start by applying the Kahler identity $[D, D^*] = i[D, [\Lambda, \bar{\partial}]]$. This latter expression is equal to $i[[D, \Lambda], \bar{\partial}] + i[\Lambda, [D, \bar{\partial}]]$. But $[D, \Lambda] = -i\bar{\partial}^*$, so the first part of this sum is $[\bar{\partial}, \bar{\partial}^*] = \overline{\Box}$. In the second part, $D^2 = \bar{\partial}^2 = 0$, so $[D, \bar{\partial}] = d^2 = \epsilon(\gamma + \Theta)$. □

In the examples we are interested in, V has $\mathfrak{g}$-action π such that $\pi|_{\mathfrak{p}}$ and $\pi|_{\overline{\mathfrak{p}}}$ are both Lie algebra actions. In this case

$$\epsilon(\Theta) = \sum_{i,j\geq 1} \epsilon(z^i)\iota(z^{-j}) \left([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}])\right).$$

on $\hat{C}^{a,b)}(V)$, while

$$\Lambda = i \sum_{i\geq 1} \epsilon(z_{-i})\iota(z_i)$$

with respect to an orthonormal basis in the Kahler metric. Now

$$\begin{aligned}
[\epsilon(z^i)\iota(z^{-j}), \epsilon(z_{-k})\iota(z_k)] &= [\epsilon(z^i), \epsilon(z_{-k})\iota(z_k)]\iota(z^{-j}) + \epsilon(z^i)[\iota(z^{-j}), \epsilon(z_{-k})\iota(z_k)] \\
&= -\delta_{ik}\epsilon(z_{-k})\iota(z^{-j}) + \delta_{jk}\epsilon(z^i)\iota(z_k).
\end{aligned}$$

Thus

$$\begin{aligned}
i[\epsilon(\Theta), \Lambda] &= -\sum_{i,j,k} [\epsilon(z^i)\iota(z^{-j}), \epsilon(z_{-k})\iota(z_k)] \left([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}])\right) && (2.1) \\
&= \sum_{i,j} \left(\epsilon(z_{-i})\iota(z^{-j}) - \epsilon(z^i)\iota(z_j)\right) \left([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}])\right) && (2.2)
\end{aligned}$$

Since the semi-infinite cocycle γ has type $(1,1)$, $(x, y) = -\gamma(x, \overline{y})$ is a Hermitian form on $\mathfrak{u}^+$ with fundamental form $\omega = i\gamma$. Since γ is a cocycle, if $(,)$ is positive definite then it is a Kahler metric for $(\mathfrak{g}, \mathfrak{g}_0)$. So $i[\epsilon(\gamma), \Lambda] = [L, \Lambda] = -H$, so we can simplify Nakano's identity to:

Corollary 2.5.5. *Suppose in addition to the hypotheses of Proposition 2.5.4 that* V *is a* $\mathfrak{p}$ *and* $\overline{\mathfrak{p}}$ *Lie algebra module, and that* $(\cdot, \cdot) = -\gamma(\cdot, \bar{\cdot})$ *is positive definite. Then*

$$\overline{\Box} = \Box + \deg + \sum_{i,j\geq 1} \left(\epsilon(z_{-i})\iota(z^{-j}) - \epsilon(z^i)\iota(z_j)\right) \left([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}])\right),$$

where deg *acts on* $\hat{C}^{a,b}$ *as multiplication by* $a + b$, *and* $\{z_i\}$ *is a basis for* $\mathfrak{u}^+$ *orthonormal in the Kahler metric.*

Chapter 3

Laplacian calculations

In this section, we make a Laplacian calculation for the cohomology rings $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{u}}^*)$ and $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*)$, where $\mathfrak{p}$ and $\mathfrak{u}$ are subalgebra of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$, as in the last chapter. Specifically, we assume that $\mathfrak{g}$ has finite-dimensional homogeneous components, an anti-linear conjugation automorphism sending $\mathfrak{g}_n$ to $\mathfrak{g}_{-n}$, and a positive-definite contragradient Hermitian form (satisfying $\mathfrak{g}_n \perp \mathfrak{g}_m$ if $m \neq n$ and $\overline{(x, y)} = (\overline{x}, \overline{y})$). The subalgebras $\mathfrak{p}$ and $\mathfrak{u}$ are equal to $\bigoplus_{n\geq 0} \mathfrak{g}_n$ and $\bigoplus_{n>0} \mathfrak{g}_n$, respectively. While we used restricted duals $\mathfrak{g}^t$ and $\mathfrak{u}^t$ for greater flexibility in the previous chapter, in this chapter we are only concerned with the algebras $\mathfrak{p}$ and $\mathfrak{u}$, and hence we return to using the continuous duals $\hat{\mathfrak{p}}^*$ and $\hat{\mathfrak{u}}^*$ of the completions of $\mathfrak{p}$ and $\mathfrak{u}$, as in the introduction. Also, since the conjugation operation will be available throughout this chapter, we prefer the notation $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ to $\mathfrak{u}^{\pm 1}$. Finally, we make a non-degeneracy assumption: we will assume that $\overline{\mathfrak{u}}^{\mathfrak{p}} = 0$, where $\overline{\mathfrak{u}}$ is regarded as the $\mathfrak{p}$-module $\mathfrak{g}/\mathfrak{p}$.

To be specific, choose a homogeneous basis $\{z_k\}$ for $\mathfrak{u}$, and let $\{z^k\}$ be the dual basis of $\hat{\mathfrak{u}}^*$. Let (V, π) be a $\hat{\mathfrak{p}}$-module. The Koszul complex for $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; V)$ is the chain complex $(C^q, \bar{\partial})$ defined by

$$C^q(\hat{\mathfrak{p}}, \mathfrak{g}_0; V) = \left(\bigwedge^q \hat{\mathfrak{u}}^* \otimes V \right)^{\mathfrak{g}_0}$$

and

$$\bar{\partial} = \sum_{k\geq 1} \epsilon(z^k) \left(\frac{1}{2} \operatorname{ad}^t_{\mathfrak{u}}(z_k) + \pi(z_k) \right).$$

If C^q is given a positive-definite Hermitian form then the cohomology H^* can be identified with the set $\ker \overline{\Box}$ of harmonic forms, where $\overline{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. The goal of this chapter is then to calculate $\ker \overline{\Box}$ for $V = S^*\hat{\mathfrak{p}}^*$ and $V = S^*\hat{\mathfrak{u}}$, in a Kahler metric that we will introduce.

In the prototypical example, $\mathfrak{g}$ is a twisted loop algebra of a semisimple Lie algebra, $\mathfrak{p}$ is some parahoric, and $\mathfrak{u}$ is the nilpotent subalgebra. When $\mathfrak{g} = L[z^{\pm 1}]$ is an untwisted loop algebra and $\mathfrak{p}$ is the current algebra $\mathfrak{p} = L[z]$, then $S^*\hat{\mathfrak{p}}^*$ and $S^*\hat{\mathfrak{u}}^*$ are isomorphic as $\hat{\mathfrak{p}}$-modules, and the relevant Laplacian calculation has been made in [FGT08]. The same

Laplacian calculation works in our more general setting, but two considerations have to be made. The first is that, for an arbitrary parahoric, the modules $S^*\hat{\mathfrak{p}}^*$ and $S^*\hat{\mathfrak{u}}^*$ are no longer isomorphic, and hence we make two Laplacian calculations to cover both cases. The second consideration is that, while $L[z]$ has a unique Kahler metric, the same is not true of an arbitrary parahoric. However, we show that the Laplacian can be calculated using the Kahler metric coming from a semi-infinite cocycle. The non-degeneracy condition $(\overline{\mathfrak{u}})^{\mathfrak{p}} = 0$ is needed to make sure that the Kahler metric constructed in this fashion is non-degenerate. Note that this condition does not hold if $\mathfrak{g}$ has a non-trivial centre. In particular, it does not hold if $\mathfrak{g} = L[z^{\pm 1}]$ where L has a non-trivial centre. However, in the latter case the centre splits off as a direct sum, and as long as this happens, the theorems in this section still hold.

3.1 The ring of harmonic forms

Lemma 3.1.1. *Let V be an $\hat{\mathfrak{p}}$-module, and J a derivation of $\mathfrak{p}$ which annihilates $\mathfrak{g}_0$. If $\phi \in \bigwedge^k \hat{\mathfrak{u}}^* \otimes V$ is $\hat{\mathfrak{p}}$-invariant then*

$$x_1 \wedge \cdots \wedge x_k \mapsto \phi(Jx_1, \ldots, Jx_k) \tag{3.1}$$

is a cocycle in $C^k(\hat{\mathfrak{p}}, \mathfrak{g}_0; V)$.

Proof. Let f be the cochain constructed as in equation (3.1). Then

$$\begin{aligned}(\bar{\partial} f)(x_0, \ldots, x_k) &= \sum_{i<j} (-1)^{i+j} f([x_i, x_j], \ldots, \check{x}_j, \ldots) + \sum_i (-1)^i x_i f(\ldots, \check{x}_i, \ldots) \\ &= \sum_{i<j} (-1)^{i+j} \phi(J[x_i, x_j], Jx_0, \ldots) + \sum_i (-1)^i x_i \phi(Jx_0, \ldots) \\ &= \sum_i (-1)^i \left(\mathrm{ad}^t(x_i)\phi\right)(Jx_0, \ldots, \check{x}_i, \ldots) + \sum_i (-1)^i x_i \phi(Jx_0, \ldots, \check{x}_i, \ldots),\end{aligned}$$

where the last equality follows from the fact that J is a derivation (ad^t denotes the adjoint action of $\hat{\mathfrak{p}}$ on $\hat{\mathfrak{u}}^*$, extended as a derivation to the exterior product). If ϕ is $\hat{\mathfrak{p}}$-invariant then the last line is zero, so f is a cocycle. That f is $\mathfrak{g}_0$-invariant is clear from the $\hat{\mathfrak{p}}$-invariance and the fact that J annihilates $\mathfrak{g}_0$. □

Definition 3.1.2. *A cochain $\omega \in \bigwedge^* \hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*$ is $\hat{\mathfrak{u}}$-*basic *if $\iota(f)\omega = 0$ for all linear functions $f : \hat{\mathfrak{p}} \to \hat{\mathfrak{u}}$ of the form $y \mapsto [x, y]$, $x \in \hat{\mathfrak{u}}$.*

The main theorem of this section allows us to identify $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*)$ with the ring of $\hat{\mathfrak{u}}$-basic $\hat{\mathfrak{p}}$-invariant cochains.

Theorem 3.1.3. *There is a positive-definite Hermitian form on $C^* = C^*(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*)$ and a derivation J of $\mathfrak{p}$ such that the harmonic forms in C^* are closed under multiplication, and furthermore the map in Lemma 3.1.1 gives an isomorphism between the ring of $\hat{\mathfrak{u}}$-basic $\hat{\mathfrak{p}}$-invariant elements of $\bigwedge^* \hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*$ and the ring of harmonic forms.*

Before proceeding to the proof, we note that Theorem 3.1.3 can be rephrased in a geometric manner. Let $\mathcal{P}$ and $\mathcal{N}$ be pro-Lie groups with Lie algebras $\hat{\mathfrak{p}}$ and $\hat{\mathfrak{u}}$ respectively. The space $\mathfrak{p}/\bigoplus_{n>k}\mathfrak{g}_n$ has the structure of an affine variety, so the pro-algebra $\hat{\mathfrak{p}}$ can be regarded as a pro-variety, with coordinate ring $S^*\hat{\mathfrak{p}}^*$.

Definition 3.1.4. *The tangent space $T\hat{\mathfrak{p}}$ is isomorphic to $\hat{\mathfrak{p}} \times \hat{\mathfrak{p}}$. Let $T_{>0}\hat{\mathfrak{p}}$ denote the subbundle of $T\hat{\mathfrak{p}}$ isomorphic to $\hat{\mathfrak{p}} \times \hat{\mathfrak{u}}$, and $T^*_{>0}\hat{\mathfrak{p}}$ the continuous dual bundle of $T_{>0}$. Let $\Omega^*_{>0}\hat{\mathfrak{p}}$ denote the ring of global sections of $\bigwedge^* T^*_{>0}\hat{\mathfrak{p}}$.*

The bundle $T_{>0}\hat{\mathfrak{p}}$ contains all tangents to $\mathcal{N}$-orbits. We will say that an element of $\Omega^*_{>0}\hat{\mathfrak{p}}$ is $\mathcal{N}$-basic *if it vanishes on all tangents to $\mathcal{N}$-orbits.*

With this terminology, we can identify the ring of $\hat{\mathfrak{u}}$-basic $\hat{\mathfrak{p}}$-invariant cochains with the ring of $\mathcal{P}$-invariant $\mathcal{N}$-basic elements of $\Omega^*_{>0}\hat{\mathfrak{p}}$.

Although Theorem 3.1.3 covers the main case of interest, a more natural result occurs if $S^*\hat{\mathfrak{p}}^*$ is replaced with $S^*\hat{\mathfrak{u}}^*$. An element ω of $\bigwedge^*\hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{u}}^*$ is $\hat{\mathfrak{p}}$-basic if $\iota(f)\omega = 0$ for all linear endomorphisms f of $\hat{\mathfrak{u}}$ of the form $y \mapsto [x,y]$, $x \in \hat{\mathfrak{p}}$.

Theorem 3.1.5. *There is a positive-definite Hermitian form on $C^*(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{u}}^*)$ and a derivation J (the same as in Theorem 3.1.3) such that the harmonic forms are closed under multiplication, and furthermore the map of Lemma 3.1.1 gives an isomorphism between the ring of $\hat{\mathfrak{p}}$-basic and invariant elements of $\bigwedge^*\hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{u}}^*$ and the ring of harmonic forms.*

In geometric language, the ring of $\hat{\mathfrak{p}}$-basic and invariant cochains is the same as the ring of $\mathcal{P}$-basic and invariant algebraic forms on $\hat{\mathfrak{u}}$.

As mentioned at the beginning of this chapter, the assumptions of Theorems 3.1.3 and 3.1.5 rule out the case that $\mathfrak{p}$ is a parahoric of $L[z^{\pm 1}]$ if L is a reductive algebra with a non-trivial centre $\mathfrak{z}$. However, in this case $L = [L,L] \oplus \mathfrak{z}$, and it is easy to deduce Theorems 3.1.3 and 3.1.5 by splitting off $\mathfrak{z}[z]$ (for instance, take J to be the identity on $\mathfrak{z}[z]$).

3.2 Construction of the Kahler metric

Proposition 3.2.1. *Let $\{,\}$ be the contragradient positive definite Hermitian form on $\mathfrak{g}$. There is a derivation of $\mathfrak{p}$, annihilating $\mathfrak{g}_0$ and positive-definite on $\mathfrak{u}$, such that $(\cdot,\cdot) = \{J\cdot,\cdot\} = \{\cdot, J\cdot\}$ is a Kahler metric for $(\mathfrak{p}, \mathfrak{g}_0)$ with fundamental form $i\gamma$, where γ is the semi-infinite cocycle.*

Proof. Let $\widetilde{\mathrm{ad}}_{\mathfrak{p}}$ denote the truncated action of $\mathfrak{p} \oplus \overline{\mathfrak{p}}$ on $\mathfrak{p} = \mathfrak{g}/\overline{\mathfrak{u}}$, and define

$$J = \sum_{k\geq 1} \widetilde{\mathrm{ad}}_{\mathfrak{p}}(x_k)\widetilde{\mathrm{ad}}_{\mathfrak{p}}(x_k)^*,$$

where $\{x_k\}_{k\geq 1}$ is a homogeneous basis for $\mathfrak{u}$, orthonormal in the contragradient metric. Define $(\cdot,\cdot) = \{J\cdot,\cdot\}$. Then J is positive semi-definite by definition, so $(,)$ is a positive semi-definite

Hermitian form. Suppose $a \in \mathfrak{g}_n$, $b \in \mathfrak{g}_{-n'}$, $n, n' \geq 0$, and assume without loss of generality that $x_1, \ldots, x_m$ is a basis of $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$. Since $\widetilde{\mathrm{ad}}_{\mathfrak{p}}(x_k)^* = -\widetilde{\mathrm{ad}}_{\mathfrak{p}}(\overline{x_k})$ we have

$$\begin{aligned}(a, \bar{b}) = \{Ja, \bar{b}\} &= \sum_{k=1}^{m} \{[\overline{x_k}, a], [\overline{x_k}, \bar{b}]\} \\ &= -\sum_{k=1}^{m} \{[b, [a, \overline{x_k}]], \overline{x_k}\} = -\sum_{l=1}^{n} \mathrm{tr}_{\mathfrak{g}_{-l}}(\mathrm{ad}(b)\,\mathrm{ad}(a)).\end{aligned}$$

Now $\mathrm{tr}_{\mathfrak{g}_{-l}}(\mathrm{ad}(b)\,\mathrm{ad}(a)) = \mathrm{tr}_{\mathfrak{g}_{n-l}}(\mathrm{ad}(a)\,\mathrm{ad}(b))$, so $-i(a, \bar{b})' = i\gamma(a, b)$. Since γ is a cocycle and $\{,\}$ is contragradient, it follows that J is a derivation. □

To give an example, suppose $\mathfrak{p}$ is a parahoric in a twisted loop algebra $\mathfrak{g} = L[z^{\pm 1}]^{\tilde{\sigma}}$ of a semisimple Lie algebra L. The parahoric $\mathfrak{p}$ comes from a grading of type d, where d_i is a non-negative integer giving the degree of the simple root vector e_i. In particular, $\mathfrak{p}$ is generated by $\mathfrak{g}_0$ and the simple root vectors e_i with $d_i > 0$. Now J annihilates $\mathfrak{g}_0$, and if $d_i > 0$ then

$$Je_i = \frac{[e_i, [f_i, e_i]]}{\{e_i, e_i\}} = 2\langle \rho, \alpha_i \rangle e_i,$$

where $f_i = -\overline{e_i}$. Since J is a derivation, this determines J on all of $\mathfrak{p}$. There is a Kac-Moody algebra $\tilde{\mathfrak{g}}$ associated to $\mathfrak{g}$, and this Kac-Moody algebra has a standard non-degenerate invariant symmetric bilinear form $\langle,\rangle$. The contragradient Hermitian form $\{,\}$ on $\mathfrak{g}$ defines a symmetric invariant bilinear form $\{\cdot, \bar{\cdot}\}$, and this symmetric form extends to a scalar multiple of the standard invariant form on $\tilde{\mathfrak{g}}$. The twisted loop algebra $\mathfrak{g}$ is also graded by the root lattice of the Kac-Moody algebra associated to $\mathfrak{g}$. Let ρ be a weight of the Kac-Moody defined on simple coroots by $\rho(\alpha_i^\vee) = 0$ if $d_i = 0$ and $\rho(\alpha_i^\vee) = 1$ if $d_i > 0$ (note that the $\alpha_i^\vee$'s are coroots of the associated Kac-Moody, not of the twisted loop algebra $\mathfrak{g}$). Then J is the derivation of $\mathfrak{p}$ acting on root spaces $\mathfrak{g}_\alpha$ as multiplication by $2\langle \rho, \alpha \rangle$.

3.3 Calculation of the curvature term

If S is a linear operator $\hat{\mathfrak{u}}^* \to \hat{\mathfrak{p}}^*$, define an operator $d_R(S)$ on $\bigwedge^* \hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*$ by

$$d_R(S)(\alpha_1 \wedge \ldots \wedge \alpha_k \otimes b) = \sum_i (-1)^{i-1} \alpha_1 \wedge \ldots \hat{\alpha}_i \ldots \wedge \alpha_k \otimes S(\alpha_i) \circ b.$$

If T is an operator $\hat{\mathfrak{p}}^* \to \hat{\mathfrak{u}}^*$, define a similar operator $d_L(T)$ by

$$d_L(T)(\alpha \otimes b_1 \circ \cdots \circ \cdots b_l) = \sum_i T(b_i) \wedge \alpha \otimes b_1 \circ \cdots \hat{b}_i \cdots \circ b_l$$

Recall that truncated actions are denoted by $\widetilde{\mathrm{ad}}$, with subscripts denoting the appropriate truncated space. By abuse of notation, let J^{-1} denote the inverse of the restriction of the derivation of Proposition 3.2.1 to $\mathfrak{u}$. We will also use J^{-1} to denote the dual operator on $\hat{\mathfrak{u}}^*$.

Proposition 3.3.1. *Let $\mathfrak{p}$ be a parahoric subalgebra of a twisted loop algebra $\mathfrak{g}$. Let $V = S^*\hat{\mathfrak{p}}^*$ with the contragradient metric. The Laplacian on $C^*(V)$ with respect to the dual Kahler metric from Proposition 3.2.1 has curvature term*

$$i[\epsilon(\gamma+\Theta),\Lambda] = \sum_{i>0} d_R\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)J^{-1}\right)^* d_R\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)J^{-1}\right),$$

where $\{x_i\}$ is a basis for $\mathfrak{u}$ orthonormal in the contragradient metric, and $\hat{\mathfrak{u}}^*$ is considered as the subset of $\hat{\mathfrak{p}}^*$ that is zero on $\mathfrak{g}_0$ (so that $\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}$ sends $\hat{\mathfrak{u}}^*$ to $\hat{\mathfrak{p}}^*$).

Proof. Applying Corollary 2.5.5, we just need to determine $R = i[\epsilon(\Theta)]$, where R is given by Equation (2.1). The action $\widetilde{\mathrm{ad}}^t$ acts as a derivation on the symmetric algebra $S^*\hat{\mathfrak{p}}^*$, so by Equation (2.1), R is a second-order operator. This means that if $\alpha_0,\ldots,\alpha_k \in \hat{\mathfrak{u}}^*$, $b_0,\ldots,b_l \in \hat{\mathfrak{p}}^*$ then

$$R(\alpha_0\wedge\cdots\wedge\alpha_k\otimes b_0\circ\cdots\circ b_l) = \sum_{i,j}(-1)^i R(\alpha_i\otimes b_j)\alpha_0\cdots\hat{\alpha}_i\cdots\hat{b}_j\cdots b_l.$$

In particular, R is determined by its action on $\hat{\mathfrak{u}}^*\otimes\hat{\mathfrak{p}}^*$.

The truncated action on V is isomorphic to the truncated action on $V' = S^*\overline{\mathfrak{p}}$ via the contragradient metric. Let $R' = i[\epsilon(\Theta_{V'}),\Lambda]$. If $f\in\hat{\mathfrak{u}}^*$ and $w\in\overline{\mathfrak{p}}$ then we claim that

$$R'(f\otimes w) = \sum_{i>0}\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(w)z^i\otimes\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i)\phi^{-1}(f),$$

where $\{z_i\}$ is any homogeneous basis of $\mathfrak{u}$, ϕ is the isomorphism $\overline{\mathfrak{u}}\to\hat{\mathfrak{u}}^*$ induced by the Kahler metric, and $\overline{\mathfrak{u}}$ is considered as a subset of $\overline{\mathfrak{p}}$, so that $\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}$ maps from $\overline{\mathfrak{u}}$ to $\overline{\mathfrak{p}}$. To prove this, let $\{z_i\}$ be orthonormal with respect to the Kahler metric, and think about $f = z^k$, w arbitrary. Observe that

$$\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z)w = \sum_{s\geq 1} y^{-s}([z,w])y_{-s},$$

where $\{y_s\}_{s\geq 1}$ is a homogeneous basis of $\mathfrak{p}$ and $y_{-s} = \overline{y_s}$. So if $z_{-j}\in\mathfrak{g}_{-m}$, then

$$\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i)\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_{-j})w = \sum_{s\geq 1} y^{-s}([z_i,[z_{-j},w]])y_{-s},$$

$$\begin{aligned}\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_{-j})\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i)w &= \sum_{s\geq 1} y^{-s}([z_i,w])[z_{-j},y_{-s}]\\ &= \sum_{n\leq 0}\sum_{y_{-s}\in\mathfrak{g}_{n-m}} y^{-s}([z_{-j},[z_i,w]])y_{-s}, \text{ and}\end{aligned}$$

$$\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}([z_i, z_{-j}])w = \sum_{s \geq 1} y^{-s}([[z_i, z_{-j}], w])y_{-s}.$$

Consequently

$$\left(\left[\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i), \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_{-j})\right] - \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}([z_i, z_{-j}])\right) w = \sum_{-m<n\leq 0} \sum_{y_{-s}\in\mathfrak{g}_n} y^{-s}([z_{-j}, [z_i, w]])y_{-s}.$$

After removing the reference to m here, we get

$$\left(\left[\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i), \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_{-j})\right] - \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}([z_i, z_{-j}])\right) w = \sum_{s\geq 1} \sum_{l\geq 1} y^{-s}([z_{-j}, z_l])z^l([z_i, w])y_{-s}.$$

Now from Equation (2.1),

$$\begin{aligned} R'(z^k \otimes w) &= -\sum_{i>0} z^i \otimes \left(\left[\widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i), \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_{-k})\right] - \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}([z_i, z_{-k}])\right) w \\ &= -\sum_{i,l,s>0} z^i \otimes y^{-s}([z_{-k}, z_l])z^l([z_i, w])y_{-s}. \end{aligned}$$

Moving the w action from z_i to z^i, the last expression becomes

$$-\sum_{i,s>0} \widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(w)z^i \otimes y^{-s}([z_{-k}, z_i])y_{-s} = \sum_{i>0} \widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(w)z^i \otimes \widetilde{\mathrm{ad}}_{\overline{\mathfrak{p}}}(z_i)z_{-k}.$$

The proof of the claim is finished by noting that this last expression is independent of the choice of basis $\{z_i\}$ for $\mathfrak{u}$ and that that $z_{-k} = \phi^{-1}(z^k)$.

Now we can translate from V' to V using the isomorphism $\psi : \overline{\mathfrak{p}} \to \hat{\mathfrak{p}}^*$ induced by the contragradient form. The operator J on $\mathfrak{u}$ has a basis $\{x_i\}$ of eigenvectors orthonormal in the contragradient metric. If $Jx_i = \lambda_i x_i$ then $\phi(\overline{x_i}) = \lambda_i x^i$, and thus $\psi \circ \phi^{-1}(x^i) = \lambda_i^{-1} x^i$. It follows that $\psi \circ \phi^{-1} = J^{-1}$ on $\hat{\mathfrak{u}}^*$. Next, $\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(w)\psi(x) = -\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x)\psi(w)$. Since $\psi(\overline{x_i}) = x^i$ we can conclude that

$$R(f \otimes g) = -\sum_{i>0} \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(\overline{x_i})g \otimes \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)J^{-1}f,$$

where $\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(\overline{x_i})$ is regarded as a map from $\hat{\mathfrak{p}}^*$ to $\hat{\mathfrak{u}}^*$.

If $S, T \in \mathrm{End}(\hat{\mathfrak{p}}^*)$, let Switch$(S,T)$ be the second order operator on $\bigwedge^* \hat{\mathfrak{p}}^* \otimes S^*\hat{\mathfrak{p}}^*$ sending $\alpha \otimes \beta \mapsto T\beta \otimes S\alpha$. Note that $\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)^* = -\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(\overline{x_i})$. We have shown that R is the restriction of the operator

$$\sum_i \mathrm{Switch}\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)J^{-1}, \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)^*\right)$$

to $\bigwedge^* \hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*$, where J^{-1} is zero on $\mathfrak{g}_0$. It is easy to see that Switch$(S,T) = d_L(T)d_R(S) - (TS)^\wedge$, where $(TS)^\wedge$ is the operator TS extended to $\bigwedge^* \hat{\mathfrak{p}}^*$ as a derivation. Also, $d_L(T)^* =$

$d_R(T^* J^{-1})$, where T^* is the adjoint of T in the contragradient metric. Note that the J^{-1} term comes from the difference between the contragradient metric on the symmetric factor and the Kahler metric on the exterior factor. Finally we have

$$R = \sum_i d_R \left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i) J^{-1} \right)^* d_R \left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i) J^{-1} \right) + \sum_i \left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(\overline{x_i}) \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i) J^{-1} \right)^{\wedge}.$$

Now $\sum_i \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(\overline{x_i}) \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)$ is the negative of the dual of the derivation J on $\mathfrak{u}$, while J^{-1} is the dual of the inverse of J. Thus on $\bigwedge^* \hat{\mathfrak{u}}^*$, this second summand is simply $-\deg$. But since we have chosen a Kahler metric with fundamental form $i\gamma$, we have $i[\epsilon(\gamma), \Lambda] = [L, \Lambda] = -H = \deg$, finishing the proof of the Proposition. □

Similarly, given endomorphisms S and T of $\hat{\mathfrak{u}}^*$ we can define operators $d_R(S)$ and $d_L(T)$ on $\bigwedge^* \hat{\mathfrak{u}}^* \otimes S^* \hat{\mathfrak{u}}^*$.

Proposition 3.3.2. *Let $\mathfrak{p}$ be a parahoric subalgebra of a twisted loop algebra $\mathfrak{g}$, and let $\mathfrak{u}$ be the nilpotent subalgebra. Let $V = S^* \hat{\mathfrak{u}}^*$ with the contragradient metric. The Laplacian on $C^*(V)$ with respect to the dual Kahler metric from Proposition 3.2.1 has curvature term*

$$i[\epsilon(\gamma + \Theta), \Lambda] = \sum_{i \geq 0} d_R \left(\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(y_i) J^{-1} \right)^* d_R \left(\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(y_i) J^{-1} \right),$$

where $\{y_i\}_{i \geq 0}$ is a basis for $\mathfrak{p}$ orthonormal in the contragradient metric.

Proof. Once again, let $R = i[\epsilon(\Theta), \Lambda]$, and let $V' = S^* \overline{\mathfrak{u}}$. The proof is similar to the proof of Proposition 3.3.1, except that if $f \in \mathfrak{u}^t$, $w \in \overline{\mathfrak{u}}$, then

$$R'(f \otimes w) = \sum_{i \geq 0} \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(w) y^i \otimes \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(y_i) \phi^{-1}(f), \tag{3.2}$$

where $\{y_i\}_{i \geq 0}$ is a basis for $\mathfrak{p}$. To prove this, let $\{z_i\}$ be orthonormal with respect to the Kahler metric, and think about $f = z^k$, w arbitrary. Observe that

$$\widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z) w = \sum_{i<0} z^i([z, w]) z_i.$$

So if $z_{-j} \in \mathfrak{g}_{-m}$, then

$$\widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_i) \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_{-j}) w = \sum_{k \geq 1} z^{-k}([z_i, [z_{-j}, w]]) z_{-k},$$

$$\begin{aligned} \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_{-j}) \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_i) w &= \sum_{k \geq 1} z^{-k}([z_i, w]) [z_{-j}, z_{-k}] \\ &= \sum_{n<0} \sum_{z_{-k} \in \mathfrak{g}_{n-m}} z^{-k}([z_{-j}, [z_i, w]]) z_{-k}, \text{ and} \end{aligned}$$

$$\widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}([z_i, z_{-j}])w = \sum_{k\geq 1} z^{-k}([[z_i, z_{-j}], w])z_{-k}.$$

Consequently

$$\left(\left[\widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_i), \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_{-j})\right] - \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}([z_i, z_{-j}])\right) w = \sum_{-m\leq n<0} \sum_{z_{-k}\in\mathfrak{g}_n} z^{-k}([z_{-j}, [z_i, w]])z_{-k}.$$

Removing the reference to m, we have

$$([\widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_i), \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_{-j})] - \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}([z_i, z_{-j}]))w = \sum_{k>0} \sum_{s\geq 0} z^{-k}([z_{-j}, y_s])y^s([z_i, w])z_{-k}.$$

From Equation (2.1),

$$\begin{aligned} i[\epsilon(\Theta), \Lambda](z^k \otimes w) &= -\sum_{i>0} z^i \otimes \left([\widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_i), \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(z_{-k})] - \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}([z_i, z_{-j}])\right) w \\ &= -\sum_{i,j>0} \sum_{s\geq 0} z^i \otimes z^{-j}([z_{-k}, y_s])y^s([z_i, w])z_{-j}. \end{aligned}$$

Now moving the z_{-l} action from z_i to z^i, we get

$$-\sum_{s\geq 0} \sum_{j>0} \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(w)y^s \otimes z^{-j}([z_{-k}, y_s])z_{-j} = \sum_{s\geq 0} \widetilde{ad}^t_{\mathfrak{p}}(w)y^s \otimes \widetilde{\mathrm{ad}}_{\overline{\mathfrak{u}}}(y_s)(z_{-k}).$$

Finally $z_{-k} = \phi^{-1}(z^k)$. □

3.4 Proof of Theorems 3.1.3 and 3.1.5

Once again let J denote the operator on $\hat{\mathfrak{u}}^*$ which is the dual of the derivation J on $\mathfrak{u}$. We give a proof of Theorem 3.1.3; the proof of Theorem 3.1.5 is identical. Let J_Δ denote the diagonal extension of J to the exterior factor of $\bigwedge^* \hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*$. The adjoint of $\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(x)$ in the Kahler metric is $-J\widetilde{\mathrm{ad}}_{\mathfrak{u}}(\overline{x})J^{-1}$. Thus we can directly calculate that

$$D^* = -\sum_{i<0} \epsilon(x_i) \left((J\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(x_{-i})J^{-1})^\wedge + \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_{-i})^{Sym}\right),$$

where $\{x_i\}$ is a basis of $\mathfrak{u}$ orthonormal in the contragradient metric and $x_{-i} = \overline{x_i}$. On $C^{0,q}(V)$ the D-Laplacian is $\Box = DD^*$, so the set of harmonic cocycles is the joint kernel of the operators D^* above and $d_R\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)J^{-1}\right)$, $i \geq 1$. The kernel of D^* on $C^{0,q}(V)$ is the joint kernel of the operators $\left(J\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(x_i)J^{-1}\right)^\wedge + \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)^{Sym}$, $i \geq 1$. Now we have

$$d_R\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)J^{-1}\right) J_\Delta = J_\Delta d_R\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)\right) \text{ and}$$

$$\left(\left(J\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(x_i)J^{-1}\right)^\wedge + \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)^{Sym}\right) J_\Delta = J_\Delta\left(\widetilde{\mathrm{ad}}^t_{\mathfrak{u}}(x_i)^\wedge + \widetilde{\mathrm{ad}}^t_{\mathfrak{p}}(x_i)^{Sym}\right).$$

Thus we see that J_Δ^{-1} identifies the set of harmonic cocycles with the joint kernels of the operators $d_R\left(\widetilde{\mathrm{ad}}_{\mathfrak{p}}^t(x_i)\right)$, $i \geq 1$, and $\left(\widetilde{\mathrm{ad}}_{\mathfrak{u}}^t(x_i)^\wedge + \widetilde{\mathrm{ad}}_{\mathfrak{p}}^t(x_i)^{Sym}\right)$, $i \geq 1$. Since the elements of $C^{0,q}(V)$ are $\mathfrak{g}_0$-invariant by definition, the kernel of the latter family of operators is the set of $\mathfrak{p}$-invariant cochains. The kernel of the former family of operators is the set of $\mathfrak{u}$-basic cochains, finishing the proof.

Chapter 4

Adjoint orbits of arc and jet groups

Let L be a reductive Lie algebra with a diagram automorphism σ of finite order k, and fix a qth root of unity. Let $\mathfrak{p}_0$ be a parabolic in L_0, and let $\mathfrak{p}$ be the standard parahoric $\{f \in L[z]^{\tilde{\sigma}} : f(0) \in \mathfrak{p}_0\}$ of the twisted loop algebra $\mathfrak{g} = L[z^{\pm 1}]^{\tilde{\sigma}}$. Let $\mathfrak{u}$ denote the nilpotent subalgebra of $\mathfrak{p}$. Grade $\mathfrak{g}$ so that $\mathfrak{p} = \bigoplus_{n \geq 0} \mathfrak{g}_n$ and $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$. Theorems 3.1.3 and 3.1.5 state that the cohomology $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*)$ is isomorphic to the ring of basic invariant elements of $\Omega_{>0}\hat{\mathfrak{p}}$, and that $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{u}}^*)$ is isomorphic to the ring of basic invariant forms on $\hat{\mathfrak{u}}$. Thus, to prove the strong Macdonald theorems in the next chapter, we need to show that basic invariant elements of $\Omega_{>0}\hat{\mathfrak{p}}$ correspond to certain types of forms on the GIT quotient of $\hat{\mathfrak{p}}$, and similarly with $\hat{\mathfrak{u}}$. In this chapter, we extend two well-known results on the orbit structure of the adjoint representation of a reductive Lie algebra to the case of a standard parahoric. The results we are interested in extending are the slice theorem for regular-semisimple orbits (addressed in subsection 4.2.1) and part of the Kostant slice theorem for regular orbits (addressed in subsection 4.2.2). These results will then be used in the next chapter to determine the ring of basic invariant forms.

This chapter is adapted from part of [Sl11b].

4.1 Twisted arc and jet schemes and the twisted arc group

This section covers background material on twisted arc and jet schemes, and proves basic facts about the twisted arc group arising from a diagram automorphism. The material in this section will be used to study adjoint orbits in Section 4.2.

4.1.1 Twisted arc and jet schemes

By a variety, we mean a separated, reduced, but not necessarily irreducible, scheme of finite type over $\mathbb{C}$. The arc scheme $J_\infty X$ of a variety X over $\mathbb{C}$ is a separated scheme of infinite type representing the functor $Y \mapsto \mathrm{Hom}(Y \times \mathrm{Spec}\,\mathbb{C}[[z]], X)$. Intuitively the arc scheme is the space of maps from the formal arc $\mathrm{Spec}\,\mathbb{C}[[z]]$ into X. The mth jet scheme J_mX $(0 \le m < +\infty)$ is a separated scheme of finite type over $\mathbb{C}$ representing the functor $Y \mapsto \mathrm{Hom}(Y \times \mathrm{Spec}\,\mathbb{C}[z]/z^m, X)$. If $m \le n$ then there is a morphism $J_nX \to J_mX$, and $J_\infty X$ is the inverse limit of the jet schemes of X. The $\mathbb{C}$-points of J_mX are m-jets, i.e. morphisms $\mathrm{Spec}\,\mathbb{C}[z]/z^m \to X$. For example, $J_0X = X$ and J_1X is the tangent scheme of X. If X is the affine subset of $\mathbb{C}^n$ cut out by the equations $f_1 = \ldots = f_k = 0$ then J_mX is the subscheme of $(\mathbb{C}[z]/z^m)^n$ cut out by the equations $f_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, f_k$, where $x_i \in \mathbb{C}[z]/z^m$ and $(\mathbb{C}[z]/z^m)^n$ is regarded as the affine space of dimension mn. The association $V \mapsto J_mV$ is functorial, so if G is an algebraic group then J_mG is an algebraic group when $m < +\infty$, and a pro-group when $m = +\infty$. The arc scheme of X is sometimes denoted by $X[[z]]$, but we use the notation $J_\infty X$ so that propositions can be stated uniformly for both arc and jet schemes.

The following well-known lemma is useful for working with jet schemes:

Lemma 4.1.1 ([Mu01])**.** *If $X \to Y$ is etale then $J_mX = X \times_Y J_mY$ for all $0 \le m \le +\infty$.*

Open immersions are etale, so if $U \subset X$ is an open subset then the pullback of U via $J_mX \to X$ is equal to J_mU. In particular, J_mX is covered by open subsets J_mU, where $U \subset X$ is an affine open (if $m = +\infty$ then we use the fact that the inverse limit of affine schemes is affine). If there is an etale map $X \to \mathbb{A}^n$ then $J_mX = X \times \mathbb{A}^{mn}$ for all $m < +\infty$, while $J_\infty X = X \times \mathbb{A}^\infty$. Consequently if X is smooth of dimension n then $J_mX \to J_kX$ is a (Zariski) locally-trivial $\mathbb{A}^{n(m-k)}$-bundle for all $k \le m < +\infty$. In particular, J_mX is smooth and the truncation morphisms $J_mX \to J_kX$ are surjective. Similarly $J_\infty X \to J_kX$ is a locally-trivial $\mathbb{A}^\infty$-bundle for all $0 \le k < +\infty$.[1]

The following lemma is likely well-known (and follows easily from formal smoothness):

Lemma 4.1.2. *If $X \to Y$ is smooth and surjective then the maps $J_mX \to J_mY$ are smooth for all $0 \le m < +\infty$, and surjective for all $0 \le m \le +\infty$*

From Lemmas 4.1.1 and 4.1.2 we get the following proposition:

Proposition 4.1.3. *Let $0 \le m < +\infty$. If $E \to M$ is an etale-locally trivial principal G-bundle then $J_mE \to J_mM$ is an etale-locally trivial principal J_mG-bundle.*

Proof. For $E \to M$ to be etale-locally trivial means that there is a surjective etale morphism $U \to M$ such that the pullback of E over U is isomorphic to the trivial G-bundle $U \times G$. Now J_m preserves etale maps (by Lemma 4.1.1 and the fact that etale maps are preserved under base change) and thus $J_mU \to J_mM$ is etale and surjective. The proof is finished by observing that J_m preserves pullbacks (which follows from the definition of the pullback via the functor of points). □

[1] The infinite-type schemes we work with are nice enough that they could be called "smooth" in their own right. However, we avoid this complication and only use smoothness for schemes of finite type. See for instance the wording of Lemma 4.1.2.

Proposition 4.1.3 on jet schemes has the following corollary:

Corollary 4.1.4. *Suppose X has a free G-action such that an etale-locally trivial quotient $X \to X/G$ exists. Then $J_m(X/G)$ is isomorphic to J_mX/J_mG, $0 \leq m < +\infty$, and $J_\infty(X/G)$ is isomorphic to $J_\infty X/J_\infty G$, where this last quotient is the pro-group quotient, i.e. the inverse limit of the quotients J_mX/J_mG, $0 \leq m < +\infty$.*

If $X \to X/G$ is etale-locally trivial then it is also surjective, so by Corollary 4.1.4 and Lemma 4.1.2 the map $J_\infty X \to J_\infty X/J_\infty G$ is surjective. If X is affine with a free G-action and G is reductive then $X/G = X//G$, the GIT quotient, and $X \to X/G$ is etale-locally trivial by Luna's slice theorem [Lu73] (the theorem applies because all orbits under a free action are closed, see the discussion on page 53 of [Bo91]). All the quotients we study will be of this type.

Now suppose that X has an automorphism σ of finite order k. This automorphism lifts to an automorphism σ of the jet and arc schemes J_mX. Choose a fixed kth root of unity q, and let $m(q)$ denote the automorphisms of $\mathbb{C}[z]/z^n$ and $\mathbb{C}[[z]]$ induced by sending $z \mapsto qz$.

Definition 4.1.5. *Let $\tilde{\sigma}$ denote the automorphism $\sigma \circ m(q)^{-1}$. The* twisted jet (resp. arc) scheme *$J_m^{\tilde{\sigma}}X$ is the equalizer of the morphisms $\mathbb{1}_{J_mX}$ and $\tilde{\sigma}$ in the category of schemes.*

In other words, if $m < +\infty$ then $J_m^{\tilde{\sigma}}X$ represents the functor $Y \mapsto \{f \in \mathrm{Hom}(Y \times \mathrm{Spec}\,\mathbb{C}[z]/z^m, X) : f \circ m(q) = \sigma \circ f\}$, while $J_\infty^{\tilde{\sigma}}X$ represents the functor $Y \mapsto \{f \in \mathrm{Hom}(Y \times \mathrm{Spec}\,\mathbb{C}[[z]], X) : f \circ m(q) = \sigma \circ f\}$.

$J_m^{\tilde{\sigma}}X$ is a closed subscheme of J_mX, and $(J_mX)^{\tilde{\sigma}}$ is separated for all m. Since $\mathrm{Spec}\,\mathbb{C}[[z]]$ is the direct limit of schemes $\mathrm{Spec}\,\mathbb{C}[z]/z^n$, it follows from the functor of points characterisation that $J_\infty^{\tilde{\sigma}}X$ is the inverse limit of schemes $J_m^{\tilde{\sigma}}X$, $0 \leq m < +\infty$. Since $J_m^{\tilde{\sigma}}X$ is a closed subscheme of J_mX, it is covered by the inverse images of the open subschemes $J_mU \subset J_mX$, for $U \subset X$ open affine. The inverse image of J_mU in $J_m^{\tilde{\sigma}}X$ is the same as the inverse image of $\tilde{\sigma}(J_mU) = J_m\sigma(U)$. Thus the inverse image of J_mU in $J_m^{\tilde{\sigma}}X$ is the same as in the inverse image of J_mV, where $V = U \cap \sigma(U) \cap \ldots \cap \sigma^{k-1}(U)$. By definition $\sigma(V) = V$, and V is affine because X is separated. Finally, the pullback of J_mV to $J_m^{\tilde{\sigma}}X$ is $J_m^{\tilde{\sigma}}V$, and $J_m^{\tilde{\sigma}}V$ is affine. We conclude that $J_m^{\tilde{\sigma}}X$ is covered by open affines $J_m^{\tilde{\sigma}}U$ where $U \subset X$ runs through open affines such that $\sigma(U) = U$.

The following lemma is an immediate consequence of the definition of tangent and jet (resp. arc) schemes via functor of points.

Lemma 4.1.6. *Let σ_* be the automorphism induced by σ on TX. Then the tangent scheme to $J_m^{\tilde{\sigma}}X$ is naturally isomorphic to the twisted jet (resp. arc) scheme $J_m^{\widetilde{\sigma_*}}(TX)$ of the tangent scheme to X.*

Using known results for finite-dimensional varieties, we can show that the twisted jet scheme of a smooth variety is also smooth.

Lemma 4.1.7. *Let $0 \le m < +\infty$. If X is a smooth variety with a finite-order automorphism σ then $J_m^{\tilde\sigma}X$ is a smooth variety. In addition, if X and Y are both smooth varieties with finite-order automorphisms σ_X and σ_Y and $X \to Y$ is a σ-equivariant smooth map then $J_m^{\tilde\sigma}X \to J_m^{\tilde\sigma}Y$ is smooth.*

Proof. We can assume that X is affine. Since X is smooth, J_mX is also a smooth variety. The twisted jet scheme $J_m^{\tilde\sigma}X$ is the fixed-point scheme of the finite group $\langle\tilde\sigma\rangle$. It is a well-known consequence of Luna's slice theorem that the fixed-point variety of a reductive algebraic group acting on a smooth variety is also smooth. This also holds for the fixed-point scheme by Proposition 7.4 of [Fo73], so $J_m^{\tilde\sigma}X$ is smooth.[2]

Since $J_m^{\tilde\sigma}X$ is a smooth variety the tangent scheme is a vector bundle. By Lemma 4.1.6, $T_xJ_m^{\tilde\sigma}X = (T_xJ_mX)^{\tilde\sigma_*}$ and similarly $T_yJ_m^{\tilde\sigma}Y = (T_yJ_mY)^{\tilde\sigma_*}$. If $X \to Y$ is smooth then $J_mX \to J_mY$ is smooth by Lemma 4.1.2, hence $TJ_mX \to TJ_mY$ is surjective on fibres, and it follows that $(T_xJ_mX)^{\tilde\sigma_*} \to (T_yJ_mY)^{\tilde\sigma_*}$ is surjective. Since both $J_m^{\tilde\sigma}X$ and $J_m^{\tilde\sigma}Y$ are smooth, $J_m^{\tilde\sigma}X \to J_m^{\tilde\sigma}Y$ is a smooth map. □

Note that $J_m^{\tilde\sigma}X$ is not necessarily irreducible, as X^σ can be disconnected.

We also have the following analogue of Lemma 4.1.1.

Lemma 4.1.8. *Let $0 \le m \le +\infty$. Suppose that X and Y have finite-order automorphisms σ_X and σ_Y. If $X \to Y$ is an etale σ-equivariant map then $J_m^{\tilde\sigma}X = X^\sigma \times_{Y^\sigma} J_m^{\tilde\sigma}Y$.*

Proof. By Lemma 4.1.1, $J_mX \cong X \times_Y J_mY$. The automorphism $\tilde\sigma_X$ on J_mX translates to the unique automorphism on the latter space which lies above σ_X on X, σ_Y on Y, and $\tilde\sigma_Y$ on J_mY. The result follows from the functor of points characterisations of the twisted jet and arc schemes and the fibre product. □

Finally, the jet structure distinguishes a subbundle of the tangent bundle of a jet or arc space.

Definition 4.1.9. *If X is a variety with finite-order automorphism σ, we let $T_{const}J_m^{\tilde\sigma}X$ denote the pullback $X^\sigma \times_{TX^\sigma} TJ_m^{\tilde\sigma}$, where $TJ_m^{\tilde\sigma}X \to TX^\sigma$ is the differential of the projection $J_m^{\tilde\sigma}X \to X^\sigma$ and $X^\sigma \to TX^\sigma$ is the zero section. Intuitively $T_{const}J_m^{\tilde\sigma}$ is the space of infinitesimal families of jets (resp. arcs) which are constant at $z = 0$.*

4.1.2 Connectedness of the twisted arc group

In this section G will be a connected algebraic group with Lie algebra L, such that the diagram automorphism σ lifts to G (for example, this occurs if G is simply-connected). H will be the torus corresponding to the chosen Cartan $\mathfrak{h}$.

We recall some basic facts about diagram automorphisms and the structure of L, using terminology and basic results from Chapter 9, Section 5 of [Ca05]. Let $\mathfrak{h}_i$ denote the q^ith

[2] If X is not smooth then the fixed-point scheme of a reductive group action can be non-reduced.

eigenspace of σ acting on $\mathfrak{h}$. By definition, there is a choice of simple roots $\alpha_1, \ldots, \alpha_l$ such that σ permutes the corresponding coroots h_{α_i} and Chevalley generators e_{α_i}. If J is an orbit the σ-action on simple roots, let $\alpha_J = \frac{1}{|J|}\sum_{\alpha \in J} \alpha$. Then the set $\{\alpha_J|_{\mathfrak{h}_0} : J \text{ is an orbit of } \sigma\}$ is a set of simple roots for L_0. Restriction to $\mathfrak{h}_0$ gives an isomorphism between the subgroup W^σ (where W is the Weyl group of L) and the Weyl group $W(L_0)$ of L_0. The simple generator s_J of $W(L_0)$ given by reflection through α_J on $\mathfrak{h}_0$ corresponds to the element of $W^\sigma \subset W(L)$ which is the maximal element in the subgroup of $W(L)$ generated by reflection through the simple roots in J. In addition, we will need:

Lemma 4.1.10. *If $N(H)$ is the normalizer of H in G, then $N(H)^\sigma = N_{G^\sigma}(H^\sigma)$, the normalizer of H^σ in G^σ. Consequently $W^\sigma = N(H)^\sigma/H^\sigma \subset W(L)$. Furthermore, the inclusion $W(L_0) \cong W^\sigma \hookrightarrow W(L)$ is length-preserving, in the sense that if $w \in W^\sigma$, then it is possible to get a reduced expression for w by first taking a reduced expression $w = s_{J_1} \cdots s_{J_r}$ for w in $W(L_0)$, and then replacing each s_{J_i} with a reduced expression in $W(L)$.*

Proof. For the first part, let $\rho \in \mathfrak{h}$ be the element such that $\alpha(\rho) = 1$ for all simple roots α of L. Then ρ is regular in $\mathfrak{h}$ and belongs to $\mathfrak{h}_0$. Any element of $N_{G^\sigma}(H^\sigma)$ sends ρ to another regular element of $\mathfrak{h}$, and hence belongs to $N(H)$.

For the second part, we we refer to the proof of Proposition 9.17 of [Ca05]. □

We can use Lemma 4.1.10 to prove:

Lemma 4.1.11. *Choose a Borel subgroup $\mathcal{B}$ of G containing H and compatible with σ and let $X = \overline{B}B$ be the big cell of the corresponding Bruhat decomposition. If $x \in G$ belongs to a Bruhat cell BwB with $w \in W^\sigma$ then there is $g \in N(H)^\sigma$ such that $gx \in X$.*

Proof. If we take for g a representative of w^{-1} in $N(H)^\sigma$, then $gBwB \subset \overline{B}B$. □

Proposition 4.1.12. *G^σ is connected.*

Proof. The connected component $(G^\sigma)^\circ$ of G^σ is a connected reductive group with Lie algebra L_0. Since σ permutes coroots, it is easy to see that H^σ is a connected torus, and in fact is a Cartan in $(G^\sigma)^\circ$. As in Lemma 4.1.11, let $\mathcal{B}$ be a Borel subgroup of G containing H and compatible with σ, and let X be the corresponding big cell. If $g \in G^\sigma$ belongs to a Bruhat cell BwB then $g \in BwB \cap \sigma(BwB)$, so $w \in W^\sigma$. By Lemma 4.1.10, every element of $N(H)^\sigma$ can be implemented by an element of $(G^\sigma)^\circ$. So by Lemma 4.1.11, we just need to prove that $G^\sigma \cap X$ is contained in $(G^\sigma)^\circ$.

Now as an algebraic variety, $X \cong \overline{U} \times H \times U$, where U is the unipotent radical of $\mathcal{B}$. The action of σ on X translates to the action of σ on each factor. Let $\mathfrak{u}$ be the Lie algebra of U. The exponential map for nilpotent Lie algebras is bijective, so U^σ is the unipotent subgroup corresponding to the nilpotent Lie algebra $\mathfrak{u}^\sigma$. In particular U^σ is connected, and similarly with $\overline{U}^\sigma$. We conclude that $X^\sigma = G^\sigma \cap X$ is connected. □

Using the fact that the exponential map for nilpotent (resp. pro-nilpotent) Lie algebras is bijective, we immediately get the following corollary.

Corollary 4.1.13. *If $0 \leq m < +\infty$ then $J_m^{\tilde{\sigma}} G$ is a connected algebraic group with Lie algebra $L[z]/z^m$. Similarly $J_\infty^{\tilde{\sigma}} G$ is a connected pro-algebraic group with Lie algebra $L[[z]]^{\tilde{\sigma}}$.*

As a scheme the Lie algebra of $J_m^{\tilde{\sigma}} G$ can be identified with $J_m^{\tilde{\sigma}} L$.

The following proposition will be crucial in the next section, since it proves that $J_m^{\tilde{\sigma}}(G/H)$ is a $J_m^{\tilde{\sigma}} G$-homogeneous space.

Proposition 4.1.14. *$J_m^{\tilde{\sigma}}(G/H) \cong J_m^{\tilde{\sigma}} G / J_m^{\tilde{\sigma}} H$, where the latter space is either the group quotient if $0 \leq m < +\infty$, or the pro-group quotient if $m = +\infty$.*

Proof. $G \to G/H$ is an etale-locally trivial principal bundle, so $J_m(G/H) \cong J_m G / J_m H$. There is an inclusion $J_m^{\tilde{\sigma}} G / J_m^{\tilde{\sigma}} H \hookrightarrow (J_m G / J_m H)^{\tilde{\sigma}}$. To prove the proposition, we will show that this inclusion is surjective for all $m < +\infty$. If $m < +\infty$ then biregularity follows from bijectivity because $(J_m G / J_m H)^{\tilde{\sigma}}$ will be a homogeneous space. Biregularity for $m = +\infty$ follows from the universal property of inverse limits.

Define $\alpha : J_m G \to J_m G$ by $g \mapsto g^{-1} \tilde{\sigma}(g)$. To show that the inclusion is surjective we need to show that every element of $(J_m G / J_m H)^{\tilde{\sigma}}$ has a representative $x \in J_m G$ such that $\alpha(x) = e$. The map α has a number of nice properties. First, the fibres of α are left $J_m^{\tilde{\sigma}} G$-cosets. Second, $g \in J_m G$ represents an element of $(J_m G / J_m H)^{\tilde{\sigma}}$ if and only if $\alpha(g) \in J_m H$. Third, if $\alpha(g) \in J_m H$ and $h \in J_m H$ then $\alpha(gh) = \alpha(g)\alpha(h)$. By these last two properties, we will have $(J_m G / J_m H)^{\tilde{\sigma}} = J_m^{\tilde{\sigma}} G / J_m^{\tilde{\sigma}} H$ if and only if $\alpha(J_m G) \cap J_m H = \alpha(J_m H)$.

Our proof depends on the Bruhat geometry of G, so pick a Borel subgroup $B \subset G$ compatible with σ. Let $X = \overline{B} B$ be the big cell. Suppose $x \in J_m G$ and $\alpha(x) \in J_m H$. Writing $x(0) = b_0 w b_1$, we get $\alpha(x(0)) = b_1^{-1} w^{-1} \alpha(b_0) \sigma(w) \sigma(b_1) \in H$. But $\alpha(b_0) \in B$, so $wB \cap B\sigma(w)B \neq \emptyset$, and thus w belongs to W^σ. Consequently there is $g_0 \in G^\sigma$ such that $g_0 x(0) \in X$, implying that $g_0 x \in J_m X$. Since $\alpha(x) = \alpha(g_0 x)$ for $g_0 \in G^\sigma$, we just need to show that $\alpha(J_m X) \cap J_m H$ is contained in $\alpha(J_m H)$.

The space X is isomorphic to $\overline{U} \times B$ via the multiplication map, where $\overline{U}$ is the unipotent subgroup of $\overline{B}$. Thus we can write any element of $J_m X$ uniquely as $a(z)b(z)$, where $a(z) \in J_m \overline{U}$ and $b(z) \in J_m B$. Suppose $\alpha(a(z)b(z)) = h(z) \in J_m H$. Since $\alpha(a(z)b(z)) = b(z)^{-1} \alpha(a(z)) \tilde{\sigma}(b(z))$, we see that $\alpha(a(z)) = b(z) h(z) \tilde{\sigma}(b(z))^{-1} \in J_m B$. Since $\alpha(a(z)) \in J_m \overline{U}$, this implies that $\alpha(a(z)) = e$ and consequently $\alpha(b(z)) = h(z)$. To finish the proof, observe that $B \cong U \times H$ via the multiplication map, where U is the unipotent subgroup of B. Writing $b(z) = b'(z) h'(z)$ for $b'(z) \in J_m U$ and $h'(z) \in J_m H$, we get $\alpha(b(z)) = h'(z)^{-1} \alpha(b'(z)) \tilde{\sigma}(h'(z))$, and hence $\alpha(b'(z))$ can be written as an element of $J_m H$. This implies that $\alpha(b'(z)) = e$, finishing the proof, since $\alpha(h'(z)) = h(z)$. □

4.2 Slice theorems for the adjoint action

We continue to use the notation from Section 4.1. In particular, G is a connected algebraic group with Lie algebra L such that σ extends to G, and the Lie algebra of $J_m^{\tilde{\sigma}} G$ is identified with $J_m^{\tilde{\sigma}} L$. In addition, we fix a standard parabolic subalgebra $\mathfrak{p}_0 \subset L_0$, and let $\mathfrak{p}_m = \{f \in J_m^{\tilde{\sigma}} : f(0) \in \mathfrak{p}_0\}$. Note that $\mathfrak{p}_\infty$ is the completion of a standard parahoric in $L[z^{\pm 1}]^{\tilde{\sigma}}$, which we also denote by $\hat{\mathfrak{p}}$. We let $\mathcal{P}_m$ be the connected algebraic (resp. pro-algebraic) subgroup of $J_m^{\tilde{\sigma}} G$ corresponding to $\mathfrak{p}_m$, and $\mathcal{N}_m$ be the nilpotent (resp. pro-nilpotent) radical of $\mathcal{P}_m$. The reductive factor $\mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$ of $\mathfrak{p}_0$ is denoted by $\mathfrak{g}_0$.

In this section we prove two slice theorems for the adjoint action of $\mathcal{P}_m$ on $\mathfrak{p}_m$. The first is an analogue of the well-known slice theorem for regular semisimple elements in L, and is given in Subsection 4.2.1. The second is an analogue of the Kostant slice theorem, and is given in Subsection 4.2.2. These theorems will be used in the next section to determine the $\mathcal{P}_\infty$-invariant $\mathcal{N}_\infty$-basic elements of $\Omega^*_{>0}\mathfrak{p}_\infty$.

The slice theorems are stated in terms of the GIT quotients $Q := L//G$ (i.e. Q is the affine variety with coordinate ring $\mathbb{C}[Q] = (S^*L^*)^G$) and $R := \mathfrak{p}_0//\mathcal{P}_0$. Recall that $\mathbb{C}[Q]$ is a free algebra generated by homogeneous elements in degrees $m_1 + 1, \ldots, m_l + 1$, where l is the rank of L and $m_1, \ldots, m_l$ are the exponents. A similar result holds for $\mathbb{C}[R]$:

Lemma 4.2.1. *Let $\mathfrak{u}_0$ the nilpotent radical of $\mathfrak{p}_0$, so that $\mathfrak{p}_0 = \mathfrak{g}_0 \oplus \mathfrak{u}_0$. If $f \in \mathbb{C}[R]$ then $f(x,y) = f(x,0)$ for all $x \in \mathfrak{g}_0$, $y \in \mathfrak{u}_0$. Consequently, if $\mathcal{M}$ is the Levi subgroup of $\mathcal{P}_0$ then $R \cong \mathfrak{g}_0//\mathcal{M} \cong \mathfrak{h}_0//W(\mathfrak{g}_0)$, where $W(\mathfrak{g}_0)$ is the Weyl group of $\mathfrak{g}_0$, and $\mathbb{C}[R]$ is a free algebra generated by homogeneous elements in degrees given by the exponents of $\mathfrak{g}_0$.*

Proof. The set of regular elements $\mathfrak{h}_0^r$ is dense in $\mathfrak{h}_0$. Since $[\mathfrak{g}_0, x] + \mathfrak{h}_0 = \mathfrak{g}_0$ for any element $x \in \mathfrak{h}_0^r$, the set $\mathcal{M}\mathfrak{h}_0^r$ is dense in $\mathfrak{g}_0$. Let $\mathcal{N}_0$ be the unipotent subgroup corresponding to $\mathfrak{u}_0$. If x belongs to $\mathfrak{h}_0^r$ then $\mathcal{N}_0 x = x + \mathfrak{u}_0$. Since $\mathcal{N}_0$ is normal in $\mathcal{P}_0$, this property extends to any $x \in \mathcal{M}\mathfrak{h}_0^r$. So if f is invariant then $f(x,y) = f(n(x,0)) = f(x,0)$ for x in an open dense subset of $\mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$. □

4.2.1 The regular semisimple slice

Let $L^{rs} \subset L$ be the subset of regular semisimple elements. L^{rs} is an affine open subset of L (its complement is the vanishing set of a single G-invariant function) and consequently the image Q^r of L^{rs} in $Q = L//G$ is open. The well-known regular semisimple slice theorem states that there is a commutative square

$$\begin{array}{ccc} G/H \times_W \mathfrak{h}^r & \longrightarrow & L^{rs} \\ \downarrow & & \downarrow \\ \mathfrak{h}^r/W & \longrightarrow & Q^r \end{array} \qquad (4.1)$$

where W is the Weyl group of L and $\mathfrak{h}^r$ is the set of regular elements in $\mathfrak{h}$. The notation $G/H \times_W \mathfrak{h}^r$ denotes the quotient of $G/H \times \mathfrak{h}^r$ under the free action of $W = N(H)/H$ acting by right multiplication on G/H and by the adjoint action on $\mathfrak{h}^r$. Both horizontal maps are isomorphisms. The top horizontal map is given by multiplication, while the bottom horizontal map is projection to Q.

Since σ is an automorphism, the sets L^{rs} and $\mathfrak{h}^r$ are closed under σ and we can apply $J_m^{\tilde{\sigma}}$ to both spaces. The image R^r of $\mathfrak{p}_0 \cap L_0^{rs}$ in R is open, since it's complement is the zero set of a single $\mathcal{P}_0$-invariant function. As usual, let $\mathcal{P}_\infty/J_\infty^{\tilde{\sigma}}H$ denote the pro-group quotient. Similarly $\mathcal{P}_\infty/J_\infty^{\tilde{\sigma}}H \times_{W(\mathfrak{g}_0)} J_\infty^{\tilde{\sigma}}\mathfrak{h}^r$ will denote the pro-group quotient of $\mathcal{P}_\infty/J_\infty^{\tilde{\sigma}}H \times J_\infty^{\tilde{\sigma}}\mathfrak{h}^r$ by $W(\mathfrak{g}_0)$, and $J_\infty^{\tilde{\sigma}}\mathfrak{h}^r/W(\mathfrak{g}_0)$ denotes the pro-group quotient of $J_m^{\tilde{\sigma}}\mathfrak{h}^r$ by $W(\mathfrak{g}_0)$. We have the following analogue of Equation (4.1) for twisted jet and arc schemes.

Theorem 4.2.2. *Let $0 \leq m \leq +\infty$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{P}_m/J_m^{\tilde{\sigma}}H \times_{W(\mathfrak{g}_0)} J_m^{\tilde{\sigma}}\mathfrak{h}^r & \longrightarrow & \mathfrak{p}_m \cap J_m^{\tilde{\sigma}}L^{rs} \\ \downarrow & & \downarrow \\ \left(J_m^{\tilde{\sigma}}\mathfrak{h}^r\right)/W(\mathfrak{g}_0) & \longrightarrow & R^r \times_{Q^\sigma} J_m^{\tilde{\sigma}}Q^r \end{array} \tag{4.2}$$

in which the horizontal maps are isomorphisms, with the top map induced by multiplication and the bottom map induced from the two projections $J_m^{\tilde{\sigma}}\mathfrak{h}^r/W(\mathfrak{g}_0) \to J_m^{\tilde{\sigma}}Q^r$ and $\mathfrak{h}_0^r/W(\mathfrak{g}_0) \cong R^r$.

To prove Theorem 4.2.2, we start with the case $m = 0$ (likely well-known, but we give the proof for completeness).

Lemma 4.2.3. *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{P}_0/H^\sigma \times_{W(\mathfrak{g}_0)} \mathfrak{h}_0^r & \longrightarrow & \mathfrak{p}_0 \cap L_0^{rs} \\ \downarrow & & \downarrow \\ \mathfrak{h}_0^r/W(\mathfrak{g}_0) & \longrightarrow & R^r \end{array}$$

in which both horizontal maps are isomorphisms. The top horizontal map is induced by multiplication, while the bottom horizontal map is induced by the projection $\mathfrak{h}_0 \to R$.

Proof. That the bottom map is an isomorphism comes from Lemma 4.2.1.

The Weyl groups of $\mathfrak{g}_0$ and L_0 can be expressed in terms of $\mathcal{M}$ and G^σ as $W(\mathfrak{g}_0) = N_{\mathcal{M}}(H^\sigma \cap \mathcal{M})/(H^\sigma \cap \mathcal{M})$ and $W(L_0) = N_{G^\sigma}(H^\sigma)/H^\sigma$. Using the Bruhat decomposition for G^σ and $\mathcal{M}$ simultaneously, as well as the Levi decomposition for $\mathcal{P}_0$, it is possible to show that $N_{G^\sigma}(H^\sigma) \cap \mathcal{P}_0 \subset N_{\mathcal{M}}(H^\sigma \cap \mathcal{M})$. The resulting inclusion $N_{G^\sigma}(H^\sigma) \cap \mathcal{P}_0/H^\sigma \subset N_{\mathcal{M}}(H^\sigma \cap \mathcal{M})/H^\sigma \cap \mathcal{M}$ is an isomorphism.

Now the commutative diagram in Equation (4.1) can be extended by adding the commutative square

$$\begin{array}{ccc} \mathcal{P}_0/H^\sigma \times_{W(\mathfrak{g}_0)} \mathfrak{h}_0^r & \longrightarrow & \mathfrak{p}_0 \cap L_0^{rs} \\ \downarrow & & \downarrow \\ G^\sigma/H^\sigma \times_{W(L_0)} \mathfrak{h}_0^r & \longrightarrow & L_0^{rs} \end{array} \tag{4.3}$$

in which the vertical maps are the natural inclusions. To show that the left vertical map is injective take two elements $([p], x)$ and $([p'], x')$ which are equal in the codomain. This means that there is $w \in N_{G^\sigma}(H^\sigma)$ with $[pw^{-1}] = [p']$ and $wx_0 = x_0'$. The former condition implies that $w \in \mathcal{P}_0 \cap N_G(H)$, so $[w] \in W(L_0)$ represents an element of $W(\mathfrak{g}_0)$, and $([p], x) = ([p'], x')$ in $\mathcal{P}_0/H_0 \times_{W(\mathfrak{g}_0)} \mathfrak{h}_0^r$.

Since the bottom map of Equation (4.3) is an isomorphism, we just need to show that $\mathcal{P}_0/H^\sigma \times_{W(\mathfrak{g}_0)} \mathfrak{h}_0^r$ maps onto $\mathfrak{p}_0 \cap L_0^{rs}$. Suppose $x \in \mathfrak{p}_0$ is semisimple in L_0. Since diagonalizability is preserved by restriction to an invariant subspace and by descent to a quotient by an invariant subspace, we can write $x = x_0 + x_1$, where x_0 is a semisimple element of $\mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$ and $x_1 \in \mathfrak{u}_0$. Conjugating x_0 by an element of the Levi factor $\mathcal{M}$ to be in $\mathfrak{h}_0$, we can assume that $x \in \mathfrak{b}_0$, a Borel subalgebra of L_0 contained in $\mathfrak{p}_0$. Thus the problem is reduced to showing that $\mathfrak{b}_0 \cap L_0^{rs} \subset \mathcal{B}_0 \mathfrak{h}_0^r$. Given x in the former set, take $g \in G^\sigma$ such that $gx = y \in \mathfrak{h}_0^r$. Then $\mathfrak{b}_0$ and $g\,\mathfrak{b}_0$ both contain $\mathfrak{h}_0$, so there is $w \in N_{G^\sigma}(H^\sigma)$ such that $w\,\mathfrak{b}_0 = g\,\mathfrak{b}_0$. Since Borel's are self-normalizing, $g^{-1}w \in B_0$ and $x = (g^{-1}w)(w^{-1}y) \in B_0 \mathfrak{h}_0^r$. □

We need two facts about diagram automorphisms and the structure of L. We use the convention from Section 4.1 to express the simple roots $\{\alpha_J\}$ of L_0 in terms of simple roots $\{\alpha\}$ of L.

Lemma 4.2.4. $\mathfrak{h}_0 \cap \mathfrak{h}^r = \mathfrak{h}_0^r$, *the set of elements in* $\mathfrak{h}_0$ *which are regular in* L_0. *Similarly,* $L_0 \cap L^{rs} = L_0^{rs}$.

Proof. The restriction map $\mathfrak{h}^* \to \mathfrak{h}_0^*$ sends roots of L to positive multiples of roots of L_0 by Proposition 9.18 of [Ca05]. All the roots of L_0 are covered by this map, so $\mathfrak{h}_0 \cap \mathfrak{h}^r = \mathfrak{h}_0^r$. An element $x \in L_0$ is semisimple in L_0 if and only if it is semisimple in L. If it is semisimple in L_0 then it can be conjugated to an element of $\mathfrak{h}_0$, so the statement for L_0 follows from the statement for $\mathfrak{h}_0$. □

Lemma 4.2.5. *There is a parabolic* $\mathfrak{p}'$ *of* L *preserved by* σ *such that* $\mathfrak{p}' \cap L_0 = \mathfrak{p}_0$. *If* $\mathfrak{m}$ *is the standard reductive factor of* $\mathfrak{p}'$ *then* $\mathfrak{m} \cap L_0 = \mathfrak{g}_0$, *the reductive factor of* $\mathfrak{p}_0$, *and* $W(\mathfrak{m})^\sigma = W(\mathfrak{g}_0)$, *where both are regarded as subgroups of* $W(L)$.

Proof. Let S be the subset of simple roots $\{\alpha_J\}$ determining $\mathfrak{p}_0$ and let S' be the subset of simple roots of L which appear in some σ-orbit J for $\alpha_J \in S$. Let $\mathfrak{p}'$ be the parabolic subalgebra determined by S'. Clearly $\mathfrak{p}'$ is σ-invariant. By Lemma 4.1.10, an element $w \in W^\sigma$ belongs to $W(\mathfrak{g}_0)$ if and only if it has a reduced expression consisting of reflections through simple roots in S', which is exactly the condition that w belongs to $W(\mathfrak{m})$. If α_J is a simple root of L_0, then the corresponding positive Chevalley generator e_J is a linear combination of the positive Chevalley generators corresponding to the simple roots of L in J, and similarly for the negative Chevalley generator f_J. Since $\mathfrak{p}_0$ is generated as a Lie algebra by $\mathfrak{h}_0$, all the e_J's, and the f_J's such that $\alpha_J \in S$, it follows that $\mathfrak{p}_0 \subset \mathfrak{p}' \cap L_0$. Since the f_J's with $\alpha_J \in S$ are the only negative generators in $\mathfrak{p}' \cap L_0$, and $\mathfrak{p}' \cap L_0$ is a parabolic subalgebra of L_0, it follows that $\mathfrak{p}' \cap L_0 = \mathfrak{p}_0$. Similarly $\mathfrak{m} \cap L_0 = \mathfrak{g}_0$. □

The real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$ is the real subspace where all roots take real values, or equivalently the real span of the coroots. If $x \in \mathfrak{h}$ let $\operatorname{Re} x$ be the projection of x to $\mathfrak{h}_{\mathbb{R}}$ under the (real-linear) splitting $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$. Note that $\operatorname{Re} \sigma x = \sigma \operatorname{Re} x$ and $\operatorname{Re} wx = w \operatorname{Re} x$ for all $w \in W$.

Proof of Theorem 4.2.2. First we show that the bottom map of Equation (4.2) is an isomorphism. Let $\mathfrak{p}'$ be the parabolic of L over $\mathfrak{p}_0$, as in Lemma 4.2.5. We start by proving that $J_m^{\tilde{\sigma}}\mathfrak{h}^r/W(\mathfrak{g}_0) \cong J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$, where $W(\mathfrak{m})$ is the Weyl group of the reductive factor of $\mathfrak{p}'$. Since $J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$ is smooth when $m < +\infty$ by Lemma 4.1.7, it is sufficient to prove that the map is bijective. By Corollary 4.1.4, $J_m\mathfrak{h}^r/W(\mathfrak{m}) \cong J_m(\mathfrak{h}^r/W(\mathfrak{m}))$, so every element of $J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$ is represented by an element of $f \in J_m\mathfrak{h}^r$ such that $w\tilde{\sigma}(f) = f$ for some $w \in W(\mathfrak{m})$. Let S' be the set of simple roots determining $\mathfrak{p}'$, let Δ' be the set of all roots of $\mathfrak{m}$, and let $D = \{x \in \mathfrak{h}_{\mathbb{R}} : \alpha(x) \neq 0, \alpha \in \Delta'\}$. The connected components of D are of the form $C \times \mathbb{R}^r$, where C is an open Weyl chamber of $\mathfrak{m}$ and $r = \dim \mathfrak{h} - |S'|$. Consequently $W(\mathfrak{m})$ acts transitively and freely on the connected components of D, so we can assume that $\operatorname{Re} f(0) \in D_0 = \{x \in \mathfrak{h}_{\mathbb{R}} : \alpha(x) > 0, \alpha \in S'\}$. But S' is σ-invariant, so D_0 is also σ-invariant, and thus $\operatorname{Re} \sigma f(0) = \sigma \operatorname{Re} f(0) \in D_0$. Since $\operatorname{Re} f(0) = w\sigma \operatorname{Re} f(0)$, this implies that $w = e$ and consequently $f \in J_m^{\tilde{\sigma}}\mathfrak{h}^r$. Thus the map $J_m^{\tilde{\sigma}}\mathfrak{h}^r \to J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$ is surjective. Suppose $f, g \in J_m^{\tilde{\sigma}}\mathfrak{h}^r$ are equal in $J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$. Then there is $w \in W(\mathfrak{m})$ such that $wf = g$. Since $f(0), g(0) \in \mathfrak{h}_0^r$, we have $\sigma(w)f(0) = g(0) = wf(0)$, and consequently $\sigma(w) = w$. Thus f and g are related by an element of $W(\mathfrak{m}) \cap W^\sigma = W(\mathfrak{g}_0)$.

As a special case of the above argument, we have $(\mathfrak{h}^r/W(\mathfrak{m}))^\sigma \cong \mathfrak{h}_0^r/W(\mathfrak{g}_0) = R^r$. Consequently $J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$ maps to R^r via evaluation at zero, and we conclude that the map $J_m^{\tilde{\sigma}}\mathfrak{h}^r/W(\mathfrak{g}_0) \to R^r \times_{Q^\sigma} J_m^{\tilde{\sigma}}Q^r$ factors through the isomorphism to $J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$. Since $\mathfrak{h}^r/W(\mathfrak{m}) \to \mathfrak{h}^r/W(L)$ is etale, the space $J_m^{\tilde{\sigma}}(\mathfrak{h}^r/W(\mathfrak{m}))$ is isomorphic to $R^r \times_{Q^\sigma} J_m^{\tilde{\sigma}}Q^r$ by Lemma 4.1.8.

We have shown that the bottom map of Equation (4.2) is an isomorphism, so we just need to do the same for the top map. Consider the case when $\mathfrak{p}_0 = L_0$, so that $\mathcal{P}_m = J_m^{\tilde{\sigma}}G$ and $W(\mathfrak{g}_0) = W(L_0)$. Combining Corollary 4.1.4 (note that H is reductive so that G/H is affine) and the isomorphism $(G/H) \times_W \mathfrak{h}^r \to L^{rs}$, we get an isomorphism $J_m(G/H) \times_W J_m\mathfrak{h}^r \to J_mL^{rs}$, where the former space is the quotient (resp. pro-quotient). The automorphism $\tilde{\sigma}$ on J_mL^{rs} translates to the diagonal action on $J_m(G/H) \times_{W(L)} J_m\mathfrak{h}^r$, and we can show that this isomorphism identifies $J_m^{\tilde{\sigma}}L^{rs}$ with $J_m^{\tilde{\sigma}}(G/H) \times_{W(L_0)} J_m^{\tilde{\sigma}}\mathfrak{h}^r$ by a similar argument to the proof of Lemma 4.2.3. Namely, $([f], g) \in J_m(G/H) \times J_m\mathfrak{h}^r$ represents an element of $J_m^{\tilde\sigma}L^{rs}$ if and only if there is $w \in W(L)$ such that $[\tilde\sigma(f)]w^{-1} = [f]$ and $w\tilde\sigma(g) = g$. Assuming that $\operatorname{Re} g(0)$ is in the open Weyl chamber we get that $w = e$ and thus $[f] \in J_m^{\tilde\sigma}(G/H)$, $g \in J_m^{\tilde\sigma}\mathfrak{h}^r$. Similarly, any two elements of $J_m^{\tilde\sigma}(G/H) \times J_m^{\tilde\sigma}\mathfrak{h}^r$ with the same image in J_mL^{rs} are $W(L_0)$-translates. Finally we can apply Proposition 4.1.14 to replace $J_m^{\tilde\sigma}(G/H)$ with $J_m^{\tilde\sigma}G/J_m^{\tilde\sigma}H$.

Now for the general case look at the square

$$\begin{array}{ccc} \mathcal{P}_m/J_m^{\tilde\sigma}H \times_{W(\mathfrak{g}_0)} J_m^{\tilde\sigma}\mathfrak{h}^r & \longrightarrow & \mathfrak{p}_m \cap J_m^{\tilde\sigma}L^{rs} \\ \downarrow & & \downarrow \\ J_m^{\tilde\sigma}G/J_m^{\tilde\sigma}H \times_{W(L_0)} J_m^{\tilde\sigma}\mathfrak{h}^r & \longrightarrow & J_m^{\tilde\sigma}L^{rs} \end{array}.$$

The group quotient (resp. pro-group quotient) $\mathcal{P}_m/J_m^{\tilde\sigma}H$ is a closed subscheme of $J_m^{\tilde\sigma}G/J_m^{\tilde\sigma}H$. As in Lemma 4.2.3, both vertical maps are inclusions and consequently the top horizontal map is injective. Every $x \in J_m^{\tilde\sigma}L^{rs}$ can be written as gy for $g \in J_m^{\tilde\sigma}G$ and $y \in J_m^{\tilde\sigma}\mathfrak{h}^r$. If $x \in \mathfrak{p}_m$ then $x(0) \in \mathfrak{p}_0$, after which Lemma 4.2.3 implies that there is $w \in W(L_0)$ such that $g(0)w^{-1} \in \mathcal{P}_0$. Consequently $gw^{-1} \in \mathcal{P}_m$ and (gw^{-1}, wy) maps to x, so the top map is surjective as required. □

4.2.2 Arcs in the regular locus

Let L^{reg} denote the open subset of regular elements in L, i.e. the set of elements x such that the stabilizer L^x has dimension equal to the rank l of L. Note that L^{reg} is σ-invariant. Kostant famously proved that the map $L^{reg} \to Q$ is surjective and smooth, and furthermore is a G-orbit map, in the sense that every fibre is a single G-orbit [Ko63b]. The proof uses the Kostant slice, an affine subspace $\nu \subset L^{reg}$ of the form $e + L^f$, where $\{h, e, f\}$ is a principal $\mathfrak{sl}_2$-triple. Kostant showed that ν intersects each regular G-orbit in a unique point, and that $\nu \hookrightarrow L^{reg} \to Q$ is an isomorphism. The following theorem extends this idea to jet and arc groups.

Theorem 4.2.6. *There is a Kostant slice ν of L which is σ-invariant and such that ν^σ is a Kostant slice for L_0. If ν is such a slice then $J_m^{\tilde\sigma}\nu \to J_m^{\tilde\sigma}Q^{\tilde\sigma}$ is an isomorphism for all $0 \le m \le +\infty$, and every $J_m^{\tilde\sigma}G$-orbit in $J_m^{\tilde\sigma}L^{reg}$ intersects $J_m^{\tilde\sigma}\nu$ in a unique point.*

At $m = 0$, Theorem 4.2.6 implies that $Q^\sigma = L_0//G^\sigma$.

For Kostant's smoothness result it is possible to incorporate a parabolic component.

Theorem 4.2.7. *The map $\mathfrak{p}_m \cap J_m^{\tilde\sigma}L^{reg} \to R \times_{Q^\sigma} J_m^{\tilde\sigma}Q$ is a surjective $\mathcal{P}_m$-orbit map for all $0 \le m \le +\infty$, and is smooth for $0 \le m < +\infty$.*

Finally, we have a technical corollary which we will need in the next section. Recall the definition of T_{const} from the previous section, and define $T_{>0}\mathfrak{p}_m$ to be the subbundle of $T\mathfrak{p}_m$

of the form $\mathfrak{p}_m \times \mathfrak{u}_m$ where $\mathfrak{u}_m$ is the nilpotent subalgebra of $\mathfrak{p}_m$, i.e. the subset of elements $f \in \mathfrak{p}_m$ with $f(0) \in \mathfrak{u}_0$, the nilpotent radical of $\mathfrak{p}_0$.

Corollary 4.2.8. *Let $0 \leq m \leq +\infty$. The differential of the map $\mathfrak{p}_m \to R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$ induces a bundle map $T_{>0}\mathfrak{p}_m \to R \times_{Q^\sigma} T_{const} J_m^{\tilde{\sigma}}$. Over $\mathfrak{p}_m \cap J_m^{\tilde{\sigma}} L^{reg}$ the bundle map is surjective on fibres.*

To prove Theorem 4.2.6, we start by proving some simple facts about regular elements in L_0, using Kostant's characterisation of regular elements (Proposition 0.4 of [Ko63b]) in L: if $x = y + z$ is the Jordan decomposition of x, so that y is semisimple, z is nilpotent, and $[y, z] = 0$, then x is regular if and only if z is a principal nilpotent in the reductive subalgebra L^y. Note that, by definition, a nilpotent element of a reductive algebra L is required to be in $[L, L]$, and if z is a nilpotent in L commuting with a semisimple element y, then z is also a nilpotent in L^y.

Lemma 4.2.9. *$L^{reg} \cap L_0 = L_0^{reg}$, the set of regular elements in L_0.*

Proof. Suppose x in L_0 has Jordan decomposition $x = y + z$ in L. Then $x = y + z$ is also the Jordan decomposition in L_0, and in particular y and z are in L_0. Now by conjugating by an element of G^σ we can assume that $y \in \mathfrak{h}_0$, and in fact that y is in the closed Weyl chamber corresponding to the Borel $L_0 \cap \mathfrak{b}$, where $\mathfrak{b}$ is the Borel in L compatible with σ. Since the simple roots of L project to positive multiples of the simple roots of L_0, y is also in the closed Weyl chamber of L corresponding to $\mathfrak{b}$. Let S be the set of simple roots α_J for L_0 that are zero on y, and similarly let S' be the set of simple roots for L that are zero on y. Since y is in the closed Weyl chamber, the stabilizer L_0^y (respectively L^y) is the reductive Lie algebra $\mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \mathbb{Z}[S]} (L_0)_\alpha$ (respectively $\mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbb{Z}[S']} L_\alpha$). Now x is regular in L_0 (respectively L) if and only if z is a principal nilpotent in L_0^y (respectively L^y). Every nilpotent element of L_0^y is contained in a Borel, and all Borels are conjugate, so we can conjugate z by an element of $(G^\sigma)^y$ to get z contained in the Borel $L_0^y \cap \mathfrak{b}$ (since it does not have a component in the centre, z will in fact be in the nilpotent radical of $L_0^y \cap \mathfrak{b}$). By Theorem 5.3 of [Ko59], z is a principal nilpotent in L_0^y if and only if the component of z in $(L_0)_\alpha$ is non-zero for all $\alpha \in S$. But by the construction of the simple Chevalley generators of L_0, this is equivalent to the component of z in L_α being non-zero for all $\alpha \in S'$. So z is a principal nilpotent in L_0^y if and only if z is a principal nilpotent in L^y, and hence x is regular in L_0 if and only if x is regular in L. □

We also need the following standard technical lemma.

Lemma 4.2.10. *Let $\mathfrak{q}$ be a $\mathbb{Z}_{\geq 0}$-graded Lie algebra, and let $\mathfrak{n}$ denote the ideal $\bigoplus_{k>0} \mathfrak{q}_k$. Suppose y is an element of $\mathfrak{q}_0$, and that $\mathfrak{r} \subset \mathfrak{n}$ is a graded subspace such that $\mathfrak{n} = [\mathfrak{n}, y] \oplus \mathfrak{r}$. Then for every x in the completion $\hat{\mathfrak{n}}$ there is g in the pro-nilpotent group $\exp(\hat{\mathfrak{n}})$ such that $g(y + x) \in y + \hat{\mathfrak{r}}$.*

Proof. Let $\{x_i\}$ be the sequence in $\hat{\mathfrak{n}}$ with $x_0 = x$ and $x_{i+1} = \exp(-z_i)(y + x_i) - y$, where $z_i \in \hat{\mathfrak{n}}$ is chosen so that $x_i = [z_i, y] + r_i$ for $r_i \in \hat{\mathfrak{r}}$. Since $\exp(-z_i)(y + x_i) = y + r_i - [z_i, x_i]$, we can show by induction that z_i and the component of x_i in $[\mathfrak{n}, y]$ are both zero below degree $i + 1$. Hence the element $g = \cdots \exp(-z_2)\exp(-z_1)\exp(-z_0)$ is a well-defined element of $\exp(\hat{\mathfrak{n}})$, and $g(y + x)$ is contained in $y + \hat{\mathfrak{r}}$ as desired. □

Proof of Theorem 4.2.6. If $m < +\infty$ then there is a homomorphism of Lie groups $J^{\tilde{\sigma}}_m G \to J^{\tilde{\sigma}}_{m-1} G$, so the induced map $J^{\tilde{\sigma}}_m L \to J^{\tilde{\sigma}}_{m-1} L$ on Lie algebras preserves semisimple (resp. nilpotent) elements. We say that an element of $J^{\tilde{\sigma}}_\infty L$ is pro-semisimple (resp. pro-nilpotent) if the image of the element is semisimple (resp. pro-nilpotent) in $J^{\tilde{\sigma}}_m L$ for every $m < +\infty$. Just as in the finite-dimensional case, every element of $J^{\tilde{\sigma}}_\infty L$ can be written uniquely as $y + z$ where y is pro-semisimple, z is pro-nilpotent, and $[y, z] = 0$.

If $y \in J^{\tilde{\sigma}}_m L$ is semisimple (resp. pro-semisimple) then $y(0)$ is semisimple in L, and hence $L = L^{y(0)} \oplus [L, y(0)]$. It follows from Lemma 4.2.10 that there is $g \in J^{\tilde{\sigma}}_m G$ such that $gy = y(0) + z$, where $z \in J^{\tilde{\sigma}}_m L^{y(0)}$ and $z(0) = 0$. Since z is nilpotent (resp. pro-nilpotent), uniqueness of the Jordan decomposition implies that $z = 0$.

More generally, if x is an arbitrary element of $J^{\tilde{\sigma}}_m L$ then there is $g \in J^{\tilde{\sigma}}_m$ such that $gx = y + z$, where $y \in L_0$ is semisimple and $z \in J^{\tilde{\sigma}}_m L^y$ is nilpotent (resp. pro-nilpotent). In particular $e = z(0)$ is nilpotent in L^y_0, so pick an $\mathfrak{sl}_2$-triple $\{h, e, f\}$ in L^y_0 containing e. Then $L^y = L^{\{y,f\}} \oplus [L^y, e]$, so applying Lemma 4.2.10 again there is $g' \in J^{\tilde{\sigma}}_m G^y$ such that $g'z \in e + J^{\tilde{\sigma}}_m L^{\{y,f\}} = J^{\tilde{\sigma}}_m (e + L^{\{y,f\}})$ and $g'(0)z(0) = e$.

Using this canonical form, we move on to the proof of the theorem statement. Pick a principal $\mathfrak{sl}_2$-triple $\{h, e, f\}$ in L_0. By Lemma 4.2.9 $\{h, e, f\}$ is also principal in L, so $\nu = e + L^f$ is a Kostant slice in L invariant under σ, and $\nu^\sigma = e + L^f_0$ is a Kostant slice in L_0. It follows immediately that $J^{\tilde{\sigma}}_m \nu \to J^{\tilde{\sigma}}_m Q$ is an isomorphism, and also that $Q^\sigma = L_0//G^\sigma$. Since $J^{\tilde{\sigma}}_m L^{reg} \to //J^{\tilde{\sigma}}_m Q$ is $J^{\tilde{\sigma}}_m G$-invariant, each orbit in $J^{\tilde{\sigma}}_m L^{reg}$ can intersect $J^{\tilde{\sigma}}_m \nu$ at most once. So we just need to show that the multiplication map $J^{\tilde{\sigma}}_m G \times J^{\tilde{\sigma}}_m \nu \to J^{\tilde{\sigma}}_m L^{reg}$ is surjective, or equivalently that every fibre of the map $J^{\tilde{\sigma}}_m L^{reg} \to J^{\tilde{\sigma}}_m Q$ is a $J^{\tilde{\sigma}}_m G$-orbit.

The projection $L^{reg} \to Q$ is smooth and every fibre is a G-orbit, so the multiplication map $G \times \nu \to L^{reg}$ is surjective and smooth. Hence by Lemma 4.1.2 the multiplication map $J_m G \times J_m \nu \to J_m L^{reg}$ is surjective. Suppose x_1 and x_2 are two points of $J^{\tilde{\sigma}}_m L^{reg}$ with the same value in $J^{\tilde{\sigma}}_m Q$. Using the $m = 0$ case and the canonical form above, we can assume that $x_1(0) = x_2(0) = y + e'$, where y is semisimple in L_0 and e' is a principal nilpotent in L^y_0, and that x_1 and x_2 are in $y + J^{\tilde{\sigma}}_m \nu'$, where ν' is the Kostant slice $e' + L^{\{y,f'\}}$ in L^y. Since x_1 and x_2 have the same image in $J_m Q$, there is $g \in J_m G$ such that $gx_1 = x_2$. Multiplication by g preserves Jordan decomposition, so $g \in (J_m G)^y$. The subgroup G^y is a connected reductive subgroup of G by Lemma 5, page 353 of [Ko63b], and the exponential map is a bijection for nilpotent (resp. pro-nilpotent) groups, so $(J_m G)^y = G^y \cdot \exp(z J_m L^y) = J_m G^y$, the connected subgroup of $J_m L$ with Lie algebra $J_m L^y$. Hence $x_1 - y$ and $x_2 - y$ are in the same regular $J_m G^y$-orbit of $J_m (L^y)^{reg}$. But $x_1 - y$ and $x_2 - y$ belong to $J^{\tilde{\sigma}}_m \nu' \subset J_m \nu'$, which we have already observed intersects each $J_m G^y$-orbit exactly once, implying that $x_1 = x_2$ as desired. □

Theorem 4.2.6 implies that the map $J_m^{\tilde{\sigma}} L^{reg} \to J_m^{\tilde{\sigma}} Q$ is surjective for $0 \leq m \leq +\infty$, and smooth for $0 \leq m < +\infty$. To prove Theorem 4.2.7, we need to account for the parabolic component. Recall that $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$.

Lemma 4.2.11. *The projection $\mathfrak{p}_0 \cap L_0^{reg} \to R$ is a surjective smooth $\mathcal{P}_0$-orbit map. In addition, if $g \in G^\sigma$ fixes an element of $\mathfrak{p}_0 \cap L_0^{reg}$ then g belongs to $\mathcal{P}_0$.*

Proof. Let $\mathfrak{b}_0$ be a Borel of L_0 contained in $\mathfrak{p}_0$ and compatible with $\mathfrak{h}_0$. Let $\mathfrak{u}_0$ be the unipotent radical of $\mathfrak{p}_0$, so that $\mathfrak{g}_0 = \mathfrak{p}_0 / \mathfrak{u}_0$. Finally let Δ be the set of roots of L_0, and let $S \subset \Delta$ be the set of simple roots. Similarly let $S_0 \subset S$ be the set of simple roots of $\mathfrak{g}_0$ corresponding to the Borel $\mathfrak{b}_0 \cap \mathfrak{g}_0$, and let $\Delta_0 = \Delta \cap \mathbb{Z}[S_0]$ be the set of roots of $\mathfrak{g}_0$.

Now suppose $y \in \mathfrak{p}_0$ is semisimple in L_0, and let $y = y_0 + y_1$ where $y_0 \in \mathfrak{g}_0$ and $y_1 \in \mathfrak{u}_0$. Then $\mathfrak{u}_0 = [\mathfrak{u}_0, y_0] \oplus \mathfrak{u}_0^{y_0}$, so by Lemma 4.2.10 there is $p \in \mathcal{P}_0$ such that $py = y_0 + z$, where $z \in \mathfrak{u}_0^{y_0}$. Since py is semisimple, we conclude that $z = 0$, and ultimately that y is conjugate by $\mathcal{P}_0$ to an element of $\mathfrak{h}_0$.

Every element $x \in \mathfrak{p}_0$ can be written as $x = y + z$ where $y, z \in \mathfrak{p}_0$, y is semisimple in L_0, z is nilpotent in L_0, and $[y, z] = 0$. By the previous paragraph, it is possible to conjugate x by an element of $\mathcal{P}_0$ so that $y \in \mathfrak{h}_0$. We can then conjugate x by an element of $\mathcal{P}_0^y$ so that z belongs to $\mathfrak{b}_0^y$. Assume x is given with $y \in \mathfrak{h}_0$ and $z \in \mathfrak{b}_0^y$. By a dimension argument, $\mathfrak{b}_0^y$ is a Borel for the reductive Lie algebra L_0^y. The corresponding simple roots are the indecomposable elements S_y of $\Delta_y^+ = \{\alpha \in \Delta^+ : \alpha(y) = 0\}$. Similarly $\mathfrak{b}_0^y \cap \mathfrak{g}_0^y$ is a Borel for $\mathfrak{g}_0^y$, and the simple roots are the elements of $S_y \cap \Delta_0$. The element x is regular in L_0 if and only if z is a principal nilpotent in L_0^y, which is true if and only if the projection to $(L_0)_\alpha$ is non-zero for all $\alpha \in S_y$. If this latter condition holds then the image of x in $\mathfrak{g}_0 = \mathfrak{p}_0 / \mathfrak{u}_0$ is regular in $\mathfrak{g}_0$. The projection $\mathfrak{p}_0 \to \mathfrak{g}_0$ is $\mathcal{P}_0$-equivariant, so we conclude that the projection sends regular elements of L_0 to regular elements of $\mathfrak{g}_0$.

Conversely, if $x \in \mathfrak{g}_0^{reg}$ then we can conjugate x by an element of the subgroup of $\mathfrak{g}_0$ to be of the form $y + z$ where $y \in \mathfrak{h}_0$ and $z \in \mathfrak{b}_0^y$ is a principal nilpotent. This means that the projection of z to $(L_0)_\alpha$ is non-zero for every $\alpha \in S_y \cap \Delta_0$. Let z' be an element of L_0 such that the projection of z' to $(L_0)_\alpha$ is non-zero if $\alpha \in S_y \setminus \Delta_0$, and is zero otherwise. Then $x + z'$ is a regular element of L_0 which projects x. Using equivariance again, we conclude that the projection $\mathfrak{p}_0 \to \mathfrak{g}_0$ induces a surjection $\mathfrak{p}_0 \cap L_0^{reg} \to \mathfrak{g}_0^{reg}$. The map $\mathfrak{g}_0^{reg} \to R$ is a smooth surjection, and $\mathfrak{p}_0 \to \mathfrak{g}_0$ is smooth, so we conclude that $\mathfrak{p}_0 \cap L_0^{reg} \to R$ is a smooth surjection.

Now suppose x_1 and x_2 in $\mathfrak{p}_0 \cap L_0^{reg}$ map to the same element of R. As in the third paragraph, we can assume without loss of generality that $x_i = y_i + z_i$ with $y_i \in \mathfrak{h}_0$ and z_i a principal nilpotent element of $L_0^{y_i}$ contained in $\mathfrak{b}_0^{y_i}$. In addition, the images of x_1 and x_2 in $\mathfrak{g}_0$ are conjugate by an element of $\mathcal{P}_0$, so in particular we can assume that $y_1 = y_2$. Thus z_1 and z_2 are both principal nilpotents of $L_0^{y_1}$ contained in $\mathfrak{b}_0^{y_1}$, and hence are conjugate by an element of the Borel subgroup of $\mathfrak{b}_0^{y_1}$. We conclude that the projection $\mathfrak{p}_0 \cap L_0^{reg} \to R$ is a $\mathcal{P}_0$-orbit map.

For the last part of the lemma, we again assume that $x \in \mathfrak{p}_0 \cap L_0^{reg}$ is of the form $y+z$ with $y \in \mathfrak{h}_0$ and $z \in \mathfrak{b}_0^y$. If x is regular then $L_0^x = (L_0^y)^z$ is contained in $\mathfrak{b}_0 \subset \mathfrak{p}_0$. By Proposition 14, page 362 of [Ko63b], $(G^\sigma)^x$ is connected, and hence a subgroup of $\mathcal{P}_0$. □

Proof of Theorem 4.2.7. Suppose $x_1, x_2 \in \mathfrak{p}_m \cap J_m^{\tilde{\sigma}} L^{reg}$ have the same image in $R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$. By Theorem 4.2.6 there is $g \in J_m^{\tilde{\sigma}} G$ such that $gx_1 = x_2$, while by Lemma 4.2.11 there is $p_0 \in \mathcal{P}_0$ such that $p_0 x_1(0) = x_2(0)$. Thus $p_0^{-1} g(0)$ fixes $x_1(0) \in \mathfrak{p}_0 \cap L_0^{reg}$, so $g \in \mathcal{P}_m$ by Lemma 4.2.11, and $\mathfrak{p}_m \cap J_m^{\tilde{\sigma}} L^{reg} \to R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$ is a $\mathcal{P}_m$-orbit map.

To show surjectivity, observe that $\mathfrak{p}_m \cap J_m^{\tilde{\sigma}} L^{reg} = (\mathfrak{p}_0 \cap L_0^{reg}) \times_{L_0} J_m^{\tilde{\sigma}} L^{reg}$. A point of $R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$ is determined by a pair of points $x \in R$ and $y \in J_m^{\tilde{\sigma}} Q$ which have the same image in Q^σ. Given a point specified in this manner, choose $x' \in \mathfrak{p}_0 \cap L_0^{reg}$ mapping to x and $y' \in J_m^{\tilde{\sigma}} L^{reg}$ mapping to y. Since x and y have the same image in Q^σ, there is $g \in G^\sigma$ such that $gy(0) = x$. Then gy belongs to $\mathfrak{p}_m$ and maps to the point $(x, y) \in R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$.

Since $\mathfrak{p}_m \cap J_m^{\tilde{\sigma}} L^{reg} \to R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$ is a $\mathcal{P}_m$-orbit map, to show smoothness it is enough to show that the map $T\mathfrak{p}_m \cap J_m^{\tilde{\sigma}} L^{reg} \to T(R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q)$ is surjective. This follows from a similar argument to the last paragraph. As mentioned in the proof of Theorem 4.2.6, if ν_0 is a Kostant slice in L_0 then $G^\sigma \times \nu_0 \to L_0^{reg}$ is smooth and surjective, so $TG^\sigma \times T\nu_0 \to TL_0^{reg}$ is also surjective, and hence if two elements of TL_0^{reg} have the same image in TQ^σ then they are conjugate by an element of TG^σ. Surjectivity of $\mathfrak{p}_0 \cap L_0^{reg} \to R$ and $J_m^{\tilde{\sigma}} L^{reg} \to J_m^{\tilde{\sigma}} Q$ follows from Lemma 4.2.11 and Theorem 4.2.6. □

Proof of Corollary 4.2.8. $R \times_{Q^\sigma} T_{const} J_m^{\tilde{\sigma}} Q$ is isomorphic to the pullback $R \times_{TQ^\sigma} T J_m^{\tilde{\sigma}} Q$, where the map $R \to TQ^\sigma$ is the composition of the zero section $R \to TR$ with the differential $TR \to TQ^\sigma$. The restriction of the differential $T\mathfrak{p}_m \to TR$ to $T_{>0}\mathfrak{p}_m$ factors through the zero section $R \to TR$, so the image of $T_{>0}\mathfrak{p}_m$ is contained in $R \times_{TQ^\sigma} T J_m^{\tilde{\sigma}} Q$. To show that this bundle map is surjective on fibres, observe that, in the argument for smoothness in the proof of Theorem 4.2.7, if $x \in TR$ is a zero tangent vector, then we can pick $x' \in T\mathfrak{p}_0 \cap TL^{reg}$ mapping to x which is also a zero tangent vector, and hence the resulting point of $T\mathfrak{p}_m$ will be contained in $T_{>0}\mathfrak{p}_m$. □

Chapter 5

Strong Macdonald theorems

In this chapter we state and prove the strong Macdonald theorems for a parahoric. We assume that $\mathfrak{p} = \{f \in L[z] : f(0) \in \mathfrak{p}_0\}$ is a standard parahoric in a twisted loop algebra $L[z^{\pm 1}]^{\tilde{\sigma}}$, where $\mathfrak{p}_0$ is a parabolic in a reductive Lie algebra L.

This chapter is adapted from part of [Sl11b].

5.1 Statement of theorems

5.1.1 Exponents and diagram automorphisms

The exponents of L are integers $m_1, \ldots, m_l$ such that $H^*(L)$ is the free super-commutative algebra generated in degrees $2m_1+1, \ldots, 2m_l+1$, where l is the rank of L. Equivalently, we can define the exponents by saying that $(S^*L^*)^L$ is the free commutative algebra generated in degrees $m_1+1, \ldots, m_l+1$. Extend the action of σ to S^*L^* by $\sigma(f)(z) = f(\sigma^{-1}z)$. This convention is chosen so that $\sigma(\mathrm{ad}^t(x)f) = \mathrm{ad}^t(\sigma(x))\sigma(f)$ for all $f \in S^*L^*$ and $x \in L$. Let $\mathfrak{M}$ be the ideal in $(S^*L^*)^L$ generated by elements of degree greater than zero. The diagram automorphism σ acts diagonalizably on the space $\mathfrak{M}/\mathfrak{M}^2$ of generators for $(S^*L^*)^L$, and consequently it is possible to find homogeneous generators of $\mathbb{C}[Q]$ which are eigenvectors of σ.

Definition 5.1.1. *Choose a set of homogeneous generators for $(S^*L^*)^L$ which are eigenvectors of σ. The exponents of L can be sorted into different sets $m_1^{(a)}, \ldots, m_{l_a}^{(a)}$, $a \in \mathbb{Z}_k$, by letting $m_1^{(a)}+1, \ldots, m_{l_a}^{(a)}+1$ be the list of degrees of homogeneous generators of $(S^*L^*)^L$ with eigenvalue q^{-a} (note the negative exponent). We call the elements of these sets the exponents of L_a.*

Recall that if V is an L_0-module and $\{h, e, f\}$ is a principal $\mathfrak{sl}_2$-triple in L_0, then the generalized exponents of V are the eigenvalues of $h/2$ on the subspace $V^{L_0^e}$ fixed by the abelian subalgebra L_0^e. The generalized exponents are always non-negative integers, and the dimension of $V^{L_0^e}$ is equal to the dimension of the zero weight space of V.

Proposition 5.1.2. *The exponents of L_a are the generalized exponents of L_a as an L_0-module.*

The proof of Proposition 5.1.2 will be given in Subsection 5.2.1. The generalized exponents of L_0 are the same as the ordinary exponents, and l_0 is the rank of L_0, so there is no conflict in our terminology. In general l_a is the dimension of $\mathfrak{h} \cap L_a$, where $\mathfrak{h}$ is a Cartan compatible with σ. If L is simple, then k is either 1 or 2, except when $L = \mathfrak{so}(8)$ in which case k can be 3 and L_1 is isomorphic to L_2. As a result, the exponents of L_a are the same as the exponents of L_{-a}. A principal $\mathfrak{sl}_2$-triple in L_0 is also principal in L (see Lemma 4.2.9), so L^e is abelian and hence $L_a^{L_0^e} = L_a^e$, simplifying the definition of generalized exponents in this case. The eigenvalues of $h/2$ give a principal grading $L_a = \bigoplus L_a^{(i)}$ of each L_a such that $L = \bigoplus_i \bigoplus_a L_a^{(i)}$ is a principal grading for L. The representation theory of $\mathfrak{sl}_2$ then implies:

Corollary 5.1.3. *The multiplicity of m in the list of exponents of L_a is $\dim L_a^{(m)} - \dim L_a^{(m+1)}$, where $L_a = \bigoplus L_a^{(i)}$ is a principal grading.*

The exponents of L_a can be easily determined when L is simple, and are given in the following table:

Type of L	k	Type of L_0	Exponents of L_0	Exponents of $L_{\pm 1}$
A_{2n}	2	B_n	$1, 3, \ldots, 2n-1$	$2, 4, \ldots, 2n$
A_{2n-1}	2	C_n	$1, 3, \ldots, 2n-1$	$2, 4, \ldots, 2n-2$
D_n	2	B_{n-1}	$1, 3, \ldots, 2n-3$	$n-1$
E_6	2	F_4	$1, 5, 7, 11$	$4, 8$
D_4	3	G_2	$1, 5$	3

5.1.2 Cohomology of superpolynomials in a standard parahoric

Let $\mathfrak{p} = \{f \in \mathfrak{g} : f(0) \in \mathfrak{p}_0\}$ be a standard parahoric in a twisted loop algebra $\mathfrak{g}$, and let $\hat{\mathfrak{p}}[s]$ denote the superpolynomial algebra in one odd variable with values in $\hat{\mathfrak{p}}$. The cohomology of the super Lie algebra $\hat{\mathfrak{p}}[s]$ can be calculated as in the ordinary case using the Koszul complex, so any grading on $\hat{\mathfrak{p}}[s]$ induces a grading on $H^*_{cts}(\hat{\mathfrak{p}}[s])$. In particular $H^*_{cts}(\hat{\mathfrak{p}}[s])$ is graded by z-degree and by s-degree.

Theorem 5.1.4. *Let $m_1^{(a)}, \ldots, m_{l_a}^{(a)}$ denote the exponents of L_a, and let $r_1, \ldots, r_{l_0}$ denote the exponents of the reductive algebra $\mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$, where $\mathfrak{p}_0$ is a parabolic in L_0. If $\mathfrak{p}$ is the standard parahoric $\{f \in L[[z]]^{\tilde{\sigma}} : f(0) \in \mathfrak{p}_0\}$ then the cohomology ring $H^*_{cts}(\hat{\mathfrak{p}}[s])$ is a free super-commutative algebra generated in degrees given in the following table:*

Cohomological degree	*s-degree*	*z-degree*	*Index set*
$2r_i + 1$	0	0	$i = 1, \ldots, l_0$
$r_i + 1$	$r_i + 1$	0	$i = 1, \ldots, l_0$
$m_i^{(-a)} + 1$	$m_i^{(-a)} + 1$	$kn - a$	$n \geq 1,\ a = 0, \ldots, k-1,\ i = 1, \ldots, l_{-a}$
$m_i^{(-a)} + 1$	$m_i^{(-a)}$	$kn - a$	$n \geq 1,\ a = 0, \ldots, k-1,\ i = 1, \ldots, l_{-a}$

To prove Theorem 5.1.4, we give an explicit description of a generating set of cocycles for the relative cohomology ring $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*)$, where $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$. Choose a set of generators I_i^a, $a \in \mathbb{Z}_k$, $i = 1, \ldots, l_a$ for $(SL^*)^*$ such that I_i^a is an eigenvector of σ with eigenvalue q^{-a}. Also choose a set of homogeneous generators $R_1, \ldots, R_{l_0}$ for $(S^*\mathfrak{g}_0^*)^{\mathfrak{g}_0}$. The polynomial functions I_k^a on L induce functions $\tilde{I}_k^a : L[[z]] \to \mathbb{C}[[z]]$, and the coefficients $[z^n]\tilde{I}_k^a$ of z^n in $\tilde{I}_k^a$ restrict to $\hat{\mathfrak{p}}$-invariant polynomial functions on $\hat{\mathfrak{p}}$. Similarly, the polynomials R_i on $\mathfrak{p}_0$ can be pulled back via the quotient map $\mathfrak{p} \to \mathfrak{g}_0$ to $\hat{\mathfrak{p}}$-invariant polynomials on $\hat{\mathfrak{p}}$. Finally, 1-cocycles can be constructed as follows. If J is a derivation of $\hat{\mathfrak{p}}$ that kills $\mathfrak{g}_0$ and $\phi \in S^k\hat{\mathfrak{p}}^*$ is $\hat{\mathfrak{p}}$-invariant then the tensor

$$\hat{\mathfrak{u}} \otimes S^{k-1}\hat{\mathfrak{p}} \to \mathbb{C} : x \otimes s_1 \circ \ldots \circ s_{k-1} \mapsto \phi(Jx \circ s_1 \circ \cdots \circ s_{k-1}). \tag{5.1}$$

is a cocycle (see Lemma 3.1.1).

Theorem 5.1.5. *Let $\mathfrak{p}$ be a standard parahoric in $\mathfrak{g}$, and let J be the derivation from Theorem 3.1.3. Then there is a metric on the Koszul complex such that the harmonic cocycles for $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*)$ form a free supercommutative ring generated by the cocycles in the following table:*

Cocycle description	*Coh. deg.*	*Sym. deg*	*z-deg.*	*Index set*
R_i	0	$\deg R_i$	0	$i = 1, \ldots, l_0$
$[z^{kn-a}]\tilde{I}_i^{-a}$	0	$\deg I_i^{-a}$	$kn - a$	$n \geq 1,\ i = 1, \ldots, l_{-a},$ $a = 0, \ldots, k-1$
$x \otimes s \mapsto [z^{kn-a}]\tilde{I}_j^{-a}(Jx \circ s)$	1	$\deg I_j^{-a} - 1$	$kn - a$	$n \geq 1,\ i = 1, \ldots, l_{-a},$ $a = 0, \ldots, k-1$

Proving Theorem 5.1.5 is the main concern of the paper; the proof is finished in Subsection 5.2.2.

Proof of Theorem 5.1.4 from Theorem 5.1.5. Since the bracket of $\hat{\mathfrak{p}}[s]$ is zero on the odd component, the Koszul complex for $\hat{\mathfrak{p}}[s]$ reduces to the Koszul complex for $\hat{\mathfrak{p}}$ with coefficients in $S^*\hat{\mathfrak{p}}^*$. Thus there is a ring isomorphism $H^*_{cts}(\hat{\mathfrak{p}}[s]) \cong H^*_{cts}(\hat{\mathfrak{p}}; S^*\hat{\mathfrak{p}}^*)$ in which $H^{n-q}_{cts}(\hat{\mathfrak{p}}; S^q\hat{\mathfrak{p}}^*)$ corresponds to the cohomology classes in $H^n_{cts}(\hat{\mathfrak{p}}[s])$ of s-degree q. This isomorphism preserves z-degree. The degree zero component of $\mathfrak{p}$ is $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$, a reductive algebra which is the quotient of $\mathfrak{p}$ by the standard nilpotent subalgebra $\mathfrak{u}$. It follows from the Hochschild-Serre spectral sequence (in particular Theorem 12 of [HS53]) that there is a ring isomorphism

$$H^*_{cts}(\hat{\mathfrak{p}}; S^*\hat{\mathfrak{p}}^*) \cong H^*(\mathfrak{g}_0) \otimes H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{p}}^*).$$

Then Theorem 5.1.4 follows from the description of relative cohomology. □

When $\mathfrak{p}_0 = L_0$, Theorem 5.1.4 states that the algebra $H^*_{cts}(L[z,s])^{\tilde{\sigma}}$ is the free supercommutative algebra with generators in tensor degree $2m_i^{(a)}+1$ and $2m_i^{(a)}+2$, and z-degree $nk+a$, for $a = 0,\ldots,k-1$, $i = 1,\ldots,l_a$, and $n \geq 0$. In addition the Hochschild-Serre spectral sequence implies that $H^*(\mathfrak{p}_0[s]) \cong H^*(\mathfrak{g}_0) \otimes (S^*\mathfrak{g}_0^*)^{\mathfrak{g}_0}$, where $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$, so $H^*(\mathfrak{p}_0[s])$ is isomorphic to the subalgebra of $H^*_{cts}(\hat{\mathfrak{p}}[s])$ of z-degree zero. In fact, the inclusion is the pullback map given by evaluation at zero, as can be seen from the explicit description of harmonic cocycles, so $H^*_{cts}(\hat{\mathfrak{p}}[s])$ is naturally isomorphic to $H^*(\mathfrak{p}_0[s]) \otimes_{H^*(L_0[s])} H^*_{cts}(L[z,s]^{\tilde{\sigma}})$.

We can also ask for an explicit description of the relative cohomology groups $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^*\hat{\mathfrak{u}}^*)$. In this case, we can only provide an answer when $\mathfrak{p}$ is an Iwahori subalgebra—that is, a standard parahoric $\{f \in L[[z]]^{\tilde{\sigma}} : f(0) \in \mathfrak{p}_0\}$ where $\mathfrak{p}_0$ is a Borel subalgebra.

Theorem 5.1.6. *Let $\mathfrak{b}$ be an Iwahori subalgebra of $\mathfrak{g}$, and let $\mathfrak{n}$ be the nilpotent subalgebra. Let J be the derivation from Theorem 3.1.3. Then there is a metric on the Koszul complex such that the harmonic cocycles for $H^*_{cts}\left(\hat{\mathfrak{b}}, \mathfrak{h}_0; S^*\hat{\mathfrak{n}}^*\right)$ form a free supercommutative ring generated by the cocycles in the following table:*

Cocycle description	*Coh. deg.*	*Sym. deg*	*z-deg.*	*Index set*
$[z^{kn-a}]\tilde{I}_i^{-a}$	0	$\deg I_i^{-a}$	$kn-a$	$n \geq 1,\ i = 1,\ldots,l_{-a}$, $a = 0,\ldots,k-1$
$x \otimes s \mapsto [z^{kn-a}]\tilde{I}_j^{-a}(Jx \circ s)$	1	$\deg I_j^{-a} - 1$	$kn-a$	$n \geq 1,\ i = 1,\ldots,l_{-a}$, $a = 0,\ldots,k-1$

Theorem 5.1.6 can be used to calculate $H^*_{cts}\left(\hat{\mathfrak{b}}; S^*\hat{\mathfrak{n}}\right)$ as in the proof of Theorem 5.1.4. With an appropriate degree shift, the cohomology ring $H^*_{cts}\left(\hat{\mathfrak{b}}, \mathfrak{h}; S^*\hat{\mathfrak{n}}\right)$ can also be regarded as the $\mathfrak{h}$-invariant part of $H^*_{cts}(\hat{\mathfrak{n}}[s])$. The proof of Theorem 5.1.6 will be completed in Subsection 5.2.3.

5.1.3 Cohomology of the truncated algebra

If N is a multiple of k then $z^N L[z]^{\tilde{\sigma}}$ is a subset of $L[z]^{\tilde{\sigma}}$, and hence $z^N\mathfrak{p}$ is an ideal of $\mathfrak{p}$. Theorem 5.1.5 can be used to determine the cohomology of the finite-dimensional Lie algebra $\mathfrak{p}/z^N\mathfrak{p}$.

Recall that the coinvariant algebra of the Weyl group $W(L_0)$ is the quotient of $S^*\mathfrak{h}_0^*$ by the ideal generated by $(S^{>0}\mathfrak{h}_0^*)^{W(L_0)}$. We define $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ to be the graded algebra which is the quotient of $(S^*\mathfrak{g}_0^*)^{\mathfrak{g}_0}$ by the ideal generated by $(S^{>0}L_0^*)^{L_0}$, where $S^*L_0^*$ acts on $S^*\mathfrak{g}_0^*$ by restriction. By the Chevalley restriction theorem, $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ is isomorphic to the subalgebra of $W(\mathfrak{g}_0)$-invariants in the coinvariant algebra of $W(L_0)$. It is well-known that the Poincare series for $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ with the symmetric grading is $\prod_{i=1}^{l_0}(1-q^{r_i+1})^{-1}\prod_{i=1}^{l_0}\left(1-q^{m_i^{(0)}+1}\right)$, where $m_i^{(0)}$ refers to the exponents of L_0 and r_i refers to the exponents of $\mathfrak{g}_0$. The dimension of $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ is $|W(L_0)|/|W(\mathfrak{g}_0)|$.

Theorem 5.1.7. *Let $m_1^{(a)}, \ldots, m_{l_a}^{(a)}$ denote the exponents of L_a, and let $r_1, \ldots, r_{l_0}$ be the exponents of the reductive Lie algebra $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$. Let* $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ *denote the coinvariant algebra, with a cohomological grading (resp. z-grading) defined by setting the cohomological degree (resp. z-degree) to twice (resp. N times) the symmetric degree.*

If $\mathfrak{p}$ is the standard parahoric $\{f \in L[[z]]^{\tilde{\sigma}} : f(0) \in \mathfrak{p}_0\}$ and N is a multiple of k then the cohomology algebra $H^*(\mathfrak{p}/z^N\mathfrak{p})$ is isomorphic to $\mathrm{Coinv}(L_0, \mathfrak{g}_0) \otimes \Lambda$, *where Λ is the free super-commutative algebra generated in degrees given by the following table:*

Cohomological degree	*z-degree*	*Index set*
$2r_i + 1$	0	$i = 1, \ldots, l_0$
$2m_i^{(a)} + 1$	$Nm_i^{(a)} + nk + a$	$a = 0, \ldots, k-1,\ i = 1, \ldots, l_a,\ 0 < nk + a < N$

As in the proof of Theorem 5.1.4, we have

$$H^*(\mathfrak{p}/z^N\mathfrak{p}) \cong H^*(\mathfrak{g}_0) \otimes H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0),$$

so we only need to compute the relative cohomology. This will be done with a spectral sequence argument in Section 5.3 (see Proposition 5.3.6).

When the parabolic component is trivial, $H^*\left(L[z]^{\tilde{\sigma}}/z^N\right)$ is simply the free super-commutative algebra with one set of generators in cohomological degree $2m_i^{(0)} + 1$ and z-degree 0 for $i = 1, \ldots, l_0$, and another set of generators in cohomological degree $2m_i^{(a)} + 1$ and z-degree $Nm_i^{(a)} + nk + a$, where $a = 0, \ldots, k-1$, $i = 1, \ldots, l_a$, and n such that $0 < nk + a < N$. Theorem 5.1.7 can be restated as saying that $H^*(\mathfrak{p}/z^N\mathfrak{p})$ is the algebra $H^*(\mathfrak{g}_0) \otimes \mathrm{Coinv}(L_0, \mathfrak{g}_0) \otimes_{H^*(L_0)} H^*(L[z]^{\tilde{\sigma}}/z^N)$.

In Lemma 5.3.7, we prove that if $\mathfrak{g}$ is untwisted and $N = 1$ then $H^*(\mathfrak{p}/z\mathfrak{p}, \mathfrak{g}_0)$ is isomorphic to $H^*(L_0, \mathfrak{g}_0)$. This algebra is in turn isomorphic to the cohomology ring of the generalized flag variety corresponding to the pair $(L_0, \mathfrak{p}_0)$. The z-grading on $H^*(\mathfrak{p}/z\mathfrak{p}, \mathfrak{g}_0)$ corresponds to the holomorphic grading appearing in the Hodge decomposition. The fact that $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ is isomorphic to $H^*(L_0, \mathfrak{g}_0)$ is a classic theorem of Borel ([Bo53], see Theorem 5.5 of [BGG73] for the parabolic case). Thus Theorem 5.1.7 can be seen as a generalization of Borel's theorem.

We can compare the cohomology of $\mathfrak{p}/z^N\mathfrak{p}$ with the cohomology of more general truncations. If $P(z)$ is a polynomial in z, then $P(z^k)L[z]^{\tilde{\sigma}}$ is a subset of $L[z]^{\tilde{\sigma}}$, and hence $P(z^k)\mathfrak{p}$ is an ideal of $\mathfrak{p}$. We can assume that P is monic, and write $P = z^d + P_0$, where d is the degree of P and P_0 contains lower degree terms. Suppose $x \in L_i$ for $i \geq 0$. Then $(z^{dk} + P_0(z^k))xz^i$ is in $P(z^k)\mathfrak{p}$ if and only if either $i > 0$ or $x \in \mathfrak{p}_0$, so the dimension of $\mathfrak{p}/P(z^k)\mathfrak{p}$ is $d \cdot \dim L$.

Lemma 5.1.8. *If P and Q are coprime then $\mathfrak{p}/P(z^k)Q(z^k)\mathfrak{p} \cong \mathfrak{p}/P(z^k)\mathfrak{p} \oplus \mathfrak{p}/Q(z^k)\mathfrak{p}$.*

By Lemma 5.1.8 the study of $\mathfrak{p}/P(z^k)\mathfrak{p}$ reduces to the case where P is the power of a linear factor. In the untwisted case, $L[z] \cong L[z-\alpha]$, so $L[z]/(z-\alpha)^N \cong L[z]/z^N$. However, in the twisted case this argument does not apply, since the automorphism $z \mapsto q^{-1}z$ is different from $z - \alpha \mapsto q^{-1}(z-\alpha)$. In particular:

Lemma 5.1.9. *If $\alpha \neq 0$ then $\mathfrak{p}/(z^k - \alpha)\mathfrak{p}$ is isomorphic to L.*

Proof. Let β be a kth root of α. Then evaluation at β defines a morphism $\mathfrak{p}/(z^k-\alpha)\mathfrak{p} \to L$. Both L and $\mathfrak{p}/(z^k-\alpha)\mathfrak{p}$ have the same dimension, so we just need to show that this map is onto. Given $x \in L$, write $x = \sum_{i=0}^{k-1} x_i$ where $x_i \in L_i$. Let $f = \sum_{i=1}^{k-1} x_i\beta^{-i}z^i + x_0\alpha^{-1}z^k$. Then $f(\beta) = x$. □

The author does not know if an analogue of Lemma 5.1.9 holds for higher powers of $(z^k - \alpha)$. The main case of interest is $\mathfrak{p}/(z^N - t)\mathfrak{p}$, which can be regarded as a deformation of $\mathfrak{p}/z^N\mathfrak{p}$. Since $z^{N/k} - t$ splits into N/k coprime linear factors, the algebra $\mathfrak{p}/(z^N - t)\mathfrak{p}$ is isomorphic to $L^{\oplus N/k}$ for $t \neq 0$. At $t = 0$, the algebra $\mathfrak{p}/z^N\mathfrak{p}$ has a large nilpotent ideal. Ignoring z-degrees, Theorem 5.1.7 tells us that $H^*\left(L[z]^{\tilde{\sigma}}/z^N\right) \cong H^*(L)^{\otimes N/k}$, so the cohomology of $L[z]^{\tilde{\sigma}}/(z^N - t)$ is independent of the value of t. On the other hand, Theorem 5.1.7 tell us that $H^*\left(\mathfrak{p}/(z^N-t)\mathfrak{p}\right)$ changes from $H^*(L)^{\otimes N/k}$ to $H^*(\mathfrak{g}_0) \otimes H^*(L_0, \mathfrak{g}_0) \otimes_{H^*(L_0)} H^*(L)^{\otimes N/k}$ as t degenerates to zero, where $H^*(L_0)$ acts on $H^*(\mathfrak{g}_0)$ via pullback. Interestingly, the cohomology of $\mathfrak{p}/z^k(z^N - t)\mathfrak{p}$ is unchanged as t degenerates to zero.

If $\mathfrak{p} = \mathfrak{b}$ is an Iwahori and $\mathfrak{n}$ is the nilpotent subalgebra, then a similar analysis can be performed for $\mathfrak{b}/z^N\mathfrak{n}$.

Theorem 5.1.10. *Let $m_1^{(a)}, \ldots, m_{l_a}^{(a)}$ denote the exponents of L_a, let $\mathfrak{b}$ be an Iwahori subalgebra of the twisted loop algebra $\mathfrak{g}$, and let $\mathfrak{n}$ be the nilpotent subalgebra. Then $H^*(\mathfrak{b}/z^N\mathfrak{n})$ is the free super-commutative algebra with a generator in cohomological degree $2m_i^{(a)} + 1$ and z-degree $Nm_i^{(a)} + nk + a$ for every $a = 0, \ldots, k-1$, $i = 1, \ldots, l_a$, and n such that $0 < nk + a \leq N$, as well as l_0 generators of cohomological degree 1 and z-degree 0.*

As with Theorem 5.1.7, the proof of Theorem 5.1.10 reduces via the Hochschild-Serre spectral sequence to the computation of the relative cohomology, which is also completed in Section 5.3 (see Proposition 5.3.8).

If $P(z)$ is a polynomial of degree d, then $\mathfrak{b}/P(z^k)\mathfrak{n}$ has dimension $d \cdot \dim L + l_0$. Furthermore, $[\mathfrak{b}, P(z^k)\mathfrak{h}_0]$ is contained in $P(z^k)\mathfrak{n}$, and there is a morphism $\mathfrak{b}/P(z^k)\mathfrak{n} \to \mathfrak{b}/P(z^k)\,\mathfrak{b}$ with kernel $P(z^k)\mathfrak{h}_0$, so $\mathfrak{b}/P(z^k)\mathfrak{n}$ is a central extension of $\mathfrak{b}/P(z^k)\,\mathfrak{b}$ of rank l_0.

Lemma 5.1.11. *If $t \neq 0$ then $\mathfrak{b}/(z^N - t)\mathfrak{n}$ is isomorphic to $L^{\oplus N/k} \oplus \mathbb{C}^{l_0}$, where the second summand is abelian.*

Proof. $\mathfrak{b}/(z^N - t)\,\mathfrak{b}$ is isomorphic to the direct sum of N/k copies of L. If L is semisimple, then so is $\mathfrak{b}/(z^N - t)\,\mathfrak{b}$, so all central extensions are trivial. The reductive case reduces to the semisimple case by splitting off the centre. □

Thus $H^*\left(\mathfrak{b}/(z^N - t)\mathfrak{n}\right)$ is also independent of t when z-degrees are disregarded.

5.1.4 The Macdonald constant term identity

Theorem 5.1.7 can be used to prove the affine version of Macdonald's constant term conjecture. If $\hat{\alpha} = \alpha + n\delta$ is an affine root, α a weight of L_0, set $e^{\hat{\alpha}} = q^{-n}e^{\alpha}$. In a slight abuse of notation, the operator $[e^0]$ will denote the sum of the $e^{n\delta}$ terms, i.e. it is $\mathbb{C}(q)$-linear. Let δ^* denote the dual element to δ. The following theorem is Conjecture 3.3 of [Ma82], and was proven for all root systems by Cherednik [Ch95].

Theorem 5.1.12 (Cherednik)**.** *Let N be a multiple of k, and let S_N be the set of real roots*[1] *$\alpha+n\delta$ of the twisted loop algebra $\mathfrak{g}$ with $0 \le n \le N$, such that α is a positive (resp. negative) root of L_0 if $n = 0$ (resp. $n = N$). Let ρ be the element of $\mathfrak{h}_0$ such that $\alpha_i(\rho) = 1$ for all simple roots $\alpha_1, \ldots, \alpha_{l_0}$ of L_0, and let $\rho_N = -N\rho + \delta^*$. Then*

$$[e^0] \prod_{\alpha \in S_N} (1 - e^{-\alpha}) = \prod_{\alpha \in S_N} \left(1 - q^{|\alpha(\rho_N)|}\right)^{\epsilon(\alpha)},$$

where $\epsilon(\alpha)$ is the sign of $\alpha(\rho_N)$.

Define a twisted q-binomial coefficient for $a \in \mathbb{Z}_k$ and multiples N, M of k by

$$\binom{N}{M}_{k,a} = \prod_{\substack{N-M<i\le N \\ i\equiv a \mod k}} (1 - q^i) \prod_{\substack{0<i\le M \\ i\equiv a \mod k}} (1 - q^i)^{-1}.$$

The right-hand side of Theorem 5.1.12 can be simplified by extending an idea of [Ma82] from the untwisted case.

Lemma 5.1.13. *The identity of Theorem 5.1.12 is equivalent to*

$$[e^0] \prod_{\alpha \in S_N} (1 - e^{-\alpha}) = \prod_{a \in \mathbb{Z}_k} \prod_{i=1}^{l_a} \binom{N(m_i^{(a)} + 1)}{N}_{k,a}. \tag{5.2}$$

Proof. Let Δ_a be the set of weights of the L_0-module L_a, and let Δ_a^+ denote the subset of $\alpha \in \Delta_a$ such that $\alpha(\rho) > 0$. If θ is an arbitrary function from positive integers to a multiplicative group, then

$$\prod_{\alpha \in \Delta_a^+} \frac{\theta(\alpha(\rho) + 1)}{\theta(\alpha(\rho))} = \prod_{i=1}^{l_a} \frac{\theta\left(m_i^{(a)} + 1\right)}{\theta(1)}.$$

[1] i.e. $\alpha \ne 0$

To prove this, note that the eigenvalues of ρ on L_a are integers giving the principal grading of L_a, so the identity follows immediately from Corollary 5.1.3 by comparing the number of times $\theta(m)$ occurs on the top versus the bottom.

Define

$$A_a = \prod_{\substack{\alpha+n\delta\in S_N \\ \alpha\in\Delta_a}} \left(1-q^{|\alpha(\rho_N)|}\right)^{\epsilon(\alpha)},$$

and set $\theta_{-a}(m) = (1-q^{Nm-a})(1-q^{Nm-k-a})\cdots(1-q^{Nm-N+k-a})$, for $a \in \mathbb{Z}_k$ represented by one of $0,\ldots,k-1$. Then

$$A_0 = \prod_{\alpha\in\Delta_0^+}\prod_{n=0}^{N/k-1}\left(1-q^{N\alpha(\rho)-nk}\right)^{-1}\prod_{n=1}^{N/k}\left(1-q^{N\alpha(\rho)+nk}\right)$$

while if $a \neq 0$ we have

$$A_a = \prod_{\alpha\in\Delta_a^+}\prod_{n=0}^{N/k-1}\left(1-q^{N\alpha(\rho)-nk-a}\right)^{-1}\left(1-q^{N\alpha(\rho)+nk+a}\right).$$

In both cases,

$$A_a = \prod_{\alpha\in\Delta_a^+}\theta_{-a}(\alpha(\rho))^{-1}\theta_a(\alpha(\rho)+1).$$

Even if $-a$ and a are not congruent, L_a and L_{-a} are still isomorphic, so

$$A_aA_{-a} = \prod_{\alpha\in\Delta_a^+}\theta_{-a}(\alpha(\rho))^{-1}\theta_a(\alpha(\rho)+1)\theta_a(\alpha(\rho))^{-1}\theta_{-a}(\alpha(\rho)+1).$$

Hence the right hand side of Theorem 5.1.12 is equal to

$$\prod_{a=0}^{k-1}A_a = \prod_{a\in\mathbb{Z}_k}\prod_{i=1}^{l_a}\frac{\theta_a\left(m_i^{(a)}+1\right)}{\theta_a(1)},$$

as required. □

Let C^* be a chain complex with an additional grading $C^* = \bigoplus C^*_n$. The weighted Euler characteristic of C^* is

$$\chi(C^*;q) = \sum_{n,i}(-1)^*\dim C^i_n q^n.$$

As in the unweighted case, the weighted Euler characteristic is invariant under taking homology. Let $\mathfrak{p} = \{f \in L[[z]]^{\tilde{\sigma}} : f(0) \in \mathfrak{p}_0\}$ be a standard parahoric and $\mathfrak{g}_0 = \mathfrak{p}_0 \cap \overline{\mathfrak{p}_0}$. Theorem 5.1.12 can be proved by comparing the z-weighted Euler characteristic for the Koszul complex of the pair $(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0)$ with the weighted Euler characteristic of the cohomology ring:

Proof. Write $\mathfrak{p}_0 = \mathfrak{g}_0 \oplus \mathfrak{u}_0$, for $\mathfrak{u}_0$ the nilpotent radical. Let K be a compact subgroup acting on L with complexified Lie algebra $\mathfrak{g}_0$, and let T be a maximal torus in K with complexified Lie algebra $\mathfrak{h}_0$. Let π_a denote the representation of K on L_a, and let ϕ and $\overline{\phi}$ denote the representation of K on $\mathfrak{u}_0$ and $\overline{\mathfrak{u}_0}$ respectively. The weighted Euler characteristic of the Koszul complex is

$$\chi(q) = \sum (-1)^i q^i \dim \left(\bigwedge^i (\mathfrak{p}/z^N \mathfrak{p})^* \right)^K .$$

By orthogonality of traces of representations with respect to Haar measure,

$$\chi(q) = \int_K \det(\mathbb{1} - \phi(k)) \det(\mathbb{1} - q^N \overline{\phi}(k)) \prod_{0<n<N} \det(\mathbb{1} - q^n \pi_n(k)) dk.$$

The integrand is conjugation invariant, so by the Weyl integral formula,

$$\begin{aligned}\chi(q) &= \frac{1}{|W(\mathfrak{g}_0)|} \int_T \det(\mathbb{1} - \phi(t)) \det(\mathbb{1} - q^N \overline{\phi}(t)) \prod_{0<n<N} \det(\mathbb{1} - q^n \pi_n(t)) \prod_{\alpha \in \Delta(\mathfrak{g}_0)} (1 - e^{\alpha(t)}) dt \\ &= \frac{1}{|W(\mathfrak{g}_0)|} [e^0] \prod_{\alpha \in \Delta(\mathfrak{g}_0)} (1 - e^{\alpha}) \cdot \Phi,\end{aligned}$$

where $\Delta(\mathfrak{g}_0)$ is the root system of $\mathfrak{g}_0$ and

$$\Phi = \prod_{\alpha \in \Delta^+(\mathfrak{g}_0)} (1 - e^{-\alpha})^{-1} (1 - q^N e^{\alpha})^{-1} \prod_{\alpha \in S_N} (1 - e^{-\alpha}) \prod_{0<n<N} (1 - q^n)^{l_n}$$

(note that the inverses divide into the other multiplicands). The coefficient of q^j in Φ is (up to sign) the character of a $\mathfrak{g}_0$-module, so Φ is $W(\mathfrak{g}_0)$ invariant. Now we use the identity

$$\sum_{w \in W(\mathfrak{g}_0)} \prod_{\alpha \in \Delta^+(\mathfrak{g}_0)} \frac{1 - q^N e^{w\alpha}}{1 - e^{w\alpha}} = \prod_{i=1}^{l_0} \frac{1 - q^{N(r_i+1)}}{1 - q^N} \tag{5.3}$$

found in [Ma82][Ma72b] to get

$$\begin{aligned}\chi(q) &= \frac{1}{|W(\mathfrak{g}_0)|} \prod_{i=1}^{l_0} \frac{1 - q^N}{1 - q^{N(r_i+1)}} \cdot [e^0] \sum_{w \in W(\mathfrak{g}_0)} \prod_{\alpha \in \Delta^+(\mathfrak{g}_0)} \frac{1 - q^N e^{w\alpha}}{1 - e^{w\alpha}} \prod_{\alpha \in \Delta(\mathfrak{g}_0)} (1 - e^{\alpha}) \cdot \Phi \\ &= \frac{1}{|W(\mathfrak{g}_0)|} \prod_{i=1}^{l_0} \frac{1 - q^N}{1 - q^{N(r_i+1)}} \cdot [e^0] \sum_{w \in W(\mathfrak{g}_0)} w \cdot \prod_{\alpha \in \Delta^+(\mathfrak{g}_0)} \frac{1 - q^N e^{\alpha}}{1 - e^{\alpha}} \prod_{\alpha \in \Delta(\mathfrak{g}_0)} (1 - e^{\alpha}) \cdot \Phi.\end{aligned}$$

Since the action of $W(\mathfrak{g}_0)$ does not change the constant term, this last sum gives

$$\chi(q) = \prod_{i=1}^{l_0} (1 - q^{N(r_i+1)})^{-1} \prod_{0<n\leq N} (1 - q^n)^{l_n} \cdot [e^0] \prod_{\alpha \in S_N} (1 - e^{-\alpha}).$$

On the other hand, Theorem 5.1.7 implies

$$\chi(q) = \prod_{i=1}^{l_0} \left(1 - q^{N(r_i+1)}\right)^{-1} \prod_{0<n\leq N} \prod_{i=1}^{l_n} \left(1 - q^{Nm_i^{(n)}+n}\right)$$

Identifying these two equations gives the identity of Lemma 5.1.13. □

Note that when $\mathfrak{p} = \mathfrak{b}$ is an Iwahori the equivalence follows without using the Weyl integration argument or identity (5.3). The z-weighted Euler characteristic identity for $H^*(\mathfrak{b}/z^N\mathfrak{n}, \mathfrak{h}_0)$, is similarly equivalent to the identity of Lemma 5.1.13.

5.2 Calculation of parahoric cohomology

In this section we finish the proofs of Theorems 5.1.5 and 5.1.6 and Proposition 5.1.2. We continue to use the notation of Section 4.2.

5.2.1 Proof of Proposition 5.1.2

Pick a principal $\mathfrak{sl}_2$-triple $\{h, e, f\}$ in L_0, and note that $\{h, e, f\}$ is principal in L by Lemma 4.2.9. We need to show that the eigenvalues of $h/2$ on L_a^e agree with the subset of the exponents defined in Definition 5.1.1. Let $L = \bigoplus L^{(i)}$ denote the principal grading of L induced by the eigenspace decomposition of $h/2$. Then $m \geq 0$ appears in the list of exponents of L with multiplicity $\dim \left(L^{(m)}\right)^e$.

Let ν denote the Kostant slice $f + L^e$. As previously mentioned, Kostant's theorem states that the restriction map $\mathbb{C}[Q] \to \mathbb{C}[\nu]$ is an isomorphism. Actually, a stronger statement is true. Identity $\mathbb{C}[\nu]$ with polynomials on L^e in the obvious way. Filter $\mathbb{C}[\nu]$ by setting $\mathbb{C}[\nu]_m$ to be the subring of polynomials on $\bigoplus_{i=0}^m \left(L^{(i)}\right)^e$. Choose homogeneous generators for $\mathbb{C}[Q] = (S^*L^*)^G$ and let $\mathbb{C}[Q]_m$ be the subring generated by generators of degree at most $m+1$. Then, by Theorem 7, page 381 of [Ko63b], the restriction map gives an isomorphism between $\mathbb{C}[Q]_m$ and $\mathbb{C}[\nu]_m$. Furthermore, if I is a generator of degree $m+1$ then the restriction of I to ν takes the form $f + I_0$ where f is in the dual space of $\left(L^{(m)}\right)^e$ and $I_0 \in \mathbb{C}[\nu]_{m-1}$ does not have constant term.

The automorphism σ acts on both $\mathbb{C}[\nu]$ and $\mathbb{C}[Q]$, preserving the filtration in both cases, and the restriction map is σ-equivariant. As before, let $\mathcal{M}$ denote the ideal in $(S^*L^*)^G$ containing all elements of degree greater than zero, so that $\mathcal{M}/\mathcal{M}^2$ is the space of generators. By definition, the multiplicity of m as an exponent is the multiplicity of q^{-a} as an eigenvalue of σ acting on the degree $m+1$ subspace of $\mathcal{M}/\mathcal{M}^2$. By the previous paragraph, this is equal to the multiplicity of q^{-a} as an eigenvalue of σ acting on the dual space of $\left(L^{(m)}\right)^e$, or equivalently the dimension of q^a as an eigenvalue of σ acting on $\left(L^{(m)}\right)^e$ itself.

5.2.2 Proof of Theorem 5.1.5

Let $\Omega^*_{const} R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ denote the sections of $\bigwedge^* R \times_{Q^\sigma} T^*_{const} J^{\tilde{\sigma}}_m Q$, where $T^*_{const} J^{\tilde{\sigma}}_m Q$ is the dual bundle to $T_{const} J^{\tilde{\sigma}}_m Q$. Similarly, let $\Omega^*_{>0}\mathfrak{p}_m$ denote the sections of $\bigwedge^* T^*_{>0}\mathfrak{p}_m$, where $T^*_{>0}\mathfrak{p}_m$ is the dual bundle to $T_{>0}\mathfrak{p}_m$. As per Theorem 3.1.3, we want to calculate the algebra of $\mathcal{P}_\infty$-invariant $\mathcal{N}_\infty$-basic elements of $\Omega^*_{>0}\mathfrak{p}_\infty$.

Proposition 5.2.1. *Pullback via the bundle map $T_{>0}\mathfrak{p}_m \to R \times_{Q^\sigma} T_{const} J^{\tilde{\sigma}}_m$ gives an isomorphism from the algebra $\Omega^*_{const} R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ to the algebra of $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic elements of $\Omega^*_{>0}\mathfrak{p}_m$.*

Proof. Every section of $\Omega^*_{>0}\mathfrak{p}_\infty$ is a pullback from $\Omega^*_{>0}\mathfrak{p}_m$ for some $m < +\infty$. By Corollary 4.2.8, the pullback map is injective, so it is enough to prove surjectivity when $m < +\infty$.

Let $\mathfrak{p}^{rs}_m$ denote the open subset $\mathfrak{p}_m \cap J^{\tilde{\sigma}}_m L^{rs}$ of $\mathfrak{p}_m$. We start by showing that the pullback map is an isomorphism from $\Omega^*_{const} R^r \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q^r$ to the algebra of $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic elements of $\Omega^*_{>0}\mathfrak{p}^{rs}_m$. By Theorem 4.2.2, $\mathfrak{p}^{rs}_m$ is isomorphic to $\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} J^{\tilde{\sigma}}_m \mathfrak{h}^r$. By Proposition 4.1.3,

$$T\left(\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} J^{\tilde{\sigma}}_m \mathfrak{h}^r\right) \cong T\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} T J^{\tilde{\sigma}}_m \mathfrak{h}^r.$$

$\mathcal{P}_m/\mathcal{N}_m$ is isomorphic to the connected reductive subgroup of $\mathcal{P}_m$ corresponding to the subalgebra $\mathfrak{g}_0 \subset J^{\tilde{\sigma}}_m L$. We work for a moment in the analytic category. Suppose γ_t is a curve in $\mathfrak{p}^{rs}_m$ representing an element of $T_{>0}\mathfrak{p}_m$, so that the image $\overline{\gamma_t}$ of γ_t in $\mathfrak{g}_0$ is constant. There are curves α_t and β_t in $\mathcal{P}_m$ and $J^{\tilde{\sigma}}_m \mathfrak{h}^r$ respectively such that $\alpha_t \beta_t = \gamma_t$. Let $\overline{\alpha_t}$ denote the image of $\alpha_t \in \mathcal{P}_m/\mathcal{N}_m$. Then $\overline{\gamma_t} = \overline{\alpha_t}\beta_t(0)$ is a constant curve in $\mathfrak{g}_0$, so $\overline{\alpha_0}^{-1}\overline{\alpha_t}\beta_t(0)$ is a constant curve in $\mathfrak{h}^r_0$. This implies that $\overline{\alpha_0}^{-1}\overline{\alpha_t} \in wH^\sigma$ for some $w \in N(H)^\sigma$, from which we can conclude that $\overline{\alpha_0}^{-1}\overline{\alpha_t}\beta_t(0) = w\beta_t(0)$, so $\beta_t(0)$ is constant, and hence represents an element of $T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$. Since $w^{-1}\overline{\alpha_0}^{-1}\overline{\alpha_t} \in H^\sigma$, the curves α_t and $\alpha_t \overline{\alpha_t}^{-1}\overline{\alpha_0} w$ are equal in $\mathcal{P}_m / J^{\tilde{\sigma}}_m H$. The latter curve projects to a constant curve in $\mathcal{P}_m/\mathcal{N}_m$, and since $\mathcal{P}_m \cong \mathcal{P}_m/\mathcal{N}_m \ltimes \mathcal{N}_m$, is tangent to a left $\mathcal{N}_m$-coset in $\mathcal{P}_m$. Since $\mathcal{N}_m$ is normal, every left $\mathcal{N}_m$-coset is a right $\mathcal{N}_m$-coset. We conclude that over $\mathfrak{p}^{rs}_m$, $T_{>0}\mathfrak{p}_m$ is isomorphic to the subbundle $T_{\mathcal{N}_m}\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ of $T\left(\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} J^{\tilde{\sigma}}_m \mathfrak{h}^r\right)$, where $T_{\mathcal{N}_m}\mathcal{P}_m / J^{\tilde{\sigma}}_m H$ is the subbundle of tangents to $\mathcal{N}_m$-orbits.

Recall from the proof of Theorem 4.2.2 that $J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$ is isomorphic to $J^{\tilde{\sigma}}_m(\mathfrak{h}^r/W(\mathfrak{m}))$, so T_{const} of the former space is well-defined. By Proposition 4.1.3 again, $(T J^{\tilde{\sigma}}_m \mathfrak{h}^r)/W(\mathfrak{g}_0) \cong T(J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0))$, so $T_{const}(J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0))$ is a subbundle of $(T J^{\tilde{\sigma}}_m \mathfrak{h}^r)/W(\mathfrak{g}_0)$. A tangent vector $v \in T J^{\tilde{\sigma}}_m \mathfrak{h}^r$ represents an element of $T_{const}(J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0))$ if and only if the projection of $v(0)$ to $\mathfrak{h}^r_0 / W(\mathfrak{g}_0) \cong (\mathfrak{h}^r/W(\mathfrak{m}))^\sigma$ is a zero tangent vector, where $v(0)$ is the image of v in $T\mathfrak{h}^r_0$. Since $\mathfrak{h}^r_0 \to \mathfrak{h}^r_0/W(\mathfrak{g}_0)$ is etale, this is true if and only if $v(0)$ is a zero tangent vector, so $T_{const}(J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)) \cong \left(T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r\right)/W(\mathfrak{g}_0)$. Similarly the isomorphism $J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0) \cong R^r \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q^r$ sends $T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$ to $R^r \times_{Q^\sigma} T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ (see the proof of Corollary 4.2.8). Applying Theorem 4.2.2, we want to show that the bundle map

$$T_{\mathcal{N}_m}\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r \to T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$$

induced by projection on the second factor gives an isomorphism from $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$ to the ring of $\mathcal{P}$-invariant $\mathcal{N}$-basic sections of $\bigwedge^* T^*_{\mathcal{N}_m} \mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} T^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$.

By pulling back to $\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times J^{\tilde{\sigma}}_m \mathfrak{h}^r$, we can identify the ring of $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic sections of $\bigwedge^* T^*_{\mathcal{N}_m} \mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} T^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ with a subring of the $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic sections of $\bigwedge^* T_{\mathcal{N}_m} \mathcal{P}_m / J^{\tilde{\sigma}}_m H \times T^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$. This latter ring is isomorphic to the ring $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ by pullback via projection on the second factor. An element of $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ descends to a section over $\mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ if and only if it is $W(\mathfrak{g}_0)$-equivariant. The splitting $TJ^{\tilde{\sigma}}_m \mathfrak{h}^r = J^{\tilde{\sigma}}_m \mathfrak{h}^r \times \mathfrak{h}_0 \oplus T_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ allows us to identify the $W(\mathfrak{g}_0)$-module $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ with the subalgebra of differential forms which vanish on $J^{\tilde{\sigma}}_m \mathfrak{h}^r \times \mathfrak{h}_0$.[2] There is a similar splitting for $TJ^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$, and thus a similar identification for $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$. The differential $TJ^{\tilde{\sigma}}_m \mathfrak{h}^r \to TJ^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$ preserves this splitting, so the pullback map $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0) \to \Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ agrees with the pullback map on differential forms. Thus pullback gives an isomorphism from $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r / W(\mathfrak{g}_0)$ to the $W(\mathfrak{g}_0)$-equivariant elements of $\Omega^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ (see, e.g., Theorem 1 of [Br98]), and this implies that all $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic sections of $\bigwedge^* T^*_{\mathcal{N}_m} \mathcal{P}_m / J^{\tilde{\sigma}}_m H \times_{W(\mathfrak{g}_0)} T^*_{const} J^{\tilde{\sigma}}_m \mathfrak{h}^r$ come from pullback on the second factor.

To finish the proof, let $\mathfrak{p}^{reg}_m = \mathfrak{p} \cap J^{\tilde{\sigma}}_m L^{reg}$, and let ϕ denote the map $\mathfrak{p}^{reg}_m \to R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$. By Theorem 4.2.7, ϕ is smooth and surjective. Hence if f is a regular function defined on an open dense subset of $R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ such that $\phi^* f$ extends to $\mathfrak{p}^{reg}_m$, then f has a unique extension to $R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$.

Suppose $\omega \in \Omega^*_{>0} \mathfrak{p}^{reg}_m$ is $\mathcal{P}_m$-invariant and $\mathcal{N}_m$-basic. Then there is $\alpha \in \Omega^*_{const} R^r \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ such that $\phi^* \alpha = \omega$ over $\mathfrak{p}^{rs}_m$. We can write $\alpha = \sum f_i \alpha_i$, where the α_i's are elements of $\Omega^*_{const} R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ which are linearly independent in fibres, and the f_i's are functions on $R^r \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q^r$. Since the bundle map is surjective on fibres, the pullbacks $\phi^* \alpha_i$ are linearly independent in fibres. Since $\phi^* \alpha = \sum_i \phi^* f_i \phi^* \alpha_i$ extends to $\mathfrak{p}^{reg}_m$, the functions $\phi^* f_i$ must extend to $\mathfrak{p}^{reg}_m$, and consequently α extends to $R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$. The pullback $\phi^* \alpha$ agrees with ω on an open dense subset, so every $\mathcal{P}_m$-invariant $\mathcal{N}_m$-basic element of $\Omega^*_{>0} \mathfrak{p}^{reg}_m$ is the pullback of an element of $\Omega^*_{const} R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ as desired. □

Proof of Theorem 5.1.5. Let I^a_i and R_i be generators for $\mathbb{C}[Q]$ and $\mathbb{C}[R]$ as in the statement of Theorem 5.1.5. Choose coordinates $\{y_{ia}\}$ for Q such that pullback of y_{ia} via the projection $L \to Q$ is I^a_i. Similarly, choose coordinates $\{r_i\}$ for R such that the pullback of r_i via the projection $\mathfrak{p}_0 \to R$ is R_i. Note that the coordinates $\{y_{ia}\}$ with a fixed correspond to the subspace Q_a of Q on which σ acts as multiplication by q^a (by previously established convention, this means that $\sigma y_{ia} = q^{-a} y_{ia}$). Consider the r_i's as functions on $R \times_{Q^\sigma} J^{\tilde{\sigma}}_\infty Q$, and let $\tilde{y}_{ia}$ denote the induced map $J_\infty Q \to Q$. Then the coordinate ring of $R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ is the free ring generated by the r_i's and the functions $[z^{nk-a}] \tilde{y}_{i,-a}$ for $a = 0, \ldots, k-1$ and $n \geq 1$. Consequently the ring $\Omega^*_{const} R \times_{Q^\sigma} J^{\tilde{\sigma}}_m Q$ is the free super-commutative ring generated by the above generators for the coordinate ring, along with the restrictions of the differential forms $d[z^{nk-a}] \tilde{y}_{i,-a}$, $a = 0, \ldots, k-1$ and $n \geq 1$. Let $\hat{\mathfrak{p}} = \mathfrak{p}_\infty$ denote the completion of a standard parahoric. Applying Proposition 5.2.1 we conclude that $\Omega^*_{>0} \hat{\mathfrak{p}}$ is the free super-commutative

[2] In contrast, there is no such identification for the $\mathcal{P}_m$-module $\Omega_{>0} \mathfrak{p}_m$.

algebra generated by the R_i's, the functions $[z^{nk-a}]\tilde{I}_i^{-a}$ for $a = 0, \ldots, k-1$ and $n \geq 1$, and the restrictions of the 1-forms $d[z^{kn-a}]\tilde{I}_i^{-a}$ to $T_{>0}\hat{\mathfrak{p}}$, again for $a = 0, \ldots, k-1$ and $n \geq 1$. Theorem 5.1.5 then follows from Theorem 3.1.3. □

5.2.3 Proof of Theorem 5.1.6

The proof of Theorem 5.1.5 can be simplified and used to prove that pullback via the map $\mathfrak{p}_m \to R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$ gives an isomorphism between algebraic forms on $R \times_{Q^\sigma} J_m^{\tilde{\sigma}} Q$ and $\mathcal{P}_m$-basic and invariant forms on $\mathfrak{p}_m$. When $\mathfrak{p}_0 = L_0$, this can be proved without Theorem 4.2.2. Namely, if ν is a Kostant slice in L then, as previously mentioned, $G \times \nu \to L^{reg}$ is surjective and smooth. By Lemma 4.1.7 and Theorem 4.2.6, the multiplication map $J_m^{\tilde{\sigma}} G \times J_m^{\tilde{\sigma}} \nu \to J_m^{\tilde{\sigma}} L^{reg}$ is surjective for all m, and smooth for $m < +\infty$. Since $J_m^{\tilde{\sigma}} \nu$ is isomorphic to $J_m^{\tilde{\sigma}} Q$, identification of algebraic forms on $J_m^{\tilde{\sigma}} Q$ with $J_m^{\tilde{\sigma}} G$-basic and invariant algebraic forms on $J_m^{\tilde{\sigma}} L$ follows by pulling back to $J_m^{\tilde{\sigma}} G \times J_m^{\tilde{\sigma}} \nu$.

This idea can be adapted to determine the algebra of $\mathcal{B}$-basic and invariant forms on $\hat{\mathfrak{n}}$, where $\mathfrak{b}$ is an Iwahori subalgebra, $\mathcal{B}$ is the subgroup corresponding to the completion $\hat{\mathfrak{b}}$, and $\hat{\mathfrak{n}}$ is the completion of the nilpotent subalgebra of $\mathfrak{b}$. More specifically, let $\mathfrak{b}_m$ be the image of $\hat{\mathfrak{b}}$ in $J_m^{\tilde{\sigma}} L$, let $\mathcal{B}_m$ be the corresponding connected subgroup of $J_m^{\tilde{\sigma}} G$, and let $\mathfrak{n}_m$ be the image of $\hat{\mathfrak{n}}$ in $J_m^{\tilde{\sigma}} L$. If X is a variety with finite order automorphism σ, and $p \in X$, let $J_{m,p}^{\tilde{\sigma}} X$ denote the subscheme $\{f \in J_m^{\tilde{\sigma}} X : f(0) = p\}$ of jets with a fixed base point.

Proposition 5.2.2. *There is a map $\mathfrak{n}_m \to J_{m,0}^{\tilde{\sigma}} Q$, and pullback via this map gives an isomorphism between the ring of algebraic forms on $J_{m,0}^{\tilde{\sigma}} Q$ and the ring of $\mathcal{B}_m$-basic and invariant algebraic forms on $\mathfrak{n}_m$.*

Proof. Once again it is sufficient to give the proof for $m < +\infty$. Let e be a principal nilpotent of L_0, contained in $\mathfrak{n}_0$. Recall that G^e is a connected subgroup of $\mathcal{B}_0$. Let $\left(J_m^{\tilde{\sigma}} G\right)_e$ denote the connected subgroup $\{f \in J_m^{\tilde{\sigma}} G : f(0) \in G^e\}$ of $J_m^{\tilde{\sigma}} G$ with Lie algebra $\{f \in J_m^{\tilde{\sigma}} L : f(0) \in L_0^e\}$. Since $f \in \mathfrak{n}_m$ belongs to $J_m^{\tilde{\sigma}} L^{reg}$ if and only if $f(0)$ is a principal nilpotent in $\mathfrak{n}_0$, and all principal nilpotents in $\mathfrak{n}_0$ are conjugate by an element of $\mathcal{B}_0$, it follows that the map

$$\mathcal{B}_m \times_{(J_m^{\tilde{\sigma}} G)_e} J_{m,e}^{\tilde{\sigma}} L \to \mathfrak{n}_m \cap J_m^{\tilde{\sigma}} L^{reg}$$

is an isomorphism. Consequently $\mathcal{B}_m$-basic and invariant forms on $\mathfrak{n}_m \cap J_m^{\tilde{\sigma}} L^{reg}$ correspond to $\left(J_m^{\tilde{\sigma}} G\right)_e$-basic and invariant forms on $J_{m,e}^{\tilde{\sigma}} L$.

The projection $L \to Q$ sends e to zero, so the restriction of $J_m^{\tilde{\sigma}} L \to J_m^{\tilde{\sigma}} Q$ to $J_{m,e}^{\tilde{\sigma}} L$ factors through $J_{m,0}^{\tilde{\sigma}} Q$. Choose a principal $\mathfrak{sl}_2$-triple $\{h, e, f\}$ in L_0 containing e, and let $\nu = e + L^f$. The isomorphism $J_m^{\tilde{\sigma}} \nu \to J_m^{\tilde{\sigma}} Q$ identifies $J_{m,e}^{\tilde{\sigma}} \nu$ with $J_{m,0}^{\tilde{\sigma}} Q$, and every $\left(J_m^{\tilde{\sigma}} G\right)_e$-orbit on $J_{m,e}^{\tilde{\sigma}} L$ intersects $J_{m,e}^{\tilde{\sigma}} \nu$ in a unique point. Consequently the map $J_{m,e}^{\tilde{\sigma}} L \to J_{m,0}^{\tilde{\sigma}} Q$ is a surjective smooth $\left(J_m^{\tilde{\sigma}} G\right)_e$-orbit map. It follows that the multiplication map $\left(J_m^{\tilde{\sigma}} G\right)_e \times J_{m,e}^{\tilde{\sigma}} \nu \to J_{m,e}^{\tilde{\sigma}} L$ is smooth and surjective. We conclude that the pullback map from algebraic forms on $J_{m,e}^{\tilde{\sigma}} L$ to algebraic forms on $\left(J_m^{\tilde{\sigma}} G\right)_e \times J_{m,e}^{\tilde{\sigma}} \nu$ is injective, and thus pullback via the map

$J^{\tilde{\sigma}}_{m,e}L \to J^{\tilde{\sigma}}_{m,0}Q$ gives an isomorphism between algebraic forms on $J^{\tilde{\sigma}}_{m,0}Q$ and $\left(J^{\tilde{\sigma}}_{m}G\right)_e$-basic and invariant forms on $J^{\tilde{\sigma}}_{m,e}L$.

Thus every $\mathcal{B}_m$-basic and invariant form on $\mathfrak{n}_m \cap J^{\tilde{\sigma}}_m L^{reg}$ is the pullback of a form from $J^{\tilde{\sigma}}_{m,0}Q$. Since $\mathfrak{n}_m \cap J^{\tilde{\sigma}}_m L^{reg}$ is dense in $\mathfrak{n}_m$, the proposition follows. □

This proof does not extend to nilpotent subalgebras of other parahorics, as $\mathfrak{u} \cap L^{reg}[[z]]$ is non-empty only in the Borel case. As in the proof of Theorem 5.1.5, Theorem 3.1.5 and Proposition 5.2.2 together imply Theorem 5.1.6.

5.3 Spectral sequence argument for the truncated algebra

In this section we finish the proof of Theorem 5.1.7 using a spectral sequence argument. We start by defining the spectral sequence, and establishing its convergence. We then prove, in Lemma 5.3.5, the key step: the collapse of the spectral sequence at the E_2-term.

Recall from Chapter 3 the definition of the operators $d_R(S)$ and $d_L(T)$ on $\bigwedge^* \hat{\mathfrak{p}} \otimes S^*\hat{\mathfrak{p}}$. The operator $d_R(S)$ is a generalized interior product, while $d_L(T)$ is a generalized exterior derivative. Hence we have the following version of Cartan's identity:

Lemma 5.3.1. *$d_R(S)d_L(T) + d_L(T)d_R(S) = (ST)^{Sym} + (TS)^{\wedge}$, where $(ST)^{Sym}$ is the extension of ST to the symmetric factor as a derivation, and $(TS)^{\wedge}$ is the extension of TS to the exterior factor as a derivation.*

Let $P : \hat{\mathfrak{p}}^* \to \hat{\mathfrak{p}}^*$ be the dual of multiplication by z^N on $\hat{\mathfrak{p}}$. Define $Q : \hat{\mathfrak{p}}^* \to \hat{\mathfrak{p}}^*$ by $(Qf)(x) = f\left(\frac{\overline{x}}{z^N}\right)$, where $\overline{x}$ is the projection to $z^N\hat{\mathfrak{p}}$ using the splitting $\hat{\mathfrak{p}} = (z^N\hat{\mathfrak{p}}) \oplus (\hat{\mathfrak{p}}/z^N\hat{\mathfrak{p}})$ suggested by the root grading. Note that $PQ = \mathbb{1}$, while QP is projection to $(z^N\hat{\mathfrak{p}})^*$ using the corresponding splitting of $\hat{\mathfrak{p}}^*$. Thus $(d_R(P)d_L(Q) + d_L(Q)d_R(P))\omega = (n+q)\omega$ if $\omega \in \bigwedge^*(\hat{\mathfrak{p}}/z^N\hat{\mathfrak{p}})^* \otimes \bigwedge^n(z^N\hat{\mathfrak{p}})^* \otimes S^q\hat{\mathfrak{p}}^*$. Then $d_R(P)^2 = 0$, and we can use Cartan's identity to show that

$$0 \longrightarrow \bigwedge^*(\hat{\mathfrak{p}}/z^N\hat{\mathfrak{p}})^* \longrightarrow \bigwedge^* \hat{\mathfrak{p}}^* \xrightarrow{d_R(P)} \bigwedge^{*-1} \hat{\mathfrak{p}}^* \otimes S^1\hat{\mathfrak{p}}^* \xrightarrow{d_R(P)} \cdots$$

is exact. Further, $d_R(P)$ commutes with the Lie algebra cohomology operator $\bar{\partial}$ with coefficients in $S^*\hat{\mathfrak{p}}^*$. Since $d_R(P)$ is $\mathfrak{p}$-equivariant and preserves the subset of cochains which vanish on $\mathfrak{g}_0$, we can restrict to the relative cochain complex to get an exact sequence

$$0 \longrightarrow \left(\bigwedge^*(\hat{\mathfrak{u}}/z^N\hat{\mathfrak{p}})^*\right)^{\mathfrak{g}_0} \longrightarrow K^{*,0} \xrightarrow{d_R(P)} K^{*,1} \xrightarrow{d_R(P)} \cdots,$$

where $K^{*,*}$ is the bigraded algebra $(\bigwedge^* \hat{\mathfrak{u}}^* \otimes S^*\hat{\mathfrak{p}}^*)^{\mathfrak{g}_0}$ graded by tensor (i.e. combined exterior and symmetric) degree and symmetric degree, regarded as a bicomplex with differentials $\bar{\partial}$ (the Lie algebra cohomology differential for $\hat{\mathfrak{u}}$ with coefficients in $S^*\hat{\mathfrak{p}}^*$) and $d_R(P)$. Note that both $\bar{\partial}$ and $d_R(P)$ are derivations of the algebra structure.

Lemma 5.3.2. *Give $K^{*,*}$ a z-grading by taking the usual z-degree for the exterior factor, and z-degree $+ N$ on $\hat{\mathfrak{p}}^*$ for the symmetric factor. This z-grading descends to $H^*(\text{total}\, K^{*,*})$, and there is an isomorphism $H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0) \to H^*(\text{total}\, K^{*,*})$ which preserves z-degrees.*

Proof. We have just shown that there is a chain map from the Koszul complex for $(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0)$ to total $K^{*,*}$. Consider the spectral sequence induced by the column-wise filtration of $K^{*,*}$, i.e. the descending filtration where the pth level contains all elements of $K^{a,b}$ with $a \geq p$. The E_1-term of this spectral sequence is

$$E_1^{p,q} = \begin{cases} \left(\bigwedge^p(\hat{\mathfrak{u}}/z^N\hat{\mathfrak{p}})^*\right)^{\mathfrak{g}_0} & q = 0 \\ 0 & q > 0 \end{cases},$$

with differential the restriction of $\bar{\partial}$. Hence

$$E_2^{p,q} = \begin{cases} H^p(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0) & q = 0 \\ 0 & q > 0 \end{cases}.$$

It follows from naturality of the spectral sequence that the induced map $H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0) \to H^*(\text{total}\, K^{*,*})$ is an isomorphism. The z-degrees on $K^{*,*}$ are preserved by $\bar{\partial}$ and $d_R(P)$, so the z-grading descends to homology and likewise is preserved by the isomorphism. □

To calculate $H^*(\text{total}\, K^{*,*})$, consider the spectral sequence of the bicomplex $K^{*,*}$ induced by the row-wise filtration, i.e. the descending filtration where the pth level contains all elements of $K^{a,b}$ with $b \geq p$. This spectral sequence has $E_1^{p,q} = H^{q-p}_{cts}(\hat{\mathfrak{p}}, \mathfrak{g}_0; S^p\hat{\mathfrak{p}})$ with differential $d_R(P)$ (note that the order of the degrees is swapped compared to $K^{*,*}$, so p is symmetric degree and q is tensor degree). Thus $E_1^{*,*}$ is a freely generated differential super-commutative algebra, with generating cocycles explicitly described in Theorem 5.1.5 as follows. If $r_1, \ldots, r_{l_0}$ is a list of exponents for $\mathfrak{g}_0$ then there is a generator in $E_1^{r_i+1, r_i+1}$ represented by a cocycle R_i. If $m_1^{(-a)}, \ldots, m_{l_{-a}}^{(-a)}$ is a list of twisted exponents then there is a generator in $E_1^{m_i^{(-a)}+1, m_i^{(-a)}+1}$ for every $n \geq 1$, represented by a cocycle $f_i^{nk-a} = [z^{nk-a}]\tilde{I}_i^{(-a)}$, and a generator in $E_1^{m_i^{(-a)}, m_i^{(-a)}+1}$ for every $n \geq 1$, represented by a cocycle $\omega_i^{nk-a} = J_\Delta d[z^{nk-a}]\tilde{I}_i^{(-a)}$. Since $d_R(P)$ is a derivation, we just need to determine its action on these generators. By degree considerations, $d_R(P)$ kills the generators R_i and f_i^{nk-a}. Note that $f_i^0 = [z^0]\tilde{I}_i^{(0)}$ lies in $E_1^{*,*}$, as it belongs to the algebra $\mathbb{C}[R]$ generated by the R_i's (apply Theorem 4.2.6 with $m = 0$). If the reductive algebra L splits as a direct sum $L = \mathfrak{z} \oplus \bigoplus L^{(i)}$, where $\mathfrak{z}$ is the centre and the $L^{(i)}$'s are σ-invariant simple components, then we can assume that the generators $I_i^{(-a)}$ of $(S^*L^*)^L$ used to construct the cochains f_i^{nk-a} belong either to $S^*\mathfrak{z}^*$ or to $\left(S^*\left(L^{(i)}\right)^*\right)^L$ for some i. With this assumption we have:

Lemma 5.3.3. *The differential $d_R(P)$ on $K^{*,*}$ sends ω_i^{nk-a} to a non-zero scalar multiple of f_i^{nk-a-N} if $nk - a \geq N$, and to zero otherwise.*

Proof. The generator ω_i^{nk-a} can be rewritten as $d_L(J)f_i^{nk-a}$. Both $d_R(P)$ and $d_L(J)$ preserve the subalgebras $(S^*(L^{(i)})^*)^L$ and $S^*\mathfrak{z}^*$, so we can assume that L is either simple or abelian. Since $d_R(P)f_i^{nk-a} = 0$, we can use Lemma 5.3.1 to get $d_R(P)\omega_i^{nk-a} = (PJ)^{Sym}f_i^{nk-a}$. As an element of the dual of $S^*\hat{\mathfrak{p}}$, $(PJ)^{Sym}f_i^{nk-a}$ is defined by

$$x_1 \circ \cdots \circ x_{m_i^{(-a)}+1} \mapsto \sum_j [z^{nk-a}] I_i^{(-a)}(\cdots \circ Jz^N x_j \circ \cdots).$$

Suppose L is abelian. Then, as noted after the statement of Theorem 3.1.3, we can assume that J is the identity, so $(PJ)^{Sym}f_i^{nk-a} = f_i^{nk-a-N}$ as required.

This leaves the case that L is simple, in which case J is defined as the derivation of $\hat{\mathfrak{p}}$ acting on weight spaces $\mathfrak{g}_\alpha$ as multiplication by $\langle \rho, \alpha \rangle$, where ρ is the weight of the associated Kac-Moody satisfying $\rho(\alpha_i^\vee) = 1$ if $d_i > 0$ in the grading of type d determining $\mathfrak{p}$, and $\rho(\alpha_i^\vee) = 0$ otherwise. Following the Kac convention in [Ka83], the Kac-Moody associated to $\mathfrak{g}$ is $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$, where c is central and d acts by $z\frac{d}{dz}$. The roots of $\tilde{\mathfrak{g}}$ belong to the dual of the Cartan $\mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$, and are defined similarly to the roots of $\mathfrak{g}$, with d^* replacing δ. If $\alpha_0 = d^* - \psi, \alpha_1, \ldots, \alpha_l$ is a list of simple roots for $\tilde{\mathfrak{g}}$, then the associated coroots are $\alpha_0^\vee = c^* - \psi_0, \alpha_1^\vee, \ldots, \alpha_l^\vee$, where ψ_0 is either $\psi^\vee$ in the untwisted case, or the element of $\mathfrak{h}_0$ such that $\langle x, \psi_0 \rangle = \psi(x)$ in the twisted case. The standard non-degenerate invariant form $\langle , \rangle$ for $\tilde{\mathfrak{g}}$ satisfies $\langle \mathfrak{h}_0, c \rangle = \langle \mathfrak{h}_0, d \rangle = \langle c, c \rangle = \langle d, d \rangle = 0$ and $\langle c, d \rangle \neq 0$.

Write $\rho = \rho_0 + Ac^*$ for some ρ_0 in $\mathfrak{h}_0^*$. If $x_j \in \mathfrak{g}_\alpha$ in the above equation then $z^N x_j \in \mathfrak{g}_{\alpha+Nd^*}$, so $Jz^N x_j = z^N(N\langle \rho, d^* \rangle + J)x_j$. Since $\langle \rho, d^* \rangle = A\langle c^*, d^* \rangle$, we have

$$d_R(P)\omega_i^{nk-a} = \begin{cases} \left(NA(m_i^{(-a)}+1)\langle c^*, d^* \rangle + J^{Sym} \right) f_i^{nk-a-N} & nk > N \\ 0 & nk \leq N \end{cases}.$$

Take a basis $\{x_{\alpha,i}\}$ for $\mathfrak{g}_\alpha$, and let x_α^i be the dual basis. Then $J^{Sym}x_\alpha^i = \langle \rho, \alpha \rangle x_\alpha^i = (\langle \rho_0, \alpha \rangle + A\langle c^*, \alpha \rangle)x_\alpha^i$. There is $\tilde{\rho}_0 \in \mathfrak{h}_0$ such that $\alpha(\tilde{\rho}_0) = \langle \rho_0, \alpha \rangle$ for all roots α, so $\mathrm{ad}^t(\tilde{\rho}_0)x_\alpha^i = -\langle \rho_0, \alpha \rangle x_\alpha^i$. Hence on the subring of $\mathfrak{h}_0$-invariant functions of $S^*\hat{\mathfrak{p}}^*$, J^{Sym} agrees with the derivation which sends x_α^i to $A\langle c^*, \alpha \rangle x_\alpha^i$. The product $\langle c^*, \alpha \rangle$ is equal to $\langle c^*, d^* \rangle$ times the z-degree of x_α^i. We conclude that $J^{Sym}f_i^{nk-a-N} = A\langle c^*, d^* \rangle(nk - a - N)f_i^{nk-a-N}$, and consequently that

$$d_R(P)\omega_i^{nk-a} = A\langle c^*, d^* \rangle \left(N(m_i^{(-a)}+1) + nk - a - N \right) f_i^{nk-a-N}$$

if $N \leq nk - a$. Since $nk - a - N \geq 0$, the coefficient is non-zero as required. □

We now have a situation parallel to when we defined $d_R(P)$. Let V_0 be the free vector space spanned by basis elements v_i^{nk+a} for $n \geq 0$, $a = 0, \ldots, k-1$, and $i = 1, \ldots, l_a$. For any integer m, let V_m be the subspace of V_0 spanned by the v_i^{nk+a}'s with $nk + a \geq m$. Identify $\bigwedge^* V_1 \otimes S^*V_0$ with a subalgebra of $E_1^{*,*}$ by sending v_i^{nk+a} to f_i^{nk+a} in the symmetric term and to ω_i^{nk+a} in the exterior term. Let P' be the linear map $V_0 \to V_0$ sending v_i^{nk+a} to v_i^{nk+a-N} if

$nk+a \geq N$, and to zero otherwise. Let Q' be the operator $V_0 \mapsto V_1$ sending $v_i^{nk+a} \mapsto v_i^{nk+a+N}$. Then $P'Q' = \mathbb{1}$, while $Q'P'$ is projection to V_N. So $d_R(P')d_L(Q') + d_L(Q')d_R(P')$ acts as multiplication by the combined symmetric degree and exterior V_N-degree. By Lemma 5.3.3, the differential on $E_1^{*,*}$ restricts to $d_R(P')$ on $\bigwedge^* V_1 \otimes S^*V_0$, and hence the homology of the differential on this subspace is the subalgebra Λ_1 of the E_2-term generated by ω_i^{nk+a}'s with $0 < nk + a < N$. To get the whole E_2-term, recall:

Lemma 5.3.4. *$(S^*\mathfrak{g}_0^*)^{\mathfrak{g}_0}$ is a free $(S^*L_0^*)^{L_0}$-module.*

Proof. Let $S = (S^*\mathfrak{g}_0^*)^{\mathfrak{g}_0}$ and $A = (S^*L_0^*)^{L_0}$. Restriction to the Cartan $\mathfrak{h}_0$ gives isomorphisms $S \cong (S^*\mathfrak{h}_0^*)^{W(\mathfrak{g}_0)}$ and $A \cong (S^*\mathfrak{h}_0^*)^{W(L_0)}$, so A is a subalgebra of S. By the Chevalley-Shephard-Todd theorem [Ch55] [Ko63b], there is a subset $H_0 \subset S^*\mathfrak{h}_0^*$ such that $S^*\mathfrak{h}_0^* \cong A \otimes H_0$ as a $W(L_0)$-module, where the isomorphism is given by multiplication, and H_0 is isomorphic to the regular representation. Hence $S \cong A \otimes H$ where $H = H_0^{W(\mathfrak{g}_0)}$. □

The algebra $\mathbb{C}[Q^\sigma]$ generated by the f_i^0's is a subalgebra of $\bigwedge^* V_1 \otimes S^*V_0$. Note that $d_R(P')$ is $\mathbb{C}[Q^\sigma]$-linear, since it kills $\mathbb{C}[Q^\sigma]$ and is a derivation. The E_1-term is isomorphic to the base extension $\mathbb{C}[R] \otimes_{\mathbb{C}[Q^\sigma]} \bigwedge^* V_1 \otimes S^*V$, with differential given by the base extension $\mathbb{1} \otimes d_R(P')$ of $d_R(P')$. Freeness implies that the E_2-term is $\mathbb{C}[R] \otimes_{\mathbb{C}[Q^\sigma]} \Lambda_1$. Since the action of $\mathbb{C}[Q^\sigma]$ on Λ_1 sends everything of symmetric degree > 0 to zero, the E_2-term is isomorphic to $\mathrm{Coinv}(L_0, \mathfrak{g}_0) \otimes \Lambda_1$.

Lemma 5.3.5. *The spectral sequence collapses at the E_2-term. Consequently the graded algebra of $H^*(\text{total}\, K^{*,*})$ with respect to the row-wise filtration is isomorphic to $\mathrm{Coinv}(L_0, \mathfrak{g}_0) \otimes \Lambda_1$.*

Proof. Λ_1 is a free algebra with a generator $\omega_i^{nk-a} \in E_2^{m_i^{(-a)}, m_i^{(-a)}+1}$ for every twisted exponent $m_i^{(-a)}$ of L and n such that $0 < nk-a < N$. The subring $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ lies in bidegrees (a, a), so the entire E_2-term is contained in bidegrees (a, b) with $a \leq b$. Suppose more generally that the E_r-term is contained in bidegrees (a, b) with $a \leq b$, and is generated in bidegrees $(a, a+1)$ and (a, a). The E_2-term differential has bidegree $(2, -1)$, and thus annihilates $\mathrm{Coinv}(L_0, \mathfrak{g}_0)$ and the generators ω_i^{nk-a}. The same argument works for higher E_r-terms as well. □

Now we just need to determine the ring structure of $H^*(\text{total}\, K^{*,*})$. The row-wise filtration of $K^{*,*}$ is the descending filtration where $F^P K^{*,*} = \bigoplus_{r \geq p} K^{*,r}$. Likewise $F^p H^*(\text{total}\, K^{*,*})$ is the subspace of homology classes which have a representative cocycle in $F^p K^{*,*}$. If $k \in K^{q,p}$ is such that $\bar{\partial} k = d_R(P)k = 0$, then k determines a homology class $[k]$ in $F^p H^{p+q}(\text{total}\, K^{*,*})$. Referring to the construction of the spectral sequence of a filtered differential module (see, e.g., pages 34-37 in [MC01]), we also see that k determines a persistent element of the spectral sequence, i.e. k represents an element in each $E_r^{p,q}$ (once again, note that the degrees are swapped between $K^{*,*}$ and $E^{*,*}$) that is killed by the rth differential, and the homology class of this element corresponds to the element represented by k in $E_{r+1}^{p,q}$. The projection $F^pH^{p+q}(\text{total}\, K^{*,*}) \to E_\infty^{p,q}$ sends $[k]$ to the element represented by k in E_∞. Finally, when $E_1^{p,q}$ is identified with $H^q(K^{*,p}, \bar{\partial})$ the element of E_1 represented by k is simply the homology class represented by k in $H^q(K^{*,p}, \bar{\partial})$, and consequently the same is true of the identification of E_2 with $H^*(H^*(K^{*,*}, \bar{\partial}), d_R(P))$. Note that this would not necessarily be true if k was not homogeneous.

We know that the E_2-term is generated by classes represented by elements R_i, $i = 1, \ldots, l_0$ and ω_i^{nk+a}, $i = 1, \ldots, l_a$ and $0 < nk + a < N$ in $K^{*,*}$. Let Λ denote the subalgebra of $K^{*,*}$ generated by the elements ω_i^{nk+a}, $i = 1, \ldots, l_a$, $0 < nk + a < N$. By Theorem 5.1.5 and Lemma 5.3.3, $\mathbb{C}[R] \otimes \Lambda \subset K^{*,*}$ is annihilated by both $\bar{\partial}$ and $d_R(P)$. Hence there is a homomorphism $\mathbb{C}[R] \otimes \Lambda \to H^*(\text{total}\, K^{*,*})$. Since $\bar{\partial}\omega_i^N = 0$, Lemma 5.3.3 implies that the image of f_i^0 in $H^*(\text{total}\, K^{*,*})$ is zero, so the homomorphism $\mathbb{C}[R] \otimes \Lambda \to H^*(\text{total}\, K^{*,*})$ descends to a map $\text{Coinv}(L_0, \mathfrak{g}_0) \otimes \Lambda \to H^*(\text{total}\, K^{*,*})$. By Lemma 5.3.5 and the argument of the last paragraph, this map is a bijection. We record this calculation in the following proposition.

Proposition 5.3.6. *Let* $\text{Coinv}(L_0, \mathfrak{g}_0)$ *denote the algebra of Lemma 5.3.4, graded by symmetric degree. Give* $\text{Coinv}(L_0, \mathfrak{g}_0)$ *a cohomological grading by doubling the symmetric grading, and a z-grading by multiplying the symmetric grading by N. Then $H^*(\mathfrak{p}/z^N\mathfrak{p}, \mathfrak{g}_0)$ is isomorphic to* $\text{Coinv}(L_0, \mathfrak{g}_0) \otimes \Lambda$*, where Λ is the free algebra generated in cohomological degree $2m_i^{(a)} + 1$, z-degree $Nm_i^{(a)} + nk + a$, for $a = 0, \ldots, k-1$, $i = 1, \ldots, l_a$, and n such that $0 < nk + a < N$.*

Consider the untwisted case where $n = 1$. In this case, $\mathfrak{p}/z\mathfrak{p}$ is the semi-direct product $\mathfrak{p}_0 \ltimes L_0/\mathfrak{p}_0$, where $L_0/\mathfrak{p}_0$ has Lie bracket equal to zero. Then Proposition 5.3.6 implies that $H^*(\mathfrak{p}_0 \ltimes L_0/\mathfrak{p}_0, \mathfrak{g}_0)$ is isomorphic to $\text{Coinv}(L_0, \mathfrak{g}_0)$. The following Lemma implies that Proposition 5.3.6 actually gives a direct Lie algebra proof of Borel's theorem that $\text{Coinv}(L_0, \mathfrak{g}_0)$ is isomorphic to $H^*(L_0, \mathfrak{g}_0)$. Note that the z-grading on $H^*(\mathfrak{p}_0 \ltimes L_0/\mathfrak{p}_0, \mathfrak{g}_0)$ is half the cohomological grading, and thus corresponds to the holomorphic grading on $H^*(L_0, \mathfrak{g}_0)$.

Lemma 5.3.7. *Let $\mathfrak{p}_0 \ltimes L_0/\mathfrak{p}_0$ be the semi-direct product where $L_0/\mathfrak{p}_0$ has Lie bracket equal to zero. The cohomology ring $H^*(\mathfrak{p}_0 \ltimes L_0/\mathfrak{p}_0, \mathfrak{g}_0)$ is isomorphic to $H^*(L_0, \mathfrak{g}_0)$.*

Proof. We use a standard Hodge theory argument. Let X be the generalized flag variety $G^\sigma/\mathcal{P}_0$, where $\mathcal{P}_0$ is the parabolic subgroup of G^σ corresponding to $\mathfrak{p}_0$. The complex-valued de Rham complex of X can be realized as the relative Koszul complex

$$C^*\left(L_0, \mathfrak{g}_0; C^\infty(K; \mathbb{C})\right) = \left(\bigwedge^*(L_0/\mathfrak{g}_0)^* \otimes C^\infty(K; \mathbb{C})\right)^{\mathfrak{g}_0},$$

where K is a compact form of X. The de Rham differential d translates to the Lie algebra cohomology boundary operator for $(L_0, \mathfrak{g}_0)$. Let $\mathfrak{u}_0$ be the nilpotent radical of $\mathfrak{p}_0$. The holomorphic structure on X gives the de Rham complex a bigrading, which can be written in terms of $C^*(L_0, \mathfrak{g}_0; C^\infty(K;\mathbb{C})$ as

$$C^{p,q}(C^\infty(K;\mathbb{C})) = \left(\bigwedge^p \overline{\mathfrak{u}_0}^* \otimes \bigwedge^q \mathfrak{u}_0^* \otimes C^\infty(K;\mathbb{C}) \right)^{\mathfrak{g}_0},$$

where p is the holomorphic degree, and q is the anti-holomorphic degree. The differential $d = \partial + \bar{\partial}$, where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic differentials respectively. On $C^{*,*}$, ∂ is the Lie algebra cohomology differential of $\overline{\mathfrak{u}_0}$ with coefficients in $\bigwedge^* \mathfrak{u}_0^* \otimes C^\infty(K;\mathbb{C})$, where $\mathfrak{u}_0$ is the $\overline{\mathfrak{u}_0}$-module $L_0/\overline{\mathfrak{p}_0}$. Similarly $\bar{\partial}$ is the Lie algebra cohomology differential of $\mathfrak{u}_0$ with coefficients in $\bigwedge^* \overline{\mathfrak{u}_0}^* \otimes C^\infty(K;\mathbb{C})$. The Kahler identities then imply that the Laplacian $dd^* + d^*d$ of d with respect to a Kahler metric is equal to twice the Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. In particular the two differentials give the same cohomology.

A theorem of Chevalley-Eilenberg implies that the de Rham complex is quasi-isomorphic to the subcomplex $C(L_0, \mathfrak{g}_0; \mathbb{C})$ of equivariant forms [CE48]. Since K acts by holomorphic maps on X, the same is true of the de Rham complex with the anti-holomorphic differential. Hence the Kahler identities imply that the cohomology of $C^*(L_0, \mathfrak{g}_0; \mathbb{C})$ is the same with respect to either d or $\bar{\partial}$. Finally $\left(C(L_0, \mathfrak{g}_0; \mathbb{C}), \bar{\partial}\right)$ can be identified with the Koszul complex for the Lie algebra cohomology of $H^*(\mathfrak{p}_0 \ltimes L_0/\mathfrak{p}_0, \mathfrak{g}_0)$. □

To finish the section, we observe that if $\mathfrak{p}$ is an Iwahori, then a similar spectral sequence calculation can be made with $S^*\hat{\mathfrak{p}}^*$ replaced by $S^*\hat{\mathfrak{u}}$. In this case the spectral sequence will converge to $H^*(\hat{\mathfrak{p}}/z^N\mathfrak{n}, \mathfrak{h}_0)$, while the E_1-term of the spectral sequence is the free super-commutative algebra $H^*_{cts}(\hat{\mathfrak{p}}, \mathfrak{h}_0; S^*\hat{\mathfrak{u}})$ generated by elements $f_i^{nk-a} \in E_1^{m_i^{(-a)}+1, m_i^{(-a)}+1}$ and $\omega_i^{nk-a} \in E^{m_i^{(-a)}, m_i^{(-a)}+1}$ for every $n \geq 1$, $a = 0, \ldots, k-1$, and $i = 1, \ldots, l_a$. The differential on the E_1-term sends ω_i^{nk-a} to f_i^{nk-a-N} if $nk - a > N$, and to zero otherwise. Thus the E_2-term will be the free algebra generated by the ω_i^{nk-a}'s with $0 < nk - a \leq N$. Since the algebra is free, the isomorphism on graded algebras lifts to give:

Proposition 5.3.8. *Let $\mathfrak{b}$ be an Iwahori subalgebra of $\mathfrak{g}$, and let $\mathfrak{n}$ be the nilpotent subalgebra. Then $H^*(\mathfrak{b}\,/z^N\mathfrak{n}, \mathfrak{h}_0)$ is a free algebra generated in cohomological degree $2m_i^{(a)} + 1$, z-degree $Nm_i^{(a)} + nk + a$, for $a = 0, \ldots, k-1$, $i = 1, \ldots, l_a$, and n such that $0 < nk + a \leq N$.*

Chapter 6

The Brylinski filtration

The point of this chapter is to define a Brylinski filtration for affine (i.e. indecomposable of affine type) Kac-Moody algebras. Throughout, $\mathfrak{g}$ will refer to a symmetrizable Kac-Moody algebra. For standard notation and terminology, we mostly follow [Ku02]. We assume a fixed presentation of $\mathfrak{g}$, from which we get a choice of Cartan $\mathfrak{h}$, simple roots $\{\alpha_i\}$, simple coroots $\{\alpha_i^\vee\}$, and Chevalley generators $\{e_i, f_i\}$. We can then grade $\mathfrak{g}$ via the principal grading, i.e. by assigning degree 1 to each e_i and degree -1 to each f_i. By choosing a real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$ we get an anti-linear Cartan involution $x \mapsto \overline{x}$, defined as the anti-linear involution sending $e_i \mapsto -f_i$ for all i and $h \mapsto -h$ for all $h \in \mathfrak{h}_\mathbb{R}$. As usual $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the standard nilpotent $\bigoplus_{n>0} \mathfrak{g}_n$. The standard Borel is the subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Associated to $\mathfrak{n}$ and $\mathfrak{b}$ are the pro-algebras $\hat{\mathfrak{n}} = \lim_{\leftarrow} \mathfrak{n}/\mathfrak{n}_k$ and $\hat{\mathfrak{b}} = \lim_{\leftarrow} \mathfrak{b}/\mathfrak{n}_k$, where $\mathfrak{n}_k = \bigoplus_{n>k} \mathfrak{g}_n$.

Recall from the introduction that the Kostant partition functions $K(\beta; q)$ are defined for weights β by

$$\sum_\beta K(\beta; q) e^\beta = \prod_{\alpha \in \Delta^+} (1 - qe^\alpha)^{-\operatorname{mult} \alpha},$$

where Δ^+ is the set of positive roots and $\operatorname{mult} \alpha = \dim \mathfrak{g}_\alpha$. The q-character of a weight space $\mathcal{L}(\lambda)_\mu$ is the function

$$m_\mu^\lambda(q) = \sum_{w \in W} \epsilon(w) K(w * \lambda - \mu; q), \tag{6.1}$$

where W is the Weyl group of $\mathfrak{g}$, ϵ is the usual sign representation of W, and $w * \lambda = w(\lambda + \rho) - \rho$ is the shifted action of W.

Let $\mathcal{L}(\lambda)$ denote the irreducible representation of $\mathfrak{g}$ of highest weight λ. We show that, as in the finite-dimensional case, there is a filtration on $\mathcal{L}(\lambda)_\mu$ such that when μ is dominant, $m_\mu^\lambda(q)$ is equal to the Poincare series of the associated graded space. Unlike the finite-dimensional case, the principal nilpotent is not sufficient to define the filtration in the affine case; instead, we use the positive part of the principal Heisenberg (this form of Brylinski's identity was first conjectured by Teleman). Brylinski's original proof of the identity $m_\mu^\lambda =$

${}^e P^\lambda_\mu$ uses a cohomology vanishing result for the flag variety. Our proof is based on the same idea, but uses the Lie algebra cohomology approach of [FGT08]. In particular we prove a vanishing result for Lie algebra cohomology by calculating the Laplacian with respect to a Kahler metric. Although we concentrate on the affine case for simplicity, our results generalize easily to the case when $\mathfrak{g}$ is a direct sum of algebras of finite or affine type. There are two difficulties in extending this result to indefinite symmetrizable Kac-Moody algebras: there does not seem to be a simple analogue of the Brylinski filtration, and the cohomology vanishing result does not extend for all dominant weights μ. We can overcome these difficulties by replacing the Brylinski filtration with an intermediate filtration, and by requiring that the root $\lambda - \mu$ has affine support. Thus we get some partial non-negativity results for the coefficients of $m^\lambda_\mu(q)$ even when $\mathfrak{g}$ is of indefinite type.

This chapter is adapted from [Sl11a].

6.1 The Brylinski filtration for affine Kac-Moody algebras

A principal nilpotent (with respect to a given presentation) of a symmetrizable Kac-Moody algebra is an element $e \in \mathfrak{g}_1$ of the form $e = \sum c_i e_i$, where $c_i \in \mathbb{C} \setminus \{0\}$ for all simple roots e_i. If $\mathfrak{g}$ is affine it is well known that the algebras $\mathfrak{s}_e = \{x \in \mathfrak{g} : [x, e] \in Z(\mathfrak{g})\}$ are Heisenberg algebras, and these algebras are called principal Heisenberg subalgebras.

Definition 6.1.1. *Let $\mathcal{L}(\lambda)$ be a highest-weight module of an affine Kac-Moody algebra $\mathfrak{g}$. Define the* Brylinski filtration *with respect to the principal Heisenberg $\mathfrak{s}$ by*

$$ {}^{\mathfrak{s}}F^i \mathcal{L}(\lambda)_\mu = \{v \in \mathcal{L}(\lambda)_\mu : x^{i+1} v = 0 \text{ for all } x \in \mathfrak{s} \cap \mathfrak{n}\}. $$

Let

$$ {}^{\mathfrak{s}}P^\lambda_\mu(q) = \sum_{i \geq 0} q^i \dim {}^{\mathfrak{s}}F^i \mathcal{L}(\lambda)_\mu \;/\; {}^{\mathfrak{s}}F^{i-1} \mathcal{L}(\lambda)_\mu. $$

be the Poincare series of the associated graded space of $\mathcal{L}(\lambda)_\mu$.

Note that the principal nilpotents form a single H-orbit, so the filtration ${}^{\mathfrak{s}}F^*$ is independent of the choice of principal Heisenberg.

Recall that a weight μ is real-valued if $\mu(h) \in \mathbb{R}$ for all $h \in \mathfrak{h}_\mathbb{R}$, and dominant if $\mu(\alpha_i^\vee) \geq 0$ for all simple coroots $\alpha_i^\vee$.

Theorem 6.1.2. *Let $\mathcal{L}(\lambda)$ be an integrable highest weight representation of an affine Kac-Moody algebra $\mathfrak{g}$, where λ is a real-valued dominant weight. If μ is a dominant weight of $\mathcal{L}(\lambda)$ then ${}^{\mathfrak{s}}P^\lambda_\mu(q) = m^\lambda_\mu(q)$.*

The dual $\hat{\mathfrak{n}}^*$ of a pro-algebra will refer to the continuous dual. If V is a $\hat{\mathfrak{b}}$-module then $H^*_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V)$ will denote the relative continuous cohomology of $(\hat{\mathfrak{b}}, \mathfrak{h})$. The proof of Theorem 6.1.2 depends on

Theorem 6.1.3. *Let $\mathcal{L}(\lambda)$ be an integrable highest weight representation of an affine Kac-Moody algebra $\mathfrak{g}$, where λ is a real-valued dominant weight. Let $V = \mathcal{L}(\lambda) \otimes S^*\hat{\mathfrak{n}}^* \otimes \mathbb{C}_{-\mu}$, where μ is a dominant weight of $\mathcal{L}(\lambda)$. Then $H^d_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V) = 0$ for $d > 0$, and in addition there is a graded isomorphism $\operatorname{gr} \mathcal{L}(\lambda)_\mu \cong H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V)$, where the latter space is graded by symmetric degree.*

Proof of Theorem 6.1.2 from Theorem 6.1.3. Let $V^p = \mathcal{L}(\lambda) \otimes S^p\hat{\mathfrak{n}}^* \otimes \mathbb{C}_{-\mu}$. By Theorem 6.1.3, $P^\lambda_\mu(q) = \sum_{p\geq 0} \dim H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V^p)q^p = \sum \chi(\hat{\mathfrak{b}}, \mathfrak{h}; V^p)q^p$, where χ is the Euler characteristic (the second equality follows from cohomology vanishing). Since $\hat{\mathfrak{n}}^*$ has finite-dimensional weight spaces and all weights belong to the negative root cone, $\bigwedge^* \hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes S^p\hat{\mathfrak{n}}^*$ has finite-dimensional weight spaces. Thus we can write

$$\begin{aligned}\sum_{p\geq 0} \chi(\hat{\mathfrak{b}}, \mathfrak{h}; V^p)q^p &= \sum_{p,k\geq 0} (-1)^k q^p \dim \left(\bigwedge^k \hat{\mathfrak{n}}^* \otimes V^p \right)^{\mathfrak{h}} \\ &= [e^\mu] \operatorname{ch} \mathcal{L}(\lambda) \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{\operatorname{mult}\alpha}(1 - qe^{-\alpha})^{-\operatorname{mult}\alpha}.\end{aligned}$$

Applying the Weyl-Kac character formula

$$\operatorname{ch} \mathcal{L}(\lambda) = \sum_{w\in W} \epsilon(w)e^{w*\lambda} \cdot \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{-\operatorname{mult}\alpha}$$

we get the result.

□

The proof of Theorem 6.1.3 will be given in Sections 6.3 and 6.4. If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is a direct sum of indecomposables of finite and affine type, the conclusions of Theorems 6.1.2 and 6.1.3 remain true with $\mathfrak{s}$ replaced by a direct sum of principal nilpotents (for the finite components) and principal Heisenbergs (for the affine components).

6.2 Examples

In the finite-dimensional case, the Brylinski filtration is defined to be the increasing filtration ${}^eF^*$, where

$${}^eF^i(\mathcal{L}(\lambda)_\mu) = \{v \in \mathcal{L}(\lambda)_\mu : e^{i+1}v = 0\},$$

for e a principal nilpotent. This definition makes sense for an arbitrary Kac-Moody algebra. Let ${}^eP^\lambda_\mu(q)$ be the Poincare series of the associated graded space of $\mathcal{L}(\lambda)_\mu$.

We now give some elementary examples to show that ${}^\mathfrak{s}F$ is different from ${}^eF$. Consider $\widehat{\mathfrak{sl}_2}$, the affine Kac-Moody algebra realized as $\mathfrak{sl}_2[z^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where c is a central element, and d is the derivation $\frac{\partial}{\partial z}$. Let $\{H, E, F\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_2$, and take principal nilpotent

$e = E + Fz$. The principal Heisenberg $\mathfrak{s}$ is spanned by the elements ez^n, $n \in \mathbb{Z}$, along with c.

The Cartan subalgebra of $\widehat{\mathfrak{sl}_2}$ is span$\{H, c, d\}$. Denote a weight $\alpha H^* + hc^* + nd^*$ by (α, h, n). The weight $\lambda = (\alpha, h, n)$ is dominant if $0 \leq \alpha \leq h$, and the corresponding irreducible highest-weight representation $L(\lambda)$ can be realized as the quotient of the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ by the $U(\mathfrak{g})$-submodule generated by $F^{\alpha+1} \otimes 1$ and $(Ez^{-1})^{h-\alpha+1} \otimes 1$. Let

$$w = (Fz^{-1})(Ez^{-1})v,$$

where v is the highest weight vector in $L(c^*)$. Note that w is a weight vector of weight $(0, 1, -2)$. It is easy to check, using the defining relations for $L(c^*)$, that $e^2 w = 0$, while $(ez)ew = 3v$, so $w \in {}^eF^2$ but is not in ${}^{\mathfrak{s}}F^2$.

The same idea can be used to calculate Poincare series. For the above example, where $\lambda = (0, 1, 0)$ and $\mu = (0, 1, -2)$, we have $\dim \mathcal{L}(\lambda)_\mu = 2$. The Poincare series for ${}^eF$ is $q + q^4$, while the Poincare series for ${}^{\mathfrak{s}}F$ is $m^\lambda_\mu(q) = q^2 + q^4$. For an example with a dominant regular weight, let $\lambda = (0, 3, 0)$ and $\mu = (2, 3, -3)$. The Poincare series of ${}^eF$ is $q + 2q^2 + q^3 + q^5$, while $m^\lambda_\mu(q) = q + q^2 + 2q^3 + q^5$.

6.3 Reduction to cohomology vanishing

In this section we introduce an equivalent filtration to the Brylinski filtration, which will allow us to reduce Theorem 6.1.3 to a cohomology vanishing statement. The line of argument is inspired by [Bry89] and [FGT08]. As usual, $\mathfrak{g}$ will be an arbitrary symmetrizable Kac-Moody algebra except where stated.

Associated to $\mathfrak{g}$ is a Kac-Moody group $\mathcal{G}$. The standard Borel subgroup $\mathcal{B}$ of $\mathcal{G}$ is a solvable pro-group with Lie algebra $\hat{\mathfrak{b}}$. The standard unipotent subgroup $\mathcal{U} \subset \mathcal{B}$ is a unipotent pro-group with Lie algebra $\hat{\mathfrak{n}}$. The Borel $\mathcal{B}$ also contains a torus H corresponding to $\mathfrak{h}$. Defining the new filtration requires two lemmas.

Lemma 6.3.1. *There are algebraic isomorphisms $\mathcal{U} \cong \mathcal{B}/H \cong \hat{\mathfrak{n}}$ giving $\mathcal{U}$ the structure of a linear space with an affine $\mathcal{B}$-action.*

Proof. Note that the spaces in question can be naturally expressed as inverse limits of affine schemes, and hence are affine schemes in their own right. Pick $\delta \in \mathfrak{h}$ acting on $\mathfrak{g}_n$ as multiplication by n, and define $\pi : \mathcal{B} \to \hat{\mathfrak{n}}$ by $\mathrm{Ad}(b)\delta = \delta + \pi(b)$. Then the composition $\mathcal{U} \hookrightarrow \mathcal{B} \to \mathcal{B}/H \to \hat{\mathfrak{n}}$ is an isomorphism. $\hat{\mathfrak{n}}$ has a linear structure, while $\mathcal{B}/H$ has a left-translation action of $\mathcal{B}$. If $b_1, b_2 \in \mathcal{B}$ then $\mathrm{Ad}(b_1 b_2)\delta = \mathrm{Ad}(b_1)(\delta + \pi(b_2)) = \delta + \pi(b_1) + \mathrm{Ad}(b_1)\pi(b_2)$, so $\pi(b_1 b_2) = \mathrm{Ad}(b_1)\pi(b_2) + \pi(b_1)$ and the resulting action of $\mathcal{B}$ on $\hat{\mathfrak{n}}$ is affine. □

Lemma 6.3.2. *Let V be a pro-representation of $\mathcal{B}$. Then evaluation at the identity gives an isomorphism $(V \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}} \to V^H$.*

Proof. Any element $v \in V^H$ extends to a $\mathcal{B}$-invariant function $\mathcal{U} \to V$ by $[b] \mapsto bv$. □

The linear structure on $\hat{\mathfrak{n}}$ and the isomorphism of $\mathcal{U}$ with $\hat{\mathfrak{n}}$ gives a $\mathcal{B}$-stable filtration of $\mathbb{C}[\mathcal{U}]$ by polynomial degree. Lemma 6.3.2 implies that if V is a pro-representation of $\mathcal{B}$ then V^H can be filtered via polynomial degree on $\mathbb{C}[\mathcal{U}]$. If μ is a weight of $\mathfrak{g}$ then extending μ by zero on $\mathcal{U}$ makes $\mathbb{C}_{-\mu}$ into a pro-representation of $\mathcal{B}$. The reason for introducing a new filtration is the following lemma, which reduces the proof of Theorem 6.1.3 to a vanishing result.

Lemma 6.3.3. *Let $W = \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}$, and filter $\mathcal{L}(\lambda)_\mu = W^H$ via the isomorphism $W^H \cong (W \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$. If $H^1_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^*\hat{\mathfrak{n}}^*) = 0$ then $H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^*\hat{\mathfrak{n}}^*) \cong \operatorname{gr} \mathcal{L}(\lambda)_\mu$.*

Proof. Let $\mathcal{F}^p$ be the subset of $\mathbb{C}[\mathcal{U}]$ of polynomials of degree at most p. Then $\operatorname{gr} \mathbb{C}[\mathcal{U}] = S^*\hat{\mathfrak{n}}^*$ as $\mathcal{B}$-modules, so there are short exact sequences

$$0 \to W \otimes \mathcal{F}^{p-1} \to W \otimes \mathcal{F}^p \to W \otimes S^p\hat{\mathfrak{n}}^* \to 0$$

of $\mathcal{B}$-modules for all p. The corresponding long exact sequence in Lie algebra cohomology is

$$\begin{aligned} H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^{p-1}) \to H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^p) \to & H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^p\hat{\mathfrak{n}}^*) \\ & \to H^{i+1}_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^{p-1}). \end{aligned}$$

Since $H^i_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^p\hat{\mathfrak{n}}^*) = 0$ for $i = 1$, the inclusion $W \otimes \mathcal{F}^{p-1} \hookrightarrow W \otimes \mathcal{F}^p$ induces a surjection in degree one cohomology for all p. Since $\mathcal{F}^{-1} = 0$, $H^1_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes \mathcal{F}^p) = 0$ for all p. The long exact sequence in degree $i = 0$ gives an isomorphism $H^0_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; W \otimes S^p\hat{\mathfrak{n}}^*) \cong (W \otimes \mathcal{F}^p)^{\mathfrak{b}} / (W \otimes \mathcal{F}^{p-1})^{\mathfrak{b}}$. This latter quotient is the graded space of $(W \otimes \mathbb{C}[U])^{\mathfrak{b}}$ as required. □

Now we show that the new filtration is equal to the Brylinski filtration when $\mathfrak{g}$ is affine.

Proposition 6.3.4. *Let $\mathcal{L}(\lambda)$ be an integrable highest-weight representation of an affine Kac-Moody $\mathfrak{g}$. Then the Brylinski filtration on a weight space $\mathcal{L}(\lambda)_\mu$ agrees with the filtration of $\mathcal{L}(\lambda)_\mu \cong (\mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu} \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$ by polynomial degree.*

The proof of Proposition 6.3.4 requires two lemmas.

Lemma 6.3.5. *If $\mathfrak{g}$ is affine and $\mathfrak{s}$ is a principal Heisenberg then* $\operatorname{Ad}(\mathcal{B})(\mathfrak{s} \cap \mathfrak{n})$ *is dense in $\hat{\mathfrak{n}}$.*

Proof. The principal nilpotents form a single orbit, so it is only necessary to prove this fact for a single principal nilpotent. We claim that there is a principal nilpotent such that $f = -\overline{e} \in \mathfrak{s}_e$, so that in particular $[e, f] \in Z(\mathfrak{g})$. Indeed, let A be the generalized Cartan matrix defining $\mathfrak{g}$, i.e. $A_{ij} = \alpha_j(\alpha_i^\vee)$. Since $\mathfrak{g}$ is affine there is a vector $c > 0$, unique up to a scalar multiple, such that $A^t c = 0$. If we pick $e = \sum \sqrt{c_i} e_i$ then $[e, f] = \sum c_i \alpha_i^\vee$, and $\alpha_j([e, f]) = \sum c_i A_{ij} = (A^t c)_j = 0$ for all simple roots α_j.

Now we show that $\mathfrak{n} = (\mathfrak{s}_e \cap \mathfrak{n}) + [\mathfrak{b}, e]$. In degree one we have $[\mathfrak{h}, e] = \mathfrak{g}_1$. For higher degrees, let $\{,\}$ denote the standard non-degenerate contragradient Hermitian form on $\mathfrak{g}$ which is positive definite on $\mathfrak{n}$. An element $x \in \mathfrak{n}$ is orthogonal to $[\mathfrak{b}, e]$ if and only if $0 = \{[e, z], x\} = \{z, [f, x]\}$ for all $z \in \mathfrak{b}$, or in other words if and only if $x \in C_{\mathfrak{g}}(f)$. Suppose $x \in \mathfrak{g}_n$, $n \geq 2$ belongs to $[\mathfrak{b}, e]^\perp$. Using the fact that $[e, f] \in Z(\mathfrak{g})$ we get that $\{[e, x], [e, x]\} = \{[f, x], [f, x]\} = 0$, and conclude that $x \in \mathfrak{s}_e$.

$(\mathfrak{s} \cap \mathfrak{n}) + [\mathfrak{b}, e] = \mathfrak{n}$ implies that the multiplication map $\mathcal{B} \times (\mathfrak{s} \cap \mathfrak{n}) \to \hat{\mathfrak{n}}$ is a submersion in a neighbourhood of $(\mathbb{1}, e)$. Since $\mathcal{B}$ acts algebraically on $\mathfrak{s} \cap \mathfrak{n} \subset \hat{\mathfrak{b}}$, the subset $\mathcal{B}(\mathfrak{s} \cap \mathfrak{n})$ is dense in $\hat{\mathfrak{n}}$. □

Lemma 6.3.6. *Let $\mathcal{L}(\lambda)$ be an integrable highest-weight module. Considered as a $\mathcal{B}$-module, $\mathcal{L}(\lambda)$ is a submodule of $\mathbb{C}[\mathcal{U}] \otimes \mathbb{C}_\lambda$.*

Proof. This statement would follow immediately from a Borel-Weil theorem for the thick flag variety of a Kac-Moody group. As we are not aware of a formal statement of the Borel-Weil theorem in this context, we recover the result from the dual of the quotient map $M_{low}(-\lambda) \to \mathcal{L}_{low}(-\lambda)$, where $M_{low}(-\lambda) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} \mathbb{C}_{-\lambda}$ is a lowest weight Verma module, and $\mathcal{L}_{low}(-\lambda)$ is the irreducible representation with lowest weight $-\lambda$. Both these spaces are $\mathfrak{g}$-modules with finite gradings induced by the principal grading of $\mathfrak{g}$. Let $M_{low}(-\lambda)^*$ and $\mathcal{L}(-\lambda)^*$ denote the finitely-supported duals, consisting of linear functions which are supported on a finite number of graded components.

Using the fact that $M_{low}(-\lambda)$ is a free $U(\mathfrak{n})$-module, we can identity $M_{low}(-\lambda)$ with $S^*\mathfrak{n} \otimes \mathbb{C}_{-\lambda}$ where $S^*\mathfrak{n}$ has the $\mathfrak{b}$-action $(y, x) \mapsto [y, \delta] \circ x + \mathrm{ad}(y)x$, the symbol $\circ$ denotes symmetric multiplication, and δ is defined as in Lemma 6.3.1 as an element of $\mathfrak{h}$ which acts on $\mathfrak{g}_n$ as multiplication by n. The finitely supported dual of $M_{low}(-\lambda)$ can be identified with $S^*\hat{\mathfrak{n}}^* \otimes \mathbb{C}_\lambda$ where $\mathfrak{b}$ acts on $S^*\hat{\mathfrak{n}}^*$ by $(y, f) \mapsto \mathrm{ad}^t(y)f + \iota([\delta, y])f$. It is not hard to check that this action integrates to the $\mathcal{B}$-action coming from identifying $S^*\hat{\mathfrak{n}}^*$ with $\mathbb{C}[\mathcal{U}]$. Since the quotient map preserves the principal grading, the dual of the surjection $M_{low}(-\lambda) \to \mathcal{L}_{low}(-\lambda)$ is an inclusion $\mathcal{L}(\lambda) = \mathcal{L}_{low}(-\lambda)^* \hookrightarrow M_{low}(-\lambda)^* = \mathbb{C}[\mathcal{U}] \otimes \mathbb{C}_\lambda$ as required. □

Proof of Proposition 6.3.4. Let $V = \mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}]$, where $\beta = \lambda - \mu$. By the last lemma, we can prove the proposition with $\mathcal{L}(\lambda)_\mu$ replaced by V^H, where the filtration on V^H is defined by $V^H \cong (V \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$. An element f of this latter set can be identified with a $\mathcal{B}$-invariant function $\mathcal{U} \times \mathcal{U} \to \mathbb{C}_\beta$. The polynomial degree on the second factor is the maximum t-degree of $f(u, tx)$ as u ranges across $\mathcal{U}$ and x ranges across $\hat{\mathfrak{n}} \cong \mathcal{U}$. Suppose this maximum is achieved at (u_0, x_0). Since $\mathcal{B}(\mathfrak{s} \cap \mathfrak{n})$ is dense in $\hat{\mathfrak{n}}$, we can assume that $x_0 = \mathrm{Ad}(b)s$ for $b \in \mathcal{B}$ and $s \in \mathfrak{s} \cap \mathfrak{n}$. Now $\mathfrak{s} \cap \mathfrak{n}$ is abelian and graded, so the graded components of s commute with each other. This allows us to find $\tilde{s} \in \mathfrak{s} \cap \mathfrak{n}$ such that $\pi(e^{t\tilde{s}}) = ts$. Since the degree of $f(u_0, \cdot)$ is achieved on the line $\mathrm{Ad}(b)\pi(e^{t\tilde{s}})$, it is also achieved on the parallel line $\mathrm{Ad}(b)\pi(e^{t\tilde{s}}) + \pi(b) = \pi(be^{t\tilde{s}})$. Thus the polynomial degree of f is equal to the t-degree of $f(u_0, b\pi(e^{t\tilde{s}})) = \beta(b)f(b^{-1}u_0, \pi(e^{t\tilde{s}}))$. Since $\beta(b)$ is a non-zero scalar, we conclude that there is $u \in \mathcal{U}$ and $s \in \mathfrak{s} \cap \mathfrak{n}$ such that the degree of f is equal to the t-degree of $f(u, \pi(e^{ts}))$.

Conversely if $s \in \mathfrak{s} \cap \mathfrak{n}$ then $\pi(e^{ts})$ is a line in $\hat{\mathfrak{n}}$, so the degree of f is equal to the t-degree of $f(u, \pi(e^{ts}))$ as u ranges across $\mathcal{U}$ and s ranges across $\mathfrak{s} \cap \mathfrak{n}$.

Given $f \in (\mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}] \otimes \mathbb{C}[\mathcal{U}])$ let $\tilde{f} \in \mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}]$ be the restriction to $\mathcal{U} \times \{\mathbb{1}\}$. The $\mathcal{B}$-action on $\mathbb{C}_\beta \otimes \mathbb{C}[\mathcal{U}]$ is defined by $(b \cdot f)(u) = \beta(b) f(b^{-1}u)$, so if f is $\mathcal{B}$-invariant then the t-degree of $f(u, \pi(e^{ts}))$ is equal to the t-degree of $(e^{ts}\tilde{f})(u)$. Since

$$e^{ts}\tilde{f} = \sum_{n \geq 0} \frac{t^n}{n!} s^n \tilde{f},$$

the degree of f is equal to the smallest n such that $s^{n+1}\tilde{f} = 0$ for all $s \in \mathfrak{s} \cap \mathfrak{n}$. □

The proof of Proposition 6.3.4 works just as well with $\mathfrak{s} \cap \mathfrak{n}$ replaced by any graded abelian subalgebra $\mathfrak{a}$ of $\hat{\mathfrak{n}}$ such that $\operatorname{Ad}(\mathcal{B})\,\mathfrak{a}$ is dense in $\hat{\mathfrak{n}}$. For example, in the finite-dimensional case we could take $\mathfrak{a} = \mathbb{C}e$. If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is a direct sum of indecomposables of finite or affine type then we can take $\mathfrak{a} = \bigoplus \mathfrak{a}_i$, where $\mathfrak{a}_i$ is either the positive part of the principal Heisenberg, or the line through the positive nilpotent, depending on whether $\mathfrak{g}_i$ is affine or finite.

6.4 Cohomology vanishing

Throughout this section $\mathfrak{g}$ will be an arbitrary symmetrizable Kac-Moody algebra. (V, π) will be a $\hat{\mathfrak{b}}$-module such that $\pi|_{\mathfrak{g}_0}$ extends to an action of $\overline{\mathfrak{b}}$ (this conjugate action will also be denoted by π). Note that since $\mathfrak{n} = \mathfrak{g}/\overline{\mathfrak{b}}$, $\hat{\mathfrak{n}}^*$ is both a $\hat{\mathfrak{b}}$-module and a $\overline{\mathfrak{b}}$-module. The space $\overline{\mathfrak{n}} = \mathfrak{g}/\,\mathfrak{b}$ has the same property. Recall that the semi-infinite chain complex $(C^{*,*}(V), \bar{\partial}, D)$ is the bicomplex

$$C^{-a,b}(V) = \left(\bigwedge^b \hat{\mathfrak{n}}^* \otimes \bigwedge^a \overline{\mathfrak{n}} \otimes V \right)^{\mathfrak{g}_0}.$$

with differentials $\bar{\partial}$ and D, where the former is the Lie algebra cohomology differential of $\hat{\mathfrak{n}}$ with coefficients in $\bigwedge^* \overline{\mathfrak{n}} \otimes V$, and the latter is the Lie algebra homology differential of $\overline{\mathfrak{n}}$ with coefficients in $\bigwedge^* \hat{\mathfrak{n}}^* \otimes V$, both restricted to $\mathfrak{g}_0$-invariants.

The semi-infinite cocycle is defined by $\gamma|_{\mathfrak{g}_m \times \mathfrak{g}_n} = 0$ if $m + n \neq 0$ and by

$$\gamma(x, y) = \sum_{0 \leq n < k} \operatorname{tr}_{\mathfrak{g}_n}(\operatorname{ad}(x)\operatorname{ad}(y))$$

for $x \in \mathfrak{g}_k$, $y \in \mathfrak{g}_{-k}$, $k \geq 0$. Since $\mathfrak{h} = \mathfrak{g}_0$ is abelian, $(x, y) = -\gamma(x, \overline{y})$ defines a Hermitian form on $\mathfrak{n}$.

Lemma 6.4.1. *Let $\langle , \rangle$ be a symmetric invariant form on $\mathfrak{g}$ (real-valued on a real-form of $\mathfrak{g}$) such that $\{\cdot,\cdot\} = -\langle \cdot, \bar{\cdot} \rangle$ is contragradient and positive-definite on $\mathfrak{n}$. Then the Hermitian form $(\cdot,\cdot) = -\gamma(\cdot, \bar{\cdot})$ on $\mathfrak{n}$ agrees with the form defined by*

$$(x, y) = 2\langle \rho, \alpha \rangle \{x, y\}, x \in \mathfrak{g}_\alpha.$$

Proof. Suppose $x, y \in \mathfrak{g}_\alpha$. If $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{h}$ with respect to $\langle,\rangle$ then

$$\begin{aligned}\operatorname{tr}_{\mathfrak{g}_0}(\operatorname{ad}(x)\operatorname{ad}(\overline{y})) &= \sum_i \langle u_i, [x, [\overline{y}, u^i]]\rangle \\ &= \langle x, \overline{y}\rangle\langle\alpha,\alpha\rangle.\end{aligned}$$

Next, let $\{e^i_\beta\}$ and $\{e^i_{-\beta}\}$ be dual bases of $\mathfrak{g}_\beta$ and $\mathfrak{g}_{-\beta}$ with respect to $\langle,\rangle$. Let $\rho \in \mathfrak{h}^*$ be such that $\rho(\alpha_i^\vee) = 1$ for all coroots $\alpha_i^\vee$. Then

$$\gamma(x,y) = \langle x,\overline{y}\rangle\langle\alpha,\alpha\rangle + \sum_{\beta\in\Delta^+}\sum_i \langle e^i_{-\beta}, [x, [\overline{y}, e^i_\beta]_-]\rangle,$$

where x_- is the projection of $x \in \mathfrak{g}$ to $\overline{\mathfrak{n}}$ using the triangular decomposition. Rearranging $\langle e^i_{-\beta}, [x, [\overline{y}, e^i_\beta]_-]\rangle = \langle x, [e^i_{-\beta}, [e^i_\beta, \overline{y}]_-]\rangle$ and applying Lemma 2.3.11 of [Ku02], we get that $\gamma(x,\overline{y}) = 2\langle\rho,\alpha\rangle\langle x,\overline{y}\rangle$. □

The result of Lemma 6.4.1 is that $(,)$ defines a $\mathfrak{g}_0$-contragradient Kahler metric on $\mathfrak{n}$. Suppose V has a positive-definite Hermitian form contragradient with respect to π. Using the Kahler metric on $\mathfrak{n}$, we can give $C^{*,*}(V)$ a positive-definite Hermitian form by defining $(\overline{x},\overline{y}) = \overline{(x,y)}$ for $x,y\in\mathfrak{n}$. Let $\overline{\Box} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ be the $\bar\partial$-Laplacian, and $\Box = DD^* + D^*D$ be the D-Laplacian. Recall that Nakano's identity states that the $\bar\partial$-Laplacian $\overline{\Box}$ and the D-Laplacian $\Box$ are related by

$$\overline{\Box} = \Box + \deg + \operatorname{Curv},$$

where deg acts on $C^{a,b}(V)$ as multiplication by $a+b$, and

$$\operatorname{Curv} = -\sum_{i,j\geq 1}\epsilon(z^i)\iota(z_j)\left([\pi(z_i),\pi(z_{-j})] - \pi([z_i,z_{-j}])\right),$$

on $C^{0,b}(V)$ for $\{z_i\}$ a homogeneous basis of $\mathfrak{n}$ orthonormal in $(,)$.

6.4.1 Laplacian calculation

Given an operator T on $\hat{\mathfrak{n}}^*$, let $d_R(T)$ and $d_L(T)$ denote the operators on $\bigwedge^*\hat{\mathfrak{n}}^*\otimes S^*\hat{\mathfrak{n}}^*$ defined by

$$\alpha_1\wedge\ldots\wedge\alpha_k\otimes\beta \mapsto \sum_{i=1}^k (-1)^i\alpha_1\wedge\ldots\check{\alpha}_i\ldots\wedge\alpha_k\otimes T(\alpha_i)\circ\beta$$

and

$$\alpha\otimes\beta_1\circ\ldots\circ\beta_l \mapsto \sum_{i=1}^l T(\beta_i)\wedge\alpha\otimes\beta_1\circ\ldots\circ\check{\beta}_i\circ\ldots\circ\beta_l$$

respectively. Define an operator J on $\hat{\mathfrak{n}}^*$ by $f\mapsto f/2\langle\rho,\alpha\rangle$ if $f\in\mathfrak{g}^*_\alpha$. As in the last section, let $\langle,\rangle$ be a real-valued symmetric invariant bilinear form such that $\{,\} = -\langle\cdot,\bar{\cdot}\rangle$ is contragradient and positive-definite on $\mathfrak{n}$.

Proposition 6.4.2. *Extend the contragradient Hermitian form* $\{,\}$ *on* $\mathfrak{n}$ *to* $V = S^*\hat{\mathfrak{n}}^*$. *On* $C^{0,b}(V)$,

$$\mathrm{Curv}_V = \sum_{s\geq 0} d_L(\mathrm{ad}^t(y'_s))d_R(\mathrm{ad}^t(y_s)J) - \mathrm{deg},$$

where $\{y_s\}$ *is a homogeneous basis for* $\mathfrak{b}$ *and* $\{y'_s\}$ *is a basis for* $\overline{\mathfrak{b}}$ *dual with respect to* $\langle,\rangle$.

Proof. Let $V' = S^*\overline{\mathfrak{n}}$, and let π denote the actions of $\mathfrak{b}$ and $\overline{\mathfrak{b}}$ on V'. From Proposition 2.5.4 we see that $\mathrm{Curv}_{V'}$ is a second-order differential operator, and thus is determined by its action on $\hat{\mathfrak{n}}^* \otimes \overline{\mathfrak{n}}$. We claim that if $f \in \hat{\mathfrak{n}}^*$ and $w \in \overline{\mathfrak{n}}$ then

$$\mathrm{Curv}_{V'}(f \otimes w) = \sum_{s\geq 0} \mathrm{ad}^t_{\mathfrak{n}}(w)y^s \otimes \mathrm{ad}_{\overline{\mathfrak{n}}}(y_s)\phi^{-1}(f),$$

where $\phi : \overline{\mathfrak{n}} \to \hat{\mathfrak{n}}^*$ is the isomorphism induced by the Kahler metric, and $\{y_s\}$ is any homogeneous basis of $\mathfrak{b}$. To prove this claim, let $\{z_i\}$ be orthonormal with respect to the Kahler metric, and think about $f = z^k$, $w = z_{-l}$. Observe that

$$\pi(z)w = \sum_{i<0} z^i([z,w])z_i.$$

Using this expression, we get that if $z_{-j} \in \mathfrak{g}_{-m}$ then

$$([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}]))w = \sum_{-m\leq n<0}\ \sum_{z_{-k}\in\mathfrak{g}_n} z^{-k}([z_{-j}, [z_i, w]])z_{-k}.$$

We can then remove the reference to m and write

$$([\pi(z_i), \pi(z_{-j})] - \pi([z_i, z_{-j}]))w = \sum_{k>0}\sum_{s\geq 0} z^{-k}([z_{-j}, y_s])y^s([z_i, w])z_{-k}.$$

Now we can calculate

$$\begin{aligned}\mathrm{Curv}_{V'}(z^k \otimes z_{-l}) &= -\sum_{i>0} z^i \otimes \left([\pi(z_i), \pi(z_{-k})] - \pi([z_i, z_{-k}])\right) z_{-l} \\ &= -\sum_{i,j>0}\sum_{s\geq 0} z^i \otimes z^{-j}([z_{-k}, y_s])y^s([z_i, z_{-l}])z_{-j}.\end{aligned}$$

By summing over $z_i \in \mathfrak{g}_n$ for fixed n, it is possible to move the z_{-l} action from z_i to z^i. The last expression becomes

$$-\sum_{s\geq 0}\sum_{j>0}(\mathrm{ad}^t(z_{-l})y^s) \otimes z^{-j}([z_{-k}, y_s])z_{-j} = \sum_{s\geq 0}(\mathrm{ad}^t(z_{-l})y^s) \otimes \pi(y_s)(z_{-k}).$$

The proof of the claim is finished by noting that $z_{-k} = \phi^{-1}(z^k)$.

Next, the contragradient metric $\{,\}$ gives an isomorphism $\psi : \overline{\mathfrak{n}} \to \hat{\mathfrak{n}}^*$ of $\mathfrak{b}$ and $\overline{\mathfrak{b}}$-modules. $J = \psi\phi^{-1}$, while $\operatorname{ad}^t(w)y^s = \operatorname{ad}^t(y'_s)\psi(w)$ where $\{y'_s\}$ is the dual basis to $\{y_s\}$. Identifying V with V' via ψ gives

$$\operatorname{Curv}_V(f \otimes g) = \sum_{s\geq 0} \operatorname{ad}^t(y'_s)g \otimes \operatorname{ad}^t(y_s)Jf.$$

Given $S, T \in \operatorname{End}(\hat{\mathfrak{n}}^*)$, define a second-order operator $\operatorname{Switch}(S,T)$ on $\bigwedge^* \hat{\mathfrak{n}}^* \otimes S^*\hat{\mathfrak{n}}^*$ by $f \otimes g \mapsto Tg \otimes Sf$. Then $\operatorname{Switch}(S,T) = d_L(T)d_R(S) - (TS)^\wedge$, where $(TS)^\wedge$ is the extension of TS to $\bigwedge^* \hat{\mathfrak{n}}^*$ as a derivation. We have shown that

$$\operatorname{Curv}_V = \sum_{s\geq 0} \operatorname{Switch}(\operatorname{ad}^t(y_s)J, \operatorname{ad}^t(y'_s)) = \sum_{s\geq 0} d_L(\operatorname{ad}^t(y'_s))d_R(\operatorname{ad}^t(y_s)J) - (TJ)^\wedge,$$

where $T = \sum_{s\geq 0} \operatorname{ad}^t(y'_s)\operatorname{ad}^t(y_s)$. It is not hard to see that $(T\psi(y))(x) = -\gamma(x,y)$ for $x \in \mathfrak{n}$, $y \in \overline{\mathfrak{n}}$, so $T = J^{-1}$ by Lemma 6.4.1. □

Note that $d_R(TJ) = d_L(T^*)$, where T^* is the adjoint of $T \in \operatorname{End}(\hat{\mathfrak{n}}^*)$ in the contragradient metric. The map J appears because the Kahler metric is used on $\bigwedge^* \hat{\mathfrak{n}}^*$ while the contragradient metric is used on $S^*\hat{\mathfrak{n}}^*$. Since the isomorphism ψ appearing in the proof is an isometry, $\operatorname{ad}^t(x)^* = -\operatorname{ad}(\overline{x})^*$ in the contragradient metric.

6.4.2 Cohomology vanishing for affine algebras

If $\mathfrak{g}$ is affine then $\mathfrak{g}$ can be realized as the algebra $(L[z^{\pm 1}] \oplus \mathbb{C}c \oplus \mathbb{C}d)^{\tilde{\sigma}}$, where L is a simple Lie algebra and $\tilde{\sigma}$ is an automorphism of $\mathfrak{g}$ defined by

$$\tilde{\sigma}(c) = c, \tilde{\sigma}(d) = d, \tilde{\sigma}(xz^n) = \zeta^{-n}\sigma(x)z^n, \quad x \in L$$

for σ a diagram automorphism of L of finite order k and ζ a fixed kth root of unity. We use the conventions of [Ka83] (see chapters 7 and 8 in particular). The bracket is defined by

$$[xz^m + \gamma_1 c + \beta_1 d, yz^n + \gamma_2 c + \beta_2 d] = \\ [x,y]z^{m+n} + \beta_1 n y z^n - \beta_2 m x z^m + \delta_{m,-n} m\langle x, y\rangle c,$$

for $x, y \in L$, $\gamma_1, \gamma_2, \beta_1, \beta_2 \in \mathbb{C}$, where $\langle , \rangle$ is the symmetric invariant bilinear form on L normalized by setting the length squared of a long root to $2k$. The diagram automorphism acts diagonalizably on L, so that

$$\mathfrak{g} = \bigoplus_{i=0}^{k-1} L_i z^i \otimes \mathbb{C}[z^{\pm k}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where L_i is the ζ^i-eigenspace of σ. The eigenspace L_0 is a simple Lie algebra, and there is a Cartan $\overset{\circ}{\mathfrak{h}} \subset L$ compatible with σ such that $\overset{\circ}{\mathfrak{h}}_0 = \overset{\circ}{\mathfrak{h}} \cap L_0$ is a Cartan in L_0. The algebra $\mathfrak{h} = \overset{\circ}{\mathfrak{h}}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Cartan for $\mathfrak{g}$. The eigenspaces L_i are irreducible L_0-modules. Choose a set of simple roots $\alpha_1, \ldots, \alpha_l$ for L_0, and let ψ be either the highest weight of L_1 (if $k > 1$), or the highest root of L_0 (if $k = 0$). Then $\alpha_0 = d^* - \psi, \alpha_1, \ldots, \alpha_l$ is a set of simple roots for $\mathfrak{g}$, and $\alpha_0^\vee = c - \nu^{-1}(\psi), \alpha_1^\vee, \ldots, \alpha_l^\vee$ is a set of simple coroots, where $\nu : \overset{\circ}{\mathfrak{h}}_0 \to \overset{\circ}{\mathfrak{h}}{}_0^*$ is the isomorphism defined by $\langle, \rangle$. There is a unique real form $\mathfrak{h}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{\alpha_i^\vee\} \oplus \mathbb{R}d$, and the anti-linear Cartan involution sends $xz^m + \alpha c + \beta d \mapsto \overline{x} z^{-m} - \overline{\alpha} c - \overline{\beta} d$, where $x \mapsto \overline{x}$ is the anti-linear Cartan involution of x in L. The real-valued symmetric invariant bilinear form $\langle, \rangle$ on $\mathfrak{g}$ is defined by

$$\begin{aligned}&\langle xz^m, yz^n \rangle = \delta_{m,-n} \langle x, y \rangle, \ \langle c, d \rangle = a_0, \text{ and} \\ &\langle xz^m, c \rangle = \langle xz^m, d \rangle = \langle c, c \rangle = \langle d, d \rangle = 0,\end{aligned}$$

where $a_0 = \langle \psi, \psi \rangle / 2$ (in fact, $a_0 = 1$ except when $L = \mathfrak{sl}(2l+1)$ and $k = 2$, in which case $a_0 = 2$). The contragradient metric $\{, \} = -\langle \cdot, \overline{\cdot} \rangle$ is positive-definite on $\mathfrak{n}$ as required.

The following lemma finishes the proof of Theorem 6.1.3.

Lemma 6.4.3. *Let μ be a dominant weight of an integrable highest weight $\mathfrak{g}$-module $\mathcal{L}(\lambda)$, where λ is a real-valued dominant weight and $\mathfrak{g}$ is affine. If μ is dominant then $H^d_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; \mathcal{L}(\lambda) \otimes S^*\hat{\mathfrak{n}}^* \otimes \mathbb{C}_{-\mu}) = 0$ for all $d > 0$.*

Proof. The result is trivial if $\lambda = \mu = 0$, so assume that λ and μ have positive level.

$S^*\hat{\mathfrak{n}}^*$ has a contragradient positive-definite Hermitian form from $\{, \}$. Since μ is a real-valued weight, $\mathbb{C}_{-\mu}$ has a contragradient positive-definite Hermitian form. Finally, $\mathcal{L}(\lambda)$ has a contragradient positive-definite Hermitian form because λ is a real-valued dominant weight. Putting everything together, $V = \mathcal{L}(\lambda) \otimes S^*\hat{\mathfrak{n}}^* \otimes \mathbb{C}_{-\mu}$ has a contragradient positive-definite Hermitian form.

The cohomology $H^*_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; V)$ can be identified with the kernel of the Laplacian $\overline{\Box}$ on the zero column $C^{0,*}(V)$ of the semi-infinite chain complex. By Nakano's identity, $\overline{\Box} = \Box + \deg + \operatorname{Curv}$. $\Box$ is positive semi-definite by definition. The curvature term splits into a sum $\operatorname{Curv} = \operatorname{Curv}_{\mathcal{L}(\lambda)} + \operatorname{Curv}_{S^*} + \operatorname{Curv}_{\mathbb{C}_{-\mu}}$. Since $\mathcal{L}(\lambda)$ is representation of $\mathfrak{g}$, $\operatorname{Curv}_{\mathcal{L}(\lambda)}$ is zero. Next consider $\operatorname{Curv}_{S^*} + \deg$. We use the realisation of $\mathfrak{g}$ via the loop algebra. The contragradient metric $\{, \}$ induces a positive-definite metric on the loop algebra $\mathfrak{g}'/\mathbb{C}c$, so we can pick a homogeneous basis for $\mathfrak{b}$ consisting of an orthonormal basis $\{y_s\}$ for the projection of $\mathfrak{b}$ to $\mathfrak{g}'/\mathbb{C}c$, as well as c and d. The dual basis to $\{c, d, y_0, \ldots, y_s, \ldots\}$ is $\{a_0^{-1}d, a_0^{-1}c, -\overline{y_0}, \ldots, -\overline{y_s}, \ldots\}$. Since c is in the centre, we have $\operatorname{ad}^t(c) = 0$, so the terms $d_L(\operatorname{ad}^t(a_0^{-1}c))$ and $d_R(\operatorname{ad}^t(c)J)$ in $\operatorname{Curv}_{S^*}$ are zero. Consequently

$$\operatorname{Curv}_{S^*} + \deg = \sum_{s \geq 0} d_L(\operatorname{ad}^t(-\overline{y_s})) d_R(\operatorname{ad}^t(y_s)J) = \sum_{s \geq 0} d_R(\operatorname{ad}^t(y_s)J)^* d_R(\operatorname{ad}^t(y_s)J)$$

is semi-positive. Finally we get that

$$\mathrm{Curv}_{\mathbb{C}_{-\mu}} = -\sum_{\alpha\in\Delta^+}\sum_{i,j}\epsilon(z_\alpha^i)\iota(z_{\alpha,j})\mu([z_{\alpha,i},\overline{z_{\alpha,j}}]),$$

where $z_{\alpha,i}$ runs through a basis for $\mathfrak{g}_\alpha$ orthonormal in the Kahler metric. Now

$$-\mu([z_{\alpha,i},\overline{z_{\alpha,j}}]) = \{z_{\alpha,i}, z_{\alpha,j}\}\langle\mu,\alpha\rangle.$$

The result is that $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$ is a derivation which multiplies occurrences of z_α^j by the non-negative number $2\langle\rho,\alpha\rangle\langle\mu,\alpha\rangle$, and thus is semi-positive.

Now we look more closely at the kernel of $\overline{\Box}$. The operator $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$ is strictly positive on $z^{\beta_1,i_1}\wedge\cdots\wedge z^{\beta_k,i_k}\otimes v$ unless all $\beta_i\in\mathbb{Z}[Y]$, where $Y=\{\alpha_i : \mu(\alpha_i^\vee)=0\}$. Let A_Y be the submatrix of the defining matrix A of $\mathfrak{g}$ with rows and columns indexed by $\{i:\alpha_i\in Y\}$. Recall that the Kac-Moody algebra $\mathfrak{g}(A_Y)$ defined by A_Y embeds in $\mathfrak{g}$. The standard nilpotent of $\mathfrak{g}(A_Y)$ is $\mathfrak{n}_Y=\bigoplus_{\alpha\in\Delta^+\cap\mathbb{Z}[Y]}\mathfrak{g}_\alpha\subset\mathfrak{g}$. Let $\mathfrak{u}_Y=\bigoplus_{\alpha\in\Delta^+\setminus\mathbb{Z}[Y]}\mathfrak{g}_\alpha$. Since μ has positive level, Y is a strict subset of simple roots, and since $\mathfrak{g}$ is affine, $\mathfrak{g}(A_Y)$ is finite-dimensional. Harmonic cocycles must belong to the kernel of $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$, so any harmonic cocycle ω must be in the $\mathfrak{h}$-invariant part of

$$\bigwedge^*\hat{\mathfrak{n}}_Y^*\otimes S^*\hat{\mathfrak{n}}^*\otimes\mathcal{L}(\lambda)\otimes\mathbb{C}_{-\mu}.$$

As a vector space, this set can be identified with $\Omega^*_{pol}\hat{\mathfrak{n}}_Y\otimes\mathbb{C}[\hat{\mathfrak{u}}_Y]\otimes\mathcal{L}(\lambda)$, where Ω^*_{pol} is the ring of polynomial differential forms and $\hat{\mathfrak{u}}_Y$ is pro-Lie algebra associated to $\mathfrak{u}_Y$. For ω to be in the kernel of $\deg+\mathrm{Curv}_{S^*}$, ω must lie in the kernel of the operators $d_R(\mathrm{ad}^t(y_s)J)$, $s\geq 0$. Since $d_R(\mathrm{ad}^t(c)J)=0$, we get that $d_R(\mathrm{ad}^t(x)J)\omega=0$ for every $x\in\mathfrak{b}_Y\subset\mathfrak{g}'\cap\mathfrak{b}$, where $\mathfrak{b}_Y$ is the standard Borel of $\mathfrak{g}(A_Y)$. Let J_Δ^{-1} denote the diagonal extension of J^{-1} to $\bigwedge^*\hat{\mathfrak{n}}^*$. Then $J_\Delta^{-1}\omega$ vanishes under contraction by the vector fields $\mathfrak{n}_Y\to T\mathfrak{n}_Y : x\mapsto(x,[x,y])$, $y\in\mathfrak{b}$. At a point $x\in\mathfrak{n}_Y$, these vector fields span the tangents to $\mathcal{B}_Y$-orbits. $\mathfrak{n}_Y$ is the positive nilpotent of a finite-dimensional Kac-Moody, so $\mathfrak{n}_Y$ has a dense $\mathcal{B}_Y$-orbit and thus ω must be of degree zero. □

The same proof applies with slight modification if $\mathfrak{g}$ is a direct sum of indecomposables of finite or affine type.

6.5 Indefinite Kac-Moody algebras

In this section $\mathfrak{g}$ will be an arbitrary symmetrizable Kac-Moody algebra. Recall from the proof of Lemma 6.4.3 that if A is the defining matrix of $\mathfrak{g}$ and Z is a subset of the simple roots then A_Z refers to the submatrix of A with rows and columns indexed by $\{i:\alpha_i\in Z\}$.

Proposition 6.5.1. *Let $\mathfrak{g}$ be the symmetrizable Kac-Moody algebra defined by the generalized Cartan matrix A, and suppose μ is a dominant weight of an integrable highest weight representation $\mathcal{L}(\lambda)$, where λ is real-valued. Write $\lambda - \mu = \sum k_i \alpha_i$, $k_i \geq 0$, and let $Z = \{\alpha_i : k_i > 0\}$. If A_Z is a direct sum of indecomposables of finite and affine type then $H^d_{cts}(\hat{\mathfrak{b}}, \mathfrak{h}; S^* \hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}) = 0$ for $d > 0$.*

Recall that the weight space $\mathcal{L}(\lambda)_\mu$ of an integrable highest weight representation is filtered via polynomial degree on the isomorphic space $(\mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu} \otimes \mathbb{C}[\mathcal{U}])^{\mathcal{B}}$. Let ${}^{deg}P^\lambda_\mu(q)$ be the corresponding Poincare polynomial. Excepting Proposition 6.3.4, the results of Sections 6.1 and 6.3 imply the following corollary:

Corollary 6.5.2. *If the hypotheses of Proposition 6.5.1 hold then $m^\lambda_\mu(q) = {}^{deg}P^\lambda_\mu(q)$*

The conclusions of Theorem 6.1.3 hold similarly, with the Brylinski filtration replaced by the degree filtration.

The requirement in Proposition 6.5.1 and Corollary 6.5.2 that $\lambda - \mu$ have affine support is a technical assumption used to prove the positive-definiteness of the $\deg + \mathrm{Curv}_{S^*}$ term in the Laplacian. It is unclear to the author whether or not this hypothesis can be dropped.

Proof of Proposition 6.5.1. We continue to use the notation of Section 6.4. For instance, $V = S^* \hat{\mathfrak{n}}^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}$. Recall that $\overline{\Box} = \Box + \deg + \mathrm{Curv}_V$, and $\mathrm{Curv}_V = \mathrm{Curv}_{\mathcal{L}(\lambda)} + \mathrm{Curv}_{\mathbb{C}_{-\mu}} + \mathrm{Curv}_{S^*}$. The operators $\Box$, $\mathrm{Curv}_{\mathcal{L}(\lambda)}$, and $\mathrm{Curv}_{\mathbb{C}_{-\mu}}$ are positive semi-definite as before, while

$$\deg + \mathrm{Curv}_{S^*} = \sum_{k \geq 1} d_R(\mathrm{ad}^t(x_k)J)^* d_R(\mathrm{ad}^t(x_k)J) + \sum_i d_L(\mathrm{ad}^t(u^i)) d_R(\mathrm{ad}^t(u_i)J),$$

where $\{x_k\}$ is a basis for $\mathfrak{n}$ orthonormal in the contragradient metric, and $\{u_i\}$ and $\{u^i\}$ are dual bases for $\mathfrak{h}$. The first summand in this equation is positive semi-definite, but the second is not if there are roots with $\langle \alpha, \alpha \rangle < 0$. Indeed, writing

$$\sum_i d_L(\mathrm{ad}^t(u^i)) d_R(\mathrm{ad}^t(u_i)J) = \sum_i \mathrm{Switch}(\mathrm{ad}^t(u^i)J, \mathrm{ad}^t(u_i)) + \sum_i (\mathrm{ad}^t(u^i)\,\mathrm{ad}^t(u_i)J)^\wedge, \quad (6.2)$$

we see that the first summand in Equation (6.2) is the second order operator defined by

$$x \otimes y \mapsto \frac{\langle \alpha, \beta \rangle}{2\langle \rho, \alpha \rangle} y \otimes x, x \in \mathfrak{g}^*_\alpha, y \in \mathfrak{g}^*_\beta,$$

while the second summand in Equation (6.2) is the derivation of $\bigwedge^* \hat{\mathfrak{n}}^*$ induced by the map

$$x \mapsto \frac{\langle \alpha, \alpha \rangle}{2\langle \rho, \alpha \rangle} x, x \in \mathfrak{g}^*_\alpha$$

on $\hat{\mathfrak{n}}^*$.

Let $\mathfrak{g}(A_Z)$ be the corresponding Kac-Moody subalgebra of $\mathfrak{g}$, and let $\mathfrak{n}_Z$ be the standard nilpotent. $\mathfrak{g}(A_Z)$ has a Cartan subalgebra $\mathfrak{h}_Z \subset \mathfrak{h}$, and the real-valued non-degenerate symmetric invariant form on $\mathfrak{g}$ restricts to such a form on $\mathfrak{g}(A_Z)$. Any $\mathfrak{h}$-invariant element of $\bigwedge^* \hat{\mathfrak{n}}^* \otimes V$ must belong to $\bigwedge^* \hat{\mathfrak{n}}_Z^* \otimes S^* \hat{\mathfrak{n}}_Z^* \otimes \mathcal{L}(\lambda) \otimes \mathbb{C}_{-\mu}$. We claim that the operator $\sum_i d_L(\mathrm{ad}^t(u^i)) d_R(\mathrm{ad}^t(u_i)J)$ on $\bigwedge^* \hat{\mathfrak{n}}^* \otimes S^* \hat{\mathfrak{n}}^*$ restricts on $\bigwedge^* \hat{\mathfrak{n}}_Z^* \otimes S^* \hat{\mathfrak{n}}_Z^*$ to the operator $\sum_i d_L(\mathrm{ad}^t(v^i)) d_R(\mathrm{ad}^t(v_i)J)$, where $\{v_i\}$ and $\{v^i\}$ are dual bases of $\mathfrak{h}_Z$. To verify this claim, note that a choice of symmetric invariant form corresponds to a choice of a diagonal matrix D with positive diagonal entries, such that DA is a symmetric matrix. If $x \in \mathfrak{h}^*$ the invariant form satisfies $\langle x, \alpha_i \rangle = D_{ii} x(\alpha_i^\vee)$. The operator in Equation (6.2) thus depends only on A and D; the claim follows from the observation that the action of the operator on $\bigwedge^* \hat{\mathfrak{n}}_Z^* \otimes S^* \hat{\mathfrak{n}}_Z^*$ depends only on A_Z and D_Z.

Now suppose A_Z is a direct sum of indecomposables of finite and affine type. The operator $\sum_i d_L(\mathrm{ad}^t(v^i)) d_R(\mathrm{ad}^t(v_i)J)$ decomposes into a summand for each component, each of which is positive semi-definite as in the proof of Lemma 6.4.3. We finish as in the proof of Lemma 6.4.3, but taking $Y = \{\alpha_i \in Z : \mu(\alpha_i) = 0\}$. □

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