Title
THE ASYMPTOTIC VERTEX FUNCTION IN A CROSSING SYMMETRIC BOOTSTRAP MODEL OF THE HADRONS

Author
Harte, John

Publication Date
1968-02-01

University of California

Ernest O. Lawrence Radiation Laboratory

THE ASYMPTOTIC VERTEX FUNCTION IN A CROSSING SYMMETRIC BOOTSTRAP MODEL OF THE HADRONS

John Harte

February 1968

Berkeley, California

DISCLAIMER

Submitted to Physical Review

UCRL-18122
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

THE ASYMPTOTIC VERTEX FUNCTION IN A CROSSING SYMMETRIC BOOTSTRAP MODEL OF THE HADRONS

John Harte

February 1968

THE ASYMPTOTIC VERTEX FUNCTION IN A CROSSING SYMMETRIC BOOTSTRAP MODEL OF THE HADRONS*

John Harte†

Department of Physics and Lawrence Radiation Laboratory

University of California, Berkeley

ABSTRACT

An asymptotic solution to an off-shell, nonlinear, crossing symmetric bootstrap equation for the vertex function is found, thus determining the explicit dependence of the asymptotic vertex function on all three variables. The equation bootstraps the entire vertex function rather than just the coupling constant and describes the constituents of a composite particle as composite particles. The asymptotic solution is characterized by a constant which we show to be universal, i.e., independent of the types of particles coupling to a given vertex. This allows us to predict the relative rates of decrease of all hadronic form factors. The predicted behavior for the $NN\gamma$ and the $N^*N\gamma$ form factors is in good agreement with experiment. Several extensions and further tests of the theory are proposed and will be discussed in future papers.

* This work was done under the auspices of the U.S. Atomic Energy Commissior

† AEC Postdoctoral Fellow under contract No. AT(11-1)-34.

2/68

I. INTRODUCTION

This paper is the second in a series devoted to the asymptotic behavior of vertex functions and scattering amplitudes in a crossing symmetric bootstrap model. Here, as in the first paper[1] (hereafter referred to as I), we shall restrict our attention to a study of the vertex function. The crossing symmetric Bethe-Salpeter bound state equation which we investigate describes the bootstrap of an entire vertex function rather than just the coupling constant which is the on-shell value of the vertex function. Because the bootstrap equation is crossing symmetric, the constituents of the composite particle are, themselves, realistically described as composite particles.[2]

The bootstrap equation is nonlinear in the vertex function and probably too difficult to employ in practical calculations of masses and coupling constants; for this reason we content ourselves with asymptotic statements. Thus we are bootstrapping the high energy behavior of the strong interactions, rather than the hadrons themselves. It is our hope that this self-consistent asymptotic behavior, which we shall see is well behaved at large momenta, may serve to regularize the field theory and thus allow the calculation of masses and coupling constants within the framework of the more standard, linear bootstrap equations with effective form factors given by Eq. (2.19) of Section II. In any case, we shall see that the theory contains a great deal of predictive power even at the asymptotic level in the information it provides about form factors and, as we shall discuss in a subsequent paper, about large angle hadron-hadron scattering.

Let us first briefly review the results of I. There we actually performed three model calculations: First we derived and found the asymptotic solution to a nonlinear Schroedinger equation describing the bound state of a composite particle in a fixed-source potential, subject to the requirement that the composite-constituent particle and the bound state of the composite-constituent particle have the same wave function. Then we investigated a relativistic Bethe-Salpeter equation bootstrap model in which the exchanged particle coupled with a form factor which was directly related to the bound state wave function. In order to simplify the equation, one of the two constituents was taken to be an elementary particle with a point interaction. This simplification allowed us to deal with vertex functions depending on only one variable rather than three as will be the case for the arbitrary vertex function which we investigate here. Finally we treated a two-body Schroedinger equation in which both of the constituent particles were composite. In all three cases we found that the wave function of the composite particle decreased exponentially in the square root of the invariant momentum transfer variable.

We begin this paper with an extension of I to the full relativistic problem with no elementary particles. Thus we are able to find the vertex function for the coupling of three composite hadrons in an asymptotic limit in which the four-momentum squared of one or more of the three legs approach infinity. We show that this asymptotic vertex function is characterized by a universal constant which is independent of the types of particles coupling at the vertex, if the bootstrap is truly reciprocal. In Section III we discuss the implications of our result and, where possible, compare with experiment. The asymptotic vertex function derived in Section II allows us to relate the nucleon form factor to that of any other hadron. The prediction for the $N^*N\gamma$ vertex is compared with experiment and is in good agreement. It is observed that a measurement of the pion form factor would provide an excellent test of the theory as the predicted behavior of the form factor of this particle at large momentum transfer is strikingly different from that for the other hadrons. Finally other applications of the theory are mentioned, although a detailed treatment of these will be given in subsequent papers.

II. THE ASYMPTOTIC VERTEX FUNCTION

The crossing symmetric, Bethe-Salpeter bootstrap equation[3] (Eq. (1.4) of I) which we shall investigate here describes the bootstrap of a scalar meson with an effective φ^3 interaction.[4] We will restrict ourselves to the single-channel, spinless problem for simplicity, although our results, as we shall argue later, are independent of this approximation. The equation reads (see I for details)

$$G_0^{-1}(p^2,(P-p)^2)\;\chi(P^2,(P-p)^2,p^2)$$

$$=\int d^4k\;\chi(P^2,(P-k)^2,k^2)\;G_0^{-1}(k^2,(P-k)^2)\;\times \tag{2.1}$$

$$\times\;\chi((P-k)^2,(p-k)^2,(P-p)^2)\,[(p-k)^2-M^2]\;\chi(k^2,(p-k)^2,p^2)$$

and is illustrated in Fig. (1). P, p and k denote four-vectors throughout the paper. χ is the Bethe-Salpeter wave function for the composite particle with mass $P^2 = M^2$, G_0 is the product of the constituent particle propagators, and we have assumed the ladder approximation for the interaction. We will show in a subsequent paper that our asymptotic results are valid for a significantly wider class of kernels than the lowest order one treated here. It should be stressed that although Eq. (2.1) has a non-crossing-symmetric appearance due to the condition $P^2 = M^2$, the equation actually contains the full symmetry because the boundary condition, $P^2 = M^2$, is symmetric in the sense that it could have been applied to any of the three external legs.

Eq. (2.1) is sufficiently complicated that we will not attempt to determine the precise asymptotic solution, as we did in I, but will only determine the exponential factor in the asymptotic solution. That is, we will not attempt to solve for the polynomial or inverse polynomial, which may multiply the exponential term in the asymptotic limit of $\chi(p_1^2, p_2^2, p_3^2)$. Since G_o is simply a polynomial (equal to the product of the bare propagators in the asymptotic limit) we can safely ignore it and the other propagator in Eq. (2.1). It will also be more convenient to work with the vertex function, Γ, rather than the wave function which is related to Γ by $\chi = G_o\Gamma$, so that Eq. (2.1) can finally be written

$$\Gamma(\underset{\sim}{P}^2,(\underset{\sim}{P}-\underset{\sim}{p})^2,\underset{\sim}{p}^2) \sim \int d^4k\, \Gamma(\underset{\sim}{P}^2,(\underset{\sim}{p}-\underset{\sim}{k})^2,\underset{\sim}{k}^2) \times \times\ \Gamma((\underset{\sim}{P}-\underset{\sim}{k})^2,(\underset{\sim}{p}-\underset{\sim}{k})^2,(\underset{\sim}{P}-\underset{\sim}{p})^2)\Gamma(\underset{\sim}{k}^2,(\underset{\sim}{p}-\underset{\sim}{k})^2,\underset{\sim}{p}^2) \quad . \tag{2.2}$$

While we have been unable to establish the validity of the Wick rotation for this equation, we shall assume it from here on and consider it as an equation in a Euclidean space. We shall use the method of steepest descent (m.o.s.d.) to find an asymptotic form for Γ which solves Eq. (2.2) in an asymptotic limit in which p^2 and/or $(P-p)^2 \to \infty$.

In order to illustrate the use of the m.o.s.d., let us digress briefly and look at a simpler problem which we treated in I - the relativistic bootstrap with one elementary-constituent and one composite-constituent. The asymptotic bootstrap equation for this problem (Eq. (3.3) of I) reads

$$p^4 \chi(p^2) \sim \int d^4k \, k^2 \, \chi(k^2) \, \chi((p-k)^2) \; . \tag{2.3}$$

Diagrammatically this equation is described by Fig. (1) except that one vertex function in the kernel is replaced by a point interaction and each vertex function is assumed to depend only on the invariant momentum-squared of the elementary particle leg attached to it rather than on all three invariants as in Eq. (2.2). In I, we showed that $\chi(p) \sim p^{-1/2} e^{-ap}$, where $p = (p^2)^{1/2}$, was an asymptotic solution to this equation. We use the m.o.s.d. now to show that if we assume a general, trial asympototic solution of the form

$$\chi(p) \sim p^{\nu} e^{-ap^{\gamma}} \tag{2.4}$$

then $\gamma = 1$.

Substituting Eq. (2.4) into Eq. (2.3) we obtain the asymptotic (Euclidean) equation

$$p^{4+\nu} \; e^{-ap^{\gamma}} \sim \int d^4k \quad k^{2+\nu} \, |k-p|^{\nu} \, e^{-ak^{\gamma}} \, e^{-a|k-p|^{\gamma}}$$

$$= 4\pi \int_0^{\infty} dk \int_{-1}^{1} dx \, (1-x^2)^{1/2} \, k^{5+\nu} (k^2 + p^2 - 2kpx)^{\nu/2} \times \tag{2.5}$$

$$\times \, e^{-ak^{\gamma}} \, e^{-a(k^2 + p^2 - 2kpx)^{\gamma/2}} \; .$$

The r.h.s. of this equation can be cast into a form suitable for the application of the m.o.s.d. by introducing the change of variable $k = yp$, and rewriting Eq. (2.5) in the form

$$p^{4+\nu}\, e^{-ap^{\gamma}} \sim 4\pi p \int_0^{\infty} dy \int_{-1}^{1} dx\, \exp\left\{- ap^{\gamma}\left[y^{\gamma} + (y^2 + 1 - 2yx)^{\gamma/2} - \frac{\ell n(1-x^2)^{1/2}}{ap^{\gamma}} - \frac{\ell n(py)^{5+\nu}}{ap^{\gamma}} - \frac{\ell n(y^2p^2 + p^2 - 2yp^2x)^{1/2}}{ap^{\gamma}}\right]\right\} . \quad (2.6)$$

In the asymptotic limit $p \to \infty$, the logarithmic terms can be dropped, as they only contribute polynomials multiplying the exponential term in the asymptotic limit of the integral, whereas our interest now is only in the exponential term. Then, by the m.o.s.d. we obtain the equation

$$e^{-ap^{\gamma}} \sim e^{-ap^{\gamma}f(y_o,x_o)} \quad (2.7)$$

where $f(y_o,x_o)$ is the minimum value of the function

$$f(y,x) = y^{\gamma} + (y^2 + 1 - 2yx)^{\gamma/2} . \quad (2.8)$$

It is straightforward to show[5] that for $\gamma \geq 1$ the minimum of $f(y,x)$ is given by $y = \frac{1}{2}$, $x = 1$ and we obtain the result

$$f(y_o,x_o) = \left(\frac{1}{2}\right)^{\gamma} + \left(\frac{1}{2}\right)^{\gamma} = 2^{1-\gamma} \quad (2.9)$$

For Eq. (2.7) to be asymptotically correct we require

$$f(y_o,x_o)' = 1 \tag{2.10}$$

or

$$\gamma = 1 \tag{2.11}$$

as we set out to prove.

We could have also performed the x integration explicitly and then carried out the m.o.s.d. in one dimension and obtained the same result. In any case, it is clear that the complexities associated with the four-dimensionality of the space were irrelevant to our final result. In addition, it can be seen that the propagators, or polynomial terms, in Eq. (2.5) did not contribute to the exponent in the asymptotic limit, but only gave rise to polynomials.

Now we turn to the relativistic bootstrap model (Eq. (2.2)) with no elementary particles and with full vertex symmetry. The additional source of complexity in this equation is the dependence of the solution on three variables. With the simpler equation discussed above, we were able to prove a theorem (see Appendix of I) implying that in order to verify the consistency of a particular asymptotic form for the wave function, it was sufficient to determine the asymptotic behavior of the r.h.s. of Eq. (2.3) with each χ replaced by χ_A, the asymptotic part of the wave function. The vertex function, here, depends on three invariants, so we make the decomposition

$$\Gamma(p_1{}^2,p_2{}^2,p_3{}^2) = f(p_1{}^2,p_2{}^2,p_3{}^2) + \Gamma_A(p_1{}^2,p_2{}^2,p_3{}^2) \tag{2.12}$$

where $\Gamma_A > f$ if one or more of the invariants p_i^2 approach infinity. The theorem proved in I can then be shown to hold and states that we can check the consistency of the asymptotic behavior, Γ_A, by replacing Γ by Γ_A in the integrand in Eq. (2.2). The proof is straightforward and proceeds analagously to that in I, so it will not be given here. We emphasize that by asymptotic we mean that Γ_A is the leading term in the vertex function if any or all of the invariant momenta p_1^2, p_2^2, p_3^2 approach infinity. We shall restrict our investigation of Eq. (2.2) to symmetric functions of the three invariants which have the property that

$$\Gamma_A(p_1^2, p_2^2, p_3^2) \sim e^{-cp_1} \tag{2.13}$$

for $p_1^2 \to \infty$ and p_2^2, p_3^2 finite, since this limiting behavior is suggested by the results of I.

A plausible first guess for the form of Γ_A might be

$$\Gamma_A(p_1^2, p_2^2, p_3^2) = e^{-a(p_1^2 + p_2^2 + p_3^2)^{1/2}} . \tag{2.14}$$

After substituting this expression into Eq. (2.2), we obtain the asymptotic equation

$$e^{-a(\underset{\sim}{p}^2 + \underset{\sim}{p}^2 + (\underset{\sim}{P}-\underset{\sim}{p})^2)^{1/2}} \sim \int d^4k \times$$

$$\times \exp[-a((\underset{\sim}{p}^2 + \underset{\sim}{k}^2 + (\underset{\sim}{P}-\underset{\sim}{k})^2)^{1/2} + ((\underset{\sim}{P}-\underset{\sim}{k})^2 + (\underset{\sim}{p}-\underset{\sim}{k})^2 + (\underset{\sim}{P}-\underset{\sim}{p})^2)^{1/2} +$$

$$+ (\underset{\sim}{k}^2 + (\underset{\sim}{k}-\underset{\sim}{P})^2 + \underset{\sim}{p}^2)^{1/2})]$$

whose consistency must be verified in the limit $p \to \infty$. To simplify this equation we shall work in one dimension since, as we saw before and have explicitly checked in this case, this does not affect the exponential term in the vertex function. Furthermore we set $P \equiv 0$ without loss of generality. If we make the substitution $k = yp$ we arrive at the equation

$$e^{-a\sqrt{2}\,p} \sim p \int_{-\infty}^{\infty} dy\, e^{-a\sqrt{2}\,p\,[\,|y| + 2(1+y^2-y)^{1/2}\,]} \qquad (2.16)$$

Then, by the m.o.s.d. we have

$$e^{-a\sqrt{2}\,p} \sim e^{-a\sqrt{2}p\,f(y_o)} \qquad (2.17)$$

where

$$f(y) = |y| + 2\,(1 + y^2 - y)^{1/2} \qquad (2.18)$$

and $f(y_o)$ is the minimum value of $f(y)$. Since $f(y_o) = 2$, Eq. (2.17) cannot be valid and we conclude that the function given by Eq. (2.14) is not a suitable asymptotic solution to Eq. (2.2).

With the use of similar techniques, a large class of trial asymptotic solutions have been tested. One asymptotic solution to Eq. (2.2) which we have discovered is

$$\Gamma_A(p_1^2, p_2^2, p_3^2) = e^{-a\,(p_1^2 p_2^2 p_3^2)^{1/2}} \qquad (2.19)$$

To verify that this is indeed a solution, we substitute into Eq. (2.2). (Now we cannot set $P \equiv 0$, clearly) and obtain the result

$$e^{-a(p^2(P-p)^2P^2)^{1/2}} \sim \int d^4k \times$$

$$\times \exp\left[-a\left[(P^2(P-k)^2k^2)^{1/2} + ((P-k)^2(p-k)^2(P-p)^2)^{1/2} + \right.\right.$$
$$\left.\left. + (k^2(p-k)^2p^2)^{1/2}\right]\right] . \tag{2.20}$$

(See appendix)

It is straightforward to verify that in the limit p^2 and/or $(P-p)^2 \to \infty$, the minimum of the function

$$g(P, p, k) = (P^2(P-k)^2k^2)^{1/2} + (P-p)^2(p-k)^2(P-p)^2)^{1/2} +$$
$$+ (k^2(p-k)^2p^2)^{1/2} \tag{2.21}$$

occurs when $(P-k)^2 = 0$ or $(p-k)^2 = 0$ or $k^2 = 0$. In other words, $g(P,p,k)$ is a minimum whenever the invariant momentum-squared of any of the three internal legs of the triangle graph (Fig. 1) vanish, and thus the saddle points in the Euclidean space are $k_o = P$, $k_o = p$ and $k_o = 0$. At each of these three values, it is easy to check that

$$g(P,p,k_o) = (P^2(P-p)^2p^2)^{1/2} \tag{2.22}$$

and hence Eq. (2.20) is satisfied. We have also verified this result in one dimension where the integral in Eq. (2.20) can be performed exactly, and we find agreement with the m.o.s.d. calculation.

Eq. (2.19) is actually only one example of a function which solves Eq. (2.2) in the asymptotic limit. The properties of Eq. (2.19) which were needed in order to verify Eq. (2.22) can be expressed in the following way. Define

$$\Gamma_A(p_1^2,p_2^2,p_3^2) \equiv e^{-ag_1(p_1^2)g_2(p_2^2)g_3(p_3^2)} \tag{2.23}$$

Then we simply require that

$$g_i(p_i^2) \sim (p_i^2)^{1/2} \tag{2.24}$$

as $p_i^2 \to \infty$, and

$$g_i(p_i^2) \to 0 \tag{2.25}$$

as $p_i^2 \to 0$, in order for Eq. (2.23) to describe an asymptotic solution to Eq. (2.2).

The function given by Eq. (2.19) would give rise to an unwanted cut in the hadronic form factors at $t = 0$ if that function described the vertex function for all p_i^2 and not just in the asymptotic limit in which one or more of the $p_i^2 \to \infty$. Of course the exact solution to Eq. (2.1) will certainly be an extraordinarily complicated function of the three invariants, but nevertheless it would be useful in practical calculations to have a simple function which satisfied the asymptotic properties which we have derived here and which did not contain unwanted singularities. Examples of such functions, satisfying Eqs. (2.24, 2.25), are

$$g_i(p_i^2) = \frac{p_i^2}{(c_i^2 + p_i^2)^{1/2} + (c_i^2)^{1/2}} \tag{2.26}$$

or

$$g_i(p_i^{\,2}) = \int_{c_i^{\,2}}^{\infty} d\mu^2 \rho(\mu^2)(\mu^2 + p_i^{\,2})^{1/2} \tag{2.27}$$

where

$$\int_{c_i^{\,2}}^{\infty} d\mu^2 \rho(\mu^2)\, \mu = 0 \tag{2.28}$$

and

$$\int_{c_i^{\,2}}^{\infty} d\mu^2 \rho(\mu^2) = 1 \tag{2.29}$$

The plus sign appears under the square root since we have written these functions in terms of the Euclidean variable $p_i^{\,2}$. In these examples, c_i is some non-zero constant related to the thresholds in the vertex function.

In the following section we will assume Eq. (2.19) describes the asymptotic vertex function and examine the consequences of this expression for the hadronic form factors at large t, remembering not to take seriously the singularity structure implied by this function at $p_i^{\,2} = 0$.

Finally it must be stressed that while we have found sufficient conditions on the vertex function in order for it to satisfy Eq. (2.2) asymptotically, we have neither demonstrated that a unique solution exists nor have we shown that if one does exist, that it have the asymptotic behavior described above. We have examined a variety of other forms for the function $\Gamma(p_1^{\,2}, p_2^{\,2}, p_3^{\,2})$ which do not satisfy Eqs. (2.23 - 2.25). Among these are the functions

$$\log \Gamma(p_1{}^2, p_2{}^2, p_3{}^2) = - a\ [(p_1{}^2 + c^2)(p_2{}^2 + c^2)(p_3{}^2 + c^2)]^{1/2}$$

$$\log \Gamma(p_1{}^2, p_2{}^2, p_3{}^2) = - a[(p_1{}^2 + c^2)^{1/2} + (p_2{}^2 + c^2)^{1/2} + (p_3{}^2 + c^2)^{1/2}]$$

and

$$\log \Gamma(p_1{}^2, p_2{}^2, p_3{}^2) = - a(p_1{}^2 + p_2{}^2 + p_3{}^2 + c^2)^{1/2} \quad ;$$

none of these are asymptotic solutions to Eq.(2.2).

III. DISCUSSION

Having established the asymptotic behavior, Eq. (2.19), for the vertex function, it is natural to ask whether one can calculate the value of the constant a.[6] While in principle it can be determined from Eq. (2.1) and the normalization condition on the Bethe-Salpeter wave function, one would have to know the entire wave function to compute it in practice; this would entail solving the complete bootstrap problem and is clearly beyond reach now. Instead, we would like to give an argument suggesting that a is a universal constant, independent of the types of particles coupling at the vertex.

In order to show this, we shall assume nuclear democracy and use the constraints imposed by the reciprocal bootstrap to show that all hadrons share the same constant a. Let us consider the subsystem of hadrons N, ρ, π, where the N is an $N\rho$ and $N\pi$ composite, the π is a $\rho\pi$ and $\bar{N}N$ composite, and the ρ is a $\pi\pi$ and $\bar{N}N$ composite. We denote the asymptotic vertex functions for the three vertices by

$$\Gamma_{NN\pi}(p_1^2, p_2^2, p_3^2) = e^{-a_{NN\pi}(p_1^2 p_2^2 p_3^2)^{1/2}} ,$$

$$\Gamma_{NN\pi}(p_1^2, p_2^2, p_3^2) = e^{-a_{NN\rho}(p_1^2 p_2^2 p_3^2)^{1/2}} , \qquad (3.1)$$

and

$$\Gamma_{NN\pi}(p_1^2, p_2^2, p_3^2) = e^{-a_{\rho\pi\pi}(p_1^2 p_2^2 p_3^2)^{1/2}} .$$

The bootstrap equations for this system read

$$\Gamma_{NN\pi} = \int \Gamma_{NN\pi}\, \Gamma_{NN\pi}\, \Gamma_{NN\pi} + \int \Gamma_{NN\rho}\, \Gamma_{NN\rho}\, \Gamma_{NN\pi} + \int \Gamma_{NN\pi}\, \Gamma_{NN\rho}\, \Gamma_{\rho\pi\pi}$$

$$\Gamma_{NN\rho} = \int \Gamma_{NN\rho}\, \Gamma_{NN\rho}\, \Gamma_{NN\rho} + \int \Gamma_{NN\pi}\, \Gamma_{NN\pi}\, \Gamma_{NN\rho} + \int \Gamma_{NN\pi}\, \Gamma_{NN\pi}\, \Gamma_{\rho\pi\pi} \qquad (3.2)$$

$$\Gamma_{\rho\pi\pi} = \int \Gamma_{\rho\pi\pi}\, \Gamma_{\rho\pi\pi}\, \Gamma_{\rho\pi\pi} + \int \Gamma_{NN\rho}\, \Gamma_{NN\pi}\, \Gamma_{NN\pi}$$

where we have dropped the momentum variables for simplicity. Now in an integral over a triangle graph with vertices given by Eq. (2.19), the leading term in the asymptotic limit will be given by the function

$$e^{-\bar{a}(p_1^2 p_2^2 p_3^2)^{1/2}} \tag{3.3}$$

where $\bar{a}$ is the minimum a_{ijk} appearing in the vertices in the integrand. That is, the integral will be dominated by the contribution which arises when the four-momentum squared of the internal line which is not connected to the vertex with the minimum value of a vanishes. Hence we have the conditions

$$\begin{aligned} a_{NN\pi} &= \text{min. } \{a_{NN\pi}, a_{NN\rho}, a_{\rho\pi\pi}\} \\ a_{NN\rho} &= \text{min. } \{a_{NN\pi}, a_{NN\rho}, \rho\pi\pi\} \\ a_{\rho\pi\pi} &= \text{min. } \{a_{\rho\pi\pi}, a_{NN\rho}, a_{NN\pi}\} \end{aligned} \tag{3.4}$$

which imply

$$a_{NN\pi} = a_{NN\rho} = a_{\rho\pi\pi} \ . \tag{3.5}$$

Clearly this argument is generalizable to an arbitrary number of channels. Thus the one-channel problem investigated in Section I easily generalizes to the many-channel bootstrap and provides us with a universal description of an arbitrary, asymptotic, hadronic vertex.

Let us look now at the implications of our results for the large momentum transfer electromagnetic form factors of the hadrons. We first point out that Eq. (2.19) also describes the vertex coupling a photon to an arbitrary hadron. This can be understood by examining particular Feynman graphs. For example, consider the ρ-dominance model which gives rise to the electromagnetic vertex

$$\Gamma_{NN\gamma}(p_1^2, p_2^2, p_3^2) = \frac{\lambda\Gamma_{NN\rho}(p_1^2, p_2^2, p_3^2)}{p_3^2 - m_\rho^2} \quad . \tag{3.6}$$

Now, since

$$\Gamma_{NN\rho}(p_1^2, p_2^2, p_3^2) \sim e^{-a(p_1^2 p_2^2 p_3^2)^{1/2}} \tag{3.7}$$

and since the constant a appearing here is the universal a in Eq. (2.19) it is clear that Eq. (2.19) describes the electromagnetic form factor. More generally, this can be made plausible by inserting photons in Fig. (1) as one of the lines with momentum $\underset{\sim}{p}$ or $\underset{\sim}{P}-\underset{\sim}{p}$ (but never as the bound state line with momentum $\underset{\sim}{P}$) and using the arguments given above to establish the universality of a.

Therefore, the electromagnetic form factor of an on-shell hadron of mass M is given by

$$F(t) \sim e^{-aM^2(-t)^{1/2}} \tag{3.8}$$

where we have set[7] p_1^2 and p_2^2 in Eq. (2.19) equal to M^2 and have set p_3^2 equal to $-t$. We have assumed[6] that the form factor is actually falling, not oscillating, in the spacelike region. From Eq. (2.2) we see that the more massive the particle, the more rapidly the form factor should decrease for large $|t|$.

In comparing Eq. (3.8) with experiment, however, we have to take into account a rather surprising feature of our model. We have shown[8] that if the vertex function given by Eq. (2.19) is used to calculate the two-photon exchange contribution to electron-hadron scattering, then because the electron is a point particle and does not couple to the photon with a composite-particle vertex, the two-photon contribution falls off as an inverse power in t rather than with an exponential as does the one-photon exchange contribution. This demonstration actually depends only on the general properties (Eqs. (2.23 - 2.25)) of the vertex function and not on the particular form of Eq. (2.19). Therefore the two-photon exchange term must eventually dominate the electron-hadron scattering amplitude. If the model is correct, then, we expect the effective form factor of the hadron to have an exponential dependence in t until $F(t)$ is of order α and then to continue to decrease with a power dependence in t. The presence of this effect makes comparison of experiment with Eq. (3.8) ambiguous at very large $|t|$, but at the same time, of course, offers a method of checking the theory as one can determine the existence of two-photon contributions experimentally.

Returning to Eq. (3.8), in Fig. 2 we compare the proton magnetic form factor to the proton-N* M1 transition form factor. The latter was extracted[9] from electroproduction data[10] and we have shown the data out to the largest measured momentum transfer. The nucleon data[11] is only shown out to $t \approx -4(\text{BeV/c})^2$ which is the region in which the exponential fit is excellent. The straight line fits[12] shown in Fig. (2) are

$$G_M(NN\gamma) = 1.5\, e^{-2.12(-t)^{1/2}} \tag{3.9}$$

$$G_{M1}(N^*N\gamma) = 1.7\, e^{-2.65(-t)^{1/2}} .$$

On the other hand our model predicts that

$$G_M(NN\gamma) \sim e^{-aM_N^2(-t)^{1/2}}$$

$$G_{M1}(N^*N\gamma) \sim e^{-aM_N M_{N^*}(-t)^{1/2}} \quad . \tag{3.10}$$

Comparing Eqs. (3.9) and Eq. (3.10) and using $m_N/m_{N^*} \approx 0.76$, we find the prediction is quite well satisfied. The value of a obtained from these fits is $a \simeq (0.75 \text{ BeV})^{-3}$.

This success of the theory may be relatively independent of the form of $\Gamma(p_1^2, p_2^2, p_3^2)$ for small p_i^2 provided Eqs. (2.23 - 2.25) are satisfied. For example, if we replace Eq. (2.19) with Eqs. (2.23, 2.26) and set $c_1^2 = c_2^2 = (m_N + m_\pi)^2$, the lowest threshold communicating with the nucleon or the N*, then we obtain essentially the same result.

If we turn now to the pion form factor, we see that the small value of the pion mass is expected to give rise to an extremely slowly decreasing asymptotic behavior. Hence, the pion form factor should not decrease as rapidly as does the nucleon's; perhaps the ρ-pole will provide a good description of the pion form factor out to large $|t|$.

In future papers we will investigate other applications of our theory. In particular, we have found a solution to a crossing symmetric off-shell equation for a hadron-hadron scattering amplitude in the asymptotic limit in which s, $|t|$, $|u| \to \infty$. In this equation, we assume each vertex is described by Eq. (2.17) and sum an infinite set of graphs which satisfy crossing symmetry and elastic unitarity. The solution enables us to relate large angle scattering cross sections to electromagnetic form factors, but unlike the situation described above for e-p scattering, there are no terms with power dependence in s, t, and u arising from higher order graphs because now all the particles are composite.

To conclude, we mention some unanswered questions. We do not know whether the condition for Eq. (2.19) to be valid is simply that p_i^2 become large or if it is that p_i^2 must become large relative to M_i^2. In other words: Does Eq. (2.19) describe the coupling constant for three on-shell particles when the mass of at least one of them is sufficiently large? Secondly, we wonder whether Eq. (2.19) describes systems like the deuteron or even heavier nuclei. While these particles are not generally considered within the framework of the reciprocal bootstrap, there is no reason not be include them at very high momentum transfers. If Eq. (2.19) describes the deuteron form factor, we expect

$$G_D(t) \sim e^{-am_D^2(-t)^{1/2}} \approx e^{-8.5(-t)^{1/2}} \quad . \tag{3.11}$$

Since this function is decreasing extremely rapidly, we expect[8] two-photon exchange effects to become important at values of $t \approx 1(\mathrm{BeV}/c)^2$. Finally there is the perplexing question of the existence and nature of the (possibly nonlocal) field theory which underlies Eq. (2.1) in a bootstrap universe; we defer speculation on this subject.

ACKNOWLEDGEMENT

The author expresses appreciation to Prof. K. Bardakci, R. C. Brower, Prof. M. B. Halpern and Dr. J. Shapiro for many helpful discussions. The hospitality extended by Dr. G. F. Chew and the Lawrence Radiation Laboratory is gratefully acknowledged.

Appendix

We wish to show here that the function

$$g.(\underset{\sim}{P}, \underset{\sim}{p}, \underset{\sim}{k}) = (\underset{\sim}{P}^2(\underset{\sim}{P}-\underset{\sim}{k})^2\ \underset{\sim}{k}^2)^{\frac{1}{2}} + ((\underset{\sim}{P}-\underset{\sim}{k})^2\ (\underset{\sim}{P}-\underset{\sim}{p})^2(\underset{\sim}{p}-\underset{\sim}{k})^2)^{\frac{1}{2}} + (\underset{\sim}{k}^2\ (\underset{\sim}{p}-\underset{\sim}{k})^2\underset{\sim}{p}^2)^{\frac{1}{2}} \tag{A.1}$$

in the limit in which $\underset{\sim}{p}^2$ and/or $(\underset{\sim}{P}-\underset{\sim}{p})^2$ approach infinity, is a minimum when $\underset{\sim}{k} = \underset{\sim}{P}$, 0 or $\underset{\sim}{p}$.

We write

$$\underset{\sim}{k} = c\,\underset{\sim}{p} + d\,\underset{\sim}{P} + \underset{\sim}{q} \tag{A.2}$$

where $\underset{\sim}{q} \cdot \underset{\sim}{p} = \underset{\sim}{q} \cdot \underset{\sim}{P} = 0$. Clearly $\underset{\sim}{q} = 0$ is a necessary condition for g to be minimum as, in a Euclidean space, $\underset{\sim}{q} \neq 0$ leads to the extra positive contribution $+ 2q^2$ under each square root sign in Eq (A.1). Consider, now, without loss of generality, the case

$$(\underset{\sim}{P}-\underset{\sim}{p})^2 = P^2 = M^2, \quad p^2 \to \infty \tag{A.3}$$

then the last term in Eq. (A.1) can be written

$$(\underset{\sim}{k}^2(\underset{\sim}{p}-\underset{\sim}{k})^2\ \underset{\sim}{p}^2)^{\frac{1}{2}} = \Big\{p^6\ (c^2 + cd)((c-1)^2 + (c-1)d) \tag{A.4}$$

$$+ p^4\ P^2\ d^2\ [(c^2 + cd) + (c-d)^2 + (c-1)d] + p^2P^4d^2\Big\}^{\frac{1}{2}}$$

Now, a necessary condition for g to be minimum is that the p^3 and p^2 terms in g vanish since at the values $\underset{\sim}{k} = \underset{\sim}{P}$, 0 and $\underset{\sim}{p}$, and under the conditions of Eq. (A.31), $g \sim p$. Setting the coefficients of these terms equal to zero in Eq. (A.4), we arrive at the four solutions

$$\begin{array}{ll} 1. & c = 0\quad d = 1 \\ 2. & c = 0,\ d = 0 \\ 3. & c = 1,\ d = 0 \\ 4. & c = 1,\ d = 1. \end{array} \tag{A.5}$$

Solutions 1., 2., and 3. are $\underset{\sim}{k} = \underset{\sim}{P}$, 0, and $\underset{\sim}{p}$ respectively. Solution 4 corresponds to $\underset{\sim}{k} = \underset{\sim}{p} - \underset{\sim}{P}$ and at this value, the last term in Eq. (A.1) equals $(\underset{\sim}{P}^2\ (\underset{\sim}{P}-\underset{\sim}{p})^2\ \underset{\sim}{p}^2)^{\frac{1}{2}}$. Solution 4 can then be shown not to yield a minimum of g by substituting $\underset{\sim}{k} = \underset{\sim}{p} - \underset{\sim}{P}$ into the first and second terms on the right hand side of Eq. (A.1). Since these two terms do not vanish for this case, we have

$$g\ (\underset{\sim}{P},\ \underset{\sim}{k},\ \underset{\sim}{p}-\underset{\sim}{P}) > (\underset{\sim}{P}^2(\underset{\sim}{P}-\underset{\sim}{p})^2\ \underset{\sim}{p}^2)^{\frac{1}{2}} = g\ (\underset{\sim}{P},\ \underset{\sim}{p},\ \underset{\sim}{k}_o) \tag{A.6}$$

where $\underset{\sim}{k}_o = \underset{\sim}{P}$, 0 or $\underset{\sim}{p}$. Hence the value $\underset{\sim}{k} = \underset{\sim}{P}$, 0, and $\underset{\sim}{p}$ yield the minima of the function g as we set out to prove.

The integral (Eq. (2.20)) which we evaluated by the m.o.s.d. is, in one dimension, of the form

$$\int_{-\infty}^{\infty} d\ x\ e^{-\ [a_3\ ((x-c_1)^2(x-c_2)^2)^{\frac{1}{2}} + a_2((x-c_1)^2(x-c_3)^2)^{\frac{1}{2}} + a_1((x-c_3)^2(x-c_2)^2)^{\frac{1}{2}}]} \tag{A.7}$$

This integral can be evaluated exactly in terms of error functions and exponentials and can be shown to have the asymptotic behavior

$$e^{-a_3\ ((c_3-c_1)^2(c_3-c_2)^2)^{\frac{1}{2}}} + e^{-a_2((c_2-c_1)^2(c_2-c_3)^2)^{\frac{1}{2}}} + e^{-a_1((c_1-c_3)^2(c_1-c_2)^2)^{\frac{1}{2}}} \tag{A.8}$$

This is precisely the asymptotic behavior we have derived in Section II.

FOOTNOTES AND REFERENCES

1. J. Harte, Phys. Rev. 165, 1557 (1968).

2. See also J. Stack, Phys. Rev. 164, 1904 (1967) for a nonrelativistic model in which one of the constituent particles is treated as a composite.

3. The group theoretic and low energy properties of this equation have been studied in detail by R. E. Cutkosky and M. Leon, Phys. Rev. 138, B667 (1965); K. Lin and R. E. Cutkosky, Phys. Rev. 140, B205 (1965); R. E. Cutkosky, Phys. Rev. 154, 1375 (1967); R. E. Cutkosky and M. A. Jacobs, Phys. Rev. 162, 1416 (1967).

4. The nature of the underlying field theory (if any) for our model is not clear to the author. We heuristically derive Eq. (2.1) from the ladder approximation to the Bethe-Salpeter equation in $\lambda\varphi^3$ theory, making the replacement $\lambda \to \Gamma(p_1^2, p_2^2, p_3^2)$, but we do not know if this corresponds to the $Z = 0$ limit of a local, Lagrangian theory or if such a theory with no elementary particle is inherently nonlocal.

5. Actually, the method of steepest descent breaks down for $\gamma < 1$ as the derivatives of $f(y)$ are divergent at the minimum values of $f(y)$, and the point $y = 1/2$ is a relative maximum of the function. A more careful treatment shows that the coefficient in the exponent is indeed a, consistent with the l.h.s. of Eq. (2.7). However, in the cases which we were able to integrate exactly, logarithmic terms arose on the r.h.s. of Eq. (2.5) multiplying the leading exponential term and thus it appears doubtful that the case $\gamma < 1$ corresponds to a possible asymptotic solution to Eq. (2.3).

6. Since we worked in a Euclidean space, we do not learn whether the vertex function decreases exponentially for spacelike momentum transfers and oscillates for timelike momentum transfers, as would be the case, for example, in a dispersive treatment of the form factor in which one sums the contribution of an infinite number of vector mesons with increasing mass. In fact, a possible solution to Eq. (2.2) would be the function

$$\Gamma_A(p_1^{\,2}, p_2^{\,2}, p_3^{\,2}) = \left[\exp(a^2 p_1^{\,2} p_2^{\,2} p_3^{\,2})^{1/2} + \exp(-b^2 p_1^{\,2} p_2^{\,2} p_3^{\,2})^{1/2}\right]^{-1} .$$

which decreases exponentially for both spacelike and timelike momentum transfers. Such a behavior is probably only possible outside of the framework of local field theory.

7. This is only approximately valid if the small $p_1^{\,2}$ and $p_2^{\,2}$ behavior of Γ is appreciably different from that given by Eq. (2.19). The qualitative result that the coefficient in the exponent of the hadronic form factor is smaller for smaller masses should still be correct, however. (Refer to the discussion following Eq. (3.10).)

8. J. Harte, University of California, Berkeley, preprint: "Possible Enhancement of Two-photon Exchange Effects in Large Momentum Transfer Electron-Proton Scattering".

9. A. J. Dufner and Y. S. Tsai, SLAC-Pub-364, November, 1967, "Phenomonological Analysis of the γNN^* Form Factors" (submitted to Physical Review).

10. H. L. Lynch, J. V. Allaby and D. M. Ritson, HEPL 404B, June 1967, Stanford University (submitted to Physical Review); F. W. Brasse, J. Engler, E. Ganssauge, M. Schwiezer, DESY preprint (1967).

11. D. H. Coward et al., Phys. Rev. Letters, 20, 292 (1968).

12. The authors of ref. (9) actually obtained a best fit to the $N^*N\gamma$ data which was $(1 + 9\sqrt{-t})^{1/2} \exp(-3.15\sqrt{-t})$ but we have chosen, for simplicity, to fit the $NN\gamma$ and $N^*N\gamma$ data with an exponential alone.

FIGURE CAPTIONS

Fig. (1) Diagrammatic representation of Eq. (2.1). The shaded circles denote the Bethe-Salpeter wave function χ.

Fig. (2) Comparison of $NN\gamma$ and $N^*N\gamma$ form factors. The data points denoted by ⊘ are experimental values for the magnetic form factor $G_M(t)$ of the proton as measured by Coward et al, ref. (11), and the date points denoted by O are the experimental values for M1 N*N transition form factor as measured by M. L. Lynch et.al. ref. (10) and analyzed by J. Dufner and Y. S. Tsai, ref. (9). The straight line fits shown in the figure are $G_{NN}(t) = 1.5 \exp[-2.12(-t)^{1/2}]$ and $G_{N^*N}(t) = 1.7 \exp[-2.65(-t)^{1/2}]$.

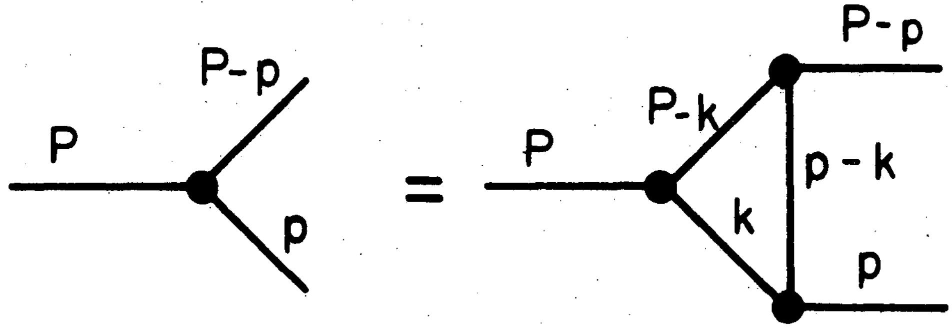

FIG. 1

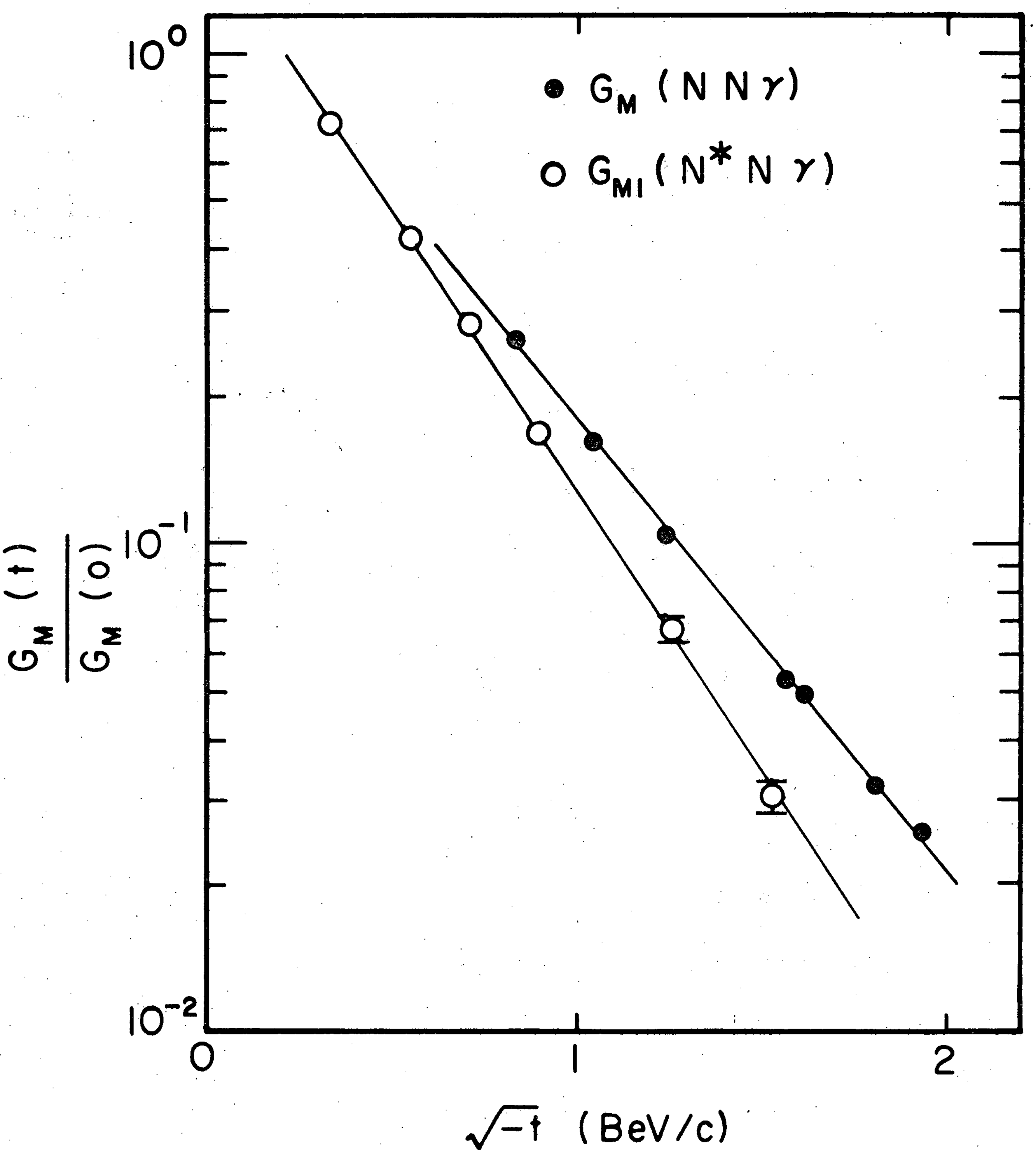

FIG. 2