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Author
Rose, Harald

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Five-dimensional Hamiltonian-Jacobi approach to relativistic quantum mechanics

H.Rose

Lawrence Berkeley National Lab, 1Cyclotron Road, Berkeley, CA 94720

Abstract

A novel theory is outlined for describing the dynamics of relativistic electrons and positrons. By introducing the Lorentz-invariant universal time as a fifth independent variable, the Hamilton-Jacobi formalism of classical mechanics is extended from three to four spatial dimensions. This approach allows one to incorporate gravitation and spin interactions in the extended five-dimensional Lagrangian in a covariant form. The universal time has the function of a hidden Bell parameter. By employing the method of variation with respect to the four coordinates of the particle and the components of the electromagnetic field, the path equation and the electromagnetic field produced by the charge and the spin of the moving particle are derived. In addition the covariant equations for the dynamics of the components of the spin tensor are obtained. These equations can be transformed to the familiar BMT equation in the case of homogeneous electromagnetic fields. The quantization of the five-dimensional Hamilton-Jacobi equation yields a five-dimensional spinor wave equation, which degenerates to the Dirac equation in the stationary case if we neglect gravitation. The quantity which corresponds to the probability density of standard quantum mechanics is the four-dimensional *mass density* which has a real physical meaning. By means of the Green method the wave equation is transformed into an integral equation enabling a covariant relativistic path integral formulation. Using this approach a very accurate approximation for the four-dimensional propagator is derived. The proposed formalism makes Dirac's hole theory obsolete and can readily be extended to many particles.

1. Introduction

It is widely believed that the effect of the spin on particle motion cannot be accurately described within the frame of validity of geometrical charged-particle optics. However, if the wave property of the particle can be neglected, there is no convincing reason which prevents one from incorporating the spin into the formalism of relativistic mechanics if an appropriate interaction Hamiltonian is found. To achieve a proper covariant formalism, it is advantageous to introduce a five-dimensional Hamiltonian-Jacobi approach for calculating the motion of particles in Minkowski space. Using $x_4 = ict$ as the fourth *"spatial"* coordinate of the four-dimensional *Euclidian* space, we avoid difficulties in constructing relativistic covariant Lagrangians and Hamiltonians. The dynamics of the particle is described by considering its coordinates x_μ, $\mu = 1, 2, 3, 4$, as functions of the independent Lorentz-invariant variable τ, which can be conceived as the *"world"* time or *universal time*, respectively. As a result, the time-like position coordinate x_4 need not to increase monotonically, as in classical mechanics.

Since the relativistic Lagrangian L_r is a scalar function, it must contain terms which are of entirely scalar nature such as a scalar function, scalar products formed by polar 4-vectors, second-rank 4-

tensors, axial 4-vectors and pseudo-scalar quantities. To be in accordance with non-relativistic classical mechanics, L_r must vanish for a particle at rest in Minkowski space. So far an appropriate Lorentz-invariant interaction term for the coupling of the spin with the electromagnetic field has not been given in the literature. Such a term will also allow one to consider the effect of the spin which acts as an additional source for the electromagnetic field.

2. Covariant Hamilton formalism for spin-1/2 particles

For reasons of simplicity we first consider the motion of a spin-1/2 charged particle in external fields. So far no satisfactory relativistic formulation for an interacting multi-particle system has been proposed for classical mechanics. However, by employing a covariant Hamilton's principle for the dynamics in the four-dimensional Minkowski space, the extension from a single particle system to a multi-particle system is straightforward.

The Lagrangian treatment of classical mechanics is based on Hamilton's principle which states that the action along the true path of a particle is an extremum, in general a minimum.

The action

$$S = Ex \int_{\tau_i}^{\tau_o} L_r \, d\tau \tag{1}$$

is the time integral of the Lagrangian

$$L_r = T - V \tag{2}$$

along the true path of the particle traveling from its initial position at universal time τ_i to its position at time of observation τ_o. The kinetic energy

$$T = T_{tr} + T_{rot} \tag{3}$$

is composed of two terms, the translational energy

$$T_{tr} = \frac{m}{2} \sum_{\nu=1}^{4} \dot{x}_\nu^{\,2} \tag{4}$$

and the kinetic energy of rotation

$$T_{rot} = \frac{1}{2} \sum_{\mu,\nu} S_{\mu\nu} \omega_{\mu\nu} \,. \tag{5}$$

The definition (4) of the translational energy is a straightforward extension of the classical kinetic energy of a point-like particle; m is its rest mass. Dots denote derivatives with respect to the universal time τ. The translational energy in Minkowski space (4) is always negative regardless of the sign of $\dot{x}_4$. A negative time-like component of the velocity 4-vector describes an electron which travels backwards in time. According to Feynman this can be conceived as a positron traveling

forward in time. The expression $-imc\dot{x}_4$ is generally defined as the total energy of the particle. This energy is positive for the electron and negative for the positron. Within the frame of our considerations this energy represents the fourth component of the kinetic moment 4-vector apart from a factor c. Although this vector is Lorentz invariant, its components are not. On the other hand our formulation requires the total energy to be a Lorentz-invariant scalar quantity.

By going from three to four dimensions, axial vectors are described by anti-symmetric second-rank tensors because a four-dimensional cubic volume has 12 surfaces, twice as many as the three-dimensional cube. Hence the four-dimensional angular velocity of the particle is defined by its 12 components $\omega_{\mu\nu}(\tau)$, each of which is the projection of the angular velocity onto the normal of the surface element $d\sigma_{\mu\nu}$. Since the normals of conjugate top and bottom surfaces have opposite directions only 6 independent components $\omega_{\mu\nu} = -\omega_{\nu\mu}$ exist. The same relations hold true for the components $S_{\mu\nu} = S_{\mu\nu}(\tau)$ of the angular momentum tensor or spin tensor, respectively. Since the absolute value of the spin is a constant of motion, the relation

$$\frac{1}{2}\sum_{\mu,\nu} S_{\mu\nu}^{\ 2} = \frac{\hbar^2}{4} \qquad (6)$$

holds. In the rest frame of the particle the time-like components $S_{\mu 4} = -S_{4\mu}$ and $\omega_{\mu 4} = -\omega_{4\mu}$ are zero. These components are purely imaginary. For the spin they can be considered as an electric moment induced by the motion of the magnetic moment. Since an elementary particle can be assumed as spherically symmetric, its spin and its angular frequency have the same direction in Minkowski space. Assuming that the absolute value of the angular velocity of the particle is a constant of motion, the rotational energy of the particle is found as

$$T_{rot} = \frac{1}{2}\sum_{\mu,\nu} S_{\mu\nu}\omega_{\mu\nu} = \vec{s}\vec{\omega} = \hbar\omega/2, \qquad (7)$$

where $\vec{s}$ and $\vec{\omega}$ denote the spin vector and the angular velocity vector, respectively, in the rest frame. The absolute value ω of the angular velocity is obtained from the condition that the total kinetic energy in Minkowski space must be zero in the absence of external fields:

$$T = T_{tr} + T_{rot} = -mc^2/2 + \hbar\omega/2 = 0. \qquad (8)$$

Hence the angular velocity of the particle $\omega = mc^2/\hbar = k_C c = \omega_C$ is identical with the Compton frequency, $\lambda_C = 2\pi/k_C$ is the Compton wave length. It readily follows from (8) that the rotational energy is half the rest energy mc^2. In the following we postulate that this behavior remains also true in the case of interacting external fields.

In the following we construct the covariant interaction energy V for a charged particle in external fields by neglecting terms which involve tensors of rank higher than two. In this case the potential energy

$$V = V_0 + V_1 + V_2 \qquad (9)$$

consists of a scalar coupling described by $V_0 = V_0(x_\nu, \tau)$, a vector coupling

$$V_1 = e \sum_{\nu=1}^{4} A_\nu \dot{x}_\nu \qquad (10)$$

and a tensor coupling

$$V_2 = -\frac{e}{2mc} \sum_{\mu,\nu} F_{\mu\nu} S_{\mu\nu}. \qquad (11)$$

The gyromagnetic ratio

$$e/2mc \qquad (12)$$

between the magnetic moment and the mechanic angular momentum of the electron equals that of a classical magnetic dipole. The Lande factor $g = 2$ need not to be considered because it is implicitly taken into account by the double summation in equation (11). The three spatial components $A_\nu = A_\nu(x_\mu, \tau)$, $\mu, \nu = 1,2,3,4$, of the electromagnetic 4-vector potential are real while the time-like component A_4 is purely imaginary. The components

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \qquad (13)$$

define the electromagnetic field tensor. The potentials V_1 and V_2 are of entirely electromagnetic nature, while the potential V_0 is not since it describes the coupling of two scalar quantities, for example masses. Accordingly, if we suppose that this potential energy accounts for the gravitation, it may be written as

$$V_0 = -m\Phi, \qquad (14)$$

where $\Phi = \Phi(x_\nu, \tau)$ is the gravitation potential. The potential energy V_2 is gauge-invariant, whereas the potential energies V_0 and V_1 are not. However, the equations of motion of the particle are invariant under gauge transformations of the potentials Φ and $\vec{A}$, respectively, although the Lagrangian (2) and hence the action (1) will change. Employing the gauges

$$\Phi \rightarrow \Phi + \Phi_0 \qquad \vec{A} \rightarrow \vec{A} + Grad\Lambda, \qquad (15)$$

the action (1) alters to

$$S \rightarrow S + m[\Phi_0(\tau_2) - \Phi_0(\tau_1)] + q[\Lambda(x_\nu(\tau_2)) - \Lambda(x_\nu(\tau_1))]. \qquad (16)$$

Here *Grad* denotes the four-dimensional gradient. The result (16) demonstrates that the action is not changed by any gauge transformation if the particle has performed a closed orbit in Minkowski space and $\Phi_0(\tau_1) = \Phi_0(\tau_2)$.

The path equations are obtained most conveniently from the action integral (1) by employing Hamilton's principle $\delta S = 0$. This condition states that among all possible paths along which the particle may move from one point to another in four-dimensional space within a certain time interval $\tau_o - \tau_i$, the actual path yields an extremum for the action. By employing the standard calculation of variation, we readily obtain from (1) the Euler-Lagrange equations for the particle motion

$$\frac{\partial L_r}{\partial x_\mu} - \frac{d}{dt}\frac{\partial L_r}{\partial \dot{x}_\mu} = 0. \qquad (17)$$

Considering the relations for the kinetic and the potential energies, the Lagrangian (2) adopts the form

$$L_r = \sum_\mu [\frac{m}{2}\dot{x}_\mu{}^2 - e\dot{x}_\mu A_\mu] - V_0 - V_2. \qquad (18)$$

The components of the momentum 4-vector are derived from this equation as

$$p_\mu = \frac{\partial L_r}{\partial \dot{x}_\mu} = m\dot{x}_\mu - eA_\mu \qquad (19)$$

These quantities are the canonical conjugate variables of the spatial coordinates x_μ. By inserting the expression (18) into the equation (17) and considering the relation (13), we eventually obtain the path equations

$$m\ddot{x}_\mu = -e\sum_\nu \dot{x}_\nu F_{\mu\nu} - e\frac{\partial A_\mu}{\partial \tau} - \frac{\partial V_0}{\partial x_\mu} - \frac{\partial V_2}{\partial x_\mu}, \qquad \mu = 1,2,3,4. \qquad (20)$$

If we multiply this equation with $\dot{x}_\mu$ and sum subsequently over the index μ, we find

$$\frac{d}{d\tau}\{\frac{m}{2}[\sum_\mu \dot{x}_\mu{}^2 - c^2] + V_0 + V_2\} = q\sum_\mu \dot{x}_\mu \frac{\partial A_\mu}{\partial \tau} + \frac{\partial V_0}{\partial \tau} + \frac{\partial V_2}{\partial \tau}. \qquad (21)$$

This equation can be written in the familiar form

$$\frac{dH_r}{d\tau} = -\frac{\partial L_r}{\partial \tau}, \qquad (22)$$

where

$$H_r = \sum_{\mu=1}^{4} \dot{x}_\mu \frac{\partial L_r}{\partial \dot{x}_\mu} - L_r = \frac{1}{2m} \sum_\mu (p_\mu - eA_\mu)^2 - \frac{mc^2}{2} + V_0 + \frac{e}{2m} \sum_{\mu,\nu} S_{\mu\nu} F_{\mu\nu} \tag{23}$$

is the relativistic four-dimensional Hamiltonian which should not be mixed up with the classical three-dimensional Hamiltonian H. If the potentials $\vec{A}$ and G are conservative, they do not depend on the universal time explicitly. In this case the terms on the right hand side of (21) vanish provided that

$$\frac{\partial V_2}{\partial \tau} = \kappa \sum_{\mu,\nu} \dot{S}_{\mu\nu} F_{\mu\nu} = 0. \tag{24}$$

As a result the Hamiltonian is then a constant of motion and equal to the total energy

$$H_r = T_{tr} - T_{rot} + V_0 + V_2 = E_0 = -mc^2. \tag{25}$$

The last relation is obtained by considering the special case where the particle is at rest in field-free space. It should be noted that the covariant energy E_0 does not depend on the sign of $\dot{x}_4$. Hence this energy must be the same for a particle and its anti-particle. Therefore, their masses are identical and positive definite. As a result the gravitational force between particles must always be attractive. It should be noted that covariant Hamiltonians suggested so far in the literature only consider the electromagnetic vector potential [1]. Moreover, the energy is either zero or $-mc^2/2$, respectively. Hence both Hamiltonians cannot be attributed to a specific energy. Our result explains the scalar nature of the covariant Hamiltonian (24) because the total energy is identical with the rest energy of the particle. This energy is a Lorentz scalar and can be conceived as the canonic conjugate "variable" of the universal time τ. Therefore, we can conclude that τ is not some meaningless Lorentz-invariant parameter, but may have a realistic physical meaning in the context of creation and annihilation of particles. Moreover, the universal time becomes obsolete or meaningless, respectively, for mass-less particles. The constraint (25) is considered as a true dynamical constraint that confines the motion of the particle to a particular three-dimensional surface in the four-dimensional non-Euclidian space. Our supposition differs from the conventional assumption that the absolute value of the velocity 4-vector is a constant of motion and equal to c. This condition is used as a definition of the *parameter* τ , which is considered as the proper time of the particle [1, 2]. In contrast to this procedure, we assume the existence of a true universal time which governs the dynamics of particles in four-dimensional space.

It is shown in the context of classical mechanics that the action S is a solution of the Hamilton-Jacobi equation apart from an arbitrary constant. By extending the formalism from three to four dimensions and considering τ as the equivalent time, the resulting five-dimensional equation is found as

$$\frac{\partial S}{\partial \tau} + H_r(x_\mu, \frac{\partial S}{\partial x_\mu}, \tau) = 0. \tag{26}$$

The canonical momentum variables in $H_r = H_r(x_\mu, p_\mu, \tau)$ have been replaced by means of the transformations

$$p_\mu = \frac{\partial S}{\partial x_\mu}, \qquad \mu = 1,2,3,4. \tag{27}$$

The Five-dimensional Hamilton-Jacobi equation (26) has the remarkable property to be both Lorentz-invariant and linear in the universal time τ. Hence we shall consider this equation as the short-wavelength limit of a covariant Schroedinger equation.
If H_r does not involve the universal time explicitly, this variable can be separated by assuming the solution

$$S = W - E\tau \tag{28}$$

resulting in the Hamilton-Jacobi equation of the reduced action W:

$$H_r(x_\mu, \frac{\partial W}{\partial x_\mu}) = E. \tag{29}$$

This equation does no longer involve the universal time.

3. Spin precession
The equations for the spin precession cannot be derived from Hamilton's principle. Therefore, we must try to construct these equations by imposing the constraints (6) and (24). In addition we require that the equations reduce to the standard form if the electric field strength vanishes,
As a suitable set of equations for the dynamics the spin in Minkowski space, we propose

$$\dot{S}_{\mu\nu} = \frac{e}{m} \sum_\lambda \{S_{\mu\lambda} F_{\lambda\nu} - S_{\nu\lambda} F_{\lambda\mu}\}, \qquad \mu,\nu = 1,2,3,4. \tag{30}$$

The equations do not alter if we exchange the subscripts μ and υ and consider $S_{\mu\nu} = -S_{\nu\mu}$. To prove the validity of the constraint (24), we multiply the equation (30) with $F_{\mu\nu}$ and sum over the indices μ und υ, yielding

$$\frac{m}{2e} \sum_{\mu,\nu} \dot{S}_{\mu\nu} F_{\mu\nu} = \sum_{\mu,\nu,\lambda} F_{\mu\lambda} F_{\mu\nu} S_{\lambda\nu} = \sum_{\mu,\nu,\lambda} F_{\mu\nu} F_{\mu\lambda} S_{\nu\lambda} = -\sum_{\mu,\nu,\lambda} F_{\mu\lambda} F_{\mu\nu} S_{\lambda\nu} = 0. \tag{31}$$

Subsequent relations have been obtained by exchanging two indices. Since this procedure is merely a change of notation, it does not affect the value of the summation. Assuming the validity of the equations (30), the result (31) demonstrates that (24) is automatically fulfilled and, hence, need not to be imposed as a constraint.
Multiplying (30) with $S_{\mu\nu}$ and employing the same procedure as in (31), we readily obtain

$$\sum_{\mu,\nu} \dot{S}_{\mu\nu} S_{\mu\nu} = \frac{1}{2} \frac{d}{d\tau} \sum_{\mu,\nu} S_{\mu\nu}{}^2 = 0. \tag{32}$$

Hence the relation (6) is also fulfilled. The spin tensor has the same structure as the electromagnetic field tensor because both tensors are Lorentz-invariant. Hence we can construct from the spatial components of the spin tensor a three-dimensional axial vector $\vec{S}_m$ with components $S_{mx} = S_{23}, S_{my} = S_{31}, S_{mz} = S_{12}$ and from the imaginary time-like components a real electric axial vector $\vec{S}_e$ with components $S_{ex} = iS_{41}, S_{ey} = iS_{42}, S_{ez} = iS_{43}$. The vector $\vec{S}_e$ accounts for the electric dipole moment induced by the motion of the magnetic dipole. If we also express the components of the electromagnetic field tensor by the components of the electric and magnetic field strengths, the equations (30) can be written as

$$\dot{\vec{S}}_m = \frac{e}{m}\{\vec{S}_m \times \vec{B} + \vec{S}_e \times \vec{E}/c\}, \qquad \dot{\vec{S}}_e = \frac{e}{m}\{\vec{S}_e \times \vec{B} - \vec{S}_m \times \vec{E}/c\}. \tag{33}$$

The coordinates of the electric and magnetic fields are given by the position of the particle. Hence the four path equations (19) and the 6 equations (30) or (33), respectively, form a coupled system of ten differential equations for the position of the particle and the orientation of its spin in the Minkowski space as functions of the universal time τ.

The equations (30) can be conceived as an alternative to the so-called BMT equation [3] which employs the spin 4-vector

$$S_\lambda = (1/2c)\varepsilon^{\lambda\kappa\mu\nu} \dot{x}_\kappa S_{\mu\nu} = \frac{1}{c}(-)^{\lambda+1} \sum_p \dot{x}_\kappa S_{\mu\nu}. \tag{34}$$

Here $\varepsilon^{\lambda\kappa\mu\nu}$ is the totally anti-symmetric fourth-rank unit tensor; denotes the cyclic permutation of the indices κ, μ, ν. These indices and λ differ from each other and each defines one of the four numbers 1, 2, 3, 4. It readily follows from the relation (34) that the scalar product $S_\lambda \dot{x}_\lambda$ vanishes identically. Hence this relation does not need to be considered as a constraint. To derive the equation for the precession of the spin, we take the derivative of the (34) with respect to the universal time τ and replace $\ddot{x}_\kappa$ by means of the path equation (19). Assuming both G=0 and $\partial_\tau A_\mu = 0$, we eventually derive at

$$\dot{S}_\lambda = (-)^{\lambda+1} \sum_p \{\ddot{x}_\kappa S_{\mu\nu} + \dot{x}_\kappa \dot{S}_{\mu\nu}\}/c = \frac{e}{m} \sum_\mu F_{\lambda\mu} S_\mu + \frac{1}{mc}(-)^{\lambda+1} \sum_p \frac{\partial V_2}{\partial x_\kappa} S_{\mu\nu}. \tag{35}$$

The last term accounts for the gradient forces. Neglecting this term, we obtain the BMT equation for a charged particle with Lande factor $g = 2$.

The vectors $\vec{S}_e$ and $\vec{S}_m$ transform from the system at rest of the particle to a system moving with velocity $\vec{v} = d\vec{r}/dt = \vec{\beta}c$ in the same as the electric and magnetic field strengths because they are

all components of second-rank tensors. Considering that $\vec{S}_e = 0$ and $\vec{S}_m = \vec{s}$ in the system at rest, we readily obtain

$$\vec{S}_e = \vec{s} \times \vec{\beta}, \qquad \vec{S}_m = \gamma \vec{s} - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta}\vec{s}), \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \qquad (36)$$

The vectors $\vec{S}_e$ and $\vec{S}_m$ are orthogonal to each other because $\vec{S}_e \vec{S}_m = 0$.

The number of variables can be reduced by consider the motion of the rest-frame spin $\vec{s} = \vec{s}(t)$ as a function of the time t. The corresponding equations for a Dirac particle are found as

$$\frac{d\vec{s}}{dt} = \frac{e}{m} \vec{s} \times [\vec{B} - \frac{\gamma}{\gamma+1} \vec{\beta}(\vec{\beta}\vec{B}) - \vec{\beta} \times \vec{E}/c] + \frac{\gamma^2}{\gamma+1} \vec{s} \times (\vec{\beta} \times \frac{d\vec{\beta}}{dt}). \qquad (37)$$

If we replace the acceleration in the laboratory frame $\vec{a} = cd\vec{\beta}/dt$ by the standard Lorentz equation, we derive at *Thomas's* equation for the spin precession in a uniform magnetic field [4]. However, we will demonstrate in the following that this substitution does not yield the complete relativistic result.

4. Reduced relativistic Lagrangian and Hamiltonian

We have shown that the Hamiltonian (25) is a constant of motion in Minkowski space if the potentials do not depend on the universal time τ explicitly. This relation can be considered as a constraint for the motion of the particle. The constraint can be used to replace the independent variable τ by one of the four coordinates x_μ, preferably the time $t = -ix_4/c$. However, this is only possible if we can express the rotational energy T_{rot} and the potential energy as functions of t Fortunately, this is possible by transforming the spin and the electromagnetic field strengths to the particle's rest frame by means of proper Lorentz transformations. In this frame the time-like components of both the angular velocity tensor and the spin tensor become zero. On the other hand we must consider that the rest frame rotates in Minkowski space if the velocity $\vec{v}$ is changed. The corresponding angular velocity

$$\vec{\omega} = \vec{\omega}_T = \frac{\gamma^2}{\gamma+1} \frac{\vec{a} \times \vec{v}}{c^2} \qquad (38)$$

is known as Thomas precession [4]. Hence the rotational energy in this frame is found as

$$T_{rot} = \vec{s}(\vec{\omega}_C + \vec{\omega}_T) = \frac{mc^2}{2} + \frac{\gamma^2}{\gamma+1} \frac{\vec{s}(\vec{a} \times \vec{\beta})}{c}. \qquad (39)$$

The second term is of entirely kinematical nature. It always arises when the velocity changes for whatever reason. The potential energy (6) can be written as

$$V_2 = -2\kappa(\vec{S}_m \vec{B} - \vec{S}_e \vec{E}/c) = \frac{e}{mc}(\vec{s} \times \vec{\beta})\vec{E} - \frac{e}{m}\gamma\{\vec{s} - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{s})\}\vec{B}. \tag{40}$$

Here $\vec{E}$ and $\vec{B}$ are the field strengths in the laboratory frame. By employing the relations (39) and (40), the Hamiltonian (25) can be written in the covariant form

$$H_r = \frac{mc^2}{2}(\frac{dt}{d\tau})^2[\vec{\beta}^2 - 1] - \frac{mc^2}{2} + U = H_{r0} = E_0 = -mc^2,$$

$$U = V_0 - \frac{\gamma^2}{\gamma+1}\vec{s}\frac{\vec{a}\times\vec{\beta}}{c} + \frac{e}{mc}\vec{s}(\vec{\beta}\times\vec{E}) - \frac{e}{m}\gamma\{\vec{s} - \frac{\gamma}{\gamma+1}\vec{\beta}(\vec{\beta}\vec{s})\}\vec{B}. \tag{41}$$

All quantities in these relations are functions of x, y, z and t. Since $H_r = E_0 = -mc^2$ is a constant of motion, we can use (41) to eliminate $d\tau/dt$ in the expression for the reduced action

$$W = Ex\int_{\tau_1}^{\tau_2}\sum_\mu p_\mu \dot{x}_\mu d\tau = Ex\int_{t_1}^{t_2}\{\frac{d\tau}{dt}m\sum_\mu \dot{x}_\mu^2 - e(\vec{v}\vec{A}_s + iA_4 c)\}dt = Ex\int_{t_1}^{t_2} L_{red} dt. \tag{42}$$

Since the expression (41) is quadratic in $dt/d\tau$, we obtain for the reciprocal differential quotient two solutions which only differ in sign; $\vec{A}_s$ is the spatial component of the 4-vector potential. It should be noted that $dt/d\tau$ only coincides with γ (36) in the case $U=0$. The positive solution can be attributed to the electron, the negative solution to the positron. Taking into account the positive solution for the electron, the corresponding reduced Lagrangian is derived as

$$L_{red} = -mc^2\sqrt{1-\beta^2}\sqrt{1+2U/mc^2} - e\vec{v}\vec{A}_s + e\varphi, \qquad A_4 = i\varphi/c. \tag{43}$$

Here φ denotes the electric potential. The function U vanishes if we neglect both scalar coupling and spin effects. In this case the Lagrangian of the reduced action and the resulting relativistic path equations adopt the well-known form listed in the literature. If we take into account the spin, the reduced Lagrangian $L_{red} = L_{red}(\vec{r}, \vec{v}, \vec{a}, t)$ becomes a function of the position $\vec{r}$, velocity $\vec{v} = d\vec{r}/dt$ and acceleration $\vec{a} = d^2\vec{r}/dt^2$ of the particle and of the time t. In this case the calculus of variation applied to the condition $\delta W = 0$ yields the modified Euler-Lagrange equations

$$\frac{d^2}{dt^2}(\frac{\partial L_{red}}{\partial \ddot{x}_\nu}) - \frac{d}{dt}(\frac{\partial L_{red}}{\partial \dot{x}_\nu}) + \frac{\partial L_{red}}{\partial x_\nu} = 0, \qquad \nu = 1,2,3. \tag{44}$$

In this equation the dots denote derivatives with respect to the conventional time t. Because the acceleration is contained in L_r in the form $\vec{a}(\vec{v}\times\vec{s})$, each of the resulting path equations has terms which are linear in the third derivative of the coordinates x_ν with respect to time and terms which are linear in the time derivatives of the spin components s_ν. So far a term $\dddot{v}$ has only been

considered phenomenologically in the context to include the reactive effects of radiation in the equations of motion for a charged particle [2]. Our covariant approach yields such terms in a straightforward way if spin effects are incorporated. Hence we may infer that the *radiative reaction force* is connected with the precession of the spin of the charged particle. This conjecture is reasonable because the spin is accompanied with a magnetic dipole whose motion induces an electric dipole in the laboratory frame.

The Lagrangian for the positron is obtained by changing the sign of the first term in the corresponding expression (43) for the electron. This differs from the common procedure which changes the sign of the charge. Because we consider in accordance with Feynman [5] the positron as an electron moving backward in time, the sign of the charge does not alter when going from the electron to the positron. However, a discrepancy exists because both procedures only yield the same equations of motion in the case $U=0$.

4.1 Non-relativistic approximation

The reduced Lagrangian L_{red} is rather involved and difficult to understand. In order to obtain a crude insight of its structure, it is advantageous to investigate the non-relativistic limit in more detail. For this purpose we expand the expression (43) in a power series with respect to $1/c^2$. Retaining only terms up to $1/c^2$, we eventually find

$$L_{red} \approx L_0 + L_1 + L_2,$$

$$L_0 = -mc^2, L_1 = \frac{m}{2}v^2 - V_0 - e\vec{v}\vec{A}_s + e\varphi + \frac{e}{m}\vec{s}\vec{B}, \qquad (45)$$

$$L_2 = \frac{m}{8c^2}v^4 + \frac{V_0^2}{2mc^2} - \frac{(\vec{s}\times\vec{v})\vec{a}}{2c^2} - \frac{e}{mc^2}\vec{s}(\vec{v}\times\vec{E}) + \frac{e}{2mc^2}(\vec{B}\times\vec{v})(\vec{s}\times\vec{v})].$$

By neglecting the constant term L_0, we obtain in the limit $c \to \infty$ the non-relativistic Lagrangian L_1 of the electron in an electromagnetic field. This Lagrangian yields the corresponding path equation

$$m\vec{a} = -e[\vec{E} + \vec{v}\times\vec{B}] + \frac{e}{m}grad(\vec{s}\vec{B}) - gradV_0 = \vec{F}. \qquad (46)$$

The force $\vec{F}$ consists of three terms, the familiar Lorentz force, the gradient force acting on the magnetic moment $e\vec{s}/m$ of the electron and the gravitational force. The non-relativistic equation for the precession of the spin is readily derived from (37) as

$$\frac{d\vec{s}}{dt} = \frac{e}{m}\vec{s}\times\vec{B}. \qquad (47)$$

Hence the precise non-relativistic motion of the electron and the precession of its spin are governed by the set (46, 47) of coupled differential equations because $\vec{B} = \vec{B}(x(t), y(t), z(t), t)$ is a function of the position of the electron. Our approach incorporates correctly the spin of the particle in the equations of motion without the need of a phenomenological g-factor and/or quantum-mechanical considerations, even in the relativistic case. This result contradicts the general belief

that a microscopic consideration of the spin is beyond the scope of classical electrodynamics. Within the frame of validity of the expansion (45), we can replace the particle acceleration $\vec{a}$ by the expression (46) yielding

$$L_2 = \frac{m}{8c^2}v^4 + \frac{V_0^2}{2mc^2} + \frac{1}{2mc^2}\vec{v}\{\vec{s}\times[\vec{E} + \frac{e}{m}grad(\vec{s}\vec{B}) - gradV_0]\}. \tag{48}$$

Using this relation together with the expressions (45) for L_0 and L_1, we obtain an expansion of the three-dimensional Hamiltonian

$$H = H(\vec{r},\vec{p},t) = \sum_{\mu=1}^{3} v_\mu \frac{\partial L_{red}}{\partial v_\mu} - L_{red}, \tag{49}$$

which is correct up to order $1/c^2$ inclusively. The somewhat lengthy yet straightforward calculation yields

$$H \approx H_0 + H_1 + H_2,$$
$$H_0 = mc^2,$$
$$H_1 = \frac{1}{2m}(\vec{p} - e\vec{A})^2 - e\varphi - \frac{e}{m}(\vec{s}\vec{B}) + V_0, \tag{50}$$
$$H_2 = -\frac{1}{8m^3c^2}(\vec{p} - e\vec{A})^2 - \frac{V_0^2}{2mc^2} - \frac{1}{2m^2c^2}(\vec{p} - e\vec{A})\{\vec{s}\times[\vec{E} + \frac{e}{m}grad(\vec{s}\vec{B}) - gradV_0]\}.$$

Here $\vec{p}$ denotes the three-dimensional canonical-momentum vector. It should be noted that $H_0 = -H_{ro} = -E_0$ has the same absolute value as the relativistic Hamiltonian. However, its sign is positive for the electron and negative for the positron, because the reduced Hamiltonian is related to the time-like component of the momentum 4-vector. Each component of this vector can be positive or negative depending on its direction in Minkowski space. As a consequence, we do not need Dirac's "hole theory" which must be introduced if $mc\dot{x}_4$ is conceived as the energy.

The Hamiltonian H_1 represents the Pauli Hamiltonian for the electron apart from the scalar potential V_0. The term H_2 coincides in the case $V_0 = 0$ with that obtained from the Dirac equation by means of the Foldy-Wouthuysen transformation if we disregard the so-called Darwin term [6]. This transformation reduces the Dirac spinor in such a way that it has a lower two-spinor identically zero for the electron. Within the frame of our non-quantum-mechanical calculations, this procedure corresponds to the transformation of the spin tensor from the laboratory frame to the particle rest frame. This behavior stirs the conjecture that the lower two-spinor should be attributed the time-like components of the spin in the laboratory system rather that to a positron contribution.

5. Properties of the action function

The action function (1) describes the propagation of an ensemble of identical particles in Minkowski space. To investigate this behavior in some detail, we assume that the H_r does not depend on the universal time explicitly. In this case the action function can be written as

$$S = \int_{\tau_1}^{\tau_2} [\sum_\mu p_\mu \dot{x}_\mu - H_r] d\tau = \int_{\vec{R}_1}^{\vec{R}_2} \vec{P} d\vec{R} - E_0(\tau_2 - \tau_1) = W(\vec{R}_1, \vec{R}_2) - E_0(\tau_2 - \tau_1). \qquad (51)$$

Here $\vec{P}$ and $\vec{R}$ denote the canonical momentum 4-vector and the position 4-vector, respectively. The integration has to be taken along the path which makes each of the two integral an extremum. Hence we have $\delta S = 0$ and $\delta W = 0$ for fixed positions of the path-defining initial and end points $\vec{R}_1$ and $\vec{R}_2$, respectively. A constant action $S = S_0$ represents a continuous set of surfaces

$$W(\vec{R}, \vec{R}_1) = E_0(\tau - \tau_1) + const. \qquad (52)$$

in the four-dimensional Minkowski space. If we vary the action S with respect to the coordinates of the endpoint $\vec{R}_2 = \vec{R}$, we readily derive from (51) the relation

$$GradS = GradW = \vec{P} = m\dot{\vec{R}} - e\vec{A}. \qquad (53)$$

This relation demonstrates that in field-free space ($\vec{A} = 0$) the trajectories of all identical particles emanating from the initial point $\vec{R}_1$ are the orthogonal trajectories to the surfaces of constant reduced action W. Since this function is attributed to an ensemble of trajectories of identical particles, it cannot distinguish the particles. As a result identical elementary particles seem to be indistinguishable. This behavior is a consequence of eliminating the universal time τ.
We can consider the condition $\delta W = 0$ as the four-dimensional extension of Maupertuis's *principle of least action*. This analogue implies that particles which start simultaneously from a given point in Minkowski space will intersect any surface of constant action W at the same universal time. Although this time is steadily increasing this is not necessarily true for the time-like spatial coordinate x_4. If the particle reverses its direction of flight with respect to this coordinate it converts to its anti-particle because an electron flying backward in the laboratory time can be considered as a positron flying forward in this time[5]. Reversal of the time-like momentum component implies that at least a change *2mc* in momentum must be transferred to the collision partner, which will emit subsequently a photon. In the Laboratory frame the observer records this event as electron-positron annihilation.
It should be noted that the action function S does not contain the initial velocity coordinates as variables explicitly. This behavior resembles the uncertainty principle of quantum mechanics according to which it is not possible to precisely determine the position and the momentum of a given state. In our case the initial canonical momentum can be obtained by varying (51) with respect to the initial coordinates of the position 4-vector $\vec{R}_1$ resulting in

$$\vec{P}_1 = -Grad_1 S = -Grad_1 W. \qquad (54)$$

Since W is a function of $\vec{R}$ and $\vec{R}_1$, we can use the relation (54) to obtain $\vec{R} = \vec{R}(\vec{R}_1, \vec{P}_1)$ as a function of the initial position and momentum of the particle, at least in principle. Owing to the existence of the action surface, the trajectories of identical particles emanating from a common

point in the four-dimensional space are correlated since their canonical momenta are orthogonal to this surface. It changes its shape as a function of the universal time, yet it will never be torn apart. However, the surface can degenerate in sheets which intersect each other forming a caustic. The caustic represents the loci of the intersections of *"rays"* which start with slightly different directions from the point source. If we take into account the wave nature of the electron, the surfaces of constant action also represent wave surfaces of constant phase. The action is a minimum for all points located in front of the caustic and a maximum if the end points are located behind the caustic. It should be noted that the particle description breaks down in the region of the caustic due to pronounced interference effects.

6. Self-action

So far we have considered the action resulting from external fields. However, the particle also produces a field. In order to guarantee that the motion of the particle is not affected by its own field, we must postulate that the *"self-action"* is a constant of motion. This constant can be put to zero because the action is only defined up to an arbitrary constant. The charge and the magnetic moment of the electron are the sources of its accompanying electromagnetic field. Any change of the velocity of the electron results in an additional radiation field. The entire electromagnetic field depends implicitly on the universal time τ, because the position coordinates $x_\mu = x_\mu(\tau)$ of the particle are functions of τ. To completely describe the self-action, we must take into account both the interaction terms and the electromagnetic field of the electron. Since this field is continuously distributed in the four-dimensional space, we must introduce a proper four-dimensional Lagrange density $l_s = l_s(x_\mu, \tau)$ and the normalized mass density $\rho_e = \rho_e(x_\mu, \tau)$ of the particle. The Lagrange density

$$l_s = \sum_\mu e\rho\dot{x}_\mu A_\mu + \sum_{\mu,\nu} \frac{e}{2m}\rho S_{\mu\nu} F_{\mu\nu} + \frac{i}{2c\mu_0}\sum_{\mu,\nu} F_{\mu\nu}\dot{F}_{\mu\nu} \tag{55}$$

consists of three terms. The first and the second term represent the self-interaction of the charge and the spin of the electron with its own electromagnetic field, whereas the third comprises the contribution of this field to the self-energy density; μ_0 is the permeability of the vacuum. The field term differs from that of standard field theory because we have extended the space from three to four dimensions. Accordingly, we must define the self-action as

$$S_s = Ex\int_{\tau_1}^{\tau_2}\{\int l_s d^4\vec{R}\}d\tau = Ex\int_{\tau_1}^{\tau_2}\{\int \sum_\mu[e\rho\dot{x}_\mu A_\mu + \sum_\nu(\frac{e}{2m}\rho S_{\mu\nu}F_{\mu\nu} + \frac{i}{2c\mu_0}F_{\mu\nu}\dot{F}_{\mu\nu})]d^4\vec{R}\}d\tau . \tag{56}$$

The equations for the electromagnetic field associated with the electron are derived by employing Hamilton's principle $\delta S_s = 0$. Since the motion of the particle is given, the variation has to be performed with respect to the four components A_μ of the 4-vector potential, where $\dot{F}_{\mu\nu}$ is considered as an independent variable. Derivatives of the deviation δA_μ with respect to x_ν are removed by partial integrations. Considering that the deviations are zero at the limits τ_1 and τ_2, we obtain

$$\delta S_s = \int_{\tau_1}^{\tau_2} \{\int \sum_\mu [j_\mu - \sum_\nu (\frac{e}{m}\rho S_{\mu\nu} + \frac{i}{c\mu_0}\frac{\partial \dot{F}_{\mu\nu}}{\partial x_\nu})]\delta A_\mu d^4\vec{R}\} d\tau = 0.$$ (57)

Here $j_\mu = e\rho \dot{x}_\mu$ is the four-dimensional current density. Because the variation can be taken at an arbitrary time $\tau_1 \le \tau \le \tau_2$ the expression in the brackets must vanish identically. Imposing the Lorentz gauge $Div\vec{A} = 0$, we derive the inhomogeneous equations

$$\sum_{\nu=1}^{4}\frac{\partial^2 \dot{A}_\mu}{\partial x_\mu^{\,2}} = ic\mu_0 [j_\mu - \frac{e}{m}\sum_\nu \rho S_{\mu\nu}], \qquad \mu = 1,2,3,4,$$ (58)

for the time derivatives $\dot{A}_\mu$ of the components of the 4-vector potential. The solution of this differential equation is accomplished most conveniently by means of the four-dimensional Green function

$$G = G(\vec{R},\vec{R}') = \frac{1}{4\pi^2(\vec{R}-\vec{R}')^2},$$ (59)

which satisfies the equation

$$\sum_\mu \frac{\partial^2 G}{\partial x_\mu^{\,2}} = -\delta^4(\vec{R}-\vec{R}').$$ (60)

Here $\delta^4(\vec{R}-\vec{R}')$ is the four-dimensional delta function; $\vec{R}'$ denotes the position of the point source By employing Green's method together with (59), we the solution

$$\dot{A}_\mu = \frac{c\mu_0}{4i\pi^2}\int \frac{1}{(\vec{R}-\vec{R}')^2}\{j_\mu(\vec{R}') - \frac{e}{m}\sum_\nu S_{\mu\nu}\frac{\partial \rho}{\partial x'_\nu}\}d^4\vec{R}'.$$ (61)

The derivative of with respect to x'_ν is removed by partial integration with respect to this coordinate. Moreover, we assume a point-like particle with density

$$\rho = \delta^4(\vec{R}'-\vec{R}_e(\tau)) \qquad .$$ (62)

By employing this density and assuming that the particle is created at the initial time $\tau_i = -\infty$, we obtain

$$A_\mu = \frac{c\mu_0}{4i\pi^2}\int_{-\infty}^{\tau_e}\{\frac{e\dot{x}_{e\mu}(\tau)}{[\vec{R}-\vec{R}_e(\tau)]^2} + \frac{2e}{m}\sum_\nu \frac{S_{\mu\nu}(\tau)[x_\nu - x_{e\nu}(\tau)]}{[\vec{R}-\vec{R}_e(\tau)]^4}\}d\tau .$$ (63)

The integration must be taken over the entire "life-time" of the particle starting at the initial time up to the universal time of observation τ_o. This time is related to the time of observation t_o in the three-dimensional laboratory frame via the relation $t_o = x_4(\tau_o)/ic$.
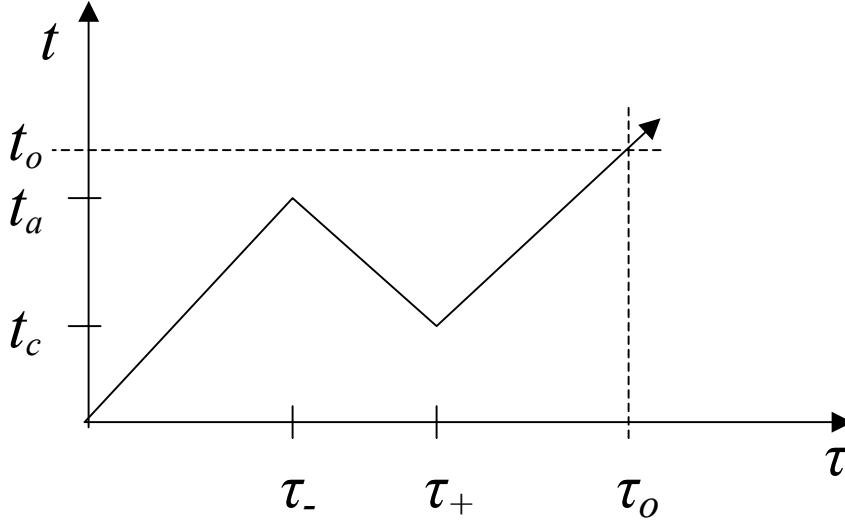

Fig.1. Laboratory time t of an electron as a function of the universal time τ.

Figure 1 shows a possible scenario for the laboratory time of an electron as a function of the universal time. In this case an interval exists for τ during which the laboratory time of the particle $t = t(\tau)$ decreases. Depending on the time of observation t_o, the observer in the laboratory frame records either a single electron or two electrons and a positron, because a negative time-like component $\dot{x}_4$ of the 4-velocity represents the anti-particle within the frame of our considerations. The diagram illustrates an interesting phenomenon. The observer records for $t_o < t_c$ a single electron. The particle seems to change the direction of its time-like velocity component at the universal time $\tau = \tau_- < \tau_+$. At $\tau = \tau_+$ the observer detects at his laboratory time $t = t_c$ the creation of a positron and an electron pair which annihilates somewhat later at laboratory time $t = t_a$. This annihilation cannot correspond to a directional change of the time-like velocity component that had occurred before the universal time τ_+ at which the direction would be reversed again because in the time interval $t_c \le t \le t_a$ the observer detects three particles at different locations in three-dimensional space. This apparent contradiction can be avoided by assuming that the positron is an electron with a negative time-like velocity component.

The main contribution to the integral (63) is delivered by the poles of its integrand. These poles are the zeros τ_v of the denominators and defined by

$$[\vec{R} - \vec{R}_e(\tau)]^2 = 0. \tag{64}$$

Accordingly, the zeros $\tau_v = \tau_v(\vec{R})$ are functions of the coordinates x_μ of the point of observation.

Depending on the path of the electron as a function τ, a distinct number N of poles exists. However, only the poles $\tau_v < \tau_o$ are contributing to the integral. If we assume that the integrand is negligibly small for $\tau = \tau_o$ and ignore the poles $\tau_v > \tau_o$, we can extent the upper integration limit to infinity. The components $x_{ev}(\tau)$ and $S_{\mu v}(\tau)$ of the electron's position and spin, respectively, are analytical functions of τ which vanish at infinity. In this case we can perform the integration analytically in the complex τ-plane by closing the contour of the integration by an infinite half circle in such a way that the loop comprises all poles $\tau_v < \tau_o$. Hence the value of the integral (63) is given by the sum of the residues. In order to find the residues we must expand the integrand in a Laurent series about each pole. The corresponding residue is given by $2ia_{-1}\pi$, where a_{-1} is the coefficient of the term $a_{-1}/(\tau - \tau_v)$ of the Laurent series. Taking into account these results, we eventually obtain

$$A_\mu \approx \oint \dot{A}_\mu d\tau = \sum_v \frac{ce\mu_0}{4\pi} \frac{\dot{x}_{e\mu}(\tau_v)}{\dot{\vec{R}}_{ev}(\vec{R}_{ev} - \vec{R})} +$$

$$\frac{ce\mu_0}{4m\pi} \sum_{v,\lambda} \{ \frac{\dot{S}_{\mu\lambda}(\tau_v)(x_\lambda - x_{e\lambda}(\tau_v)) - S_{\mu\lambda}(\tau_v)\dot{x}_{e\lambda}(\tau_v)}{[\dot{\vec{R}}_{ev}(\vec{R}_{ev} - \vec{R})]^2} + \frac{S_{\mu\lambda}(\tau_v)[x_\lambda - x_{e\lambda}(\tau_v)][\dot{\vec{R}}_{ev}^2 - \ddot{\vec{R}}_{ev}(\vec{R}_{ev} - \vec{R})]}{[\dot{\vec{R}}_{ev}(\vec{R}_{ev} - \vec{R})]^3} \}.$$

(65)

The acceleration $\ddot{\vec{R}}_{ev} = \ddot{\vec{R}}_e(\tau_v)$ and $\dot{S}_{\mu\lambda}(\tau_v)$ vanish if the electron moves in a field-free region. To demonstrate the validity of the expression (65), we consider an electron in its field-free rest frame. In this case we also have $\dot{x}_{e1} = \dot{x}_{e2} = \dot{x}_{e3} = 0, \dot{x}_{e4} = ic$, $S_{4\lambda} = 0$ and $\tau_o = t$. Then the equation (64) adopts the simple form

$$[\vec{R} - \vec{R}_e]^2 = (\vec{r} - \vec{r}_e)^2 - c^2(t - \tau)^2 = 0$$

(66)

with the two solutions $\tau_1 = t - |\vec{r} - \vec{r}_e|/c$, $\tau_2 = t + |\vec{r} - \vec{r}_e|/c > \tau_o$. The second solution, which does not contribute to the integral (63), violates causality in the laboratory frame. In the rest frame the spin tensor degenerates to a three-dimensional axial vector $\vec{s} = s_x\vec{e}_x + s_y\vec{e}_x + s_z\vec{e}_z$ with components $s_x = S_{23}, s_y = S_{31}, s_z = S_{12}$. By taking into account these considerations together with the relation $\mu_0\varepsilon_0 = 1/c^2$, we eventually find

$$\varphi = icA_4 = -\frac{e}{4\pi\varepsilon_0|\vec{r} - \vec{r}_e|}, \qquad \vec{A}_s = \frac{e\mu_0}{4m\pi} \frac{(\vec{r} - \vec{r}_e) \times \vec{s}}{|\vec{r} - \vec{r}_e|^3}.$$

(67)

These familiar relations represent the electrostatic potential φ of a charge $q = -e$ and the magnetic vector potential of a magnetic moment $\vec{\mu} = e\vec{s}/m$, respectively. If we apply the same procedure to a steadily moving electron and neglect the spin, we obtain the Lienard-Wiechert potentials [7] which represent the electric and the magnetic field of charge moving in the laboratory frame.

7. Multi-particle system

Our covariant Lagrangian procedure is well suited for a relativistic formulation of the dynamics of a many-particle system in Minkowsi space. In our description τ is not considered as a meaningless Lorentz-invariant parameter, but has the property of a steadily increasing universal time. Since it is connected with the mass, we may conclude that it started at the *big bang*. Moreover, the universal time can be considered as a hidden Bell parameter [8] with a realistic physical property. The incorporation of the universal time as the independent Lorentz-invariant variable also avoids the need for statistical or probability descriptions, because it becomes possible to describe the motion of the constituent particles of an ensemble separately as long as interference effects can be neglected. To elucidate this behavior, we consider the classical motion of particles emanating from a point source in a static three-dimensional field. By solving the equations of motion, we obtain the position of each particle as a function of the laboratory time t. Hence if the velocity vectors of the particles are given at some initial time $t = t_i$, we can precisely determine their position at any later time $t > t_i$. Since the forces are conservative, we can use the relation for the conservation of energy to substitute a spatial coordinate for the time. In this case the particle ensemble is described by a homo-centric bundle of trajectories, each of which represents the path of a particle. However, by using this procedure, we have lost information because we can no longer distinguish particles traveling along the same trajectory nor determine the position of the particles at a given time. Using the number of trajectories per unit area as a measure, we can determine the probability to find a particle at a given position if the current density of the source is known. The same situation arises in Minkowski space if we substitute the laboratory time t for the universal time τ by means of the relation (41) because in this space t has the role of a spatial coordinate.

The universal time should not be mixed up with the proper time of a given particle. Therefore, the universal time is the proper invariant parameter to describe the evolution of the system in the four-dimensional space. Within the frame our approach, the time-like spatial coordinate $x_{n4} = x_{n4}(\tau)$ of each member n of an ensemble of N particles is a function of the universal time. Accordingly, these coordinates will in general differ from each other for a given τ. Conversely, the requirement $x_{n4} = x_4$ will result in a universal time τ_n which is different for each particle n. Moreover, our preceding results enable a straightforward treatment of covariant interactions between particles. The corresponding Lagrangian is readily obtained by extending (18) from a-single particle to many particles. Owing to the linearity of the Maxwell equations, the electromagnetic 4-vector potential produced by the N particles

$$\vec{A} = \vec{A}(\vec{R}, \vec{R}_1, \vec{R}_2, ..., \vec{R}_N) = \sum_{n=1}^{N} \vec{A}_n(\vec{R}, \vec{R}_n) \tag{68}$$

is the sum of the 4-vector potentials (65) of all N particles. Their total kinetic energy in the four-dimensional space is given by

$$T = \frac{1}{2} \sum_{n=1}^{N} m_n \{ \sum_{\mu=1}^{4} \dot{x}_{n\mu}^2 - c^2 \}, \tag{69}$$

where m_n denotes the rest mass of the nth particle. We suppose that the scalar potential $\Phi_n = \Phi_n(\vec{R}, \vec{R}_n)$ can be derived in the same way as the electromagnetic potential by imposing the condition that the gravitational self-action vanishes. We eventually obtain for a point-like particle the result

$$\Phi_n = \frac{icm_c\kappa}{4\pi^2}\int_{-\infty}^{\tau_0}\frac{d\tau}{[\vec{R}-\vec{R}_n]^2} \approx \frac{m_nc\kappa}{4\pi\dot{\vec{R}}_n(\vec{R}-\vec{R}_n)|\tau = \tau_-} . \tag{70}$$

Here κ is the gravitational constant and $\tau_- < \tau_o$ represents the retarded solution of equation (64), where we have assumed that only a single retarded solution exists. Since the gravitational force is negligibly small, it will not be considered in the following. Then the potential energy of the system is given by

$$V = \sum_n\left(q_n\sum_\mu\dot{x}_{n\mu}\sum_{m\approx n}^N A_{m\mu} + \frac{e(1+\varepsilon_n)}{2m_n}\sum_{\mu,\nu}S_{n\mu\nu}\sum_{m\neq n}^N F_{m\mu\nu}\right), \tag{71}$$

where q_n denotes the charge and ε_n accounts for the *anomalous* magnetic moment of the nth particle, respectively. The sum over the index m has to be taken over all particles $m \neq n$ because the self-action does not contribute. The variation of the corresponding Lagrangian with respect to the coordinates $x_{n\mu}$ of each particle yields *4N* equations of motion. However, since these equations also depend on the spin motions, we must in addition incorporate the equations for the spin precession. These equations are readily derived by applying the relation (30) to a system of many particles. Their charges and spins produce the external field which affects the precession of the spin of each particle. The generalization of (30) from a single-particle system to a many-particle system yields for the spin precession the set of equations

$$\dot{S}_{n\mu\nu} = \sum_{\lambda=1}^4\{S_{n\mu\lambda}\sum_{m\neq n}^N F_{m\lambda\nu} - S_{n\nu\lambda}\sum_{m\neq n}^N F_{m\lambda\mu}\}, n = 1,..,N; \mu,\nu = 1,2,3,4. \tag{72}$$

This set consists of *6N* equations because the spin components are anti-symmetric. Together with the *4N* equations for the motion of the particles we have to solve *10N* equations for obtaining a deterministic description for evolution of the system in Minkowski space. Multiplying (72) with $S_{n\mu\nu}$ and performing the same manipulations as in (31), we find that the absolute value of the spin is a constant of motion for each particle. This behavior does not hold true for the energy since it can be shown that only the total energy $H_{0t} = -\sum_{n=1}^N m_nc^2$ is a constant of motion provided that the electromagnetic field does not depend on τ explicitly. Hence it is only possible to substitute the time-like coordinate of any single particle for the universal time τ. The *N+1* conserved dynamical quantities act as constraints for the motion of the *N* particles in Minkowski space. Our covariant treatment of the multi-particle system has demonstrated that it is possible, at least in principle, to determine the motion of relativistic particles by employing the same methods as in classical mechanics, although the amount of expenditure increases significantly.

8. Quantization of the five-dimensional Hamilton-Jacobi equation

The Hamilton-Jacobi (HJ) equation of classical mechanics offers the most appropriate procedure for incorporating the wave nature of the particles in the dynamics of the system. The reason for this behavior is due to the fact that the HJ equation for a single particle represents the particle analogue of the eikonal equation of light optics which is the wave surface or the phase, respectively, of a monochromatic wave in the limit of an infinitely short wave length. According to this analogue, we can assume that the action S represents the phase of the wave ψ associated with the particle. Since a constant phase represents the surface which is orthogonal to all trajectories emanating from a common point, the eikonal cannot directly be attributed to the path of a distinct particle. The scalar optical wave equation can be retrieved from the corresponding eikonal equation by substituting the operator $-i\lambda_v \partial / \partial x_\mu$ for the partial derivative $\partial S / \partial x_\mu$, $\mu = 1,2,3,4$, where $\lambda_v = \lambda_v / 2\pi = 1/k_v$ is the vacuum wavelength. The resulting operator then acts on the wave function ψ yielding the wave equation. Employing the same procedure to the three-dimensional HJ equation together with the substitution $-i\hbar \partial \psi / \partial t$ for $\partial S / \partial t$ and replacing the wave number k_v by $\hbar$, we readily derive the time-dependent Schroedinger equation. Since this equation is of first order with respect to the time derivative, it only allows propagation in the positive direction of the time. This is reasonable if the time t is considered as a steadily increasing variable.

Within the frame of our relativistic considerations, we must consider t as a fourth time-like spatial coordinate which can adopt positive and negative values. The role of the time is taken by the universal time τ. The classical non-relativistic HJ equation does not account for the spin. As a result the spin term must be incorporated into the Hamiltonian of the resulting Schroedinger equation a posteriori yielding the Pauli equation. On the other hand the four-dimensional Hamiltonian (28) contains the spin term. Its structure suggests that it must be quantized in such a way that the resulting term coincides with that of the iterated Dirac equation [9]. By imposing this condition we must express the wave function as a 4-component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \overline{\psi} = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix}, \tag{73}$$

where $\overline{\psi}$ is the adjoint spinor. However, within the frame of our considerations the components $\psi_\mu = \psi_\mu(x_v, \tau)$ are function of the four spatial coordinates x_v and the universal time τ. Moreover, the wave function must be normalized in four-dimensional space rather than in three-dimensional space as in Dirac theory. According to this theory, the components of the spin tensor are operators represented by

$$S_{\varpi v} = \hbar \sigma_{\mu v} / 2, \qquad \sigma_{\mu v} = -i\gamma_\mu \gamma_v, \tag{74}$$

where γ_μ, $\mu = 1,2,3,4$, are the *gamma matrices*. By taking into account these relations and applying the *quantization "rules"* to the five-dimensional HJ equation (26) with Hamiltonian (23), we readily derive the covariant wave equation

$$i\hbar \frac{\partial \psi}{\partial \tau} = H_r \psi,$$

$$H_r = -\frac{\hbar^2}{2m}\sum_{\mu=1}^{4}\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar}A_\mu\right)^2 - \frac{mc^2}{2} + V_0 + \frac{e\hbar}{4m}\sum_{\mu,\nu}\sigma_{\mu\nu}F_{\mu\nu}.$$

(75)

This equation has a similar structure as the Schroedinger equation and it fulfills the requirement to be linear in the time derivative. Due to the asymmetry with respect to τ, the wave equation differentiates between past and future. As a consequence an event at time τ can only be influenced by events ar previous times $\tau' \le \tau$. In order that (75) represents a suitable equation, it must be possible to form a positive-definite scalar density. In the Schroedinger theory the wave function ψ is a complex scalar function whose adjoint is the conjugate complex function ψ^*. The probability density is the product of these functions. Accordingly, it seems to be appropriate to define the product

$$\rho = \overline{\psi}\psi$$

(76)

as the probability density in Minkowski space. In order that this definition is reasonable, the probability density must be positive definite and satisfy a continuity equation. This equation is obtained by multiplying (75) on the left by $\overline{\psi}$, the adjoint equation by ψ from the right and subtracting the resulting equations from each other. It should be noted that the adjoint equation is obtained by taking the conjugate complex of equation (75) and by placing $\overline{\psi}$ in front of the double sum in the last term. This is required because $\overline{\psi} = \gamma_4\psi^+$ does not commute with $\sigma_{\mu\nu}$, where $\psi^+ = \left(\psi_1^* \quad \psi_2^* \quad \psi_3^* \quad \psi_4^*\right)$ is the *Hermitian conjugate* spinor. Considering the relation

$$\gamma_4\sigma_{\mu\nu}F_{\mu\nu}^* = \sigma_{\mu\nu}F_{\mu\nu}\gamma_4,$$

(77)

we find

$$\frac{\partial(\overline{\psi}\psi)}{\partial \tau} = \frac{\hbar}{2im}\sum_{\mu=1}^{4}\frac{\partial}{\partial x_\mu}\left(\Psi\frac{\partial\overline{\psi}}{\partial x_\mu} - \overline{\psi}\frac{\partial\psi}{\partial x_\mu} + \frac{2ie}{\hbar}A_\mu\overline{\psi}\psi\right).$$

(78)

This relation represents the four-dimensional continuity equation

$$\frac{\partial\rho}{\partial \tau} + Div\vec{J} = 0,$$

(79)

where *Div* denotes the four-dimensional divergence and

$$\vec{J} = \frac{\hbar}{m}\Im\left(\overline{\psi}Grad\psi + \frac{2ie}{\hbar}\vec{A}\overline{\psi}\psi\right)$$ (80)

represents the current density which does not contain the spin. This current density coincides with that derived from the Klein-Gordon equation and should either be interpreted as mass-current density or as charge-current density rather than as the probability flux [10]. It should be mentioned that a continuity equation of the form (79) does not exist for $\psi^+\psi$. Since the current density (80) does not vanish in the case $e = 0$, it is more appropriate to interpret (80) as mass-current density and ρ as the mass density in four-dimensional space. In this case we should be able to incorporate the creation or annihilation of particles in our formalism in a deterministic way.

By assuming that $\vec{J}$ vanishes on the surface of the infinite hyper-sphere, the four-dimensional Gauss theorem applied to the continuity equation (78) or (79) yields

$$\int(\overline{\psi}\psi)d^4\vec{R} = const .$$ (81)

Accordingly, the mass density

$$\rho = \overline{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$ (82)

can never change sign. This result is very satisfactory because it avoids the unphysical interpretation of ρ as a probability density. Moreover, it supports the conjecture that $\psi = \psi(x_\mu,\tau)$ defines a *"real physical situation"*. Since present quantum mechanics does not include the universal time τ , this formalism is incomplete, as argued by Einstein [11].

As a further test for the validity of our approach, we determine the evolution of the Hermitian spin operators $\sigma_{\mu\nu}$ with respect to the universal time τ. The expectation value of a Hermitian operator Ω in the four-dimensional space must satisfy the relation

$$<\Omega> = \int\overline{\psi}\Omega\Psi d^4\vec{R} = \int\psi\Omega^+\overline{\psi}d^4\vec{R} = <\overline{\Omega}> .$$ (83)

By taking the derivative of this equation with respect to τ and utilizing the relativistic wave equation (75), we obtain the Heisenberg equation of motion in four-dimensional space:

$$\frac{d\Omega}{d\tau} = \frac{i}{\hbar}[H_r,\Omega] + \frac{\partial\Omega}{\partial\tau} .$$ (84)

We apply this equation to the spin operator $\sigma_{\mu\nu}$ which does not depend explicitly on time. In this case the last term in (84) vanishes. It follows from the expression (75) that all terms of the Hamilton operator H_r commute with $\sigma_{\mu\nu}$ apart from the spin term. Considering in addition the commutation relation for the spin operators

$$[\sigma_{\mu\lambda},\sigma_{\mu\kappa}] = 2i\sigma_{\lambda\kappa} ,$$ (85)

we find

$$\frac{d\sigma_{\mu\nu}}{d\tau} = \frac{ie}{4m}\sum_{\kappa,\lambda}F_{\kappa\lambda}\left(\sigma_{\kappa\lambda}\sigma_{\mu\nu} - \sigma_{\mu\nu}\sigma_{\kappa\lambda}\right) =$$

$$\frac{ie}{2m}\sum_{\lambda}\left[F_{\mu\lambda}\left(\sigma_{\mu\lambda}\sigma_{\mu\nu} - \sigma_{\mu\nu}\sigma_{\mu\lambda}\right) - F_{\nu\lambda}\left(\sigma_{\nu\lambda}\sigma_{\nu\mu} - \sigma_{\nu\mu}\sigma_{\nu\lambda}\right)\right] = \frac{e}{m}\sum_{\lambda}\left[\sigma_{\mu\lambda}F_{\lambda\nu} - \sigma_{\nu\lambda}F_{\lambda\mu}\right]$$

(86)

The comparison of this quantum-mechanical operator equation with the corresponding "mechanical" equation (30) reveals that this equation forms the exact *"classical"* counterpart to (86).

9. Free-particle solutions

For a free particle ($A_\mu = 0, V_0 = 0$) the wave equation (75) reduces to

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial\tau} = \frac{\hbar^2}{2m}\sum_{\mu}\frac{\partial^2\psi}{\partial x_\mu^2} + \frac{mc^2}{2}\psi \ ,$$

(87)

which has the stationary solution $\psi = e^{-iE_0\tau/\hbar}\psi_s, E_0 = -mc^2$. The function $\psi_s = \psi_s(x_\mu)$ depends only on the four spatial coordinates and satisfies the field-free Klein-Gordon equation

$$\sum_{\mu}\frac{\partial^2\psi_s}{\partial x_\mu^2} - k_C^2\psi_s = 0, \qquad k_C = mc/\hbar \ .$$

(88)

This equation has plane-wave solutions

$$\psi_s = \psi_e + \psi_p,$$
$$\psi_e = C_1 e^{-i\vec{K}\vec{R}}, \psi_p = C_2 e^{i\vec{K}\vec{R}},$$

(89)

where the wave 4-vector fulfills the relation

$$\vec{K}^2 = k_4^2 + \vec{k}^2 = -k_C^2 \ .$$

(90)

The time-like component of this vector

$$k_4 = p_4/\hbar = ik_0 = iE/c\hbar = \pm i\sqrt{\vec{k}^2 + k_C^2}$$

(91)

is purely imaginary and proportional to the "conventional" energy *E*, which is defined by the time-like component p_4 of the momentum 4-vector. Accordingly, the two terms of the wave function ψ have phases

$$\Phi_{1,2} = k_C c\tau \mp [|E|t/\hbar - \vec{k}\vec{r}]. \tag{92}$$

The surfaces of constant phase $\Phi_1 = \Phi_2 = 0$ taken at different universal times $\tau_n = n\Delta\tau, n = 1,2,....$, form two sets of hyper-planes in four-dimensional space, one associated with particles moving forward in time t, the other with particles moving backward in this time. The paths of the particles, which are the orthogonal trajectories to these planes, form straight lines. According to these considerations we can attribute the term ψ_e to the electron and the term ψ_p to the positron. It should be mentioned that the phase Φ_2 for the positron part of the wave function (89) differs from that given in the literature by the minus sign in front of the last term .

In order to determine the unknown "constants" C_1 and C_2, we must consider that both electron and positron are spin-1/2 particles whose wave functions must also satisfy the *Dirac equation*. Hence these factors are spinors rather than scalars. Since the direction of the momentum 4-vector of the electron is opposite to that of the positron, we suppose that ψ_e and ψ_p satisfy separate equations

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + k_C\right)\psi_e = 0, \qquad \left(-\gamma_\mu \frac{\partial}{\partial x_\mu} + k_C\right)\psi_p = 0. \tag{93}$$

Here summation has to be taken over equal indices. The first equation is the familiar Dirac equation which is supposed to describe both the electron and the positron. We do not follow this route and propose a slightly different equation for the positron. It should be noted that the mass is positive definite for both particles. By adding and subtracting the two equations we get two coupled first-order differential equations for $\psi_s = \psi_e + \psi_p$ and $\psi_a = \psi_e - \psi_p$, respectively.

Eliminating ψ_a yields directly the *Klein-Gordon equation* for ψ_s.

In order to determine the constant four-component spinors, we insert each of the wave functions (89) into the corresponding equation (93). Since we consider positron as an electron with a negative time-like momentum component, the spinors are the same for both particles contrary to the results found in textbooks [10]. Moreover, because we consider ρ (82) as mass density, it cannot adopt negative values. Therefore, the factors C_1 and C_2 must have the form :

$$C_1 = a_1 u^{(1)}(\vec{K}) + a_2 u^{(2)}(\vec{K}). \qquad C_2 = a_3 u^{(1)}(\vec{K}) + a_4 u^{(2)}(\vec{K}), \tag{94}$$

where each of the four coefficients a_μ is a complex scalar constant. The two spinors are normalized eigen-vectors which are orthogonal to each other. They are found as

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \dfrac{k_3}{k_0 + k_C} \\ \dfrac{k_1 + ik_2}{k_0 + k_C} \end{pmatrix}, u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \dfrac{k_1 - ik_2}{k_0 + k_C} \\ \dfrac{-k_3}{k_0 + k_C} \end{pmatrix}; N = \sqrt{\dfrac{k_0 + k_C}{2k_C}}. \tag{95}$$

These spinors coincide with those given in the literature for the electron [10]. Our approach enables us to attribute the lower two components of the spinors to the time-like components of the expectations values $<\sigma_{\mu\nu}>$ of the spin operators for the electron and the positron. Moreover, the mass density is positive definite for both particles.

The wave equation (75) governs the evolution of the wave function $\psi = \psi(\vec{R}, \tau)$ in Minkowski space with respect to the universal time. Since this equation is of first order in τ, the wave function at time $\tau > \tau_0$ can be derived from the wave equation (75) if ψ is known at any previous time τ_0. This equation can be considered formally as a four-dimensional inhomogeneous diffusion equation with an imaginary diffusion coefficient D. In order to demonstrate this equivalence, we rewrite (75) as follows:

$$D \frac{\partial \psi}{\partial \tau} + \sum_{\mu} \left(\frac{\partial}{\partial x_{\mu}} + \frac{ie}{\hbar} A_{0\mu} \right)^2 \psi + k_C^2 \psi = \Sigma, \quad D = \frac{2mi}{\hbar}, \tag{96}$$

where $A_{0\mu}$ are the components of an arbitrary constant 4-vector potential. The *"source term"* $\Sigma = \sigma\psi = (2m/\hbar^2) H_{int}\psi$ with operator

$$\sigma = \sigma_1 + \sigma_2,$$

$$\sigma_1 = \frac{2ie}{\hbar} \sum_{\mu} (A_{0\mu} - A_{\mu}) \frac{\partial}{\partial x_{\mu}} + \frac{e^2}{\hbar^2} \left(\vec{A}^2 - \vec{A}_0^2 \right), \sigma_2 = \frac{e}{2\hbar} \sum_{\mu,\nu} \sigma_{\mu\nu} F_{\mu\nu} + \frac{2m}{\hbar^2} V_0 \tag{97}$$

accounts for the interaction Hamiltonian H_{int}. The form (95) of the wave equation offers the possibility to transform (96) to an integral equation by employing the Green's method for solving the diffusion equation [12]. The Green function $G = G(\vec{R}, \tau; \vec{R}', \tau')$ is the solution of (96) for a point source $\Sigma_P = -\delta(\vec{R} - \vec{R}')\delta(\tau - \tau')$ which is turned on at time τ' at position $\vec{R}_0$. The solution for $\tau \geq \tau'$ is found as

$$G = \frac{m}{8i\pi^2 \hbar(\tau - \tau')^2} e^{iS/\hbar} = \frac{\hbar}{2im} \frac{\sqrt{D_V}}{(2\hbar\pi)^2} e^{iS/\hbar}, \tag{98}$$

where

$$S = S(\vec{R} - \vec{R}', \tau - \tau') = \frac{m}{2}\frac{(\vec{R} - \vec{R}')^2}{\tau - \tau'} + \frac{mc^2}{2}(\tau - \tau') - e\vec{A}_0(\vec{R} - \vec{R}') \qquad (99)$$

is the solution of the field-free relativistic Hamilton-Jacobi equation (26) in the case of a non-vanishing constant 4-vector potential for trajectories originating at the common point $\vec{R} = \vec{R}'$. The gauge of the constant 4-vector potential is chosen such that $\vec{A}_0 = \vec{A}(\vec{R}')$. The four-dimensional van Vleck determinant

$$D_V = \det\left|\frac{\partial^2 S}{\partial x_\mu \partial x'_\nu}\right| = \frac{m^4}{(\tau - \tau')^4} \qquad (100)$$

represents *the trajectory density* apart from a constant factor. The solution (98) is the elementary four-dimensional wave emanating from the point $\vec{R}'$ at universal time τ'. Since the total energy

$$E_0 = -\frac{\partial S}{\partial \tau} = -\frac{m}{2}\left(\frac{(\vec{R} - \vec{R}')^2}{(\tau - \tau')^2} + c^2\right) \qquad (101)$$

is not conserved, the elementary wave cannot be attributed to a single particle with a fixed rest energy. To survey the property of this wave, we investigate the propagation of its wave surface S=0 in the case $\vec{A}_0 = 0$. The resulting equation

$$c^2(t - t')^2 - (\vec{r} - \vec{r}')^2 = c^2(\tau - \tau')^2. \qquad (102)$$

describes hyperboloids which propagate from the point $t', \vec{r}'$ in opposite t-directions. At the starting time $\tau = \tau'$ the wave surface degenerates to the light cone. Each trajectory starts from the apex of this cone. Its starting direction is given by the initial momentum

$$\vec{P}' = -Grad'\,S = m\frac{\vec{R} - \vec{R}'}{\tau - \tau'}. \qquad (103)$$

Hence each path forms a straight line

$$\vec{R} = \vec{R}' + \frac{\vec{P}'}{m}(\tau - \tau') \qquad (104)$$

in radial direction originating at the starting point. This equation demonstrates that for a given starting point either the end point or the initial momentum must be given in order to define the actual path. Since the wave surface is a function of the starting point and the point of observation, the solution of the wave equation defines the path by these points.

10. Integral equation and path integral

The Green function (98) is an elementary wave which has the same structure as the Fresnel propagator of light optics. According to the Huyghens principle the wave at time $t = t' + \Delta t$ is formed by the sum of all elementary waves, each of which originates from a point of the wave at time t. Since the Feynman path integral has the same property, we suppose that the Green function (98) transforms the four-dimensional wave function given at time τ' to the wave function at time τ in field-free space. To prove this conjecture, we transform the differential wave equation to an integral equation. Since the boundary conditions are incorporated in this equation, it yields a unique solution. Extending the Green's function method for the diffusion equation [12] from three to four dimensions, we eventually derive the integral equation

$$\psi(\vec{R},\tau) = \frac{2m}{i\hbar}\int G(\vec{R}-\vec{R}',\tau-\tau_i)\psi(\vec{R}',\tau_i)d^4\vec{R}' + \int\limits_{\tau_i}^{\tau}\int G(\vec{R}-\vec{R}',\tau-\tau')\Sigma(\vec{R}',\tau')\psi(\vec{R}',\tau')d^4\vec{R}'d\tau'. \quad (105)$$

The volume integration has to be taken over the entire four-dimensional space. The source is turned on at the initial time $\tau' = \tau_i$. Therefore, the integration over τ' in the second term has to be taken from the initial time up to the time of observation τ. The first term on the right describes the "non-scattered" incident wave at the point of observation $\vec{R},\tau$, whereas the second term accounts for the scattered wave. Hence we can conceive (105) as the equivalence of the Lippman-Schwinger equation [13] for the Minkowski space.

We can also utilize this equation to obtain a covariant path integral representation for the evolution of the wave function in Minkowsi space in the case of interactions. Our procedure will yield a straightforward derivation of the path integral without the need of employing Feynman's "intuitive" procedure. According to a remark by Dirac, he assumed that the operator $\exp(i\Delta S/\hbar)$ transforms the wave at time t to the wave at time $t+\Delta t$. Unfortunately, Feynman's space-time approach is non-relativistic [14]. So far a covariant approach has not been found. In order to derive a covariant path-integral representation, we subdivide in accordance with Feynman the time interval $\tau-\tau_i = N\Delta\tau$ into N infinitesimally short time *"slices"* of duration $\Delta\tau$. This approach resembles the multi-slice method employed for calculating the propagation of fast electrons in crystalline objects [15].

Employing this method, it suffices to derive the transformation of the wave function after propagating through a single slice. The final wave function is then obtained by successive iteration.

In the limit $\Delta\tau = \tau-\tau_i \to 0$, the Green function (98) approaches a four-dimensional delta function:

$$\lim_{\Delta\tau\to 0} G(\vec{R}-\vec{R}',\Delta\tau) = \frac{i\hbar}{2m}\delta^4(\vec{R}-\vec{R}'). \quad (106)$$

By inserting this relation into (105) and considering $\sigma_1(\vec{R}') = 0$, we readily derive at

$$\psi(\vec{R},\tau) = \psi(\vec{R},\tau_i) + \frac{i\hbar}{2m}\int_{\tau_i}^{\tau}\sigma_2(\vec{R}\tau')\psi(\vec{R},\tau')d\tau'. \tag{107}$$

This integral equation has the solution

$$\psi(\vec{R},\tau) = \psi(\vec{R},\tau_i)\exp\left(\frac{i\hbar}{2m}\int_{\tau_i}^{\tau}\sigma_2(\vec{R},\tau')d\tau'\right), \tag{108}$$

which is obtained by differentiating (107) with respect to τ and integrating the resulting rather simple first-order differential equation. We can use this result to obtain an improved approximation for the wave function at time $\tau + \Delta\tau$ by substituting (108) for $\psi(\vec{R}',\tau')$ and by replacing $G(\vec{R} - \vec{R}';\tau - \tau') \rightarrow G(\vec{R} - \vec{R}';\tau - \tau_i)$ in the second integral of equation (105). We further assume that the 4-vector potential does not change appreciably along a distance $\Delta R \leq \sqrt{2\hbar\Delta\tau/m}$. In this case we can put $\sigma_1 = 0$ and perform the integration over τ' analytically:

$$\psi(\vec{R},\tau_i + \Delta\tau) \approx \frac{2m}{i\hbar}\int\psi(\vec{R}',\tau_i)G(\vec{R} - \vec{R}',\Delta\tau)\exp\left(\frac{i\hbar}{2m}\int_{\tau_i}^{\tau}\sigma_2(\vec{R}',\tau')d\tau'\right)d^4\vec{R}'$$

$$\approx \frac{m^2}{(2i\hbar\pi\Delta\tau)^2}\int\psi(\vec{R}',\tau_i)\exp\left(\frac{i\Delta S}{\hbar}\right)d^4\vec{R}'. \tag{109}$$

This result proves the validity of Dirac's conjecture and can be considered as a relativistic extension of Feynman's non-relativistic path integral. The wave function at time $\tau = \tau_i + N\Delta\tau$ is obtained by successive iteration of (109) through all N slices. The differential action is given by

$$\Delta S = \frac{m}{2}\frac{(\vec{R} - \vec{R}')^2}{\Delta\tau} - e\vec{A}(\vec{R}')(\vec{R} - \vec{R}') + \Delta\tau\left(\frac{mc^2}{2} + \frac{e\hbar}{4m}\sum_{\mu,\nu}\sigma_{\mu\nu}F_{\mu\nu} + V_0\right). \tag{110}$$

This covariant expression differs significantly from the non-relativistic expression which does not account for the spin and gravitation.
If the interaction Hamiltonian does not depend explicitly on τ the wave function becomes stationary with respect to this time. In this case the solutions of the wave equation (75) adopt the stationary form

$$\psi(\vec{R},\tau) = \psi_s(\vec{R})\exp(iH_0\tau/\hbar), H_0 = mc^2. \tag{111}$$

Stationary conditions imply a steadily radiating source which has been turned on at the initial time $\tau_i = -\infty$. In this case we can perform the integration with respect to τ' in (105) analytically. Considering the relations (98) and (99) for the Green function G and inserting the expression (111) in the integrand, we obtain

$$\int_{-\infty}^{\tau} G \exp(iH_0\tau'/\hbar)d\tau' = \frac{ik_C}{16\pi} \frac{H_1^{(2)}(sk_C)}{s} \exp\frac{i}{\hbar}\left(H_0\tau - e\vec{A}_0(\vec{R} - \vec{R}')\right),$$

$$H_1^{(2)}(sk_C) = -H_1^{(1)}(-sk_C) = -\frac{2s}{\pi}\int_0^{\infty} \frac{1}{u^2}\exp\frac{-ik_C}{2}\left(u + \frac{s^2}{u}\right)du. \tag{112}$$

Here $H_1^{(2)}(x)$ and $H_1^{(1)}(x)$ are the first-order Hankel functions of second and first kind, respectively [16]. The result has been derived by substituting $u = c(\tau - \tau')$ for τ' as integration variable. The variable

$$s = \pm\sqrt{-(\vec{R} - \vec{R}')^2} \tag{113}$$

is real for time-like separations of the points $\vec{R}$ and $\vec{R}'$, it is imaginary for space-like separations. The positive sign has to be taken for $t > t'$, the negative sign for $t < t'$. By considering relation (111) for the total wave function, the asymptotic form

$$\lim_{x\to\infty} H_1^{(2)}(x) \approx \sqrt{\frac{2}{ix\pi}}\exp(-ix) \tag{114}$$

demonstrates that the expression (112) represents an outgoing hyper-spherical elementary wave in the case for $t > t'$ .and an incident hyper-spherical wave in the other case. Hence we can attribute the positive sign to the electron, the negative sign to the positron part of the total wave function. For space-like separations of the points $\vec{R}$ and $\vec{R}'$ the variable s becomes imaginary $s = -i|\vec{R} - \vec{R}'|$ for $t > t'$, $s = +i|\vec{R} - \vec{R}'|$ for $t < t'$. In this case the Hankel functions decrease exponentially if s increases. As a result the charge density is non-vanishing in the space-like region violating causality for space-like distances $|\vec{R} - \vec{R}'| \le \lambda_C$.

If we insert the relation (112) into the integral equation (105) and put the $\vec{A}_0 = 0$, we eventually obtain the integral equation for the stationary wave

$$\psi_s(\vec{R}) = \psi_{si}(\vec{R}) - \frac{imk_C}{8\hbar^2\pi}\int \frac{H_1^{(2)}(sk_C)}{s}H_{int}(\vec{R}')\psi_s(\vec{R}')d^4\vec{R}, s = \begin{cases} > 0 \Rightarrow t > t' \\ < 0 \Rightarrow t < t' \end{cases}. \tag{115}$$

The first factor of the integrand represents the four-dimensional Green function for stationary conditions. In this case the mass is conserved. It is interesting to note that the integration (112) alters the phase $S/\hbar$ of the Green function G according to a Legendre transformation of S that replaces the variable τ by the energy H_0. This energy should not be mixed up with the time-like component of the momentum 4-vector. Within the frame of our considerations stationary in τ does not imply stationary in the laboratory time t. Hence equation (115) also accounts for time-dependent interactions in a covariant form. This equation is completely equivalent to that obtained by Feynman based on the Dirac wave equation [17].

If the interaction Hamiltonian does neither depend on the universal time nor on the time t, $H_{int} = H_{int}(\vec{r})$, the system is defined as *static*. In this case the energies E_0 and

$$E = -ic\hbar K_4 = -imc\dot{x}_4 = -ic(\hbar k_4 + eA_4) \tag{116}$$

are conserved. Accordingly, the wave function adopts the form

$$\psi(\vec{R},\tau) = e^{i\chi}\psi_r(\vec{r}), \chi = ck_C\tau + K_4 x_4 = -(E_0\tau + Et)/\hbar. \tag{117}$$

If we insert this expression into the integral equation (105) and consider that Σ is only a function of $\vec{r}$, we can perform the integrations over τ' and x'_4 analytically. In order to simplify the calculations, it is advantageous to integrate over x'_4 first and subsequently over $u = c(\tau - \tau')$. Putting $\vec{A}_0 = 0$ and considering the integral representation of the Hankel functions of order ½ [16], we eventually find

$$\int_{-\infty}^{\tau}\int_{-\infty}^{\infty} G e^{iK_4 x'_4} e^{ick_C\tau'} dx'_4 \, d\tau' = \frac{1}{4\pi}\sqrt{\frac{mc}{2i\hbar\pi}} e^{i\chi} \int_0^{\infty} u^{-3/2} \exp\frac{i}{2}\left(k_C \frac{(\vec{r}-\vec{r}')^2}{u} - u\frac{K_4^2 + k_C^2}{k_C}\right) du$$

$$= \frac{1}{4\pi} e^{i\chi} \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}. \tag{118}$$

The wave number

$$k = \pm\sqrt{-(K_4^2 + k_C^2)} = \pm\frac{1}{\hbar c}\sqrt{E^2 - m^2 c^4} \tag{119}$$

is positive for *E>0* and negative for *E<0*. Using the result (118), we readily derive from (105) the integral equation

$$\psi_r(\vec{r}) = \psi_{ri}(\vec{r}) - \frac{1}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\Sigma(\vec{r}')\psi_r(\vec{r}')d^3\vec{r}' \tag{120}$$

for the three-dimensional spatial part $\psi_r(\vec{r})$ of the total wave function (117). The second term on the right represents the elastically scattered wave. If we substitute the expression (120) for $\psi_r(\vec{r})$ in (117), it readily follows that in the three-dimensional subspace the scattered wave represents an outgoing wave for both the electron *(E>0)* and the positron *(E<0)*. The equation (120) is formally identical with the Lippmann-Schwinger equation for elastic electron scattering derived from the Schroedinger equation. However, a decisive difference exists in so far that the interaction Hamiltonian $\Sigma(\vec{r}')$ is a covariant operator comprising spin interactions. Since ψ_r is a four component spinor, the integral equation (120) represents a set of four coupled integral equations for the four components $\psi_{r\mu}$ of the spinor ψ_r.

11. Eikonal approximation of the relativistic propagator.

The *eikonal* approximation of light optics is the equivalent of the WKB approximation of wave mechanics. The Greek word εικον means image and the eikonal governs the course of the rays emanating from a common point. An ideal image is formed at the position where these rays intersect each other again. In this case the eikonal approximation fails because the eikonal can no longer be attributed to a distinct ray of the homo-centric bundle of rays. Since a plane partial wave can be attributed to each ray, strong interference effects occur in the region of the caustic. The caustic degenerates to a point in the case of ideal imaging. In the region outside the caustic, the eikonal approximation describes the particle wave rather accurately. In this case it is possible to evaluate approximately the path integral for the propagator. The propagator is a hyper-spherical elementary wave in the absence of fields. If the wave propagates into the field region the wave surfaces will be deformed. As long as the curvature of the wave surface does not diverge at any point, we can employ the propagator for determining the propagation of an arbitrary wave from its initial time τ_i to the time of observation τ. In this case we can evaluate the Feynman path integral approximately because only a single classical path connects any two given points $\vec{R}$ and $\vec{R}'$. Since this path satisfies $\delta S(\vec{R},\tau;\vec{R}',\tau_i) = 0$, paths which only slightly deviate from the classical path contribute the most to the path integral. For all other paths its integrand strongly oscillates yielding no significant contribution to the integral.

Therefore, it suffices if we add all differential phases $\Delta S / \hbar$ along the classical path. The resulting total phase $S / \hbar$ is an operator because the differential phases (110) depend on the spin matrices $\sigma_{\mu\nu}$. The propagator $Q = Q(\vec{R},\tau;\vec{R}_i,\tau_i)$ represents an elementary wave which emanates from the point $\vec{R}_i$ of the initial wave function at time τ_i in the presence of external fields. It transfers the initial wave to the wave at time τ in the same way as the Green function in the field-free case:

$$\psi(\vec{R},\tau) = \int Q(\vec{R},\tau;\vec{R}_i,\tau_i)\psi(\vec{R}_i,\tau_i)d^4\vec{R}_i. \tag{121}$$

The semi-classical approximation for the propagator has the form

$$Q = \Gamma \exp(iS/\hbar), \tag{122}$$

where the action

$$S = S_t + S_s,$$

$$S_t = \int_{\tau_i}^{\tau}\left(\sum_\mu [\frac{m}{2}\dot{x}_\mu^2 - eA_\mu\dot{x}_\mu] + \frac{mc^2}{2} - V_0\right)d\tau', \tag{123}$$

$$S_s = \frac{e\hbar}{4m}\int_{\tau_i}^{\tau}\sum_{\mu,\nu}\sigma_{\mu\nu}F_{\mu\nu}d\tau' = \hbar\sum_{\mu,\nu}\sigma_{\mu\nu}\alpha_{\mu\nu}$$

consist of a scalar term S_t connected with the trajectories and an operator term S_s which effects the orientation of the spin. The integration over τ' has to be performed along the classical path. This is

achieved by substituting in the integrands the solutions $x_\mu(\tau')$ of the path equations for the coordinate x_μ. The exponential operator $\exp(iS_s/\hbar)$ can be written as

$$\exp(iS_s/\hbar) = \exp\left(i\sum_{\mu,\nu}\alpha_{\mu\nu}\sigma_{\mu\nu}\right) = \cos\Theta + \frac{i}{\Theta}\sum_{\mu,\nu}\sigma_{\mu\nu}\alpha_{\mu\nu}\sin\Theta, \tag{124}$$

where

$$\Theta = \sum_{\mu,\nu}\alpha_{\mu\nu}^{\;2}. \tag{125}$$

The representation (124) is most appropriate for applying the exponential operator. The deformation of the wave surfaces $S_t = const.$ by the external fields is connected with a variation of the propagator amplitude $\Gamma = \Gamma(\vec{R},\tau;\vec{R}_i,\tau_i)$. To determine this amplitude, we utilize the fact that the absolute value of the wave propagator is proportional to the square root of the mass density (82). For obtaining this density along the classical path, we consider the differential volume element

$$\delta V_i = (\Delta\tau)^4 \dot{R}_i^{\,3}\delta\dot{R}_i\delta\Omega_i = \left(\frac{\Delta\tau}{m}\right)^4\prod_{\mu=1}^{4}\delta(p_{\mu i}+eA_{\mu i}) \tag{126}$$

formed at time $\tau'=\tau_i+\Delta\tau$ by the trajectories of particles which start from the point $\vec{R}_i$ at time τ_i on the cone of the four-dimensional differential solid angle $\delta\Omega_i$. The enclosed initial volume element δV_i travels with the particles thereby changing its volume in such a way that the enclosed differential mass

$$\delta m_i = \rho_i\delta V_i = \delta q = \rho\delta V = \rho(\tau)\delta x_1\delta x_2\delta x_3\delta x_4 = const. \tag{127}$$

is conserved at any later time $\tau > \tau_i$ at which the variable volume element is located at the point $\vec{R} = \vec{R}(\tau)$. The conservation of the mass is a consequence of the continuity equation (79). The time interval $\Delta\tau$ at the start is chosen such that the propagator can be approximated in this region with a sufficient degree of accuracy by the undisturbed propagator (98) up to a factor $2m/i\hbar$. In this case the initial mass density is found as

$$\rho_i = \overline{\psi}_i\psi_i = |\Gamma_i|^2 = \frac{4m^2}{\hbar^2}|G(\Delta\tau)|^2 = \frac{m^4}{(2\pi\hbar\Delta\tau)^4}. \tag{128}$$

Using this result together with the relations (109) and (110) and considering that $p_{\mu i} = -\partial S_t/\partial x_{\mu i}$ and $\partial A_{\mu i}/\partial x_\nu = \partial A_\mu(\vec{R}_i)/\partial x_\nu = 0$, we readily obtain

$$\rho = \left|\Gamma\right|^2 = \rho_i \frac{\delta V_i}{\delta V} = \frac{D_V}{(2\hbar\pi)^4}, D_V = \det\left|\frac{\partial^2 S_t(\vec{R},\vec{R}_i)}{\partial x_\mu \partial x_{vi}}\right|. \tag{127}$$

This relation determines Γ up to a constant phase factor. This factor must coincide with that of the field-free propagator (98) in the absence of external fields. As a result we find

$$\Gamma(\vec{R},\tau;\vec{R}_i,\tau_i) \approx \frac{\sqrt{D_V}}{i(2\hbar\pi)^2}. \tag{128}$$

By inserting this result and the relation (123) for S into the expression (121), we obtain a very accurate approximation for the propagator Q provided that the point of observation is located in front of the caustic. If this point is located on the far side of the caustic, we must employ two propagators, one for the propagation of the initial wave to a hyper-plane in front of the caustic and the other for the propagation from each point of this hyper-plane to the point of observation located on the far side of the caustic. Subsequently we must integrate over all points of the hyper-plane. Hence even in this case we can avoid the numerous integrations required by the path-integral procedure.

11. Conclusion

The introduction of the universal time opens a new avenue for describing the dynamics of elementary particles in a deterministic way. The resulting five-dimensional approach enables a better understanding of the physical situation on an atomic scale, such as the motion of the spin in an electromagnetic field. Moreover, our approach does not require an infinite sea of negative energy states because the standard energy corresponds to the fourth component of the momentum 4-vector whose components can have arbitrary signs. In classical three-dimensional mechanics the position and the momentum variables are conjugate quantities, while the energy E is conjugate to the time t. In the frame of quantum mechanics each pair of conjugate variables must satisfy the uncertainty relation. By extending the three-dimensional space to the four-dimensional Minkowski space, the universal time τ takes over the role of the conventional time t and the rest energy E_0 the role of the standard energy E. Therefore, an additional uncertainty relation

$$\left|\Delta E_0 \Delta\tau\right| \geq \hbar \tag{129}$$

must exist between the conjugate quantities $E_0 = -mc^2$ or the mass, respectively, and τ. As a consequence, the universal time becomes meaningless without any mass. Moreover, all radiation would then be confined to the hyper-surface $\vec{R}^2 = \vec{r}^2 - c^2t^2 = 0$. This scenario may have been the case at the *"big bang"*. We can speculate that this event represents the origin of the universal time.

The five-dimensional Hamilton-Jacobi approach enables an improved physical understanding of relativistic quantum mechanics and yields the Schroedinger, Pauli and Dirac equations as special cases. Contrary to the Dirac equation, the five-dimensional single-particle wave equation (75) can readily be extended to a many-particle equation in the same way as the Schroedinger equation. This extension may prove to be a promising alternative to present quantum field theory because the

creation and annihilation of particles can be considered in the frame of our approach as *"inelastic"* scattering in Minkowski space. By including the electromagnetic field energy in the five-dimensional formalism, it should be possible to describe the annihilation and creation of particles as inelastic scattering where mass energy is transferred into radiation energy and vice versa. Such effects can only be described by the non-stationary solutions of the five-dimensional wave equation. These solutions are not attainable with the present *"stationary"* formalism which relies on creation and annihilation operators in order to incorporate the effect of the non-stationary processes. Since the method of second quantization cannot distinguish between individual particles, it must describe a many-particle system in terms of the occupation number of a particular state. On the other hand the proposed five-dimensional formalism describes the many-particle system in the configuration space without the need of creation and annihilation operators introduced by second quantization.

Acknowledgement

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