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by

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Abstract

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Professor Xin Guo, Chair

This thesis studies a class of mean field games (MFG) with singular controls of bounded velocity. By relaxing absolute continuity of control processes, it generalizes the MFG framework of Lasry and Lions [58] and Huang, Malhamé, and Caines [46]. It provides a unique solution to the MFG with singular controls of bounded velocity and its explicit optimal control policy establishes an ϵ-Nash equilibrium of the corresponding stochastic differential N player game with singular controls. It also includes MFGs on an infinite time horizon. Our method to approach MFGs with singular controls is from bounded velocity processes, and we analyse the relationship between singular controls with finite variation processes and singular controls with bounded velocity.

Finally, it illustrates particular MFGs with explicit solutions in a systemic risk model originally formulated by Carmona, Fouque, and Sun [23] in a regular control setting and an optimal partially reversible investment problem with N players originally formulated by Guo and Pham [39] in a single player setting.

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Chapter 1

Introduction of MFG

The theory of Mean Field Games (MFG) studies stochastic differential N player games of a large population with small interactions. Although stochastic differential N player games are useful and applicable in various fields, as N becomes large number, the game has high complexity, and finding a Nash equilibrium of the game is intractable in general. Because of these difficulties, the theory of MFGs suggests a new approach to approximate Nash equilibria of stochastic differential N player games. The key idea of MFGs is to approximate the dynamics and the objective function from population's probability distributions. Instead of considering all other players' behaviour directly, considering a probability distribution of all other players' behaviour decreases complexity of the game. As such, the MFG provides a simple and elegant analytical approach to approximate the Nash equilibrium of N player games. The MFG is first introduced by Lasry and Lions [58] and independently studied by Huang, Malhamé, and Caines [46]. In additions to the PDEs and control approach, there is an alternative probabilistic approach of Carmona, Delarue, and Lacker ([19], [20], [22]). There are many applications of MFGs in economics and finance, such as the systemic risk by Carmona, Fouque, and Sun [23] and by Garnier, Papanicolaou, and Yang [33], and the growth theory by Lasry, Lions, and Gueant [59], high frequency trading by Jaimungal and Nourian [47], and by Lachapelle, Lasry, Lehalle, and Lions [56], non-renewable resources by Bauso, Tembine, and Basar [5] and by Gueant, Lasry, and Lions [36], and finally the queueing theory by Manjrekar, Ramaswamy, and Shakkottai [64], by Wiecek, Altman, and Ghosh [80], and by Bayraktar, Budhiraja, and Cohen [6].

1.1 Toy model of MFG

Let us start with a toy model to illustrate the main idea and solution technique behind the MFG. This toy model is "the meeting start time" in [36]. Suppose that there are identical $N = 10K$ agents distributed on the negative half-line according to the probability distribution m_0. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t<\infty}, P)$, and let $\{W_t^i\}$ for $i = 1, 2, \cdots, N$ be independent and identically distributed (i.i.d.) standard Brownian motions on this probability space. Let $m(t, x)$ be a probability distribution of agents' positions at time t and $m(0, x) = m_0(x) = 0$ for $\forall x \geq 0$. The location of the meeting is

$x = 0$. Let X_t^i be a position of agent i at time t and α_t^i be an agent i's control variable at time t with dynamics $dX_t^i = \alpha_t^i dt + \sigma dW_t^i$. Although the announced meeting time is $t_0 > 0$, the meeting actually starts at T, when 90% of agents present at the meeting place 0. τ_i is the time at which agent i would like to arrive but in reality the agent i will arrive at time $\tilde{\tau}^i = \tau^i + \sigma\epsilon^i = \min_t\{t : X_t^i = 0\}$ where ϵ^i for $i = 1, 2, \cdots, N$ are i.i.d. normal distributions with variance 1. The cost function for each agent consists of four parts: a cost by controls $\int_0^{\tilde{\tau}^i} \frac{1}{2}\alpha_t^{i2} dt$, a cost of lateness (reputation effect) $c_1(t_0, T, \tilde{\tau}^i) = \alpha[\tilde{\tau}^i - t]_+$, a cost of lateness (personal inconvenience) $c_2(t_0, T, \tilde{\tau}^i) = \beta[\tilde{\tau}^i - T]_+$, and a waiting time cost $c_3(t_0, T, \tilde{\tau}^i) = \gamma[T - \tilde{\tau}^i]_+$. Denote $c(t_0, T, \tilde{\tau}^i) = c_1(t_0, T, \tilde{\tau}^i) + c_2(t_0, T, \tilde{\tau}^i) + c_3(t_0, T, \tilde{\tau}^i)$. Then, each agent tries to minimize its cost by controlling α_t^i based on all other agents' arrival time. There are two different methods: the N player game approach and the MFG approach.

In the N player game approach, each agent chooses its controls in the view of N identical distributions $(\epsilon^1, \cdots, \epsilon^N)$. One can simplify the problem using order statistics for 90% percentile of arrival times $(\tilde{\epsilon}_{(9k-1)}, \tilde{\epsilon}_{(9k)})$. A given player's optimal strategy τ^* satisfies

$$\begin{aligned} \tau^* &= arg \min_{\tau} E[C(\tau^* + \sigma\epsilon^1, \tau^* + \sigma\epsilon^2, \cdots, \tau^* + \sigma\epsilon^N)] \\ &= arg \min_{\tau} E[C(\tau^* + \sigma\epsilon^1, \tau^* + \sigma\tilde{\epsilon}_{(9k-1)}, \tau^* + \sigma\tilde{\epsilon}_{(9k)})], \end{aligned}$$

where C is the cost function for the player 1 with respect to other players' arrival times.

1.2 From N player game to MFG

Let's formulate general N player game first, and then heuristically derive the MFG from the N player game. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, P)$ be a probability space with $W^i = \{W_t^i\}_{0\le t\le T}$ independent and identically distributed (i.i.d.) standard Brownian motions in this space for $i = 1, 2, \ldots, N$. Fix a finite time T and assume that there are N identical players. The state process for the ith player is X_t^i satisfying the dynamics for $t \in [0, T]$

$$dX_t^i = \frac{1}{N}\sum_{j=1}^N b_0(t, X_t^i, X_t^j, \alpha_t^i)dt + \frac{1}{N}\sum_{j=1}^N \sigma_0(t, X_t^i, X_t^j, \alpha_t^i)dW_t^i, \qquad X_0^i = x^i \in \mathbb{R}^d, \quad (1.1)$$

where α_t^i is the control process for the ith player. $b_0 : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_0 : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous functions. Then, the stochastic differential N player game is for any $i = 1, 2, \cdots, N$,

$$\inf_{\alpha^i \in \mathcal{A}} E\left[\int_0^T \frac{1}{N}\sum_{j=1}^N f_0(t, X_t^i, X_t^j, \alpha_t^i)dt + \frac{1}{N}\sum_{j=1}^N h_0(X_T^i, X_T^j)\right], \qquad (1.2)$$

subject to (1.1) where $f_0 : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $h_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz continuous functions. The ith player chooses its optimal control process based on all N stochastic processes $X_t^1, X_t^2, \cdots, X_t^N$. Hence, the optimal control α_t^i is a function of $X_t^1, X_t^2, \cdots, X_t^N$.

The strong law of large numbers is for any i.i.d. random variables $Y_1, Y_2, \cdots, Y_N$ with $E[Y_i] < \infty$,

$$\frac{1}{N}\sum_{i=1}^{N} Y_i \to E[Y_i] \quad a.s. \quad \text{as} \quad N \to \infty.$$

Consequently, with some technical condition on f_0, h_0, b_0, σ_0, as N goes to infinity, the strong law of large numbers applies to $\phi_0 = f_0, h_0, b_0, \sigma_0$ as

$$\frac{1}{N}\sum_{j=1}^{N} \phi_0(t, X_t^i, X_t^j, \alpha_t^i) \to E[\phi_0(t, X_t^i, X_t^j, \alpha_t^i)] = \int \phi_0(t, X_t^i, x, \alpha_t^i)\mu_t(dx) = \phi(t, X_t^i, \mu_t, \alpha_t^i),$$

where μ_t is the probability measure for $\{X_t^j\}_{j=1,\cdots,N}$. Define $f, b, \sigma : [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\times\mathbb{R}^d \to \mathbb{R}^d$ and $h : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ as corresponding functions if $\mathcal{P}(\mathbb{R}^d)$ is a set of all probability measures on $\mathbb{R}^d$. Then, the mean field game with regular controls (MFG) can be formulated as following:

$$\begin{aligned} &\inf_{\alpha\in\mathcal{A}} E\left[\int_0^T f(t, X_t, \mu_t, \alpha_t)dt + h(X_T, \mu_T)\right] \\ &\text{subject to} \\ &dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t, \quad \forall t \in [0,T], \qquad X_0 = x, \mu_0 = \mu. \end{aligned} \tag{1.3}$$

where μ_t is the probability measure of X_t for any $t \in [0,T]$. Here $\{\alpha_t\}$ is the control process in an appropriate admissible control set $\mathcal{A}$. The player determines its control based on its current state X_t and the probability measure μ_t rather than based on the state of all players. The solution to the MFG can be defined as below.

Definition 1. *A solution of the MFG* (3.1) *is defined as a pair of the optimal control* $\alpha_t^* \in \mathcal{A}$ *and the flow of probability measures* $\{\mu_t^*\}$ *if they satisfy*
$v(s, X_s^*) = E\left[\int_s^T f(t, X_t^*, \mu_t^*, \alpha_t^*)dt + h(X_T^*, \mu_T^*)\right]$ *for all* $(s, X_s^*) \in [0,T] \times \mathbb{R}^d$ *and for each* $t \in [0,T]$ μ_t^* *is a probability measure of the optimal controlled process* X_t^* *which is*

$$dX_t^* = b(t, X_t^*, \mu_t^*, \alpha_t^*)dt + \sigma(t, X_t^*, \mu_t^*, \alpha_t)dW_t, \quad \forall t \in [0,T], \qquad X_0^* = x, \mu_0^* = \mu.$$

1.3 PDEs/control approach

Because of probability measures μ_t, a MFG is different from a stochastic control problem. To use the PDE/control method, one needs to fix μ_t as a deterministic function first.

PDE/controls approach to MFGs The PDE/control approach of [58, 18, 46] consists of three steps. The first step is to fix a deterministic mean information process and to analyse the corresponding stochastic control problem. Given the solution to the optimal control, the second step is to analyse the optimal controlled process, i.e., the McKean–Vlasov equation or stochastic differential equation (SDE). The third step is to show that the iterations of previous two steps converge to a fixed point solution to the MFG. In this approach, the MFG is essentially analysed by studying two coupled PDEs, the backward Hamilton–Jacobi–Bellman (HJB) equation and the forward McKean–Vlasov SDE. This approach requires three main technical questions. The first question is existence and uniqueness of optimal controls $\{\alpha_t\}$ and the value function v of the stochastic control problem in the first step. The second question is existence and uniqueness of the flow of probability measures $\{\mu'_t\}$ which is the solution to the stochastic differential equation in the second step. The last question is existence and uniqueness of the fixed point solution in the third step.

For the first step, fix a deterministic function $t \in [0,T] \to \mu_t \in \mathcal{P}(\mathbb{R}^d)$, and then the MFG (3.1) is equivalent to the stochastic control problem:

$$\begin{aligned} v(s,x) &= \inf_{\alpha \in \mathcal{A}} E\left[\int_s^T f(t,X_t,\mu_t,\alpha_t)dt + h(X_T,\mu_T)\right] \\ s.t. \quad dX_t &= b(t,X_t,\mu_t,\alpha_t^i)dt + \sigma(t,X_t,\mu_t,\alpha_t)dW_t \quad \text{and} \quad X_s = x, \end{aligned}$$

where $\mathcal{A}$ is the admissible set which contains all progressively measurable processes satisfying the condition $E\int_0^T |\alpha_t|^2 dt < \infty$. There are two approaches to solve the stochastic control problem: the Dynamic Programming Principle (DPP) which refers to the PDE/control method and the stochastic maximum principle which refers to the probabilistic method. The DPP established by Bellman is a technique to solve the optimization problem from considering sub-optimization problems in different time and states. Suppose the value function $v(s,x)$ provides the optimized value for time $s \in [0,T]$ and state $x \in \mathbb{R}^d$. Then, the DPP relies on Bellman's principle of optimality: for any $s \le S \le T$,

$$v(s,x) = \inf_{\alpha \in \mathcal{A}_{[s,S]}} E\left[\int_s^S f(t,X_t,\mu_t,\alpha_t)dt + v(S,X_S)\right],$$

where $\mathcal{A}_{[s,S]}$ denotes the admissible set over $[s,S]$. One can derive an infinitesimal version of Bellman's principle of optimality, the HJB equation:

$$\begin{aligned} -\partial_t v(t,x) - \inf_{\alpha \in \mathcal{A}} \{ &\frac{1}{2} tr(\sigma(t,x,\mu,\alpha)\sigma(t,x,\mu,\alpha)^T D_{xx} v(t,x)) \\ &+ b(t,x,\mu,\alpha) \cdot D_x v(t,x) + f(t,x,\mu,\alpha)\} = 0 \\ \text{with} \quad v(T,x) &= h(x,\mu_T). \end{aligned} \tag{1.4}$$

Because the HJB equation (1.4) could have multiple solutions or does not have smooth solutions, one needs to verify the solution. If the HJB equation (1.4) has a $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ solution, w, then using the Itô's formula, one can prove $w(s,x) = v(s,x)$ for any $(s,x) \in [0,T]\times\mathbb{R}^d$. If it does not have a $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ solution, one can use the viscosity solution to (1.4). [1] By proving that the viscosity solution to (1.4) is the value function v and the viscosity solution to (1.4) is unique, one concludes that the value function v is the viscosity solution to (1.4). Within the verification step, one can also show that there is a unique

[1] The definition of the viscosity solution to (1.4) is in appendix A.2.

optimal control $\alpha_t(\mu_t) \in \mathcal{A}$ if there is a unique value function to the problem. Therefore, by the uniqueness of the value function v and optimal control $\alpha_t(\mu_t)$ of (3.1) under fixed $\{\mu_t\}$, one can define a mapping Γ_1 such that $\Gamma_1(\{\mu_t\}) = \{\alpha_t(\mu_t)\}$.

The second step is the consistency part. Under the optimal control $\alpha_t(\mu_t)$ from the first step, one will update the flow of probability measures μ'_t using P_{X_t} which is a probability measure of X_t. There are two different methods in this step. In [46], they approach the second step with the McKean-Vlasov SDE:

$$dX_t = b(t, X_t, P_{X_t}, \alpha_t(\mu_t))dt + \sigma(t, X_t, P_{X_t}, \alpha_t(\mu_t))dW_t, \quad \forall t \in [0, T], \qquad X_0 = x \in \mathbb{R}^d.$$

Under fixed $\alpha_t(\mu_t)$, with some technical assumptions, the McKean-Valsov SDE has a unique solution pair (X_t, P_{X_t}) where P_{X_t} is a probability measure of X_t. Another method is to use the SDE under fixed μ_t. With fixed μ_t and $\alpha_t(\mu_t)$ from the first step, the second step is equivalent to solve the SDE:

$$dX_t = b(t, X_t, \mu_t, \alpha_t(\mu_t))dt + \sigma(t, X_t, \mu_t, \alpha_t(\mu_t))dW_t, \quad \forall t \in [0, T], \quad X_0 = x \in \mathbb{R}^d. \tag{1.5}$$

The Kolmogorov forward equation for (1.5) is for any $x \in \mathbb{R}^d$ and $t \in [0, T]$

$$\partial_t P(t, x) = -\sum_{j=1}^{d} \partial_{x_j}(b(t, x, \mu_t, \alpha_t(\mu_t))P(t, x)) \tag{1.6}$$

$$+ \frac{1}{2} \sum_{k=1}^{d} \sum_{j=1}^{d} \partial_{x_k} \partial_{x_j} (\sigma(t, x, \mu_t, \alpha_t(\mu_t))\sigma(t, x, \mu_t, \alpha_t(\mu_t))^T P(t, x)). \tag{1.7}$$

The solution to the Kolmogorov forward equation, $P(t, x)$, is the probability measure for X_t with $\alpha_t(\mu_t)$. So, one can update new fixed flow of probability measures $\{\mu'_t\}$ using the solution to the McKean-Vlasov SDE or the Kolmogorov forward equation. Therefore, with the uniqueness of probability measure under fixed $\{\mu_t\}$ and $\{\alpha_t(\mu_t)\}$, one can define a mapping Γ_2 such that $\Gamma_2(\{\alpha_t(\mu_t)\}) = \{\mu'_t\}$.

The third step is the fixed point part. One can define Γ_1 and Γ_2 in previous two steps, and the MFG framework repeats $\Gamma_2 \circ \Gamma_1$ until convergence. By Schauder fixed point theorem, if the mapping $\Gamma_2 \circ \Gamma_1$ is continuous mapping, there exists a fixed point solution, and under some technical conditions it is unique [58, 18, 46].

Therefore, the PDE/controls approach to MFGs is of solving these three main technical questions. After all, the MFG framework can be converted as two time conflict PDEs: the HJB equation (1.4) and the Kolmogorov forward equation (1.6). The HJB equation is associated with the stochastic control part, so it is backward in time. The Kolmogorov forward equation is associated with the consistency part of the statical distribution, so it is forward in time. Because these two differential equations are nonlinear PDE and conflicting direction of time, solving these two equations is not easy in general.

Let us illustrate the PDE/control approach to the toy model in section 1.1.

The MFG framework for the toy example Fix an actual meeting start time as deterministic T. Then, with fixed T, the problem is equivalent to the stochastic control problem:

$$v(x_0) = \inf_{\alpha} E[c(t_0, T, \tilde{\tau}) + \frac{1}{2}\int_0^{\tilde{\tau}} \alpha_t^2 dt]$$
$$s.t. \quad dX_t = \alpha_t dt + \sigma dW_t, \quad X_0 = x_0, \quad \tilde{\tau} = \min\{s : X_s = 0\}.$$

Then, the Hamilton-Jacobi-Bellman (HJB) equation for the value function v is

$$0 = \partial_t v + \min_{\alpha}\{\alpha \partial_x v + \frac{1}{2}\alpha^{i2}\} + \frac{\sigma^2}{2}\partial_{xx}^2 v.$$

However, this solution is the optimal solution under fixed T, finding the proper T is also crucial to solve the MFG. Let's use fixed point method for finding the solution T. The Kolmogorov forward equation for the distribution of agents' position at time t, $m(t, x)$, is

$$\partial_t m + \partial_x((-\partial_x v)m) = \frac{\sigma^2}{2}\partial_{xx}^2 m.$$

Define the cumulative distribution function of arrival times as $F(s) = -\int_0^s \partial_x m(\tau, 0) d\tau$. Then, one can update $T' = F^{-1}(0.9)$ as new actual meeting start time. With new fixed T', one can repeat the same iterations, the HJB equation and the Kolmogorov forward equation. Repeat these iterations until the solution converges to a fixed point solution of T^* and v^*. Then, this is the solution to the MFG.

1.4 Probabilistic approach

The stochastic maximum principle is also one of the most common approaches to solve stochastic control problems as well as the PDE/control approach. Because the HJB equation only works for Markovian controls, the stochastic maximum principle is suitable to solve the stochastic controls with non-Markovian controls or open loop controls. The stochastic maximum principle is a probabilistic approach using the duality principle and an adjoint process. It gives necessary conditions for optimality, so by solving adjoint backward stochastic differential equations (BSDE) it suggests an optimal solution to the problem.

The theory of BSDE is introduced by Bismut [9, 10] and it is widely studied in 1990s. Pardoux and Peng [66] prove the existence of uniqueness of general BSDEs with Lipschitz conditions, and there are works connecting BSDEs and stochastic control problems [67, 68, 55]. Beyond BSDEs, the forward backward stochastic differential equations (FBSDE) which are coupled BSDEs is introduced by Antonelli [1], and Ma, Protter, and Yong [62] study more general FBSDEs. Because of connection between stochastic control problems and (F)BSDEs, it has many applications in finance such as [29, 73].

BSDEs are stochastic differential equations in backward time with terminal conditions. For any fixed time interval $[0, T]$, define $\mathcal{F}_t$-adapted processes $\{y_t\}, \{z_t\}$ with dynamics

$$dy_t = h(t, y_t, z_t)dt + z_t dW_t, \quad t \in [0, T), \quad y_T = \xi,$$

where ξ is in $L^2(\mathcal{F};\mathbb{R}^d)$. The solution to the BSDE is a pair of (y_t, z_t) satisfying the above BSDE with terminal condition. If $h(t, y, z)$ is Lipschitz continuous in y and z, the BSDE has a unique solution which has finite second moments. In general, because the proof of existence and uniqueness of the solution to BSDEs relies on the second order norm and the fixed point method, the space of processes is with L^2 norm. Based on the relationship between BSDEs and the stochastic maximum principle, BSDE is widely used to solve stochastic control problems.

Canonically, interest on alternative probabilistic approach on the MFG with stochastic maximum principles arises. Buckdahn et al. [13, 14] and Carmona, Delarue, and Lacker [19, 20, 22] propose probabilistic approaches to directly analyse the combined FBSDE. For the following MFG

$$\begin{aligned}&\inf_{\alpha\in\mathcal{A}} E\left[\int_0^T f(t, X_t, \mu_t, \alpha_t)dt + h(X_T, \mu_T)\right]\\&\text{subject to } dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma dW_t, \quad \forall t \in [0, T], \qquad X_0 = x, \mu_0 = \mu,\end{aligned} \tag{1.8}$$

with proper admissible set $\mathcal{A}$, define the Hamiltonian H as $H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha)$ for any $(t, x, \mu, y) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. In stochastic controls, because a minimizer of the Hamiltonian with respect to the control process is the minimizer of the stochastic control problem, it is important to analyse the Hamiltonian. If the drift b is an affine function in control processes and the cost function f is convex and Lipschitz continuous derivatives, then, with some technical assumptions the Hamiltonian H has a unique minimizer $\hat{\alpha}(t, x, \mu, y)$.

As in the PDE/control approach, with fixed $\{\mu_t\}$, the MFG (1.8) is equivalent to the stochastic control problem. From the Hamiltonian H and its minimizer $\hat{\alpha}$, one can apply the stochastic maximum principle to find necessary and sufficient conditions for the problem. With adjoining process $\{Y_t\}$, if the FBSDE

$$\begin{aligned}dX_t &= b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + \sigma dW_t, \quad X_0 = x \in \mathbb{R}^d,\\dY_t &= -\partial_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + Z_t dW_t, \quad Y_T = \partial_x h(X_T, \mu_T),\end{aligned}$$

has a solution with finite second moments, then the minimizer $\hat{\alpha}$ of the Hamiltonian H is the optimal controls and X_t is the optimal controlled process. Since the stochastic maximum principle provides an optimal solution only under fixed $\{\mu_t\}$, the consistency part which studies updating probability measures $\{\mu_t\}$ needs to be analysed. In [19], Carmona and Delarue derive the FBSDE of the McKean-Vlasov type such as

$$\begin{aligned}dX_t &= b(t, X_t, P_{X_t}, \hat{\alpha}(t, X_t, P_{X_t}, Y_t))dt + \sigma dW_t, \quad X_0 = x \in \mathbb{R}^d,\\dY_t &= -\partial_x H(t, X_t, P_{X_t}, Y_t, \hat{\alpha}(t, X_t, P_{X_t}, Y_t))dt + Z_t dW_t, \quad Y_T = \partial_x h(X_T, P_{X_T}),\end{aligned} \tag{1.9}$$

where P_{X_t} is the probability distribution of X_t. Under some technical conditions such as functions are Lipschitz continuous and the drift is affine, (1.9) has a unique solution, and furthermore it is the solution to the MFG (1.8). Therefore, there exists a unique solution to the MFG (1.8).

There are many recent works on the probabilistic MFG theory: Carmona and Lacker [22], who study a weak formulation of MFGs, Lacker [57], who analyses MFGs with controlled martingale problems, and Carmona, Delarue, and Lacker [21], who add common noise to MFGs.

1.5 MFG solution and ϵ-Nash equilibrium

Another main result of the theory of MFG is that the optimal control to the MFG is actually an ϵ-Nash equilibrium of the corresponding N player game.[2] In [46], Huang, Malhamé, and Caines prove that the optimal control to the MFG (3.1) is an ϵ_N-Nash equilibrium of the corresponding N player game (1.2) for some special class of games; the dimension of the space d is 1, the terminal cost function h is 0, and σ is constant. Furthermore, ϵ_N has a bound $c\frac{1}{N}$ for some constant $c > 0$, ϵ_N converges to 0 as N goes to infinity.

In the MFG probabilistic approach [19], Carmona and Delarue also show relationship between the solution to the MFG and an ϵ-Nash equilibrium. Consider the MFG (1.8) and let α_t^* is the optimal control to the MFG (1.8). The corresponding N player game of the MFG (1.8) is formulated as, for any $i = 1, 2, \cdots, N$,

$$J^{N,i}(\alpha^1, \cdots, \alpha_t^N) = \inf_{\alpha \in \mathcal{A}} E\left[\int_0^T f(t, X_t^i, \delta_t^N, \alpha_t^i)dt + h(X_T^i, \delta_T^N)\right],$$

subject to the dynamics

$$dX_t^i = b(t, X_t^i, \delta_t^N, \alpha_t^i)dt + \sigma dW_t^i, \quad \forall t \in [0, T], \qquad X_0^i = x^i \in \mathbb{R}^d,$$

where $\delta_t^N = \frac{1}{N}\sum_{i=1}^N X_t^i$ is the empirical distribution of $X_t^1, \cdots, X_t^N$ at time t, $\mathcal{A}$ is the same admissible set, and W_t^i are i.i.d. standard Brownian motions. Then, with some technical assumptions, for any $\beta^i \in \mathcal{A}$ the following inequality holds:

$$J^{N,i}(\alpha^{*1}, \cdots, \beta^i, \cdots, \alpha_t^{*N}) \geq J^{N,i}(\alpha^{*1}, \cdots, \alpha_t^{*N}) - \epsilon_N,$$

with $\epsilon_N = O(N^{-\frac{1}{d+4}})$.

Therefore, the optimal control of the MFG is an ϵ_N-Nash equilibrium of the corresponding N player game and ϵ_N goes to 0 as N goes to infinity.

The toy model case In [36], there exist explicit solutions to the toy model by the MFG approach and by the N player game approach. Denote τ_N^* as the solution to the toy model by the N player game approach and τ_{MFG}^* as the solution to the toy model by the MFG approach, then it provides the relation as

$$\tau_N^* = \tau_{MFG}^* - \frac{1}{N}G + o(\frac{1}{N}),$$

where G is a some function which is independent to N. So, the MFG solution τ_{MFG}^* converges to the solution of N player game τ_N^* as N goes to infinity, and it is an ϵ_N-Nash equilibrium where $\epsilon_N = \frac{1}{N}G + o(\frac{1}{N})$.

[2] The definition of ϵ-Nash equilibrium is in appendix A.1.

1.6 MFG with common noise

In the previous MFG (3.1), there were only i.i.d. noises for each player. However, because in reality there are many possible common impacts to players within the industry, studying games with common noise as well as individual noises is important. Let W_t^0 be another Brownian motion in the same space and W_t^0 is independent and identical to W_t^i for $i = 1, \cdots, N$. Then, the MFG with common noise is

$$\begin{aligned} v(s,x) &= \inf_{\alpha \in \mathcal{A}} E\left[\int_s^T f(t, X_t, \mu_t, \alpha_t)dt + h(T, X_T, \mu_T)\right] \\ s.t. \quad dX_t &= b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t)dW_t + \sigma_0(t, X_t, \mu_t)dW_t^0, \quad \text{and} \quad X_s = x. \end{aligned}$$

With conditioning on common noise W_t^0, Carmona, Delarue, and Lacker [21] develop the theory from weak formations and prove the uniqueness and existence solution of the MFG with common noise. In economics literatures, idiosyncratic noises can be eliminated by the exact law of large numbers [26, 53, 71, 75]. Using the exact law of large numbers and conditional exact law of large numbers, Nutz [65] studies MFGs of optimal stopping problems with common noise, and Carmona, Fouque, and Sun [23] apply the MFG with common noise to systemic risk. Because of the common noise term, the HJB equation and Kolmogorov forward equation for MFGs have different form. Consider the following example in [23]

$$\begin{aligned} v(s,x,m) &= \inf_{\alpha \in \mathcal{A}} E\left[\int_s^T f(t, X_t, m_t, \alpha_t)dt + h(X_T, m_T)\right] \\ s.t. \quad dX_t &= b(t, X_t, m_t, \alpha_t)dt + \sigma\sqrt{1-\rho^2}dW_t + \sigma\rho dW_t^0, \quad \text{and} \quad X_s = x, m_s = m, \end{aligned}$$

where $m_t = \int x P_{X_t}(dx)$ is the mean of X_t with conditioning on W^0. Then, by conditioning on the common noise, the HJB equation and Kolmogorov forward equation for this MFG become following stochastic PDEs:

$$dP_{X_t} = \left[-\partial_x\left(b(t, X_t, m_t, \alpha_t)P_{X_t}\right) + \frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}P_{X_t}\right]dt - \rho\sigma\partial_x P_{X_t} dW_t^0,$$

and

$$\begin{aligned} dv &+ \left[\frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}v + \mathcal{L}^m v + \partial_{xm}v\frac{d<m,x>}{dt}\right]dt \\ &+ \inf_{\alpha \in \mathcal{U}}\left[b(t,x,m,\alpha)\partial_x v + f(t,x,m,\alpha)\right]dt \\ &+ \rho\sigma\partial_m v dW_t^0 + \rho\sigma\partial_x v dW_t^0 = 0, \end{aligned}$$

where $\mathcal{L}^m + \rho\sigma\partial_m dW_t^0$ is an infinitesimal generator for m_t.

1.7 Mean–field SDE and McKean–Vlasov type controls

After Lions introduces differentiability with respect to the probability measure in his MFG course [18], there are many works in stochastic controls with mean field interaction using derivatives with respect to the probability measure. For any random variable $Y \in L^2(\mathcal{F};\mathbb{R}^d)$, let P_Y be a probability measure of Y, then one can define the lift function as $\tilde{\Phi}(Y) = \Phi(P_Y)$ for any function $\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Then, differentiability of Φ with respect to P_Y can be defined as differentiability in Fréchet sense of $\tilde{\Phi}$ in [18, 15].

Definition 2. *If $\tilde{\Phi}$ is differentiable in Fréchet sense at $Y_0 \in L^2(\mathcal{F};\mathbb{R}^d)$, there exists a unique $h_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi(P_Y) - \Phi(P_{Y_0}) = E[h_0(Y_0)\cdot(Y - Y_0)] + o(|Y - Y_0|_{L^2})$ for any $Y \in L^2(\mathcal{F};\mathbb{R}^d)$. Then, define $\partial_\mu \Phi(P_{Y_0}, y) = h_0(y)$ for any $y \in \mathbb{R}^d$.*

Using the derivative with respect to the probability measure, Buckdahn, Li, Peng, and Rainer [15] derive the generalized Itô's formula for mean-field SDEs

$$dX_t = b(X_t, P_{X_t})dt + \sigma(X_t, P_{X_t})dW_t, \quad X_0 = x, P_{X_0} = P_\xi,$$

and for the function $\Phi : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $\mathcal{C}_b^{1,(2,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$ as following:

$$\begin{aligned}
&\Phi(s, X_s, P_{X_s}) - \Phi(r, X_r, P_{X_r}) \\
=& \int_r^s \partial_t \Phi(t, X_t, P_{X_t}) + \sum_{i=1}^d \partial_{x_i}\Phi(t, X_t, P_{X_t}) b_i(X_t, P_{X_t}) \\
&+ \frac{1}{2}\sum_{i,j,k=1}^d \partial^2_{x_i x_j}\Phi(t, X_t, P_{X_t})\sigma_{i,k}\sigma_{k,j}(X_t, P_{X_t}) + \bar{E}\Bigg[\sum_{i=1}^d (\partial_\mu \Phi)_i(t, X_t, P_{X_t}, \bar{X}_t) b_i(\bar{X}_t, P_{X_t}) \\
&+ \frac{1}{2}\sum_{i,j,k=1}^d \partial_{x_i}\left((\partial_\mu\Phi)_j(t, X_t, P_{X_t}, \bar{X}_t)\right)\sigma_{i,k}\sigma_{k,j}(\bar{X}_t, P_{X_t}\Bigg]dt \\
&+ \int_r^s \sum_{i,j=1}^d \partial_{x_i}\Phi(t, X_t, P_{X_t})\sigma_{i,j}(X_t, P_{X_t})dW_t^j, \quad 0 \le r \le s \le T,
\end{aligned} \tag{1.10}$$

where $\bar{X}_t$ is independent and identical copy of X_t. Using the generalized Itô's formula, recently, Pham and Wei [69, 70] study following stochastic McKean–Vlasov control problems:

$$\begin{aligned}
\inf_{\alpha\in\mathcal{A}} J(\alpha) = \inf_{\alpha\in\mathcal{A}} E\int_0^T f(t, X_t, \alpha_t, P_{X_t,\alpha_t})dt + g(X_T, P_{X_T}) \\
\text{s.t.} \quad dX_t = b(t, X_t, \alpha_t, P_{X_t,\alpha_t})dt + \sigma(t, X_t, \alpha_t, P_{X_t,\alpha_t})dW_t, \quad X_0 = x_0,
\end{aligned}$$

where b, σ are Lipschitz continuous, f, g satisfies the quadratic condition in [69], and $\mathcal{A}$ is the admissible set which includes progressively measurable and square integrable processes with closed loop in feedback forms. Then, the dynamic programming principle holds for this stochastic McKean–Vlasov control problem, and they derive the Bellman equation and prove the verification theorem using results in [15].

1.8 Notations

Throughout the thesis, use the following notation, unless otherwise specified.

- $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, P)$ is a probability space and $W^i = \{W_t^i\}_{0\le t\le T}$ are i.i.d. standard Brownian motions in this space for $i = 1, 2, \dots, N$;
- $\mathcal{P}(\mathbb{R})$ is the set of all probability measures on $\mathbb{R}$;
- $\mathcal{P}_p(\mathbb{R})$ is the set of all probability measures of p order on $\mathbb{R}$ such that $\mu \in \mathcal{P}_p(\mathbb{R})$ if $(\int |x|^p \mu(dx))^{\frac{1}{p}} < \infty$;
- $Lip(f)$ is a Lipschitz coefficient of f for any Lipschitz function f. That is, $|f(x) - f(y)| \le Lip(f)|x - y|$ for all $x, y \in \mathbb{R}$;
- D^p is the pth order Wasserstein metric on $\mathcal{P}_p(\mathbb{R})$ between two probability measures μ and μ', defined as $D^p(\mu, \mu') = \inf_{\tilde{\mu}} \left(\int |y - y'|^p \tilde{\mu}(dy, dy')\right)^{\frac{1}{p}}$, where $\tilde{\mu}$ is a coupling of μ and μ';
- $\mathcal{M}_{[0,T]} \subset \mathcal{C}([0, T] : \mathcal{P}_2(\mathbb{R}))$ is a class of flows of probability measures $\{\mu_t\}$ on $[0, T]$ and contains all $\{\mu_t\}$ so that

$$\sup_{s\neq t} \frac{D^1(\mu_t, \mu_s)}{|t - s|^{\frac{1}{2}}} \le c_1, \qquad \sup_{t\in[0,T]} \int |x|^2 \mu_t(dx) \le c_1$$

  where c_1 is a positive constant. $\mathcal{M}_{[0,T]}$ is a metric space endowed with the metric $d_{\mathcal{M}}$ such that

$$d_{\mathcal{M}}(\{\mu_t\}, \{\mu'_t\}) = \sup_{0\le t\le T} D^1(\mu_t, \mu'_t); \tag{1.11}$$

- $\mathcal{L}\psi(x) = b(x)\partial_x\psi(x) + \frac{1}{2}\sigma^2(x)\partial_{xx}\psi(x)$ for any stochastic process $dx_t = b(x_t)dt + \sigma(x_t)dW_t$ and any $\psi(x) \in \mathcal{C}^2$;
- A function f is said to satisfy a polynomial growth condition if $f(x) \le c(|x|^k + 1)$ for some constants c, k, for all x.
- P_ϑ is a probability distribution of the random variable ϑ;
- P_{X_∞} is a limiting stationary distribution of the process $\{X_t\}_{t\ge0}$ if $\{X_t\}_{t\ge0}$ has a limiting distribution and a stationary distribution and they are same.

Chapter 2

Singular controls

2.1 Singular controls

The stochastic maximum principle and the dynamic programming principle are most commonly used approaches to solve stochastic control problems. Because the dynamic programming principle only works for Markovian controls, for stochastic control problems with non-Markovian controls the stochastic maximum principle is widely used to solve it. However, since the stochastic maximum principle relies on a unique maximizer or minimizer of the Hamiltonian and on boundedness of the Hamiltonian, it fails to give a solution when the Hamiltonian does not converge or maximizer or minimizer is not unique. It is called a bang-bang situation.

Let's look at an example of the bang-bang situation in [7]. Let $x_t = x + W_t + \xi_t$ be a state process where x is an initial position, W_t is a Brownian motion, and ξ_t is a control process which is a finite variation process. Since ξ_t is finite variation, it can be decomposed as the difference of two nondecreasing processes $\xi_t = \xi_t^+ - \xi_t^-$. The objective of controls is to minimize the following function:

$$J(\xi_t) = E\int_0^\infty e^{-\alpha t}\{d\xi_t^+ + d\xi_t^- + h(x + w_t + \xi_t)dt\}.$$

The Hamiltonian for this model is

$$H(x,p) = \inf_{\xi_t}\{p \cdot d\xi_t + d\xi_t^+ + d\xi_t^-\} = \inf_{\xi_t}\{(p+1)d\xi_t^+ - (p-1)d\xi_t^-\}.$$

According to the stochastic maximum principle, the optimal control is a minimizer of the Hamiltonian H, but if $p > 1$, the minimizer is $d\xi_t^+ = 0, d\xi_t^- = +\infty$. If $p < -1$, the minimizer is $d\xi_t^+ = +\infty, d\xi_t^- = 0$. If $p \in (-1, 1)$, the minimizer is $d\xi_t = 0$. In this case, the Hamiltonian diverges and one can not use the classical stochastic maximum principle.

Because of these situations, one needs to consider the singular control which control processes are not absolutely continuous but finite variation processes. This class of controls was introduced by Bather and Chernoff [2], and Beneš, Shepp, and Witzenhausen suggest some singular control problems with explicit solutions. In 1970s and 1980s, Harrison and Taylor [43], Karatzas [48, 49], and Karatzas and Shreve [50, 51] develop the theory of singular controls in one dimensional space. For any finite variation process ξ_t, by the Lebesgue decomposition theorem one can decompose uniquely ξ_t as the pure jump part $\xi_t^{jp} = \sum_{0<s<t} \Delta\xi_s$ and the continuous part $\xi_t^c = \xi_t - \xi_t^{jp}$. The continuous part consists of the absolutely continuous part $\xi_t^{ac} = \int_0^t \dot{\xi}_s^c ds$, and the singularly continuous part $\xi_t^{sc} = \xi_t^c - \xi_t^{ac}$. The regular control is a special case of the stochastic control with $\xi^{sc} \equiv \xi^{jp} \equiv 0$.

Because the Hamiltonian may diverge for singular controls, the HJB equation for singular controls also does not have a solution in general. However, singular controls can be solvable when one analyses the variational inequality. Consider the following stochastic control problem:

$$\begin{aligned} &v(s,x) = \sup_{\alpha \in \mathcal{A}} J(s,x,\alpha) = \sup_{\alpha \in \mathcal{A}} E\left[\int_s^T f(X_t,\alpha_t)dt + g(X_T)\right], \\ &s.t. \quad dX_t = b(X_t,\alpha_t)dt + \sigma(X_t,\alpha_t)dW_t, \quad X_s = x \in \mathbb{R}^d. \end{aligned}$$

The Hamiltonian is $H(x,Dv,D^2v) = \sup_{\alpha \in \mathcal{A}} \left[b(x,\alpha)Dv + \frac{1}{2}tr(\sigma\sigma^T D^2 v) + f(x,\alpha)\right]$, and $F(t,x,v,v_t,Dv,D^2v) = -v_t - H(x,Dv,D^2v)$ is the HJB equation for this problem. Because H may diverge, define $\mathbb{H} = \{(x,p,M) \in (\mathbb{R}^d,\mathbb{R}^d,\mathcal{S}_+^d) | H(x,p,M) < \infty\}$ where $\mathcal{S}_+^d$ is the set of $d \times d$ semi positive definite matrices. Assume that there exists a continuous function $G : (\mathbb{R}^d,\mathbb{R}^d,\mathcal{S}_+^d) \to \mathbb{R}$ such that $(x,p,M) \in \mathbb{H}$ if and only if $G(x,p,M) \geq 0$. Then, the variational inequality for this stochastic control problem is

$$\min\{F(t,x,v,v_t,Dv,D^2v), G(x,Dv,D^2v)\} = 0. \tag{2.1}$$

Then, as in regular controls, the solution to (2.1) is one of the solution candidates, so one needs to verify that the solution to (2.1) is actually the value function of the problem using the verification theorem or the uniqueness of the viscosity solution to the equation. Regularity of the solution to the HJB equation or the variational inequality is useful to prove the verification theorem. For the variational inequality, because of the existence of G, the $\mathcal{C}^2$ regularity is not an easy problem in general. There are previous papers which study the smooth fit regularity of the value function for singular controls by Beneš, Shepp, and Witsenhausen [7], by Harrison [42], by Harrison and Taylor [43], by Karatzas [48, 49], by Taksar [78], by Guo and Pham [39], and by Guo and Tomecek [41].

Unlike regular controls, the rates of optimal control processes in singular controls are not Lipschitz continuous and the control space can be divided into two regions. On the set $\mathbb{H}$, the optimal control is equal to 0, "do nothing", or on the set $\mathbb{H}^c$, the optimal control is singularly continuous or has a jump, "action". It is interpreted as the optimal control is either pushing the object with maximum force to keep the object in some specific region if the object is out of that region, or doing nothing if the object is within the region.

2.2 Singular controls of bounded velocity

Because of difficulties on singular controls, approaching the singular controls from controls with bounded velocity process is one of the useful techniques. The process ξ_t is called to the bounded velocity process if $|\frac{d\xi_t}{dt}| \leq \theta$ for some constant $\theta \in [0, \infty)$. Since a bounded velocity process is absolutely continuous, the control problem with bounded velocity is again regular control problem. However, the solution behaviour of the control problem with bounded velocity is similar to the solution behaviour of the singular control problem, and furthermore the solution to the singular control problem can be approximated by the solution to the control problem with bounded velocity as θ go to ∞,.

The relationship between singular controls and singular controls with bounded velocity is previously studied in [44]. In [44], under some technical assumptions with one direction of controls, the value function and optimal controls of convex stochastic control problems converge to some function and optimal controls as bounds going to ∞, and resulting function and optimal controls satisfy the optimality of the corresponding control problem.

Therefore, we approach the MFG with singular controls from bounded velocity in this thesis. First, we formulate the MFG with singular controls of bounded velocity (with a bound $\theta < \infty$) and prove uniqueness and existence of the solution to the MFG. Then, we show that optimal controls of the solution to the MFG is an ϵ-Nash equilibrium to the corresponding N player game and, furthermore, ϵ converges to 0 as the number of players N and the bound θ go to infinity.

Chapter 3

MFG with singular controls of bounded velocity

3.1 Introduction to MFG with singular controls of bounded velocity

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, P)$ be a probability space with $W = \{W_t\}_{0\le t\le T}$ standard Brownian motion in this space. Fix a finite time T and a probability measure μ. This paper introduces and analyses the following class of stochastic games:
for any $(s, x) \in [0, T] \times \mathbb{R}$,

$$v(s, x) = \inf_{\xi_\cdot^+, \xi_\cdot^- \in \mathcal{U}} E\left[\int_s^T \left(f(x_t, \mu_t)dt + g_1(x_t)d\xi_t^+ + g_2(x_t)d\xi_t^-\right)\right], \tag{3.1}$$

subject to

$$dx_t = b(x_t, \mu_t)dt + d\xi_t^+ - d\xi_t^- + \sigma dW_t, \quad \forall t \in [s, T], \qquad x_s = x, \mu_s = \mu. \tag{3.2}$$

Here (ξ_t^+, ξ_t^-) is a pair of non-decreasing càdlàg processes in an appropriate admissible control set $\mathcal{U}$, μ_t is a probability measure of x_t, and f, g_1, g_2 are functions satisfying some technical assumptions to be specified in Section 3.2.

This kind of problems belongs to a broad class of stochastic games known as the *mean field games* (MFGs). The theoretical development of MFGs is led by the pioneering work of [58] and [46], who studied stochastic games of a large population with small interactions. MFG avoids directly analyzing the notoriously hard N-player stochastic games when N is large. Instead, it approximates the dynamics and the objective function under the notion of population's probability distribution flows, a.k.a., mean information processes. (This idea can be traced to physics on weakly interacting particles.) As such, MFG leads to an elegant and analytically feasible framework to approximate the Nash equilibrium (equilibria) of N-player stochastic games.

Our work with singular controls. Most research on MFG theory focuses on regular controls where controls are absolutely continuous and rates of optimal controls are usually

Lipschitz continuous. In practice, controls are not necessarily absolutely continuous and/or the control rate might be constrained. These types of controls are called *singular controls* or impulse controls depending on the degree of discontinuity of the control. Generally, singular and impulse controls are much harder to analyse. For instance, studying singular controls involves analyzing fully nonlinear PDEs with additional gradient constraints, an important and difficult subject in PDE theory especially in terms of the regularity property. On the other hand, the subject of singular controls has fascinated control theorists, with its distinct "bang-bang" type control policy (Beneš, Shepp, and Witsenhausen [7]) and its connection to optimal stopping and switching (Karatzas and Shreve [50, 51], Boetius [12], Guo and Tomecek [40]).

Our thesis studies MFGs with singular controls with a bounded velocity for which the rates of optimal controls are no longer Lipschitz continuous. For a class of MFGs in the form of Eqn. (3.5), it shows that under appropriate technical conditions,

- the MFG admits a unique optimal control, and
- the value function of the MFG is an ϵ-Nash equilibrium to the corresponding N-player game, with $\epsilon = O\left(\frac{1}{\sqrt{N}}\right)$.

These results are analogous to those for MFGs with regular controls. Furthermore, our paper provides an MFG with singular control with an *explicit* analytic solution. This case study illustrates a curious connection between MFGs with and without common noise, under some "symmetric" problem structure.

Solution approach. Our solution approach is built on the PDE/control methodology of [58] and [46]. However, the analysis is more difficult for both the HJB equation and the SDE: not only the HJB equation is with additional state constraints, but also the rate of optimal controls is no longer Lipschitz continuous. Our analysis technique is inspired by the work of El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [28] for reflected BSDEs. The key element is to impose the rationality of players. Mathematically, it means that the control is non-increasing with respect to the player's current state. Intuitively, it says that the better off the state of the individual player, the less likely the player exercises controls (in order to minimize cost).

Related work. Some early work on MFG with singular controls includes Zhang [82] and Hu, Oksendal, and Sulem [45]. Both establish the stochastic maximal principle while the latter also proves the existence of optimal control policies for a class of MFGs with singular controls. The work of Fu and Horst [32] adopts the notion of relaxed controls to prove the existence of the solution of MFG with singular controls. Their problem setting and solution approach, however, are different from ours. In addition, our thesis establishes both the uniqueness and existence of the solution for MFGs, with explicit structures for the optimal control.

Outline of the chapter This chapter is organized as follows. Section 3.2 defines the MFG with singular controls of bounded velocity, and presents the main results regarding the existence and uniqueness of the solution to the MFG, as well as its ϵ-Nash equilibrium to the N-player game. Section 3.3 provides detailed proofs and Section 3.5 analyses a MFG in a systemic risk model with explicit solutions.

3.2 Problem formulations and main results

N-player games with singular controls. Fix a finite time T and suppose there are N identical players in the game. Denote $\{x_t^i\}_{0\leq t\leq T}$ as the state process in $\mathbb{R}$ for the ith player ($i = 1, 2, \ldots, N$), with $x_s^i = x^i$ starting from time $s \in [0, T]$. Now assume that the dynamics of x_t^i follows, for $t \in [s, T]$,

$$dx_t^i = \frac{1}{N}\sum_{j=1}^N b_0(x_t^i, x_t^j)dt + \sigma dW_t^i + d\xi_t^i, \quad x_s^i = x^i,$$

where $b_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and σ is a positive constant. Here $\{\xi_t^i\}_{s\leq t\leq T}$ is the control by the ith player with $i = 1, 2, \ldots, N$, assumed to be a càdlàg process and of a finite variation with $\xi_s^i = 0$.

The finite variation process $\{\xi_t^i\}_{s\leq t\leq T}$ can be decomposed into two nondecreasing processes $\{\xi_t^{i+}\}_{s\leq t\leq T}, \{\xi_t^{i-}\}_{s\leq t\leq T}$ such that $\xi_t^i = \xi_t^{i+} - \xi_t^{i-}$ with $\xi_s^{i+} = \xi_s^{i-} = 0$. Therefore the dynamics of x_t^i can be rewritten as

$$dx_t^i = \frac{1}{N}\sum_{j=1}^N b_0(x_t^i, x_t^j)dt + \sigma dW_t^i + d\xi_t^{i+} - d\xi_t^{i-}, \quad x_s^i = x^i.$$

The objective of the ith player is to minimize a cost function $J_\infty^{i,N}(s, x, \xi_\cdot^{i+}, \xi_\cdot^{i-}; \xi_\cdot^{-i})$ where $\xi_\cdot^{-i}$ is all other players' control processes $\{\xi_t^{j+}, \xi_t^{j-}\}_{j=1,j\neq i}^n$:

$$\begin{aligned}
&\inf_{\xi_t^{i+},\xi_t^{i-}\in\mathcal{U}_\infty} J_\infty^{i,N}(s, x^i, \xi_\cdot^{i+}, \xi_\cdot^{i-}; \xi_\cdot^{-i}) \\
&= \inf_{\xi_t^{i+},\xi_t^{i-}\in\mathcal{U}_\infty} E\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(x_t^i, x_t^j)dt + g_1(x_t^i)d\xi_t^{i+} + g_2(x_t^i)d\xi_t^{i-}\right], \quad (3.3)\\
&\text{subject to} \quad dx_t^i = \frac{1}{N}\sum_{j=1}^N b_0(x_t^i, x_t^j)dt + \sigma dW_t^i + d\xi_t^{i+} - d\xi_t^{i-}, \quad x_s^i = x^i,
\end{aligned}$$

for Lipschitz continuous functions $f_0 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, and over an appropriate admissible control set

$$\mathcal{U}_\infty = \{\{\xi_t^i\} | \xi_t^i \text{ is } \mathcal{F}_t\text{-progressively measurable, nondecreasing, and } \xi_s^i = 0\},$$

where $\mathcal{F}_t$ is a sigma algebra of $\sigma(x_t^1, \cdots, x_t^N)$. We consider that controls are closed loop in feedback form. That is, $d\xi_t^i = d\xi_t^{i+} - d\xi_t^{i-} = d\psi_1^i(t, x_t^i; x_t^1, \cdots, x_t^N) - d\psi_2^i(t, x_t^i; x_t^1, \cdots, x_t^N)$ for some function ψ_1^i, ψ_2^i.

Definition 3 (ϵ-Nash equilibrium). $\{\xi_t^{i*+}, \xi_t^{i*-}\}_{i=1}^n$ *is called an* ϵ-Nash equilibrium *to* (3.3) *if for any* $i \in \{1, 2, \ldots, n\}$*, any* $(s, x) \in [0, T] \times \mathbb{R}$*, and any* $\xi_t^{i'+}, \xi_t^{i'-} \in \mathcal{U}_\infty$*,*

$$J_\infty^{i,N}(s, x, \xi_\cdot^{i'+}, \xi_\cdot^{i'-}; \xi_\cdot^{*-i}) \geq J_\infty^{i,N}(s, x, \xi_\cdot^{i*+}, \xi_\cdot^{i*-}; \xi_\cdot^{*-i}) - \epsilon.$$

Note that in this N-player game, both the drift term in (3.3) for the dynamics and the first term in (3.3) for the objective function are affected by both the local information (i.e., the state of the ith player itself) and the global information (i.e., the states of other players). In general, this type of stochastic game is difficult to analyse: although the work of Uchida [79] shows the existence of Nash equilibrium for such an N-player game, finding a Nash equilibrium of the N-player game is in general intractable.

Now assume that the controls ξ_t^{i+}, ξ_t^{i-} are with bounded velocity so that $d\xi_t^{i+} = \dot{\xi}_t^{i+} dt$ and $d\xi_t^{i-} = \dot{\xi}_t^{i-} dt$ with $0 \leq \dot{\xi}_t^{i+}, \dot{\xi}_t^{i-} \leq \theta$ for a constant $\theta > 0$. Corresponding N player game with bounded velocity can be formulated as:

$$\begin{aligned} &\inf_{\xi_t^{i+}, \xi_t^{i-} \in \mathcal{U}_\theta} J_\theta^{i,N}(s, x^i, \xi_\cdot^{i+}, \xi_\cdot^{i-}; \xi_\cdot^{-i}) \\ &= \inf_{\xi_t^{i+}, \xi_t^{i-} \in \mathcal{U}_\theta} E\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(x_t^i, x_t^j) dt + g_1(x_t^i)\dot{\xi}_t^{i+} dt + g_2(x_t^i)\dot{\xi}_t^{i-} dt\right], \end{aligned} \tag{3.4}$$

$$\text{subject to} \quad dx_t^i = \frac{1}{N}\sum_{j=1}^N b_0(x_t^i, x_t^j) dt + \sigma dW_t^i + \dot{\xi}_t^{i+} dt - \dot{\xi}_t^{i-} dt, \quad x_s^i = x^i.$$

Similarly with the game (3.3), the admissible control set is

$$\mathcal{U}_\theta = \{\{\xi_t\} | \{\xi_t\} \text{ is } \mathcal{F}_t\text{-progressively measurable, nondecreasing, and } \xi_0 = 0, \dot{\xi}_t \in U = [0, \theta]\},$$

where $\mathcal{F}_t$ is a sigma algebra of $\sigma(x_t^1, \cdots, x_t^N)$, and we will again restrict ourselves to closed loop controls in feedback form.

A heuristic derivation to the MFG formulation. Assume that all N players are identical. Then, for each time $t \in [0, T]$, all x_t^i for $i = 1, 2, \cdots, N$ have same probability distribution. If $\varepsilon_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_t^i}$ is an empirical distribution of x_t^i for $i = 1, 2, \cdots, N$, under appropriate technical conditions, one can approximate via ε_t, according to SLLN, the drift function and cost function of the ith player game when $N \to \infty$, so that

$$\begin{aligned} \frac{1}{N}\sum_{j=1}^N b_0(x_t, x_t^j) &\to \int b_0(x_t, y)\varepsilon_t(dy) = b(x_t, \varepsilon_t), \\ \frac{1}{N}\sum_{j=1}^N f_0(x_t, x_t^j) &\to \int f_0(x_t, y)\varepsilon_t(dy) = f(x_t, \varepsilon_t). \end{aligned}$$

where $b, f : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ and $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are functions satisfying following assumptions;

Assumptions

(A1) $b(x,\mu), f(x,\mu), g_1(x)$, and $g_2(x)$ are Lipschitz continuous in x and μ (i.e. $|b(x_1,\mu^1) - b(x_2,\mu^2)| \leq Lip(b)(|x_1 - x_2| + D^1(\mu^1,\mu^2))$ for some $Lip(b) > 0$ and $|f(x_1,\mu^1) - f(x_2,\mu^2)| \leq Lip(f)(|x_1 - x_2| + D^1(\mu^1,\mu^2))$ for some $Lip(f) > 0$);

(A2) $f(x,\mu)$ has first order derivatives, and f and $\partial_x f(x,\mu)$ satisfy the polynomial growth condition;

(A3) $b(x,\mu)$, $g_1(x)$, and $g_2(x)$ have first and second order derivatives with respect to x, and derivatives are uniformly continuous and bounded in x;

(A4) $-g_1(x) \leq g_2(x)$ and $g_1(x), g_2(x) \neq 0$ for any $x \in \mathbb{R}$.

Note that now the drift term in the dynamics and the objective function rely only on the local information x_t^i and the aggregated mean information μ_t. This leads to an MFG formulation of (3.4). The MFG problem of Eqn. (3.1) can be defined precisely as

$$\begin{aligned}
v_\theta(s,x) &= \inf_{\xi^+_\cdot,\xi^-_\cdot \in \mathcal{U}_\theta} J^\infty_\theta(s,x,\xi^+_\cdot,\xi^-_\cdot) \\
&= \inf_{\xi^+_\cdot,\xi^-_\cdot \in \mathcal{U}_\theta} E\left[\int_s^T \left(f(x_t,\mu_t) + g_1(x_t)\dot{\xi}_t^+ + g_2(x_t)\dot{\xi}_t^-\right) dt\right], \qquad (3.5)\\
\text{subject to} \quad dx_t &= \left(b(x_t,\mu_t) + \dot{\xi}_t^+ - \dot{\xi}_t^-\right) dt + \sigma dW_t, \qquad x_s = x, \mu_s = \mu.
\end{aligned}$$

where μ_t is a probability measure of x_t for any $t \in [s,T]$ and controls are closed loop in feedback form over the admissible set $\mathcal{U}_\theta$. Since controls are closed loop in feedback form and the control process ξ_t is bounded velocity, we can define the control function φ as $\dot{\xi}_t = \dot{\xi}_t^+ - \dot{\xi}_t^- = \varphi(t,x_t;\{\mu_t\}) = \varphi_1(t,x_t;\{\mu_t\}) - \varphi_2(t,x_t;\{\mu_t\})$ where $\varphi_1 = \max\{\varphi,0\}$ and $\varphi_2 = -\max\{-\varphi,0\}$.

Assumptions.

(A5) (Monotonicity of the cost function) Either i) f satisfies the following condition

$$\int (f(x,\mu^1) - f(x,\mu^2))(\mu^1 - \mu^2)(dx) > 0, \text{ for any } \mu^1 \neq \mu^2 \in \mathcal{P}_2(\mathbb{R}),$$

or ii) f satisfies the following condition

$$\int (f(x,\mu^1) - f(x,\mu^2))(\mu^1 - \mu^2)(dx) \geq 0, \text{ for any } \mu^1, \mu^2 \in \mathcal{P}_2(\mathbb{R}),$$

and $H(x,p) = \inf_{\dot{\xi}^+,\dot{\xi}^- \in [0,\theta]} \{(\dot{\xi}^+ - \dot{\xi}^-)p + g_1(x)\dot{\xi}^+ + g_2(x)\dot{\xi}^-\}$ satisfies the following condition for any $x,p,q \in \mathbb{R}$

$$\text{if } H(x,p+q) - H(x,p) - \partial_p H(x,p)q = 0, \text{ then } \partial_p H(x,p+q) = \partial_p H(x,p);$$

(A6) (Rationality of players) For any control functions φ, any $t \in [0,T]$, and any $x, y \in \mathbb{R}$, $(x-y)(\varphi(t,x)-\varphi(t,y)) \leq 0$.

The assumption (A4) ensures the finiteness of the value function. For the game (3.3), if $-g_1(x) > g_2(x)$, by letting $d\xi_t^{i+} = d\xi_t^{i-} = M$ and $M \to \infty$, $J_\infty^{i,N}$ goes to $-\infty$. The assumption (A5) which is used in the proof of proposition 4 is for the uniqueness of the fixed point as in [58, 18] . The assumption (A6) which is used for the proof of theorem 2 is that closed loop in feedback type control functions are nonincreasing in current states because tendency of increasing state decreases as the current state is higher.

Solution approach and main results. Our solution approach is in the spirit of [58, 46], and consists of three steps. Fix a flow of probability measures $\{\mu_t\}$ which is in $\mathcal{M}_{[0,T]}$ [1]. The first step is to analyse a stochastic control problem under the fixed flow of probability measures $\{\mu_t\}_{0\leq t\leq T}$. If such a control problem has a unique optimal control, denoted the optimal control as $\dot{\xi}_t dt = \varphi(t, x_t|\{\mu_t\})dt$, then one can proceed to define a mapping Γ_1 from the class of flows of probability measures $\mathcal{M}_{[0,T]}$ to the space of optimal control functions and $\mathcal{M}_{[0,T]}$ so that $\Gamma_1(\{\mu_t\}) = (\varphi(t,x|\{\mu_t\}), \{\mu_t\})$. The second step is to analyse the optimal controlled process, the SDE, given the optimal control function φ. If this SDE allows for a unique flow of probability measures solution in $\mathcal{M}_{[0,T]}$, denoted as $\{\tilde{\mu}_t\}_{0\leq t\leq T}$, then one can define another mapping Γ_2 from the space of optimal control functions and $\mathcal{M}_{[0,T]}$ to $\mathcal{M}_{[0,T]}$ so that $\Gamma_2(\varphi(t,x|\{\mu_t\}), \{\mu_t\}) = \{\tilde{\mu}_t\}$. Then, repeat the first and second step under fixed flow of probability measures $\{\tilde{\mu}_t\}$. Keep repeat these steps until these iterations converge to a fixed point of the flow of probability measures $\{\mu_t^*\}$. We will check if $\Gamma = \Gamma_1 \circ \Gamma_2$ is a continuous mapping to allow for a fixed point solution and if Γ has at most one fixed point solution, leading to the solution of the MFG.

Definition 4. *A solution of the MFG (3.5) is defined as a pair of an optimal control $\{\xi_t^*\}$ and a flow of probability measures $\{\mu_t^*\} \in \mathcal{M}_{[0,T]}$ if they satisfy $v(s,x) = J_\theta^\infty(s,x,\xi_\cdot^{*+},\xi_\cdot^{*-})$ for all $(s,x) \in [0,T]\times\mathbb{R}$ and μ_t^* is a probability measure of the optimal controlled process x_t^* for all $t \in [0,T]$ where the dynamics of x_t^* is*

$$dx_t^* = \left(b(x_t^*, \mu_t^*) + \dot{\xi}_t^{*+} - \dot{\xi}_t^{*-}\right)dt + \sigma dW_t, \qquad x_s^* = x,$$

for $s \leq t \leq T$.

Now, we are ready to state the main results of the paper.

Theorem 1. *Under (A1)–(A5), there exists a unique solution $(\xi_t^*, \{\mu_t^*\})$ to the MFG (3.5). Moreover, the corresponding value function v for the MFG (3.5) is a function in $C^{1,2}([0,T]\times\mathbb{R})$, of a polynomial growth.*

Theorem 2. *Assume (A1)–(A6). Then,*

[1] By proposition 2 in later section, a flow of probability measures for the optimally controlled state process $\{\mu_t\}_{0\leq t\leq T}$ is in $\mathcal{M}_{[0,T]}$.

a) *for any fixed* $\mu_t \in \mathcal{M}_{[0,T]}$*, the value function to the stochastic control problem with bounded velocity processes* (3.5) *converges to the value function to the stochastic control problem with finite variations:*

$$\begin{aligned} v_\infty(s,x:\{\mu_t\}) &= \inf_{\xi_\cdot^+,\xi_\cdot^- \in \mathcal{U}_\infty} E\left[\int_s^T f(x_t,\mu_t)dt + g_1(x_t)d\xi_t^+ + g_2(x_t)d\xi_t^-\right], \\ \textit{subject to} \quad dx_t &= b(x_t,\mu_t)dt + d\xi_t^+ - d\xi_t^- + \sigma dW_t, \qquad x_s = x, \end{aligned} \tag{3.6}$$

as θ *goes to infinity;*

b) *the optimal control to the MFG with bounded velocity processes* (3.5) *is an* ϵ_N*-Nash equilibrium to the corresponding* N*-player game with bounded velocity processes* (3.4)*, with* $\epsilon_N = O(\frac{1}{\sqrt{N}})$*;*

c) *the optimal control to the MFG with bounded velocity processes* (3.5) *is an* $(\epsilon_N + \epsilon_\theta)$*-Nash equilibrium to the corresponding* N*-player game with finite variations* (3.3)*, with* $\epsilon_N = O(\frac{1}{\sqrt{N}})$ *and* $\epsilon_\theta \to 0$ *as* $\theta \to \infty$*.*

3.3 Proof of existence and uniqueness of solutions to MFG

The stochastic control problem

Let $\{\mu_t\} \in \mathcal{M}_{[0.T]}$ be a fixed exogenous flow of probability measures with $\mu_s = \mu$. Then, (3.5) is the following control problem,

$$v_\theta(s,x;\{\mu_t\}) = \inf_{\xi_\cdot^+,\xi_\cdot^- \in \mathcal{U}_\theta} E\left[\int_s^T \left(f(x_t,\mu_t) + g_1(x_t)\dot{\xi}_t^+ + g_2(x_t)\dot{\xi}_t^-\right)dt\right], \tag{3.7}$$

subject to

$$dx_t = \left(b(x_t,\mu_t) + \dot{\xi}_t^+ - \dot{\xi}_t^-\right)dt + \sigma dW_t, \qquad x_s = x.$$

This is a classical stochastic control problem, and the corresponding HJB equation with the terminal condition is given by

$$\begin{aligned} -\partial_t v_\theta &= \inf_{\dot{\xi}^+,\dot{\xi}^- \in [0,\theta]} \left\{\left(b(x,\mu) + (\dot{\xi}^+ - \dot{\xi}^-)\right)\partial_x v_\theta + \left(f(x,\mu) + g_1(x)\dot{\xi}^+ + g_2(x)\dot{\xi}^-\right)\right\} + \frac{\sigma^2}{2}\partial_{xx}v_\theta \\ &= \inf_{\dot{\xi}^+,\dot{\xi}^- \in [0,\theta]} \left\{(\partial_x v_\theta + g_1(x))\dot{\xi}^+ + (-\partial_x v_\theta + g_2(x))\dot{\xi}^-\right\} + b(x,\mu)\partial_x v_\theta + f(x,\mu) + \frac{\sigma^2}{2}\partial_{xx}v_\theta \\ &= \min\left\{(\partial_x v_\theta + g_1(x))\theta, (-\partial_x v_\theta + g_2(x))\theta, 0\right\} + b(x,\mu)\partial_x v_\theta + f(x,\mu) + \frac{\sigma^2}{2}\partial_{xx}v_\theta, \\ &\text{with } v_\theta(T,x) = 0, \quad \forall x \in \mathbb{R}. \end{aligned} \tag{3.8}$$

The existence and uniqueness of a $C^{1,2}([0,T]\times\mathbb{R})$ solution to (3.8) is clear by Theorem 6.2. in Chapter VI. [31]. Moreover, we can show that such a solution to (3.8) is the value function of (3.7).

Before establishing this result, let us recall the viscosity solution to (3.8).

Definition 5. *$\hat{v}$ is called a* viscosity solution *to (3.8) if $\hat{v}$ is both a viscosity supersolution and a viscosity subsolution, with the following definitions,*
(i) viscosity supersolution: for any $(t_0,x_0)\in[0,T]\times\mathbb{R}$ and any $\vartheta\in\mathcal{C}^{1,2}$, if (t_0,x_0) is a local minimum of $\hat{v}-\vartheta$ with $\hat{v}(t_0,x_0)-\vartheta(t_0,x_0)=0$, then

$$\begin{aligned}&-\inf_{\dot{\xi}^+,\dot{\xi}^-\in[0,\theta]}\left\{\left(b(x_0,\mu)+(\dot{\xi}^+-\dot{\xi}^-)\right)\partial_x\vartheta(t_0,x_0)+\left(f(x_0,\mu)+g_1(x_0)\dot{\xi}^++g_2(x_0)\dot{\xi}^-\right)\right\}\\&-\partial_t\vartheta(t_0,x_0)-\frac{\sigma^2}{2}\partial_{xx}\vartheta(t_0,x_0)\geq 0,\qquad \textit{and}\quad \vartheta(T,x_0)\geq 0;\end{aligned}$$

(ii) viscosity subsolution: for any $(t_0,x_0)\in[0,T]\times\mathbb{R}$ and any $\vartheta\in\mathcal{C}^{1,2}$, if (t_0,x_0) is a local maximum of $\hat{v}-\vartheta$ with $\hat{v}(t_0,x_0)-\vartheta(t_0,x_0)=0$, then

$$\begin{aligned}&-\inf_{\dot{\xi}^+,\dot{\xi}^-\in[0,\theta]}\left\{\left(b(x_0,\mu)+(\dot{\xi}^+-\dot{\xi}^-)\right)\partial_x\vartheta(t_0,x_0)+\left(f(x_0,\mu)+g_1(x_0)\dot{\xi}^++g_2(x_0)\dot{\xi}^-\right)\right\}\\&-\partial_t\vartheta(t_0,x_0)-\frac{\sigma^2}{2}\partial_{xx}\vartheta(t_0,x_0)\leq 0,\qquad \textit{and}\quad \vartheta(T,x_0)\leq 0.\end{aligned}$$

Proposition 1. *Assume a fixed $\{\mu_t\}$ in $\mathcal{M}_{[0,T]}$ for $0\leq t\leq T$. Under the assumptions (A1)-(A5), the HJB Eqn. (3.8) with a terminal condition $v_\theta(T,x)=0$ for any $x\in\mathbb{R}$ has a unique solution v in $C^{1,2}([0,T]\times\mathbb{R})$, of a polynomial growth. Furthermore, the solution is the value function to the problem (3.7), with the optimal control given by*

$$\varphi_\theta(t,x_t|\{\mu_t\})=\dot{\xi}^+_{t,\theta}-\dot{\xi}^-_{t,\theta}=\begin{cases}\theta & \textit{if}\quad \partial_x v_\theta(t,x_t)\leq -g_1(x_t),\\ 0 & \textit{if}\quad -g_1(x_t)\leq\partial_x v_\theta(t,x_t)\leq g_2(x_t),\\ -\theta & \textit{if}\quad g_2(x_t)\leq\partial_x v_\theta(t,x_t).\end{cases}$$

Proof. By theorem 6.2. in Chapter VI. [31], there exists a unique solution w which is in $C^{1,2}([0,T]\times\mathbb{R})$ with polynomial growth to (3.8) and let v_θ be the value function of (3.7).

First, w is a viscosity subsolution to (3.8). That is, for any $(s,x)\in[0,T]\times\mathbb{R}$, $w(T,x)\leq 0$ and

$$-\partial_t w-\inf_{\dot{\xi}^+,\dot{\xi}^-\in[0,\theta]}\left\{\left(b(x,\mu)+(\dot{\xi}^+-\dot{\xi}^-)\right)\partial_x w+\left(f(x,\mu)+g_1(x)\dot{\xi}^++g_2(x)\dot{\xi}^-\right)\right\}-\frac{\sigma^2}{2}\partial_{xx}w\leq 0.$$

On one hand, for any $\dot{\xi}^+_t,\dot{\xi}^-_t\in\mathcal{U}_\theta$, let x_t be a controlled process with $\dot{\xi}^+_t,\dot{\xi}^-_t$. Then, by the Itô's formula on $w(s,x)$,

$$\begin{aligned}0&\geq E[w(T,x_T)]\\&=w(s,x)+E\left[\int_s^T\partial_t w(t,x_t)+(b(x_t,\mu_t)+(\dot{\xi}^+_t-\dot{\xi}^-_t))\partial_x w(t,x_t)+\frac{\sigma^2}{2}\partial_{xx}w(t,x_t)dt\right]\\&\geq w(s,x)-E\left[\int_s^T f(x_t,\mu_t)+g_1(x_t)\dot{\xi}^+_t+g_2(x_t)\dot{\xi}^-_t dt\right].\end{aligned}$$

Hence, for any $\dot{\xi}_t^+, \dot{\xi}_t^- \in \mathcal{U}_\theta$,

$$E\left[\int_s^T f(x_t,\mu_t) + g_1(x_t)\dot{\xi}_t^+ + g_2(x_t)\dot{\xi}_t^- dt\right] \geq w(s,x).$$

Therefore,

$$v(s,x) = \inf_{\dot{\xi}_t^+,\dot{\xi}_t^- \in \mathcal{U}} E\left[\int_s^T f(x_t,\mu_t) + g_1(x_t)\dot{\xi}_t^+ + g_2(x_t)\dot{\xi}_t^- dt\right] \geq w(s,x).$$

On the other hand, let $\dot{\xi}_{t,\theta}^+, \dot{\xi}_{t,\theta}^- \in \mathcal{U}_\theta$ be a minimizer of the Hamiltonian:

$$\inf_{\dot{\xi}^+,\dot{\xi}^- \in [0,\theta]} \left\{ \left(b(x,\mu) + (\dot{\xi}^+ - \dot{\xi}^-)\right)\partial_x w + \left(f(x,\mu) + g_1(x)\dot{\xi}^+ + g_2(x)\dot{\xi}^-\right) \right\}.$$

Then, since $w(s,x)$ is the solution to (3.8), $w(T,x) = 0$ for any $x \in \mathbb{R}$ and with controls $\dot{\xi}_{t,\theta}^+, \dot{\xi}_{t,\theta}^-$

$$-\partial_t w - \left\{ \left(b(x,\mu) + (\dot{\xi}_{t,\theta}^+ - \dot{\xi}_{t,\theta}^-)\right)\partial_x w + \left(f(x,\mu) + g_1(x)\dot{\xi}_{t,\theta}^+ + g_2(x)\dot{\xi}_{t,\theta}^-\right) \right\} - \frac{\sigma^2}{2}\partial_{xx} w = 0.$$

Let $x_{t,\theta}$ be the controlled process with controls $\dot{\xi}_{t,\theta}^+, \dot{\xi}_{t,\theta}^-$. Then, applying the Itô's formula to $w(t,x)$,

$$\begin{aligned}
0 &= E[w(T,x_{T,\theta})] \\
&= w(s,x) + E\left[\int_s^T \left(\partial_t w(t,x_{t,\theta}) + (b(x_{t,\theta},\mu_t) + (\dot{\xi}_{t,\theta}^+ - \dot{\xi}_{t,\theta}^-)\partial_x w(t,x_{t,\theta}) + \frac{\sigma^2}{2}\partial_{xx} w(t,x_{t,\theta})\right) dt\right] \\
&= w(s,x) - E\left[\int_s^T \left(f(x_{t,\theta},\mu_t) + g_1(x_{t,\theta})\dot{\xi}_{t,\theta}^+ + g_2(x_{t,\theta})\dot{\xi}_{t,\theta}^-\right) dt\right] \\
&\leq w(s,x) - v(s,x).
\end{aligned}$$

Hence, $v(s,x) \leq w(s,x)$.

Combined, $v(s,x) = w(s,x)$, and $\dot{\xi}_{t,\theta}^+ - \dot{\xi}_{t,\theta}^- = \begin{cases} \theta & \text{if} \quad \partial_x v_\theta(t,x_{t,\theta}) \leq -g_1(x_{t,\theta}), \\ 0 & \text{if} \quad -g_1(x_{t,\theta}) \leq \partial_x v_\theta(t,x_{t,\theta}) \leq g_2(x_{t,\theta}), \\ -\theta & \text{if} \quad g_2(x_{t,\theta}) \leq \partial_x v_\theta(t,x_{t,\theta}). \end{cases}$

is the optimal controls.

□

Now, one can define Γ_1 from the class of flows of probability measures $\mathcal{M}_{[0,T]}$ so that $\Gamma_1(\{\mu_t\}) = (\varphi(t,x|\{\mu_t\}),\{\mu_t\})$ which is a pair of the optimal control function under fixed $\{\mu_t\}$ and the fixed flow of probability measures $\{\mu_t\}$.

Consistency part

Now with fixed $\{\mu_t\}$ and the optimal control function $\varphi(t,x|\{\mu_t\})$, the dynamics of the optimal controlled process x_t follows

$$dx_t = (b(x_t,\mu_t) + \varphi(t,x_t|\{\mu_t\}))\,dt + \sigma dW_t, \qquad x_0 = x \tag{3.9}$$

Since the function $|b(x,\mu)+\varphi(t,x|\{\mu_t\})| \le M(1+|x|)$ for some positive M and σ is a positive constant, the SDE (3.9) has weak solutions and it is unique in law by the Stroock-Varadhan theorem (from the chapter V.19 and V.24 in [72]).

Consequently, one can define Γ_2 so that $\Gamma_2(\varphi(t,x|\{\mu_t\}),\{\mu_t\}) = \{\tilde{\mu}_t\}$ which is an updated mean information probability measure flow under the fixed control function φ.

The fixed point method

Define a mapping Γ as $\Gamma(\{\mu_t\}) = \Gamma_2 \circ \Gamma_1(\{\mu_t\}) = \{\tilde{\mu}_t\}$. We will use the Schauder fixed point theorem to show the existence of fixed point.

Lemma 1 (Schauder fixed point theorem)**.** *If K is a nonempty conves subset of a normed space V and Γ is a continuous mapping of K into K such that the range $\Gamma(K)$ is compact in K, then Γ has a fixed point.*

Let's prove that Γ is a mapping of $\mathcal{M}_{[0,T]}$ into $\mathcal{M}_{[0,T]}$, and it is continuous and relatively compact.

Proposition 2. *Γ is the function from $\mathcal{M}_{[0,T]}$ to $\mathcal{M}_{[0,T]}$.*

Proof. For any $\{\mu_t\}$ in $\mathcal{M}_{[0,T]}$, let's prove that $\{\tilde{\mu}_t\} = \Gamma(\{\mu_t\})$ is also in $\mathcal{M}_{[0,T]}$. Without loss of generality, suppose $s > t$, and $x_s = x_t + \int_t^s (b(x_r,\mu_r) + \varphi(r,x_r))dr + \int_t^s \sigma dW_r$. Since $b(x,\mu)$ is Lipschitz and $|\varphi(s,x_s)| \le \theta$,

$$\begin{aligned}
D^1(\tilde{\mu}_s,\tilde{\mu}_t) &\le E|x_s - x_t| \\
&\le E\int_t^s |b(x_r,\mu_r) + \varphi(r,x_r)|dr + \sigma E \sup_{r\in[t,s]} |W_r - W_t| \\
&\le E\int_t^s |b(x_r,\mu_r) - b(x_t,\mu_t) + b(x_t,\mu_t)|dr + \theta|s-t| + \sigma E \sup_{r\in[t,s]} |W_r - W_t| \\
&\le E\int_t^s |b(x_r,\mu_r) - b(x_t,\mu_t)|dr + |s-t|E|b(x_t,\mu_t)| + \theta|s-t| + \sigma|s-t|^{\frac{1}{2}} \\
&\le E\int_t^s Lip(b)(|x_r - x_t| + D^1(\mu_r,\mu_t))dr + |s-t|(E|b(x_t,\mu_t)| + \theta) + \sigma|s-t|^{\frac{1}{2}} \\
&\le Lip(b)E\int_t^s |x_r - x_t|dr + \int_t^s c_1|r-t|^{\frac{1}{2}}dr + |s-t|(E|b(x_t,\mu_t)| + \theta) + \sigma|s-t|^{\frac{1}{2}} \\
&\le Lip(b)E\int_t^s |x_r - x_t|dr + \frac{2}{3}c_1|s-t|^{\frac{3}{2}} + |s-t|(E|b(x_t,\mu_t)| + \theta) + \sigma|s-t|^{\frac{1}{2}}
\end{aligned}$$

By Gronwall's inequallity,

$$D^1(\tilde{\mu}_s, \tilde{\mu}_t) \le E|x_s - x_t| \le \left(\frac{2}{3}c_1|s-t|^{\frac{3}{2}} + |s-t|(E|b(x_t,\mu_t)| + \theta) + \sigma|s-t|^{\frac{1}{2}}\right) e^{\int_t^s Lip(b)dr}$$
$$= \left(\frac{2}{3}c_1|s-t|^{\frac{3}{2}} + |s-t|(E|b(x_t,\mu_t)| + \theta) + \sigma|s-t|^{\frac{1}{2}}\right) e^{Lip(b)|t-s|}$$

Since b is Lipschitz, for any $t \in [0,T]$, $E|b(x_t,\mu_t)| < M$ for some positive $M > 0$. Therefore,

$$\sup_{s\neq t} \frac{D^1(\tilde{\mu}_t, \tilde{\mu}_s)}{|t-s|^{\frac{1}{2}}} \le c,$$

For any $t \in [0,T]$,

$$\int |x|^2 \tilde{\mu}_t(dx) \le 2E[\int |x|^2 d\tilde{\mu}_0 + c_2^2 t^2 + \sigma^2 t] \le 2E[\int |x|^2 d\tilde{\mu}_0 + c_2^2 T^2 + \sigma^2 T]$$

$$\therefore \sup_{t\in[0,T]} \int |x|^2 \tilde{\mu}_t(dx) \le c$$

□

Proposition 3. Γ *is continuous.*

Proof. Let $\{\mu_t^n\} \in \mathcal{M}_{[0,T]}$ for $n = 1, 2, \cdots$ be a sequence of measure flows which converges to $\{\mu_t\} \in \mathcal{M}_{[0,T]}$ as $n \to \infty$ in the sense of metric $d_{\mathcal{M}}$ (1.11) i.e. $d_{\mathcal{M}}(\{\mu_t^n\}, \{\mu_t\}) \to 0$ as $n \to \infty$. For each n, denote $\varphi^n(t,x)$ as the optimal control function of the MFG (3.5) under fixed $\{\mu_t^n\}$, and $\{x_t^n\}$ is corresponding optimal controlled process:

$$dx_t^n = (b(x_t^n, \mu_t^n) + \varphi^n(t, x_t^n))dt + \sigma dW_t, \quad x_0^n = x.$$

Let $\{\tilde{\mu}_t^n\}$ be a flow of probability measures of $\{x_t^n\}$, then $\Gamma(\{\mu_t^n\}) = \{\tilde{\mu}_t^n\}$.

Similarly, define $\varphi(t,x)$ as the optimal control function with respect to $\{\mu_t\}$, $\{x_t\}$ is corresponding optimal controlled process:

$$dx_t = (b(x_t, \mu_t) + \varphi(t, x_t))dt + \sigma dW_t, \quad x_0 = x,$$

and $\{\tilde{\mu}_t\}$ is a flow of probability measures of $\{x_t\}$. Let's prove that $d_{\mathcal{M}}(\{\tilde{\mu}_t^n\}, \{\tilde{\mu}_t\}) \to 0$ as $n \to \infty$.

Step 1: Find a relation between $D^2(\{\tilde{\mu}_t^n\}, \{\tilde{\mu}_t\})$ and $D^2(\{\mu_t^n\}, \{\mu_t\})$.

For arbitrary $t \in [0,T]$, for any s such that $0 \le s \le t$,

$$d(x_s - x_s^n) = (b(x_s, \mu_s) - b(x_s^n, \mu_s^n) + \varphi(s, x_s) - \varphi^n(s, x_s^n))ds.$$

By the chain rule,

$$
\begin{aligned}
|x_t - x_t^n|^2 &= 2\int_0^t (b(x_s,\mu_s) - b(x_s^n,\mu_s^n) + \varphi(s,x_s) - \varphi^n(s,x_s^n))(x_s - x_s^n)ds \\
&\le 2\int_0^t \Big|\int_{\mathbb{R}} b_0(x_s,y)\mu_s(dy) - \int_{\mathbb{R}} b_0(x_s^n,y^n)\mu_s^n(dy^n)\Big||x_s - x_s^n| \\
&\quad + (\varphi(s,x_s) - \varphi^n(s,x_s^n))(x_s - x_s^n)ds \\
&\le 2\int_0^t \int_{\mathbb{R}\times\mathbb{R}} |b_0(x_s,y) - b_0(x_s^n,y^n)||x_s - x_s^n|\Xi_s(dy,dy^n) \\
&\quad + (\varphi(s,x_s) - \varphi^n(s,x_s^n))(x_s - x_s^n)ds
\end{aligned}
$$

where Ξ_s is an arbitrary coupling probability measure of μ_s and μ_s^n for any $s \in [0,T]$.

$$
\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}} |b_0(x_s,y) - b_0(x_s^n,y^n)||x_s - x_s^n|\Xi_s(dy,dy^n) \\
=&\int_{\mathbb{R}\times\mathbb{R}} |b_0(x_s,y) - b_0(x_s^n,y) + b_0(x_s^n,y) - b_0(x_s^n,y^n)||x_s - x_s^n|\Xi_s(dy,dy^n) \\
\le&\int_{\mathbb{R}\times\mathbb{R}} (Lip(b_0)|x_s - x_s^n|^2 + Lip(b_0)|y - y^n||x_s - x_s^n|)\Xi_s(dy,dy^n) \\
\le&\int_{\mathbb{R}\times\mathbb{R}} \left(Lip(b_0)|x_s - x_s^n|^2 + \frac{Lip(b_0)}{2}(|y - y^n|^2 + |x_s - x_s^n|^2)\right)\Xi_s(dy,dy^n)
\end{aligned}
$$

and, since $\varphi^n(s,x)$ is nonincreasing function in x,

$$
\begin{aligned}
&(\varphi(s,x_s^n) - \varphi^n(s,x_s^n))(x_s - x_s^n) \\
\le&(\varphi(s,x_s) - \varphi^n(s,x_s) + \varphi^n(s,x_s) - \varphi^n(s,x_s^n))(x_s - x_s^n) \\
\le&(\varphi(s,x_s) - \varphi^n(s,x_s))(x_s - x_s^n) \\
\le&\frac{1}{2}(|\varphi(s,x_s) - \varphi^n(s,x_s)|^2 + |x_s - x_s^n|^2)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&|x_t - x_t^n|^2 \\
\le&\int_0^t (3Lip(b_0)+1)|x_s - x_s^n|^2 + Lip(b_0)\int_{\mathbb{R}\times\mathbb{R}} |y - \tilde{y}|^2\Xi_s(dy,dy^n) + |\varphi(s,x_s) - \varphi^n(s,x_s)|^2 ds.
\end{aligned}
$$

Because the inequality holds for any coupling measure Ξ_s and by the Gronwall's inequality,

$$
(D^2(\tilde{\mu}_t,\tilde{\mu}_t^n))^2 \le c_1\int_0^t Lip(b_0)(D^2(\mu_s,\mu_s^n))^2 + |\varphi(s,x_s) - \varphi^n(s,x_s)|^2 ds \tag{3.10}
$$

for some constant c_1 depending on T and $Lip(b_0)$.

Step 2: Using Itô's formula, find useful equations and inequalities.

By proposition 1, for fixed $\{\mu_t\}$ the stochastic control problem (3.7) has a unique value function $v(s,x)$ and optimal controls given by

$$\varphi(s,x) = \begin{cases} \theta & \text{if} \quad \partial_x v(s,x) \le -g_1(x), \\ 0 & \text{if} \quad -g_1(x) \le \partial_x v(s,x) \le g_2(x), \\ -\theta & \text{if} \quad g_2(x) \le \partial_x v(s,x), \end{cases} \quad \text{and}$$

for fixed $\{\mu_t^n\}$ the stochastic control problem (3.7) has a unique value function $v^n(s,x)$ and optimal controls given by

$$\varphi^n(s,x) = \begin{cases} \theta & \text{if} \quad \partial_x v^n(s,x) \le -g_1(x), \\ 0 & \text{if} \quad -g_1(x) \le \partial_x v^n(s,x) \le g_2(x), \\ -\theta & \text{if} \quad g_2(x) \le \partial_x v^n(s,x). \end{cases}$$

Let's prove that for any $(t,x) \in [0,T] \times \mathbb{R}$ as n goes to infinity, $\partial_x v^n(t,x)$ converges to $\partial_x v(t,x)$.

Fix $t \in [0,T]$ again and let $0 \le t \le s \le T$. For each n, $\{x_s^n\}_{t \le s \le T}$ is the optimal controlled process:

$$dx_s^n = (b(x_s^n, \mu_s^n) + \varphi^n(s, x_s^n))ds + \sigma dW_s, \quad x_t^n = x.$$

Similarly, $\{x_s\}_{t \le s \le T}$ is the optimal controlled process:

$$dx_s = (b(x_s, \mu_s) + \varphi(s, x_s))ds + \sigma dW_s, \quad x_t = x.$$

By proposition 1, v and v^n are the solution to the HJB equation (3.8). Denote $\varphi_1(s,x) = \max\{\varphi(s,x), 0\}$, $\varphi_2(s,x) = -\max\{-\varphi(s,x), 0\}$, $\varphi_1^n(s,x) = \max\{\varphi^n(s,x), 0\}$ and $\varphi_2^n(s,x) = -\max\{-\varphi^n(s,x), 0\}$.

By the Itô's formula and the HJB equation (3.8),

$$\begin{aligned} & v(T, x_T) - v(t, x_t) \\ &= \int_t^T \partial_t v(s,x_s) + (b(x_s,\mu_s) + \varphi(x_s))\partial_x v(s,x_s) + \frac{\sigma^2}{2}\partial_{xx} v(s,x_s) ds + \int_t^T \sigma \partial_x v(s,x_s) dW_s \\ &= -\int_t^T [f(x_s,\mu_s) + g_1(x_s)\varphi_1(s,x_s) + g_2(x_s)\varphi_2(s,x_s)] ds + \int_t^T \sigma \partial_x v(s,x_s) dW_s \end{aligned}$$

Since $v(T,x) = 0$ for any $x \in \mathbb{R}$,

$$v(t,x) = \int_t^T [f(x_s,\mu_s) + g_1(x_s)\varphi_1(s,x_s) + g_2(x_s)\varphi_2(s,x_s)] ds - \int_t^T \sigma \partial_x v(s,x_s) dW_s. \quad (3.11)$$

Similarly, for any $n \in \mathbb{N}$, by the Itô's formula for $v^n(s,x)$ and $\{x_s\}$,

$$\begin{aligned}
&v^n(T,x_T) - v^n(t,x_t) \\
&= \int_t^T \partial_t v^n(s,x_s) + (b(x_s,\mu_s) + \varphi(s,x_s))\partial_x v^n(s,x_s) + \frac{\sigma^2}{2}\partial_{xx} v^n(s,x_s) ds \\
&+ \int_t^T \sigma \partial_x v^n(s,x_s) dW_s \\
&= \int_t^T \partial_t v^n(s,x_s) + (b(x_s,\mu_s^n) + \varphi^n(s,x_s))\partial_x v^n(s,x_s) + \frac{\sigma^2}{2}\partial_{xx} v^n(s,x_s) ds \\
&+ \int_t^T \sigma \partial_x v^n(s,x_s) dW_s - \int_t^T (b(x_s,\mu_s^n) - b(x_s,\mu_s) + \varphi^n(s,x_s) - \varphi(s,x_s))\partial_x v^n(s,x_s) ds \\
&= -\int_t^T [f(x_s,\mu_s^n) + g_1(x_s)\varphi_1^n(s,x_s) + g_2(x_s)\varphi_2^n(s,x_s)] ds + \int_t^T \sigma \partial_x v^n(s,x_s) dW_s \\
&- \int_t^T (b(x_s,\mu_s^n) - b(x_s,\mu_s) + \varphi^n(s,x_s) - \varphi(s,x_s))\partial_x v^n(s,x_s) ds
\end{aligned}$$

The last equality is due to the HJB equation (3.8).

$$\begin{aligned}
v^n(t,x) = &\int_t^T [f(x_s,\mu_s^n) + g_1(x_s)\varphi_1^n(s,x_s) + g_2(x_s)\varphi_2^n(s,x_s)] ds \\
&- \int_t^T \sigma \partial_x v^n(s,x_s) dW_s + \int_t^T (b(x_s,\mu_s^n) - b(x_s,\mu_s) + \varphi^n(s,x_s) - \varphi(s,x_s))\partial_x v^n(s,x_s) ds.
\end{aligned} \tag{3.12}$$

From equations (3.11) and (3.12),

$$\begin{aligned}
&v(t,x) - v^n(t,x) \\
=&E[v(t,x) - v^n(t,x)] \\
=&E\left[\int_t^T f(x_s,\mu_s) - f(x_s,\mu_s^n) + g_1(x_s)(\varphi_1(s,x_s) - \varphi_1^n(s,x_s)) + g_2(x_s)(\varphi_2(s,x_s) - \varphi_2^n(s,x_s)) ds\right] \\
&+ E\left[\int_t^T (b(x_s,\mu_s) - b(x_s,\mu_s^n) + \varphi(s,x_s) - \varphi^n(s,x_s))\partial_x v^n(s,x_s) ds\right] \\
=&E[\int_t^T f(x_s,\mu_s) - f(x_s,\mu_s^n) + (g_1(x_s) + \partial_x v^n(s,x_s))(\varphi_1(s,x_s) - \varphi_1^n(s,x_s)) \\
&+ (g_2(x_s) - \partial_x v^n(s,x_s))(\varphi_2(s,x_s) - \varphi_2^n(s,x_s)) + (b(x_s,\mu_s) - b(x_s,\mu_s^n))\partial_x v^n(s,x_s) ds]
\end{aligned}$$

Hence,

$$
\begin{aligned}
&|v(t,x)-v^n(t,x)|+E[\int_t^T |f(x_s,\mu_s)-f(x_s,\mu_s^n)|+|(b(x_s,\mu_s)-b(x_s,\mu_s^n))\partial_x v^n(s,x_s)|ds] \\
\geq &\left|v(t,x)-v^n(t,x)-E[\int_t^T f(x_s,\mu_s)-f(x_s,\mu_s^n)+(b(x_s,\mu_s)-b(x_s,\mu_s^n))\partial_x v^n(s,x_s)ds]\right| \\
= &\Big|E[\int_t^T (g_1(x_s)+\partial_x v^n(s,x_s))(\varphi_1(s,x_s)-\varphi_1^n(s,x_s)) \\
&\quad +(g_2(x_s)-\partial_x v^n(s,x_s))(\varphi_2(s,x_s)-\varphi_2^n(s,x_s))ds]\Big|
\end{aligned}
$$

By definition of φ_1^n, if $\varphi_1^n(s,x_s)=\theta$, then $g_1(x_s)+\partial_x v^n(s,x_s)\leq 0$ (Because the controls are Markovian, a closed loop in feedback form, $\varphi^n(s,x_s)$ does not depend on the process $\{x_s\}$ but only depend the value $x_s\in\mathbb{R}$), and if $\varphi_1^n(s,x_s)=0$, then $g_1(x_s)+\partial_x v^n(s,x_s)\geq 0$ (same here; because the controls are Markovian, a closed loop in feedback form, $\varphi^n(s,x_s)$ does not depend on the process $\{x_s\}$ but only depend the value "$x_s\in\mathbb{R}$"). So, by definitions of φ_1 and φ_1^n, $(g_1(x_s)+\partial_x v^n(s,x_s))(\varphi_1(s,x_s)-\varphi_1^n(s,x_s))\geq 0$.
By definition of φ_2^n, if $\varphi_2^n(s,x_s)=\theta$, then $g_2(x_s)-\partial_x v^n(s,x_s)\leq 0$, and if $\varphi_2^n(s,x_s)=0$, then $g_2(x_s)-\partial_x v^n(s,x_s)\geq 0$. So, by definitions of φ_2 and φ_2^n, $(g_2(x_s)-\partial_x v^n(s,x_s))(\varphi_2(s,x_s)-\varphi_2^n(s,x_s))\geq 0$.
Hence,

$$
\begin{aligned}
&E[\int_t^T |(g_1(x_s)+\partial_x v^n(s,x_s))(\varphi_1(s,x_s)-\varphi_1^n(s,x_s))| \\
&\quad +|(g_2(x_s)-\partial_x v^n(s,x_s))(\varphi_2(s,x_s)-\varphi_2^n(s,x_s))|ds] \\
\leq &|v(t,x)-v^n(t,x)|+E[\int_t^T |f(x_s,\mu_s)-f(x_s,\mu_s^n)|+|(b(x_s,\mu_s)-b(x_s,\mu_s^n))\partial_x v^n(s,x_s)|ds] \\
\leq &|v(t,x)-v^n(t,x)|+E[\int_t^T Lip(f)D^1(\mu_s,\mu_s^n)+Lip(b)D^1(\mu_s,\mu_s^n)|\partial_x v^n(s,x_s)|ds],
\end{aligned} \tag{3.13}
$$

because f and b are Lipschitz continuous.

Since for any $s\in[t,T]$ $D^1(\mu_s^n,\mu_s)\to 0$ as $n\to\infty$, $|b(x_s,\mu_s)-b(x_s,\mu_s^n)|\leq Lip(b)D^1(\mu_s,\mu_s^n)\to 0$ as $n\to\infty$. By proposition 4.1. in chapter 4 in [81], $v^n(t,x)\to v(t,x)$ for any $(t,x)\in[0,T]\times\mathbb{R}$ as $n\to\infty$. By definition of v^n and boundedness of g_1,g_2, $|\partial_x v^n(s,x)|\leq \sup\{x: |g_1(x)|,|g_2(x)|\}<M$ for some M. Hence, as n goes to infinity, the last term in the inequality (3.13) goes to 0.

Consequently, for any $x\in\mathbb{R}$, as n goes to infinity

$$|(g_1(x)+\partial_x v^n(s,x))(\varphi_1(s,x)-\varphi_1^n(s,x))|+|(g_2(x)-\partial_x v^n(s,x))(\varphi_2(s,x)-\varphi_2^n(s,x))|\to 0.$$

Step 3: Prove that $\varphi^n(s,x)$ converges to $\varphi(s,x)$ for any $s,x\in[0,T]\times\mathbb{R}$ as n goes to infinity.

From equations (3.11) and (3.12), by the Cauchy-Schwartz inequality and the Itô's isometry,

$$
\begin{aligned}
&(v(t,x) - v^n(t,x))^2 + \sigma^2 E[\int_t^T (\partial_x v(s,x_s) - \partial_x v^n(s,x_s))^2 ds] \\
\leq &4(T-t)E\Bigg[\int_t^T \{f(x_s,\mu_s) - f(x_s,\mu_s^n)\}^2 + \{(b(x_s,\mu_s) - b(x_s,\mu_s^n))\partial_x v^n(s,x_s)\}^2 \\
&+ \{(g_1(x_s) + \partial_x v^n(s,x))(\varphi_1(s,x_s) - \varphi_1^n(s,x_s)) \\
&+ (g_2(x) - \partial_x v^n(s,x))(\varphi_2(s,x_s) - \varphi_2^n(s,x_s))\}^2 ds\Bigg] \\
\leq &4(T-t)E\Bigg[\int_t^T \{Lip(f)D^1(\mu_s,\mu_s^n)\}^2 + \{Lip(b)D^1(\mu_s,\mu_s^n)\partial_x v^n(s,x_s)\}^2 \\
&+ \{(g_1(x_s) + \partial_x v^n(s,x_s))(\varphi_1(s,x_s) - \varphi_1^n(s,x_s)) \\
&+ (g_2(x_s) - \partial_x v^n(s,x_s))(\varphi_2(s,x_s) - \varphi_2^n(s,x_s))\}^2 ds\Bigg]
\end{aligned}
$$

Because of the result in step 2 and $D^1(\mu_s,\mu_s^n) \to 0$, the last term goes to 0 as n goes to infinity. Therefore, $\partial_x v^n(s,x)$ converges to $\partial_x v(s,x)$ for any $(s,x) \in [t,T] \times \mathbb{R}$. Furthermore, by definition of φ^n and φ, $\varphi^n(s,x)$ converges to $\varphi(s,x)$ for any $s,x \in [t,T] \times \mathbb{R}$. This holds for any $t \in [0,T]$. Therefore, $\varphi^n(s,x)$ converges to $\varphi(s,x)$ for any $s,x \in [0,T] \times \mathbb{R}$.

Step 4: Prove $d_{\mathcal{M}}(\{\tilde{\mu}_t\},\{\tilde{\mu}_t^n\}) \to 0$ as $n \to \infty$.

From previous steps, as n goes to infinity, for any $s,x \in [0,T] \times \mathbb{R}$, $\varphi^n(s,x)$ converges to $\varphi(s,x)$ and $D^2(\mu_s,\mu_s^n)$ converges to 0. Hence, by the inequality (3.10), $D^2(\tilde{\mu}_t,\tilde{\mu}_t^n)$ converges to 0 for any $t \in [0,T]$. Since $D^1(\tilde{\mu}_t,\tilde{\mu}_t^n) \leq D^2(\tilde{\mu}_t,\tilde{\mu}_t^n)$, $D^1(\tilde{\mu}_t,\tilde{\mu}_t^n)$ converges to 0 for any $t \in [0,T]$, and $d_{\mathcal{M}}(\{\tilde{\mu}_t\},\{\tilde{\mu}_t^n\}) \to 0$ as $n \to \infty$. Therefore, Γ is continuous. □

Proposition 4. *Γ has a fixed point, and the MFG (3.5) has a unique solution.*

Proof. As the proof in lemma 5.7 in [18], the range of the mapping Γ is relatively compact, and by proposition 3, Γ is a continuous mapping. Hence, due to the Schauder fixed point theorem, Γ has a fixed point such that $\Gamma(\{\mu_t\}) = \{\mu_t\} \in \mathcal{M}_{[0,T]}$. By the assumption (A5), the fixed point is at most one ([58],[18]). Therefore, there exists a unique fixed point solution of flow of probability measures $\{\mu_t^*\}$. Then, consider the MFG (3.5) with fixed $\{\mu_t^*\}$. It is again stochastic control problem and by proposition 1, there exists a unique optimal controls $\{\xi_t^*\}$ for the MFG (3.5) with fixed $\{\mu_t^*\}$. by definition of the solution to a MFG, $\{\xi_t^*\}$ is a optimal control solution to the MFG (3.5) and by proposition 1, it is unique. (If there is another optimal controls, there are two optimal controls for the the MFG (3.5) with fixed $\{\mu_t^*\}$. So, it is contradiction.) □

3.4 Proof of ϵ-Nash equilibrium

Suppose $(\{\xi_{t,\theta}\},\{\mu_{t,\theta}\})$ is a MFG solution to the MFG (3.5) with bound θ, $v_\theta(s,x:\{\mu_t\})$ is the value function of the MFG (3.5) with any fixed flow of probability measures $\{\mu_t\}$ (i.e.

$v_\theta(s,x:\{\mu_{t,\theta}\})$ is the value function of the MFG (3.5)), and $x_{t,\theta}$ is the optimal controlled process:

$$dx_{t,\theta} = (b(x_{t,\theta},\mu_{t,\theta}) + \varphi_{1,\theta}(t,x_{t,\theta}|\{\mu_{t,\theta}\}) - \varphi_{2,\theta}(t,x_{t,\theta}|\{\mu_{t,\theta}\}))dt + \sigma dW_t, \quad x_{s,\theta} = x,$$

where $\dot{\xi}_{t,\theta} = \varphi_\theta(t,x|\{\mu_t\}) = \varphi_{1,\theta}(t,x|\{\mu_t\}) - \varphi_{2,\theta}(t,x|\{\mu_t\})$ is the optimal control function.

Proof of theorem 2 a) For any fixed $\{\mu_t\} \in \mathcal{M}_{[0,T]}$, $v_\theta(s,x:\{\mu_t\})$ converges to $v_\infty(s,x:\{\mu_t\})$ as $\theta \to \infty$.

Proof. Fix $\{\mu_t\} \in \mathcal{M}_{[0,T]}$. For any $\zeta^+_{t,\infty}, \zeta^-_{t,\infty} \in \mathcal{U}_\infty$, since each path of a finite variation process is almost everywhere differentiable, there exists a sequence of bounded velocity functions which converges to the path as bounds go to infinity. Hence, there exists a sequence $\{\zeta^+_{t,\theta}\}_{\theta\in[0,\infty)}, \{\zeta^-_{t,\theta}\}_{\theta\in[0,\infty)}$ such that $\zeta^+_{t,\theta}, \zeta^-_{t,\theta} \in \mathcal{U}_\theta$ and $E\int_0^T |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| \to 0, E\int_0^T |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}| \to 0$ as $\theta \to \infty$.

Denote

$$\begin{aligned} d\hat{x}_{t,\theta} &= (b(\hat{x}_{t,\theta},\mu_t) + \dot{\zeta}^+_{t,\theta} - \dot{\zeta}^-_{t,\theta})dt + \sigma dW_t, \quad \hat{x}_{s,\theta} = x, \text{ and} \\ d\hat{x}_t &= b(\hat{x}_t,\mu_t)dt + \sigma dW_t + d\zeta^+_{t,\infty} - d\zeta^-_{t,\infty}, \qquad \hat{x}_s = x. \end{aligned}$$

Then, for any $\tau \in [s,T]$,

$$\begin{aligned} |\hat{x}_{\tau,\theta} - \hat{x}_\tau| &\le \int_s^\tau |b(\hat{x}_{t,\theta},\mu_t) - b(\hat{x}_t,\mu_t)|dt + \int_s^\tau |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \int_s^\tau |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}| \\ &\le \int_s^\tau Lip(b)|\hat{x}_{t,\theta} - \hat{x}_t|dt + \int_s^\tau |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + \int_s^\tau |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}| \end{aligned}$$

By the Gronwall's inequality,

$$E|\hat{x}_{\tau,\theta} - \hat{x}_\tau| \le O\left(E\int_s^\tau |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + E\int_s^\tau |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}|\right).$$

Consequently,

$$\begin{aligned} &|J^\infty_\infty(s,x,\zeta^+_{t,\infty},\zeta^-_{t,\infty}:\{\mu_t\}) - J^\infty_\theta(s,x,\zeta^+_{t,\theta},\zeta^-_{t,\theta}:\{\mu_t\})| \\ \le& E[|\int_s^T f(\hat{x}_t,\mu_t) - f(\hat{x}_{t,\theta},\mu_t) + g_1(\hat{x}_t)d\zeta^+_{t,\infty} + g_2(\hat{x}_t)d\zeta^-_{t,\infty} - g_1(\hat{x}_{t,\theta})\dot{\zeta}^+_{t,\theta}dt - g_2(\hat{x}_{t,\theta})\dot{\zeta}^-_{t,\theta}dt|] \\ \le& E[\int_s^T Lip(f)|\hat{x}_t - \hat{x}_{t,\theta}| + Lip(g_1)|\hat{x}_t - \hat{x}_{t,\theta}|d\zeta^+_{t,\infty} + Lip(g_2)|\hat{x}_t - \hat{x}_{t,\theta}|d\zeta^-_{t,\infty} \\ &+ g_1(\hat{x}_{t,\theta})|d\zeta^+_{t,\infty} - \dot{\zeta}^+_{t,\theta}dt| + g_2(\hat{x}_{t,\theta})|d\zeta^-_{t,\infty} - \dot{\zeta}^-_{t,\theta}dt|] \\ \le& O\left(E\int_s^T |\dot{\zeta}^+_{t,\theta}dt - d\zeta^+_{t,\infty}| + E\int_s^T |\dot{\zeta}^-_{t,\theta}dt - d\zeta^-_{t,\infty}|\right). \end{aligned}$$

Hence, $|v_\infty(s,x:\{\mu_t\}) - v_\theta(s,x:\{\mu_t\})| \to 0$ as $\theta \to 0$ □

Define ϵ_θ as $\sup_{s,x,\{\mu_t\}\in[0,T]\times\mathbb{R}\times\mathcal{M}_{[0,T]}} |v_\infty(s,x:\{\mu_t\}) - v_\theta(s,x:\{\mu_t\})| \le \epsilon_\theta$ with $\epsilon_\theta \to 0$ as $\theta \to \infty$.

Consider the stochastic control problem (3.6) with $\{\mu_{t,\theta}\}$. $v_\infty(s,x:\{\mu_{t,\theta}\})$ is the value function, and let $x_{t,\infty}$ be the optimal controlled process:

$$dx_{t,\infty} = b(x_{t,\infty}, \mu_{t,\infty})dt + \sigma dW_t + d\xi^+_{t,\infty} - d\xi^-_{t,\infty}, \quad x_{s,\infty} = x.$$

The optimal controls $\xi_{t,\infty}$ is also of feedback form. Hence, denote $d\varphi_\infty(t,x|\{\mu_{t,\theta}\}) = d\varphi_{1,\infty}(t,x|\{\mu_{t,\theta}\}) - d\varphi_{2,\infty}(t,x|\{\mu_{t,\theta}\}) = d\xi^+_{t,\infty} - d\xi^-_{t,\infty}$ as the optimal control function for the stochastic control problem (3.3) with fixed $\{\mu_{t,\theta}\}$.

Denote,[2] for $i = 1,\cdots,N$,

$$\begin{aligned}
dx^i_{t,\theta} &= (b(x^i_{t,\theta}, \mu_{t,\theta}) + \varphi_{1,\theta}(t, x^i_{t,\theta}) - \varphi_{2,\theta}(t, x^i_{t,\theta}))dt + \sigma dW^i_t, \quad x^i_{s,\theta} = x, \\
dx^i_{t,\infty} &= b(x^i_{t,\infty}, \mu_{t,\theta})dt + d\varphi_{1,\infty}(t, x^i_{t,\infty}) - d\varphi_{2,\infty}(t, x^i_{t,\infty}) + \sigma dW^i_t, \quad x^i_{s,\infty} = x, \\
dx^{i,N}_{t,\theta} &= (\frac{1}{N}\sum_{j=1,\cdots,N} b_0(x^{i,N}_{t,\theta}, x^{i,N}_{t,\theta}) + \varphi_{1,\theta}(t, x^{i,N}_{t,\theta}) - \varphi_{2,\theta}(t, x^{i,N}_{t,\theta}))dt + \sigma dW^i_t, \quad x^{i,N}_{s,\theta} = x,
\end{aligned}$$

Since $(\mu_{t,\theta}, \varphi_\theta)$ is the solution to the MFG (3.5), $\mu_{t,\theta}$ is the probability measure of $x^i_{t,\theta}$ for any $i = 1,\cdots,N$ ($x^i_{t,\theta}$ for $i = 1,\cdots,N$ are i.i.d.) .

Lemma 2. *For any* $1 \le i \le n$, $E \sup_{s\le t\le T} |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 = O(\frac{1}{N})$.

Proof.

$$d(x^i_{t,\theta} - x^{i,N}_{t,\theta}) = \left(\int b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) + \varphi_\theta(t, x^i_{t,\theta}) - \varphi_\theta(t, x^{i,N}_{t,\theta})\right)dt,$$

and

$$\begin{aligned}
d(x^i_{t,\theta} - x^{i,N}_{t,\theta})^2 = \Bigg(&2(x^i_{t,\theta} - x^{i,N}_{t,\theta})(\int b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) \\
&+ \varphi^\theta(t, x^i_{t,\theta}) - \varphi^\theta(t, x^{i,N}_{t,\theta}))\Bigg)dt.
\end{aligned}$$

[2]In this proof, omit $\{\mu_{t,\theta}\}$ for notations simplicity. Denote $\varphi_{i,\theta}(t,x) = \varphi_{i,\theta}(t,x|\{\mu_{t,\theta}\})$ and $\varphi_{i,\infty}(t,x) = \varphi_{i,\infty}(t,x|\{\mu_{t,\theta}\})$ for $i = 1,2$.

Since $\varphi_\theta(t,x)$ is nonincreasing in x, $(x^i_{t,\theta} - x^{i,N}_{t,\theta})(\varphi_\theta(t,x^i_{t,\theta}) - \varphi_\theta(t,x^{i,N}_{t,\theta})) \le 0$. So,

$$
\begin{aligned}
|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 &\le \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}|| \int b_0(x^i_{t,\theta},y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta})|dt \\
&\le \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}|| \int b_0(x^i_{t,\theta},y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta})|dt \\
&\quad + \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}|| \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta}) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta})|dt \\
&\le \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}|| \int b_0(x^i_{t,\theta},y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta})|dt \\
&\quad + \int_s^T 2|x^i_{t,\theta} - x^{i,N}_{t,\theta}| \frac{1}{N}\sum_{j=1}^N Lip(b_0)|x^j_{t,\theta} - x^{j,N}_{t,\theta}|dt \\
&\le \int_s^T |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + | \int b_0(x^i_{t,\theta},y)\mu^\theta_t(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta})|^2 dt \\
&\quad + \int_s^T \frac{1}{N} Lip(b_0)\sum_{j=1}^N (|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + |x^j_{t,\theta} - x^{j,N}_{t,\theta}|^2)dt \\
&\le (1 + Lip(b_0))\int_s^T |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt \\
&\quad + \int_s^T | \int b_0(x^i_{t,\theta},y)\mu t,\theta(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta})|^2 dt \\
&\quad + \int_s^T \frac{1}{N} Lip(b_0)\sum_{j=1}^N |x^j_{t,\theta} - x^{j,N}_{t,\theta}|^2 dt
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 &\le (1 + Lip(b_0))E\int_s^T |x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt \\
&\quad + E\int_s^T | \int b_0(x^i_{t,\theta},y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^{j}_{t,\theta})|^2 dt \\
&\quad + E\int_s^T Lip(b_0)|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left|\int b_0(x^i_{t,\theta}, y)\mu_{t,\theta}(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta})\right|^2 \\
&\leq 2E\left|\int b_0(x^i_{t,\theta}, y) - b_0(x^{i,N}_{t,\theta}, y)\mu_t(dy)\right|^2 + 2E\left|\int b_0(x^{i,N}_{t,\theta}, y)\mu_t(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta})\right|^2 \\
&\leq 2E\left|\int Lip(b_0)|x^i_{t,\theta} - x^{i,N}_{t,\theta}|\mu_t(dy)\right|^2 + 2E\left|\int b_0(x^{i,N}_{t,\theta}, y)\mu_t(dy) - \frac{1}{N}\sum_{j=1}^N b_0(x^{i,N}_{t,\theta}, x^j_{t,\theta})\right|^2 \\
&= 2Lip(b_0)^2 E|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + \epsilon^2,
\end{aligned}
$$

with $\epsilon = O(\frac{1}{\sqrt{N}})$ by the central limit theorem. Consequently,

$$
E|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 \leq E\int_s^T (1 + 2Lip(b_0))|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 dt + E\int_s^T \left(2Lip(b_0)^2|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 + \epsilon^2\right) dt.
$$

By the Gronwall's inequality,

$$
\begin{aligned}
E\sup_{s\leq t\leq T}|x^i_{T,\theta} - x^{i,N}_{T,\theta}|^2 &\leq \int_s^T \epsilon^2 dt \cdot E[\exp(\int_s^T (1 + 4Lip(b_0)^2)dt)] \\
&= \int_s^T \epsilon^2 dt \cdot e^{(1+4Lip(b_0)^2)T} = O\left(\frac{1}{N}\right).
\end{aligned}
$$

Therefore, $E\sup_{s\leq t\leq T}|x^i_{t,\theta} - x^{i,N}_{t,\theta}|^2 = O\left(\frac{1}{N}\right)$. □

Proof of theorem 2 b)

Proof. Suppose that the first player chooses a different control function $\xi'_t \in \mathcal{U}_\infty$ which is bounded velocity and all other players $i = 2, 3, \ldots, N$ choose to stay with the optimal control function $\{\xi_{t,\theta}\}$. Denote $d\xi'_t = \dot{\xi}'_t dt = \varphi'(t,x)dt$ and $d\xi_{t,\theta} = \dot{\xi}_{t,\theta}dt = \varphi_\theta(t,x)dt$. Then the corresponding dynamics for the MFG is

$$
d\tilde{x}^1_{t,\theta} = (b(\tilde{x}^1_{t,\theta}, \mu_{t,\theta}) + \varphi'(t, \tilde{x}^1_{t,\theta}))dt + \sigma dW^1_t,
$$

and the corresponding dynamics for N-player game are

$$
\begin{aligned}
d\tilde{x}^{1,N}_{t,\theta} &= \left(\frac{1}{N}\sum_{j=1}^N b_0(\tilde{x}^{1,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta}) + \varphi'(t, \tilde{x}^{1,N}_{t,\theta})\right) dt + \sigma dW^1_t, \\
d\tilde{x}^{i,N}_{t,\theta} &= \left(\frac{1}{N}\sum_{j=1}^N b(\tilde{x}^{i,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta}) + \varphi_\theta(t, \tilde{x}^{i,N}_{t,\theta})\right) dt + \sigma dW^i_t, \qquad 2 \leq i \leq N.
\end{aligned}
$$

We can show

Lemma 3. $\sup_{2\le i\le N} E \sup_{0\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| \le O(\frac{1}{\sqrt{N}})$.

Proof. For any $2 \le i \le N$,

$$d(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}) = \left[\frac{1}{N}\sum_{j=1}^{N}\Big(b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) - b_0(\tilde{x}^{i,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta})\Big) + \varphi_\theta(t, x^{i,N}_{t,\theta}) - \varphi_\theta(t, \tilde{x}^{i,N}_{t,\theta})\right] dt.$$

Because $\varphi_\theta(t,x)$ is nonincreasing in x,

$$\begin{aligned}
|x^{i,N}_{T,\theta} - \tilde{x}^{i,N}_{T,\theta}|^2 &\le \int_s^T 2(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta})\left(\frac{1}{N}\sum_{j=1}^{N}\Big(b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) - b_0(\tilde{x}^{i,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta})\Big)\right) dt \\
&\le \int_s^T 2(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta})\frac{1}{N}\sum_{j=1}^{N} Lip(b_0)(|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| + |x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|)dt \\
&\le 2Lip(b_0)\int_s^T |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|\frac{1}{N}\sum_{j=1}^{N}|x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|dt \\
&\le 2Lip(b_0)\int_s^T |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + \frac{1}{2N}\sum_{j=1}^{N}(|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + |x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|^2)dt \\
&\le Lip(b_0)\int_s^T 3|x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 + \frac{1}{N}\sum_{j=1}^{N}|x^{j,N}_{t,\theta} - \tilde{x}^{j,N}_{t,\theta}|^2)dt,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 \\
&\le Lip(b_0)\int_s^T [\sup_{2\le i\le N} E \sup_{s\le t'\le t} 3|x^{i,N}_{t',\theta} - \tilde{x}^{i,N}_{t',\theta}|^2 \\
&\qquad + \frac{N-1}{N}\sup_{2\le j\le N} E \sup_{s\le t'\le t}|x^{j,N}_{t',\theta} - \tilde{x}^{j,N}_{t',\theta}|^2 + \frac{1}{N}E|x^{1,N}_{t,\theta} - \tilde{x}^{1,N}_{t,\theta}|^2]dt \\
&= Lip(b_0)\int_s^T \left[\frac{4N-1}{N}\sup_{2\le i\le N} E \sup_{s\le t'\le t}|x^{i,N}_{t',\theta} - \tilde{x}^{i,N}_{t',\theta}|^2 + \frac{1}{N}E|x^{1,N}_{t,\theta} - \tilde{x}^{1,N}_{t,\theta}|^2\right] dt.
\end{aligned}$$

By the Gronwall's inequality,

$$\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}|^2 \le Lip(b_0)\int_s^T \frac{1}{N}E|x^{1,N}_{t,\theta} - \tilde{x}^{1,N}_{t,\theta}|^2 dt \cdot e^{\int_0^T Lip(b_0)\frac{4N-1}{N}dt} = O\left(\frac{1}{N}\right).$$

So, $\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| = O(\frac{1}{\sqrt{N}})$. □

From the Lemma 2, for any $2 \le i \le N$, $\sup_{s\le t\le T} E|x^i_{t,\theta} - x^{i,N}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right)$, and by the triangle inequality, $\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^i_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| = O(\frac{1}{\sqrt{N}})$. Therefore,

$$\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^i_{t,\theta} - \tilde{x}^{i,N}_{t,\theta}| + \sup_{1\le i\le N} E \sup_{s\le t\le T} |x^i_{t,\theta} - x^{i,N}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right).$$

Define

$$d\bar{x}^{1,N}_{t,\theta} = \left(\frac{1}{N}\sum_{j=1}^{N} b_0(\bar{x}^{1,N}_{t,\theta}, x^j_{t,\theta}) + \varphi'(t, \bar{x}^{1,N}_{t,\theta})\right) dt + \sigma dW^1_t,$$

Since $(x-y)(\varphi'(t,x)-\varphi'(t,y)) \le 0$ by the assumption (A6), then an approach as in Lemma 2 shows $E \sup_{0\le t\le T} |\tilde{x}^{1,N}_{t,\theta} - \bar{x}^{1,N}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right)$ and $E \sup_{0\le t\le T} |\bar{x}^{1,N}_{t,\theta} - \tilde{x}^{1}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right)$. Therefore,

$$\begin{aligned}
J^{i,N}_\theta(s, x^1, \xi'^+_\cdot, \xi'^-_\cdot; \xi^{-1}_{\cdot,\theta}) &= E\left[\int_s^T \frac{1}{N}\sum_{j=1}^{N} f_0(\tilde{x}^{1,N}_{t,\theta}, \tilde{x}^{j,N}_{t,\theta})dt + g(\tilde{x}^{1,N}_{t,\theta})\varphi'(t, \tilde{x}^{1,N}_{t,\theta})dt\right] \\
&\ge E\left[\int_s^T \frac{1}{N}\sum_{j=1}^{N} f_0(\tilde{x}^{1,N}_{t,\theta}, x^{j}_{t,\theta})dt + g(\tilde{x}^{1,N}_{t,\theta})\varphi'(t, \tilde{x}^{1,N}_{t,\theta})dt\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge E\left[\int_s^T \frac{1}{N}\sum_{j=1}^{N} f_0(\bar{x}^{1,N}_{t,\theta}, x^{j}_{t,\theta})dt + g(\bar{x}^{1,N}_{t,\theta})\varphi'(t, \bar{x}^{1,N}_{t,\theta})dt\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge E\left[\int_s^T \int f_0(\tilde{x}^{1}_{t,\theta}, y)\mu_{t,\theta}(dy) + g(\tilde{x}^{1}_{t,\theta})\varphi'(t, \tilde{x}^{1}_{t,\theta})dt\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge E\left[\int_s^T \int f_0(x^{1}_{t,\theta}, y)\mu_{t,\theta}(dy) + g(x^{1}_{t,\theta})\varphi_\theta(t, x^{1}_{t,\theta})dt\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&= E\left[\int_s^T \frac{1}{N}\sum_{j=1}^{N} f_0(x^{1,N}_{t,\theta}, x^{j,N}_{t,\theta})dt + g(x^{1,N}_{t,\theta})\varphi(t, x^{1,N}_{t,\theta})dt\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&= J^{i,N}_\theta(s, x^1, \xi^+_{\cdot,\theta}, \xi^-_{\cdot,\theta}; \xi^{-1}_{\cdot,\theta}) - O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}$$

the last inequality is due to the optimality of φ as the optimal control function of the MFG (3.5), and the last equality is due to the central limit theorem. This completes the proof. □

Proof of theorem 2 c)

Proof. Similarly, let the player 1 choose any other controls $\xi'_t \in \mathcal{U}_\infty$ which is a finite variation process but all other players choose same optimal controls $\xi_{t,\theta}$. Denote $d\xi'_t = d\varphi'(t,x) =$

$d\varphi_1'(t,x) - d\varphi_2'(t,x)$.

$$d\tilde{x}^1_{t,\infty} = b(\tilde{x}^1_{t,\infty}, \mu^\theta_t)dt + d\varphi_1'(t,\tilde{x}^1_{t,\infty}) - d\varphi_2'(t,\tilde{x}^1_{t,\infty}) + \sigma dW^1_t \quad \tilde{x}^1_{s,\infty} = x,$$
$$d\tilde{x}^{1,N}_{t,\infty} = \frac{1}{N}\sum_{j=1,\cdots,N} b_0(\tilde{x}^{1,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty})dt + d\varphi_1'(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi_2'(t,\tilde{x}^{1,N}_{t,\infty}) + \sigma dW^1_t, \quad \tilde{x}^{1,N}_{s,\infty} = x,$$
$$d\tilde{x}^{i,N}_{t,\infty} = (\frac{1}{N}\sum_{j=1,\cdots,N} b_0(\tilde{x}^{i,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty}) + \varphi_{1,\theta}(t,\tilde{x}^{i,N}_{t,\infty}) - \varphi_{2,\theta}(t,\tilde{x}^{i,N}_{t,\infty}))dt + \sigma dW^i_t, \quad x^{i,N}_{s,\infty} = x,$$

for $i = 2,\cdots,N$,

Lemma 4. $\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| = O(\frac{1}{\sqrt{N}})$.

Proof. For any $2 \le i \le N$,

$$d(x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}) = \left[\frac{1}{N}\sum_{j=1}^N \left(b_0(x^{i,N}_{t,\theta}, x^{j,N}_{t,\theta}) - b_0(\tilde{x}^{i,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty})\right) + \varphi_\theta(t, x^{i,N}_{t,\theta}) - \varphi_\theta(t,\tilde{x}^{i,N}_{t,\infty})\right] dt.$$

By the definition, $\varphi_\theta(t,x)$ is nonincreasing in x. Hence, similarly with the proof of the lemma 3, we can prove $\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^{i,N}_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| = O(\frac{1}{\sqrt{N}})$. □

From the Lemma 2, for any $2 \le i \le N$, $\sup_{s\le t\le T} E|x^i_{t,\theta} - x^{i,N}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right)$. Then, by the triangle inequality, $\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^i_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| = O(\frac{1}{\sqrt{N}})$. Therefore,

$$\sup_{2\le i\le N} E \sup_{s\le t\le T} |x^i_{t,\theta} - \tilde{x}^{i,N}_{t,\infty}| + \sup_{1\le i\le N} E \sup_{s\le t\le T} |x^i_{t,\theta} - x^{i,N}_{t,\theta}| = O\left(\frac{1}{\sqrt{N}}\right).$$

Since $d\varphi'(t,x)$ is also nonincreasing in x, then an approach as in Lemma 2 shows $E \sup_{s\le t\le T} |\tilde{x}^{1,N}_{t,\infty} - \tilde{x}^1_{t,\infty}| = O\left(\frac{1}{\sqrt{N}}\right)$. Therefore, due to Lipschitz continuity of f, f_0, g_1, g_2,

$$\begin{aligned}
&J^{1,N}_\infty(s,x,\xi'^+_\cdot,\xi'^-_\cdot;\xi^{-1}_{\cdot,\theta}) \\
&= E\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(\tilde{x}^{1,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty})dt + g_1(\tilde{x}^{1,N}_{t,\infty})d\varphi_1'(t,\tilde{x}^{1,N}_{t,\infty}) + g_2(\tilde{x}^{1,N}_{t,\infty})d\varphi_2'(t,\tilde{x}^{1,N}_{t,\infty})\right] \\
&\ge E\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(\tilde{x}^{1,N}_{t,\infty}, x^j_{t,\theta})dt + g_1(\tilde{x}^{1,N}_{t,\infty})d\varphi_1'(t,\tilde{x}^{1,N}_{t,\infty}) + g_2(\tilde{x}^{1,N}_{t,\infty})d\varphi_2'(t,\tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge E\left[\int_s^T \int f(\tilde{x}^{1,N}_{t,\infty}, y)\mu_{t,\theta}(dy)dt + g_1(\tilde{x}^{1,N}_{t,\infty})d\varphi_1'(t,\tilde{x}^{1,N}_{t,\infty}) + g_2(\tilde{x}^{1,N}_{t,\infty})d\varphi_2'(t,\tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right) \\
&\ge E\left[\int_s^T \int f(\tilde{x}^{1}_{t,\infty}, y)\mu_{t,\theta}(dy)dt + g_1(\tilde{x}^{1}_{t,\infty})d\varphi_1'(t,\tilde{x}^{1,N}_{t,\infty}) + g_2(\tilde{x}^{1}_{t,\infty})d\varphi_2'(t,\tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right).
\end{aligned}$$

By definition of φ'_1, φ'_2,

$$E\left|d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty}) - d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right|$$
$$\leq Ed|\tilde{x}^{1,N}_{t,\infty} - \tilde{x}^{1}_{t,\infty}| + E\left|\frac{1}{N}\sum_{j=1,\cdots,N} b_0(\tilde{x}^{1,N}_{t,\infty}, \tilde{x}^{j,N}_{t,\infty}) - b(\tilde{x}^{\infty 1}_{t}, \mu_{t,\theta})\right| dt = O\left(\frac{1}{\sqrt{N}}\right).$$

By definition of φ'_i,

$$\left|\left(d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty})\right) + \left(-d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right)\right|$$
$$= \left|d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty})\right| + \left|-d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right|.$$

Consequently, $E\sup_{s\leq t\leq T}\left|d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) - d\varphi'_1(t,\tilde{x}^{1}_{t,\infty})\right| = O\left(\frac{1}{\sqrt{N}}\right)$ and
$E\sup_{s\leq t\leq T}\left|-d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty}) + d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right| = O\left(\frac{1}{\sqrt{N}}\right)$.
Therefore, since g_1, g_2 are bounded,

$$E\left[\int_s^T \int f(\tilde{x}^{1}_{t,\infty}, y)\mu_{t,\theta}(dy)dt + g_1(\tilde{x}^{1}_{t,\infty})d\varphi'_1(t,\tilde{x}^{1,N}_{t,\infty}) + g_2(\tilde{x}^{1}_{t,\infty})d\varphi'_2(t,\tilde{x}^{1,N}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right)$$
$$\geq E\left[\int_s^T \int f(\tilde{x}^{1}_{t,\infty}, y)\mu_{t,\theta}(dy)dt + g_1(\tilde{x}^{1}_{t,\infty})d\varphi'_1(t,\tilde{x}^{1}_{t,\infty}) + g_2(\tilde{x}^{\infty 1}_{t})d\varphi'_2(t,\tilde{x}^{1}_{t,\infty})\right] - O\left(\frac{1}{\sqrt{N}}\right)$$
$$\geq E\left[\int_s^T \int f(x^{1}_{t,\infty}, y)\mu_{t,\theta}(dy)dt + g_1(x^{1}_{t,\infty})d\varphi_{1,\infty}(t,x^{1}_{t,\infty}) + g_2(x^{1}_{t,\infty})d\varphi_{2,\infty}(t,x^{1}_{t,\infty})\right]$$
$$- O\left(\frac{1}{\sqrt{N}}\right)$$
$$= v_\infty(s,x:\{\mu_{t,\theta}\}) - O\left(\frac{1}{\sqrt{N}}\right),$$

the last inequality is due to the optimality of φ_∞. By theorem 2 a), $|v_\theta(s,x:\{\mu_{t,\theta}\})) - v_\infty(s,x:\{\mu_{t,\theta}\}))| \leq O(\epsilon_\theta)$.

Hence, by $E\sup\limits_{s\leq t\leq T}|x^i_{t,\theta} - x^{i,N}_{t,\theta}| = O(\frac{1}{\sqrt{N}})$ and similarly with previous steps, we could derive

$$
\begin{aligned}
&J^{1,N}_{\infty}(s,x,\xi'^{+}_{\cdot},\xi'^{-}_{\cdot};\xi^{-1}_{\cdot,\theta}) = v_{\infty}(s,x:\{\mu_{t,\theta}\}) - O\left(\frac{1}{\sqrt{N}}\right)\\
&\geq v_{\theta}(s,x:\{\mu_{t,\theta}\}) - O\left(\frac{1}{\sqrt{N}}+\epsilon_{\theta}\right)\\
&\geq E\left[\int_s^T\int f(x^1_{t,\theta},y)\mu_{t,\theta}(dy)dt + g_1(x^1_{t,\theta})d\varphi_{1,\theta}(t,x^1_{t,\theta}) + g_2(x^1_{t,\theta})d\varphi_{2,\theta}(t,x^1_{t,\theta})\right]\\
&\quad - O\left(\frac{1}{\sqrt{N}}+\epsilon_{\theta}\right)\\
&\geq E\left[\int_s^T \frac{1}{N}\sum_{j=1}^N f_0(x^{1,N}_{t,\theta},x^{j,N}_{t,\theta})dt + g_1(x^{1,N}_{t,\theta})d\varphi^{\theta}_1(t,x^{1,N}_{t,\theta}) + g_2(x^{1,N}_{t,\theta})d\varphi^{\theta}_2(t,x^{1,N}_{t,\theta})\right]\\
&\quad - O\left(\frac{1}{\sqrt{N}}+\epsilon_{\theta}\right)\\
&= J^{1,N}_{\infty}(s,x,\xi^{+}_{\cdot,\theta},\xi^{-}_{\cdot,\theta};\xi^{-1}_{\cdot,\theta}).
\end{aligned}
$$

□

3.5 An MFG with singular controls of bounded velocity: systemic risk

In this section, we study a particular MFG and provide explicit solutions. For comparison purposes, we present a singular control counterpart of the MFG originally formulated with regular controls by [23] for systemic risk. We will see that our solution structure is consistent with theirs, despite the differences in problem settings.

The basic idea behind the interbank systemic risk model of [23] is as follows. (A similar model can also be found in [33].) There are N banks in the system that borrow and lend money among each other. Each bank controls its rates of borrowing and lending to minimize a cost function. There are common noise and individual noise for each bank. Define x^i_t to be the log-monetary reserve for bank i with $i = 1, 2, \ldots, N$. Then, the dynamics of x^i_t is assumed to be

$$
\begin{aligned}
dx^i_t &= \frac{a}{N}\sum_{j=1}^N (x^j_t - x^i_t)dt + \dot{\xi}^i_t dt + \sigma(\rho dW^0_t + \sqrt{1-\rho^2}dW^i_t),\\
&= a(m_t - x^i_t)dt + \dot{\xi}^i_t dt + \sigma(\rho dW^0_t + \sqrt{1-\rho^2}dW^i_t), \quad x^i_s = x^i.
\end{aligned}
\tag{3.14}
$$

Here, $\{W^i_t\}_{0\leq t\leq T}$ represents the individual noise for the ith player with $i = 1, 2, \ldots, N$, and $\{W^0_t\}_{0\leq t\leq T}$ is another independent Brownian motion representing the common noise, $m_t = \frac{1}{N}\sum\limits_{j=1}^N x^j_t$ with $m_s = m$, ξ^i_t is the control by bank i, a is a mean-reversion rate, and σ, ρ, q, c, and ϵ are nonnegative constants.

The objective is to solve this stochastic game over an admissible control set $\mathcal{A}$, which includes adapted processes satisfying proper integrability condition. That is to solve

$$v(s,x) = \inf_{\xi_\cdot \in \mathcal{A}} E_{s,x,m}\left[\int_s^T (\frac{1}{2}\dot{\xi}_t^2 - q\dot{\xi}_t(m_t - x_t) + \frac{\epsilon}{2}(m_t - x_t)^2)dt + \frac{c}{2}(m_T - x_T)^2\right], \quad (3.15)$$

subject to $dx_t = a(m_t - x_t)dt + \dot{\xi}_t dt + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t), \quad x_s = x, m_s = m.$

Our model. Now consider the model with singular controls of bounded velocity. Assuming (realistically) that the rate of bank borrowing and lending is bounded, the MFG takes the following form

$$v(s,x) = \inf_{\dot{\xi}_\cdot \in \mathcal{U}_\theta} E_{s,x,m}\left[\int_s^T (r|\dot{\xi}_t| + \frac{\epsilon}{2}(m_t - x_t)^2)dt + \frac{c}{2}(m_T - x_T)^2\right], \quad (3.16)$$

subject to

$$\begin{aligned} dx_t &= a(m_t - x_t)dt + d\xi_t + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t), \\ &= \left[a(m_t - x_t) + \dot{\xi}_t\right] dt + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t), \quad x_s = x, m_s = m. \end{aligned} \quad (3.17)$$

Here $m_t = \int x\mu_t(dx)$ is a mean information process with μ_t a probability measure of x_t^i. $r, \epsilon, c, a, \sigma, \rho$ are nonnegative constants, and the admissible control set is given by

$$\mathcal{U}_\theta = \{\{\dot{\xi}_t\} \mid \{\xi_t\} \text{ is } \mathcal{F}_t\text{-progressively measurable, finite variation, } \xi_0 = 0, \dot{\xi}_t \in [-\theta, \theta]\}.$$

Remark 1. *It is worth noting that the choice of $q = 0$ here is mainly for exposition simplicity and does not change the general solution structure. Also, instead of the quadratic form, $|\dot{\xi}_t^i|$ is used. Technically, it could be replaced by any convex and symmetric function as far as explicit solution is concerned, as demonstrated in Karatzas [49] which generalizes the earlier work of [7] for singular control problems.*

This particular MFG appears different from the general problem setting presented in Eqn. (3.1), with the additional term of common noise. We will show, nevertheless, that appropriate conditioning argument coupled with the symmetric structure in the problem will reduce this MFG to to the case without common noise.

Solution for $\rho = 0$ (no common noise). Step 1. Fix m_t as a deterministic process with $m_s = m \in \mathbb{R}$. The problem now is a stochastic control problem. The HJB equation is

$$\partial_t v + \frac{1}{2}\sigma^2 \partial_{xx} v + \inf_{\dot{\xi} \in \mathcal{U}_\theta}\{(a(m-x) + \dot{\xi})\partial_x v + r|\dot{\xi}| + \frac{\epsilon}{2}(m-x)^2\} = 0,$$

with the terminal condition $v(T,x) = \frac{c}{2}(m_T - x)^2$. By Proposition 1, the optimal control is

$$\dot{\xi}_t(x|\{m_t\}) = \begin{cases} \theta & \text{if} \quad \partial_x v(t,x) \le -r, \\ 0 & \text{if} \quad -r < \partial_x v(t,x) < r, \\ -\theta & \text{if} \quad r \le \partial_x v(t,x). \end{cases}$$

By symmetry of the model, $v(t, m_t - h) = v(t, m_t + h)$ for any $t \in [0, T]$ and any $h > 0$. Hence, $\partial_x v(t, m_t - h) = -\partial_x v(t, m_t + h)$ for any $h > 0$. So, we can denote the optimal controls as

$$\dot{\xi}_t(x|\{m_t\}) = \begin{cases} \theta, & \text{if} \quad x \le m_t - h_t, \\ 0, & \text{if} \quad m_t - h_t < x < m_t + h_t, \\ -\theta, & \text{if} \quad m_t + h_t \le x, \end{cases} \tag{3.18}$$

for some deterministic $h_t > 0$.

Step 2. Solve the McKean–Vlasov equation: [3]

$$dx_t = \Big[a(m'_t - x_t) + \dot{\xi}_t(x_t|\{m'_t\})\Big]\, dt + \sigma dW_t, \quad x_s = x, m'_s = m,$$

where m'_t is an updated mean information process with $\dot{\xi}_t$ such that $m'_t = \int x\mu_t(dx)$, with μ_t a probability measure of optimal state process x_t. The Kolmogorov forward equation for μ_t is

$$\partial_t \mu_t = -\partial_x \Big((a(m'_t - x_t) + \dot{\xi}_t)\mu_t \Big) + \frac{1}{2}\sigma^2 \partial_{xx}\mu_t. \tag{3.19}$$

Then, by (3.19) and $m'_t = \int x\mu_t(dx)$,

$$dm'_t = \theta \Big(P(\dot{\xi}_t = -\theta) - P(\dot{\xi}_t = \theta) \Big)\, dt = \theta \left(P(x_t > m'_t + h_t) - P(x_t < m'_t - h_t) \right) dt = 0,$$

because x_t is symmetric to m'_t. Hence, $dm'_t = 0$ and $m'_t = m$ for $t \in [s, T]$.

From step 1 and 2, updated mean information process is always $m'_t = m$ for $t \in [s, T]$. So, let's solve the associated HJB for the value function with fixed $m_t = m$:

$$\partial_t v + \frac{\epsilon}{2}(m - x)^2 + a(m - x)\partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v + \theta \min\{0, r + \partial_x v, r - \partial_x v\} = 0, \tag{3.20}$$

with the terminal condition $v(T, x) = \frac{c}{2}(m - x)^2$.

One can solve for the value function explicitly. Indeed, since the value function $v(s, \cdot)$ is convex, define

$$f_1(s, m) = \sup\{x : \partial_x v(s, x) = -r\},$$

and

$$f_2(s, m) = \inf\{x : \partial_x v(s, x) = r\}.$$

Then, on $f_1(s) \le x \le f_2(s)$,

$$\partial_t v + \frac{\epsilon}{2}(m - x)^2 + a(m - x)\partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v = 0, \qquad v(T, x) = \frac{c}{2}(m - x)^2. \tag{3.21}$$

[3] In previous section 2, we use the stochastic differential equation for the second step (consistency part) instead of the McKean-Vlasov equation. With the existence and uniqueness of the fixed point solution to the MFG, both methods suggest same fixed point solution, so either method gives the solution to the MFG. So, we use the McKean-Vlasov method for this example.

The Laplace transform of v is $\tilde{v}(\lambda, x) = \int_{-\infty}^{T} e^{-\lambda t} v(t,x)dt$ for $\lambda < 0$, which satisfies

$$\begin{aligned}
\widetilde{\partial_t v}(\lambda, x, m) &= \int_{-\infty}^{T} e^{-\lambda t}\partial_t v(t,x)dt \\
&= e^{-\lambda t} v(t,x)|_{t=-\infty}^{T} + \lambda \int_{-\infty}^{T} e^{-\lambda t} v(t,x)dt \\
&= e^{-\lambda T}\frac{\epsilon}{2}(m-x)^2 + \lambda \int_{-\infty}^{T} e^{-\lambda t} v(t,x)dt \\
&= e^{-\lambda T}\frac{\epsilon}{2}(m-x)^2 + \lambda \tilde{v}(\lambda, x).
\end{aligned}$$

Thus,

$$\left(1 - \frac{1}{\lambda}\right)\frac{\epsilon}{2}e^{-\lambda T}(m-x)^2 + \lambda\tilde{v} + a(m-x)\partial_x\tilde{v} + \frac{1}{2}\sigma^2\partial_{xx}\tilde{v} = 0. \tag{3.22}$$

A particular solution to (3.22) is given by

$$\frac{\epsilon}{4a - 2\lambda}\left(1 - \frac{1}{\lambda}\right)e^{-\lambda T}(m-x)^2 - \frac{\sigma^2\epsilon}{\lambda(4a-2\lambda)}\left(1 - \frac{1}{\lambda}\right)e^{-\lambda T}.$$

The fundamental solutions to (3.22) are sums of two parabolic cylinder functions

$$\begin{aligned}
\tilde{\phi}_1(\lambda, m-x) &= e^{\frac{a(x-m)^2}{2\sigma^2}} D_{\frac{\lambda}{a}}\left(-\frac{x-m}{\sigma}\sqrt{2a}\right), \\
\tilde{\psi}_1(\lambda, m-x) &= e^{\frac{a(x-m)^2}{2\sigma^2}} D_{\frac{\lambda}{a}}\left(\frac{x-m}{\sigma}\sqrt{2a}\right) = \tilde{\phi}_1(\lambda, x-m),
\end{aligned}$$

where $D_\alpha(x) = \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}e^{-\frac{t^2}{2}-xt}dt$. Therefore, solution to (3.22) is

$$\begin{aligned}
\tilde{v}(\lambda, x, m) =& \frac{\epsilon}{4a-2\lambda}\left(1 - \frac{1}{\lambda}\right)e^{-\lambda T}(m-x)^2 - \frac{\sigma^2\epsilon}{\lambda(4a-2\lambda)}\left(1 - \frac{1}{\lambda}\right)e^{-\lambda T} \\
&+ c_1\tilde{\phi}_1(\lambda, m-x) + c_2\tilde{\phi}_1(\lambda, x-m) \\
=& \frac{\epsilon}{4a-2\lambda}\left(1 - \frac{1}{\lambda}\right)e^{-\lambda T}(m-x)^2 - \frac{\sigma^2\epsilon}{\lambda(4a-2\lambda)}\left(1 - \frac{1}{\lambda}\right)e^{-\lambda T} \\
&s + \frac{c_1}{\Gamma(-\frac{\lambda}{a})}\int_0^\infty z^{-\frac{\lambda}{a}-1}e^{-\frac{z^2}{2}+\frac{x-m}{\sigma}\sqrt{2a}z}dz + \frac{c_2}{\Gamma(-\frac{\lambda}{a})}\int_0^\infty z^{-\frac{\lambda}{a}-1}e^{-\frac{z^2}{2}-\frac{x-m}{\sigma}\sqrt{2a}z}dz.
\end{aligned}$$

for some constant c_1, c_2. That is,

$$\tilde{v}(\lambda, x) = \tilde{\eta}_1(\lambda)(m-x)^2 + \tilde{\eta}_3(\lambda) + c_1\tilde{\phi}_1(\lambda, m-x) + c_2\tilde{\phi}_1(\lambda, x-m).$$

Inverting this function yields the solution to the original PDE (3.21),

$$v(s,x) = \eta_1(s)(m-x)^2 + \eta_3(s) + c_1\phi_1(s, m-x) + c_2\phi_1(s, x-m).$$

where ϕ_1 is the inverse Laplace transform of $\tilde{\phi}_1$. Here $\eta_1(s), \eta_2(s), \eta_3(s)$ are solutions to ODEs

$$\begin{aligned} \partial_t \eta_1 - 2a\eta_1 + \frac{\epsilon}{2} &= 0, \qquad & \eta_1(T) &= \frac{c}{2}, \\ \partial_t \eta_3 + \sigma^2 \eta_1 &= 0, \qquad & \eta_3(T) &= 0, \end{aligned}$$

and can be expressed explicitly as

$$\eta_1(s) = \left(\frac{c}{2} - \frac{\epsilon}{4a}\right) e^{2a(s-T)} + \frac{\epsilon}{4a},$$

and

$$\eta_3(s) = -\sigma^2 \frac{1}{2a}\left(\frac{c}{2} - \frac{\epsilon}{4a}\right) e^{2a(s-T)} - \sigma^2 \frac{\epsilon}{4a}(s-T) + \sigma^2 \frac{1}{2a}\left(\frac{c}{2} - \frac{\epsilon}{4a}\right).$$

Similarly, on $x < f_1(s)$, the HJB equation is

$$\partial_t v + \frac{\epsilon}{2}(m-x)^2 + r\theta + (a(m-x)+\theta)\partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v = 0, \qquad v(T,x) = \frac{c}{2}(m-x)^2.$$

The solution is

$$v(s,x) = \zeta_1(s)(m-x)^2 + \zeta_2(s)(m-x) + \zeta_3(s) + c_3\phi_2(s, m-x) + c_4\psi_2(s, m-x),$$

where ϕ_2, ψ_2 are the inverse Laplace transforms of $\tilde{\phi}_2$ and $\tilde{\psi}_2$ respectively, with

$$\tilde{\phi}_2(\lambda, m-x) = e^{\frac{a(x-m-\frac{\theta}{a})^2}{2\sigma^2}} D_{\frac{\lambda}{a}}\left(-\frac{x-m-\frac{\theta}{a}}{\sigma}\sqrt{2a}\right),$$

and

$$\tilde{\psi}_2(\lambda, m-x) = e^{\frac{a(x-m-\frac{\theta}{a})^2}{2\sigma^2}} D_{\frac{\lambda}{a}}\left(\frac{x-m-\frac{\theta}{a}}{\sigma}\sqrt{2a}\right).$$

Here $\zeta_1(s), \zeta_2(s), \zeta_3(s)$ satisfy

$$\begin{aligned} \partial_t \zeta_1 - 2a\zeta_1 + \frac{\epsilon}{2} &= 0, \qquad & \zeta_1(T) &= \frac{c}{2}, \\ \partial_t \zeta_2 - a\zeta_2 - 2\theta\zeta_1 &= 0, \qquad & \zeta_2(T) &= 0, \\ \partial_t \zeta_3 + r\theta - \theta\zeta_2 + \sigma^2\zeta_1 &= 0, \qquad & \zeta_3(T) &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \zeta_1(s) &= \left(\frac{c}{2} - \frac{\epsilon}{4a}\right) e^{2a(s-T)} + \frac{\epsilon}{4a}, \\ \zeta_2(s) &= -\frac{\theta}{a}\left(c - \frac{\epsilon}{a}\right) e^{a(s-T)} + \frac{\theta}{a}\left(c - \frac{\epsilon}{2a}\right) e^{2a(s-T)} - \frac{\theta\epsilon}{2a^2}, \end{aligned}$$

and

$$\zeta_3(s) = \left(-r\theta - \frac{\theta^2\epsilon}{2a^2} - \frac{\epsilon\sigma^2}{4a}\right)(s-T) - \frac{\theta^2}{a^2}\left(c - \frac{\epsilon}{a}\right)(e^{a(s-T)} - 1)$$
$$+ \left(\frac{\theta^2}{2a^2}\left(c - \frac{\epsilon}{2a}\right) - \frac{\sigma^2}{2a}\left(\frac{c}{2} - \frac{\epsilon}{4a}\right)\right)(e^{2a(s-T)} - 1).$$

On $f_2(s) < x$, the HJB equation is

$$\partial_t v + \frac{\epsilon}{2}(m-x)^2 + r\theta + (a(m-x) - \theta)\partial_x v + \frac{1}{2}\sigma^2\partial_{xx}v = 0, \qquad v(T,x) = \frac{c}{2}(m-x)^2.$$

Therefore,

$$v(s,x) = \Lambda_1(s)(m-x)^2 + \Lambda_2(s)(m-x) + \Lambda_3(s) + c_5\psi_2(s, x-m) + c_6\phi_2(s, x-m).$$

where $\Lambda_1, \Lambda_2, \Lambda_3$ satisfy

$$\partial_t\Lambda_1 - 2a\Lambda_1 + \frac{\epsilon}{2} = 0, \qquad \Lambda_1(T) = \frac{c}{2},$$
$$\partial_t\Lambda_2 - a\Lambda_2 + 2\theta\Lambda_1 = 0, \qquad \Lambda_2(T) = 0,$$
$$\partial_t\Lambda_3 + r\theta + \theta\Lambda_2 + \sigma^2\Lambda_1 = 0, \qquad \Lambda_3(T) = 0.$$

That is,

$$\Lambda_1(s) = \left(\frac{c}{2} - \frac{\epsilon}{4a}\right)e^{2a(s-T)} + \frac{\epsilon}{4a},$$
$$\Lambda_2(s) = \frac{\theta}{a}\left(c - \frac{\epsilon}{a}\right)e^{a(s-T)} - \frac{\theta}{a}\left(c - \frac{\epsilon}{2a}\right)e^{2a(s-T)} + \frac{\theta\epsilon}{2a^2},$$

and

$$\Lambda_3(s) = \left(-r\theta - \frac{\theta^2\epsilon}{2a^2} - \frac{\epsilon\sigma^2}{4a}\right)(s-T) - \frac{\theta^2}{a^2}\left(c - \frac{\epsilon}{a}\right)(e^{a(s-T)} - 1)$$
$$+ \left(\frac{\theta^2}{2a^2}\left(c - \frac{\epsilon}{2a}\right) - \frac{\sigma^2}{2a}\left(\frac{c}{2} - \frac{\epsilon}{4a}\right)\right)(e^{2a(s-T)} - 1).$$

Note that $\eta_1(s) = \zeta_1(s) = \Lambda_1(s)$, $-\zeta_2(s) = \Lambda_2(s)$, and $\zeta_3(s) = \Lambda_3(s)$. Hence, the value function v is also symmetric to m, meaning $c_1 = c_2$, $c_3 = c_6$ and $c_4 = c_5$. Moreover, the regularity and convexity of $v(s, \cdot, m)$ implies that $\partial_x v$ is nondecreasing and that there are $x_1 < x_2$ satisfying

$$x_1 = \sup\{x : \partial_x v(s,x) = -r\},$$

and

$$x_2 = \inf\{x : \partial_x v(s,x) = r\}.$$

Now, the symmetry of $v(s, \cdot)$ with respect to m implies that $x_1 = m - h$ and $x_2 = m + h$ for some $h > 0$.

In fact, one can solve for c_1, c_3, c_4 and h from the $\mathcal{C}^2$-smoothness of $v^i(s,\cdot)$ at x_2, and get

$$\begin{aligned}
c_1 &= \frac{2\Lambda_1(s)h - r}{-\phi_1'(s,h) + \phi_1'(s,-h)},\\
c_3 &= \frac{1}{\phi_2(s,-h)}((2\Lambda_1(s)h - r)\frac{\phi_1(s,h)+\phi_1(s,-h)}{-\phi_1'(s,h)+\phi_1'(s,-h)} - \Lambda_2(s)h - \Lambda_3(s) + \eta_2(s))\\
&\quad - \frac{\phi_2(s,-h)}{\psi_2(s,-h)}\frac{1}{\psi_2'(s,-h)\phi_2(s,-h) - \psi_2(s,-h)\phi_2'(s,-h)}(\phi_2(s,-h)(2\Lambda_1(s)h + \Lambda_2(s) - r)\\
&\quad - \phi_2'(s,-h)((2\Lambda_1(s)h - r)\frac{\phi_1(s,h)+\phi_1(s,-h)}{-\phi_1'(s,h)+\phi_1'(s,-h)} - \Lambda_2(s)h - \Lambda_3(s) + \eta_2(s))),\\
c_4 &= \frac{1}{\psi_2'(s,-h)\phi_2(s,-h) - \psi_2(s,-h)\phi_2'(s,-h)}(\phi_2(s,-h)(2\Lambda_1(s)h + \Lambda_2(s) - r)\\
&\quad - \phi_2'(s,-h)((2\Lambda_1(s)h - r)\frac{\phi_1(s,h)+\phi_1(s,-h)}{-\phi_1'(s,h)+\phi_1'(s,-h)} - \Lambda_2(s)h - \Lambda_3(s) + \eta_2(s))),
\end{aligned}$$

and eventually

$$c_4\psi_2''(s,-h) + c_3\phi_2''(s,-h) = c_1\phi_1''(s,h) + c_1\phi_1''(s,-h).$$

By definitions of x_1 and x_2 and convexity of $v(s,\cdot,m)$, $x_1 = m-h$ and $x_2 = m+h$ with

$$h = \inf\{\kappa : c_4\psi_2''(s,-\kappa) + c_3\phi_2''(s,-\kappa) = c_1\phi_1''(s,\kappa) + c_1\phi_1''(s,-\kappa)\}. \tag{3.23}$$

Note that the degenerate case of $h=\infty$ means that there is no action region and $\xi_t^* = 0$ for all $x \in \mathbb{R}$.

In summary, the solution to the MFG (3.16) with $\rho = 0$ is given by Eqn. (3.18) for the optimal control, and

$$dm_t^* = 0 \quad \forall t \in [s,T], \qquad m_s^* = m, \tag{3.24}$$

$$\begin{aligned}
v(s,x) &= a_1(s)(m-x)^2 + a_2(s)(m-x) + a_3(s)\\
&\quad + a_4(s)\phi_1(s,m-x) + a_5(s)\phi_1(s,x-m) + a_6(s)\phi_2(s,m-x) + a_7(s)\psi_2(s,m-x).
\end{aligned} \tag{3.25}$$

for deterministic functions $a_j(s) = \begin{cases} \zeta_j(s), & \text{if } x < m-h,\\ \eta_j(s), & \text{if } m-h \le x \le m+h, \quad \text{for } j = 1,3,\\ \zeta_j(s), & \text{if } m+h < x,\end{cases}$

$$a_2(s) = \begin{cases} \zeta_2(s), & \text{if } x < m-h,\\ 0, & \text{if } m-h \le x \le m+h,\\ -\zeta_2(s), & \text{if } m+h < x,\end{cases}$$

$$a_j(s) = \begin{cases} 0, & \text{if } x < m-h,\\ c_1, & \text{if } m-h \le x \le m+h, \quad \text{for } j = 4,5,\\ 0, & \text{if } m+h < x,\end{cases}$$

$$a_6(s) = \begin{cases} c_3, & \text{if } x < m - h, \\ 0, & \text{if } m - h \leq x \leq m + h, \text{ and} \\ c_3, & \text{if } m + h < x, \end{cases}$$
$$a_7(s) = \begin{cases} c_4, & \text{if } x < m - h, \\ 0, & \text{if } m - h \leq x \leq m + h, \\ c_4, & \text{if } m + h < x. \end{cases}$$

Solution for $\rho \neq 0$ (with common noise). As the alternative method to deal with the presence of common noise [21, 23], we approach this problem with conditioning on W_t^0.

Step 1: stochastic control part Let's derive the HJB equation for (3.16) with conditioning on W_t^0 and fixed m_t.

$$\begin{aligned} v(s,x) &= \inf_{\dot{\xi}_\cdot \in \mathcal{U}_\theta} E_{s,x,m}\left[\int_s^T (r|\dot{\xi}_t| + \frac{\epsilon}{2}(m_t - x_t)^2)dt + \frac{c}{2}(m_T - x_T)^2\right] \\ &\leq \int_s^{s+\delta} \left(r|\dot{\xi}_t| + \frac{\epsilon}{2}(m_t - x_t)^2\right) dt + v(s+\delta, x_{s+\delta}) \end{aligned}$$

Hence, for any ξ_t,

$$\frac{v(s,x) - v(s+\delta, x_{s+\delta})}{\delta} \leq \frac{1}{\delta}\left(\int_s^{s+\delta} (r|\dot{\xi}_t| + \frac{\epsilon}{2}(m_t - x_t)^2)dt\right).$$

Let $\delta \to 0$, then $v(s,x) - v(s+dt, x_{s+dt}) \leq \left(r|\dot{\xi}_s| + \frac{\epsilon}{2}(m-x)^2\right) dt$. By the Itô's formula for x_t, we can derive

$$dv + \left[\frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}v + \frac{\epsilon}{2}(m-x)^2\right] dt + \inf_{\dot{\xi} \in \mathcal{U}_\theta} \left[(a(m-x) + \dot{\xi})\partial_x v + r|\dot{\xi}|\right] dt + \rho\sigma\partial_x v dW_t^0 = 0, \tag{3.26}$$

Similar to the special case without common noise, the value function with a fixed m_t, $v(t,x)$ is also symmetric with respect to m_t. That is, $v(t, m_t - h) = v(t, m_t + h)$ for any $h \geq 0$. From the HJB (3.26), the optimal control is given by Eqn. (3.18), which is independent of the common noise, and $P(x_t > m_t + h_t) = P(x_t < m_t - h_t)$ for some deterministic $h_t > 0$.

Step 2: Consistency part Let's solve the Mckean-Vlasov equation:

$$dx_t = \left[a(m'_t - x_t) + \dot{\xi}_t\right] dt + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t), \quad x_s = x, m'_s = m. \tag{3.27}$$

Kramers-Moyal Expansion of the master equation for the SDE (3.27) with conditioning on W_t^0 is

$$d\mu_t(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial x^m}[a^{(m)}(t,x)\mu_t(x)]dt, \tag{3.28}$$

where $a^{(m)}(t,x) = \frac{1}{dt}E[(dx_t)^m|W_t^0]|_{x_t=x}$ and μ_t is the probability measure of x_t.
Then, $a^{(1)}(t,x) = a(m'_t - x) + \dot{\xi}_t + \rho\sigma\frac{dW_t^0}{dt}$,
$a^{(2)}(t,x) = \sigma^2(1-\rho^2)dt$, and $a^{(m)}(t,x) = 0$ for any $m \geq 3$.

$$\therefore d\mu_t = \left[-\partial_x\left((a(m'_t - x) + \dot{\xi}_t)\mu_t\right) + \frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}\mu_t\right]dt - \rho\sigma\partial_x\mu_t dW_t^0,$$

and from $m'_t = \int x\mu_t(dx)$,

$$dm'_t = \theta(P(x_t > m'_t + h_t) - P(x_t < m'_t - h_t)dt + \rho\sigma dW_t^0 = \rho\sigma dW_t^0.$$

Hence, conditioning on W^0, m'_t satisfies $dm'_t = \rho\sigma dW_t^0$ with $m'_s = m$ for $t \in [s,T]$.

Step 3: Fixed point part As in the previous case, for any give fixed m_t, the updated mean information process which is the result from step 1 and 2 is always $dm'_t = \rho\sigma dW_t^0$ with $m'_s = m$ for $t \in [s,T]$. Therefore, with conditioning on W_t^0, there exists a unique fixed point solution and it is

$$dm_t^* = \rho\sigma dW_t^0 \quad \forall t \in [s,T], \qquad m_s^* = m. \tag{3.29}$$

Let's prove that m_t^* is a unique optimal solution to the MFG (3.16) subject to (3.17). Suppose there is a better or another solution m_t^{**} to the MFG. Since m_t^{**} is also solution to the MFG, if we iterate steps with initial fixed m_t^{**}, then the fixed point solution would be m_t^{**}. However, according to the previous steps, resulting fixed point solution m_t^* is independent to the initial fixed m_t and it is always $dm_t^* = \rho\sigma dW_t^0$. Hence, $m_t^* = m_t^{**}$ and it is a unique solution to the MFG.

Because of common noise, we rewrite $v(t,x_t) = w(t,x_t,m_t^*)$. Then,

$$dv = \partial_t v dt = \partial_t w dt + \partial_m w dm_t^* = dw + \partial_m w\rho\sigma dW_t^0.$$

$$\begin{aligned} dw + \left[\frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}w + \frac{\epsilon}{2}(m-x)^2\right]dt + \inf_{\dot{\xi}\in\mathcal{U}_\theta}\left[(a(m-x)+\dot{\xi})\partial_x w + r|\dot{\xi}|\right]dt \\ + \rho\sigma\partial_m w dW_t^0 + \rho\sigma\partial_x w dW_t^0 = 0, \end{aligned} \tag{3.30}$$

By the model structure, $w(t,x,m) = w(t,x+h,m+h)$ for any $h > 0$. So, $\partial_m w = -\partial_x w$ and the HJB equation (3.30) is same as the HJB equation without common noise. Therefore, value functions are same.

Chapter 4

MFG with singular controls over infinite time

4.1 Problem Formulations and Main Results

Background: Single player optimal partially reversible investment

Consider the optimal partially reversible investment model which is formulated as singular controls in [39]. There is a player who produces a single product, and the player controls its production capacity to maximize its profit on an infinite time horizon. The player can expand or contract its capacity, but it is partially reversible.

Mathematically, let $\{K_t\}_{0\leq t<\infty}$ be a càdlàg process representing the production capacity of the player. We assume that given initial position $k \in \mathbb{R}^+$ the dynamics of K_t is

$$dK_t = K_t(\delta dt + \gamma dW_t) + dL_t - dM_t, \quad K_{0-} = k,$$

for constants $\gamma > 0$ and δ. $\{L_t\}_{0\leq t<\infty}$ and $\{M_t\}_{0\leq t<\infty}$ are càdlàg control processes in an admissible set $\mathcal{U}$ which includes all nondecreasing $\sigma(K_t)$-adapted processes satisfying the integrability condition $E[\int_0^\infty e^{-rt}dL_t] < \infty$ or $E[\int_0^\infty e^{-rt}dM_t] < \infty$ and $L_0 = M_0 = 0$. There is a nonnegative discount rate r satisfying $r > \delta$. $\Pi_1 : \mathbb{R} \to \mathbb{R}$ is the production output function which is continuous, nondecreasing, and concave. The typical example of Π_1 function is the Cobb-Douglass production function: $\Pi_1(k) = ck^\alpha$ with some positive constants $c, \alpha \in (0, 1)$. We also assume that one unit of increasing capacity needs p units of investment cost and one unit of decreasing capacity generates $(1-\lambda)p$ units of profit. Because of $0 < \lambda < 1$, it is partially reversible. Then, the optimal partially reversible investment with singular control on an infinite time horizon in [39] can be formulated as

$$\begin{aligned} w(k) = \sup_{L_t, M_t \in \mathcal{U}} E\left[\int_0^\infty e^{-rt}[\Pi_1(K_t)dt - pdL_t + p(1-\lambda)dM_t]\right] \\ s.t. \quad dK_t = K_t(\delta dt + \gamma dW_t) + dL_t - dM_t, \qquad K_{0-} = k. \end{aligned} \tag{4.1}$$

The HJB equation associated with (4.1) is

$$\min\{rw - \mathcal{L}w - \Pi_1, \partial_k w - (1-\lambda)p, p - \partial_k w\} = 0,$$

and according to results in [39], this equation has a unique $\mathcal{C}^2$-solution, which is the value function of the model. For the case of $\Pi_1(k) = k^\alpha$, the value function $w(k)$ is

$$w(k) = \begin{cases} A_1 + pk & k \leq k_b \\ A_2 k^m + A_3 k^n + H k^\alpha & k_b < k < k_s \\ A_4 + p(1-\lambda)k & k_s \leq k \end{cases}$$

for some constants $k_b < k_s$ and $m, n, H, A_1, A_2, A_3, A_4$ which are defined in [39]. The optimally controlled process K_t^* is a reflected geometric Brownian motion with two boundaries $[k_b, k_s]$ and the optimal controls $\{L_t^*\}$ and $\{M_t^*\}$ are processes in such Skorohod type problem for K_t^*:

$$dK_t^* = K_t^*(\delta dt + \gamma dW_t) + dL_t^* - dM_t^*, \quad K_{0-}^* = k, \quad K_t^* \in [k_b, k_s] \quad a.s.$$
$$\int_0^\infty 1_{\{K_t^* > k_b\}} dL_t^* = 0, \quad \int_0^\infty 1_{\{K_t^* < k_s\}} dM_t^* = 0.$$

This model is a single player model, so by adding other players' behaviour and interaction among players one could generalize the control problem to a game.

Optimal partially reversible investment stationary MFG

Suppose that there are identical N players in same industry and they produce the same product. For $i = 1, 2, \ldots, N$, let $\{K_t^i\}_{0 \leq t < \infty}$ be the production capacity process for ith player which is a càdlàg process satisfying the dynamics for any $k^i \in \mathbb{R}$

$$dK_t^i = K_t^i(\delta dt + \gamma dW_t^i) + dL_t^i - dM_t^i \qquad K_{0-}^i = k^i,$$

where δ, γ are nonnegative constants. Let $\mu_t \in \mathcal{P}_2(\mathbb{R})$ be a probability measure of $\{K_t^i\}_{i=1,2,\cdots,N}$ at time t. Then, the optimal partially reversible investment for N players can be formulated as

$$\begin{aligned} \sup_{L_t^i, M_t^i \in \mathcal{U}} J_N^i(k^i, L_t^i, M_t^i) &= \sup_{L_t^i, M_t^i \in \mathcal{U}} E\left[\int_0^\infty e^{-rt}[\frac{1}{N}\sum_{j=1}^N \Pi_0(K_t^i, K_t^j)dt - pdL_t^i + p(1-\lambda)dM_t^i]\right] \\ &= \sup_{L_t^i, M_t^i \in \mathcal{U}} E\left[\int_0^\infty e^{-rt}[\Pi(K_t^i, \epsilon_t^N)dt - pdL_t^i + p(1-\lambda)dM_t^i]\right] \\ s.t. \quad dK_t^i &= K_t^i(\delta dt + \gamma dW_t^i) + dL_t^i - dM_t^i, \qquad K_{0-}^i = k^i, \end{aligned}$$

where ϵ_t^N is an empirical distribution of $\{K_t^i\}_{i=1,2,\cdots,N}$ at time t and can be consider as price information at time t. Each player's admissible control processes are $\{L_t^i\}_{0 \leq t < \infty}$ and $\{M_t^i\}_{0 \leq t < \infty}$ which are càdlàg nondecreasing processes with $L_0^i = M_0^i = 0$. Let's assume that our controls are closed loop feedback type. Let $\mathcal{U}$ be an admissible set including such admissible controls. Let $r > 0$ be a discount rate and $\lambda \in (0,1), p$ be nonnegative constants. Let $\Pi : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a revenue function for the ith player which depends on the production output of the ith player (K_t^i) and the price (ϵ_t^N).

As N goes to infinity, we can approximate the N player game using a MFG;

$$\sup_{L_t, M_t \in \mathcal{U}} J_\infty(k, L_t, M_t) = \sup_{L_t, M_t \in \mathcal{U}} E\left[\int_0^\infty e^{-rt}[\Pi(K_t, \mu_t)dt - pdL_t + p(1-\lambda)dM_t]\right]$$
$$s.t. \quad dK_t = K_t(\delta dt + \gamma dW_t) + dL_t - dM_t, \qquad K_{0-} = k.$$

where μ_t is a probability measure of K_t at time t and $\mathcal{U}$ is the admissible set and controls are closed loop feedback form adapted in $\sigma(K_{t-}, \mu_{t-})$.

μ_t is the price information at time t in this model. The model can be with short run price which is changing as time or with long run average price which is constant over time. Let's consider the model with stationary $\mu_t = \mu$ first which determines the long run average price in this section, and then consider the model with general nonstationary μ_t which determines short run prices in next section.

Then, the stationary MFG (SMFG) can be formulated as

$$\begin{aligned} w(k) = \sup_{L_t, M_t \in \mathcal{U}} J^s_\infty(k, L_t, M_t) &= \sup_{L_t, M_t \in \mathcal{U}} E\left[\int_0^\infty e^{-rt}[\Pi(K_t, \mu)dt - pdL_t + p(1-\lambda)dM_t]\right] \\ s.t. \quad dK_t =& K_t(\delta dt + \gamma dW_t) + dL_t - dM_t, \qquad K_{0-} = k, \end{aligned} \tag{4.2}$$

where μ is a limiting stationary distribution of $\{K_t\}$ if it exists, and the admissible set of controls $\mathcal{U}$ is

$$\begin{aligned} \mathcal{U} =& \{L_t, M_t | L_t, M_t \text{ are } \mathcal{F}^{K_{t-}}\text{-adapted, cádlág, and nondecreasing with} \\ & L_0 = M_0 = 0, \text{ and } E\int_0^\infty e^{-rt} dL_t < \infty, E\int_0^\infty e^{-rt} dM_t < \infty\}. \end{aligned}$$

Again controls are closed loop feedback form.

Assumption

(A1) $\Pi(k, \mu)$ is Lipschitz continuous, nondecreasing, bounded and concave over $k \in (0, \infty)$. It also satisfies two conditions: $\lim\limits_{k \downarrow 0} \frac{\Pi(k,\mu)}{k} = \infty$ and the Legendre-Fenchel transform of Π is finite: $\tilde{\Pi}(z) = \sup\limits_{k>0}[\Pi(k, \mu) - kz] < \infty$.

Let's define a solution to the SMFG (4.2).

Definition 6. *A solution of the SMFG* (4.2) *is defined as a pair of optimal control processes* $\{L^*_t\}, \{M^*_t\}$ *and a probability measures* $\mu^* \in \mathcal{P}_2(\mathbb{R})$ *if the optimal controlled process* $\{K^*_t\}$ *has a limiting stationary distribution* μ^* *and* $w(k) = J^s_\infty(k, L^*_t, M^*_t)$ *for all* $k \in \mathbb{R}$.

SMFG framework We can solve the SMFG (4.2) by following three steps.

- Step 1: (stochastic control part) Fix $\{\mu\}$ as deterministic first, then (4.2) becomes a stochastic control problem. Solve this stochastic control problem, and let $\{L_t\}, \{M_t\}$ be optimal controls and $\{K_t\}$ be the corresponding optimal controlled process.
  Define the mapping $\Gamma(\mu) = (\{L_t\}, \{M_t\})$.
- Step 2: (consistency part) Given the optimal controls $(\{L_t\}, \{M_t\})$ from step 1, solve stochastic differential equation:

$$dK_t = K_t(\delta dt + \gamma dW_t) + dL_t - dM_t \qquad K_{0-} = k. \tag{4.3}$$

  Update fixed flow of probability measures μ' as a limiting stationary probability distributions of $\{K_t\}$ if it exists.
  Define the mapping $\Gamma_0(\{L_t\}, \{M_t\}) = \mu'$.
- Step 3: (fixed point part) Under new fixed $\{\mu'\}$, repeat step 1 and 2 until convergence to a fixed point solution $\mu^*, \{L_t^*\}, \{M_t^*\}$.

Theorem 3. *Under (A1),*

1) for any fixed μ, there exists a unique optimal controls $\{L_t\}, \{M_t\}$ for the step 1 in the SMFG framework. The mapping Γ is well-defined;

2) for any optimal controls $\{L_t\}, \{M_t\}$ from the step 1, the optimal controlled process $\{K_t\}$ following the dynamics (4.3) has a limiting stationary distribution. The mapping Γ_0 is well-defined;

3) if $\Gamma \circ \Gamma_0$ is a contraction mapping, the SMFG (4.2) has a unique solution $(\mu^*, \{L_t^*\}, \{M_t^*\})$.

Proof. 1) Step 1: well-definedness of Γ. Fix μ as deterministic first. With deterministic μ, the SMFG (4.2) is equivalent to the stochastic control problem (4.1). As we have seen in previous section, there exists a unique value function and optimal controls. Hence, the mapping Γ is well-defined.
2) Step 2: well-definedness of Γ_0. Under given $\{L_t\}, \{M_t\}$, we use the limiting stationary distribution of K_t from (4.3). For any Markov chain, if it is irreducible, aperiodic, and positive recurrent, then there exists a unique stationary distribution and it is a limiting distribution. The optimal controlled process K_t is a geometric Brownian motion with two reflected constant boundaries by definition of L_t, M_t. Hence, K_t has a unique limiting stationary distribution. Consequently, the mapping Γ_0 is also well-defined.
3) Step 3: The space of probability measures $\mathcal{P}_2(\mathbb{R})$ with 2nd order Wasserstein metric is complete. By Banach fixed-point theorem, if $\Gamma \circ \Gamma_0$ is a contraction mapping, the SMFG framework has a unique fixed point $(\mu^*, \{L_t^*\}, \{M_t^*\})$. By definition of each mapping, the fixed point solution satisfies definition 6. □

Remark By the theorem, whether $\Gamma \circ \Gamma_0$ is contraction or not is critical for the solution to the MFG. Actually it depends on coefficients of the model so we will illustrate explicit numerical examples when $\Gamma \circ \Gamma_0$ is a contraction mapping in section 4.3.

Optimal partially reversible investment MFG

Let's generalize the model with a general drift function and with a mean information process $\{\mu_t\}$ which does not need to be stationary. The model can be formulated as

$$\begin{aligned}
&\sup_{L_t, M_t \in \mathcal{U}} E\left[\int_0^\infty e^{-rt}[\Pi(K_t,\mu_t)dt - pdL_t + p(1-\lambda)dM_t]\right] \\
&s.t. \quad dK_t = b(K_t,\mu_t)dt + \sigma dW_t + dL_t - dM_t \qquad K_{0-} = k, \mu_{0-} = \mu.
\end{aligned} \tag{4.4}$$

where $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is Lipschitz continuous, μ_t is the probability measure of K_t, and the admissible set $\mathcal{U}$. This game is a MFG with singular controls on an infinite time horizon. Because of difficulties when we approach MFG with singular controls using PDE method directly, we will approach this game through MFG with singular controls of bounded velocity. We will discuss the relation between singular controls and singular controls of bounded velocity in the later section.

MFG with singular controls of bounded velocity For any fixed $\theta > 0$, we assume that $\{L_t\}$ and $\{M_t\}$ are bounded velocity processes which means $dL_t = \dot{L}_t dt$ and $dM_t = \dot{M}_t dt$ with $0 \le \dot{L}_t \le \theta$ and $0 \le \dot{M}_t \le \theta$. Then, the value function for any fixed $\mu \in \mathcal{P}_2(\mathbb{R})$ is

$$\begin{aligned}
v(s,k) &= \sup_{L_t, M_t \in \mathcal{U}_\theta} J(s,k,L_t,M_t) \\
&= \sup_{L_t, M_t \in \mathcal{U}_\theta} E\left[\int_s^\infty e^{-r(t-s)}[\Pi(K_t,\mu_t)dt - p\dot{L}_t dt + p(1-\lambda)\dot{M}_t dt]\right] \\
s.t. \quad & dK_t = b(K_t,\mu_t)dt + \sigma dW_t + \dot{L}_t dt - \dot{M}_t dt \qquad K_{s-} = k, \mu_{s-} = \mu,
\end{aligned} \tag{4.5}$$

for any $s \in [0,\infty)$ and $k \in \mathbb{R}$. We impose assumption (A1) and additional assumptions as follows

Assumptions

(A2) For any fixed $\mu \in \mathcal{P}_2(\mathbb{R})$, $b(k,\mu)$ are measurable and linear function in k. So, there exists some constant $\delta_0 > 0$ such that $|b(k,\mu)| + |\sigma| \le \delta_0(1+k)$. Furthermore, we assume $r > \delta_0$.

Admissible set

$$\begin{aligned}
\mathcal{U}_\theta = &\{\xi_t | \xi_t \text{ is } \mathcal{F}^{(K_{t-},\mu_{t-})}\text{-adapted, cadlag, and nondecreasing with} \\
&0 \le \dot{\xi}_t \le \theta, \xi_0 = 0, \text{ and } E\int_0^\infty e^{-rt}\dot{\xi}_t dt < \infty\}
\end{aligned}$$

MFG solution Let's define the solution to the MFG (4.5).

Definition 7. *A solution of the MFG* (4.5) *is defined as a pair of optimal control processes* $\{L_t^*\}, \{M_t^*\}$ *and a flow of probability measures* $\{\mu_t^*\} \in \mathcal{M}_{[0,\infty)}$ *if they satisfy* $v(s,k) = J(s,k,L_t^*,M_t^*)$ *for all* $(s,k) \in [0,\infty) \times \mathbb{R}$ *and* μ_t^* *is a probability measure of the optimal controlled process* $\{K_t^*\}$ *for all* $t \in [0,\infty)$.

MFG framework We can solve the MFG (4.5) by following three steps.

- Step 1: (stochastic control part) Fix $\{\mu_t\}$ as deterministic first, then (4.5) becomes a stochastic control problem. Solve this stochastic control problem, and let $\{L_t^*\}, \{M_t^*\}$ be optimal controls and $\{K_t^*\}$ be the corresponding optimal controlled process.
- Step 2: (consistency part) Solve stochastic differential equation:

  $$dK_t = b(K_t,\mu_t)dt + \sigma dW_t + dL_t - dM_t \qquad K_{0-} = k, \mu_{0-} = \mu.$$

  Let P_{K_t} be the weak solution, then update fixed flow of probability measures $\{\mu_t'\}$ as probability distributions of $\{K_t\}$: $\mu_t'(k) = P_{K_t}(k)$.
- Step 3: (fixed point part) Under new fixed $\{\mu_t'\}$, repeat step 1 and 2 until convergence to a fixed point solution $\{\mu_t^*\}, \{L_t^*\}, \{M_t^*\}$.

Before approaching the MFG (4.5) on an infinite time horizon, let's define value functions for the MFG (4.5) on a finite time horizon as in [37]. For any finite time $T > 0$, denote

$$\begin{aligned} \bar{v}(s,T,k) = \sup_{L_t,M_t \in \mathcal{U}_\theta} & E\left[\int_s^T e^{-r(t-s)}[\Pi(K_t,\mu_t)dt - p\dot{L}_t dt + p(1-\lambda)\dot{M}_t dt]\right] \\ s.t. \quad & dK_t = b(K_t,\mu_t)dt + \sigma dW_t + \dot{L}_t dt - \dot{M}_t dt \qquad K_{s_-} = k, \mu_{s_-} = \mu \end{aligned} \tag{4.6}$$

for any $(s,k) \in [0,\infty) \times \mathbb{R}$. Then, the HJB equation associate with the MFG (4.6) with fixed $\{\mu_t\}$ is

$$\begin{aligned} & -\partial_t \bar{v} + r\bar{v} - \Pi - \frac{1}{2}\sigma^2 \partial_{kk}\bar{v} - b\partial_k \bar{v} + \theta \min\{0, \partial_k \bar{v} - p(1-\lambda), p - \partial_k \bar{v}\} = 0 \\ & \text{with terminal conditions } \bar{v}(T,T,k) = 0 \quad \forall k \in \mathbb{R} \end{aligned} \tag{4.7}$$

According to the theorem 1 in [37], (4.7) has a unique $\mathcal{C}^{1,2}$ solution, and this solution is the value function of (4.6) with fixed $\{\mu_t\}$.

Definition 8 (Supersolution and subsolution)**.** *The* v *is a continuous viscosity solution to (4.7) on* $[0,T] \times (0,\infty)$ *if* v *satisfies*
1) Viscosity supersolution: for any $(s_0,k_0) \in [0,T) \times \mathbb{R}$ *and for any function* $\psi \in \mathcal{C}^{1,2}$ *such that* (s_0,k_0) *is a local minimum of* $v - \psi$ *with* $v(s_0,T,k_0) = \psi(s_0,T,k_0)$*,*

$$-\partial_t \psi + r\psi - \Pi - \frac{1}{2}\sigma^2 \partial_{kk}\psi - b\partial_k \psi + \theta \min\{0, \partial_k \psi - p(1-\lambda), p - \partial_k \psi\} \geq 0,$$

and for any $k_0 \in \mathbb{R}$, $\psi(T,T,k_0) \geq 0$;
2) Viscosity subsolution: for any $(s_0,k_0) \in [0,T) \times \mathbb{R}$ and for any function $\psi \in \mathcal{C}^{1,2}$ such that (s_0,k_0) is a local maximum of $\upsilon - \psi$ with $\upsilon(s_0,T,k_0) = \psi(s_0,T,k_0)$,

$$-\partial_t\psi + r\psi - \Pi - \frac{1}{2}\sigma^2\partial_{kk}\psi - b\partial_k\psi + \theta\min\{0, \partial_k\psi - p(1-\lambda), p - \partial_k\psi\} \leq 0$$

and for any $k_0 \in \mathbb{R}$, $\psi(T,T,k_0) \leq 0$.

Let's look at main results of the model. Proofs of propositions and theorems are in the next section.

Proposition 5. *Under fixed $\{\mu_t\}$,*

$$v(s,k) = \lim_{T\to\infty} \bar{v}(s,T,k) \quad \forall (s,k) \in [0,\infty) \times \mathbb{R},$$

and for any fixed $s \in [0,\infty)$ the value function $v(s,k)$ is concave and differentiable in k. Then, the MFG (4.5) with fixed $\{\mu_t\}$ has a unique value function v and unique optimal control processes $\{L_t^*\}, \{M_t^*\}$.

The optimal control process will have two threshold functions $k_b(\tau), k_s(\tau) : [0,\infty) \to \mathbb{R}$ such that $k_b(\tau) = \sup\{k : \partial_k v(\tau,k) = p\}$ $k_s(\tau) = \inf\{k : \partial_k v(\tau,k) = p(1-\lambda)\}$ and the optimal control processes are

$$L_t = \int_0^t \theta 1_{\{K_\tau < k_b(\tau)\}} d\tau \qquad M_t = \int_0^t \theta 1_{\{K_\tau > k_s(\tau)\}} d\tau.$$

So, we can define a mapping from the fixed flow of probability measures to optimal control processes under fixed μ_t as $\Gamma_1(\{\mu_t\}) = (\{L_t\}, \{M_t\})$.

Proposition 6. *Under fixed $\{\mu_t\}, \{L_t\}, \{M_t\}$, dynamics of K_t*

$$dK_t = b(K_t,\mu_t)dt + \sigma dW_t + \dot{L}_t dt - \dot{M}_t dt, \quad t \geq 0, \qquad K_{0-} = k \tag{4.8}$$

has a unique weak solution $P_{K_t^*}$. Furthermore, the corresponding Kolmogorov forward equation is

$$\partial_t P_{K_t}(k) = -\partial_k[(b(k,\mu_t) + \dot{L}_t - \dot{M}_t)P_{K_t}(k)] + \frac{1}{2}\partial_{kk}[\sigma^2 P_{K_t}(k)], \quad \forall k \in \mathbb{R}. \tag{4.9}$$

Update μ_t' using P_{K_t}. Then, define a mapping $\Gamma_2(\{L_t\}, \{M_t\}) = \{\mu_t'\}$.

Theorem 4. *If $\Gamma_1 \circ \Gamma_2$ is a contraction mapping, then the MFG (4.5) has a unique solution $(\{\mu_t^*\}, \{L_t^*\}, \{M_t^*\})$.*

Remark Whether $\Gamma_1 \circ \Gamma_2$ is a contraction mapping or not depends on the coefficients of functions in the model. We will illustrate explicit numerical examples when $\Gamma_1 \circ \Gamma_2$ is a contraction mapping in section 4.3.

Relationship between MFGs and N-player games One of the goals of this paper is showing that optimal control processes $\{L_t^*\}, \{M_t^*\}$ of the MFG with singular controls of bounded velocity is a ϵ-Nash equilibrium of corresponding N-player game. The corresponding N-player game of the MFG (4.5) is

$$\begin{aligned} &\sup_{L_t^i, M_t^i \in \mathcal{U}} E\left[\int_s^\infty e^{-r(t-s)}[\frac{1}{N}\sum_{j=1,\cdots,N}\Pi_0(K_t^i, K_t^j)dt - p\dot{L}_t^i dt + p(1-\lambda)\dot{M}_t^i dt]\right] \\ &s.t. \quad dK_t^i = \frac{1}{N}\sum_{j=1,\cdots,N} b_0(K_t^i, K_t^j)dt + \sigma dW_t^i + \dot{L}_t^i dt - \dot{M}_t^i dt, \qquad K_{s-}^i = k^i. \end{aligned} \tag{4.10}$$

Under some technical condition, we can approximate the objective function and drift function by the strong law of large numbers:
$\frac{1}{N}\sum_{j=1,\ldots,N} \Pi_0(K_t^i, K_t^j) \to \int \Pi_0(K_t^i, y)\mu_t(dy) = \Pi(K_t^i, \mu_t)$ and
$\frac{1}{N}\sum_{j=1,\ldots,N} b_0(K_t^i, K_t^j) \to \int b_0(K_t^i, y)\mu_t(dy) = b(K_t^i, \mu_t)$ as N goes to ∞ where $\{\mu_t\}$ is a flow of probability measures of $\{K_t^j\}_{j=1,\cdots,N}$. So, we can consider that the MFG (4.5) is a approximated game of the N-player game (4.10), and they have following relation.

Theorem 5. *The optimal control processes of the solution to the MFG* (4.5) *is an ϵ-Nash equilibrium of the N-player game* (4.10) *with $\epsilon = O(\frac{1}{\sqrt{N}})$.*

Remark If the game has a well-known form of stochastic controls problems such as a linear-quadratic form, we could find an exact Nash equilibrium for some cases. However, in general N-player games, finding an exact Nash equilibrium is intractable. Therefore, studying MFG for ϵ-Nash equilibrium for N-player game is useful, and it is an approximated Nash equilibrium as N goes to infinity. We prove the general results under our formulation, and we will illustrate examples of solutions of MFGs.

4.2 Proofs of the MFG with singular controls of bounded velocity on an infinite time horizon

Existence and uniqueness of solutions to the MFG

Fix $\{\mu_t\}$ as deterministic. Then, the MFG (4.5) and (4.6) are stochastic problems with singular controls of bounded velocity.

$$\begin{aligned} v(s,k) &= \sup_{L_t, M_t \in \mathcal{U}_\theta} J(s,k,L_t,M_t) \\ &= \sup_{L_t, M_t \in \mathcal{U}_\theta} E\left[\int_s^\infty e^{-r(t-s)}[\Pi(K_t,\mu_t)dt - p\dot{L}_t dt + p(1-\lambda)\dot{M}_t dt]\right] \\ s.t. \quad dK_t &= b(K_t,\mu_t)dt + \sigma dW_t + \dot{L}_t dt - \dot{M}_t dt \qquad K_{s-} = k, \mu_{s-} = \mu. \end{aligned}$$

$$\bar{v}(s,T,k) = \sup_{L_t,M_t\in\mathcal{U}_\theta} E\left[\int_s^T e^{-r(t-s)}[\Pi(K_t,\mu_t)dt - p\dot{L}_t dt + p(1-\lambda)\dot{M}_t dt]\right]$$
$$s.t. \quad dK_t = b(K_t,\mu_t)dt + \sigma dW_t + \dot{L}_t dt - \dot{M}_t dt \qquad K_{s_-} = k, \mu_{s_-} = \mu.$$

Proof on proposition 5

Proof.

Lemma 5. *Let $\varphi(s,k)$ be a nonnegative $\mathcal{C}^{1,2}$ supersolution to (4.7) with fixed $T > 0$, then $\bar{v}(s,T,k) \le \varphi(s,k)$ for any $(s,k) \in [0,\infty) \times \mathbb{R}$.*

Proof. Let $\varphi \in \mathcal{C}^{1,2}([0,\infty)\times\mathbb{R})$ and $\{L_t\}, \{M_t\}$ be control processes in the admissible set $\mathcal{U}_\theta$. Let $\{K_t\}$ be a corresponding controlled process under $\{L_t\}, \{M_t\}$. Define $\tau_n = \inf\{t \ge 0; K_t \ge n\} \wedge (n \wedge T), \quad n \in \mathbb{N}$
Then, for $[s,\tau_n]$, by the Itô's formula,

$$\begin{aligned} E[e^{-r\tau_n}\varphi(\tau_n, K_{\tau_n})] =& e^{-rs}\varphi(s,k) + E\left[\int_s^{\tau_n} e^{-rt}(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(t,K_t)dt\right] \\ &+ E\left[\int_s^{\tau_n} e^{-rt}\partial_k\varphi(t,K_t)(\dot{L}_t dt - \dot{M}_t dt)\right] \end{aligned}$$

Let's prove $\forall t,k \in [s,\infty] \times \mathbb{R}$,

$$(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(t,k) + \partial_k\varphi(t,k)\dot{L}_t - \partial_k\varphi(t,k)\dot{M}_t \le -\Pi(k,\mu) + p\dot{L}_t - (1-\lambda)p\dot{M}_t.$$

1) If $(1-\lambda)p \le \partial_k\varphi(t,k) \le p$, the inequality is true since $\dot{L}_t, \dot{M}_t \ge 0$.
2) If $p \le \partial_k\varphi(t,k)$, since φ is the supersolution, $r\varphi - \partial_t\varphi - \Pi - \mathcal{L}\varphi + \theta(p - \partial_k\varphi) \ge 0$. So,

$$\begin{aligned} &(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(t,k) + \partial_k\varphi(t,k)\dot{L}_t - \partial_k\varphi(t,k)\dot{M}_t \\ &\le -\Pi(k,\mu) + \theta(p - \partial_k\varphi(t,k)) + \partial_k\varphi(t,k)\dot{L}_t - \partial_k\varphi(t,k)\dot{M}_t \\ &\le -\Pi(k,\mu)dt + p\dot{L}_t - (1-\lambda)p\dot{M}_t \end{aligned}$$

3) If $\partial_k\varphi(t,k) \le (1-\lambda)p$, similarly the inequality is ture.
Therefore,

$$E\left[\int_s^{\tau_n} e^{-rt}\{\Pi(K_t,\mu_t)dt - pdL_t + (1-\lambda)pdM_t\}\right] + E[e^{-r\tau_n}\varphi(K_{\tau_n})] \le e^{-rs}\varphi(s,k)$$

Since φ is nonnegative,

$$E\left[\int_s^{\tau_n} e^{-rt}\{\Pi(K_t,\mu_t)dt + (1-\lambda)pdM_t\}\right] - E\left[\int_s^T e^{-rt}pdL_t\right] \le e^{-rs}\varphi(s,k)$$

By Fatou's lemma with taking n goes to ∞,

$$E\left[\int_s^T e^{-r(t-s)}\{\Pi(K_t,\mu_t)dt - pdL_t + (1-\lambda)pdM_t\}\right] \le \varphi(s,k)$$

for any admissible L_t and M_t. Hence, $\bar{v}(s,T,k) \le \varphi(s,k) \qquad \forall s,k > 0.$ □

From [37], the HJB equation (4.7) with fixed T has a unique solution, and the solution is the value function $\bar{v}(s,T,k)$. Let's prove that $\bar{v}(s,T,k)$ pointwisely converges to some function as T goes to infinity. Denote $\bar{v}(s,k) = \limsup_{T\to\infty} \bar{v}(s,T,k)$.

Lemma 6. *$\bar{v}(s,T,k)$ converges to $\bar{v}(s,k)$ as T goes to ∞ for any $(s,k) \in [0,\infty)\times\mathbb{R}$.*

Proof. Let $\{T_n\}_{n\in N}$ be any monotonic increasing positive sequence satisfying $\lim_{n\to\infty} T_n = \infty$. By definition, $\bar{v}(s,T_i,k)$ are monotonic increasing as T_i increases.
Let's prove that $\bar{v}(s,T_n,k)$ converges to $\bar{v}(s,k)$ as $n\to\infty$. To prove this, we need to prove enough that for any $\epsilon > 0$, there exists n_0 such that for any $m > n \geq n_0$, $\bar{v}(s,T_m,k) - \bar{v}(s,T_n,k) < \epsilon$.
For any $q \in ((1-\lambda)p, p)$, let $q_0 = \frac{\tilde{\Pi}((r-\delta_0)q)}{r}$ where $\tilde{\Pi}$ is the Legendre-Fenchel transform, then $\varphi(k) = qk + q_0$ is a supersolution of (4.7) with T_m. By lemma 5, $\bar{v}(\cdot,T_m,k) \leq qk+q_0$. Then,

$$\begin{aligned}\bar{v}(s,T_m,k) &\leq \bar{v}(s,T_n,k) + e^{-r(T_n-s)}E[\bar{v}(T_n,T_m,K_{T_n})]\\ &\leq \bar{v}(s,T_n,k) + e^{-r(T_n-s)}E[qK_{T_n}+q_0]\end{aligned}$$

By assumption 2, for any fixed $\mu \in \mathcal{P}_2(\mathbb{R})$, $b(k,\mu) \leq \delta_0(1+k)$. Then, by the Itô's formula with $f(t,k) = ke^{-\delta_0 t}$, $E[K_{T_n}] \leq k_0\frac{\delta_0+\theta}{\delta_0}e^{\delta_0 T_n}(1-e^{-\delta_0 T_n})$.

$$\bar{v}(s,T_m,k) - \bar{v}(s,T_n,k) \leq qk_0\frac{\delta_0+\theta}{\delta_0}e^{-(r-\delta_0)(T_n-s)} + q_0e^{-r(T_n-s)} < \epsilon$$
$$\text{for large enough } T_n \text{ since } r > \delta_0.$$

Therefore, $\bar{v}(s,T_n,k)$ pointwisely converges to $\bar{v}(s,k)$ as n goes to ∞. □

Now, let's prove that the limit of solutions for finite interval, $\bar{v}(s,k)$, is the value function $v(s,k)$ of (4.5).

Lemma 7. *(Verification theorem) $\bar{v}(s,k) = v(s,k)$ for any $(s,k)\in[0,\infty)\times\mathbb{R}$*

Proof. Fix $(s,k)\in[0,\infty)\times\mathbb{R}$

1) $v(s,k) \leq \bar{v}(s,k)$
Fix $T>0$, $\bar{v}(s,T,k)$ is a classical $\mathcal{C}^{1,2}$ solution to (4.7) with T, so it is a supersolution. So,

$$-\partial_t\bar{v} + r\bar{v} - \Pi(k,\mu) - \frac{1}{2}\sigma^2\partial_{kk}\bar{v} - b\partial_k\bar{v} + \theta\min\{0, \partial_k\bar{v} - p(1-\lambda), p - \partial_k\bar{v}\} \geq 0$$

For any $L_t, M_t \in \mathcal{U}_\theta$, let $\{K_t\}_{t\in[s,\infty)}$ be the state process with control processes L_t, M_t. By the Itô's formula,

$$
\begin{aligned}
&E[e^{-r(T-s)}\bar{v}(T,T,K_T)] \\
&\quad =\bar{v}(s,T,K_s) + E[\int_s^T e^{-r(t-s)}(-r\bar{v} + \partial_t\bar{v} + \mathcal{L}\bar{v})(t,T,K_t)dt] \\
&\qquad + E[\int_s^T e^{-r(t-s)}\sigma\partial_k\bar{v}(t,T,K_t)dW_t] + E[\int_s^T e^{-r(t-s)}\partial_k\bar{v}(t,T,K_t)(\dot{L}_t - \dot{M}_t)dt] \\
&\quad =\bar{v}(s,T,K_s) + E[\int_s^T e^{-r(t-s)}((-r\bar{v} + \partial_t\bar{v} + \mathcal{L}\bar{v})(t,K_t) + \partial_k\bar{v}(t,T,K_t)(\dot{L}_t - \dot{M}_t))dt] \\
&\quad \le\bar{v}(s,T,K_s) + E[\int_s^T e^{-r(t-s)}(-\Pi(K_t,\mu) + \theta\min\{0, \partial_k\bar{v} - p(1-\lambda), p - \partial_k\bar{v}\} \\
&\qquad + (\dot{L}_t - \dot{M}_t)\partial_k\bar{v}(t,T,K_t))dt] \\
&\quad \le\bar{v}(s,T,K_s) + E[\int_s^T e^{-r(t-s)}(-\Pi(K_t,\mu) + p\dot{L}_t - p(1-\lambda)\dot{M}_t)dt
\end{aligned}
$$

Since $\bar{v}(T,T,k) = 0$ for any $T > 0, k \in \mathbb{R}$,

$$
E[\int_s^T e^{-r(t-s)}(\Pi(K_t,\mu) - p\dot{L}_t + p(1-\lambda)\dot{M}_t)dt \le\bar{v}(s,T,K_s)
$$

Hence, as $T \to \infty$,

$$
E[\int_s^\infty e^{-r(t-s)}(\Pi(K_t,\mu) - p\dot{L}_t + p(1-\lambda)\dot{M}_t)dt \le\bar{v}(s,K_s)
$$

This is true for any $L_t, M_t \in \mathcal{U}_\theta$. So,

$$
v(s,k) = \sup_{L_t,M_t\in\mathcal{U}_\theta} E[\int_s^\infty e^{-r(t-s)}(\Pi(K_t,\mu) - p\dot{L}_t + p(1-\lambda)\dot{M}_t)dt \le \bar{v}(s,k)
$$

Therefore, $v(s,k) \le \bar{v}(s,k)$ for any $(s,k) \in [0,\infty) \times \mathbb{R}$.

2) $v(s,k) \ge \bar{v}(s,k)$
Fix $T > s$,

Let $\dot{L}_t^{*T} - \dot{M}_t^{*T} = \begin{cases} \theta & \text{if} \quad p \le \partial_k\bar{v}(t,T,K_t) \\ 0 & \text{if} \quad p(1-\lambda) < \partial_k\bar{v}(t,T,K_t) < p \\ -\theta & \text{if} \quad \partial_k\bar{v}(t,T,K_t) \le p(1-\lambda) \end{cases}$

Then, the HJB equation (4.7) is

$$
-\partial_t\bar{v} + r\bar{v} - \Pi(k,\mu) - \frac{1}{2}\sigma^2\partial_{kk}\bar{v} - b\partial_k\bar{v} - \Big((\dot{L}_t^{*T} - \dot{M}_t^{*T})\partial_k\bar{v} - p\dot{L}_t^{*T} + p(1-\lambda)\dot{M}_t^{*T}\Big) = 0
$$

Similarly with 1),

$$
\begin{aligned}
&E[e^{-r(T-s)}\bar{v}(T,T,K_T)] \\
&\quad =\bar{v}(s,T,k)+E[\int_s^T e^{-r(t-s)}((-r\bar{v}+\partial_t\bar{v}+\mathcal{L}\bar{v})(t,K_t)+\partial_k\bar{v}(t,T,K_t)(\dot{L}_t^{*T}-\dot{M}_t^{*T}))dt] \\
&\quad =\bar{v}(s,T,k)+E[\int_s^T e^{-r(t-s)}\left(-\Pi(K_t,\mu_t)+p\dot{L}_t^{*T}-p(1-\lambda)\dot{M}_t^{*T}\right)dt
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\bar{v}(s,T,k) &= E[e^{-r(T-s)}\bar{v}(T,T,K_T)] \\
&\quad + E[\int_s^T e^{-r(t-s)}\left(\Pi(K_t,\mu_t)-p\dot{L}_t^{*T}+p(1-\lambda)\dot{M}_t^{*T}\right)dt \\
&= E[\int_s^T e^{-r(t-s)}\left(\Pi(K_t,\mu_t)-p\dot{L}_t^{*T}+p(1-\lambda)\dot{M}_t^{*T}\right)dt \\
&\le v(s,k)
\end{aligned}
$$

Therefore, as $T\to\infty$, $\bar{v}(s,k)\le v(s,k)$.
By 1) and 2), $\bar{v}(s,k)=v(s,k)$ □

We proved that the value function $v(s,k)$ is the limit of $v(s,T,k)$. Now, let's prove that there exists a unique optimal control process L_t^*, M_t^*. As in finite time horizon models, if $v(s,k)$ is differentiable in k, we can define optimal controls as
$\dot{L}_t^*-\dot{M}_t^* = \begin{cases} \theta & \text{if} \quad p\le\partial_k v \\ 0 & \text{if} \quad p(1-\lambda)<\partial_k v<p \\ -\theta & \text{if} \quad \partial_k v\le p(1-\lambda)\end{cases}$. Let's prove that $v(s,k)$ is differentiable in k.

Lemma 8. *For any fixed $s\in[0,\infty)$, the value function $v(s,k)$ is concave and continuous on $k\in\mathbb{R}$.*

Proof. Because Π is nondecreasing, $v(s,k)$ is also nondecreasing. By using concavity of Π over k and linearity of dynamics with L_t and M_y, the value function v is concave. Since $v(s,k)$ is finite and concave on $(0,\infty)$, it is continuous. □

We will use the viscosity solution method. The HJB equation associated with the MFG (4.5) with fixed $\{\mu_t\}$ is for any $(t,k)\in[0,\infty)\times\mathbb{R}$ and for any $\mu\in\mathcal{P}_2(\mathbb{R})$,

$$
\begin{aligned}
-\partial_t v(t,k)+rv(t,k)-\Pi(k,\mu)-\frac{1}{2}\sigma^2\partial_{kk}v(t,k)-b(k,\mu)\partial_k v(t,k)& \\
+\theta\min\{0,\partial_k v(t,k)-p(1-\lambda),p-\partial_k v(t,k)\}=0.&
\end{aligned}
\tag{4.11}
$$

Define the viscosity solution to (4.11) as

Definition 9. *(Viscosity solution) The v is a continuous viscosity solution to (4.11) on $[0,\infty)\times(0,\infty)$ if v satisfies*

1) Viscosity supersolution: for any $(s_0, k_0) \in [0, \infty) \times \mathbb{R}$ and for any function $\psi \in \mathcal{C}^{1,2}$ such that (s_0, k_0) is a local minimum of $\upsilon - \psi$ with $\upsilon(s_0, k_0) = \psi(s_0, k_0)$,

$$-\partial_t\psi + r\psi - \Pi - \frac{1}{2}\sigma^2\partial_{kk}\psi - b\partial_k\psi + \theta\min\{0, \partial_k\psi - p(1-\lambda), p - \partial_k\psi\} \geq 0$$

2) Viscosity subsolution: for any $(s_0, k_0) \in [0, \infty) \times \mathbb{R}$ and for any function $\psi \in \mathcal{C}^{1,2}$ such that (s_0, k_0) is a local maximum of $\upsilon - \psi$ with $\upsilon(s_0, k_0) = \psi(s_0, k_0)$,

$$-\partial_t\psi + r\psi - \Pi - \frac{1}{2}\sigma^2\partial_{kk}\psi - b\partial_k\psi + \theta\min\{0, \partial_k\psi - p(1-\lambda), p - \partial_k\psi\} \leq 0$$

Lemma 9. *The value function $v(s, k)$ is a continuous viscosity solution to* (4.11)*.*

Proof. 1) v is a viscosity supersolution.
Let k_0 be any point and $\varphi \in \mathcal{C}^{1,2}$ satisfying $v(s_0, k_0) = \varphi(s_0, k_0)$ and $v(s, k) \geq \varphi(s, k)$ on $(s, k) \in B_\epsilon(s_0, k_0)$ for small enough ϵ. Define $\tau_\epsilon = \inf\{t \geq 0; (s_0, K_{s_0+t}) \notin B_\epsilon(s_0, k_0)$ for $K_{s_0} = k_0\}$. Let $L_t(\theta) = \theta t$ for $t \in (0, h)$ where $0 \leq \theta_1 \leq \theta$ for any $h > 0$, and $M_t^i = 0$.

$$\begin{aligned}
\varphi(s_0, k_0) &= v(s_0, k_0) \\
&\geq E\left[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}(\Pi(K_t, \mu_t)dt - p\theta dt) + e^{-r(s_0+\tau_\epsilon\wedge h)}v(s_0 + \tau_\epsilon \wedge h, K_{s_0+\tau_\epsilon\wedge h})\right] \\
&\geq E\left[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}(\Pi(K_t, \mu_t)dt - p\theta dt) + e^{-r(s_0+\tau_\epsilon\wedge h)}\varphi(s_0 + \tau_\epsilon \wedge h, K_{s_0+\tau_\epsilon\wedge h})\right]
\end{aligned}$$

By the Itô's formula,

$$\begin{aligned}
&E[e^{-r(\tau_\epsilon\wedge h)}\varphi(s_0 + \tau_\epsilon \wedge h, K_{s_0+\tau_\epsilon\wedge h})] \\
&\qquad =\varphi(s_0, k_0) + E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(t, K_t)dt] \\
&\qquad\quad + E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}\partial_k\varphi(t, K_t)\theta dt] \\
&\qquad \geq E\left[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}(\Pi(K_t, \mu_t)dt - p\theta dt) + e^{-r(\tau_\epsilon\wedge h)}\varphi(s_0 + \tau_\epsilon \wedge h, K_{s_0+\tau_\epsilon\wedge h})\right] \\
&\qquad\quad + E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(t, K_t)dt] \\
&\qquad\quad + E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}\partial_k\varphi(t, K_t)\theta dt]
\end{aligned}$$

Hence,

$$\begin{aligned}
&E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}\{(r\varphi - \partial_t\varphi - \mathcal{L}\varphi)(t, K_t) - \Pi(K_t, \mu_t)\}dt] \\
&\qquad\qquad + E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}p\theta dt] - E[\int_{s_0}^{s_0+\tau_\epsilon\wedge h} e^{-r(t-s_0)}\partial_k\varphi(K_t)\theta dt] \geq 0
\end{aligned}$$

Let $\theta_1 = 0$. Then,

$$E[\int_{s_0}^{s_0+\tau_\epsilon \wedge h} e^{-r(t-s_0)}\{(r\varphi - \partial_t\varphi - \mathcal{L}\varphi)(t, K_t) - \Pi(K_t, \mu_t)\}dt] \geq 0$$

Letting $h \to 0$.

$$(r\varphi - \partial_t\varphi - \mathcal{L}\varphi)(s_0, k_0) - \Pi(k_0, \mu_{s_0}) \geq 0$$

Let $\theta_1 = \theta$ and $h \to 0$. Then,

$$(r\varphi - \partial_t\varphi - \mathcal{L}\varphi)(s_0, k_0) - \Pi(k_0, \mu_{s_0}) + \theta(p - \partial_k\varphi(s_0, k_0)) \geq 0$$

Similarly, let $\dot{M}_t = \theta$ and $\dot{L}_t = 0$ for $t \in (0, h)$ for some $h > 0$. Then, as letting h goes to 0, then

$$(r\varphi - \partial_t\varphi - \mathcal{L}\varphi)(s_0, k_0) - \Pi(k_0, \mu_{s_0}) + \theta(\partial_k\varphi(s_0, k_0) - (1 - \lambda)p) \geq 0$$

$$\therefore r\varphi - \partial_t\varphi - \Pi - \mathcal{L}\varphi + \theta \min\{0, \partial_k\varphi - p(1 - \lambda), p - \partial_k\varphi\} \geq 0 \quad at \quad (s_0, k_0)$$

Hence, v is a viscosity supersolution to (4.11).
2) v is a viscosity subsolution.
Suppose not. Suppose that v is not a viscosity subsolution.
There exists a $(s_0, k_0) \in [0, \infty) \times (0, \infty)$ and $\varphi \in \mathcal{C}^{1,2}$ satisfying $\varphi(s_0, k_0) = v(s_0, k_0)$ and $\varphi(s, k) \geq v(s, k)$ for any $(s, k) \in \bar{B}_\epsilon(s_0, k_0)$, but

$$r\varphi - \partial_t\varphi - \Pi - \mathcal{L}\varphi + \theta \min\{0, \partial_k\varphi - p(1 - \lambda), p - \partial_k\varphi\} > 0 \quad at \quad (s_0, k_0).$$

Then, there exists $\alpha > 0$ such that

$$r\varphi - \partial_t\varphi - \Pi - \mathcal{L}\varphi + \theta \min\{0, \partial_k\varphi - p(1 - \lambda), p - \partial_k\varphi\} \geq \alpha$$

for all $(s, k) \in \bar{B}_\epsilon(s_0, k_0)$ for small enough $0 < \epsilon$, because it is continuous.
Define $\tau_\epsilon = \inf\{t \geq 0; (s_0, K_{s_0+t}) \notin \bar{B}_\epsilon(s_0, k_0) \quad \text{where} \quad K_{s_0} = k_0\}$.
For any admissible L_t, M_t, by the Itô's formula,

$$\begin{aligned} &E[e^{-r\tau_\epsilon}\varphi(s_0 + \tau_\epsilon, K_{s_0+\tau_\epsilon})] \\ &\quad = \varphi(s_0, k_0) + E[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(t, K_t)dt] \\ &\quad + E[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}\partial_k\varphi(t, K_t)(\dot{L}_t - \dot{M}_t)dt] \end{aligned}$$

For any $(t, K_t) \in \bar{B}_\epsilon(s_0, k_0)$ and μ_t,
if $(1 - \lambda)p \leq \partial_k\varphi(t, K_t) \leq p$,
because of $r\varphi - \partial_t\varphi - \Pi - \mathcal{L}\varphi \geq \alpha$, then $(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(K_t) \leq -\Pi(K_t, \mu_t) - \alpha$
By $\varphi_k(t, K_t)(\dot{L}_t - \dot{M}_t) \leq p\dot{L}_t - (1 - \lambda)p\dot{M}_t$,

$(-r\varphi + \varphi_t + \mathcal{L}\varphi)(t, K_t) + \partial_k\varphi(t, K_t)(\dot{L}_t - \dot{M}_t) \leq -\Pi(K_t, \mu_t) + p\dot{L}_t - (1-\lambda)p\dot{M}_t - \alpha$
If $(1-\lambda)p < p < \varphi_k(t, K_t)$,
$(-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(K_t) + \varphi_k(t, K_t)(\dot{L}_t - \dot{M}_t)$
$\leq (-r\varphi + \partial_t\varphi + \mathcal{L}\varphi)(K_t) + \varphi_k(t, K_t)\dot{L}_t - (1-\lambda)p\dot{M}_t$
$\leq -\Pi(K_t, \mu_t) - \alpha + \theta(p - \partial_k\varphi(t, K_t)) + \partial_k\varphi(t, K_t)\dot{L}_t - (1-\lambda)p\dot{M}_t$
$\leq -\Pi(K_t, \mu_t) - \alpha + (p - \partial_k\varphi(t, K_t))\dot{L}_t + \partial_k\varphi(t, K_t)\dot{L}_t - (1-\lambda)p\dot{M}_t$
$\leq -\Pi(K_t, \mu_t) + p\dot{L}_t - (1-\lambda)p\dot{M}_t - \alpha$
If $\partial_k\varphi(t, K_t) < (1-\lambda)p < p$, similarly $(-r\varphi + \varphi_t + \mathcal{L}\varphi)(t, K_t) + \partial_k\varphi(t, K_t)(\dot{L}_t - \dot{M}_t)$
$\leq -\Pi(K_t, \mu_t) + p\dot{L}_t - (1-\lambda)p\dot{M}_t - \alpha$.
So, $(-r\varphi + \varphi_t + \mathcal{L}\varphi)(t, K_t) + \partial_k\varphi(t, K_t)(\dot{L}_t - \dot{M}_t) \leq -\Pi(K_t^i, \mu_t) + p\dot{L}_t - (1-\lambda)p\dot{M}_t - \alpha$, and

$$\begin{aligned} &E[e^{-r\tau_\epsilon}\varphi(s_0 + \tau_\epsilon, K_{s_0+\tau_\epsilon})] \\ &\quad \leq \varphi(s_0, k_0) + E[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}(-\Pi(K_t, \mu_t)dt + p\dot{L}_t dt - (1-\lambda)p\dot{M}_t dt - \alpha dt\}] \end{aligned}$$

By definition of φ,

$$\begin{aligned} &E[e^{-r\tau_\epsilon}v(s_0 + \tau_\epsilon, K_{s_0+\tau_\epsilon})] \\ &\quad \leq v(s_0, k_0) + E[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}\{-\Pi(K_t, \mu_t)dt + p\dot{L}_t dt - (1-\lambda)p\dot{M}_t dt - \alpha dt\}] \end{aligned}$$

$$\begin{aligned} E\left[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}\{\Pi(K_t, \mu_t)dt - p\dot{L}_t dt + (1-\lambda)p\dot{M}_t dt\} + e^{-r\tau_\epsilon}v(s_0 + \tau_\epsilon, K_{s_0+\tau_\epsilon})\right] \\ + \alpha E[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}dt] \leq v(s_0, k_0) \end{aligned}$$

Taking the supremum over all admissible L_t, M_t, then

$$v(s_0, k_0) + \alpha E[\int_{s_0}^{s_0+\tau_\epsilon} e^{-r(t-s_0)}dt] \leq v(s_0, k_0)$$

Since $\alpha > 0$, it is contradiction. Therefore, v is a viscosity subsolution.
So, the value function v is a continuous viscosity solution on $(0, \infty)$ to (4.11). □

Lemma 10. *For any fixed $s \in [0, \infty)$, the value function $v(s, k)$ is differentiable in k.*

Proof. Fix $s \in [0, \infty)$. By previous lemma, the value function v is a viscosity solution to the HJB equation (4.11), and $v(s, k)$ is continuous, nondecreasing, and concave in k. Hence, for any $k \in (0, \infty)$, $\partial_{k+}v(s, k) \leq \partial_{k-}v(s, k)$. Suppose that $v(s, \cdot)$ is not $\mathcal{C}^1$ at k_0. Then, there exists q such that $\partial_{k+}v(s, k_0) < q < \partial_{k-}v(s, k_0)$. Define $\varphi^\epsilon(t, k) = v(t, k_0) + q(k - k_0) - \frac{1}{2\epsilon}(k - k_0)^2 - \frac{1}{2\epsilon}(s - t)^2$. Then, $\varphi^\epsilon(s, k_0) = v(s, k_0)$ and $\partial_k\varphi^\epsilon(s, k_0) = q$. Since $\partial_{k+}v(s, k_0) < q < \partial_{k-}v(s, k_0)$ and $\partial_k\varphi^\epsilon(s, k_0) = q$, then $v(t, k) - \varphi^\epsilon(t, k)$ has a local maximum at (s, k_0). Because v is a viscosity subsolution for the HJB equation (4.11),
$r\varphi^\epsilon - \partial_t\varphi^\epsilon - \Pi - \mathcal{L}\varphi^\epsilon + \theta\min\{0, \partial_k\varphi^\epsilon - p(1-\lambda), p - \partial_k\varphi^\epsilon\} \leq 0$

Case1 : $p(1-\lambda) < \partial_k \varphi^\epsilon(s,k_0) = q < p$
Hence,

$$\begin{aligned} & r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(s,k_0) - \mathcal{L}\varphi^\epsilon(s,k_0) \\ =& r\varphi^\epsilon - \Pi - \mathcal{L}\varphi^\epsilon + \theta \min\{0, \partial_k \varphi^\epsilon - p(1-\lambda), p - \partial_k \varphi^\epsilon\} \leq 0 \end{aligned}$$

$\partial_k \varphi^\epsilon(s,k_0) = q$ and $\partial_{kk} \varphi^\epsilon(s,k_0) = -\frac{1}{2\epsilon} \Rightarrow \mathcal{L}\varphi^\epsilon(s,k_0) = \frac{1}{2}\sigma^2(-\frac{1}{2\epsilon}) + b(k_0,\cdot)q$
$\therefore r\varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) + \frac{1}{4\epsilon}\sigma^2 - b(k_0,\cdot)q = r\varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) - \mathcal{L}\varphi^\epsilon(s,k_0) \leq 0$
Since k_0, σ are constants, and φ^ϵ and Π are continuous function and bounded on the bounded set, as letting $\epsilon \to \infty$, it is contradiction.
Case2 : $\partial_k \varphi^\epsilon(s,k_0) = q \geq p$
Hence,

$$\begin{aligned} & r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) - \mathcal{L}\varphi^\epsilon(s,k_0) + \theta(p - \partial_k \varphi^\epsilon(s,k_0)) \\ =& r\varphi^\epsilon \partial_t \varphi^\epsilon - \Pi - \mathcal{L}\varphi^\epsilon + \theta \min\{0, \partial_k \varphi^\epsilon - p(1-\lambda), p - \partial_k \varphi^\epsilon\} \leq 0 \end{aligned}$$

$\partial_k \varphi^\epsilon(s,k_0) = q$ and $\partial_{kk} \varphi^\epsilon(s,k_0) = -\frac{1}{2\epsilon} \Rightarrow \mathcal{L}\varphi^\epsilon(s,k_0) = \frac{1}{2}\sigma^2(-\frac{1}{2\epsilon}) + b(k_0,\cdot)q.$
$\therefore r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) + \frac{1}{4\epsilon}\sigma^2 - b(k_0,\cdot)q + \theta(p-q)$
$= r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) - \mathcal{L}\varphi^\epsilon(s,k_0) + \theta(p - \partial_k \varphi^\epsilon(s,k_0)) \leq 0$
Since k_0, θ, σ are constants, and φ^ϵ and Π are continuous function and bounded on the bounded set, as letting $\epsilon \to \infty$, it is contradiction.
Case3 : $\partial_k \varphi^\epsilon(s,k_0) = q \leq p(1-\lambda)$
Hence,

$$\begin{aligned} & r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) - \mathcal{L}\varphi^\epsilon(s,k_0) + \theta_0(\partial_k \varphi^\epsilon(s,k_0) - p(1-\lambda)) \\ =& r\varphi^\epsilon - \partial_t \varphi^\epsilon - \Pi - \mathcal{L}\varphi^\epsilon + \theta \min\{0, \partial_k \varphi^\epsilon - p(1-\lambda), p - \partial_k \varphi^\epsilon\} \leq 0 \end{aligned}$$

$\partial_k \varphi^\epsilon(s,k_0) = q$ and $\partial_{kk} \varphi^\epsilon(s,k_0) = -\frac{1}{2\epsilon} \Rightarrow \mathcal{L}\varphi^\epsilon(s,k_0) = \frac{1}{2}\sigma^2(-\frac{1}{2\epsilon}) + b(k_0,\cdot)q$
$\therefore r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) + \frac{1}{4\epsilon}\sigma^2 - b(k_0,\cdot)q + \theta(q - p(1-\lambda))$
$= r\varphi^\epsilon(s,k_0) - \partial_t \varphi^\epsilon(s,k_0) - \Pi(k_0,\cdot) - \mathcal{L}\varphi^\epsilon(s,k_0) + \theta(\partial_k \varphi^\epsilon(s,k_0) - p(1-\lambda)) \leq 0$
Since k_0, θ_0, σ are constants, and φ_ϵ and Π are continuous function and bounded on the bounded set, as letting $\epsilon \to \infty$, it is contradiction.
Hence, $\partial_{k-} v(s,k) = \partial_{k+} v(s,k) \quad \forall k \in (0,\infty)$.
$\therefore v(s,k)$ is differentiable in k.

□

Optimal controls Since $v(s,k)$ is differentiable and concave in k, for fixed $t \in [s,\infty)$, we can define $k_b(t) < k_s(t)$ where $k_b(t) = \sup\{k | \partial_k v(t,k) = p\}$ and $k_s(t) = \inf\{k | \partial_k v(t,k) = (1-\lambda)p\}$, and define

$$L_t = \int_s^t \theta 1_{\{K_r \leq k_b(r)\}} dr, \quad M_t = \int_s^t \theta 1_{\{K_r \geq k_s(r)\}} dr.$$

Then, because v is differentiable and concave in k, we apply the Tanaka's formula which is generalized Itô's formula for v with L_t, M_t. Similarly with the proof in lemma 7, it is the optimal controls. □

Proof of proposition 6

Proof. Under fixed $\{\mu_t\}, \{L_t\}, \{M_t\}$, the existence and uniqueness of the martingale problem (4.8) is equivalent to the existence and uniqueness of weak solution to (4.8), and by the Stroock-Varadhan theorem, since $|b(k, \mu_t)| + \theta + |\sigma| \leq \delta_2(1 + |k|)$ for some $\delta_2 > 0$, (4.8) has a unique weak solution (from the chapter 4.19 and 4.24 in [72]). Hence, (4.8) has a unique probability distribution P_{K_t} and update the mean process from this distribution for consistency part.
Furthermore, using Kramers-Moyal Expansion of the master equation

$$\partial_t P_{K_t}(k) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \partial_{k^m}[a^{(m)}(t,k)P_{K_t}(k)]$$

where $a^{(m)}(t,k) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[(K_{t+\Delta t} - K_t)^m]|_{K_t = k}$, it is easy to derive the Kolmogorov forward equation (4.9). □

Remark. For general finite variation processes L_t, M_t, $a^{(m)}(t,k)$ may not converge for some m.

Approximation to N-player games and its Nash equilibrium

Proof of theorem 5

Proof. In this paper, we only prove the case which the drift function is $b(k)$. For general $b(k, \mu)$, using the proof in [37] and combining this proof for the model on an infinite time horizon, we can prove this theorem.

Let $\dot{L}_t^{i*}$ and $\dot{M}_t^{i*}$ be optimal control processes to (4.5). Let only player 1 choose other control $\dot{L}_t^{1'}$ and $\dot{M}_t^{1'}$ but all other players choose $\dot{L}_t^{i*}$ and $\dot{M}_t^{i*}$ as controls. K_t^i does not change because dynamics of K_t^i only depends on player's own state and controls. μ_t is a distribution of K_t^i for $i = 2, 3, \cdots, N$. Let $K_t^{1'}$ be a new state process for player 1 under controls $\dot{L}_t^{1'}$ and $\dot{M}_t^{1'}$.
Because of lemma 5 and lemma 6 in previous section, if $\varphi(k)$ is a supersolution of the HJB equation (4.11), $v^i(s,k) \leq \varphi(k) = qk + q_0$ for same q and q_0 in the proof of lemma 6. Hence,

$$\frac{1}{N} E\left[\int_s^{\infty} e^{-rt}\Pi(K_s^1, \mu_t)dt\right] \leq \frac{1}{N} v^1(K_s^1) \leq \frac{1}{N}(qK_s^1 + q_0) = O(\frac{1}{N})$$

$$\frac{1}{N} E\left[\int_s^{\infty} e^{-rt}\Pi_0(K_t^{1'}, K_t^{1'})dt\right] \leq \frac{1}{N} v^1(K_s^1) \leq \frac{1}{N}(qK_s^1 + q_0) = O(\frac{1}{N})$$

$$\frac{1}{N-1} \sum_{j \neq i} \Pi_0(K_t^i, K_t^j) = \Pi(K_t^i, \mu_t) + O(\frac{1}{\sqrt{N}})$$

$$
\begin{aligned}
&E\left[\int_s^\infty e^{-rt}[\frac{1}{N}\sum_{j=1,\cdots,N}\Pi_0(K_t^{1'},K_t^j)dt-p\dot{L}_t^{1'}dt+p(1-\lambda)\dot{M}_t^{1'}dt]\right]\\
=&E\left[\int_s^\infty e^{-rt}[\frac{1}{N}\Pi_0(K_t^{1'},K_t^{1'})+\frac{1}{N}\sum_{j\neq 1}\Pi_0(K_t^{1'},K_t^j)dt-p\dot{L}_t^{1'}dt+p(1-\lambda)\dot{M}_t^{1'}dt]\right]\\
=&E\left[\int_s^\infty e^{-rt}[\frac{1}{N}\Pi_0(K_t^{1'},K_t^{1'})+\frac{N-1}{N}\Pi(K_t^{1'},\mu_t)dt-p\dot{L}_t^{1'}dt+p(1-\lambda)\dot{M}_t^{1'}dt]\right]+O(\frac{1}{\sqrt{N}})\\
\leq&E\left[\int_s^\infty e^{-rt}[\frac{N-1}{N}\Pi(K_t^{1'},\mu_t)dt-p\dot{L}_t^{1'}dt+p(1-\lambda)\dot{M}_t^{1'}dt]\right]+O(\frac{1}{\sqrt{N}})+O(\frac{1}{N})\\
\leq&E\left[\int_s^\infty e^{-rt}[\Pi(K_t^{1'},\mu_t)dt-p\dot{L}_t^{1'}dt+p(1-\lambda)\dot{M}_t^{1'}dt]\right]+O(\frac{1}{\sqrt{N}})\\
\leq&E\left[\int_s^\infty e^{-rt}[\Pi(K_t^{1},\mu_t)dt-p\dot{L}_t^{1*}dt+p(1-\lambda)\dot{M}_t^{1*}dt]\right]+O(\frac{1}{\sqrt{N}})\\
\leq&E\left[\int_s^\infty e^{-rt}[\frac{1}{N}\sum_{j=1,\cdots,N}\Pi_0(K_t^{1},K_t^j)dt-p\dot{L}_t^{1*}dt+p(1-\lambda)\dot{M}_t^{1*}dt]\right]+O(\frac{1}{\sqrt{N}})
\end{aligned}
$$

Last two inequality is due to the optimality of $\dot{L}_t^{1*}, \dot{M}_t^{1*}$, and due to the central limit theorem. This inequality holds for any $(\dot{L}_t', \dot{M}_t')$. Hence, optimal controls of the MFG (4.5) is an ϵ-Nash equilibrium for this $N-$player game (4.10), and $\epsilon = O(\frac{1}{\sqrt{N}})$. □

4.3 Stationary optimal partially reversible investment MFG

Let's solve the toy model of optimal partially reversible investment SMFG. We will solve one toy model with singular controls and another toy model with singular controls of bounded velocity. Then, we shall compare both models and study limit behaviours as the bound goes to infinity.

Problems

SMFG with singular controls Consider the toy model which is a SMFG model with singular controls as follows:

$$
\begin{aligned}
v(k) = &\sup_{L_t,M_t\in\mathcal{U}} E\left[\int_0^\infty e^{-rt}[\Pi(K_t,\mu)dt-pdL_t+p(1-\lambda)dM_t]\right]\\
s.t.\quad dK_t =&K_t(\delta dt+\gamma dW_t)+dL_t-dM_t,\qquad K_{0-}=k,\quad \mu=P_{K_\infty}
\end{aligned}
\tag{4.12}
$$

where δ, γ are nonnegative constants and P_{K_∞} is a limiting stationary distribution of K_t.

SMFG with singular controls of bounded velocity The toy SMFG model which is singular controls of bounded velocity is for any fixed $\theta > 0$

$$\begin{aligned}\tilde{v}(k:\theta) = &\sup_{L_t, M_t \in \mathcal{U}_\theta} E\left[\int_0^\infty e^{-rt}[\Pi(K_t,\mu)dt - pK_t\dot{L}_t dt + p(1-\lambda)K_t\dot{M}_t dt]\right] \\ s.t. \quad dK_t = &K_t(\delta dt + \gamma dW_t) + K_t\dot{L}_t dt - K_t\dot{M}_t dt, \quad K_{0-} = k, \quad \mu = P_{K_\infty}.\end{aligned} \tag{4.13}$$

In this model, we re-parametrize control variables $\dot{L}_t dt \to K_t\dot{L}_t dt$ for calculation. Assume that $\theta > \delta$.

Revenue function $\Pi(k,\mu)$ is the revenue function which is the long run average price ρ times the production output level. By the Cobb-Douglass model, we assume $\Pi(k,\mu) = \rho c k^\alpha$ for constant $c > 0$ and $\alpha \in (0,1)$. The limiting stationary distribution μ of $\{K_t\}$ determines the long run average price. According to the inverse demand function, we set $\rho = a_0 - a_2(ck^\alpha)^{\frac{1-\alpha}{\alpha}}$ for some positive constant a_0, a_2 where ρ is the long run average price and ck^α is the production output level. So, $\rho = a_0 - a_1 k^{1-\alpha}$ for some constant a_1. Then, $\Pi(k,\mu) = ck^\alpha\rho = ck^\alpha \int (a_0 - a_1 k^{1-\alpha})\mu(dk)$

Solution to the SMFG model (4.13)

Let's solve it through the MFG framework.

Step 1 Fix μ as deterministic. Then, the HJB equation to the SMFG model (4.13) under fixed μ is

$$0 = r\tilde{v} - ck^\alpha\rho - \delta k\partial_k\tilde{v} - \frac{1}{2}\gamma^2k^2\partial_{kk}\tilde{v} + \theta k\min\{0, p - \partial_k\tilde{v}, \partial_k\tilde{v} - p(1-\lambda)\} \tag{4.14}$$

Then, the solution $\tilde{v}$ is

$$\tilde{v}(k:\theta) = \begin{cases} B_1k^{n'} + H_1k^\alpha + H_2k & k \le \tilde{k}_b \\ B_2k^m + B_3k^n + Hk^\alpha & \tilde{k}_b < k < \tilde{k}_s \\ B_4k^{m''} + H_3k^\alpha + H_4k & \tilde{k}_s \le k \end{cases}$$

where $\tilde{k}_b = \inf\{k : \partial_k\tilde{v}(k:\theta) = p\}, \quad \tilde{k}_s = \sup\{k : \partial_k\tilde{v}(k:\theta) = p(1-\lambda)\}$ and

$$\begin{aligned} m', n' &= -\frac{\delta+\theta}{\gamma^2} + \frac{1}{2} \mp \sqrt{(-\frac{\delta+\theta}{\gamma^2} + \frac{1}{2})^2 + \frac{2r}{\gamma^2}} \\ m'', n'' &= -\frac{\delta-\theta}{\gamma^2} + \frac{1}{2} \mp \sqrt{(-\frac{\delta-\theta}{\gamma^2} + \frac{1}{2})^2 + \frac{2r}{\gamma^2}} \\ H_1 &= \frac{2c\rho}{\gamma^2(n'-\alpha)(\alpha-m')}, \quad H_2 = -\frac{2\theta p}{\gamma^2(n'-1)(1-m')} \\ H_3 &= \frac{2c\rho}{\gamma^2(n''-\alpha)(\alpha-m'')}, \quad H_4 = \frac{2\theta(1-\lambda)p}{\gamma^2(n''-1)(1-m'')}. \end{aligned}$$

Then, $\tilde{k}_b = c_4\rho^{\frac{1}{1-\alpha}}$ and $\tilde{k}_s = c_5\rho^{\frac{1}{1-\alpha}}$
where the numerator of $c_4^{1-\alpha}$ is

$$
\begin{aligned}
&(\frac{n'(n'-1)-m(m-1)}{m}\frac{y_2^n-y_2^\alpha}{y_2^n-y_2^m}+\frac{n(n-1)-n'(n'-1)}{n}\frac{y_2^m-y_2^\alpha}{y_2^n-y_2^m})\frac{2c\alpha}{\gamma^2(n-\alpha)(\alpha-m)}\\
&+(\frac{2c(\alpha(\alpha-1)-n'(n'-1))}{\gamma^2(n-\alpha)(\alpha-m)}-\frac{2c(\alpha(\alpha-1)-n'(n'-1))}{\gamma^2(n'-\alpha)(\alpha-m')}),
\end{aligned}
$$

and the denominator of $c_4^{1-\alpha}$ is

$$
\frac{n'(n'-1)-m(m-1)}{m}\frac{y_2^n-(1-\lambda)y_2}{y_2^n-y_2^m}p+\frac{n(n-1)-n'(n'-1)}{n}\frac{y_2^m-(1-\lambda)y_2}{y_2^n-y_2^m}p+n'(n'-1)H_2,
$$

and the numerator of of $c_5^{1-\alpha}$ is

$$
\begin{aligned}
&(\frac{m''(m''-1)-n(n-1)}{n}\frac{y_2^{-m}-y_2^{-\alpha}}{y_2^{-m}-y_2^{-n}}+\frac{m(m-1)-m''(m''-1)}{m}\frac{y_2^{-n}-y_2^{-\alpha}}{y_2^{-m}-y_2^{-n}})\frac{2c\alpha}{\gamma^2(n-\alpha)(\alpha-m)}\\
&+(\frac{2c(\alpha(\alpha-1)-m''(m''-1))}{\gamma^2(n-\alpha)(\alpha-m)}-\frac{2c(\alpha(\alpha-1)-m''(m''-1))}{\gamma^2(n''-\alpha)(\alpha-m'')}),
\end{aligned}
$$

and the denominator of $c_5^{1-\alpha}$ is

$$
\begin{aligned}
&\frac{m''(m''-1)-n(n-1)}{n}\frac{y_2^{-1}-(1-\lambda)y_2^{-m}}{y_2^{-n}-y_2^{-m}}p\\
&+\frac{m(m-1)-m''(m''-1)}{m}\frac{y_2^{-1}-(1-\lambda)y_2^{-n}}{y_2^{-n}-y_2^{-m}}p+m''(m''-1)H_4.
\end{aligned}
$$

Hence, c_4, c_5 are independent to ρ.
B_1, B_2, B_3, B_4, are:

$$
B_2=\frac{p(y_2^n-(1-\lambda)y_2)-H\alpha k_b^{\alpha-1}(y_2^n-y_2^\alpha)}{mk_b^{m-1}(y_2^n-y_2^m)},\quad B_3=\frac{p(y_2^m-(1-\lambda)y_2)-H\alpha k_b^{\alpha-1}(y_2^m-y_2^\alpha)}{nk_b^{n-1}(y_2^m-y_2^n)},
$$
$$
B_1=\frac{B_2\tilde{k}_b^m+B_3\tilde{k}_b^n+H\tilde{k}_b^\alpha-H_1\tilde{k}_b^\alpha-H_2\tilde{k}_b}{\tilde{k}_b^{n'}},\quad B_4=\frac{B_2\tilde{k}_s^m+B_3\tilde{k}_s^n+H\tilde{k}_s^\alpha-H_3\tilde{k}_s^\alpha-H_4\tilde{k}_s}{\tilde{k}_s^{m''}},
$$

and $y_2 > 1$ is the solution on $(1, \infty)$ to the equation: $F(y_2) = \frac{F_1(y_2)}{F_2(y_2)} - \frac{F_3(y_2)}{F_4(y_2)} = 0$ such that

$$\begin{aligned}
F_1(y_2) =& \Big(\frac{n'(n'-1) - m(m-1)}{m} \frac{y_2^n - y_2^\alpha}{y_2^n - y_2^m} + \frac{n(n-1) - n'(n'-1)}{n} \frac{y_2^m - y_2^\alpha}{y_2^n - y_2^m}\Big) \frac{2c\alpha}{\gamma^2(n-\alpha)(\alpha-m)} \\
&+ \Big(\frac{2c(\alpha(\alpha-1) - n'(n'-1))}{\gamma^2(n-\alpha)(\alpha-m)} - \frac{2c(\alpha(\alpha-1) - n'(n'-1))}{\gamma^2(n'-\alpha)(\alpha-m')}\Big) \\
F_2(y_2) =& y_2^{\alpha-1}\Big(\frac{n'(n'-1) - m(m-1)}{m} \frac{y_2^n - (1-\lambda)y_2}{y_2^n - y_2^m} p \\
&+ \frac{n(n-1) - n'(n'-1)}{n} \frac{y_2^m - (1-\lambda)y_2}{y_2^n - y_2^m} p + n'(n'-1)H_2\Big) \\
F_3(y_2) =& \Big(\frac{m''(m''-1) - n(n-1)}{n} \frac{y_2^{-m} - y_2^{-\alpha}}{y_2^{-m} - y_2^{-n}} + \frac{m(m-1) - m''(m''-1)}{m} \frac{y_2^{-n} - y_2^{-\alpha}}{y_2^{-m} - y_2^{-n}}\Big) \\
&\times \frac{2c\alpha}{\gamma^2(n-\alpha)(\alpha-m)} + \Big(\frac{2c(\alpha(\alpha-1) - m''(m''-1))}{\gamma^2(n-\alpha)(\alpha-m)} - \frac{2c(\alpha(\alpha-1) - m''(m''-1))}{\gamma^2(n''-\alpha)(\alpha-m'')}\Big) \\
F_4(y_2) =& \Big|\frac{m''(m''-1) - n(n-1)}{n} \frac{y_2^{-1} - (1-\lambda)y_2^{-m}}{y_2^{-n} - y_2^{-m}} p \\
&+ \frac{m(m-1) - m''(m''-1)}{m} \frac{y_2^{-1} - (1-\lambda)y_2^{-n}}{y_2^{-n} - y_2^{-m}} p + m''(m''-1)H_4\Big|.
\end{aligned}$$

Because of $n > 1, 0 > m$ and $m'' < 0, n' > 0$, $\lim_{y_2 \to \infty} F(y_2) = +\infty$ and $\lim_{y_2 \downarrow 1} F(y_2) = \frac{1}{\lambda p(n'(n'-1)(\frac{1}{m} - \frac{1}{n}) + n - m)} - \frac{1}{\lambda p|m''(m''-1)(\frac{1}{n} - \frac{1}{m}) - n + m)|} < 0$. So, there exists a solution such that $F(y_2) = 0$ on $(1, \infty)$ and by uniqueness of solutions of $B_1, B_2, B_3, B_4, \tilde{k}_b, \tilde{k}_s$, it is unique. We will give numerical examples for this in section 4.3.

Step 2 In [11] and [63], for a diffusion process $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$ with reflecting boundaries $k_b < k_s$, the scale density is $s(x) = \exp[-\int_{k_b}^x \frac{2\mu(y)}{\sigma(y)} dy]$ and the scale density is $m(x) = \frac{2}{\sigma(x)^2 s(x)}$. Then, the limiting stationary distribution for X_t is $P_{X_\infty} = \frac{m(x)}{M(k_s)}$ where $M(x) = \int_{k_b}^x m(y)dy$. The drift function of K_t is $\mu(y) = \begin{cases} (\delta+\theta)y & y \leq \tilde{k}_b \\ \delta y & \tilde{k}_b < y < \tilde{k}_s \\ (\delta-\theta)y & \tilde{k}_s \leq y \end{cases}$. Then, the limiting stationary distribution is

$$P_{K_\infty}(y) = \begin{cases} \frac{1}{M} y^{\frac{2(\delta+\theta)}{\gamma^2} - 2} & y \leq \tilde{k}_b \\ \frac{1}{M} \tilde{k}_b^{\frac{2\theta}{\gamma^2}} y^{\frac{2\delta}{\gamma^2} - 2} & \tilde{k}_b < y < \tilde{k}_s \\ \frac{1}{M} \tilde{k}_b^{\frac{2\theta}{\gamma^2}} \tilde{k}_s^{\frac{2\theta}{\gamma^2}} y^{\frac{2(\delta-\theta)}{\gamma^2} - 2} & \tilde{k}_s \leq y \end{cases}$$

where $M = (\frac{\gamma^2}{2(\delta+\theta)-\gamma^2} - \frac{\gamma^2}{2\delta-\gamma^2}) \tilde{k}_b^{\frac{2}{\gamma^2}(\delta+\theta)-1} + (\frac{\gamma^2}{2\delta-\gamma^2} - \frac{\gamma^2}{2(\delta-\theta)-\gamma^2}) \tilde{k}_b^{\frac{2\theta}{\gamma^2}} \tilde{k}_s^{\frac{2\delta}{\gamma^2}-1}$.

Proof. The scale density with arbitrary $\Delta > 0$ is $s(y) = \exp\{-\int_\Delta^y \frac{2\mu(y)}{\gamma^2 y^2} dy\}$.

$$s(y) = \begin{cases} \Delta^{\frac{2(\delta+\theta)}{\gamma^2}} y^{-\frac{2(\delta+\theta)}{\gamma^2}} & y \in (0, \tilde{k}_b] \\ \Delta^{\frac{2(\delta+\theta)}{\gamma^2}} \tilde{k}_b^{-\frac{2\theta}{\gamma^2}} y^{-\frac{2\delta}{\gamma^2}} & y \in (\tilde{k}_b, \tilde{k}_s) \\ \Delta^{\frac{2(\delta+\theta)}{\gamma^2}} \tilde{k}_b^{-\frac{2\theta}{\gamma^2}} k_s^{-\frac{2\theta}{\gamma^2}} y^{-\frac{2(\delta-\theta)}{\gamma^2}} & y \in [\tilde{k}_s, +\infty) \end{cases}$$

The speed density is $m(y) = \frac{2}{\gamma^2 y^2 s(y)}$ and the density $\pi(y)$ is proportional to $m(y)$. Since $\pi(y)$ is proportional to $m(y)$, we can ignore Δ and $\frac{2}{\gamma^2}$. Then,

$$m(y) = \begin{cases} y^{\frac{2(\delta+\theta)}{\gamma^2}-2} & y \in (0, \tilde{k}_b] \\ \tilde{k}_b^{\frac{2\theta}{\gamma^2}} y^{\frac{2\delta}{\gamma^2}-2} & y \in (\tilde{k}_b, \tilde{k}_s) \\ \tilde{k}_b^{\frac{2\theta}{\gamma^2}} k_s^{\frac{2\theta}{\gamma^2}} y^{\frac{2(\delta-\theta)}{\gamma^2}-2} & y \in [\tilde{k}_s, +\infty) \end{cases}$$

$P_{K_\infty}(y) = \frac{m(y)}{\int_0^\infty m(z)dz}$

$$\begin{aligned} M = \int_0^\infty m(z)dz &= \int_0^{\tilde{k}_b} z^{\frac{2(\delta+\theta)}{\gamma^2}-2} dz + \int_{k_b}^{\tilde{k}_s} \tilde{k}_b^{\frac{2\theta}{\gamma^2}} z^{\frac{2\delta}{\gamma^2}-2} dz + \int_{\tilde{k}_s}^\infty \tilde{k}_b^{\frac{2\theta}{\gamma^2}} \tilde{k}_s^{\frac{2\theta}{\gamma^2}} z^{\frac{2(\delta-\theta)-2}{\gamma^2}} dz \\ &= (\frac{\gamma^2}{2(\delta+\theta)-\gamma^2} - \frac{\gamma^2}{2\delta-\gamma^2}) \tilde{k}_b^{\frac{2}{\gamma^2}(\delta+\theta)-1} + (\frac{\gamma^2}{2\delta-\gamma^2} - \frac{\gamma^2}{2(\delta-\theta)-\gamma^2}) \tilde{k}_b^{\frac{2\theta}{\gamma^2}} \tilde{k}_s^{\frac{2\delta}{\gamma^2}-1} \end{aligned}$$

□

Hence, updated long run average price is

$$\begin{aligned} \rho' &= \int (a_0 - a_1 y^{1-\alpha}) P_{K_\infty}(dy) \\ &= a_0 - a_1 \frac{\left(\frac{1}{2(\delta+\theta)-\alpha\gamma^2} - \frac{1}{2\delta-\alpha\gamma^2}\right) \tilde{k}_b^{\frac{2\delta}{\gamma^2}-\alpha} + \left(\frac{1}{2\delta-\alpha\gamma^2} - \frac{1}{2(\delta-\theta)-\alpha\gamma^2}\right) \tilde{k}_s^{\frac{2\delta}{\gamma^2}-\alpha}}{(\frac{1}{2(\delta+\theta)-\gamma^2} - \frac{1}{2\delta-\gamma^2}) \tilde{k}_b^{\frac{2\delta}{\gamma^2}-1} + (\frac{1}{2\delta-\gamma^2} - \frac{1}{2(\delta-\theta)-\gamma^2}) \tilde{k}_s^{\frac{2\delta}{\gamma^2}-1}} \end{aligned}$$

From step 1, $\tilde{k}_b = c_4 \rho^{\frac{1}{1-\alpha}}, \tilde{k}_s = c_5 \rho^{\frac{1}{1-\alpha}}$ and $\frac{\tilde{k}_b}{\tilde{k}_s} = \frac{c_4}{c_5}$ where c_4, c_5 do not depend on ρ. Consequently,

$$\begin{aligned} \rho' =& a_0 - a_1 \frac{\left(\frac{1}{2(\delta+\theta)-\alpha\gamma^2} - \frac{1}{2\delta-\alpha\gamma^2}\right) (\frac{c_4}{c_5}\tilde{k}_s)^{\frac{2\delta}{\gamma^2}-\alpha} + \left(\frac{1}{2\delta-\alpha\gamma^2} - \frac{1}{2(\delta-\theta)+-\alpha\gamma^2}\right) \tilde{k}_s^{\frac{2\delta}{\gamma^2}+\alpha-1}}{(\frac{1}{2(\delta+\theta)-\gamma^2} - \frac{1}{2\delta-\gamma^2})(\frac{c_4}{c_5}\tilde{k}_s)^{\frac{2\delta}{\gamma^2}-1} + (\frac{1}{2\delta-\gamma^2} - \frac{1}{2(\delta-\theta)-\gamma^2}) \tilde{k}_s^{\frac{2\delta}{\gamma^2}-1}} \\ =& a_0 - a_1 \frac{\left(\frac{1}{2(\delta+\theta)-\alpha\gamma^2} - \frac{1}{2\delta-\alpha\gamma^2}\right) (\frac{c_4}{c_5})^{\frac{2\delta}{\gamma^2}-\alpha} + \left(\frac{1}{2\delta-\alpha\gamma^2} - \frac{1}{2(\delta-\theta)-\alpha\gamma^2}\right)}{(\frac{1}{2(\delta+\theta)-\gamma^2} - \frac{1}{2\delta-\gamma^2})(\frac{c_4}{c_5})^{\frac{2\delta}{\gamma^2}-1} + (\frac{1}{2\delta-\gamma^2} - \frac{1}{2(\delta-\theta)-\gamma^2})} \tilde{k}_s^{1-\alpha} \end{aligned}$$

Let $\rho' = a_0 - a_1 c_6 \tilde{k}_s^{1-\alpha}$ where c_6 is independent to $\rho, \tilde{k}_b, \tilde{k}_s$.

Step 3 Fixed point method
Fixed ρ
step 1: $\tilde{k}_b = c_4\rho^{\frac{1}{1-\alpha}}, \tilde{k}_s = c_5\rho^{\frac{1}{1-\alpha}}$
step 2: $\rho' = a_0 - a_1c_6k_s^{1-\alpha} = a_0 - a_1c_6(c_5\rho^{\frac{1}{1-\alpha}})^{1-\alpha} = a_0 - a_1c_6c_5^{1-\alpha}\rho$
c_3, c_4, c_5 are independent to ρ. Hence, for any ρ_1, ρ_2,

$$|\rho_1' - \rho_2'| = a_1c_6c_5^{1-\alpha}|\rho_1 - \rho_2|$$

If it is contraction mapping (i.e. $|a_1c_6c_5^{1-\alpha}| < 1$), it has a fixed point solution. Moreover, the long run average price of the fixed point solution is $\rho^* = \frac{a_0}{1+a_1c_6c_5^{1-\alpha}}$. We will illustrate numerical examples when $|a_1c_6c_5^{1-\alpha}| < 1$ with numerical solutions.

Solution to the SMFG model (4.12)

Let's solve it through the mean field game framework.

Step 1 Fix μ as deterministic. As in [39], the HJB equation to the MFG (4.12)

$$\min\{rv - ck^\alpha\rho - \frac{1}{2}\gamma^2k^2\partial_{kk}v - \delta k\partial_k v, \partial_k v - p(1-\lambda), p - \partial_k v\} = 0$$

has a unique viscosity solution and it is the value function of the stochastic control problem. The optimal state process under optimal controls K_t^* is a geometric Brownian motion with the reflecting boundaries $k_b < k_s$. Let $y_1 = \frac{k_s}{k_b}$ and

$$m = -\frac{\delta}{\gamma^2} + \frac{1}{2} - \sqrt{(-\frac{\delta}{\gamma^2} + \frac{1}{2})^2 + \frac{2r}{\gamma^2}} < 0, \quad n = -\frac{\delta}{\gamma^2} + \frac{1}{2} + \sqrt{(-\frac{\delta}{\gamma^2} + \frac{1}{2})^2 + \frac{2r}{\gamma^2}} > 1$$
$$H = \frac{2c\rho}{\gamma^2(n-\alpha)(\alpha-m)}.$$

Then,

$$k_b = \left(\frac{2c\alpha(y_1^n - y_1^\alpha)}{\gamma^2(n-\alpha)p(1-m)(y_1^n - (1-\lambda)y_1)}\right)^{\frac{1}{1-\alpha}} \rho^{\frac{1}{1-\alpha}} = c_1\rho^{\frac{1}{1-\alpha}}$$
$$k_s = y_1\left(\frac{2c\alpha(y_1^n - y_1^\alpha)}{\gamma^2(n-\alpha)p(1-m)(y_1^n - (1-\lambda)y_1)}\right)^{\frac{1}{1-\alpha}} \rho^{\frac{1}{1-\alpha}} = c_2\rho^{\frac{1}{1-\alpha}}$$

where c_1, c_2 are independent to ρ and $y_1 > 1$ is the solution to the equation

$$\frac{(n-\alpha)(1-m)y_1^{n-1}(y_1^\alpha - y_1^m) + (\alpha-m)(n-1)y_1^{m-1}(y_1^n - y_1^\alpha)}{(n-\alpha)(1-m)(y_1^\alpha - y_1^m) + (\alpha-m)(n-1)(y_1^n - y_1^\alpha)} = 1-\lambda.$$

(By [39], this equation has a unique solution y_1 on $(1, \infty)$.)

The value function is

$$v(k) = \begin{cases} A_1 + pk & k \le k_b \\ A_2 k^m + A_3 k^n + H k^\alpha & k_b < k < k_s \\ A_4 + p(1-\lambda)k & k_s \le k \end{cases}$$

where A_1, A_2, A_3, A_4 are constants which are uniquely determined by continuity and $\mathcal{C}^2$-ness of the value function, and the conditions $k_b = \sup\{k : \partial_k v(k) = p\}, \quad k_s = \sup\{k : \partial_k v(k) = p(1-\lambda)\}$.

$$A_2 = \frac{(n-1)pk_b + \alpha H(\alpha - n)k_b^\alpha}{m(n-m)k_b^m}, \quad A_3 = \frac{(1-m)pk_b - \alpha H(\alpha - m)k_b^\alpha}{n(n-m)k_b^n},$$
$$A_1 = A_2 k_b^m + A_3 k_b^n + H k_b^\alpha - pk_b, \quad A_4 = A_2 k_s^m + A_3 k_s^n + H k_s^\alpha - p(1-\lambda)k_s$$

Step 2 From [39], the optimal state process K_t^* is the reflected geometric Brownian motion in $[k_b, k_s]$ of drift δ and diffusion γ with two boundaries $k_b < k_s$.
Similarly with the step 2 in the previous section, for K_t^*,

$$\begin{aligned} s(x) &= \exp\{-\int_{k_b}^{x} \frac{2\delta y}{\gamma^2 y^2} dy\} = \exp\{-\frac{2\delta}{\gamma^2}\int_{k_b}^{x} \frac{1}{y} dy\} \\ &= \exp\{-\frac{2\delta}{\gamma^2}\log(\frac{x}{k_b}) dy\} = k_b^{\frac{2\delta}{\gamma^2}} x^{-\frac{2\delta}{\gamma^2}} \quad , \quad x \in (k_b, k_s) \end{aligned}$$

$$m(x) = \frac{2}{\gamma^2 x^2 s(x)} = \frac{2k_b^{-\frac{2\delta}{\gamma^2}}}{\gamma^2} x^{\frac{2\delta}{\gamma^2}-2} \quad , \quad x \in (k_b, k_s)$$

$$M(k_s) = \int_{k_b}^{k_s} \frac{2k_b^{-\frac{2\delta}{\gamma^2}}}{\gamma^2} x^{\frac{2\delta}{\gamma^2}-2} dx = \frac{2k_b^{-\frac{2\delta}{\gamma^2}}}{\gamma^2(\frac{2\delta}{\gamma^2}-1)}(k_s^{\frac{2\delta}{\gamma^2}-1} - k_b^{\frac{2\delta}{\gamma^2}-1})$$

Consequently,

$$\begin{aligned} \rho' &= \int (a_0 - a_1 x^{1-\alpha}) P_{K_\infty^*}(dx) \\ &= a_0 - a_1 \int_{k_b}^{k_s} \frac{2\delta - \gamma^2}{\gamma^2(k_s^{\frac{2\delta}{\gamma^2}-1} - k_b^{\frac{2\delta}{\gamma^2}-1})} x^{\frac{2\delta}{\gamma^2}-1-\alpha} dx \\ &= a_0 - a_1 \frac{2\delta - \gamma^2}{\gamma^2(\frac{2\delta}{\gamma^2} - 1 + \alpha)} \frac{k_s^{\frac{2\delta}{\gamma^2}-\alpha} - k_b^{\frac{2\delta}{\gamma^2}-\alpha}}{k_s^{\frac{2\delta}{\gamma^2}-1} - k_b^{\frac{2\delta}{\gamma^2}-1}} \\ &= a_0 - a_1 \frac{2\delta - \gamma^2}{2\delta + (\alpha - 1)\gamma^2} \frac{1 - (\frac{c_1}{c_2})^{\frac{2\delta}{\gamma^2}-\alpha}}{1 - (\frac{c_1}{c_2})^{\frac{2\delta}{\gamma^2}-1}} k_s^{1-\alpha} \\ &= a_0 - a_1 c_3 k_s^{1-\alpha} \end{aligned}$$

where $c_3 = \frac{2\delta-\gamma^2}{2\delta+(\alpha-1)\gamma^2}\frac{1-(\frac{c_1}{c_2})^{\frac{2\delta}{\gamma^2}-\alpha}}{1-(\frac{c_1}{c_2})^{\frac{2\delta}{\gamma^2}-1}}$ does not depend on ρ,k_b, k_s. Therefore,

$$\rho' = a_0 - a_1 c_3 k_s^{1-\alpha} = a_0 - a_1 c_3 c_2^{1-\alpha}\rho$$

$$|\rho_1' - \rho_2'| = a_1 c_3 c_2^{1-\alpha}|\rho_1 - \rho_2|$$

Step 3 As we have seen above, if $a_1 c_3 c_2^{1-\alpha} < 1$, this process has a fixed point solution. Moreover, the long run average price of the fixed point solution is $\rho^* = \frac{a_0}{1+a_1 c_3 c_2^{1-\alpha}}$.

Limit behaviour of value functions

Proposition 7. *$\tilde{v}(k:\theta)$ converges to $v(k)$ as θ goes to ∞ pointwisely.*

Proof. Under any fixed ρ, as $\theta \to \infty$,

$$m' = -\frac{\delta+\theta}{\gamma^2} + \frac{1}{2} - \sqrt{(-\frac{\delta+\theta}{\gamma^2} + \frac{1}{2})^2 + \frac{2r}{\gamma^2}} \to -\infty, \qquad n' = -\frac{\frac{2r}{\gamma^2}}{m'} \to 0$$
$$n'' = -\frac{\delta-\theta}{\gamma^2} + \frac{1}{2} + \sqrt{(-\frac{\delta-\theta}{\gamma^2} + \frac{1}{2})^2 + \frac{2r}{\gamma^2}} \to \infty, \qquad m'' = -\frac{\frac{2r}{\gamma^2}}{n''} \to 0.$$

Hence, as $\theta \to \infty$,

$$H_1 = \frac{2c\mu}{\gamma^2(n'-\alpha)(\alpha-m')} \to 0$$
$$H_2 = -\frac{2\theta p}{\gamma^2(n'-1)(1-m')} = \frac{2p}{1+\frac{\delta}{\theta}-\frac{\gamma^2}{2\theta}+\sqrt{(-\frac{\delta}{\theta}+1+\frac{\gamma^2}{2\theta})^2+\frac{2r\gamma^2}{\theta^2}}} \to p$$
$$H_3 = \frac{2c\mu}{\gamma^2(n''-\alpha)(\alpha-m'')} \to 0$$
$$H_4 = \frac{2\theta(1-\lambda)p}{\gamma^2(n''-1)(1-m'')} = (1-\lambda)p.$$

Therefore,

$$\tilde{v}(k:\infty) = \begin{cases} B_1 + pk & k \le \tilde{k}_b \\ B_2 k^m + B_3 k^n + H k^\alpha & \tilde{k}_b < k < \tilde{k}_s \\ B_4 + p(1-\lambda)k & \tilde{k}_s \le k \end{cases}$$

Originally,

$$v(k) = \begin{cases} A_1 + pk & k \le k_b \\ A_2 k^m + A_3 k^n + H k^\alpha & k_b < k < k_s \\ A_4 + p(1-\lambda)k & k_s \le k \end{cases}$$

Constants in both cases are uniquely determined by same conditions. Furthermore, we can see $\tilde{k}_b \to k_b$, $\tilde{k}_s \to k_s$, $B_1 \to A_1$, $B_2 \to A_2$, $B_3 \to A_3$, $B_4 \to A_4$ as θ goes to ∞. Hence, $\tilde{v}$ converges to v as θ goes to ∞ pointwisely. With fixed ρ, the value function of singular controls of bounded velocity problem converges to the value function of corresponding singular controls problem. Furthermore, c_4 and c_5 are also converging to c_1 and c_2 because of $\tilde{k}_b \to k_b$, $\tilde{k}_s \to k_s$. Hence, the fixed point solution price ρ^* of the MFG with singular controls of bounded velocity converges to the fixed point solution price ρ^* of the MFG with singular controls. Therefore, the value function $\tilde{v}(k,\theta)$ of the MFG with singular controls of bounded velocity converges to the value function $v(k)$ of the MFG with singular controls as $\theta \to \infty$. □

Numerical examples

Numerical solutions of SMFGs Set $p = 0.5, \lambda = 0.6, \alpha = 0.6, \delta = 1, \gamma = 2, r = 3, c = 1$. Denote $\theta = \infty$ as the case of singular controls.

1) SMFG model with singular controls (4.12) (i.e. $\theta = \infty$) ; start with the initial price $\rho = 1$. Then, as in [39], $k_b = 0.069$ and $k_s = 4.83$. In this case, $c_2 = 4.89$ and $c_3 = 0.36$. Then, $c_3 c_2^{1-\alpha} = 0.673 < 1$. a_1 is the ratio between one unit of the production level and one unit of the price, so it will be very small number. So, $a_1 < 1$ and $a_1 c_3 c_2^{1-\alpha} < 1$. Set $a_0 = 1$ and $a_1 = 0.1$. Then, this model has a unique solution and the long run average price is 0.937 and $k_b^* = 0.058, k_s^* = 4.10$.

2) SMFG with bounded velocity (4.13); start with the price $\rho = 1$. Then, repeat step 2 and 3 until fixed point solutions. In this case, $c_5 = 11.46$ and $c_6 = 0.29$. Then, $c_6 c_5^{1-\alpha} = 0.78 < 1$. a_1 is the ratio between one unit of the production level and one unit of the price, so it will be very small number: $|a_1| < 1$ and $|a_1 c_3 c_2^{1-\alpha}| < 1$. Set $a_0 = 1$ and $a_1 = 0.1$. Then, $a_1 c_3 c_2^{1-\alpha} = 0.078 < 1$. Therefore, there exists a unique fixed point solution and the long run average price of the fixed point solution is $\rho^* = \frac{a_0}{1+a_1 c_6 c_5^{1-\alpha}}$

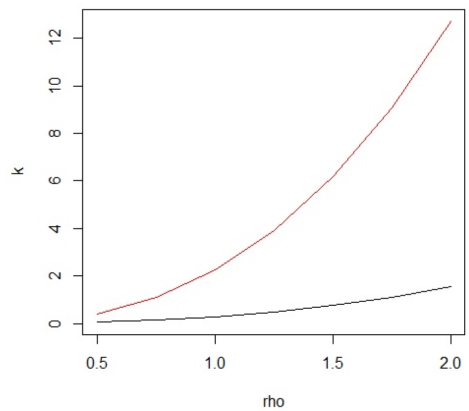
θ	50	100	200	300	400	500	1000	∞
$\tilde{k}_b^*$	0.0680	0.0634	0.0609	0.0600	0.0596	0.0593	0.0588	0.0583
$\tilde{k}_s^*$	9.4900	6.0819	4.9565	4.6412	4.4932	4.4074	4.2419	4.100
ρ^*	0.928	0.9325	0.9347	0.9355	0.9359	0.9361	0.9365	0.9369

Table 1: Change of optimal controls and prices of the MFG as $\theta \to \infty$

Single player models vs SMFG Changes of optimal controls with respect to the price ρ; set $p = 0.5, \lambda = 0.6, \alpha = 0.6, \delta = 1, \gamma = 2, r = 3, c = 1$, and $\theta = 1.1$. Optimal controls of the fixed point solution have two thresholds $\tilde{k}_b$ and $\tilde{k}_s$ as $\tilde{k}_b = c_4 \rho^{\frac{1}{1-\alpha}}$ and $\tilde{k}_s = c_5 \rho^{\frac{1}{1-\alpha}}$.

ρ	0.5	0.75	1	1.25	1.5	1.75	2
$\tilde{k}_b$	0.0484	0.1334	0.2739	0.4785	0.7548	1.1096	1.5493
$\tilde{k}_s$	0.3966	1.0928	2.2434	3.9190	6.1820	9.0885	12.6904

Table 2: Change of optimal controls as the price changes



The upper line is k_s and the lower line is k_b

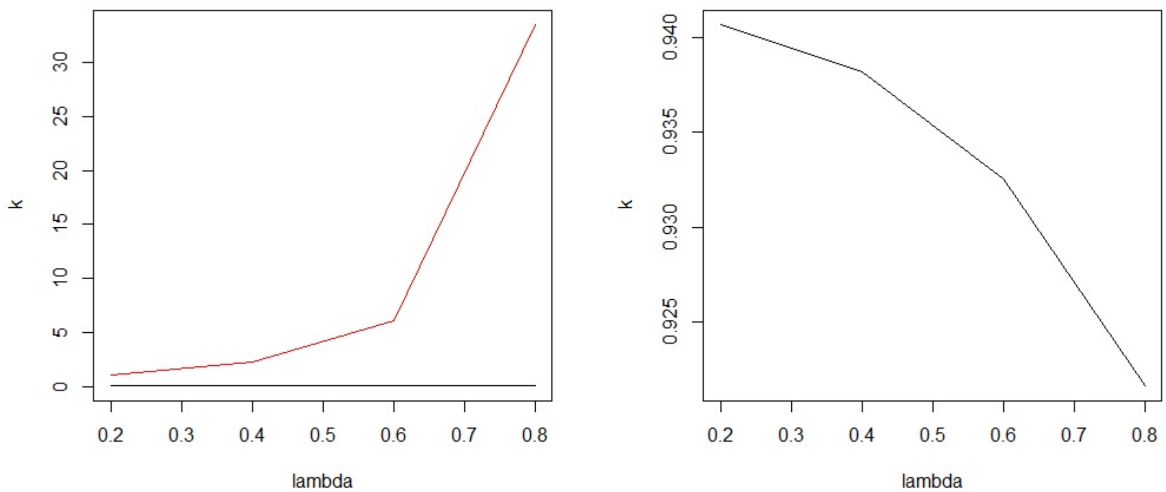
Figure 4.1: Change of optimal controls as the price changes

Hence, as ρ increases, $\tilde{k}_b, \tilde{k}_s$ also increase. Hence, the expansion region $[0, \tilde{k}_b)$ is larger and the contraction region (k_s, ∞) is smaller.
Furthermore, as ρ increases (i.e. the price goes up), the region of "no action" $[\tilde{k}_b, \tilde{k}_s]$ is larger. So, $\tilde{k}_s - \tilde{k}_b = (c_5 - c_4)\rho^{\frac{1}{1-\alpha}}$ and $c_5 > c_4$ are independent to ρ.) It can be interpreted in terms of the benefit of good price for the player. For higher price, there are more the region of "no action" $[\tilde{k}_b, \tilde{k}_s]$. So, the player does not need to expand or contract its capacity in many situations. Vice versa, if the price is low, then the "no action" region is small. So, the player has to control its capacity (expansion or contraction) more frequently.

Sensitivity analysis with respect to λ for the SMFG model (4.13) Set $p = 0.5, \alpha = 0.6, \delta = 1, \gamma = 2, r = 3, c = 1$, and $\theta = 100$. If λ is close to 1, the model is more irreversible. Hence, the player unlikely controls its capacity since the investment is irreversible. In contrast, if λ is close to 0, the model is more reversible. Hence, the player more likely controls its capacity. So, this case is more favourable to the player, and the optimal price will be "good price" for the player.

λ	0.2	0.4	0.6	0.8
$\tilde{k}_b^*$	0.0927	0.0726	0.0634	0.0583
$\tilde{k}_s^*$	1.0807	2.2331	6.0820	33.4333
ρ^*	0.9406	0.9382	0.9325	0.9217

Table 3: Change of optimal controls and prices as λ changes



(a) The upper line is k_s^* and the lower line is k_b^* (b) ρ^* vs λ

Figure 4.2: Change of optimal controls and prices as λ changes

The red line in the left graph represents $\tilde{k}_s^*$ and the black line in the left graph represents $\tilde{k}_b^*$. So, as λ is close to 1, the waiting region $[\tilde{k}_b, \tilde{k}_s]$ is larger and the optimal price decreases.

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Appendix

A.1. Viscosity solution

The definition of viscosity solution to (1.4) is as follows

Definition 10. *$\hat{v}$ is called a* viscosity solution *to* (1.4) *if $\hat{v}$ is both a viscosity supersolution and a viscosity subsolution, with the following definitions,*
(i) viscosity supersolution: for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ and any $\vartheta \in \mathcal{C}^{1,2}$,
if (t_0, x_0) is a local minimum of $v^i - \vartheta$ with $v(t_0, x_0) - \vartheta(t_0, x_0) = 0$, then

$$-\partial_t \vartheta(t_0, x_0) - \inf_{\alpha \in \mathcal{A}} \{ \frac{1}{2} tr(\sigma(t_0, x_0, \mu, \alpha) \sigma(t_0, x_0, \mu, \alpha)^T D_{xx} \vartheta(t_0, x_0)) \\ + b(t_0, x_0, \mu, \alpha) \cdot D_x \vartheta(t, x) + f(t, x, \mu, \alpha) \} \geq 0,$$

and $\vartheta(T, x_0) \geq h(x_0, \mu_T)$ for $\forall x_0 \in \mathbb{R}^d$;
(ii) viscosity subsolution: for any $(t_0, x_0) \in [0, T] \times \mathbb{R}$ and any $\vartheta \in \mathcal{C}^{1,2}$,
if (t_0, x_0) is a local maximum of $\hat{v} - \vartheta$ with $\hat{v}(t_0, x_0) - \vartheta(t_0, x_0) = 0$, then

$$-\partial_t \vartheta(t_0, x_0) - \inf_{\alpha \in \mathcal{A}} \{ \frac{1}{2} tr(\sigma(t_0, x_0, \mu, \alpha) \sigma(t_0, x_0, \mu, \alpha)^T D_{xx} \vartheta(t_0, x_0)) \\ + b(t_0, x_0, \mu, \alpha) \cdot D_x \vartheta(t, x) + f(t, x, \mu, \alpha) \} \leq 0,$$

and $\vartheta(T, x_0) \leq h(x_0, \mu_T)$ for $\forall x_0 \in \mathbb{R}^d$.

A.2. Pareto optimality and Nash equilibrium

Consider the N player game

$$\inf_{\alpha^i \in \mathcal{A}} J_N^i(\alpha^1, \cdots, \alpha^N) = \inf_{\alpha^i \in \mathcal{A}} E \left[\int_0^T \frac{1}{N} \sum_{j=1}^N f_0(t, X_t^i, X_t^j, \alpha_t^i) dt + \frac{1}{N} \sum_{j=1}^N h_0(T, X_T^i, X_T^j) \right], \tag{4.15}$$

with dynamics

$$dX_t^i = \frac{1}{N} \sum_{j=1}^N b_0(t, X_t^i, X_t^j, \alpha_t^i) dt + \frac{1}{N} \sum_{j=1}^N \sigma_0(t, X_t^i, X_t^j, \alpha_t^i) dW_t^i, \qquad X_0^i = x^i \in \mathbb{R}^d,$$

where $\mathcal{A}$ is the admissible set of controls. $\mathcal{A} = \{\alpha_t : \alpha_t \text{ is } \mathcal{F}-\text{adapted and square integrable.}\}$. One needs to precise to define the filtration $\mathcal{F}$ of controls for adeptness. If the controls are open loop, then $\{\alpha_t\}$ is $\mathcal{F}^{W_{[0,t]}}$-adapted and α_t is the function of t and $W_{[0,t]}$. If the controls are closed loop, then $\{\alpha_t\}$ is $\mathcal{F}^{X_{[0,t]}}$-adapted and α_t is the function of t and $X_{[0,t]}$. If the controls are closed loop in feedback form, then $\{\alpha_t\}$ is $\mathcal{F}^{X_t}$-adapted and α_t is the function of t and X_t which α_t is Markovian control. In this thesis, we consider the controls as closed loop in feedback form, so we could write $\alpha_t = \phi(t, X_t)$ for some function ϕ.

There are two types of equilibrium in this game: Pareto optimality and Nash equilibrium.

Definition 11 (Pareto optimality)**.** *Then, $\{\alpha_t^{i*}\}_{i=1}^n$ is called a* Pareto optimal *of the N player game if for any i*

$$J_N^i(\alpha^{1*}, \cdots, \alpha^{N*}) = \inf_{\alpha^i \in \mathcal{A}} J_N^i(\alpha^1, \cdots, \alpha^N).$$

Definition 12 (Nash equilibrium)**.** *Then, $\{\alpha_t^{i*}\}_{i=1}^n$ is called a* Nash equilibrium *of the N player game if for any $i \in \{1, 2, \ldots, n\}$ and any $\alpha_t^{i'} \in \mathcal{A}$, $J_N^i(\alpha_t^{i'}, \alpha^{*-i}) \geq J_N^i(\alpha_t^{i*}, \alpha^{*-i})$, where J_N^i is the cost function for the ith player and α^{*-i} is the control processes $\{\alpha_t^{j*}\}_{j=1, j\neq i}^N$ by all players except the ith player.*

Pareto optimality is an optimal solution based on all players' behaviour, but Nash equilibrium is an optimal solution for a player given that all other players behaviours remain the same. A Pareto optimality is also one of the Nash equilibria, but a Nash equilibrium is not necessary to be a Pareto optimality. ϵ-Nash equilibrium is an approximation to the Nash equilibrium. The definition of ϵ-Nash equilibrium is following.

Definition 13 (ϵ-Nash equilibrium)**.** *Then, $\{\alpha_t^{i*}\}_{i=1}^n$ is called an* ϵ-Nash equilibrium *of the N player game if for any $i \in \{1, 2, \ldots, n\}$ and any $\alpha_t^{i'} \in \mathcal{A}$, $J_N^i(\alpha_t^{i'}, \alpha^{*-i}) \geq J_N^i(\alpha_t^{i*}, \alpha^{*-i}) - \epsilon$, where J_N^i is the cost function for the ith player and α^{*-i} is the control processes $\{\alpha_t^{j*}\}_{j=1, j\neq i}^N$ by all players except the ith player.*

Existence of the Nash equilibrium Because of N player game's complexity, finding a Nash equilibrium or a Pareto optimality is not simple. There are previous works on existence of the Nash equilibrium in N player stochastic differential games. In [79], the Nash equilibrium exists if the N player stochastic differential game satisfies the Nash condition. The N player game (4.15) satisfies the Nash condition, so it has a Nash equilibrium.

Definition 14 (Nash condition)**.** *Consider the following N player game which has a cost function for the player i as*

$$J(\alpha^1, \cdots, \alpha^N) = E[\int_0^T f_i(t, x_t, \alpha^1, \cdots, \alpha^N)dt + h(x_T)],$$

and dynamics as $dx_t = b(t, x_t, \alpha^1, \cdots, \alpha^N)dt + \sigma dW_t, \quad x_0 = x_0$. Controls $\alpha^i(t, x)$ are Markov feedback controls. Define the Hamiltonian for each $i = 1, 2, \cdots, N$ as

$$H_i(t, x, p_i : \alpha_t^1, \cdots, \alpha_t^N) = p_i b(t, x_t, \alpha^1, \cdots, \alpha^N) + f_i(t, x_t, \alpha^1, \cdots, \alpha^N).$$

The Nash condition holds if there exists processes α_t^{i*} *such that for all* $i = 1, \cdots, N$,

$$H_i(t, x, p_i : \alpha^{1*}, \cdots, \alpha^{N*}) \leq H_i(t, x, p_i : \alpha^{1*}, \cdots, \alpha^{i-1*}, \alpha^i, \alpha^{i+1*}, \cdots, \alpha^{N*}),$$

for all $(t, x, \alpha^1, \cdots, \alpha^N, p_1, \cdots, p_N)$.

A.3. Schauder fixed point theorem

Proposition 8 (Schauder fixed point theorem)**.** *Let* M *be a nonempty convex subset in the locally convex topological vector space. Then, any continuous mapping* $F : M \to M$ *has a fixed point* x *such that* $F(x) = x$ *if* $F(M) \subset M$ *is compact. Furthermore, if the mapping is contraction i.e. for all* $x, y \in M$, $d(F(x), F(y)) \leq \lambda d(x, y)$ *with* $0 \leq \lambda < 1$, *then the fixed point is unique.*

A.4. Relationship between singular controls and controls with bounded velocity - approximation of the value function

We study the relationship between singular controls and singular controls of bounded velocity which is previously studied in [44]. Define two HJB equation F_θ and F as

$$\begin{aligned} 0 &= F_\theta(\partial_{xx}v, \partial_x v, \partial_t v, v, t, x) = \partial_t v + \mathcal{L}v + f + \theta \min\{0, g_1 + \partial_x v, -\partial_x v + g_2\} \\ 0 &= F(\partial_{xx}v, \partial_x v, \partial_t v, v, t, x) = \min\{\partial_t v + \mathcal{L}v + f, g_1 + \partial_x v, -\partial_x v + g_2\} \end{aligned}$$

where $\mathcal{L}$ is a some second order uniformly elliptic partial differential operator, f is a bounded and Lipschitz continuous function, and two positive constants $p > q$. g is continuous function.

We will study the relationship between solutions to F_θ and F. Consider new sequence of functions

$$\begin{aligned} &\tilde{F}_\theta(\partial_{xx}v, \partial_x v, \partial_t v, v, t, x) \\ &= [\partial_t v + \mathcal{L}v + f] \cdot [\frac{\partial_t v + \mathcal{L}v + f}{\theta} + g_1 + \partial_x v] \cdot [\frac{\partial_t v + \mathcal{L}v + f}{\theta} - \partial_x v + g_2], \\ &\tilde{F}(\partial_{xx}v, \partial_x v, \partial_t v, v, t, x) = [\partial_t v + \mathcal{L}v + f] \cdot [g_1 + \partial_x v] \cdot [-\partial_x v + g_2]. \end{aligned}$$

Then, $\tilde{F}_\theta$ goes to $\tilde{F}$ as $\theta \to \infty$ uniformly on compact sets, and there is a large enough $\Theta > 0$ such that for any $\theta > \Theta$, $\tilde{F}_\theta, \tilde{F}$ are uniformly elliptic partial differential equations.

Theorem 6. *Let* $\Theta > 0$ *be a large enough constant such that for any* $\theta > \Theta$, $\tilde{F}_\theta$ *are uniformly elliptic partial differential equations. Assume that* $\tilde{F}_\theta$ *has at most one viscosity solution for any* $\theta > \Theta$ *and* $\tilde{F}$ *also has at most one viscosity solution. If for each* $\theta > 0$ $F_\theta = 0$ *has a unique classical* $\mathcal{C}^{1,2}$ *solution* v_θ *and* $F = 0$ *also has a unique classical* $\mathcal{C}^{1,2}$ *solution* v, *then* v_θ *converges to* v *uniformly on compact set as* $\theta \to 0$.

Proof. $\tilde{F}_\theta$ has at most one viscosity solution and F_θ has a classical $\mathcal{C}^{1,2}$ solution v_θ. Then, $0 = F_\theta(v_\theta) = \partial_t v_\theta + \mathcal{L}v_\theta + f + \theta \min\{0, g_1 + \partial_x v, -\partial_x v + g_2\}$. So, $0 = [\partial_t v_\theta + \mathcal{L}v_\theta - \Pi] \cdot [\frac{\partial_t v_\theta + \mathcal{L}v_\theta + f}{\theta} + g_1 + \partial_x v)] \cdot [\frac{\partial_t v_\theta + \mathcal{L}v_\theta + f}{\theta} + (-\partial_x v + g_2)] = \tilde{F}_\theta(v_\theta)$. Hence, v_θ is also a classical $\mathcal{C}^{1,2}$ solution to $\tilde{F}_\theta$, and it is a unique viscosity solution to $\tilde{F}_\theta$. Similarly, v is also a classical $\mathcal{C}^{1,2}$ solution to $\tilde{F}$, and it is a unique viscosity solution to $\tilde{F}$.
Let's use following lemmas.

Lemma 11. *(stability of viscosity solutions) Let $\{G_n(M, p, q, r, t, x)\}_{n=1}^{\infty}$ be a sequence of functions and assume $G_n \to G$ uniformly on compact sets for some function G. Let v_n be a viscosity solution to $G_n(\partial_{xx}v_n, \partial_x v_n, \partial_t v_n, v_n, t, x) = 0$. If $v_n \to u$ uniformly on compact sets, then v is a viscosity solution to $G(\partial_{xx}v_n, \partial_x v_n, \partial_t v, v, t, x) = 0$.*

Lemma 12. *v_θ are classical $\mathcal{C}^{1,2}$ solutions to $\tilde{F}_\theta = 0$. Then, v_θ converges to some function w uniformly on compact sets.*

Proof. Since $\mathcal{L}$ is uniformly elliptic, we can use the maximum principle.
Hence, $\| v_\theta \|_\infty \leq \| f \|_\infty$. Let's replace $f(x)$ to $H(x) = f(x + h)$ for any $h > 0$. Then, $u_\theta(x + h)$ is a solution to $\mathcal{L}u_\theta + H + \theta \min\{0, g_1 + \partial_x v, -\partial_x v + g_2\} = 0$ and

$$\| v_\theta(\cdot) - v_\theta(\cdot + h) \|_\infty = \| v_\theta - u_\theta \|_\infty \leq \| f - H \|_\infty = \| f(\cdot) - f(\cdot + h) \|_\infty \leq C|h|$$

because f is Lipschitz. Therefore, $\{v_\theta\}$ is bounded and equicontinuous. By Arzela-Ascoli theorem, there exists w such that $v_\theta \to w$ as $\theta \to \infty$. □

By lemma 12, v_θ converges to some function w. Then, by lemma 11, w is a viscosity solution to $\tilde{F}$. However, $\tilde{F}$ has a unique viscosity solution v, so $w = v$. Therefore, v_θ converges to v. □

F_θ is the HJB equation for stochastic controls of bounded velocity and F is the HJB equation for singular controls. Hence, if we know that two HJB equations have classical solutions and if they are value functions, the value function of bounded velocity converges to the value function of singular controls.