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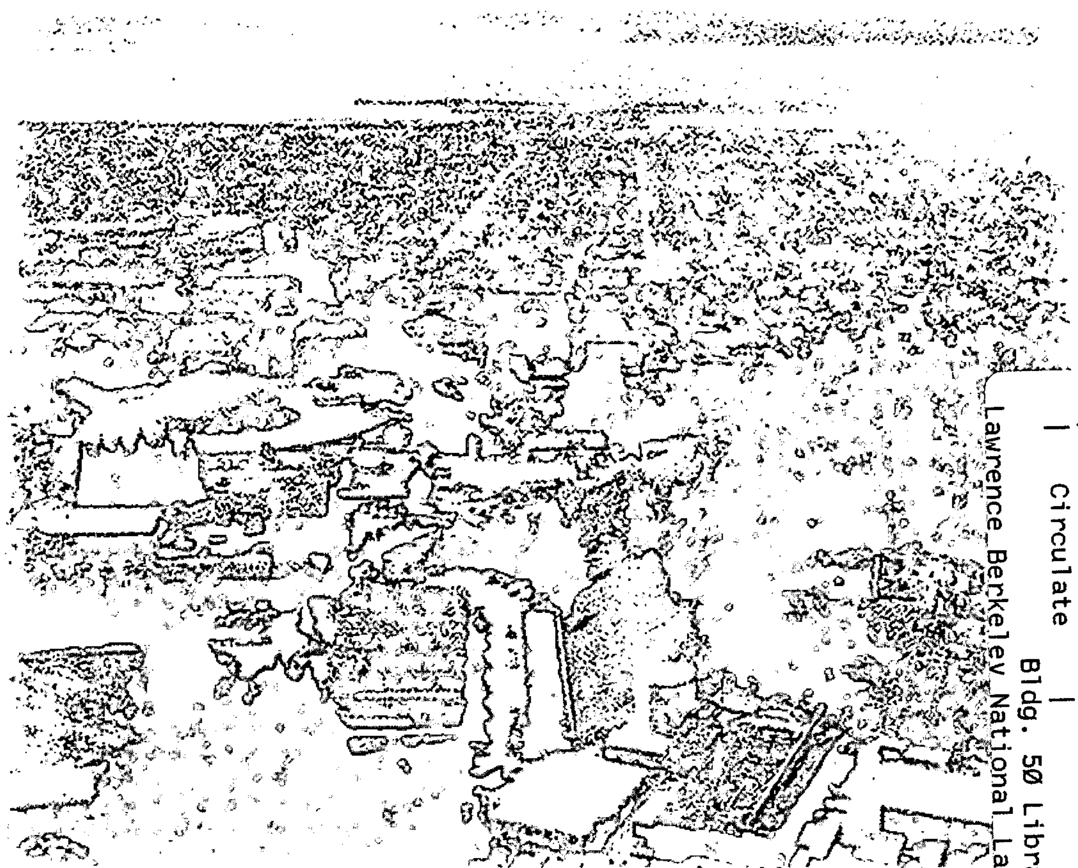

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ABSTRACT

Starting from the contributions of Ohm, Fick and Joule during the nineteenth century, an integral expression is derived for a steady-state groundwater flow system. In general, this integral statement gives expression to the fact that the steady-state groundwater system is characterized by two dependent variables, namely, flow geometry and fluid potential. As a consequence, solving the steady-state flow problem implies the finding of optimal conditions under which flow geometry and the distribution of potentials are compatible with each other, subject to the constraint of least action. With the availability of the digital computer and powerful graphics softwares, this perspective opens up possibilities of understanding the groundwater flow process without resorting to the traditional differential equation. Conceptual difficulties arise in extending the integral expression to a transient groundwater flow system. These difficulties suggest that the foundations of groundwater hydraulics deserve to be reexamined.

INTRODUCTION

This paper has been prepared to honor Shlomo Neuman on his sixtieth birthday, in recognition of his many insightful contributions to hydrogeology. On an occasion such as this it is useful to give ourselves the opportunity to address intriguing issues and ideas that we find it difficult to fit into our normal schedule of research work. In this spirit, this paper takes a look at the conceptual-mathematical foundations of groundwater movement from a perspective which differs from that of the traditional differential equation. Accordingly, we start with the contributions of Ohm, Fick and Joule during the nineteenth century and, invoking the principle of least action, we physically formulate an integral statement of the steady-state groundwater flow process. Although the integral statement coincides with the variational statement for steady-state diffusion, the derivations and discussions presented offer useful insights into appreciating the groundwater flow process from an unusual perspective.

HISTORICAL BACKGROUND

It is widely known that the equation governing the steady and non-steady flow of groundwater stems from Fourier's heat conduction equation, by an analogy between the flow of heat in solids and the flow of water in porous materials. Fourier proposed the differential equation of heat diffusion in an unpublished report to the French Academy of Science in 1807. Surprisingly, Fourier's work was met with resistance and would not become available to the general scientific community until some fifteen years later (Fourier, 1822). Once it became known, Fourier's work developed to be one of the most important contributions of modern mathematical physics (Narasimhan, 1998). Specifically, Ohm (1827) and Fick (1855) were directly inspired by Fourier in formulating their equations for the steady flow of direct current and the diffusion of molecules in liquids respectively. Yet, Ohm and Fick introduced their own creativity in the way they took Fourier's mathematical statement in applying Fourier's model to the physical systems they were dealing with.

At a philosophical level, two schools of thought prevailed during the early nineteenth century concerning the description of physical systems. These schools were inspired by two giants of modern science, Newton and Leibniz. The physical foundations of force and momentum and the infinitesimal calculus of Newton led to the mechanistic school which was committed to the notion that the physical world could be fully described and understood by analyzing all the forces which occur at a point. One of the major proponents of this view was Laplace. Clearly, forces at a point needed to be resolved in three dimensions and this need led to multidimensional differential equations and the tensor calculus. On the other hand Leibniz led the analytic school which addressed the total system behavior in terms of work and action, leading to integrals involving scalar quantities. Lagrange and Hamilton belonged to this school. Although differing fundamentally in the way they perceived the statement of the problem, these approaches led ultimately to the same result, but needed different tools of analysis. In particular the approach of the analytic school led to variational principles (Lanczos, 1970).

THE THREE LAWS

Ohm's Law

In 1827 Georg Simon Ohm published a lengthy pamphlet on the steady flow of electric current in galvanic circuits which is considered to be a work of fundamental importance in electricity. In this work, Ohm conducted a large number of experiments on the steady flow of electric current through different electrical conductors, whose length and area of cross section were variable. Each conductor experimented had a uniform cross sectional area throughout its length. Although he was admittedly inspired by Fourier's heat conduction equation in interpreting his experimental results, Ohm departed from Fourier in the way he presented his equation. Fourier (1807, 1822) defined thermal conductivity from mathematical considerations in terms of a *thermal gradient*,

(1) $$Q_x = -K\frac{\partial T}{\partial x}A$$

where Q_x is the steady of heat per unit time in the x direction, T is temperature and A is area of cross section. In deriving the equation governing heat flux, Fourier was concerned with infinitesimals and was naturally led to a flux law (1) in terms of a gradient. However, Ohm was an experimentalist who dealt with specimens of finite shape and size. In presenting and interpreting the observations, Ohm chose to express the physical property of the sample as an entity. Thus he stated,

(2) $$I = \frac{\Delta V}{R},$$

where, I is the current, ΔV is the drop in electrical potential over the length of the conductor and R is the electrical resistance of the conductor. The resistance R in (2) is a function of the shape and size of the conductor as well as the material making up the conductor. In other words R is an integrated quantity. In a sense, therefore, Ohm's Law is intrinsically an integral statement of the flux law.

Fick's Law

Adolf Fick (1855) saw in Fourier's heat conduction equation a conceptual model for describing the diffusion of solutes in dilute solutions. Fick went on to demonstrate applicability of Fourier's equation to liquid diffusion through a novel diffusion experiment conducted on a vessel with the shape of an inverted cone with apex truncated. In other words, Fick dealt with a flow tube of variable cross sectional area. To interpret the experiment, Fick (1855) wrote the non-steady state diffusion equation for a tube of non-uniform cross sectional area thus,

(3) $$D\left(\frac{\partial^2 c}{\partial x^2} + \frac{1}{A}\frac{dA}{dx}\frac{\partial c}{\partial x}\right) = \frac{\partial c}{\partial t} \quad ,$$

where D is chemical diffusivity, c is volumetric aqueous concentration and A is area of cross section perpendicular to the flow path. If we consider a steady-state system in which the right-hand side of (3) is zero, then, the left-hand side of (3) may be viewed as a differential form of Ohm's Law, valid for a flow tube whose cross sectional area is variable along the flow path.

Joule's Law

An important development in physics during the early nineteenth century was the recognition that all forms of energy (heat, electricity, magnetism, mechanical work) were the same. A major contribution in this regard was that of Joule, who expressed the equivalence between electrical energy expended in the flow of electricity through a resistor and mechanical work. Maxwell (1888), expressed Joule's Law in a very intuitive way as follows,

(4) Heat generated measured in dynamical units =

Square of current X Resistance X Time.

Or, in view of (2) and (4) we equivalently have,

(5) $$W = R I^2 \Delta t \quad ,$$

where, W is the work done over an interval of time Δt. In view of (2), we may rewrite (5) conveniently as,

(6) $$W = \frac{(\Delta V)^2}{R} \Delta t \quad .$$

We will see how Joule's Law expressed in the form of (6) can help us formulate an integral statement of the steady-state groundwater flow process.

PRINCIPLE OF LEAST ACTION

In view of Ohm's Law and Joule's Law described above, let us proceed to consider a groundwater system in which water is flowing in a steady state, compatible with the existing boundary conditions. Following Hubbert (1940) we assume that groundwater flows in the direction of decreasing potential Φ, where the potential is defined as mechanical energy contained in a unit mass of water under isothermal conditions. Consider now a steady groundwater flow system with potentials prescribed over different segments of its boundary. On some boundary segments the potential will be higher, and on some, it will be lower. Water entering the system across a boundary segment will bring energy into the system at a steady rate and water which leaves the system across other boundaries will carry energy out of the system at a steady rate. Let Q_k the *mass flux* (with dimensions of mass per unit time) and Φ_k be the potential at the k^{th} boundary segment. Then, by virtue of the definition of potential, the energy E_k brought into the system across the k^{th} boundary segment over an interval of time is given by,

(7)
$$E_k = Q_k \Phi_k \Delta t \quad .$$

Note that Q is positive if water is flowing into the system and negative if otherwise. Thus, in a steady state system, if we algebraically sum up the energy entering the system over all the boundary segments, we will get a net excess energy. This net excess energy is the work done by water as it moves through the groundwater system, overcoming frictional resistance. This work is expended as heat and we will assume that the effect of the temperature rise over the system on the movement of water is negligible.

We now postulate that in response to the forces acting on the boundaries, the steady-state system has adjusted itself in such a way that the work done over a given interval of time (that is, the sum of E_k over all the k segments) is an extremum. We say "an extremum" because of two possibilities: (i) if potentials are prescribed on the all the boundary segments, a maximum amount of water will flow through the system, resulting in a maximum amount of work being done in response to the imposed forces, and, (ii) if fluxes are prescribed on all the boundaries, the system will organize itself in such a way that the mass flux through the system will be accomplished with a minimum amount of work. This is the principle of least action.

We now need to translate this principle of least action into a mathematical statement in the form of an integral for steady groundwater flow. To do this we carry out a volume integration in the interior of the flow domain rather than a surface integration over the surface the encloses the domain. Alternatively, we may also divide the steady-state flow region into a number of flow tubes and sum up the work done in each flow tube in light of Joules Law.

Consider a steady-state flow domain as shown in Figure 1 with potentials prescribed on five boundary segments. Let the flow domain be divided into i = 1, 2, 3 .... I flow tubes. Let each flow tube be divided into j = 1, 2, 3 ..... J isopotential intervals. Let $\Delta\Phi_{ij}$ denote the drop in potential over the j[th] of the i[th] flow tube. Then, according to Joule's Law, which is also applicable to the flow of groundwater, the work done W_{ij} is given by,

(8)
$$W_{ij} = \frac{(\Delta\Phi_{ij})^2}{R_{ij}} .$$

If the length of the flow tube segment ij, Δx_{ij} is small, R_{ij} can be approximated by,

(9)
$$R_{ij} \cong \frac{\Delta x_{ij}}{\rho K A_{ij}} ,$$

where ρ is density of water K is a coefficient related to hydraulic conductivity and A_{ij} is the area of cross section. In (9) the appearance of the density of water is due to the fact that we are dealing with mass flux (rather than volumetric flux) of water. Therefore, the total work done over the region, W is,

(10)
$$W = \sum_i^I \sum_j^J \frac{(\Delta\Phi_{ij})^2}{R_{ij}} = \sum_i^I \sum_j^J \frac{\rho K (\Delta\Phi_{ij})^2 A_{ij}}{\Delta x_{ij}} .$$

Noting that $A_{ij}\ \Delta x_{ij} = \Delta V_{ij}$, the volume of the segment ij, and letting I and J tend to infinity, the summation in (10) can be replaced by an integral,

(11) $$W = \int_V \rho K (\nabla\Phi)^2 dV \quad .$$

We recognize that (11) is the variational principle pertaining to the steady flow of groundwater .

DISCUSSION

An interesting aspect of the above derivation is that we have described the behavior of the steady-state groundwater flow system purely in terms of an integral, without reference to a differential equation. Fundamental to this derivation are the concepts of resistance and potential, both of which are experimentally measurable quantities. Thus, we have gone directly from measurable, experimental quantities, through the notions of work and energy to a governing integral statement of the groundwater flow problem. However, the end product (11) is not an equation (such as the differential equation) but a quantity that needs to be minimized.

It is pertinent here to examine why we get an equation when the steady-state flow problem is stated in terms of infinitesimals while we get an optimization principle when the same problem is stated in the form an integral. Note that the differential equation of steady state groundwater flow has only one dependent variable, namely potential. For purposes of obtaining analytic solutions, we make the tacit assumption that the flow geometry is known. In general, however, when the flow domain has non-trivial geometry (shape and size) and is occupied by more than one material (heterogeneous), the flow pattern will be characterized by converging and diverging flow lines whose dispositions are not known a priori. In these cases, the steady state groundwater flow problem is distinguished by two dependent variables, flow geometry and fluid potential. When the problem is characterized by more than one mutually dependent variables, it is not any more possible to write an explicit equation. Rather, one has to find an optimum situation under which the mutually dependent variables are in harmony with each other.

Let us examine how, in practice, such an optimization of two mutually dependant variables may be achieved. Consider a steady-state groundwater flow system of arbitrary shape with potentials prescribed on several segments of the boundary. An example is shown in Figure 1 in which the flow region is subject to prescribed boundary potentials on five segments. Given sufficient time, the system can organize itself into an infinite number of flow configurations; four of these are schematically shown in Figures 2a through 2d. In Figure 2a water enters the flow region through one inlet and leaves the flow region at four outlets. In Figure 2d, on the contrary, water enters the flow region across four different boundary segments and exits at one outlet. The particular flow geometry preferred by the system will be dictated by the least-action postulate. Note, in figures 2a through 2d, that the flow system comprises three or four subsystems, each being a large flow tube of arbitrary shape. Recall that the rate of work done in a flow tube is equal to the product of the mass flux through the tube and the potential drop over the tube. Therefore, the least-action postulate requires that,

$$(12) \qquad W^* = \sum_i Q_i \Delta\Phi_i = \frac{(\Delta\Phi_i)^2}{R_i}, \quad i = I, II, III, IV$$

be a maximum, given that the potential, Φ, has been prescribed on the boundary. Note that the magnitude of hydraulic resistance depends on the *geometry of a flow tube* as well as the physical nature of the materials occupying the flow tube. Thus, given a certain material distribution within the flow region, the system has the freedom to adjust the geometry of the flow tubes in such a way that the self-organization postulate is satisfied. It follows therefore that the particular flow geometry preferred by the system (figures 2a through 2d) will depend on the spatial distribution of materials of varying hydraulic resistivity (heterogeneity) occupying the flow region. In heterogeneous media, flow lines will refract at the interface between materials of contrasting hydraulic resistivity according to a law of tangents (Hubbert, 1940).

Thus, if we wish to use (11) as the basis of solving the steady-state groundwater flow problem, we need to follow a solution strategy that is very different from what we are normally used to. Essentially, we have to start with a set of assumed flow tubes and calculate the resistance of each tube between its inlet and outlet as dictated by its geometry and material make up. Using the resistances so calculated and the potentials at the ends of the tubes, it is easy to calculate the total work done on the system pertaining to the particular flow geometry considered. The task now is to progressively adjust the flow geometry and recalculate the total work done, until the calculated value of total work done attains an extremum value.

Furthermore, because we are concerned here with work and its relation to resistance, it stands to reason that the derivation presented above is valid for heterogeneous systems as well as systems in which the hydraulic conductivity is a known function of the potential.

If the ideas presented above look remarkably similar to the construction of flow nets pioneered by Forcheimer (1886) in the late nineteenth century, they are. The primary difference is that in drawing the flow nets, one ensures by trial and error that the isopotentials are everywhere perpendicular to the flow lines. It does seem logical to conclude that a steady-state flow problem solved either drawing a flow net or by minimizing work done should lead to the same result. However, to prove this by rigorous mathematics may not be an easy task. Also we must recognize that flow nets are drawn only in two dimensions, whereas the work minimization method is, in principle, applicable in two and three dimensions.

From a practical point of view, one could argue that although the work minimization method appears interesting, it is not easy to implement. Such an argument would have been valid a quarter of a century ago, when our ability to store and retrieve large quantities of information as well as carry out computations with rapidity were very limited. In the absence of fast computational devices, our best course of action was to analytically solve the partial differential equations, tractable solutions being limited to simple flow systems within which the flow pattern is known *a priori* (e.g. unidimensional, radial, spherical).

With the availability of exceptionally powerful desk-top computers and graphics capabilities, we are now in a position to consider for obtaining mathematical solutions, complex systems within which the flow pattern may be arbitrarily complex. In these systems, the central task of the solution process is to calculate flow resistances along arbitrarily-shaped convergent and divergent flow tubes. One could argue therefore that the next great challenge in computationally solving problems of groundwater flow is to harness geometry, by storing and retrieving information on flow geometry and use that information on calculating resistances. Indeed, our ability to harness complex geometries may enable us to negate one of the paradigms of current numerical modeling practice, namely, that finer and finer mesh discretization leads to better and better accuracy. This paradigm is valid if our focus is the evaluation of gradients because, by definition, gradient is an infinitesimal concept. However, if we focus attention on flow geometry and resistances, the notion of a gradient is unnecessary. Therefore, the ideas presented above lead us to a computational basis in which finer and finer discretization of space and time are not needed to obtain more and more accurate solutions.

Although the groundwater flow equation is formulated by analogy with the heat equation of Fourier, we cannot but fail to note a paradoxical difference. Note that the notion of potential is very well defined in the case of groundwater. Therefore, we have been naturally led to understand the flow of water in terms of work and energy. However, in the case of heat, temperature is not truly a potential, although it is an intensive quantity. We do not think of heat as a material permeant and we do not consider temperature to be energy per unit entity of the permeant. Thus, the import of a variational principle analogous to (11) for heat must be physically interpreted in a different way from the one we have followed above.

So far, the physics and the mathematics have proceeded nicely, hand in hand. However, fundamental problems arise when we go beyond steady-state groundwater systems to non-steady systems. In the steady-state case, the variational principle and the partial differential equation are equivalent in describing the same physical system. Not so in the case of non-steady flow. We do not as yet have a principle analogous to the least action postulate to describe why a transient groundwater system evolves in time in one particular way rather than any other. Stated differently, we do not as yet have a physically meaningful variational principle for the transient groundwater flow process in terms of work and energy. It is recognized here that a variational principle has been proposed for the transient groundwater flow problem (Gurtin, 1964; linear diffusion equation), purely on mathematical considerations. Although Gurtin's variational principle, upon minimization, yields the linear heat conduction equation, its physical meaning is not entirely obvious. Thus, even as we use mathematical methods to solve practical problems in groundwater hydrology, it is worth our energies to examine the foundations on which these methods rest.

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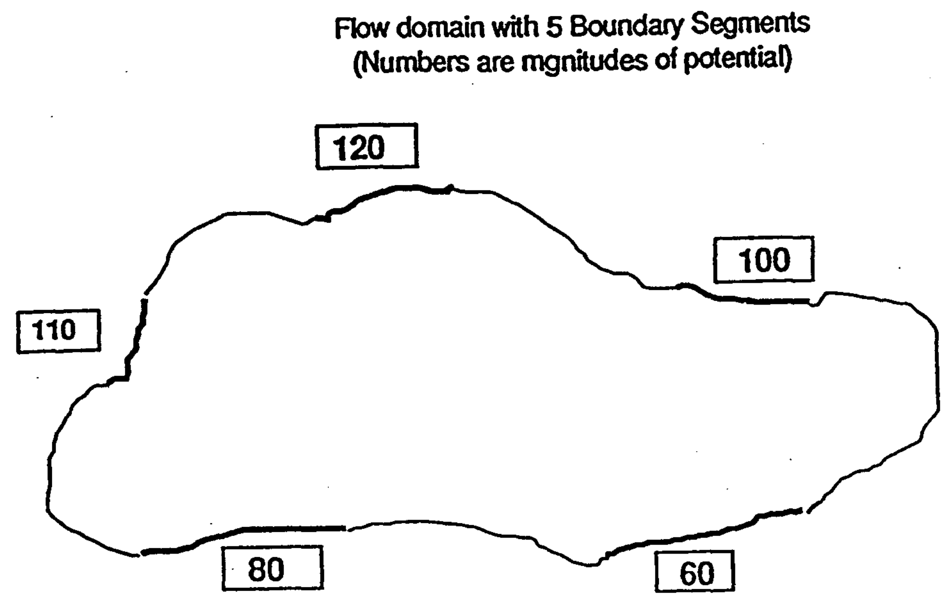

Figure 1: A flow domain with 5 boundary segments on which potentials are prescribed

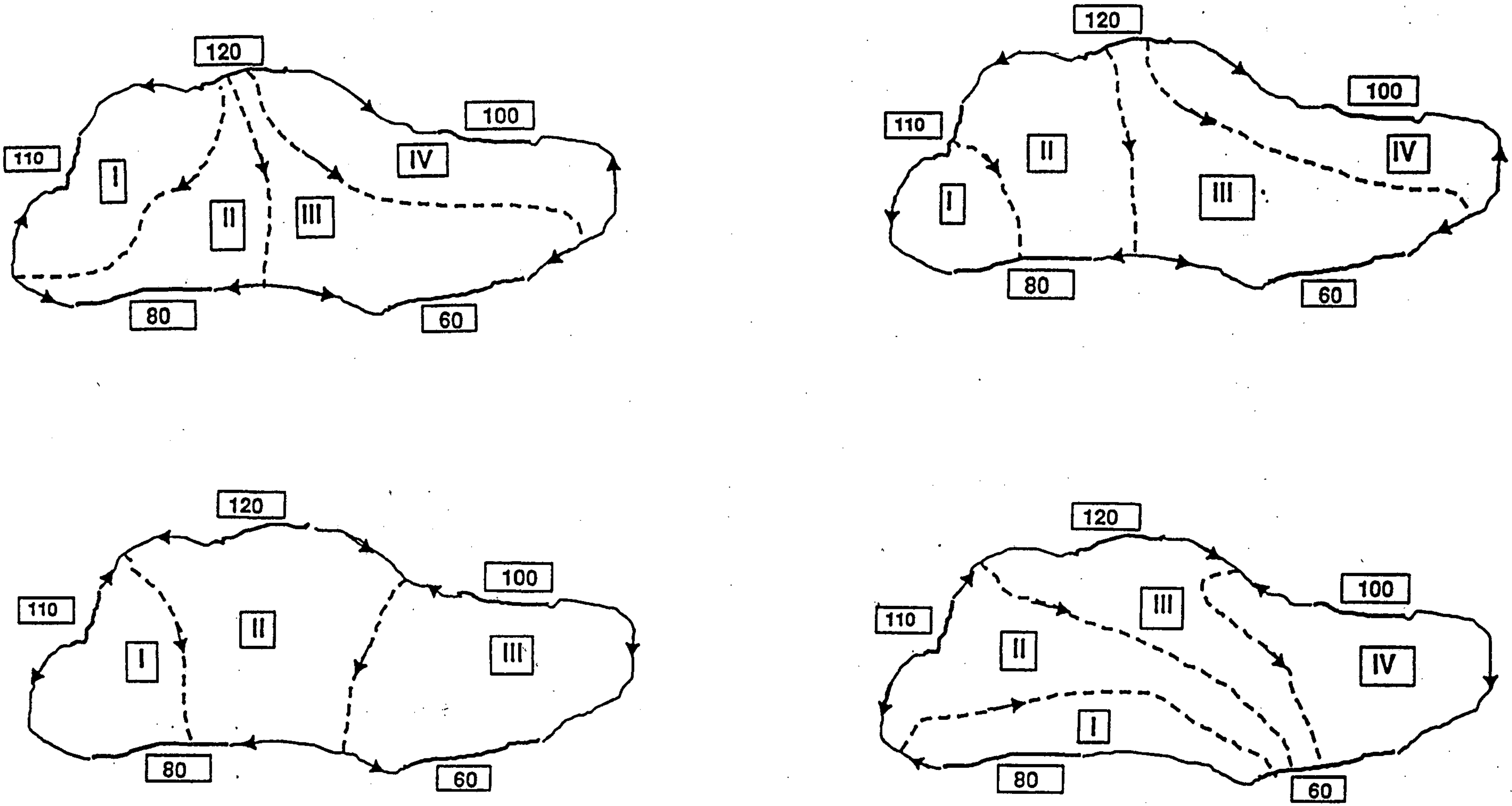

Figure 2: Four possible flow configurations, (A) One inlet and four outlets; (B) Two inlets and 3 outlets; (C) Three inlets and two outlets, and, (D) Four inlets and one outlet

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