DROPPING BODIES

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Drop three bodies. Where can they go?

When someone says they've dropped a cup, we imagine it falling down to the ground, attracted to the earth by gravity. But please, in this thought experiment, take away the earth. There's no direction "down". Our three bodies are alone in the universe, attracted only to each other.

Take each body to be a point mass. Dropping the bodies means letting them go from rest, subject only to the rules of Newtonian mechanics and the assumption that the only forces acting on a body are the gravitational $1/r^2$ pulls of the other two. Each body will then sweep out a plane curve, subject to the attractive pull of the other two moving bodies. Taken together, these three parameterized plane curves form a solution to the Newtonian three-body problem.

Below we have depicted a half-dozen answers to our question - indications of where the bodies went. The answers vary widely depending on the starting triangle. We urge the reader to view some of the animations which can be found at [5] and [4]. In the sampled pictures and animations the three masses are equal. (The answers seem to be prettier that way.) The solution depicted in figure 1 was found by Lagrange. The other figures depict periodic solutions which are solutions for which the three bodies shuttle back and forth between two "brake triangles"- configurations of the three bodies at which they are instantaneously at rest or "braked". Either brake triangle can be supposed to be the initial configuration from which the bodies are dropped. Figures 2, 3, and 4 have been selected from a database of thirty collision-free equal mass 'dropped' solutions found by Li and Liao ([9]) and available to view at ([10]). They are all collision-free. Figures 5, and 6 display two of the infinitely many periodic isosceles brake solutions found by Nai-Chia Chen ([3]), all of which suffer binary collisions and whose existence she established in her PhD thesis. Chen's discoveries remind me of Paul Klee's drawings. Her animations are as if Klee's drawings had children with the mobiles of Alexander Calder.

Just above, we used the term brake orbit as a synonym for dropped solution. A brake orbit is thus a solution to the N-body problem for which, at some instant, all N bodies are instantaneously at rest, or in automotive parlance, have "braked" to a stop. Another synonym for brake orbit is "free-fall solution".

The observer will see various symmetries within the orbits. For example, figure (2) has a reflectional symmetry. If you watch the animation that reflectional symmetry become spatio-temporal. When we drop the bodies at time $t = 0$ and if T is the period, then the other brake triangle arises at time $T/2$. At time $T/4$ all three bodies lie on the symmetry line of the figure. Why?

Key words and phrases. Brake orbits, three-body problem, Newton's equations.

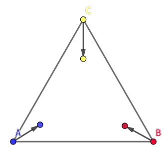

FIGURE 1. Dropping an equilateral triangle. Lagrange showed the triangle shrinks to triple collision, remaining equilateral at each instant.

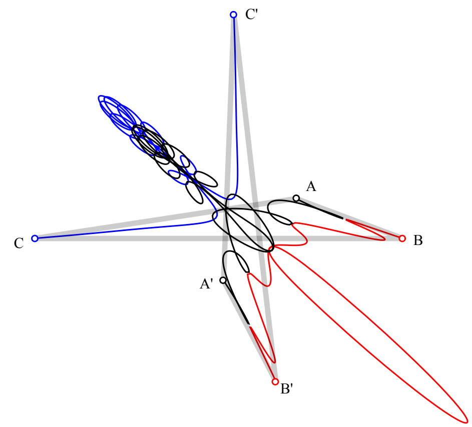

FIGURE 2. Orbit 18 or "$F_{18}(1,1,1)$" in the Li-Liao database [10]. Drop triangle A B C. A half-period later we will arrive at its reflection, triangle A' B' C' . The two labelled triangles are related by a reflection. As a consequence, at the half-way time between, the three bodies must line up along the reflectional line of symmetry of the figure. See the section 'Symmetry Puzzles".

1. History

Lagrange showed that a dropped equilateral triangle stays equilateral all the way until it has collapsed to triple collision. This is not a surprise when the masses are equal but his result holds true for unequal masses. Euler, five years before Lagrange, uncovered a similar fact by placing three masses judiciously on a line: he could get their shape to remain the same upon dropping them. It is easy to get the placement right for three equal masses. Place one at the midpoint of the other two. The center of mass of the triple is then this midpoint and the extremal two collapse symmetrically onto this midpoint. For general unequal masses Euler needed to solve a quintic to get the mass placements right.

In the 1893 a person named Meissel asserted, with little evidence, that if masses in the ratio of $3 : 4 : 5$ are placed at the vertices of a Pythagorean $3 : 4 : 5$ triangle, with mass 5 placed at the vertex opposite the side of length 5 and so

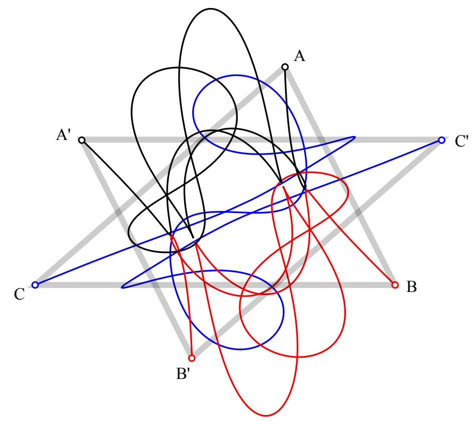

FIGURE 3. Orbit 14 or "$F_{14}(1,1,1)$" in the Li-Liao database [10]. The two *unlabelled* triangles are related by central inversion, i.e. a 180 degree rotation. That inversion takes C to B' rather than C'.

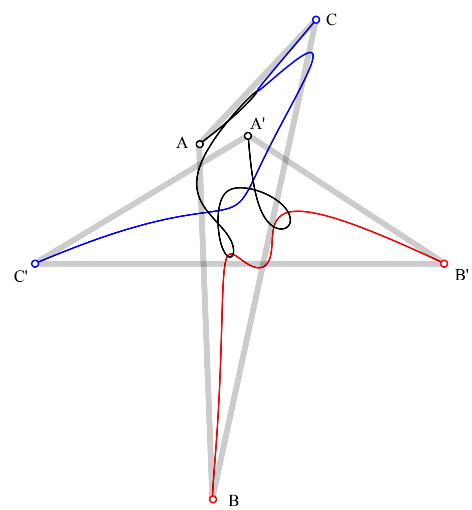

FIGURE 4. Orbit 1 or "$F_1(1,1,1)$" in the Li-Liao database [10]. No isometries relate the two braked triangles.

on, and the bodies are dropped, then the resulting solution is periodic. Verifying Meissel's assertion became a benchmark problem for numerical computation in celestial mechanics and was christened the "Pythagorean three-body problem" or "Burrau's three-body problem". Carl Burrau's name ([2]) became attached to the problem after he took up Meissel's challenge and published the inconclusive results of hand-cranked numerical integration for the problem in 1913. Let's all raise our glasses to the long-suffering degree candidate Sigurd Kristensen who, 109 years ago, did an essential part -"Ein wesentlicher Teil der Rechernarbeit " of Burrau's computational work (one guesses all) but whose name has faded from view, no co-authorship, only an honorable mention within the article. In 1967 the celestial mechanician Szebehely and his team proved Meissel wrong using the new-fangled

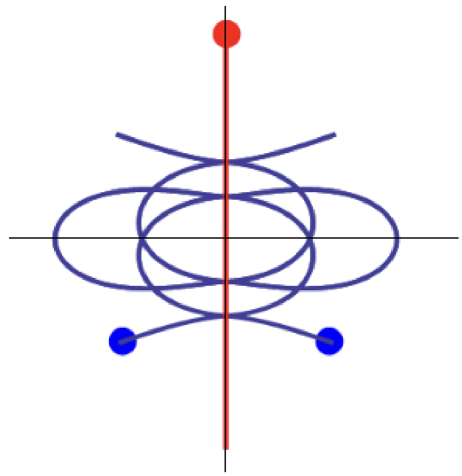

FIGURE 5. A periodic brake orbit in which the three bodies form an isosceles triangle at each instant. The orbit discovered by Nai Chia Chen attempts to redraw an early Paul Klee painting.

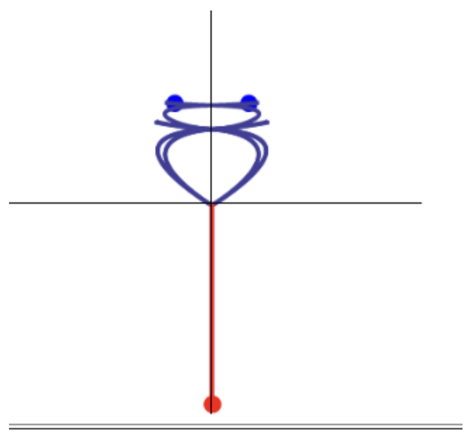

FIGURE 6. Another one of the $6 * \infty$ periodic brake orbits found by Chen.

digital computers. They showed that the smaller of the two masses eventually form a binary pair which escapes to infinity. Before the big escape many interchanges and several near collision incidents occur, incidents requiring Szebehely and team to implement Levi-Civita's regularization of binary collisions in order to get an accurate integration and proceed to follow the orbit to its end [1]

Wait! What is Levi-Civita doing here? Didn't he work on connections and Riemannian geometry and write a book with his advisor Ricci on tensor calculus? A few years after Levi-Civita uncovered the famous connection now bearing his name, he established a suprisingly simple change of variables which "regularizes" the binary collisions of the N-body problem. Gravitational forces blow up at collisions,

[1]Nowadays, high order Taylor methods can recreate Szebehely's discovery without requiring regularization. See [5] and the final section of this paper.

making it seem impossible to continue solutions through collision. However Levi-Civita's change of both time and space variables renders the defining ODEs analytic near isolated binary collisions. Chen's solutions have binary collisions and to make sense of them, or the isosceles three-body problem in general, requires a variant of Levi-Civita's regularization.

Soon after Szehebely had established that Meissel was wrong he decided to tweak Meissel's initial conditions and found, remarkably, near to Meissel's initial conditions, a periodic brake orbit. At the half-period this solution suffers a binary collision in which the non-colliding mass has braked to a stop, and as a consequence of a closer look at Levi-Civita, the solution reverses its path, returing to the starting point. This solution provided inspiration for many. At my own university, the astronomer Greg Laughlin (sadly for me, now moved to Yale) collaborated with a dance professor to choreograph Szehebely's near Pythagorean periodic brake orbit. Three humans played the falling bodies. See [7] for more.

The Newtonian N-body problem is a special case of a Hamiltonian dynamical system whose energy, the Hamiltonian, has the form of kinetic plus potential. See equation (1) below, allowing V to be general. Brake orbits makes sense in this general context. Seifert, the topologist, wrote a beautiful paper ([15]) on brake orbits in this more general context. His paper provided some of the fuel for Weinstein to make a now-rather-famous conjecture known as the Weinstein conjecture - a kind of contact version of the Arnol'd conjectures which drove the development of the field of symplectic topology. The Weinstein conjecture was proved for the case of 3-dimensional contact geometries by Taubes [18] about a decade ago.

Weinstein's student Ruiz ([13], [14]) coined the name "brake orbit" in his thesis extending Seifert's results. Ruiz insisted brake orbits be periodic. We prefer to use the word in our less restrictive sense, requiring only one brake instant while Ruiz definition requires two. Periodic brake orbits must of necessity shuttle back and forth between two brake configurations.

Nothing's sacred here about "three". Drop N bodies. The analogues of the solutions found by Euler and Lagrange are nowadays called central configurations. In other words, if upon being dropped, the N bodies maintain their shape while collapsing to total collision then that initial configuration is called a "central configuration". If we apply an isometry or a scaling to a central configuration we get another one. Modulo such isometries and scaling, is it true that the number of central configurations is finite? This problem made it onto Smale's list of mathematical problems for the 21st century ([16]), published in this same esteemed journal. In 2006 ([6]) the problem was finally answered 'yes' for the case $N = 4$. We have an "almost yes" for $N = 5$ (see [1]) and a "we're basically clueless beyond numerical experiments which indicate yes" for $N > 5$.

2. The problem

The ODE defining the three-body problem, some of whose solutions are depicted in our figures, can be written

$$\ddot{q} = -\nabla V(q). \tag{1}$$

Here $q = (q_1, q_2, q_3)$ records the positions of the three bodies, so that $q_a \in \mathbb{R}^2 \cong \mathbb{C}$, $a = 1, 2, 3$. The curves $q_a(t)$ are parameterized by Newtonian time t. The double dots over q denotes acceleration - the second derivative with respect to time. ∇V

denotes the gradient of the potential function (2) with respect to an inner product on the space $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ of q's called the "mass inner product":

$$\langle \dot{q}, \dot{q} \rangle = \Sigma m_a |\dot{q}_a|^2.$$

A dropped solution, or brake orbit, is one for which, at the initial time $t = 0$ we have $\dot{q}_a = 0, a = 1, 2, 3$.

The Newtonian potential is

$$V = -G\Sigma \frac{m_a m_b}{r_{ab}}, \qquad \text{where } r_{ab} = |q_a - q_b| \tag{2}$$

is the distance between the bodies. $G > 0$ is the gravitational constant, needed if the units on both sides of (1) are to match up. The positive constants m_a are the masses. We set them equal to each other for the equal mass three-body problem.

The dynamics defined by equation (1) leaves the center-of-mass subspace $\Sigma m_a q_a = 0$ invariant. This means is that if we start with initial conditions $q_a(0), \dot{q}_a(0), a = 1, \ldots, N$ for which $\Sigma m_a q_a(0) = 0$ and $\Sigma m_a \dot{q}_a(0) = 0$ then, for all time the resulting solution $q_a(t)$ lies in the subspace: $\Sigma m_a q_a(t) = 0$. There is a standard undergraduate physics trick which reduces any solution to such a center-of-mass zero solution. We invoke this reduction to center-of-mass several times below, perhaps once or twice without saying so.

The mass inner product was built so that the usual kinetic energy is written as $K(\dot{q}) = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle$. The total energy $K(\dot{q}) + V(q)$ is conserved, meaning constant along solutions to equation (1). In addition to this energy, the linear momentum, and angular momentum are conserved. The linear momentum we've already seen. It is $\Sigma m_a \dot{q}_a$. The angular momentum is given by $\Sigma m_a q_a \wedge \dot{q}_a$ where the "$\wedge$" denotes the two-dimensional version of the cross product: $(x, y) \wedge (\dot{x}, \dot{y}) = x\dot{y} - y\dot{x}$, for $(x, y), (\dot{x}, \dot{y}) \in \mathbb{R}^2$. The linear and angular momenta are linear in velocities, while the energy is of the form (kinetic) + (potential) where the kinetic energy is positive definite in velocities and the potential is an everywhere negative function. It follows that all our dropped solutions have zero linear momentum, zero angular momentum, and negative energy.

The physically literate reader may have protested at my potential and said "The potential you wrote down is the potential coming from the fundamental solution of the Laplacian in $\mathbb{R}^3$, not $\mathbb{R}^2$! It yields the gravitational force for bodies moving in space, not in the plane. Your q_a must lie in Euclidean space $\mathbb{R}^3$. Your insistence $q_a \in \mathbb{R}^2$ is unsightly and wrong. " I will counter by reminding my literate reader that since dropped solutions have zero angular momentum they necessarily remain in the plane containing their initial triangle with vertices $q_a(0)$, $a = 1, 2, 3$ at time $t = 0$. Identify this plane with $\mathbb{R}^2$ and the dynamics are correct.

3. Symmetry Puzzles

The seed of this paper were two mysteries and a paradox which Alex Gofen brought to my attention. Gofen had been going through a database of 30 periodic collision-free brake orbits for the equal-mass three body problem which are compiled in [9]. Gofen was verifying and exploring these solutions using his own "Taylor Center" [5] integration scheme. A periodic brake orbit must shuttle back and forth between two distinct brake configurations, which is to say two distinct brake triangles for the three body problem. Gofen noticed that in 12 out of the 30 cases the two brake triangles were related by an isometry. Thus mystery number one: why such a large number? And is the number “right”? For example, if we could make a database of the ‘next’ 300 or 3,000 equal mass periodic collision-free brake orbits would we continue to find roughly 1 : 3 of them had congruent brake triangles? Now to mystery two. One observes in the database that a symmetry relating the two ‘end’ brake triangles induces a symmetry of the entire spatio-temporal structure of the orbit. Why? An instance of mystery number two was discussed above in relation to figure 2. Finally, to the paradox. Gofen observed that the solution depicted in figure 3 shuttles back and forth between two brake triangles which are congruent by a rotation, indeed by rotation by 180 degrees. My own “shape space” perspective (see [12]) on the three-body problem told me that what he observed was impossible. Hence the paradox.

To resolve the paradox, let’s begin by understanding the shuttling back-and-forth. Newton’s equations (1) enjoy time reversal symmetry: if $q(t)$ solves, so does $q(-t)$. Now if $q(t)$ is a brake orbit with brake instant $t = 0$ then $q(-t)$ has precisely the same initial conditons - same configuration and same zero velocity – at time $t = 0$ as $q(t)$. It follows by the unique dependence of solutions on initial conditions that we must have $q(t) = q(-t)$. Suppose in addition that the brake orbit is periodic with period T . Then we have the additional temporal symmetry $q(t+T) = q(t)$. By substituting $t = h-T/2$ into this periodicity relation we get $q(h+T/2) = q(h-T/2)$. Use the time reversal symmetry to get $q(h+T/2) = q(T/2-h)$. Differentiate with respect to h at $h = 0$ to see that $t = T/2$ is another brake instant. Thus periodic brake orbits must shuttle back and forth between two brake configurations.

This second brake configuration must be different from the first. If not, we can cut our period in half and repeat the argument. Eventually by this process we either arrive at a fundamental minimal period with two distinct configurations or the periods go to zero which means our original brake orbit was in fact a fixed point - a critical point of the potential. For Newton’s potential this last possibility is excluded. The potential has no critical points. N stars cannot just sit in space, attracting each other but not moving.

Let me proceed now to shape space thinking. The Newtonian N-body problem enjoys spatial symmetries in addition to its temporal symmetries. These are the isometries of space, or, in the case of the planar N-body problem, the isometries of the plane. What this means is that if R is any isometry of the plane and $q(t) = (q_1(t), q_2(t), q_3(t))$ solves the planar three-body problem, then so does $Rq(t)$ where by $R(q_1, q_2, q_3)$ we mean $(Rq_1(t), Rq_2(t), Rq_3(t))$. We can use these symmetries to push down the ODEs defining the three-body problem down to a space I call “shape space”. The points of shape space are *oriented* congruence classes of planar triangles. Two triangles represent the same “shape,” or oriented congruence class if there is an orientation preserving isometry R, i.e. a rotation composed with a translation, taking one to the other.

Here is the salient point of the paradox: when the angular momentum is zero this pushed-down dynamics is also of Newtonian type, so the argument of the preceding paragraph holds. And brake orbits all have zero angular momentum. A periodic brake orbit, viewed in shape space, is a periodic brake orbit down in shape space, and so must shuttle back and forth between two shapes and *these shapes must be distinct*. But Gofen told me he had found orbits where the two brake triangles were related by rotation hence down in shape space it was shuttling back and forth between the same point!

The resolution of this paradox is that Gofen was viewing his triangles as unlabelled while my triangles have to be labelled triangles in order for me to construct shape space with its dynamics. When a rotation R takes a labelled triangle to another it must, by definition, preserve the (mass or vertex) labellings. But in the example depicted in figure (3) the two brake triangles are congruent as *unlabelled* triangles, not as labelled one. A 180 degree rotation does take one triangle to the other but in so doing it messes up their labellings.

When the masses are all equal, the N-body problem enjoys additional symmetries beyond the Galilean symmetries of time and space isometries. We may interchange any two masses: if $(q_1(t), q_2(t), q_3(t))$ solves the equal mass three-body problem then so does $(q_2(t), q_1(t), q_3(t))$, etcetera. The operation of interchanging masses defines a representation of the permutation group on the configuration space so that we could write the above interchange of masses 1 and 2 as σ_{12}. The brake triangles of figure (3) are related by a symmetry of the form $F = R \circ \sigma$ where σ is one of the transpositions. Although R acts as the identity on shape space, such an F does not. The shape of $q(0)$ and $F(q(0))$ are different, allowing us to avoid the paradox.

Having extricated ourselves from our mathematical paradox, we move on to Mystery Two. If we have a brake orbit whose ends– the two brake triangles - are related by a symmetry F as above we can use that symmetry to extract non-trivial information about the configuration $q(T/4)$ at the quarter period, $T/4$ being half-way between the two brake times of $t = 0$ and $t = T/2$. Suppose then, that $F(q(0)) = q(T/2)$. Consider the new solution $F(q(t))$ at time $t = 0$ its initial condition is shared with that of $q(t)$ at time $t = T/2$: namely, it is brake with configuration $q(T/2)$. But $q(t + T/2)$ solve Newton's equations and has precisely these initial conditions. (Like any autonomous ODE (equation (1)) enjoys the time translational symmetry: If $q(t)$ solves so does $q(t + t_0)$ for any time t_0.) It follows that

$$q(t + T/2) = F(q(t)). \tag{3}$$

Now take $t = -T/4$ and use the time reversal symmetry to conclude that

$$q(T/4) = F(q(T/4)) \tag{4}$$

The midway point must a fixed point of our symmetry F!

Return to figure 2. A reflection F relates its two brake triangles. Reflections are symmetries of the three-body problem. Take F to be this reflection and ℓ its line of reflection. F's fixed points are collinear “triangles” in which all three masses lie on ℓ. In this way we have solved the mystery around that figure: why all three masses form a syzygy at the mid time, and indeed at the same time a moment's contemplation of equation (3) shows that we have also accounted for the overall reflectional symmetry of that orbit as being a consequence of the symmetric relation between its endpoints.

In figure 3 the two brake triangles are related by $F = R \circ \sigma$ where R is rotation by 180 degrees and σ interchanges two of the masses. The fixed point set of such an F is the set of “Euler configurations” : degenerate collinear triangles with the non-interchanged mass forming the midpoint of the other two. This fact, and of course equation (3) matches Gofen's data.

4. A hole in shape space and harmonic oscillators

Our mathematical hero Arnol'd had a saying he was fond of sprinkling into his lectures which was a variation of the phrase "the exception that proves the rule". We present our exception.

Observe that we can run our shape space argument for any potential invariant under isometries. One such potential is the harmonic oscillator potential $V = \Sigma k_{ab} r_{ab}^2$ with $k_{ab} > 0$ spring constants. The act of replacing Newton's potential (equation (2)) by this quadratic potential corresponds to replacing the gravitational force by Hooke's spring forces. The corresponding ODEs are linear of the form $\ddot{q} = -Aq$ where A is a matrix depending on the masses and springs and which is positive definite on the center-of-mass subspace $\Sigma m_a q_a = 0$ and which leaves this subspace invariant. Choose an eigenbasis E_i for A restricted to this subspace so that $AE_i = -\omega_i^2 E_i$ with $\omega_i^2 > 0$ the eigenvalues. Then $q(t) = \cos(\omega_i t) E_i$ is a brake solution shuttling back and forth between E_i and $-E_i$. But $-E_i$ corresponds to rotating E_i by 180 degrees. Pushed down to shape space, this brake solution connects the shape corresponding to E_i to itself, contradicting my alleged "shape-space thinking" theorem that such a solution is impossible. This is the exception that proves the rule.

Another paradox. What's happening? Is our theorem true or not?

The resolution of this apparent paradox involves the projection of 0 to shape space, 0 representing total collision. The map from configuration space to shape space, upon restriction to the center-of-mass subspace, fails to be a submersion exactly at 0. We should view the shape 0 of total collision as a singularity in shape space. (Indeed, for $N > 3$ it is a topological singularity since the shape space for the planar N-body problem is the cone over complex projective space of complex dimension $N - 2$.) The relation between the dynamics upstairs and downstairs breaks down at total collision. Our eigenvector-based solution above passes through 0 at time $t = \pi/(2\omega_i)$, and our shape-space argument fails for solutions passing through 0.

We can derive an alternative resolution to this paradox by following the implications of equation (3). A rotation $F = R$ is a symmetry and so that equation hold for any brake solution whose end triangles are related by a rotation R. Then equation (4) asserts that $q(T/4)$ is a fixed point of the rotation. But the only centered configuration invariant under a nontrivial rotation is the zero configuration $0 = (0, 0, 0)$, the configuration representing total collision. Our periodic brake solution must pass through total collision half-way between its two ends! That's fine for the harmonic oscillator. No problem. For the gravitational N-body problem total collision acts like an essential singularity - a hole in shape space if you will - through which there is no consistent way to travel beyond and we have to stop the dynamics at total collision and call it quits. So there's no such brake orbit for the planar N-body problem.

More information can be extracted from equation (3). Evaluating the equation at $t = -T/2$ and $t = 0$ to get $q(0) = F(q(T/2))$ and $q(T/2) = F(q(0))$ so that $q(0) = F^2(q(0))$. If the triangle $q(0)$ is in general position, or even if it is a degenerate collinear triangle but F is of the form R or $R \circ \sigma$, then this fixed point relation implies that $F^2 = Id$. (In other terms, the rotation group acts freely on configuration space away from triple collision, so that $R^2(q(0)) = q(0)$ implies that $R^2 = Id$.) When $F = R \circ \sigma$ we have $F^2 = R^2$, so either way, when $F = R$ or $R \circ \sigma$, we get $R^2 = Id$.

The only R's that solve the identity $R^2 = Id$ are rotations by 180 degrees! This R is also known as central inversion: $Rq_a = -q_a$. The original solution Gofen showed me, figure 3 above, has its brake triangles related by central inversion, but again, related as *unlabelled* triangles. Again, this matches Gofen's data: all his nontrivial brake triangles, when related by symmetries, are related by central inversion.

I sure hope Gofen does not wander back out into the land of equal mass periodic brake orbits this summer and comes back to me with a solution whose brake triangles are related by an F of the form $R\sigma$ with the R being a 45 degree rotation! I will not know how to resolve the resulting paradox. The remainder of my summer vacations with family would be threatened with ruin! Wait till next spring, please, Alex, for alerting me to such a paradox.

Mystery one, the mystery of 12 out of 30 of these "first" equal mass collision-free periodic brake orbits having extra symmetries, remains a mystery.

5. End Note. Gofen's Taylor Center

Alex Gofen, whose figures grace this paper, asked me to say a few things about the "Taylor Center" that he runs and which generated three of the displayed figures.

The name Taylor Center stands for two things:

- a comprehensive resource dedicated to particular mathematical problems; and...
- a software for Windows - the advanced ODE solver [b] based on the modern Taylor integration. This ODE solver offers several unique features: numerical and graphical.

As a numerical tool, it employs the most accurate Intel float point type called extended with 63-bit mantissa, integrating with order 30 or higher, and providing several methods of accuracy control up to all available 63 binary digits.

As a graphical tool, it offers high resolution graphics, plotting the trajectories as real time animation: both in 2D and 3D stereo (viewable via red/blue glasses). Thanks to such graphics, this software may serve as Lab-works in various fields of applied mathematics. A few of such lab topics were already posted [c]: for example, the three types of the rigid body motion; selected samples in celestial mechanics. The Lab-works library keeps growing.

[a] `http://taylorcenter.org/`

[b] `http://taylorcenter.org/Gofen/TaylorMethod.htm`

[c] `http://taylorcenter.org/Exploratorium/`

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