Title
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Permalink
https://escholarship.org/uc/item/1d04f586

Journal
Manufacturing & Service Operations Management, 18(4)

ISSN
1523-4614

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Publication Date
2016-10-01

DOI
10.1287/msom.2016.0587

Peer reviewed

Submitted to *Manufacturing & Service Operations Management*
manuscript MSOM-13-215.R2

The Time-Money Trade-Off for Entrepreneurs: When to Hire the First Employee?

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For many early-stage entrepreneurs, hiring the first employee is a critical step in the firm's growth. Doing so often requires significant time and monetary investments. To understand the trade-offs involved in deciding when to hire the first employee and understand how hiring differs in entrepreneurial settings from more established firm settings, we present a simple growth model that depends on two critical inputs for revenue generation: the entrepreneur's time and money. We show that without hiring, the entrepreneur's time eventually becomes more valuable than money in contributing to the firm's growth. In that context, the value of the employee is driven by how much relief s/he provides to the entrepreneur. We characterize the optimal timing of hiring in terms of the firm's cash position and how it is affected if it requires an upfront fixed investment in time and/or money. We find that an upfront investment in time needed for hiring cannot be converted to an equivalent upfront investment in money and that mis-timing hiring can be very costly, especially when these upfront investments are high.

Key words: Entrepreneurial operations, hiring, optimal stopping problem.

1. Introduction

Consider the following scenario typical in early-stage entrepreneurial firms. A company is founded by a single individual. In the early days the entrepreneur is involved in all aspects of the company's operations. However, as the company does well and grows, the number of tasks requiring the entrepreneur's time and attention multiplies, and it becomes increasingly difficult for the entrepreneur to manage everything. At some point the entrepreneur needs to hire an employee to delegate tasks and sustain the growth of the firm. This leads to a decision faced by most early-stage entrepreneurs: When to hire that first employee?

Our focus is on "bootstrapping" entrepreneurs who have an operating business that they seek to grow over a planning horizon by investing their own time and re-investing their earnings (Ebben and Johnson 2006). Specifically, according to Mills's (2015) classification of 28 million "small businesses" in America, our context is the subset of 23 million "sole proprietorships" who plan to become either one of the 4 million "main street" businesses or one of the 1 million "suppliers." Businesses that already have a small number of employees but none who participate in managerial activities also fit within our context. For instance, a contractor may have a few carpenters on the payroll but is still essentially managing her firm single-handedly, just like a restaurateur who may have a few waitstaff and line cooks but who still basically runs the restaurant by himself.

For our type of entrepreneur, hiring the very first managerial employee is a particularly important decision because it requires, in addition to the ongoing salary payments, significant upfront time and money investments. This is because working with an employee entails fundamental changes to how the firm operates, e.g., the entrepreneur must set up an organizational structure and codify the work processes and assign roles (Churchill and Lewis 1983, Gerber 2001, Burgstone and Murphy 2012, Hess 2012). However, it also frees up some of the entrepreneur's time, which can then be invested in continued growth. As such, hiring the first employee marks the transition from an entrepreneur-dominated firm to a phase of rapid growth in the company's life cycle (Hambrick and Crozier 1985).

This paper has two goals: first, to provide insights into when that first hire should occur, and second, to highlight how this decision is different for an entrepreneur than for an established firm. To do so, we present a simple growth model of an entrepreneurial firm based on the assumptions that (a) the entrepreneur's time, in addition to money, is an important input for generating revenue and growth (Lévesque and MacCrimmon 1997), and (b) the firm is constrained in both time (McCarthy et al. 1990, Gifford 1992) and money (Hambrick and Crozier 1985, Ebben and Johnson 2006). Hiring allows the entrepreneur to trade off one constrained resource (money) to gain another constrained resource (time). This is different from a more established firm, where hiring is done by an HR department or by others that are less directly linked to the revenue-generating activities of the firm.

Within the context of that growth model, we formulate the decision of when to hire the first employee as an optimal stopping problem, we characterize the optimal timing in terms of a cash threshold, and analyze the sensitivity of that decision, both analytically and numerically. Our analysis generates the following four take-aways, which have, in varying degrees, managerial implications for entrepreneurs and shed insights onto the differences between entrepreneurial and more established firms.

First, we find that, without hiring, the entrepreneur's time eventually becomes more valuable than money in contributing to the firm's growth. For entrepreneurs, this key observation implies that their time is likely to become more valuable over time. That is, even if today an entrepreneur may prefer \$10,000 over 10 hours of time, there will come a time when the entrepreneur will prefer the 10 hours. A direct implication of that result is that investing money and/or time today to save time later is likely to be a good investment. In addition, this result suggests that an employee's value increases over time in an entrepreneurial setting because the entrepreneur's time constraint becomes tighter as the firm grows, whereas it is more stable in established firms that operate in stationary environments.

Second, we show that the optimal timing of hiring can be characterized in terms of the firm's total cash position. Moreover, the optimal timing of hiring can occur before or after the entrepreneur's time becomes the chief bottleneck of the firm, depending, among others, on the employee's wage and the amount of time relief the employee provides to the entrepreneur. As a result, entrepreneurs should not necessarily wait with hiring until they feel that they have to hire, but they should consider hiring as soon as they can afford to. In contrast, in established firms, the decision to hire an additional employee is typically independent of the firm's cash position.

Third, we find that the required upfront time investment and money investment associated with hiring the first employee have different effects on the timing of hiring. When greater monetary investment is required, the entrepreneur should delay hiring. In contrast, when greater time investment is required, it may be optimal to either expedite or delay hiring. The intuition behind this non-monotone effect is as follows. On the one hand, more time (and money) diverted away from revenue-generating activities impacts growth more severely earlier in the growth phase, making it desirable to delay hiring. On the other hand, because the opportunity cost of time becomes more valuable later, it is desirable to incur a larger time outlay earlier when time is less valuable. In other words, upfront setup time and cost are not interchangeable in entrepreneurial firms. Hence, if an entrepreneur realizes that she underestimated the monetary cost of hiring, she should delay hiring relative to her original plan; but if she underestimated the time cost of hiring, she may have to hire earlier rather than delaying. The different effect of hiring setup time and hiring setup cost is specific to entrepreneurial settings. In more established firm settings, setup time and setup cost are typically considered as interchangeable because a dedicated human resource function exists to incur the upfront time required for hiring (e.g. screening, assigning roles, and training) on the employer's behalf (see e.g. Gans and Zhou 2002). And unlike the entrepreneur's time, which has a direct effect on the entrepreneurial firm's growth, the time spent by that dedicated human resource department has typically no direct impact on generating or fulfilling demand.

Fourth, we numerically observe that the timing of hiring is a significant factor in determining the growth trajectory, especially when hiring entails large upfront investments. Whereas hiring too soon is costly because growth is stunted more heavily, hiring too late is costly because of the missed growth opportunity. Because the negative effects of mistimed hiring appear to be smaller when it requires less upfront investments of time and money, entrepreneurs who, in practice, may not be able to hire at the optimal time, should strive to reduce these hiring setup costs and time. For instance, entrepreneurs could always be on the lookout for that first employee from the moment they start the firm, so that, when the time comes that they actually need to hire, their setup time will be lower. Although mistiming hiring is costly in both entrepreneurial and established firms, the effect may be more severe in entrepreneurial firms because it stunts growth in addition to deferring future profits.

2. Literature Review

Our work builds on several streams of literature. In the human resource management literature, Cardon and Stevens (2004) report a limited understanding of many important human resource issues specific to entrepreneurial firms. Apart from a few descriptive studies that explore the entrepreneur's difficulty in hiring or their non-traditional recruitment practices (Aldrich and Fiol 1994, Williamson 2000, Collins and Clark 2003), the lack of formal theory is widely noted (Tansky and Heneman 2006). Due to the advent of data analytics, the theory and practice of recruitment by established firms are increasingly being guided by quantifying an employee's financial value to the firm. However, applying the same approach is difficult in the entrepreneurial context (Hayton 2003). Taking an operations management perspective we derive a simple guideline for how an employee should be valued based on the level of relief s/he provides to the entrepreneur's time constraint. To the best of our knowledge, such a formalization has not been made in this context.

Our paper relates to studies that recognize the importance of managing time as a key resource for entrepreneurs, such as McCarthy et al. (1990), Cooper et al. (1997), Lévesque et al. (2002), Lévesque and Schade (2005), Mueller et al. (2012), and Burmeister-Lamp et al. (2012)). This literature argues that having more time available is beneficial for the venture's success, but its treatment of the cost of that time is usually limited to lost wages or reduced leisure time and it does not consider how to increase the supply of time. In contrast, we formalize the evolving value of time during the growth horizon and value when the entrepreneur should buy additional time.

Our model's focus on time and money inputs shares some similarity to Lévesque and MacCrimmon (1997), who examine how an entrepreneur should transition from being an employee in a wage job to founding a new venture by diverting time and money from the former to the latter. They examine how the work tolerance of the entrepreneur and the nature of the new venture determine the optimal allocation of time between the two competing jobs over a planning horizon and explain the empirically observed patterns in entrepreneurs' shifting time allocation. In contrast, we focus on full-time entrepreneurs who invest all of their available time in their business and who wish to acquire more time by hiring. Our aim is to attain insights into when the first hiring should occur.

Managing the workforce has long been an important problem in operations management, from the early literature on workforce planning (Holt et al. 1960, Orrbeck et al. 1968, Ebert 1976) to more recent staffing in service systems (Pinker and Shumsky 2000, Gans and Zhou 2002, Arlotto et al. 2014). These hiring-related studies have traditionally focused on large firm settings with exogenous demand. In these contexts, the hiring setup time is often used interchangeably with setup costs (Gans and Zhou 2002). We contribute to the literature by concentrating on an early-stage entrepreneurial firm, in which revenue and growth depend on how the entrepreneur manages both time and money, and where hiring changes the growth trajectory of the firm. We show that, in our context, hiring setup time and cost have a different effect on the timing of hiring.

The timing of hiring the first employee is analogous to the timing of irreversible decisions such as investing in capacity (Luss 1982, Van Mieghem 2003) or adopting a new technology (Pindyck 1988, Dixit and Pindyck 1994, McDonald and Siegel 1986). Similar to these papers, we adopt an optimal stopping problem framework to examine the decision. However, there are key differences in the underlying trade-offs that we study. Unlike the capacity expansion setting where the decision primarily concerns a fixed upfront investment of capital (or money), we examine a fixed investment of two complementary resources, namely the hiring setup cost and time, which influence the availability of time and money in all future periods. Moreover, unlike the technology adoption settings where the timing depends on learning and resolution of technological uncertainty, the timing decision here depends mainly on increasing the value of time relative to money.

Finally, there is a growing literature on operations of entrepreneurial firms. The common goal of this research is to identify strategies to improve the chance of success of newly formed companies. For example, Archibald et al. (2002) show how to make conservative inventory decisions to maximize survival; Babich and Sobel (2004) examine how to maximize the expected value of payoff in IPO; Swinney et al. (2011) study how to invest in capacity when competing against an established firm; Tanrisever et al. (2012) examine how to invest in process improvement under the threat of bankruptcy. Focusing specifically on the time allocation decision, Yoo et al. (2015) examine how entrepreneurial firms should allocate their time to process improvement activities, while Huang et al. (2015) examine how to allocate time to maximize short-term sales. Focusing on financing challenges, Buzacott and Zhang (2004) examine how startups can leverage asset-based financing. We contribute to this stream of research by focusing on the entrepreneur's hiring decision.

3. A Model of Entrepreneurial Growth without Hiring

We first present a stylized model of revenue generation and growth of an entrepreneurial firm that reinvests its earnings to fund its growth (Ebben and Johnson 2006). We consider two inputs, time and money, which are frequently cited as the most important ones for entrepreneurial firms (Lévesque and MacCrimmon 1997) to show that, as the firm grows, its chief bottleneck shifts from money to time. This model will serve as the basis for our subsequent analysis of when to hire the first employee, in §4.

3.1. Model and Notation

In each period t (e.g., two months) the entrepreneur has time (T_t) and money (M_t) available to invest in the firm, which generate revenue $R(M_t, T_t)$ for period t for simplicity. The revenue function $R(M_t, T_t)$ is assumed to be the same each period throughout the planning horizon N. Investments M_t and T_t in period t may also have longer-lasting effects beyond period t. For instance, time and money spent on advertising may also effect future sales, although this effect will decline over time (Mahajan et al. 1984). We model this using exponentially decaying residual revenues $\gamma^k R(M_t, T_t)$ ($0 \le \gamma < 1$) in future periods $t+k$, $\forall k \ge 0$ as a result of period t's investments. If $\gamma = 0$, then the effects of investments accrue only in the current period.

We assume that more resources invested in the firm leads to greater revenue, and that money exhibits diminishing returns. That is, $R(M_t, T_t)$ is increasing in (M_t, T_t) and concave in M_t; however, $R(M_t, T_t)$ need not be concave in T_t. We further assume that investing more money in the business can never generate less revenue (e.g. because the additional money can be put aside in the firm's account), i.e. $\frac{\partial R(M,T)}{\partial M} \ge 1$. Time and money are complementary so the marginal return on investment in one is increasing in the investment in the other, i.e., $R(M_t, T_t)$ has increasing differences in (M_t, T_t). Finally, $R(0, T) \ge 0 \ \forall T \ge 0$. (Throughout the paper, we use the terms increasing/decreasing in a weak sense, i.e., as nondecreasing/nonincreasing.)

Without hiring, the entrepreneur's time supply remains fixed ($\forall t$, $T_t = T$), all of which is invested into the venture. We ignore learning effects that may influence the entrepreneur's productivity. Accordingly we drop the subscript and refer to T throughout.

The firm's cash level is exposed to additive random shocks, which can impact future periods. This can result from fire-fighting, for example, giving unhappy customers rebate in future dealings to limit negative publicity, or having to pay premiums to expedite shipments in the future to make up for delays. We use additive shocks because businesses are typically less vulnerable to risk, relative to their size, as they grow, see e.g., Phillips and Kirchhoff (1989).

We model the stochastic shocks by a Markov process $\{Z_t\}$, where $Z_{t+1} = \phi(Z_t, \zeta_t)$, in which $\{\zeta_t\}$ are independently distributed random variables with finite support. In particular, one can define $Z_{t+1} = \rho Z_t + (1-\rho)\zeta_t$, in which $\{\zeta_t\}$ are independent and identically distributed (iid); in that case $\{Z_t\}$ is a stationary sequence of correlated shocks. Our model can also capture cumulative shocks, when $Z_{t+1} = \rho Z_t + \zeta_t$ and $\{\zeta_t\}$ are iid. If $\rho > 0$ ($\rho < 0$), the next period's random shock Z_{t+1} is positively (negatively) correlated with the shock experienced in the current period (Z_t) and a random component (ζ_t). If $\rho = 0$, we have state-independent random shocks $\phi(Z_t, \zeta_t) = \zeta_t$.

If the entrepreneur starts in period 1 with a cash level M_1 and shock state Z_1, the firm's cash position at the end of period t, M_{t+1}, is equal to

$$M_{t+1} = R(M_t, T) + \sum_{s=1}^{t-1} \gamma^{t-s} R(M_s, T) - \phi(Z_t, \zeta_t). \tag{1}$$

The presence of stochastic shocks allows the sequence $\{M_t\}$ to decrease, and it is in principle possible for the firm to lose all its money during the horizon. Given our focus on growth-oriented firms, we are interested in cases where M_t remains positive throughout the planning horizon N; that is, despite the stochastic shocks, enough revenue $R(M,T)$ is generated each period to maintain a positive cash level over the length of the planning horizon N. For mathematical completeness, we consider the extended value extension of $R(M,T)$ in our analysis, i.e., $R(M,T) = -\infty$ if $M < 0$ (Boyd and Vandenberghe 2004), to ensure that the entrepreneur's hiring decisions do not lead to a negative cash position.

As is common in the entrepreneurial OM literature, we assume risk neutrality (Archibald et al. 2002, Buzacott and Zhang 2004, Tanrisever et al. 2012). We consider the objective of maximizing the firm's cash position at the end of a planning horizon of N periods (e.g., a couple of years), i.e., maximizing M_N. This objective (after subtracting initial cash) represents the total earnings over the planning horizon, which is a common goal for early-stage startups (Gerber 2001, Burgstone and Murphy 2012). Moreover, because $R(M,T)$ is increasing in M and cash accumulation (1) increases in $R(M_t,T)$ for all t, any alternative objective that is an increasing transformation of M_N (e.g. maximize the revenue $R(M,T)$ at period $t = N$) would lead to same insights.

3.2. Bottleneck Shift

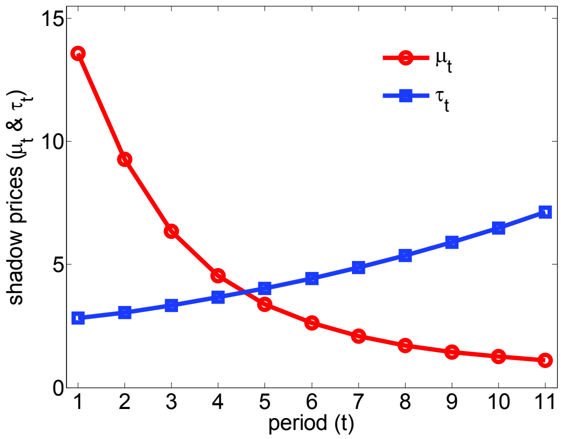
Given our time- and money-dependent growth model, we next present how the firm's bottleneck shifts from money to time over a given planning horizon. The following result shows a fundamental characteristic of early-stage bootstrapping entrepreneurial firms, experiencing deterministic growth, i.e., when $Z_t = 0\ \forall t$. (Later, we relax the monotone growth assumption.)

PROPOSITION 1. *Let $\mu_t = \frac{\partial M_N}{\partial M_t}$ and $\tau_t = \frac{\partial M_N}{\partial T_t}$ denote the effect on period-N cash position of an investment of money and time in period t when $Z_t = 0, \forall t$. Then, μ_t/τ_t decreases in t and there exists at most one time period $\hat{t}$ at which the entrepreneur's bottleneck switches from money to time, i.e., $\tau_t > \mu_t$ if and only if $t \geq \hat{t}$.*

Proposition 1 shows a simple but fundamental consequence of growth in the entrepreneurial firms we consider: an incremental unit of the entrepreneur's time will eventually become more valuable than an incremental unit of money. This is illustrated in Figure 1, where the bottleneck shift occurs in period $t = 5$. (To help illustrate our results, we use a running example throughout the paper, which is introduced formally in Section 5. The parameters of that example appear in the note in Figure 1.) Hiring can be seen as a mechanism for trading money for additional time, and Proposition 1 shows that this trade-off becomes more attractive in later periods.

Suppose an employee can bring y additional units of time and costs a wage w each period. (Here, y is expressed in equivalent units of the entrepreneur's time. For instance, an employee

Figure 1 **Evolution of shadow prices as the firm grows over time. The entrepreneur's time eventually replaces money as the chief bottleneck of the firm.**



Note. Parameters: $R(M,T) = 1.2M^{0.9}T^{0.4}$, $\gamma = 0.1$, $M_1 = 1$, $T = 2$, $Z_t = 0\ \forall t$.

who is 75% as productive as the entrepreneur and works the same total number of hours T would contribute $y = 0.75T$ units of time to the firm.) Then, after hiring, the total available time per period increases from T to $T + y$, whereas the available money M_t decreases by w, in all future periods t. This employee, although not necessarily valuable early on, may become increasingly valuable for the entrepreneur as the firm grows. In contrast, in established firms that operate in stationary environments, the value of an employee remains stable over time.

The next corollary characterizes how the shadow prices of money and time relate to when the entrepreneur should hire a "marginal" employee, meaning an employee who adds marginal time y for marginal wage w.

Corollary 1. *Consider a potential employee who costs w and contributes y each period after being hired. When $Z_t = 0\ \forall t$, as $w \downarrow 0$ and $y \downarrow 0$, it is optimal to hire the employee in period t if and only if $y \cdot \tau_t > w \cdot \mu_t$.*

The corollary implies that the optimal timing of hiring can occur before or after the entrepreneur's time becomes the chief bottleneck of the firm. For example, when the time relief (y) provided by the employee is sufficiently high relative to the employee's wage (w), hiring can be optimal before the entrepreneur's time becomes the chief bottleneck of the firm (i.e. when $\tau_t < \mu_t$).

The corollary can further be applied to approximate the wage the entrepreneur would be willing to pay for an employee contributing y units of time from period t onwards. To illustrate this, let us consider a Cobb-Douglas revenue function, i.e., $R(M,T) = \kappa M^\alpha T^\beta$ with $0 < \alpha < 1$, $\kappa > 0$ and $\beta > 0$. The Cobb-Douglas revenue function allows us to represent different types of businesses. For example, having $\beta \approx 1$ is similar to someone billing for time, while having $\beta \ll 1$ might be more a fixed-cost business where an additional hour of the entrepreneur's time is less valuable than the first hour. Similarly, having $\alpha \approx 1$ might involve the entrepreneur buying and selling inventory, while having $\alpha \ll 1$ might correspond to the entrepreneur investing in advertising but not buying goods for resale. With this Cobb-Douglas revenue function, the ratio of shadow prices μ_t/τ_t turns out to simplify to $\alpha T/(\beta M_t)$ (see the proof of Proposition 1 for details), and by Corollary 1, for sufficiently small w and y, the entrepreneur should hire the employee if

$$w \leq \left(\frac{\beta}{\alpha}\right)\left(\frac{M_t}{T}\right) y.$$

Hence, an employee's value increases if s/he is more available and/or more productive (greater y), if the firm has more money (greater M_t) or less time (smaller T), or if the revenue is more elastic to time than money (greater β/α). Although our model is too stylized for this result to offer a precise quantitative rule for entrepreneurs to use in practice, it highlights that the value of an employee should be put in the context of the type of business (α, β) and the growth stage (M_t) of the firm.

This section considered how the bottleneck shifts under monotone growth and no hiring. We next analyze a more general case with hiring that requires upfront investments, and examine how the interplay between time and money influences the optimal timing of hiring.

4. Hiring the First Employee

In practice, hiring the first employee requires significant upfront investment of time and money because the entrepreneur needs to create a structure for the organization, such as codifying the work processes and assigning roles to accommodate this and subsequent employees. Thus, hiring may require the entrepreneur to divert resources away from revenue-generating activities, which may involve a temporary slowdown in revenue generation. To formalize these required upfront investments, we assume that a setup cost S_M and a setup time S_T must be incurred in the hiring period, just before the employee joins. Hence, the available cash in the hiring period decreases to $M_t - S_M$ and the available time decreases to $T - S_T$. We assume S_T such that $T - S_T \geq 0$. The employee, once hired, contributes y units of additional time and costs a wage w in all periods after being hired.

We examine when the entrepreneur should hire the first employee in this more general setting by using an optimal stopping time analysis. We first introduce the model (§4.1), characterize the solution via a threshold rule (§4.2), and examine the comparative statics with respect to the upfront time and money investment required in hiring the first employee (§4.3).

4.1. Optimal Stopping Time Framework

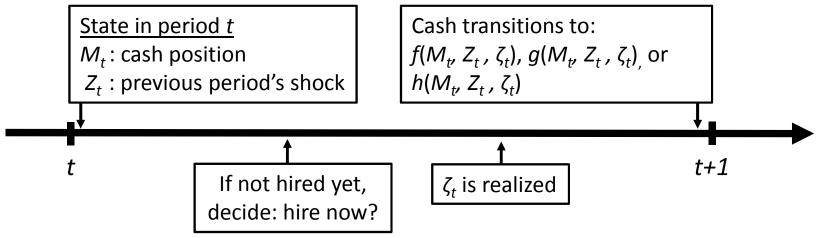
Because hiring creates different flows of cash and time, the cash transition in (1) needs to be modified. Specifically, the cash transitions during and after hiring account for the hiring setup cost and time (S_M and S_T) and the employee's time contribution and wage (y and w) respectively.

The following lemma characterizes the cash transitions before, during, and after hiring, and the sequence of events is shown in Figure 2.

Lemma 1. *In period $t > 1$, for state (M_t, Z_t, ζ_t), the cash available in period $t+1$ when no employee has been hired $(f(M_t, Z_t, \zeta_t))$, when an employee is being hired $(g(M_t, Z_t, \zeta_t))$, or when an employee has been hired in a previous period $(h(M_t, Z_t, \zeta_t))$, are given by respectively:*

$$\begin{aligned} f(M_t, Z_t, \zeta_t) &\triangleq \bar{f}(M_t) - \phi(Z_t, \zeta_t) + \gamma Z_t \triangleq [\gamma M_t + R(M_t, T)] - \phi(Z_t, \zeta_t) + \gamma Z_t, \\ g(M_t, Z_t, \zeta_t) &\triangleq \bar{g}(M_t) - \phi(Z_t, \zeta_t) + \gamma Z_t \triangleq [\gamma M_t + R(M_t - S_M, T - S_T)] - \phi(Z_t, \zeta_t) + \gamma Z_t, \\ h(M_t, Z_t, \zeta_t) &\triangleq \bar{h}(M_t) - \phi(Z_t, \zeta_t) + \gamma Z_t \triangleq [\gamma M_t + R(M_t - w, T + y)] - \phi(Z_t, \zeta_t) + \gamma Z_t. \end{aligned} \tag{2}$$

Figure 2 Sequence of events.



Note that $h(M_t, Z_t, \zeta_t) - f(M_t, Z_t, \zeta_t) = \bar{h}(M_t) - \bar{f}(M_t)$ is independent of ζ_t and Z_t, and it is increasing in M_t since $R(M_t, T)$ is concave in M_t and has increasing differences. In other words, having hired becomes more attractive as the firm's cash position increases. Throughout, we assume the following on the hiring setup cost and time.

Assumption 1. *(S_M, S_T) are such that $R(M_t, T) - R(M_t - S_M, T - S_T)$ is decreasing in M_t.*

The assumption states that the loss of revenue in the hiring setup period due to the diversion of resources, i.e. $R(M_t, T) - R(M_t - S_M, T - S_T)$, will be smaller when the firm has more cash on hand at the beginning of that period (M_t). In particular when $S_T = 0$, this condition is automatically satisfied since $R(M, T)$ is concave in M.

Denoting whether or not an employee has been hired by a binary state variable E_t ($E_t = 1$ if an employee has been hired in or before period t, and $E_t = 0$ if not), we have the following optimal stopping problem.

$$\begin{aligned} V_t(0, M_t, Z_t) &= \max\Big\{ \underbrace{\mathbb{E}_{\zeta_t} V_{t+1}(0, f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))}_{NotHire},\ \underbrace{\mathbb{E}_{\zeta_t} V_{t+1}(1, g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))}_{Hire} \Big\}, \quad 1 < t < N, \\ V_t(1, M_t, Z_t) &= \mathbb{E}_{\zeta_t} V_{t+1}(1, h(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t)), \qquad t < N, \\ V_1(0, M_1, Z_1) &= \mathbb{E}_{\zeta_1} V_2(0, R(M_1, T) - \phi(Z_1, \zeta_1), \phi(Z_1, \zeta_1)), \\ V_N(\cdot, M_N, Z_N) &= M_N. \end{aligned} \tag{3}$$

In the next section, we examine the optimal timing of hiring.

4.2. Optimal Threshold Policy

We characterize the optimal hiring time in terms of a cash threshold. The threshold may, for example, correspond to the slack in capital or the number of months of employee's salary the firm must have before hiring. An increase in the threshold corresponds to, all else being equal, a delay in the timing of hiring because the firm must accumulate a higher cash level. We first establish a lower bound on this optimal cash threshold.

PROPOSITION 2. *Suppose Assumption 1 holds. If* $M_t \leq M^{lb}(Z_t) \triangleq \max\{M \mid \bar{g}(f(M_t, Z_t, \zeta_t)) \geq \bar{h}(g(M_t, Z_t, \zeta_t)), \forall \zeta_t\}$*, then it is optimal not to hire in period* t.

The proposition identifies a lower bound on the threshold below which the entrepreneur should never hire. The lower bound is constructed by comparing hiring in the current period (t) and in the following period ($t+1$). The lower bound is useful because it denotes a necessary condition for hiring; that is, if the entrepreneur has less cash than $M^{lb}(Z_t)$, then s/he should not hire. The following corollary shows that this lower bound may correspond to the optimal threshold under specific circumstances.

COROLLARY 2. *Suppose Assumption 1 holds. If (i)* $Z_t = 0$ $\forall t$*, and (ii) hiring must occur before the end of the planning horizon (e.g., by modifying the boundary condition in (3) so that* $V_N(0, f(M), Z) < V_N(1, g(M), Z)$ $\forall M$*), then it is optimal to hire in period* t *if and only if* $M_t \geq M^{lb}(Z_t)$*, for any* $t \leq N-2$.

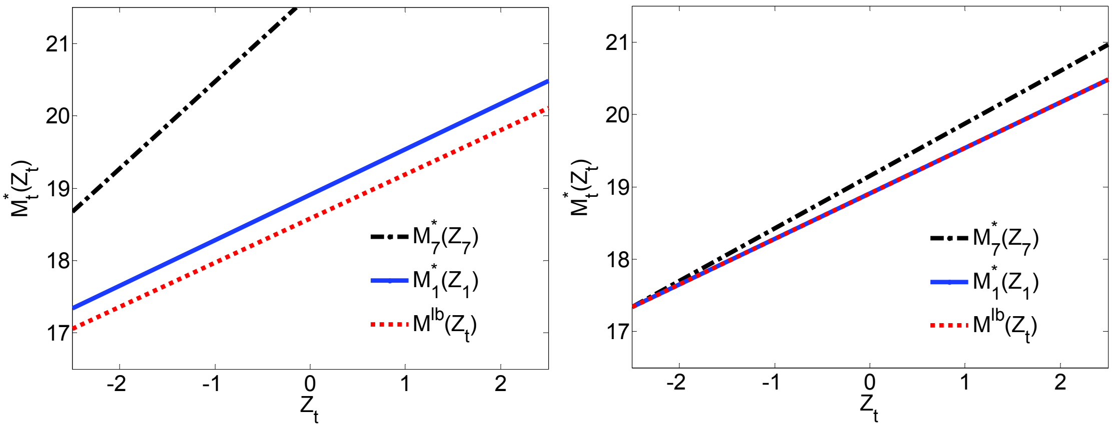
In general, however, the lower bound need not be tight under stochastic growth. If we assume that $R(M,T)$ is affine in M, the optimal hiring timing can be expressed in terms of a cash threshold.

PROPOSITION 3. *Suppose that Assumption 1 holds and that* $R(M,T) = a(T) + b(T)M$*, with* $a(T)$*,* $b'(T) \geq 0$*, and* $b(T) \geq 1$*. Then,*
(i) there exists a unique cash threshold $M_t^*(Z_t)$ *such that it is optimal to hire in period* t *if and only if* $M_t \geq M_t^*(Z_t)$.
(ii) If $\phi(Z_t, \zeta_t) = \rho Z_t + \zeta_t$*, with* $\{\zeta_t\}$ *independent,* $M_t^*(Z_t)$ *is increasing in* Z_t *if and only if* $\rho \geq \gamma$.

The first part of Proposition 3 translates to a clear prescription on the timing of hiring: the entrepreneur should hire as soon as $M_t^*(Z_t)$ units of money are available and not hire otherwise. The second part of Proposition 3 confirms the intuition that entrepreneurs should be more cautious and delay hiring when they expect greater cash drains in the future, i.e., when Z_t is larger.

The proof of Proposition 3 consists of analyzing the value functions in (3), which involve the composition of functions and are difficult to analyze for nonlinear revenue functions. Although the proof that a unique necessary and sufficient cash threshold exists requires that $R(M,T)$ is affine

Figure 3 **Lower bound $M^{lb}(Z_t)$ and optimal hiring thresholds $M_t^*(Z_t)$ for $t=1$ and $t=7$ as a function of Z_t, under larger (left) and smaller (right) shocks.**



Note. $R(M,T)=1.2M^{0.9}T^{0.4}$, $\gamma=0.1$, $T=2$, $(w,y)=(0.5,1.5)$, $(S_M,S_T)=(12,0.2)$. In the left panel, $\phi(Z_t,\zeta_t)=Z_t+\zeta_t$, where $\zeta_t=0\pm0.5$ with equal probability $\forall t$. In the right panel, $\zeta_t=0\pm0.05$ with equal probability $\forall t$.

in M, we have not observed any case with multiple cash thresholds in our extensive numerical analysis with more general revenue functions.

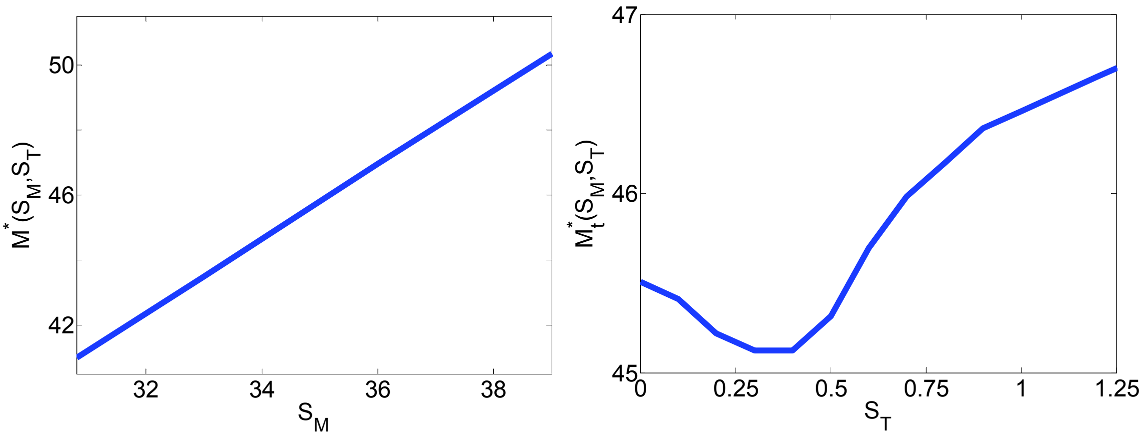
Building on the numerical example first shown in Figure 1, Figure 3 depicts the hiring cash threshold when the linearity assumption is relaxed. In this case, we found that the hiring thresholds were unique. The dotted curves in both panels indicate the lower bounds $M^{lb}(Z_t)$ characterized in Proposition 2, and the solid and dash-dot curves indicate the optimal hiring thresholds $M_t^*(Z_t)$ in periods $t=1$ and $t=7$ respectively. All thresholds are plotted as a function of Z_t (i.e., Z_1 or Z_7) and appear to be increasing in Z_t, consistent with Proposition 3(ii).

The left panel shows that the lower bound $M^{lb}(Z_t)$ and the optimal hiring threshold $M_t^*(Z_t)$ are in general far apart, and that the optimal hiring threshold in period $t=7$ is greater than in $t=1$, i.e., $M_7^*(Z_t)\geq M_1^*(Z_t)$. This is because with less periods to go in the planning horizon, the long-term benefits of hiring are less likely to justify the upfront costs. Comparing the right panel to the left panel shows that when the shock process $\{\zeta_t\}$ approaches a deterministic and nonincreasing sequence, the gaps between $M_t^*(Z_t)$ and $M^{lb}(Z_t)$ shrink towards zero for early periods (e.g., $t=1$), consistent with Corollary 2. This suggests that more volatile stochastic shocks may also contribute to delays in the optimal timing of hiring.

4.3. Sensitivity to Upfront Investment of Money and Time (S_M,S_T)

We next examine how the hiring decision is influenced by the required investment of money (S_M) and time (S_T) associated with hiring. An increase in either S_M or S_T means that the entrepreneur must use more resources for hiring, which would otherwise be used to generate revenue. Since an increase in either cash or time investment decreases the revenue in the hiring setup period and

Figure 4 **Optimal hiring thresholds with respect to S_M and S_T.**



Note. $R(M,T) = 1.2M^{0.9}T^{0.4}$, $\gamma = 0.1$, $T = 2$, $(w, y) = (0.5, 1.5)$. Left panel: $S_T = 0.2$, Right panel: $S_M = 35$, $Z_t = 0$ $\forall t$.

temporarily stunts the growth trajectory, it may seem intuitive at first that it would lead to a delay of hiring. However, we show next that the effect is more nuanced.

PROPOSITION 4. *Suppose Assumption 1 holds, and that* $\{\zeta_t\}$ *is deterministic* $\forall t$. *Then, if there exists a unique hiring threshold* $M_t^*(S_M, S_T)$,
(i) $M_t^*(S_M, S_T)$ *is increasing in* S_M,
(ii) $M_t^*(S_M, S_T)$ *need not be monotone in* S_T.

Proposition 4 highlights that the effects of hiring setup cost (S_M) and setup time (S_T) are in general different in an entrepreneurial setting. This difference is illustrated in Figure 4, which continues the numerical example from Figures 1 and 3. As expected, if the monetary investment required for hiring S_M increases (left panel), it is optimal to postpone the resource drain to a period where money is less critical, i.e., delaying hiring is optimal. By contrast, the effect of an increase in S_T need not be monotone (right panel). Two opposing effects are at play. Suppose that the time investment required for hiring S_T increases. On the one hand, delaying hiring would be desirable to postpone the resource drain and preserve the growth trajectory. On the other hand, the opportunity cost of time increases during the planning horizon (see Proposition 1), so the entrepreneur would prefer to hire earlier in order to incur the increased time investment (S_T) earlier when the value of time is lower.

In classical staffing models of large firms, no distinction is made between time and money investments associated with hiring, and the investment of time is often converted to costs (Gans and Zhou 2002). In an established firm, the upfront time investment is typically bore by a dedicated human resource department that typically has no direct impact on generating or fulfilling demand. In such setting, if hiring takes twice as long as expected, it doesn't have a direct effect on output, while for the entrepreneur it takes away that much productive time from the entrepreneur, so it does have a direct effect on output. If hiring required primarily monetary investment, then an increase in that cost would mean the entrepreneur should delay hiring. If, on the other hand, the investment is primarily of time, then it may be that, the greater that time investment, the earlier the entrepreneur should hire. Proposition 4 thus underscores the importance of distinguishing between time and money investments in early-stage growth-oriented entrepreneurial firms, due to the different paths of the shadow prices of time and money.

5. Numerical Illustrations

In this section, we complement the analysis of the previous section and numerically visualize the effect of hiring on the firm's growth (§5.1) and explore the sensitivity of the exact timing of hiring (§5.2).

Throughout this section, we consider a Cobb-Douglas revenue function, $R(M,T) = \kappa M^{\alpha} T^{\beta}$, with $\kappa = 1.2$, $\alpha = 0.9$, and $\beta = 0.4$, which satisfy the assumptions of the revenue function. (We obtain similar insights for other parameter values.) The relatively high α and low β indicates that additional money available brings a greater increase in revenue than additional time. We assume that $\gamma = 0.1$, i.e., that 10% of the revenue generated today will recur next period. This may represent a business that involves the entrepreneur buying and selling inventory to consumers, some of whom become repeat customers. We set the entrepreneur's available time each period as $T = 2$ (representing two months/period), and consider a planning horizon of $N = 12$ (two years).

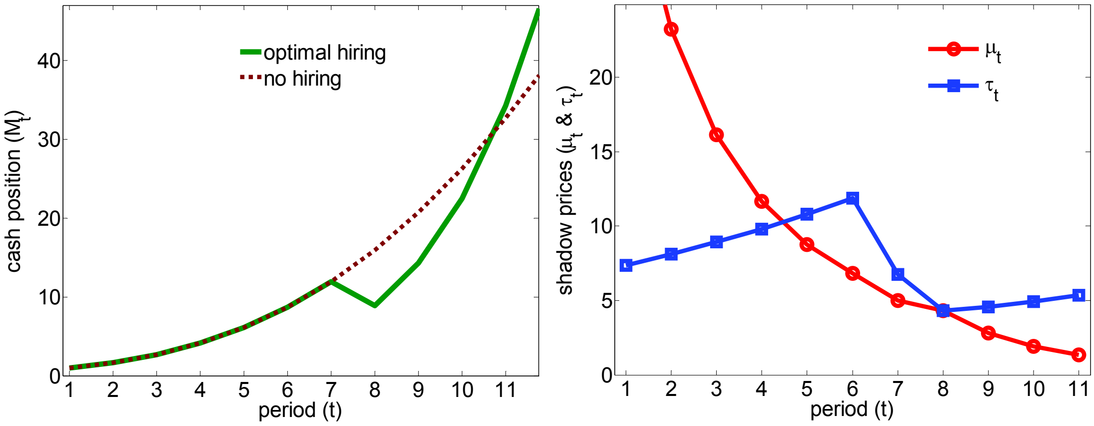
For the stochastic shock process $\{\zeta_t\}$, we assume that $\phi(Z_t, \zeta_t) = Z_t + \zeta_t$, where ζ_t are independent and identically distributed with the following 2-point distribution: $\zeta_t = d + \sigma$ or $d - \sigma$ with equal probability for all t, with d the drift and σ the volatility. A positive drift $d > 0$ implies that $\{Z_t\}$ will increase in t on average. All code is written in Matlab and is available from the authors upon request.

5.1. Effect of Hiring on the Firm's Growth

We illustrate the behavior of the optimal timing of hiring when the employee terms are $(w, y) = (0.5, 1.5)$, and the required upfront investments associated with hiring are $(S_M, S_T) = (6, 0.1)$. Identical to Figure 1, we assume $Z_t = 0\ \forall t$, i.e., $(d, \sigma) = (0, 0)$.

The left panel of Figure 5 shows a sample evolution of the cash position with and without hiring (solid curve and dotted curve respectively) starting from $M_1 = 1$. Examining the growth trajectory without hiring (dotted curve), we observe that the cash position accelerates approximately at a constant rate throughout. Examining the growth trajectory with hiring (solid curve), we observe that hiring optimally takes place in period $t = 7$. During this period, the entrepreneur's time and money are diverted from revenue-generating activities, and as a result, the cash positions for several

Figure 5 Effect of Hiring: Evolution of firm's cash position (M_t) and shadow prices of money and time (μ_t, τ_t).



Note. $R(M,T) = 1.2M^{0.9}T^{0.4}$, $\gamma = 0.1$, $M_1 = 1$, $T = 2$, $(w,y) = (0.5, 1.5)$, $(S_M, S_T) = (6, 0.1)$, $Z_t = 0\ \forall t$.

periods following hiring is less than what it would be without hiring. However, because additional time is available with hiring, growth is accelerated and the firm eventually reaches a higher cash position M_N than without hiring.

The right panel of Figure 5 illustrates the corresponding evolution of the shadow prices when hiring occurs. In contrast to Figure 1, when the shadow price of time continues to increase throughout the planning horizon without hiring (Proposition 1), we observe a marked decrease in the shadow price of time (and a slight increase in that of money) in period $t = 7$ and $t = 8$ after hiring occurs. This is because hiring makes more time and less money available to the firm. After hiring, extra time again continues to become more valuable while extra money continues to become less valuable, suggesting the need for more hiring in the future.

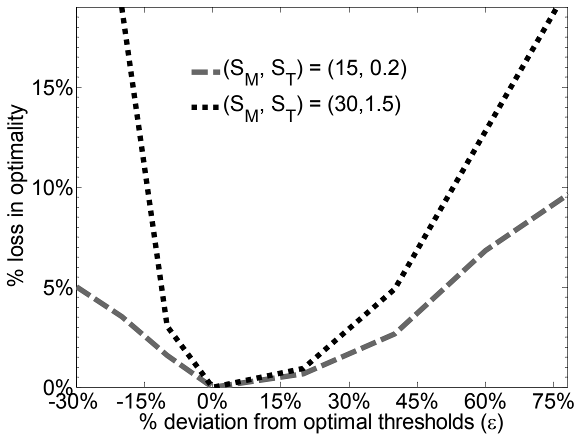
5.2. Sensitivity of the Timing of Hiring

In this section, we examine the consequences of sub-optimal timing of hiring. We determine the ϵ%-deviation from the optimal thresholds $M_t^*(Z_t)$ – i.e., $(1+\epsilon)M_t^*(Z_t)$ – and examine the percentage loss of optimality ($\frac{V^{opt}-V^{sim}}{V^{opt}}$) based on the average of 5,000 simulation runs, where V^{opt} denotes the computed optimal value and V^{sim} the simulation average based on applying the suboptimal hiring threshold policy.

When the ϵ%-deviation is negative, it indicates that hiring occurs too early, which is costly because it stunts the growth trajectory. When it is positive, it indicates that hiring takes place too late, which is costly because of missed opportunity in growth. Figure 6 illustrates the percentage loss of optimality when the hiring cash thresholds deviate from the optimal cash thresholds for varying levels of upfront cost S_M and time S_T, while keeping constant $w = 0.5$, $y = 1.5$, and $(d, \sigma) = (0, 0.5)$.

The curves show that timing of hiring can have a significant effect on the firm's growth objective, and that it depends on the setup costs and time. Hiring too early or too late can cause significant

Figure 6 **Loss of optimality due to hiring below or above the optimal cash threshold.**



Note. $R(M,T) = 1.2M^{0.9}T^{0.4}$, $\gamma = 0.1$, $T = 2$, $w = .5$, $y = 1.5$, and $M_1 = 20$, $Z_1 = 0$, $(d,\sigma) = (0, 0.5)$.

loss in optimality. The percentage loss in optimality is smaller when the setup cost and time are relatively low as indicated by the dashed curve ($(S_M, S_T) = (15, 0.2)$) than when they are relatively high as indicated by the dotted curve ($(S_M, S_T) = (30, 1.5)$). In the dashed curve, a drop in the optimal threshold by 30% leads to a 5%-loss in expected profit compared to the optimal; in the dotted curve, a drop of 15%, leads to a 20%-loss in the expected profit compared to the optimal. This suggests that when hiring requires significant upfront costs, mis-timing of hiring causes greater damage to the firm's growth compared to when it does not. This implies that for entrepreneurs, reducing the hiring setup time and cost (e.g., through process improvement) could lessen the negative effect of mistimed hiring.

We have focused our attention to the growth context and so limited our study to settings where firms do not run out of cash during the horizon (i.e., do not go bankrupt). In a separate numerical study, we have examined the effect of bankruptcy on the hiring decision. We found, as expected, that hiring too early can increase the probability of bankruptcy while hiring too late has little effect on further reducing that probability relative to the optimal hiring period. Therefore, when bankruptcy is a concern, erring on the side of hiring late would be more desirable than erring on the side of hiring too early.

6. Conclusion

In this paper, we consider an early-stage entrepreneur who is seeking growth by bootstrapping. We characterize when to hire the first employee, and assess how this decision is different than it would be in an established firm. To examine this, we analyze a stylized growth model for an early-stage entrepreneurial firm based on the assumptions that two inputs, namely, time and money, are required to generate revenue, and that both are limited. We conceptualize hiring as a time-money tradeoff; in particular, hiring can accelerate growth by trading away less valuable money to gain more valuable time, but timing is crucial. Within that growth model, we characterize the optimal timing of hiring as an optimal stopping rule in terms of the firm's cash position and analyze the sensitivity of that decision to the hiring setup time and setup costs.

Our results lead to several managerial implications for entrepreneurs. First, entrepreneurs should realize that their time will become more valuable as their firm grows and, as a result, that early investments to save time in the future are worth considering. Second, entrepreneurs should not necessarily wait with hiring until they feel that they have to hire, but they should consider hiring as soon as they can afford to. Third, if an entrepreneur realizes that she underestimated the monetary cost of hiring, she should delay hiring relative to her original plan; but if she underestimated the time cost of hiring, she may have to hire earlier rather than delaying. Fourth, entrepreneurs who may not be able to hire at the optimal time, should strive to reduce the hiring setup costs and time, e.g., by being on the lookout for that first employee from the moment they start the firm. Overall, our conceptual framework can help entrepreneurs determine the value and the timing of hiring an employee in terms of the associated time savings. For example, entrepreneurs can examine more closely their own time usage, and also assess the potential employees based on the time savings they provide.

Our results also highlight some key differences between entrepreneurial settings and more established firm settings, as far as the hiring decision is concerned. First, an employee's value increases over time in an entrepreneurial setting because the entrepreneur's time constraint becomes tighter as the firm grows, whereas it is more stable in established firms. Second, the timing of hiring depends on the firm's cash position in an entrepreneurial setting, whereas it is typically independent of the firm's cash position in a more established firm. Third, the optimal timing of hiring is influenced differently by the required upfront monetary investment and time investment during the hiring setup period, in contrast to more established firms, where hiring setup time and hiring setup costs are typically interchangeable. Fourth, the cost of mistiming hiring is more severe in entrepreneurial firms than in established firms because if stunts their growth in addition to deferring future profits. As we have shown, operations management insights that stem from large firms may not be relevant to the entrepreneurial setting. Our work considers only a subset of different types of entrepreneurial settings and exogenous uncertainties, and we hope that our work will stimulate further research on other important operational questions for entrepreneurs.

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Appendix

This appendix contains the proofs of the results in the paper as well as several intermediate Lemmas (A-1 through A-5) that we need in deriving those proofs.

Proof of Proposition 1. The proof proceeds by first deriving the expressions for μ_t and τ_t, and then using the expressions prove the result.

(i) We first show by induction that for any given N, μ_t and τ_t for $t < N$ are characterized by:

$$\mu_t = \frac{\partial M_N}{\partial M_t} = \sum_{i=t}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_t,T)}{\partial M_t}, \tag{A-1}$$

$$\tau_t = \frac{\partial M_N}{\partial T_t} = \sum_{i=t}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_t,T)}{\partial T}. \tag{A-2}$$

Start with $t = N-1$. Since from (1),

$$M_{t+1} = R(M_t,T) + \sum_{s=1}^{t-1}\gamma^{t-s}R(M_s,T) - \phi(Z_t,\zeta_t),$$

we have that when $Z_t = 0\ \forall t$,

$$M_N = R(M_{N-1},T) + \sum_{s=1}^{N-2}\gamma^{N-1-s}R(M_s,T),$$

and we obtain that $\mu_{N-1} = \frac{\partial M_N}{\partial M_{N-1}} = \frac{\partial R(M_{N-1},T)}{\partial M_{N-1}}$, and $\tau_{N-1} = \frac{\partial M_N}{\partial T_{N-1}} = \frac{\partial R(M_{N-1},T)}{\partial T}$.

Now, for the induction, suppose the expressions hold for $t+1$, i.e.,

$$\mu_{t+1} = \frac{\partial M_N}{\partial M_{t+1}} = \sum_{i=t+1}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+2}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_{t+1},T)}{\partial M_{t+1}},$$

$$\tau_{t+1} = \frac{\partial M_N}{\partial T_{t+1}} = \sum_{i=t+1}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+2}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_{t+1},T)}{\partial T}.$$

By (1), $M_N = \sum_{s=t+1}^{N-1}\gamma^{N-1-s}R(M_s,T) + \gamma^{N-1-t}R(M_t,T) + \sum_{s=1}^{t-1}\gamma^{N-1-s}R(M_s,T)$. Because the first term depends on M_t only through M_{t+1} and the last term is independent of M_t, we obtain, using the chain rule and using $\frac{\partial M_{k+1}}{\partial M_k} = \frac{\partial R(M_k,T)}{M_k}\ \forall k$,

$$\frac{\partial M_N}{\partial M_t} = \frac{\partial}{\partial M_{t+1}}\left[\sum_{s=t+1}^{N-1}\gamma^{N-1-s}R(M_s,T)\right]\frac{\partial M_{t+1}}{\partial M_t} + \gamma^{N-1-t}\frac{\partial R(M_t,T)}{\partial M_t}$$

$$
\begin{aligned}
&= \sum_{i=t+1}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial M_{t+1}}{\partial M_t}+\gamma^{N-1-t}\frac{\partial R(M_t,T)}{\partial M_t}\\
&= \sum_{i=t+1}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_t,T)}{\partial M_t}+\gamma^{N-1-t}\frac{\partial R(M_t,T)}{\partial M_t}\\
&= \sum_{i=t}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_t,T)}{\partial M_t},
\end{aligned}
$$

where the second equality is by the induction assumption. Similarly, using $\frac{\partial M_{k+1}}{\partial T_k}=\frac{\partial R(M_k,T)}{T_k}$ $\forall k$,

$$
\begin{aligned}
\frac{\partial M_N}{\partial T_t} &= \frac{\partial}{\partial M_{t+1}}\left[\sum_{s=t+1}^{N-1}\gamma^{N-1-s}R(M_s,T)\right]\frac{\partial M_{t+1}}{\partial T_t}+\gamma^{N-1-t}\frac{\partial R(M_t,T)}{\partial T}\\
&= \sum_{i=t+1}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial M_{t+1}}{\partial T_t}+\gamma^{N-1-t}\frac{\partial R(M_t,T)}{\partial T}\\
&= \sum_{i=t+1}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_t,T)}{\partial T}+\gamma^{N-1-t}\frac{\partial R(M_t,T)}{\partial T}\\
&= \sum_{i=t}^{N-1}\left(\gamma^{N-1-i}\prod_{k=t+1}^{i}\frac{\partial R(M_k,T)}{\partial M_k}\right)\cdot\frac{\partial R(M_t,T)}{\partial T},
\end{aligned}
$$

where the second equality is by the induction assumption.

(ii) Based on expressions (A-1)-(A-2), we have:

$$
\frac{\mu_t}{\tau_t}=\frac{\frac{\partial R(M_t,T)}{\partial M_t}}{\frac{\partial R(M_t,T)}{\partial T}}. \tag{A-3}
$$

When $Z_t=0$, $M_{t+1}\geq M_t$ since $R(M,T)=R(0,T)+\int_0^M\frac{\partial R(m,T)}{\partial m}dm\geq R(0,T)+M\geq M$. Therefore, $\frac{\partial R(M_{t+1},T)}{\partial T}\geq\frac{\partial R(M_t,T)}{\partial T}$ due to increasing differences of $R(M,T)$ and $\frac{\partial R(M_{t+1},T)}{\partial M}\leq\frac{\partial R(M_t,T)}{\partial M}$ by concavity. As a result, $\mu_t/\tau_t\geq\mu_{t+1}/\tau_{t+1}$, i.e., $\{\mu_t/\tau_t\}$ is a monotone decreasing sequence. Thus, $\{\mu_t/\tau_t\}$ crosses 1 at most once, and if it does, it does so from above. $\square$

Proof of Corollary 1. Consider an employee who costs $w=\epsilon$ and contributes $y=c\epsilon$ each period after being hired. As $\epsilon\to 0$, if that employee is hired in period t, his/her value to the firm is equal to $c\left(\sum_{s=t}^{N-1}\tau_s\right)-\left(\sum_{s=t}^{N-1}\mu_s\right)$. Hence, it is worthwhile to hire that employee from period t onwards when $c\left(\sum_{s=t}^{N-1}\tau_s\right)\geq\left(\sum_{s=t}^{N-1}\mu_s\right)$. By Proposition 1, μ_t/τ_t is decreasing in t. Hence if $c\cdot\tau_t\geq\mu_t$, then $c\cdot\tau_s\geq\mu_s$ $\forall s>t$, and therefore $c\sum_{s=t}^{N-1}\tau_s\geq\sum_{s=t}^{N-1}\mu_s$. $\square$

Proof of Lemma 1. For $f(M_t,Z_t,\zeta_t)$, we have from (1),

$$
\begin{aligned}
f(M_t,Z_t,\zeta_t) &= R(M_t,T)+\gamma R(M_{t-1},T)+\sum_{s=1}^{t-2}\gamma^{t-s}R(M_s,T)-\phi(Z_t,\zeta_t)\\
&= R(M_t,T)+\gamma\left(R(M_{t-1},T)+\sum_{s=1}^{t-2}\gamma^{t-1-s}R(M_s,T)\right)-\phi(Z_t,\zeta_t)\\
&= R(M_t,T)+\gamma\left(R(M_{t-1},T)+\sum_{s=1}^{t-2}\gamma^{t-1-s}R(M_s,T)-\phi(Z_{t-1},\zeta_{t-1})\right)+\gamma\phi(Z_{t-1},\zeta_{t-1})-\phi(Z_t,\zeta_t)\\
&= R(M_t,T)+\gamma M_t+\gamma Z_t-\phi(Z_t,\zeta_t).
\end{aligned}
$$

Similarly, for $g(M_t, Z_t, \zeta_t)$ and $h(M_t, Z_t, \zeta_t)$, we have

$$\begin{aligned} g(M_t, Z_t, \zeta_t) &= R(M_t - S_M, T - S_T) + \gamma R(M_{t-1}, T) + \sum_{s=1}^{t-2} \gamma^{t-s} R(M_s, T) - \phi(Z_t, \zeta_t) \\ &= R(M_t - S_M, T - S_T) + \gamma \left(R(M_{t-1}, T) + \sum_{s=1}^{t-2} \gamma^{t-1-s} R(M_s, T) \right) - \phi(Z_t, \zeta_t) \\ &= R(M_t - S_M, T - S_T) + \gamma M_t + \gamma Z_t - \phi(Z_t, \zeta_t), \\ h(M_t, Z_t, \zeta_t) &= R(M_t - w, T + y) + \gamma R(M_{t-1}, T) + \sum_{s=1}^{t-2} \gamma^{t-s} R(M_s, T) - \phi(Z_t, \zeta_t) \\ &= R(M_t - w, T + y) + \gamma \left(R(M_{t-1}, T) + \sum_{s=1}^{t-2} \gamma^{t-1-s} R(M_s, T) \right) - \phi(Z_t, \zeta_t) \\ &= R(M_t - w, T + y) + \gamma M_t + \gamma Z_t - \phi(Z_t, \zeta_t). \quad \square \end{aligned}$$

LEMMA A-1. *Under Assumption 1, $\bar{g}(f(M_t, Z_t, \zeta_t)) - \bar{h}(g(M_t, Z_t, \zeta_t))$ is decreasing in M_t.*

Proof of Lemma A-1. For all M_t, by (2), we have:

$$\begin{aligned} &\bar{g}(f(M_t, Z_t, \zeta_t)) - \bar{h}(g(M_t, Z_t, \zeta_t)) \\ &= \gamma f(M_t, Z_t, \zeta_t) + R(f(M_t, Z_t, \zeta_t) - S_M, T - S_T) - (\gamma g(M_t, Z_t, \zeta_t) + R(g(M_t, Z_t, \zeta_t) - w, T + y)) \\ &= \gamma [f(M_t, Z_t, \zeta_t) - g(M_t, Z_t, \zeta_t)] + R(f(M_t, Z_t, \zeta_t) - S_M, T - S_T) - R(g(M_t, Z_t, \zeta_t) - w, T + y). \end{aligned}$$

By Assumption 1, $f(M_t, Z_t, \zeta_t) - g(M_t, Z_t, \zeta_t)$ is decreasing in M_t, $\forall \zeta_t$. We next show that $R(f(M_t, Z_t, \zeta_t) - S_M, T - S_T) - R(g(M_t, Z_t, \zeta_t) - w, T + y)$ is decreasing in M_t, $\forall \zeta_t$. For $\epsilon > 0$, we have

$$\begin{aligned} &R(g(M_t + \epsilon, Z_t, \zeta_t) - w, T + y) - R(g(M_t, Z_t, \zeta_t) - w, T + y) \\ &\geq R(g(M_t + \epsilon, Z_t, \zeta_t) - w, T - S_T) - R(g(M_t, Z_t, \zeta_t) - w, T - S_T) \\ &\geq R(f(M_t + \epsilon, Z_t, \zeta_t) + g(M_t, Z_t, \zeta_t) - f(M_t, Z_t, \zeta_t) - w, T - S_T) - R(g(M_t, Z_t, \zeta_t) - w, T - S_T) \\ &\geq R(f(M_t + \epsilon, Z_t, \zeta_t) - S_M, T - S_T) - R(f(M_t, Z_t, \zeta_t) - S_M, T - S_T), \end{aligned}$$

in which the first inequality follows because $g(M_t, Z_t, \zeta_t)$ is increasing in M_t and because $R(M, T)$ has increasing differences in (M, T); the second is because $R(M, T)$ is increasing in M and $g(M_t, Z_t, \zeta_t) - f(M_t, Z_t, \zeta_t)$ is increasing in M_t by Assumption 1; and the last inequality is because $R(M, T)$ is concave in M and

$$\begin{aligned} f(M_t, Z_t, \zeta_t) - g(M_t, Z_t, \zeta_t) + w &= R(M_t, T) - R(M_t - S_M, T - S_T) + w \geq R(M_t, T - S_T) - R(M_t - S_M, T - S_T) \\ &= \int_{M_t - S_M}^{M_t} \frac{\partial R(m, T - S_T)}{\partial m} dm \geq \int_{M_t - S_M}^{M_t} dm = S_M. \quad \square \end{aligned}$$

LEMMA A-2. *Define for all $t < N - 1$,*

$$\begin{aligned} \Delta_t(M_t, Z_t) &= \mathbb{E}_{\zeta_t} \mathbb{E}_{\zeta_{t+1}} \{ V_{t+2}(1, g(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1})) \\ &\quad - V_{t+2}(1, h(g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1})) \} \end{aligned} \tag{A-4}$$

$$\beta_t(M_t, Z_t) = \Delta_t(M_t, Z_t) + \mathbb{E}_{\zeta_t} \max\{0, \beta_{t+1}(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\}, \tag{A-5}$$

$$\begin{aligned} \beta_{N-1}(M_{N-1}, Z_{N-1}) &= \mathbb{E}_{\zeta_{N-1}} \{ V_N(0, f(M_{N-1}, Z_{N-1}, \zeta_{N-1}), \phi(Z_{N-1}, \zeta_{N-1})) \\ &\quad - V_N(1, g(M_{N-1}, Z_{N-1}, \zeta_{N-1}), \phi(Z_{N-1}, \zeta_{N-1})) \}. \end{aligned} \tag{A-6}$$

Then hiring is optimal in period t if and only if $\beta_t(M_t, Z_t) < 0$.

Proof of Lemma A-2 The proof proceeds by induction by showing that

$$\beta_t(M_t, Z_t) = \mathbb{E}_{\zeta_t}[V_{t+1}(0, f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t)) - V_{t+1}(1, g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))], \ \forall t.$$

By (A-6), this holds when $t = N-1$. Next fix $t < N-1$, and suppose that

$$\beta_{t+1}(M_{t+1}, Z_{t+1}) = \mathbb{E}_{\zeta_{t+1}}[V_{t+2}(0, f(M_{t+1}, Z_{t+1}, \zeta_{t+1}), \phi(Z_{t+1}, \zeta_{t+1})) - V_{t+2}(1, g(M_{t+1}, Z_{t+1}, \zeta_{t+1}), \phi(Z_{t+1}, \zeta_{t+1}))].$$

Using (3), (A-4), and (A-5), together with the induction hypothesis, we obtain

$$\begin{aligned}
&\beta_t(M_t, Z_t)\\
&= \Delta_t(M_t, Z_t) + \mathbb{E}_{\zeta_t} \max\{0, \beta_{t+1}(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\}\\
&= \{\mathbb{E}_{\zeta_t}\mathbb{E}_{\zeta_{t+1}} V_{t+2}(1, g(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1}))\\
&\quad -\mathbb{E}_{\zeta_t}\mathbb{E}_{\zeta_{t+1}} V_{t+2}(1, h(g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1}))\}\\
&\quad +\mathbb{E}_{\zeta_t} \max\{0, \beta_{t+1}(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\}.\\
&= \mathbb{E}_{\zeta_t} \max\Big\{\mathbb{E}_{\zeta_{t+1}} V_{t+2}(1, g(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1})),\ \beta_{t+1}(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\\
&\quad +\mathbb{E}_{\zeta_{t+1}} V_{t+2}(1, g(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1}))\Big\} - \mathbb{E}_{\zeta_t} V_{t+1}(1, g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\\
&= \mathbb{E}_{\zeta_t} \max\Big\{\mathbb{E}_{\zeta_{t+1}} V_{t+2}(1, g(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1})),\\
&\qquad \mathbb{E}_{\zeta_{t+1}} V_{t+2}(0, f(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1}))\Big\} - \mathbb{E}_{\zeta_t} V_{t+1}(1, g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\\
&= \mathbb{E}_{\zeta_t}[V_{t+1}(0, f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t)) - V_{t+1}(1, g(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))],
\end{aligned}$$

which completes the induction step. Because the last expression denotes the difference in expected value-to-go between continuing without and with hiring in period t, it is desirable to hire the employee in period t if and only if $\beta_t(M_t, Z_t) < 0$. $\square$

LEMMA A-3. *If $R(M_t, T) = a(T) + b(T) \cdot M_t$, then*

$$\Delta_t(M_t, Z_t) = (\gamma + b(T+y))^{N-t-2}\mathbb{E}_{\zeta_t}\Big(\bar{g}(f(M_t, Z_t, \zeta_t)) - \bar{h}(g(M_t, Z_t, \zeta_t))\Big), \tag{A-7}$$

where $\Delta_t(M_t, Z_t)$ is defined in (A-4).

Proof of Lemma A-3 If $R(M_t, T) = a(T) + b(T) \cdot M_t$, then by (2),

$$\begin{aligned}
h(M_t, Z_t, \zeta_t) &= \gamma M_t + a(T+y) + b(T+y) \cdot (M_t - w) - \phi(Z_t, \zeta_t) + \gamma Z_t\\
&= (\gamma + b(T+y))M_t + a(T+y) - b(T+y)w - \phi(Z_t, \zeta_t) + \gamma Z_t.\\
h(h(M_t, Z_t, \zeta_t), Z_{t+1}, \zeta_{t+1}) &= (\gamma + b(T+y))h(M_t, Z_t, \zeta_t) + a(T+y) - b(T+y)w - \phi(Z_{t+1}, \zeta_{t+1}) + \gamma Z_{t+1}\\
&= (\gamma + b(T+y))^2 M_t + (1 + (\gamma + b(T+y)))[a(T+y) - b(T+y)w]\\
&\qquad +[-\phi(Z_{t+1}, \zeta_{t+1}) + \gamma Z_{t+1}] + (\gamma + b(T+y))[-\phi(Z_t, \zeta_t) + \gamma Z_t].\\
&\ \vdots\\
\underbrace{h(\cdots(h(M_t, Z_t, \zeta_t))\cdots)}_{N-t} &= (\gamma + b(T+y))^{N-t} M_t + \sum_{k=0}^{N-t-1} (\gamma + b(T+y))^k [a(T+y) - b(T+y)w]
\end{aligned}$$

$$+\sum_{k=0}^{N-t-1}(\gamma+b(T+y))^{N-t-1-k}[-\phi(Z_{t+k},\zeta_{t+k})+\gamma Z_{t+k}].$$

Thus, by (3), for any $t\le N-1$,

$$\begin{aligned}V_t(1,M_t,Z_t) &= E_{\zeta_t\cdots\zeta_{N-1}}h(\cdots h(h(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1})\cdots)\\ &= (\gamma+b(T+y))^{N-t}M_t+\sum_{k=0}^{N-t-1}(\gamma+b(T+y))^k[a(T+y)-b(T+y)w]\\ &\quad +E_{\zeta_t\cdots\zeta_{N-1}}\left\{\sum_{k=0}^{N-t-1}(\gamma+b(T+y))^{N-t-1-k}[-\phi(Z_{t+k},\zeta_{t+k})+\gamma Z_{t+k}]\right\}.\end{aligned}$$

Note that the expression for $V_t(1,M_t,Z_t)$ consists of a term that is linear in M_t and remaining terms that are independent of M_t. Thus, by (A-4) we have

$$\begin{aligned}\Delta_t(M_t,Z_t) &= \mathbb{E}_{\zeta_t}\mathbb{E}_{\zeta_{t+1}}\Big\{V_{t+2}(1,g(f(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1}),\phi(\phi(Z_t,\zeta_t),\zeta_{t+1}))\\ &\qquad -V_{t+2}(1,h(g(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1}),\phi(\phi(Z_t,\zeta_t),\zeta_{t+1}))\Big\}\\ &= \mathbb{E}_{\zeta_t}\mathbb{E}_{\zeta_{t+1}}[(\gamma+b(T+y))^{N-t-2}g(f(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1})\\ &\qquad -(\gamma+b(T+y))^{N-t-2}h(g(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1})]\\ &= (\gamma+b(T+y))^{N-t-2}\mathbb{E}_{\zeta_t}\Big(\bar g(f(M_t,Z_t,\zeta_t))-\bar h(g(M_t,Z_t,\zeta_t))\Big),\end{aligned}$$

where the second equality is due to the cancelation of the identical trailing terms from period $t+2,\ldots,N$, and the final equality is because of (2). $\square$

LEMMA A-4. *Suppose that Assumption 1 holds. If $R(M_t,T)=a(T)+b(T)\cdot M_t$ with $a(T),b(T)$, $b'(T)\ge 0$, and $b(T)\ge 1$, then $\beta_t(M_t,Z_t)$ is decreasing in M_t, $\forall Z_t$, where $\beta_t(M_t,Z_t)$ is defined by (A-5) and (A-6).*

Proof of Lemma A-4 We proceed by induction. For $t=N-1$ from (A-6) and (3), $\beta_{N-1}(M_{N-1},Z_{N-1})=\mathbb{E}_{\zeta_{N-1}}[f(M_{N-1},Z_{N-1},\zeta_{N-1})-g(M_{N-1},Z_{N-1},\zeta_{N-1})]$, which is decreasing in M_{N-1} for all Z_{N-1} by Assumption 1. Now suppose that $\beta_{t+1}(M_{t+1},Z_{t+1})$ is decreasing in M_{t+1}. By (A-5), we have $\beta_t(M_t,Z_t)=\Delta_t(M_t,Z_t)+\mathbb{E}_{\zeta_t}\max\{0,\beta_{t+1}(f(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t))\}$. Since $f(M_t,Z_t,\zeta_t)$ is increasing in M_t by (2), the second term is non-increasing in M_t. Moreover, $\Delta_t(M_t,Z_t)$ given by (A-7) in Lemma A-3 is decreasing in M_t because $\bar g(f(M_t,Z_t,\zeta_t))-\bar h(g(M_t,Z_t,\zeta_t))$ is decreasing in M_t by Lemma A-1 under Assumption 1. As a result, $\beta_t(M_t,Z_t)$ is decreasing in M_t. $\square$

Proof of Proposition 2. The proof relies on Lemma A-2. By this lemma, hiring is not optimal if $\beta_t(M_t,Z_t)>0$ where $\beta_t(M_t,Z_t)$ is defined in (A-5). Given that $\beta_t(M_t,Z_t)>0$ if $\Delta_t(M_t,Z_t)>0$, where Δ_t is defined by (A-4), it suffices to show that $\Delta_t(M_t,Z_t)>0$.

By Lemma A-1, for all ζ_t, $\bar g(f(M_t,Z_t,\zeta_t))-\bar h(g(M_t,Z_t,\zeta_t))$ is decreasing in M_t and therefore $\bar g(f(M_t,Z_t,\zeta_t))>\bar h(g(M_t,Z_t,\zeta_t))$ for all $M<M^{lb}(Z_t)$. Because $V_{t+2}(1,M_{t+2},Z_{t+2})$ is increasing in M_{t+2}, we obtain that for all $M_t<M^{lb}(Z_t)$, $V_{t+2}(1,g(f(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1}),\phi(\phi(Z_t,\zeta_t),\zeta_{t+1}))>V_{t+2}(1,h(g(M_t,Z_t,\zeta_t),\phi(Z_t,\zeta_t),\zeta_{t+1}),\phi(\phi(Z_t,\zeta_t),\zeta_{t+1}))$, $\forall\zeta_t,\zeta_{t+1}$. Therefore, $\Delta_t(M_t,Z_t)>0$, where $\Delta_t(M_t,Z_t)$ is defined in (A-4), which implies $\beta_t(M_t,Z_t)>0$ by (A-5). Thus, by Lemma A-2, hiring is not optimal. $\square$

Proof of Corollary 2. The proof relies on Lemma A-2. In particular, we show that it is optimal to hire if and only if $\beta_t(M_t, Z_t) > 0$, which we will show will be equivalent to having $\Delta_t(M_t, Z_t) > 0$, which we will show will be equivalent to having $M_t \geq M^{lb}$.

Since $Z_t = 0$ for all t, we suppress the arguments ζ_t and Z_t throughout the proof. By (A-4) we have,

$$\Delta_t(M_t) = V_{t+2}(1, g(f(M_t))) - V_{t+2}(1, h(g(M_t))).$$

Because $\bar{g}(f(M_t)) - \bar{h}(g(M_t))$ is decreasing in M_t by Lemma A-1, $M^{lb}(Z_t)$ is such that $\bar{g}(f(M^{lb})) = \bar{h}(g(M^{lb}))$. Since $V_{t+2}(1, M)$ is increasing in M for all t, we have that $\Delta_t(M^{lb}) = 0$ for all t, and $\Delta_t(M_t) > 0$ if $M_t < M^{lb}$ and $\Delta_t(M_t) < 0$ if $M_t > M^{lb}$.

We will show by induction that for any $t \leq N-2$, $\beta_t(M_t, Z_t) > 0$ if and only if $\Delta_t(M_t, Z_t) > 0$ $\forall t$.

Start with the base case. By assumption that $V_N(0, f(M)) < V_N(1, g(M))$ and (A-6), $\beta_{N-1}(M_{N-1}) < 0$. Hence by (A-5), $\beta_{N-2}(M_{N-2}) = \Delta_{N-2}(M_{N-2})$.

Next, the induction step. Fix $t < N-2$. Now suppose that $\beta_{t+1}(M_{t+1}, Z_{t+1}) > 0$ if and only if $\Delta_{t+1}(M_{t+1}, Z_{t+1}) > 0$. From (A-5), it is clear that if $\Delta_t(M_t) \geq 0$ then $\beta_t(M_t) \geq 0$. Next, assume that $\Delta_t(M_t) < 0$. Since $\Delta_t(M_t) < 0$, $M_t > M^{lb}$. Because $f(M_t) > M_t$ when $Z_t = 0$, $f(M_t) > M^{lb}$. Hence $\Delta_{t+1}(f(M_t)) < 0$. By induction hypothesis, we thus have that $\beta_{t+1}(f(M_t)) < 0$, which implies, by (A-5), that $\beta_t(M_t) = \Delta_t(M_t) < 0$. This completes the induction step. By Lemma A-2, it is thus optimal to hire in period $t \leq N-2$ if and only if $\Delta_t(M_t) < 0$ if and only if $M_t > M^{lb}$. $\square$

Proof of Proposition 3. (i) By Lemma A-2, hiring is optimal if and only if $\beta_t(M_t, Z_t) < 0$. By Lemma A-4 if $R(M, T) = a(T) + b(T)M$, $\beta_t(M_t, Z_t)$ is decreasing in M_t for all Z_t. Hence, there exists a unique cash level $M_t^*(Z_t)$ such that $\beta_t(M_t^*, Z_t) = 0$, for which hiring is optimal if and only if $M_t \geq M_t^*(Z_t)$.

(ii) We will show by induction that $\beta_t(M_t, Z_t)$ is increasing in Z_t. From (A-5) and (3), $\beta_{N-1}(M_{N-1}, Z_{N-1}) = \mathbb{E}_{\zeta_{N-1}}[f(M_{N-1}, Z_{N-1}, \zeta_{N-1}) - g(M_{N-1}, Z_{N-1}, \zeta_{N-1})] = \bar{f}(M_{N-1}) - \bar{g}(M_{N-1})$, which is independent of Z_{N-1}. Next, suppose that $\beta_{t+1}(M_{t+1}, Z_{t+1})$ is increasing in Z_{t+1}. By (A-5), we have $\beta_t(M_t, Z_t) = \Delta_t(M_t, Z_t) + E_{\zeta_t} \max\{0, \beta_{t+1}(f(M_t, Z_t, \zeta_t), \phi(Z_t, \zeta_t))\}$. We know that $f(M_t, Z_t, \zeta_t)$ is decreasing in Z_t by (2) since $\frac{\partial \phi(Z_t, \zeta_t)}{\partial Z_t} = 1$ and $\gamma < 1$, that $\beta_{t+1}(M_{t+1}, Z_{t+1})$ is decreasing in M_{t+1} by Lemma A-4, and that $\phi(Z_t, \zeta_t) = \rho Z_t + \zeta_t$ is increasing in Z_t. Therefore, by the induction assumption, the second term is increasing in Z_t. It is clear from (A-7) in Lemma A-3 that $\Delta_t(M_t, Z_t)$ increases in Z_t if and only if $\bar{g}(f(M_t, Z_t, \zeta_t)) - \bar{h}(g(M_t, Z_t, \zeta_t))$ increases in Z_t. Since by (2),

$$\begin{aligned}
&\bar{g}(f(M_t, Z_t, \zeta_t)) - \bar{h}(g(M_t, Z_t, \zeta_t)) \\
&= (\gamma + b(T - S_T))f(M_t, Z_t, \zeta_t) + a(T - S_T) - b(T - S_T)S_M \\
&\quad - [(\gamma + b(T + y))g(M_t, Z_t, \zeta_t) + a(T + y) - b(T + y)w] \\
&= (\gamma + b(T - S_T))\bar{f}(M_t) + (\gamma + b(T - S_T))[-\phi(Z_t, \zeta_t) + \gamma Z_t] + a(T - S_T) - b(T - S_T)S_M \\
&\quad - [(\gamma + b(T + y))\bar{g}(M_t) + (\gamma + b(T + y))[-\phi(Z_t, \zeta_t) + \gamma Z_t] + a(T + y) - b(T + y)w] \\
&= (\rho - \gamma)[b(T + y) - b(T - S_T)]Z_t + \Big\{(\gamma + b(T - S_T))\bar{f}(M_t) + a(T - S_T) - b(T - S_T)S_M \\
&\quad - [(\gamma + b(T + y))\bar{g}(M_t) + a(T + y) - b(T + y)w]\Big\} + [b(T + y) - b(T - S_T)]\zeta_t,
\end{aligned}$$

which is increasing in Z_t since $b(T+y) \geq b(T-S_T)$ if and only if $\rho \geq \gamma$. Hence, $\Delta_t(M_t, Z_t)$ is increasing in Z_t. Therefore, $\beta_t(M_t, Z_t)$ is increasing in Z_t. Since $\beta_t(M_t, Z_t)$ is decreasing in M_t (Lemma A-4), we have that for all t, $M_t^*(Z_t)$ is increasing in Z_t. □

LEMMA A-5. *Suppose that Assumption 1 holds. Let $\{\zeta_t\}$ be a deterministic sequence and M_t^* be such that $\beta_t(M_t^*, Z_t) = 0$.*
(i) If $\bar{g}(f(M_t^*, Z_t, \zeta_t)) - \bar{h}(g(M_t^*, Z_t, \zeta_t))$ is increasing in (S_M, S_T) $\forall t$, then $\Delta_t(M_t^*, Z_t)$ is increasing in (S_M, S_T) $\forall t$.
(ii) If $\Delta_t(M_t^*, Z_t)$ is increasing in (S_M, S_T) $\forall t$, then $\beta_t(M_t^*, Z_t)$ is increasing in (S_M, S_T) $\forall t$.

Proof of Lemma A-5. (i) Let $x \in \{S_M, S_T\}$. By (A-4) when $\{\zeta_t\}$ is deterministic, we obtain, using the chain rule,

$$\begin{aligned}
\frac{\partial \Delta_t(M_t^*, Z_t)}{\partial x} &= \frac{\partial}{\partial x}\Big\{V_{t+2}(1, g(f(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1})) \\
&\qquad - V_{t+2}(1, h(g(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1}), \phi(\phi(Z_t, \zeta_t), \zeta_{t+1}))\Big\} \\
&= \left(\frac{\partial V_{t+2}(1, M, Z_{t+2})}{\partial M}\Big|_{M = g(f(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1})}\right)\left(\frac{\partial}{\partial x} g(f(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1})\right) \\
&\qquad - \left(\frac{\partial V_{t+2}(1, M, Z_{t+2})}{\partial M}\Big|_{M = h(g(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1})}\right)\left(\frac{\partial}{\partial x} h(g(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1})\right) \\
&\geq \left(\frac{\partial V_{t+2}(1, M, Z_{t+2})}{\partial M}\Big|_{M = g(f(M_t^*, Z_t, \zeta_t), \phi(Z_t, \zeta_t), \zeta_{t+1})}\right)\left(\frac{\partial}{\partial x}\Big(\bar{g}(f(M_t^*, Z_t, \zeta_t)) - \bar{h}(g(M_t^*, Z_t, \zeta_t))\Big)\right),
\end{aligned}$$

where the inequality is because (a) by (3),

$$V_{t+2}(1, M_{t+2}, Z_{t+2}) = E_{\zeta_{t+2} \cdots \zeta_{N-1}} h(\cdots h(h(M_{t+2}, Z_{t+2}, \zeta_{t+2}), \phi(Z_{t+2}, \zeta_{t+2}), \zeta_{t+3}) \cdots),$$

is a composition of concave increasing functions, and is therefore concave increasing (Boyd and Vandenberghe 2004) and (b) $\bar{g}(f(M_t^*, Z_t, \zeta_t)) < \bar{h}(g(M_t^*, Z_t, \zeta_t))$ by Lemma A-1 since $M_t^*(Z_t) \geq M^{lb}(Z_t)$ by Proposition 2 and since $\{\zeta_t\}$ is deterministic. The result then follows from the fact that $\frac{\partial V_{t+2}}{\partial M} > 0$.

(ii) We prove the result by induction. (Base case). From (A-6), (2), and (3), $\beta_{N-1}(M_{N-1}; S_M, S_T) = f(M_{N-1}, Z_{N-1}, \zeta_{N-1}) - g(M_{N-1}, Z_{N-1}, \zeta_{N-1}) = \bar{f}(M_{N-1}) - \bar{g}(M_{N-1}) = R(M_t, T) - R(M_t - S_M, T - S_T)$, which is increasing in S_M and S_T. Next fix $t < N-1$ and suppose that $\beta_{t+1}(M_{t+1}; S_M, S_T)$ is increasing in (S_M, S_T). By (A-5), we have $\beta_t(M_t; S_M, S_T) = \Delta_t(M_t; S_M, S_T) + \max\{0, \beta_{t+1}(f(M_t, Z_t, \zeta_t); S_M, S_T)\}$. We know that $f(M_t, Z_t, \zeta_t)$ is independent of (S_M, S_T) by (2) and so the second term is increasing in (S_M, S_T). Thus, if $\Delta_t(M_t; S_M, S_T)$ is increasing in (S_M, S_T), then $\beta_t(M_t; S_M, S_T)$ is also increasing in (S_M, S_T). □

Proof of Proposition 4. Because it is optimal to hire in period t if and only if $M_t \geq M_t^*(S_M, S_T)$ by assumption, Lemma A-2 implies that $\beta_t(M_t; S_M, S_T)$ crosses 0 only once at $M_t = M_t^*(S_M, S_T)$ and the crossing is from above. Thus, if $\beta_t(M_t; S_M, S_T)$ is increasing (or decreasing) in $x \in \{S_M, S_T\}$, this implies that $M_t^*(S_M, S_T)$ is increasing (or decreasing).

The proof consists in showing that $\bar{g}(f(M, Z_t, \zeta_t)) - \bar{h}(g(M, Z_t, \zeta_t))$ is increasing in S_M, which implies by Lemma A-5(i), that $\Delta_t(M_t^*, Z_t)$ is increasing in S_M, which implies by Lemma A-5(ii) that $\beta_t(M_t^*, Z_t)$ is increasing in S_M.

(i) Consider the derivative of the function $\bar{g}(f(M,Z_t,\zeta_t)) - \bar{h}(g(M,Z_t,\zeta_t))$ with respect to S_M. For any ζ_t, by the chain rule and (2), we have

$$\begin{aligned}
&\frac{\partial\{\bar{g}(f(M_t,Z_t,\zeta_t)) - \bar{h}(g(M_t,Z_t,\zeta_t))\}}{\partial S_M} \\
&= \left(-\frac{\partial R(f(M_t,Z_t,\zeta_t) - S_M, T - S_T)}{\partial M}\right) - \left(\gamma + \frac{\partial R(g(M_t,Z_t,\zeta_t) - w, T+y)}{\partial M}\right)\left(-\frac{\partial R(M_t - S_M, T - S_T)}{\partial M}\right) \\
&= \left(\gamma + \frac{\partial R(g(M_t,Z_t,\zeta_t) - w, T+y)}{\partial M}\right)\left(\frac{\partial R(M_t - S_M, T - S_T)}{\partial M}\right) - \left(\frac{\partial R(f(M_t,Z_t,\zeta_t) - S_M, T - S_T)}{\partial M}\right) \\
&\geq \frac{\partial R(g(M_t,Z_t,\zeta_t) - w, T+y)}{\partial M} - \frac{\partial R(f(M_t,Z_t,\zeta_t) - S_M, T - S_T)}{\partial M} \geq 0,
\end{aligned}$$

in which the first inequality follows because $\gamma \geq 0$ and because $\frac{\partial R(M,T)}{\partial M} \geq 1$ by assumption, and the second inequality is because $R(g(M_t,Z_t,\zeta_t) - w, T+y) - R(f(M_t,Z_t,\zeta_t) - S_M, T - S_T)$ is increasing in M_t (See the proof of Lemma A-1).

Because $\bar{g}(f(M_t^*,Z_t,\zeta_t)) - \bar{h}(g(M_t^*,Z_t,\zeta_t))$ is increasing in S_M, Lemma A-5 implies that $\Delta_t(M_t^*,Z_t)$ is increasing in S_M (part (i)), which in turn implies that $\beta_t(M_t^*,Z_t)$ is increasing in S_M (part (ii)). Since $\beta_t(M_t^*,Z_t)$ is decreasing in M and crosses 0 once by assumption, $M_t^*(S_M,S_T)$ increases in S_M.

(ii) See right panel of Figure 4. $\square$