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# COLLUSION WITH CAPACITY CONSTRAINTS OVER THE BUSINESS CYCLE

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## Abstract

This paper investigates the effect of capacity constraints on the sustainability of collusion in markets subject to cyclical demand fluctuations. In the absence of capacity constraints (i.e. a limiting case of our model), Haltiwanger and Harrington (1991) show that firms find it more difficult to collude during periods of decreasing demand. We find that this prediction can be overturned if firms' capacities are sufficiently small. Capacity constraints imply that punishment profits move procyclically, so that periods of increasing demand may lead to lower losses from cheating even if collusive profits are rising. Haltiwanger and Harrington's main prediction remains valid for sufficiently large capacities.

**Keywords:** Collusion, Capacity constraints, Business cycles.

**JEL No:** C73, L13, E32.

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# 1 Introduction

The ability of firms to collude over the business cycle has been a major topic of research in theoretical and empirical industrial organization over the last two decades. The literature has commonly used an infinitely repeated game where firms try to sustain the highest level of profits with credible threats to punish defectors. As firms have a short-run temptation to cheat from the collusive agreement, collusion is stable only if the one-shot deviation gains do not exceed the losses of future collusive profits, net of the value of punishment profits. Changes in demand conditions affect both the gains and losses from cheating, implying that the balance between the two need not remain constant as demand moves over time. Therefore, the state of the business cycle has a crucial effect on the sustainability and profitability of collusive outcomes.

In this paper, we revisit the classical question of whether firms find it more difficult to collude during booms or during recessions. Our point of departure is the model developed by Haltiwanger and Harrington (1991) (hereafter, HH). Holding constant the level of current demand, they show that firms' incentives to deviate are strengthened when future demand is falling, given that the value of the forgone collusive profits is smaller as compared to when demand is rising. Therefore, it is more difficult to sustain collusion during periods of decreasing demand. This result crucially depends on marginal costs being constant in output and symmetric across firms, as this means that punishment profits are zero and therefore invariant to demand movements. If a weakening of demand conditions also leads to a drop in punishment profits, it is no longer clear whether firms would lose less by deviating in periods of falling demand. What this implies for our current purposes is that the effects of future demand movements on the sustainability of collusion are not unambiguous, as they are under the assumption of constant (and symmetric) marginal costs.<sup>1</sup>

By introducing capacity constraints into HH's formulation, we show that the issue of whether firms find it more difficult to collude during booms or recessions is unambigu-

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<sup>1</sup>Typically, most of the industries which have been subject to empirical analyses of collusive behavior are characterized either by cyclical cost movements (as the gasoline market analyzed by Borenstein and Shepard (1996) ), or by tight capacity constraints (as the aluminum industry analyzed by ? and Bresnahan and Suslow (1989) ), or the cement industry analyzed by Iwand and Rosenbaum (1991) and Rosenbaum and Sukharomana (2001) , among others).

ously linked to the value of firms' capacities. When capacity constraints are sufficiently tight, firms find it more difficult to collude during booms, whereas the contrary is true for larger capacity values. Intuitively, when capacity constraints are severe enough, the lack of excess capacity during a boom implies that the future costs of being punished are low. Thus, the losses from cheating decrease even if collusive profits are rising. In contrast, the emergence of excess capacity during a recession makes the punishment threat more severe, and thereby induces an increase in the losses from cheating even if collusive profits decline.

## 1.1 Review of related papers

This paper also contributes to highlight the importance of the assumptions made in some of the previous papers on collusion. For our current purposes, two assumptions are crucial: first, whether firms are capacity constrained (or more generally, whether production costs exhibit some degree of decreasing returns to scale); and second, whether there is some link between current and future demand conditions. The literature on collusion is vast, so we will just review here the papers that are most related to our work.

In a seminal paper, Rotemberg and Saloner (1986) explore optimal collusive pricing assuming that demand is subject to (observable) independent and identically distributed (*i.i.d.*) shocks and that firms' marginal costs are constant in output. Under these assumptions, the current level of demand only affects the sustainability of collusion through its positive effect on firms' short-run temptation to cheat: deviations are more profitable in periods of high demand given that undercutting allows the deviator to capture a larger share of the market. However, the level of current demand has no effect on firms' expectations of future demand, and thus the expected losses from cheating are independent of the level of current demand. Associating a boom (recession) with a period of high (low) demand, Rotemberg and Saloner find that it is more difficult to sustain collusion during booms, when the incentive to deviate is the greatest.<sup>2</sup>

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<sup>2</sup>Given the *i.i.d.* assumption, expected future demand at a period with a high demand realization is lower than current demand. Therefore, in Rotemberg and Saloner's model, a boom (current demand is high) is also a period in which future demand is falling. This should be noted to avoid confusion

By introducing capacity constraints into Rotemberg and Saloner (1986)'s model, Staiger and Wolak (1992) show that the *price wars during booms relationship* can be reversed. The main reason is that capacity constraints, by limiting the size of the market that a firm can capture by itself, reduce the profitability of defections when demand is sufficiently high. However, by retaining the assumption that the shocks in demand are *i.i.d.*, Staiger and Wolak omit an equally important factor: namely, that the existence of capacity constraints also alters the value of the future losses from cheating through their impact on the severity of future punishments. ? and Compte and Rey (2002) highlight the importance of capacity constraints in shaping punishment possibilities. However, since these models assume fixed demand over time, they cannot be used to address the issue of whether booms or recessions are critical for the sustainability of collusion.

Haltiwanger and Harrington (1991) replace the *i.i.d.* assumption by assuming instead that demand is subject to (deterministic) cyclical demand fluctuations.<sup>3</sup> This approach is better suited to understand the influence of the business cycle on firms' pricing behavior since "stronger (weaker) demand tomorrow" is exactly what firms expect if they believe that the economy is in an upturn (downturn). Hence, even if, in the absence of capacity constraints, it is still true that the greatest deviation gains are achieved at the peak of the cycle, it is no longer clear whether collusion will be weaker during booms if the greater incentives to deviate are offset by the increasing value of the forgone collusive profits.

Given that our paper is closely related to HH's, it is worthwhile understanding its main insights through the following thought experiment. Consider two points on the cycle with equal demand, but such that demand is increasing in one and decreasing in the other. Clearly, the losses from cheating are greater at the point at which demand is rising, since the near-term profits, which are more heavily weighted, are expected to be

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with our (and HH's) terminology, according to which a boom is a period followed by larger demand levels.

<sup>3</sup>Kandori (1991) assumes correlated demand shocks. More recently, Bagwell and Staiger (1997) assume that the level of market demand alternates stochastically between states of slow (recessions) and fast (boom) growth rates. They show that collusive pricing is weakly procyclical or countercyclical depending on whether market demand growth rates are positively or negatively correlated through time.

higher. Thus, the high cost that would be induced by a price war acts as a deterrent to firms' incentives to cheat. Since such a deterrent is weaker when demand is expected to fall, collusion is more vulnerable during recessions than during booms. However, as already mentioned, the constant marginal cost assumption hides the possibility that future demand movements may also affect future punishment profits, and thus provides an incomplete picture of collusion possibilities in industries where this assumption is not satisfied.

Our model relaxes both the assumption that demand shocks are *i.i.d.* and the assumption that marginal costs are constant in output. By allowing demand to move in cycles (as opposed to Rotemberg and Saloner (1986) and Staiger and Wolak (1992)), we can shed some light on the link between the state of the business cycle and the sustainability of collusion. Furthermore, by introducing capacity constraints, we can provide an answer to the question of whether firms find it more difficult to collude in booms or in recessions for all capacity values, and not only for the limiting case in which capacities tend to infinity (which is equivalent to the assumption of capacity-unconstrained firms, as in HH). By capturing these two elements at a time, our model is able to highlight new results that, although previously conjectured by some authors, have not been so far formalized.<sup>4</sup>

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 provides the analysis and main results, and Section 4 concludes. Proofs are relegated to the Appendix.

## 2 The Model

Consider an industry with  $n$  infinitely lived firms, where  $n \geq 2$  and finite, which compete in every period  $t \geq 1$  by making simultaneous pricing decisions. Firms are symmetric as they offer homogenous products, and face identical cost functions with constant marginal costs (normalized to zero) up to their (exogenously given) symmetric capacity,  $k$ . Production above capacity is impossible, i.e. infinitely costly. Market demand in period  $t$  is represented by the demand function  $D(p, \theta_t)$ , where  $p$  denotes

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<sup>4</sup>See, for instance, Borenstein and Shepard (1996), Cowling (1983), Iwand and Rosenbaum (1991), and Rosenbaum (1989).

price and  $\theta_t$  is a demand parameter. Furthermore, we make the following standard assumptions about demand:

A 1 :  $D(\cdot, \theta_t)$  is a continuous and bounded function,  $\forall t$ ;

A 2 : There exists a price  $\bar{p}(\theta_t) > 0$  such that  $D(p, \theta_t) = 0$  if and only if  $p \geq \bar{p}(\theta_t)$ ,  $\forall t$ ;

A 3 :  $D(\cdot, \theta_t)$  is decreasing in  $p$ ,  $\forall p \in [0, \bar{p}(\theta_t)]$ ,  $\forall t$ ;

A 4 :  $pD(p, \theta_t)$  is strictly quasi-concave in  $p$ ,  $\forall p \in [0, \bar{p}(\theta_t)]$ ,  $\forall t$ ;

A 5 : The parameter  $\theta_t$  defines a family of demand functions such that  $\theta_{t'} > \theta_{t''}$  implies  $\bar{p}(\theta_{t'}) \geq \bar{p}(\theta_{t''})$  and  $D(p, \theta_{t'}) > D(p, \theta_{t''})$  for all  $p < \bar{p}(\theta_{t'})$ .

We must specify how customers are rationed when the firms offering the lowest prices have insufficient capacity to serve all demand. We follow the specification used by Kreps and Scheinkman (1983) and Osborne and Pitchik (1986), among others, and assume that demand is rationed according to the efficient rationing rule.<sup>5</sup>

An implication of A 4 is that, for given  $\theta_t$ , there exists a unique monopoly price  $p^m(\theta_t)$  that maximizes total industry profits, i.e.

$$p^m(\theta_t) = \arg \max_p \{p \min [D(p, \theta_t), nk]\}.$$

Furthermore, whenever  $D(0, \theta_t) > (n-1)k$ , there exists a unique price  $p^r(\theta_t, k)$ , referred to as the ‘residual monopoly price’, that maximizes a firm’s profits from serving the residual demand when their competitors are selling at capacity, i.e.

$$p^r(\theta_t, k) = \arg \max_p \{p \min [D(p, \theta_t) - (n-1)k, k]\}.$$

It will be convenient to write

$$\pi^m(\theta_t) = p^m(\theta_t) D(p^m(\theta_t); \theta_t) \tag{1}$$

$$\pi^r(\theta_t, k) = p^r(\theta_t, k) [D(p^r(\theta_t, k); \theta_t) - [n-1]k] \tag{2}$$

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<sup>5</sup>The efficient rationing rule specifies that consumers buy first from the low-priced firms, until their capacities are exhausted. The residual demand faced by the high-priced firms equals total demand net of the capacity of the low-priced firms. Davidson and Deneckere (1986) discuss alternative rationing rules.

to denote monopoly profits and the residual monopolist's profits. From A 5 it follows that both  $p^m(\theta_t)$ ,  $p^r(\theta_t, k)$ , and the profit function evaluated at these prices, are increasing in  $\theta_t$ . Furthermore, it is straightforward to see that both  $p^r(\theta_t, k)$  and  $\pi^r(\theta_t, k)$  are strictly decreasing in  $k$  for all  $k$  such that  $D(0, \theta_t) > (n - 1)k$ .

To investigate the impact of demand fluctuations on the sustainability of collusion, we place a similar structure on the intertemporal movement of demand as that of HH. The demand parameter  $\theta_t$  is assumed to fluctuate in cycles of length  $\bar{t}$  according to (3),

$$\theta_t = \begin{cases} \theta_1 & \text{if } t \in \{1, \bar{t} + 1, 2\bar{t} + 1, \dots\}, \\ \vdots & \vdots \\ \theta_{\hat{t}} & \text{if } t \in \{\hat{t}, \bar{t} + \hat{t}, 2\bar{t} + \hat{t}, \dots\}, \\ \vdots & \vdots \\ \theta_{\bar{t}} & \text{if } t \in \{\bar{t}, 2\bar{t}, 3\bar{t}, \dots\}. \end{cases} \quad (3)$$

We only impose the restriction that this cycle must be single-peaked. That is, starting at period 1 of the cycle, the demand function is assumed to shift out over time, up to some period  $\hat{t}$ , and to shift back until it reaches its minimum level at  $t = \bar{t} + 1$ . We thus have the following assumption:

$$A\ 6 : \theta_1 < \dots < \theta_{\hat{t}} > \dots > \theta_{\bar{t}} > \theta_1.$$

An implication of A 6 is that monopoly profits and the residual monopolist's profits move in the same direction as market demand. That is, monopoly profits increase from period 1 to period  $\hat{t}$ , and then shift down from period  $\hat{t} + 1$  to  $\bar{t} + 1$ . Similarly, the residual monopolist's profits increase from the first period at which demand at marginal costs exceeds the aggregate capacity of  $[n - 1]$  firms up to period  $\hat{t}$ , and then shifts down from period  $\hat{t} + 1$  until the last point of the cycle at which demand at marginal costs exceeds the aggregate capacity of  $[n - 1]$  firms. For all other periods, the residual monopolist's profits equal zero, and are therefore invariant to demand movements.

Other than A 6, no further restrictions are imposed on the demand cycle. It can be symmetric or asymmetric both in the length of the recession and boom, or in the speed at which demand grows during booms or declines during recessions.<sup>6</sup>

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<sup>6</sup>Implicit in this formulation is the assumption that demand movements are not so strong so as to induce exit or entry, nor capacity expansions or contractions. Endogenizing market structure and



Given this demand and cost structure, firms make simultaneous pricing decisions so as to meet demand. With an infinite horizon, a strategy for firm  $i$  is an infinite sequence of action functions,  $\{S_{it}\}_{t=1}^{\infty}$ , where  $S_{it} \in [0, p^m(\theta_t)]$  specifies the price to be charged by firm  $i$  in period  $t$  as a function of the prices charged by all firms in all previous periods.<sup>7</sup> The payoff function for firm  $i$  is the sum of discounted profits, where firms' common discount factor is  $\delta \in (0, 1)$ . All firms are assumed to be risk neutral, and hence aim to maximize their expected payoff. All aspects of the game are assumed to be common knowledge.

### 3 Analysis and Results

The aim of this paper is to highlight the effect of capacity constraints on the sustainability of perfect collusion over the cycle. Therefore, we will first characterize the necessary and sufficient conditions for the path of monopoly prices,  $\{p^m(\theta_t)\}_{t=1}^{\infty}$ , to be a subgame perfect equilibrium outcome of the infinitely repeated game described above.

In a general setting that encompasses ours, Lambson (1988) shows that firms can be driven down to their security levels (the discounted sum of the stream of minmax profits in every period) through credible punishments. Since more severe punishment threats would not be credible, the price path of monopoly prices can be supportable by subgame perfect equilibria if and only if it is supportable by the threat to revert to a security level penal code. It is easy to see that firm  $i$  receives its minmax profits when all its rivals price at zero and firm  $i$  maximizes its profits over the residual demand. Hence, at period  $t$ , the value of the security level penal code is given by  $\sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \pi^r(\theta_{\tau}, k)$ .

To analyze the sustainability of perfect collusion, we consider the following symmetric strategy profile: each firm is called to price at the monopoly price in every period as long as no firm has deviated in previous periods, and to revert to the punishment path driving the deviator's profits to its security level in the period immediately after

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capacity levels is out of the scope of the paper. See Rotemberg and Woodford (1992) for a general equilibrium approach and Staiger and Wolak (1992) and Compte and Rey (2002) for analysis of collusion with endogenous capacities.

<sup>7</sup>Note that we are only considering pure-strategy equilibria. The analysis could nevertheless be extended in a natural way to allow for non-degenerate mixed strategies. See footnote 8.

the one in which a deviation takes place.<sup>8</sup>

The path of monopoly prices is a subgame perfect equilibrium outcome if and only if the following condition is satisfied,

$$L^m(t; \delta) \geq G^m(t) \quad \forall t, \quad (4)$$

where

$$\begin{aligned} L^m(t; \delta) &= \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} L^m(\tau) \\ &= \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} \left[ \frac{1}{n} \pi^m(\theta_\tau) - \pi^r(\theta_\tau, k) \right], \end{aligned} \quad (5)$$

and

$$G^m(t) = p^m(\theta_t) \min \{ D(p^m(\theta_t); \theta_t), k \} - \frac{1}{n} \pi^m(\theta_t). \quad (6)$$

In words, when a firm deviates from the monopoly price in period  $t$ , it gives up the difference between its share of monopoly profits and the profits that it attains along the optimal punishment path in all periods following the deviation. Therefore,  $L^m(\tau)$  represents the one-shot losses from cheating in period  $\tau \geq t + 1$ , and  $L^m(t; \delta)$  denotes the present discounted value of the losses from cheating from period  $t + 1$  onwards. Since the optimal deviation in period  $t$  is to slightly undercut  $p^m(\theta_t)$  (and this results in profits  $p^m(\theta_t) \min \{ D(p^m(\theta_t); \theta_t), k \}$  rather than  $\frac{1}{n} \pi^m(\theta_t)$ ),  $G^m(t)$  represents the one-shot deviation gain in period  $t$ .

The incentive compatibility constraint, (4), can be solved in terms of the discount factor,  $\delta$ . As is already standard, the path of monopoly prices is subgame perfect if and only if the discount factor is sufficiently large.

**Proposition 1** *There exists  $\widehat{\delta} \in (0, 1)$  such that the price path  $\{p^m(\theta_t)\}_{t=1}^{\infty}$  is supportable by subgame perfect equilibria if and only if  $\delta \in [\widehat{\delta}, 1)$ .*

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<sup>8</sup>The same results would be obtained if we relied on grim-trigger strategies that involve an infinite reversion to the one-shot Nash equilibrium after a deviation takes place. Since for some demand values, the one-shot Nash equilibrium is in non-degenerate mixed strategies, one should extend the strategy space to include the set of mixed strategies (cumulative distribution functions over  $S_{it}$ ). See, among others, Kreps and Scheinkman (1983) and Deneckere and Kovenock (1996), p.4.

Proposition 1 is used to implicitly define the period of the cycle when firms find it more difficult to sustain perfect collusion. When the discount factor exceeds  $\widehat{\delta}$ , the incentive compatibility constraint (4) is satisfied with an strict inequality in all periods. When it equals  $\widehat{\delta}$ , there exists some point of the cycle, which we will denote  $t^*$ , at which the incentive compatibility constraint (4) is satisfied with an strict equality, whereas it continues to be satisfied with slack at all other points of the cycle. Therefore, as the discount factor is slightly reduced below  $\widehat{\delta}$ ,  $t^*$  is the first period at which the monopoly price cannot not be sustained. We thus refer to  $t^*$  as *the critical point of the cycle*.

In order to asses whether  $t^*$  belongs to the boom or to the recession, we first need to investigate how the value of the one-shot losses and gains from cheating depends on the level of firms' capacities.

For this purpose, let us first assess how the one-shot losses from cheating vary as a function of the demand parameter  $\theta_t$ , for a given level of firms' capacities. Consider a situation in which demand is so low relative to firms' capacities that punishment profits are driven down to zero. In this case, the losses from cheating are just equal to the value of the forgone monopoly profits, which are clearly increasing in demand. However, for higher demand values, capacity constraints start to play a role in limiting the scope for punishing defectors, i.e. in partly offsetting the increase in the losses from cheating. When demand is high enough, the increase in collusive profits is not sufficient to completely outweigh the increase in punishment profits, and the losses from cheating start to decrease as demand conditions strengthen.

Therefore, the comparison of the one-shot losses from cheating across two periods with lower and higher demand depends on the value of each firm's capacity. That is, on whether it is large enough so that the one-shot losses from cheating are always increasing in demand, or alternatively, on whether it is low enough so that the one-shot losses from cheating are always decreasing in demand. The following Lemma characterizes these two critical values.

**Lemma 1** *There exists  $\bar{k}_L \geq \underline{k}_L$  such that for all  $t', t'' \in \{1, \dots, \bar{t}\}$  satisfying  $\theta_{t'} \geq \theta_{t''}$ ,*

- (i)  $L^m(t') \geq L^m(t'')$  if  $k \geq \bar{k}_L$ ; and
- (ii)  $L^m(t') \leq L^m(t'')$  if  $k \leq \underline{k}_L$ .

We can perform the same analysis to understand the impact of capacity constraints on the value of the one-shot deviation gains. Consider first a situation in which demand is so low relative to each firm's capacity that a defector would have enough capacity to serve all demand at the monopoly price. Thus, the larger demand, the stronger the one-shot deviation gains. For higher demand levels, the deviator would be capacity-constrained to expand its production up to the monopoly quantity. Since, as a function of demand, the increase in the deviator's profits is of lower-order magnitude than the increase in monopoly profits, the rate of growth of the one-shot deviation gains starts to slow down. If demand is high enough, the former effect dominates the latter, which implies that the one-shot deviation gains are decreasing in demand. Therefore, the comparison of the one-shot deviation gains across periods depends, as before, on the value of firms' capacities.

**Lemma 2** *There exists  $\bar{k}_G \geq \underline{k}_G$  such that for all  $t', t'' \in \{1, \dots, \bar{t}\}$  satisfying  $\theta_{t'} \geq \theta_{t''}$ ,*

- (i)  $G^m(t') \geq G^m(t'')$  if  $k \geq \bar{k}_G$ ; and
- (ii)  $G^m(t') \leq G^m(t'')$  if  $k \leq \underline{k}_G$ .

Building on these insights, we can now assess whether the critical point of the cycle for perfect collusion belongs to the boom or to the recession, and how this depends on the value of firms' capacities.

**Theorem 1** *There exists a unique  $k^* \in [\min\{\underline{k}_L, \underline{k}_G\}, \max\{\bar{k}_L, \bar{k}_G\}]$  such that the critical point of the cycle belongs to the boom, i.e.  $t^* \in \{1, \dots, \hat{t} - 1\}$  if and only if  $k \leq k^*$ , and to the recession, i.e.  $t^* \in \{\hat{t}, \dots, \bar{t}\}$ , otherwise.*

Theorem 1 shows that the critical period of the cycle belongs to the boom when each firm's capacity is small enough, and to the recession otherwise.

As shown in Lemmas 1 and 2, parts (i), when capacities are large enough, i.e.  $k \geq \max\{\bar{k}_L, \bar{k}_G\}$ , both the losses from cheating and the one-shot deviation gains are larger in periods of greater demand. Hence, the same logic as in HH applies. Note that the result that recessions are critical for collusion still holds even without excluding the possibility that the defector might be capacity constrained to capture the whole market or that punishment profits might be increasing in demand.

On the other hand, just the opposite occurs when capacities are small enough, i.e. when  $k \leq \min \{k_L, k_G\}$ . In this case, as shown in Lemmas 1 and 2, parts (ii), the one-shot losses and gains from cheating are larger in periods of lower demand. Therefore, for any point at which demand is falling,  $t^R$ , one can always find a point at which demand is rising,  $t^B$ , that yields at least as high a one-shot gain from defection. Now, the losses from cheating would be greater at  $t^R$ , since the near term losses, which are more heavily weighted, exceed those at  $t^B$ . Therefore, as  $\delta$  is slightly reduced below  $\widehat{\delta}$ , the first point at which the monopoly price cannot be sustained belongs to the boom, and not to the recession.

For the remaining capacity values, a continuity argument implies that there exists a monotonic relationship between the level of firms' capacity and the location of the critical point for perfect collusion, which moves from the boom to the recession as the value of capacity goes up.

One would hope to say more about the critical point of the cycle. For instance, whether it is located at periods of higher or lower demand, and how its exact location depends on the value of firms' capacities. However, given the general class of cycles and demand functions considered, it is not possible to find the exact relationship between current demand, firms' capacities and the location of the critical point of the cycle for firms to perfectly collude. It is possible to find reasonable demand specifications for which the critical point for collusion lays at the peak of the boom, at the last period of the cycle, or at any point between these two, if capacities are large enough; or at the trough of the cycle, at the period just before the peak, or at any point between these two, for low enough capacities. The following section presents numerical solutions that provide some examples of these possibilities.

## 4 Numerical Solutions

We have parameterized the model described above and computed its numerical solutions. We have considered two linear demand functions: under demand function 1 demand shifts are additive, whereas under demand function 2 demand shifts are

multiplicative. Specifically,

$$\text{Demand function 1: } D_1(p, \theta_t) = \theta_t - p$$

$$\text{Demand function 2: } D_2(p, \theta_t) = \theta_t [1 - p]$$

We have further assumed that the demand parameter  $\theta_t$  takes eight values over the cycle, the peak occurring at period  $t = 5$ , periods  $t \in \{1, 2, 3, 4\}$  belonging to the boom and periods  $t \in \{5, 6, 7, 8\}$  belonging to the recession. We have considered three different patterns of demand: demand pattern  $A$  is symmetric, demand pattern  $B$  involves a slower rate of decline during the recession, and demand pattern  $C$  involves a slower rate of growth during the boom. Specifically,

$$\text{Demand pattern } A: \theta_t = \{160, 170, 180, 190, 200, 190, 180, 170\}$$

$$\text{Demand pattern } B: \theta_t = \{160, 170, 180, 190, 200, 195, 190, 185\}$$

$$\text{Demand pattern } C: \theta_t = \{160, 165, 170, 175, 200, 190, 180, 170\}$$

We have considered variations of  $k$  to compute, for each capacity level, the critical period for perfect collusion  $t^*$ , and the critical discount factor for perfect collusion  $\widehat{\delta}(t^*)$ . The results, for each demand function and demand pattern, are depicted in Figures 1 to 6.

This exercise also helps to identify, for each case, the critical value  $k^*$  above (below) which  $t^*$  belongs to the recession (boom),<sup>9</sup> the value of the critical discount factor for perfect collusion when  $k = k^*$ ,  $\widehat{\delta}(k^*)$ , the minimum discount factor,  $\min \widehat{\delta}(k)$ , and the value of capacity for which the critical discount factor is minimum. The results, for each demand function and demand pattern, are summarized in Table 1.

The results illustrate our theoretical findings. Namely, there exists an unambiguous relationship between capacity levels and the issue of whether the critical point belongs to the boom (small capacities), or to the recession (large capacities). Furthermore, these results show, as stated in the text, that it is not possible to provide a general result concerning the issue of whether the critical point is located at periods of higher or lower demand, or how its exact location depends on the value of firms' capacities. Also, as shown in Brock and Scheinkman (1985), these figures depict a non-monotonic relationship between the critical discount factor and the value of firms' capacities, whose value first decreases as capacity grows (when the boom is critical), and then

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<sup>9</sup>This value is also plotted in the figures as a vertical line.

Demand Pattern-Function	$[\widehat{\delta}(k^*), k^*]$	$[\min \widehat{\delta}(k), k]$
$A - 1$	[0.5776; 45.0]	[0.5776; 45.0]
$A - 2$	[0.6272; 57.0]	[0.5988; 45.6]
$B - 1$	[0.5787; 45.2]	[0.5787; 45.2]
$B - 2$	[0.7251; 81.2]	[0.5988; 46.1]
$C - 1$	[0.5786; 44.9]	[0.5786; 44.9]
$C - 2$	[0.7007; 71.0]	[0.5923; 46.2]

Table 1: Critical capacity values and discount factors

increases up to the point in which capacities play no role in limiting the defector's profits.

From the inspection of the numerical solutions, there are some recurrent features that seem to suggest more general results. First, unless capacities are extremely low, the critical discount factor for perfect collusion is larger when the recession is critical than when the boom is critical. Second, when the critical point belongs to the recession, increases in capacities monotonically lead to increases in the discount factor. Third, when the demand parameter enters the demand function additively, the minimum discount factor that allows for perfect collusion is always attained at the critical capacity value  $k^*$ . And last, the critical point moves down the recession as capacity constraints become more severe, and never up. However, we have not been able to prove these results analytically. Further work should evaluate the theoretical plausibility of these observations.

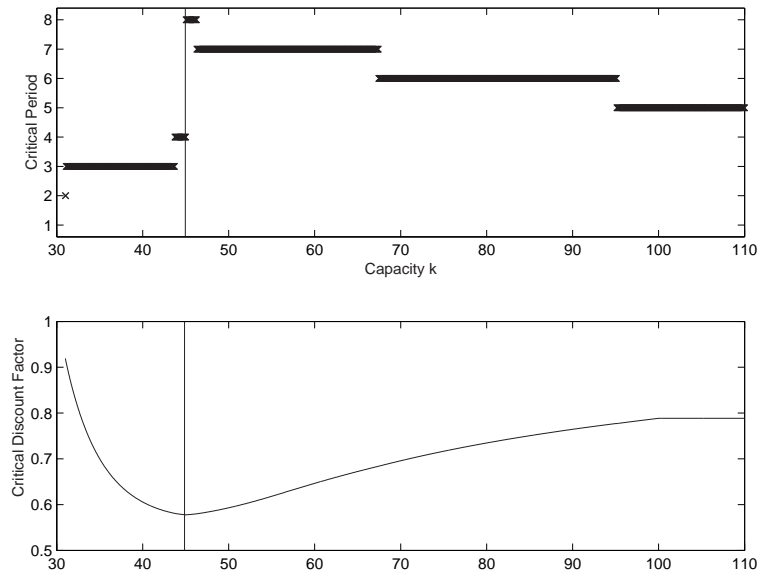


Figure 1: The critical point,  $t^*$ , and critical discount factor for perfect collusion,  $\delta^*$ , as a function of  $k$ , Demand Function 1, Demand Pattern A

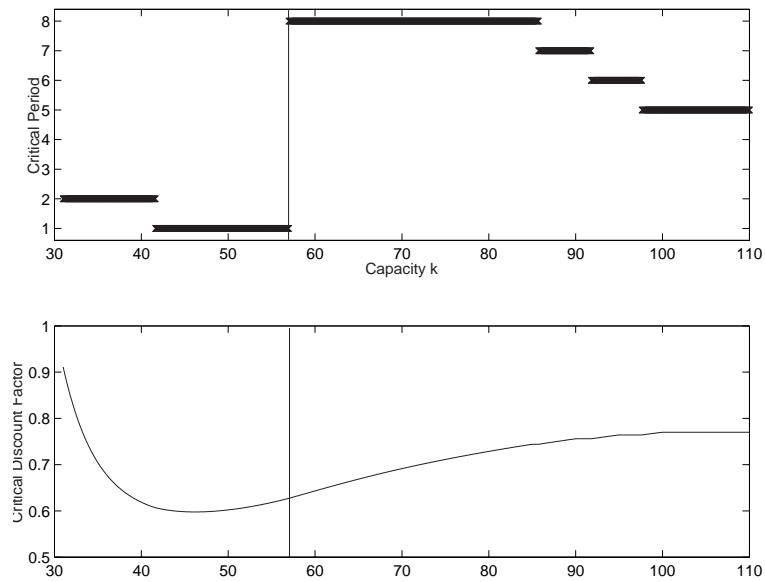


Figure 2: The critical point,  $t^*$ , and critical discount factor for perfect collusion,  $\widehat{\delta}(t^*)$ , as a function of  $k$ , Demand Function 2, Demand Pattern A



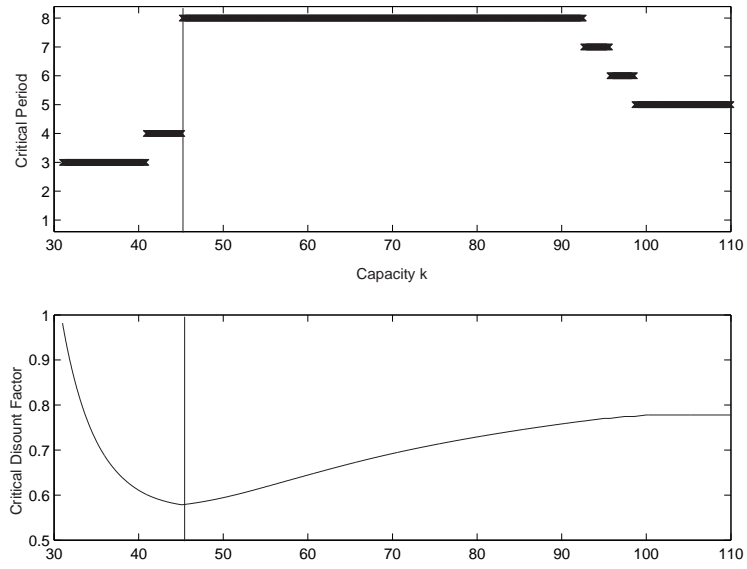


Figure 3: The critical point,  $t^*$ , and critical discount factor for perfect collusion,  $\widehat{\delta}(t^*)$ , as a function of  $k$ , Demand Function 1, Demand Pattern B

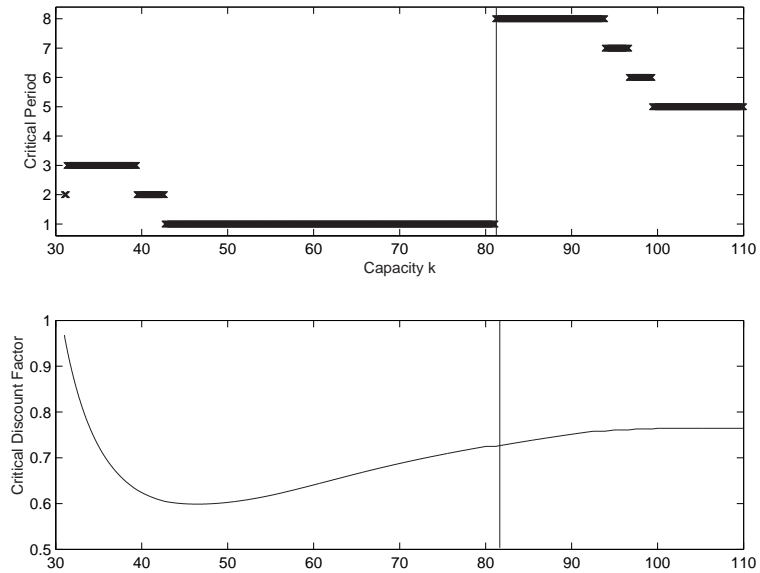


Figure 4: The critical point,  $t^*$ , and critical discount factor for perfect collusion,  $\widehat{\delta}(t^*)$ , as a function of  $k$ , Demand Function 2, Demand Pattern B

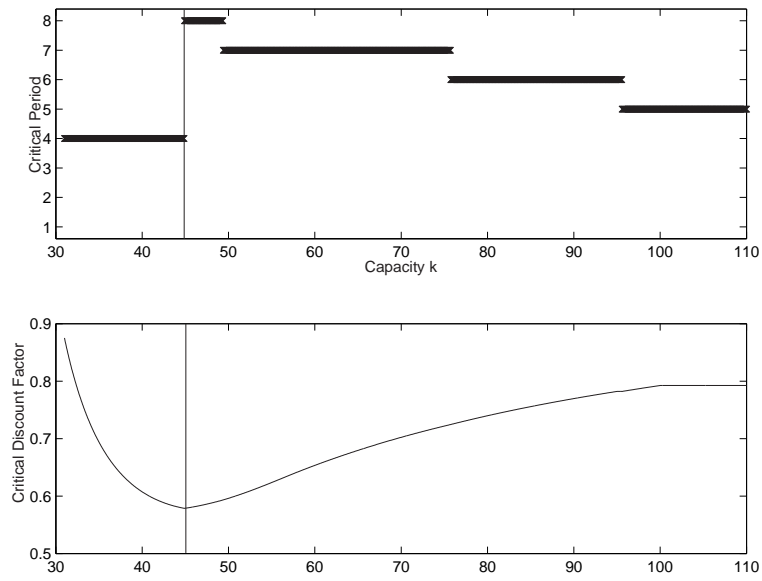


Figure 5: The critical point,  $t^*$ , and critical discount factor for perfect collusion,  $\widehat{\delta}(t^*)$ , as a function of  $k$ , Demand Function 1, Demand Pattern C

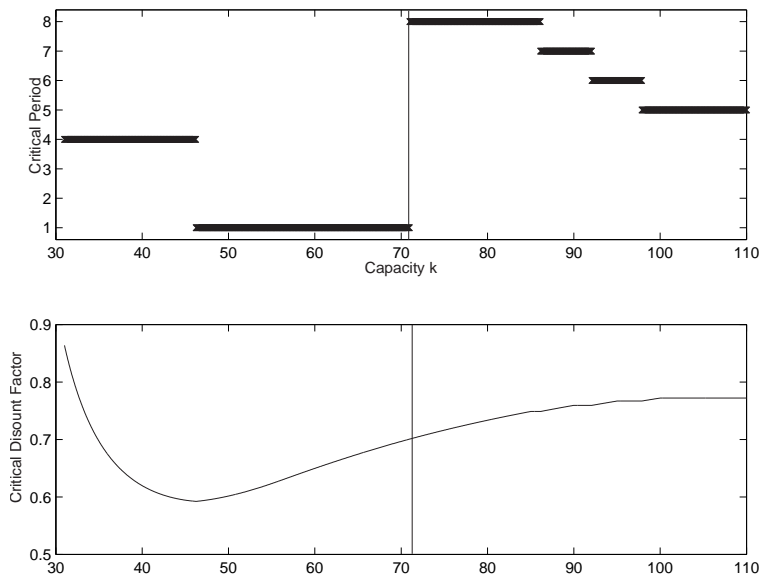


Figure 6: The critical point,  $t^*$ , and critical discount factor for perfect collusion,  $\widehat{\delta}(t^*)$ , as a function of  $k$ , Demand Function 2, Demand Pattern C

## 5 Conclusions

The main objective of this paper has been to identify whether firms find it more difficult to collude during booms or during recessions, and to assess how this depends on the level of firms' capacities. In a model that extends that of Haltiwanger and Harrington (1991) by introducing capacity constraints, we have shown that there exists an unambiguous relationship between the level of firms' capacities and the location of the period where firms find it more difficult to perfectly collude. This point moves from the recession to the boom as capacity constraints become more severe. The reason underlying this result goes as follows: when firms face severe capacity constraints, the impact of demand fluctuations on the value of future punishment profits is greater than its effect on the value of the forgone collusive profits; hence, periods of expanding demand give rise to lower losses from cheating, which make collusion more difficult during booms rather than recessions. When capacity constraints are not severe enough, the increase in the value of future punishment profits during a boom is not sufficient to outweigh the faster increase in firms' collusive profits. Thus, firms find it more difficult to collude during recessions even if capacity constraints play a role in reducing the one-shot deviation gains and the severity of optimal punishments.

The main implication of this analysis for the empirical work is that the signs of the effects of future demand on current prices are not unambiguous. This suggests that the projected link between the level of future demand and the value of firms' capacities could be used as an additional determinant of the intertemporal price path in collusive industries subject to cyclical demand fluctuations.

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## Appendix: Proofs

### Proof of Proposition 1:

The subgame perfect equilibrium conditions (as expressed in (4)) take the following form:

$$L^m(t; \delta) = \frac{1}{1 - \delta^{\bar{t}}} \left[ \delta L^m(t+1) + \dots + \delta^{\bar{t}-t} L^m(\bar{t}) + \dots + \delta^t L^m(t) \right] \geq G^m(t).$$

First notice that  $L^m(t; 0) = 0 < G^m(t)$ ,  $\lim_{\delta \rightarrow 1} L^m(t; \delta) = \infty > G^m(t)$  and  $\frac{\partial L^m(t; \delta)}{\partial \delta} > 0 = \frac{\partial G^m(t)}{\partial \delta}$ . By the continuity of  $L^m(t; \delta)$  in  $\delta$ , there exists  $\hat{\delta}(t) \in (0, 1)$  such that  $L^m(t, \delta) \geq G^m(t)$  if and only if  $\delta \geq \hat{\delta}(t)$ . Hence, the price path of monopoly prices is a subgame perfect equilibrium outcome iff  $\delta \geq \hat{\delta} = \max \{ \hat{\delta}(1), \dots, \hat{\delta}(\bar{t}) \}$ . Since  $\hat{\delta}(t) \in (0, 1) \forall t$ , then  $\hat{\delta} \in (0, 1)$ .

### Proof of Lemma 1:

The difference between the one-shot losses from cheating in periods  $t'$  and  $t''$ , such that  $\theta_{t'} \geq \theta_{t''}$ , is given by

$$L^m(t') - L^m(t'') = \frac{1}{n} [\pi^m(\theta_{t'}) - \pi^m(\theta_{t''})] - [\pi^r(\theta_{t'}, k) - \pi^r(\theta_{t''}, k)] \quad (7)$$

First note that for  $k \geq \frac{D(0, \theta_{t'})}{n-1}$ ,  $\pi^r(\theta_{t'}, k) = \pi^r(\theta_{t''}, k) = 0$ . Given that  $\pi^m(\theta_{t'}) \geq \pi^m(\theta_{t''})$ , it follows that  $L^m(t') \geq L^m(t'')$ . Second note that for  $k \leq \frac{D(p^m(\theta_{t'}), \theta_{t'})}{n}$ ,  $\frac{\pi^m(\theta_{t'})}{n} = \pi^r(\theta_{t'}, k)$ , which implies  $L^m(t') = 0$ . Given that  $L^m(t'') \geq 0$ , it follows that

$L^m(t') \leq L^m(t'')$ . Last, equation (7) is monotonically increasing in  $k \in \left( \frac{D(p^m(\theta_{t'}), \theta_{t'})}{n}, \frac{D(0, \theta_{t'})}{n-1} \right)$ .

To see this, take its total derivative with respect to  $k$ ,

$$\frac{d[L^m(t') - L^m(t'')]}{dk} = - \left[ \frac{d\pi^r(\theta_{t'}, k)}{dk} - \frac{d\pi^r(\theta_{t''}, k)}{dk} \right]$$

Note that

$$\begin{aligned} \frac{d\pi^r(\theta_t, k)}{dk} &= \frac{\partial \pi^r(\theta_t, k)}{\partial p} \frac{\partial p^r(\theta_t)}{\partial k} + \frac{\partial \pi^r(\theta_t, k)}{\partial k} \\ &= \frac{\partial \pi^r(\theta_t, k)}{\partial k} \Big|_{p=p^r(\theta_t)} \\ &= -[n-1] p^r(\theta_t) \end{aligned}$$

where the second equality follows from the envelope theorem, and the third from the definition of  $\pi^r(\theta_t, k)$ , as in (2). Therefore,

$$\frac{d[L^m(t') - L^m(t'')]}{dk} = -[n-1][p^r(\theta_{t''}) - p^r(\theta_{t'})] \geq 0 \quad (8)$$

By the continuity of  $L^m(t)$  in  $k$ , it follows that there exists a critical capacity value  $\widehat{k}_L(\theta_{t'}, \theta_{t''}) \in \left( \frac{D(p^m(\theta_{t'}), \theta_{t'})}{n}, \frac{D(0, \theta_{t'})}{n-1} \right)$  such that  $L^m(t') \geq L^m(t'')$  if and only if  $k \geq \widehat{k}_L(\theta_{t'}, \theta_{t''})$ . Therefore,  $L^m(t') \leq L^m(t'')$  for all  $t', t'' \in \{1, \dots, \bar{t}\}$  iff  $k \geq \max_{t', t''} \left\{ \widehat{k}_L(\theta_{t'}, \theta_{t''}) \right\} = \bar{k}_L$  and  $L^m(t') \geq L^m(t'')$  for all  $t', t'' \in \{1, \dots, \bar{t}\}$  iff  $k \leq \min_{t', t''} \left\{ \widehat{k}_L(\theta_{t'}, \theta_{t''}) \right\} = \underline{k}_L$ . Since  $\widehat{k}_L(\theta_{t'}, \theta_{t''}) \in \left( \frac{D(p^m(\theta_{t'}), \theta_{t'})}{n}, \frac{D(0, \theta_{t'})}{n-1} \right)$ , then  $\bar{k}_L, \underline{k}_L \in \left( \frac{D(p^m(\theta_1), \theta_1)}{n}, \frac{D(0, \theta_{\bar{t}})}{n-1} \right)$

### Proof of Lemma 2:

The method of proof is similar to that of Lemma 1. The difference between the one-shot gains from cheating in periods  $t'$  and  $t''$ , such that  $\theta_{t'} \geq \theta_{t''}$ , is given by

$$\begin{aligned} G^m(t') - G^m(t'') &= p^m(\theta_{t'}) \min \{ D(p^m(\theta_{t'}); \theta_{t'}), k \} - \frac{1}{n} \pi^m(\theta_{t'}) \\ &\quad - \left[ p^m(\theta_{t''}) \min \{ D(p^m(\theta_{t''}); \theta_{t''}), k \} - \frac{1}{n} \pi^m(\theta_{t''}) \right] \end{aligned} \quad (9)$$

First note that for  $k \geq D(p^m(\theta_{t'}); \theta_{t'})$ ,

$$G^m(t') - G^m(t'') = \frac{n-1}{n} [\pi^m(\theta_{t'}) - \pi^m(\theta_{t''})] > 0$$

Second note that for  $k \leq \frac{D(p^m(\theta_{t'}), \theta_{t'})}{n}$ , we have that  $\frac{1}{n} \pi^m(\theta_{t'}) = p^m(\theta_{t'}) k$ , which implies  $G^m(t') = 0$ . Given that  $G^m(t'') > 0$  for such a  $k$ , it follows that  $G^m(t') < G^m(t'')$ .

Last, equation (9) is monotonically increasing in  $k$ . To see this, take its total derivative with respect to  $k$ ,

$$\frac{d[G^m(t') - G^m(t'')]}{dk} = \begin{cases} 0 & \text{if } k > D(p^m(\theta_{t'}); \theta_{t'}) \\ p^m(\theta_{t'}) > 0 & \text{if } D(p^m(\theta_{t''}); \theta_{t''}) \leq k \leq D(p^m(\theta_{t'}); \theta_{t'}) \\ p^m(\theta_{t'}) - p^m(\theta_{t''}) \geq 0 & \text{if } k > D(p^m(\theta_{t''}); \theta_{t''}) \end{cases}$$

By the continuity of  $G^m(t)$  in  $k$ , it follows that there exists a unique capacity value  $\widehat{k}_G(\theta_{t'}, \theta_{t''}) \in \left( \frac{D(p^m(\theta_{t''}), \theta_{t''})}{n}, D(p^m(\theta_{t'}); \theta_{t'}) \right)$  such that  $G^m(t') \geq G^m(t'')$  if and only if  $k \geq \widehat{k}_G(\theta_{t'}, \theta_{t''})$ . Therefore,  $G^m(t') > G^m(t'')$  for all  $t', t'' \in \{1, \dots, \bar{t}\}$  iff  $k \geq \max_{t', t''} \left\{ \widehat{k}_G(\theta_{t'}, \theta_{t''}) \right\} = \bar{k}_G$  and  $G^m(t') \leq G^m(t'')$  for all  $t', t'' \in \{1, \dots, \bar{t}\}$  iff  $k \leq \min_{t', t''} \left\{ \widehat{k}_G(\theta_{t'}, \theta_{t''}) \right\} = \underline{k}_G$ . Since  $\widehat{k}_G(\theta_{t'}, \theta_{t''}) \in \left( \frac{D(p^m(\theta_{t''}), \theta_{t''})}{n}, D(p^m(\theta_{t'}); \theta_{t'}) \right)$ , then  $\bar{k}_G, \underline{k}_G \in \left( \frac{D(p^m(\theta_1), \theta_1)}{n}, D(p^m(\theta_{\hat{t}}); \theta_{\hat{t}}) \right)$ .

**Proof of Theorem 1:**

We follow HH's proof of Theorem 5, and introduce several changes where needed. Let  $t^*$  be defined by  $\widehat{\delta} = \widehat{\delta}(t^*)$ . To prove Theorem 1, we then need to show that there exists a unique  $k^* \in [\min\{\underline{k}_L, \underline{k}_G\}, \max\{\bar{k}_L, \bar{k}_G\}]$  such that  $\widehat{\delta} > \widehat{\delta}(t)$  for all  $t \in \{\widehat{t}, \dots, \bar{t}\}$  iff  $k \leq k^*$ . Since  $t^*$  exists, then it must lie in  $\{1, \dots, \widehat{t} - 1\}$  iff  $k \leq k^*$  and in  $\{\widehat{t}, \dots, \bar{t}\}$  iff  $k > k^*$ .

Define  $f(t)$  as follows:

$$f(t) \equiv \max \left\{ \tau \mid \theta_\tau \geq \theta_t, \tau \in \{t+1, \dots, \bar{t}\} \right\}, t \in \{1, \dots, \widehat{t} - 1\}.$$

$f(t)$  is the latest point of the cycle at which the demand parameter is at least as great as the demand parameter at a period  $t$  belonging to the boom. Given that the single peak of the cycle is attained at  $\widehat{t}$ , it is clear that  $f(t)$  belongs to the recession.

The method of proof will be to show that the difference

$$[L^m(t; \delta) - G^m(t)] - [L^m(f(t); \delta) - G^m(f(t))] \quad (10)$$

is negative if  $k \leq \min\{\underline{k}_L, \underline{k}_G\}$ , positive if  $k \geq \max\{\bar{k}_L, \bar{k}_G\}$ , and that it is monotonically increasing in  $k \in [\min\{\underline{k}_L, \underline{k}_G\}, \max\{\bar{k}_L, \bar{k}_G\}]$ ,  $\forall t \in \{1, \dots, \widehat{t} - 1\}$ . This implies that there exists a unique  $k^* \in [\min\{\underline{k}_L, \underline{k}_G\}, \max\{\bar{k}_L, \bar{k}_G\}]$  such that (10) is negative iff  $k < k^*$ . Since this implies  $\widehat{\delta}(t) > \widehat{\delta}(f(t))$  iff  $k < k^*$ , this will be sufficient to prove the Theorem.

These results are stated and proved in the following Lemma.

**Lemma 3**

(i) If  $k \leq \min \{\underline{k}_L, \underline{k}_G\}$ , then  $[L^m(t; \delta) - G^m(t)] < [L^m(f(t); \delta) - G^m(f(t))]$   $\forall t \in \{1, \dots, \hat{t} - 1\}$ .

(ii) If  $k \geq \max \{\bar{k}_L, \bar{k}_G\}$ , then  $[L^m(t; \delta) - G^m(t)] > [L^m(f(t); \delta) - G^m(f(t))]$   $\forall t \in \{1, \dots, \hat{t} - 1\}$ .

(iii) The difference  $[L^m(t; \delta) - G^m(t)] - [L^m(f(t); \delta) - G^m(f(t))]$ , is monotonically increasing in  $k \in [\min \{\underline{k}_L, \underline{k}_G\}, \max \{\bar{k}_L, \bar{k}_G\}]$ ,  $\forall t \in \{1, \dots, \hat{t} - 1\}$ .

**Proof of Lemma 3:**

(i) Let  $t^B \in \{1, \dots, \hat{t} - 1\}$  and  $t^R = f(t^B)$ . Then,

$$[L^m(t^B; \delta) - G^m(t^B)] = \frac{1}{1 - \delta^{\bar{t}}} [\delta L^m(t^B + 1) + \dots + \delta^{\bar{t}} L^m(t^B)] - G^m(t^B) \quad (11)$$

$$[L^m(t^R; \delta) - G^m(t^R)] = \frac{1}{1 - \delta^{\bar{t}}} [\delta L^m(t^R + 1) + \dots + \delta^{\bar{t}} L^m(t^R)] - G^m(t^R) \quad (12)$$

By the definition of  $f(t)$ , we know by Lemma 2, that if  $k \leq \min \{\underline{k}_L, \underline{k}_G\}$ , then  $G^m(t^B) > G^m(t^R)$ . Hence, the difference between (11) and (12) is negative if

$$\delta L^m(t^B + 1) + \dots + \delta^{\bar{t}} L^m(t^B) < \delta L^m(t^R + 1) + \dots + \delta^{\bar{t}} L^m(t^R) \quad (13)$$

Define:

$$A \equiv \delta L^m(t^B + 1) + \dots + \delta^{t^R - t^B} L^m(t^R) \quad (14)$$

$$B \equiv \delta L^m(t^R + 1) + \dots + \delta^{\bar{t} - t^R + t^B} L^m(t^B) \quad (15)$$

By these definitions, condition (13) is equivalent to

$$A + \delta^{t^R - t^B} B < B + \delta^{\bar{t} - t^R + t^B} A$$

Rearranging terms,

$$\frac{A}{1 - \delta^{t^R - t^B}} < \frac{B}{1 - \delta^{\bar{t} - t^R + t^B}} \quad (16)$$

The expression on the left hand side of (16) is the present discounted value of the stream of the losses from cheating  $\{L^m(t^B + 1), \dots, L^m(t^R)\}$  made every  $t^R - t^B$  periods, and the right hand side is the present discounted value of the stream of the losses from cheating  $\{L^m(t^R + 1), \dots, L^m(t^B)\}$  made every  $\bar{t} - t^R + t^B$  periods. Since  $t^R = f(t^B)$  and  $k \leq \bar{k}_L$ , by Lemma 1 it is then true that  $L^m(t') < L^m(t'') \forall t' \in \{t^B + 1, \dots, t^R\}, \forall t'' \in$



$\{t^R + 1, \dots, t^B\}$ , which implies that  $L^m(t^B, \delta) < L^m(t^R, \delta)$ . This proves that if  $k \leq \min\{\bar{k}_L, \bar{k}_G\}$ , then  $L^m(t; \delta) - G^m(t) < L^m(f(t); \delta) - G^m(f(t)) \forall t \in \{1, \dots, \hat{t} - 1\}$ .

(ii) It follows the same lines of the proof of part (i), with a change in the sign of the inequalities.

(iii) Again, let  $t^B \in \{1, \dots, \hat{t} - 1\}$  and  $t^R = f(t^B)$ . From the proof of Lemma 2,

$$\frac{dG^m(t^B)}{dk} < \frac{dG^m(t^R)}{dk}$$

Hence, the difference  $[L^m(t^B; \delta) - G^m(t^B)] - [L^m(t^R; \delta) - G^m(t^R)]$  is monotonically increasing in  $k$ , if

$$\frac{dL^m(t^B; \delta)}{dk} > \frac{dL^m(t^R; \delta)}{dk} \quad (17)$$

Using the definitions (14) and (15), condition (17) is equivalent to

$$\frac{dA}{dk} > \frac{dB}{dk}$$

Since  $t^R = f(t^B)$ , by the proof of Lemma 1, equation (8), it is then true that  $\frac{dL^m(t')}{dk} > \frac{dL^m(t'')}{dk} \forall t' \in \{t^B + 1, \dots, t^R\}, \forall t'' \in \{t^R + 1, \dots, t^B\}$ , which implies that condition (17) is satisfied. This proves that  $L^m(t; \delta) - G^m(t) > L^m(f(t); \delta) - G^m(f(t)) \forall t \in \{1, \dots, \hat{t} - 1\}$  is monotonically increasing in  $k$ , which completes the proof.