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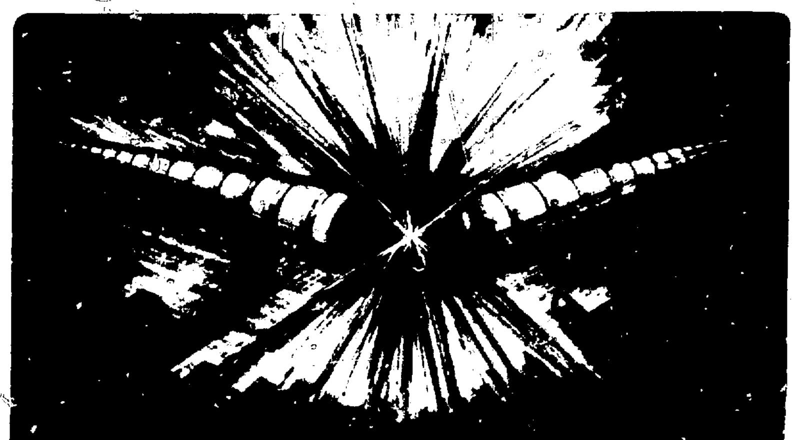

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Noncanonical Hamiltonian Methods in Plasma Dynamics*

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I. Motivation

A Hamiltonian approach to plasma dynamics has numerous advantages over equivalent formulations which ignore the underlying Hamiltonian structure. In addition to achieving a deeper understanding of processes, Hamiltonian methods yield concise expressions (such as the Kubo form for linear susceptibility), greatly shorten the length of calculations, expose relationships (such as between the ponderomotive Hamiltonian and the linear susceptibility), determine invariants in terms of symmetry operations, and cover situations of great generality. In addition, they yield the Poincaré invariants, in particular Liouville volume and adiabatic actions.

II. Myth: canonical variables

The belief is prevalent, fostered by textbooks, that Hamiltonian methods require the use of canonically conjugate pairs of dynamical variables (q_i, p_i), whose Poisson brackets (PB) are either unity or zero. There is still the freedom to make canonical transformations to new variables $(q,p \to Q,P)$, while preserving the PB relations.

This myth has two great difficulties: in the first place, canonical variables are often not known, as in the case of the Vlasov field. Secondly, even when they are known, they may be unphysical, such as canonical momentum, which is not gauge-invariant.

III. Reality: Poisson structure [1]

Let g(z) be an observable, expressed as a function on phase space, with an arbitrary coordinate system. The PB of two observables g_1, g_2 is given by the expression

$$\{g_1, g_2\} = (\partial g_1/\partial z^\mu)(\partial g_2/\partial z^\nu)\, J^{\mu\nu}(z), \qquad (1)$$

where the antisymmetric tensor J need not be constant, but must be such that the Jacobi condition:

$$\{\{g_1, g_2\}, g_3\} \quad + \text{ cyclic permutations} \quad = \quad 0 \qquad (2)$$

is satisfied. This creates a Poisson structure; examples will be shown below. If one knows the relation between (physical) noncanonical variables z and (unphysical) canonical variables $\bar{z}$, J can be obtained by elementary tensor analysis:

$$J^{\mu\nu} = (\partial z^\mu/\partial \bar{z}^\lambda)(\partial z^\nu/\partial \bar{z}^\sigma)\, \bar{J}^{\lambda\sigma} \quad , \qquad (3)$$

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where the elements of $\bar{J}$ are zero or unity.

The evolution in time of an observable g is then given by

$$\dot{g} = \{g, H\} \quad , \tag{4}$$

in terms of the PB (1) and a Hamiltonian H(z). These are the only elements needed for a Hamiltonian theory.

IV. Guiding-Center Representation

Although the guiding-center description of particle motion is conventionally an asymptotic theory, in the small parameter $\varepsilon \sim$ gyroradius/scale length, the remarkable Poisson structure found by Littlejohn is exact.[2] The six coordinates of phase space are $\underline{R}$ (guiding-center position in three-dimensional physical space), P(guiding-center parallel momentum), μ (magnetic moment = gyromomentum), and θ (gyrophase). The latter two are conjugate, while the first four are noncanonical. The PB is

$$\{g_1, g_2\} = (\partial g_1/\partial\theta\ \partial g_2/\partial\mu - g_1 \leftrightarrow g_2) - \nabla g_1 \times \nabla g_2 \cdot b(\underline{R})/B^*(\underline{R})$$
$$+ (\nabla g_1\ \partial g_2/\partial P - g_1 \leftrightarrow g_2) \cdot \underline{B}^*(\underline{R})/B^*(\underline{R}). \tag{5}$$

The modified field $\underline{B}^*$ is the curl of the modified potential $\underline{A}^* = \underline{A} + Pb$ (for the case $\underline{E}=0$); b is the unit vector of $\underline{B}$, and $B^* = b \cdot \underline{B}^*$.

Observables (such as particle velocity) and the Hamiltonian are expressed as series in ε; for the latter, one has

$$H = \tfrac{1}{2}P^2 + \mu B(\underline{R}) + O(\varepsilon^2),$$

in the case $\underline{E} = 0$.

Littlejohn presents the general case $\underline{E} \neq 0$ in his paper. More recently, he has used the action principle for a Lagrangian approach[3]:

$$L = \mu\dot{\theta} + \dot{\underline{R}} \cdot \underline{A}^*(\underline{R},P,t) - \phi^*(\underline{R},P,\mu,t),$$

where

$$\underline{A}^* = \underline{A}(\underline{R},t) + Pb(\underline{R},t) + \underline{u}_E(\underline{R},t) \quad ,$$
$$\phi^* = \phi(\underline{R},t) + \tfrac{1}{2}P^2 + \tfrac{1}{2}u_E^2(\underline{R},t) + \mu B(\underline{R},t),$$
$$\underline{u}_E = \underline{E} \times b/B.$$

The Euler-Lagrange equations yield the concise form

$$b(b \cdot \ddot{\underline{R}}) = \underline{E}^* + \dot{\underline{R}} \times \underline{B}^* \ ,$$

with

$$\underline{E}^* = -\nabla\phi^* - \partial\underline{A}^*/\partial t, \tag{6}$$
$$\underline{B}^* = \nabla \times \underline{A}^*.$$

The guiding-center variables are by no means limited in their utility to adiabatic conditions. In the presence of a small-amplitude electromagnetic wave of high frequency and short wave length [4]:

$$\underline{A}(\underline{x},t) = \underline{A}_0(\underline{x}) + \delta\underline{A}(\underline{x},t),$$

$$\delta\underline{A}(\underline{x},t) = \underline{\tilde{A}}\,(\underline{x}) \exp i\,[\Psi(\underline{x}) - \omega t] + \text{c.c.}, \tag{7}$$

one uses $\underline{A}_0$ to define the variables. With the Hamiltonian perturbation

$$\delta H(z,t) = -\int d^3x\, \underline{j}(\underline{x}|z) \cdot \delta\underline{A}(\underline{x},t), \tag{8}$$

one uses Kubo response theory to express the two-point susceptibility $\underset{\sim}{\chi}(x, x';\omega)$ in terms of the PB of $\underline{j}(\underline{x}|z)$, and the unperturbed distribution $f_s(z)$:

$$\underset{\sim}{\chi}^s(\underline{x},\underline{x}';\omega) = -(\omega_s^2(\underline{x})/\omega^2)\, \underset{\sim}{I}\delta(\underline{x}-\underline{x}') - \frac{4\pi}{\omega^2}\int_0^\infty d\tau\, e^{i\omega\tau}\langle\{\underline{j}(\underline{x}|z_t), \underline{j}(\underline{x}'|z_{t-\tau})\}\rangle, \tag{9}$$

where

$$\langle \cdots \rangle = \int d^6z\, f_s(z)(\cdots), \qquad \omega_s^2(\underline{x}) = 4\pi n_s(\underline{x})\, e_s^2/m_s.$$

Knowing $\underset{\sim}{\chi}$, one uses the $\underset{\sim}{\chi}$-K theorem [5] to obtain the ponderomotive Hamiltonian of the oscillation center:

$$K_2(z) = -[\delta/\delta f_s(z)] \iint d^3x d^3x' \underline{E}^*(\underline{x})\cdot\underset{\sim}{\chi}'\,(\underline{x},\underline{x}';\omega)\cdot\underline{E}(\underline{x}')/4\pi, \tag{10}$$

$$K_2(\underline{R},P,\mu) = |\underline{\tilde{E}}(\underline{R})|^2/\omega^2 + \sum_\ell D_\ell(|H_\ell|^2\,\omega_d^{-1}), \tag{11}$$

$$D_\ell = \ell\partial/\partial\mu + b(\underline{R})\cdot\underline{k}(\underline{R})\,\partial/\partial P, \qquad \underline{k}\,(\underline{x}) = \nabla\Psi(\underline{x}),$$

$$\omega_d = \omega - \ell\Omega(\underline{R}) - \underline{k}(\underline{R})\cdot\dot{\underline{R}}\,(\underline{R},P,\mu),$$

$$|H_\ell| = |[Pb + (\ell\Omega/k_\perp)\hat{k}_\perp + (2i\Omega\mu/k_\perp)b \times \hat{k}\,\partial/\partial\mu]\, J_\ell(k_\perp(2\mu\Omega)^{1/2})\cdot\underline{\tilde{E}}(\underline{R})|/\omega.$$

By differentiating K_2 with respect to $\underline{R}$, P, μ, one obtains the ponderomotive drift and force, the wave-induced oscillation-center velocity, and the gyrofrequency shift. The Lie transform used here also yields the wave-induced increment to the gyromomentum, which is still an adiabatic invariant.

V. Vlasov Field Theory

In the Coulomb model for a Vlasov system, the dynamical variable is the Vlasov distribution f(z). The Hamiltonian functional is evidently

$$H(f) = \int d^6z\, H_1(z)\, f(z) + \tfrac{1}{2}\iint d^6z\, d^6z'\, H_2(z,z')f(z)\, f(z'), \tag{12}$$

suppressing species summation. The PB can be found by sophisticated mathematics [6], inspired guess [7], or heuristic methods [8]. On observable functionals A(f), the PB is

$$[A_1, A_2] = \int d^6z\, f(z)\, \{\delta A_1/\delta f, \delta A_2/\delta f\}\,. \tag{13}$$

The evolution of f is thus

$$\dot{f} = [f,H]$$
$$= -\{f, H_V\},$$

where H_V is the usual Vlasov Hamiltonian:

$$H_V(z) = H_1(z) + \int d^6z'\, H_2(z,z')f(z'). \tag{14}$$

The coupling of f(z) to the Maxwell field $\underline{E}(\underline{x})$, $\underline{B}(\underline{x})$ appears in the PB, not in the Hamiltonian:

$$H(f,\underline{E},\underline{B}) = \int d^6z\, \tfrac{1}{2}mv^2\, f(z) + \int d^3x (E^2 + B^2)/8\pi \ . \qquad (15)$$

On functionals A(f, $\underline{E}$, $\underline{B}$), the PB $[A_1, A_2]$ consists of three terms: the first is (13) above, the second is the standard PB of electromagnetic field theory:

$$\int d^3x\ (\delta A_1/\delta\underline{E} \cdot \nabla\times\ \delta A_2/\delta\underline{B} \quad - \quad A_1 \leftrightarrow A_2) \ .$$

while the third provides the coupling:

$$(e/m) \int d^6z (\delta f/\delta\underline{v}) \cdot (\delta A_1/\delta\underline{E}(\underline{r}) \quad \delta A_2/\delta f(z) - A_1 \leftrightarrow A_2) \ .$$

The evolution equations for f, $\underline{E}$, B yield the Vlasov-Maxwell system.

In contrast to the guiding-center representation, where useful results have already been obtained, the field theory has not yet been applied to practical problems.

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VII. Acknowledgement

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