A NOTE ON THE EXPECTED NUMBER OF SINGLETON CYCLES IN A PERMUTATION OF 1,2, ... ,N.

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One of the important constants which arises during the analysis of algorithms is S_n, the expected number of singletons in a permutation of n distinct objects where each of the n! possible permutations occurs with equal probability. In (1) Knuth describes two techniques for deriving S_n, one of which depends on the Principle of Inclusion and Exclusion and the other on the manipulation of binomial coefficients. In this note a method is presented which depends on an interesting decomposition of the set of permutations of the integers 1,2,...,n. The method is also interesting in that the constant S_n is derived by proving its independence of n.

Let $p_{n,k}$ = the probability that a permutation of 1,2,..., n contains k singletons

$P_{n,k}$ = the set of all n - permutations having exactly k singletons

$C_{n,k,i}$ = the set of all n - permutations in which there are exactly k singletons and in which n is a member of an i - cycle in the permutation

Clearly $P_{n,k} = \bigcup_{i=1}^{n} C_{n,k,i}$

Let $|P|$ denote the number of elements in the set P and use the convention that

$$p_{o,o} = |P_{o,o}| = 1 \text{ and } p_{0,k} = |P_{o,k}| = 0 \text{ for all } k \neq 0.$$

Then for all $k \geq 0$, $n \geq i \geq 2$

$$|C_{n,k,i}| = (n-1) \cdot (n-2) \cdots (n-i+1) \quad |P_{n-i,k}|,$$

since there can be (n-1) different elements 'following' n in the i-cycle; (n-2) different elements 'following' each of these elements, and so on. For each possible i-cycle which contains n there are n-i remaining elements which can be permuted in any way that results in k singletons in that n-i permutation.

If we also define $p_{n,k} = |P_{n,k}| = 0$ for all $k<0$, then for

$$k \geqslant 0,\ n \geqslant 1,$$

$$|C_{n,k,1}| = |P_{n-1,\ k-1}|$$

since the number of n permutations with k singletons in which n is a singleton is precisely the number of (n-1)-permutations in which there are k-1 singletons.

Hence

$$p_{n,k} = \frac{|P_{n,k}|}{n!} = \frac{\sum_{i=1}^{n} |C_{n,k,i}|}{n!}$$

$$= \frac{|P_{n-1,\ k-1}|}{n!} + \frac{\sum_{i=2}^{n} |C_{n,k,i}|}{n!}$$

$$= \frac{1}{n}\ p_{n-1,\ k-1} + \sum_{i=2}^{n} \frac{(n-1)\cdots(n-i+1)\ |P_{n-i,k}|}{n!}$$

$$= \frac{1}{n}\ p_{n-1,\ k-1} + \sum_{i=2}^{n} \frac{1}{n}\ \frac{|P_{n-i,k}|}{(n-i)!}$$

$$= \frac{1}{n}\ (p_{n-1,k-1} + \sum_{i=2}^{n} p_{n-i,k})$$

If we also define $p_{n,k} = 0$ for all $n<0$, then for all $n \geq 1$ $k \geq 1$, the above term is

$$= \frac{1}{n} (p_{n-1,\ k-1} + (n-1) \cdot p_{n-1,k} + p_{n-2,k} - p_{n-2,\ k-1})$$

Consider now the generating function

$$G_n(z) = \sum_{k=0}^{n} p_{n,k}\ z^k = p_{n,0} + \sum_{k=1}^{n} p_{n,k}\ z^k$$

$$= p_{n,0} + \frac{1}{n} \sum_{k=1}^{n} (p_{n-1,\ k-1} + (n-1) \cdot p_{n-1,k} + p_{n-2,k} - p_{n-2,\ k-1})\ z^k$$

$$= p_{n,0} - \frac{1}{n} ((n-1) \cdot p_{n-1,0} + p_{n-2,0})$$

$$+ \frac{1}{n} (z \cdot G_{n-1}(z) + (n-1) \cdot G_{n-1}(z) + G_{n-2}(z) - z \cdot G_{n-2}(z)$$

Hence

$$G'_n(z) = \frac{1}{n} (G_{n-1}(z) + z \cdot G'_{n-1}(z) + (n-1) \cdot G'_{n-1}(z)$$

$$+ G'_{n-2}(z) - G_{n-2}(z) - z \cdot G'_{n-2}(z))$$

and

$$G'_n(1) = \frac{1}{n} (G_{n-1}(1) + n \cdot G'_n(1) - G_{n-2}(1))$$

$$= \frac{1}{n} (n \cdot G'_{n-1}(1)) = G'_{n-1}(i)$$

S_n is thereforeindependent of n. Since

$$S_1 = 1\ ,\ S_n = 1 \text{ for all } n.$$

A similar technique can be used to derive a recurrence relation for $p_{n,k}$ for k=0, if desired.

Reference: (1) Knuth, Donald E., The Art of Computer Programming, Vol. 1, Addison - Wesley, 1969.