Title

Technical Note: Derivation of Earth-Rotation Correction (Sagnac)and Analysis of the Effect of Receiver Clock Bias

Authors
Hu, Wang
Farrell, Jay A

Publication Date
2019-11-19

Technical Note: Derivation of Earth-Rotation Correction (Sagnac) and Analysis of the Effect of Receiver Clock Bias

Wang Hu (whu027@ucr.edu) and Jay A. Farrell (farrell@ee.ucr.edu)
Department of Electrical and Computer Engineering
University of California, Riverside, 92521

***Abstract*—This document derives and analyzes two theoretically equivalent methods to account for the fact that the ECEF frame rotates between the time that a signal leaves a satellite t_s and the time that it is received at a receiver t_r. The first approach rotates the computed satellite position vector to a desired frame (see for example Appendix Section C.3 in [1]). The second approach is the Sagnac correction (see for example page 340 in [2]).**

In particular, the report analyzes the effect of uncompensated receiver clock error on the accuracy of the computed range.

I. Notation

Let E_r represent the ECEF frame at time t_r. Let E_s represent the ECEF frame at time t_s. Frame E_r is distinct from E_s due to the earth rotation $\boldsymbol{\omega}_{ie}\,(t_r - t_s)$.

Because E_r is distinct from E_s, the satellite position vector at time t_s in the E_r frame, denoted as $\mathbf{P}_s^{E_r}$, is distinct from the satellite position vector at time t_s in the E_s, denoted as $\mathbf{P}_s^{E_s}$. These two position vectors are related by

$$\mathbf{P}_s^{E_r} = \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}. \tag{1}$$

Given that the time-of-transit is $t_T = \frac{R}{c}$, where c is the speed of light, and the earth rotation vector is $\boldsymbol{\omega}_{ie} = [0, 0, \omega_{ie}]^\top$, the rotation matrix $\mathbf{R}_{E_s}^{E_r}$ can be expressed as

$$\mathbf{R}_{E_s}^{E_r} = \begin{bmatrix} cos(\omega_{ie}t_T) & sin(\omega_{ie}t_T) & 0 \\ -sin(\omega_{ie}t_T) & cos(\omega_{ie}t_T) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Vectors can only be properly difference when they are in the same frame. Therefore, a valid definition of the true geometric range is

$$R = ||\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_r}||. \tag{3}$$

Herein, the symbol R denotes a true geometric range. The symbol ρ is reserved for pseudorange which is geometric range plus a clock bias.

The Sagnac correction states that the geometric range can also be computed as:

$$R = ||\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s}|| + R_{corr}(\mathbf{P}_s^{E_s}, \mathbf{P}_r^{E_r}) + e_1 + e_2 \tag{4}$$

where $R_{corr}(\mathbf{P}_s^{E_s}, \mathbf{P}_r^{E_r}) = \frac{\omega_{ie}}{c}(xb - ay)$ using the coordinate representations: $\mathbf{P}_r^{E_r} = [a, b, d]^T$ and $\mathbf{P}_s^{E_s} = [x, y, z]^T$. The symbols e_1 and e_2 represent small error terms defined subsequently.

II. Derivation of Equivalence

The true range defined in eqn. (3) can be represented as

$$R = ||\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}||. \tag{5}$$

Because both ω_{ie} and t_T are small, the angle $\omega_{ie}t_T$ is also small. Applying small angle approximations yields:

$$\mathbf{R}_{E_s}^{E_r} = \begin{bmatrix} 1 & \omega_{ie}\frac{R}{c} & 0 \\ -\omega_{ie}\frac{R}{c} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

Eqn. (5) can be manipulated as follows:

$$\begin{aligned}
R &= ||\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}|| && (7)\\
&= \frac{\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}}{||\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}||} \cdot (\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) && (8)\\
&= \mathbf{1}_r \cdot (\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s} + \mathbf{P}_s^{E_s} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) && (9)\\
&= \mathbf{1}_s \cdot (\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s}) + \mathbf{1}_r \cdot (\mathbf{P}_s^{E_s} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) + e_1 && (10)\\
&= ||\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s}|| + \mathbf{1}_r \cdot (\mathbf{P}_s^{E_s} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) + e_1 && (11)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{1}_r &= \frac{\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}}{||\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}||} && (12)\\
\mathbf{1}_s &= \frac{\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s}}{||\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s}||} && (13)\\
e_1 &= (\mathbf{1}_r - \mathbf{1}_s) \cdot (\mathbf{P}_r^{E_r} - \mathbf{P}_s^{E_s}), && (14)
\end{aligned}$$

e_1 has order of magnitude 10^{-5} to 10^{-3}.

The Sagnac correction is derived from the second term:

$$\mathbf{1}_r \cdot (\mathbf{P}_s^{E_s} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}).$$

Note that $\mathbf{R}_{E_s}^{E_r} = \left(\mathbf{I} - \frac{R}{c}[\boldsymbol{\omega}_{ie}\times]\right)$. Therefore,

$$\begin{aligned}
(\mathbf{P}_s^{E_s} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) &= (\mathbf{I} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s} && (15)\\
&= \left(\frac{R}{c}[\boldsymbol{\omega}_{ie} \times \mathbf{P}_s^{E_s}]\right) && (16)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&\mathbf{1}_r \cdot \left(\frac{R}{c}[\boldsymbol{\omega}_{ie} \times \mathbf{P}_s^{E_s}]\right) = \frac{\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}}{R} \cdot (\mathbf{P}_s^{E_s} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s})\\
&\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} - \begin{bmatrix} x + \omega_{ie}\frac{R}{c}y \\ y - \omega_{ie}\frac{R}{c}x \\ 0 \end{bmatrix}\right) \cdot \left(\frac{R}{c}[\boldsymbol{\omega}_{ie} \times \mathbf{P}_s^{E_s}]\right)\frac{1}{R}\\
&\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} - \begin{bmatrix} x + \omega_{ie}\frac{R}{c}y \\ y - \omega_{ie}\frac{R}{c}x \\ 0 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}\right)\frac{\omega_{ie}}{c} && (17)\\
&= \frac{\omega_{ie}}{c}(xb - ay) + e_2\\
&= \frac{\omega_{ie}}{c}(xb - ay) + e_2
\end{aligned}$$

where $e_2 = \frac{\omega_{ie}}{c^2}(x^2 + y^2)$ which has order of magnitude 10^{-6}. This completes the derivation of eqn. (4) from eqn. (3).

III. Analysis of the Effect of Receiver Clock Bias

When the designer chooses to implement the rotation matrix approach of eqn. (3), instead of the Sagnac correction approach of eqn. (4), the implementation must compute the signal time-of-transit t_T to determine the angle of rotation. After eqn. (1) this time is defined as $t_T = \frac{R}{c}$. Therefore, the implementation requires the range R, but only the pseudorange $\rho = R + ct_r$ is available at the start of the data processing for a given epoch. This section considers the effect of the receiver clock bias ct_r on the accuracy of the range computed by eqn. (3).

The corrected pseudorange is related to the range by

$$\rho = R + c(t_r - \hat{t}_r) + c(t^s - \hat{t}^s) + I_a + T_a + m_p + \eta \tag{18}$$

where $\hat{t}_r$ represents the available estimate of the receiver clock bias (if any), t^s and $\hat{t}^s$ are the satellite clock bias and its estimate computed from ephemeris data, I_a is ionospheric delay, T_a is tropospheric delay, m_p represents multi-path, and η represents measurement noise. In the analysis to follow, we will write this as $\rho = R + \delta$, where

$$\delta = c(t_r - \hat{t}_r) + c(t^s - \hat{t}^s) + I_a + T_a + m_p + \eta.$$

If the implementation computes the transit time as $\hat{t}_T = \frac{\rho}{c}$, then

$$\begin{aligned} \hat{t}_T &= \frac{\rho}{c} = \frac{R+\delta}{c} \\ &= t_T + (t_r - \hat{t}_r) + (t^s - \hat{t}^s) + \frac{1}{c}\big(I_a + T_a + m_p + \eta\big). \end{aligned}$$

of the three error terms on the right, $(t_r - \hat{t}_r)$ may be at the millisecond level, while $(t^s - \hat{t}^s)$ and $\frac{1}{c}\big(I_a + T_a + m_p + \eta\big)$ are at the tens of nanosecond level. Therefore, the receiver clock error, which is correctable through $\hat{t}_r$, is the dominant error. The presentation that follows focuses only on the receiver clock bias error

$$t_c = (t_r - \hat{t}_r). \tag{19}$$

The analysis for the other terms in δ follows an identical approach.

The computed rotation matrix is

$$\begin{aligned} \hat{\mathbf{R}}_{E_s}^{E_r} &= \begin{bmatrix} 1 & \omega_{ie}\frac{\rho}{c} & 0 \\ -\omega_{ie}\frac{\rho}{c} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (20) \\ &= \begin{bmatrix} 1 & \omega_{ie}\frac{R+ct_c}{c} & 0 \\ -\omega_{ie}\frac{R+ct_c}{c} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (21) \\ &= \begin{bmatrix} 1 & \omega_{ie}(t_T + t_c) & 0 \\ -\omega_{ie}(t_T + t_c) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (22) \\ &= \mathbf{R}_{E_s}^{E_r} + \delta_{\mathbf{R}}. && (23) \end{aligned}$$

where

$$\delta_{\mathbf{R}} = \begin{bmatrix} 0 & \omega_{ie}t_c & 0 \\ -\omega_{ie}t_c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = t_c\,[\boldsymbol{\omega}_{ie}\times]$$

The computed range is

$$\begin{aligned} \hat{R} &= \|\mathbf{P}_r^{E_r} - \hat{\mathbf{R}}_{E_s}^{E_r}\mathbf{P}_s^{E_s}\| && (24) \\ &= \frac{\mathbf{P}_r^{E_r} - \hat{\mathbf{R}}_{E_s}^{E_r}\mathbf{P}_s^{E_s}}{\|\mathbf{P}_r^{E_r} - \hat{\mathbf{R}}_{E_s}^{E_r}\mathbf{P}_s^{E_s}\|} \cdot (\mathbf{P}_r^{E_r} - \hat{\mathbf{R}}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) && (25) \\ &= \hat{\mathbf{1}}_r \cdot (\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s} - \delta_{\mathbf{R}}\mathbf{P}_s^{E_s}) && (26) \\ &= \mathbf{1}_r \cdot (\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}) - \hat{\mathbf{1}}_r \cdot \delta_{\mathbf{R}}\mathbf{P}_s^{E_s} + e_4 && (27) \\ &= R - \big(\hat{\mathbf{1}}_r \cdot [\boldsymbol{\omega}_{ie} \times \mathbf{P}_s^{E_s}]\big)t_c + e_4 && (28) \\ &= R + e_3 + e_4 && (29) \end{aligned}$$

where

$$\begin{aligned} \hat{\mathbf{1}}_r &= \frac{\mathbf{P}_r^{E_r} - \hat{\mathbf{R}}_{E_s}^{E_r}\mathbf{P}_s^{E_s}}{\|\mathbf{P}_r^{E_r} - \hat{\mathbf{R}}_{E_s}^{E_r}\mathbf{P}_s^{E_s}\|} && (30) \\ e_3 &= -\big(\hat{\mathbf{1}}_r \cdot [\boldsymbol{\omega}_{ie} \times \mathbf{P}_s^{E_s}]\big)t_c && (31) \\ e_4 &= \big(\hat{\mathbf{1}}_r - \mathbf{1}_r\big) \cdot (\mathbf{P}_r^{E_r} - \mathbf{R}_{E_s}^{E_r}\mathbf{P}_s^{E_s}). && (32) \end{aligned}$$

The magnitude of the error e_3 is less than $\Big(\|\boldsymbol{\omega}_{ie}\|\,\|\mathbf{P}_s^{E_s}\|\,t_c\Big)$ which is approximately $1400t_c$.

Fig. 1 shows examples of the biases e_3 and e_4 as a function of the uncorrected receiver clock error t_c, for various satellites. The figure shows that the range error e_3 causes range errors at the meter level if the clock error is not properly compensated.

To exemplify how receiver clock bias affects the positioning error, Figs. 2.a and 2.b show the DGNSS positioning error versus time for two different algorithms. The algorithm used to produces Fig. 2.a involves a recursive implementation of nonlinear least squares that estimates receiver antenna position and clock bias, where the clock bias estimate is used in eqn. (19) to compensate eqn. (20). Fig. 2.b uses the rotation matrix of eqn. (20) to rotate the satellite position from E_s to E_r (i.e., $t_c = t_r$ with $\hat{t}_r = 0$) before estimating the rover position. Fig. 2.c shows the actual receiver clock bias. Comparing the first two figures shows that failure to compensate for the receiver clock bias decreases the positioning accuracy (i.e., increases the position error) in a patern comparable to the magnitude of receiver clock bias.

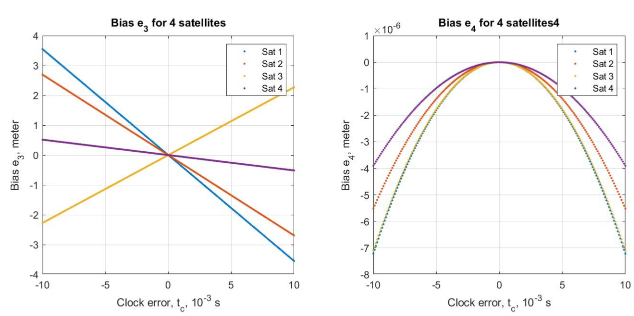

Fig. 1: Examples for e_3 and e_4 versus clock error.

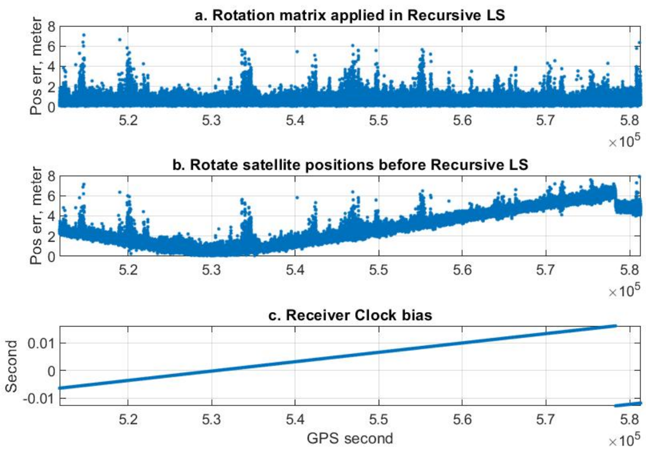

Fig. 2: The effect of rotation matrix

REFERENCES

[1] J. Farrell, *Aided Navigation: GPS with High Rate Sensors*. McGraw-Hill, Inc., 2008.

[2] P. D. Groves, *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems*. Artech house, 2013.