Twisted Gromov-Witten invariants and applications to quantum K-theory

by

Valentin Tonita

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Committee in charge:

Professor Alexander Givental, Chair
Professor Constantin Teleman
Professor Ori J. Ganor

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Abstract

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Given a projective smooth complex variety X, one way to associate to it numerical invariants is by taking holomorphic Euler characteristics of interesting vector bundles on $X_{0,n,d}$ - the moduli spaces of genus 0, degree d stable maps with n marked points to X. We call these numbers *genuine* quantum K-theoretic invariants of X. Their generating series is called the genus 0 K-theoretic descendant potential of X and can be viewed as a function on a suitable infinite dimensional vector space $\mathcal{K}_+$. Its graph is a uniruled Lagrangian cone in $T^*\mathcal{K}_+$.

We give a complete characterisation of points on the cone, proving a Hirzebruch Riemann Roch type theorem for the genuine K-theory of X. In particular, our result can be used to recursively express all genus 0 K-theoretic invariants of X in terms of cohomological ones (usually known as *Gromov-Witten invariants*). The main technical tool we use is the Kawasaki Riemann Roch theorem of [Ka], which reduces the computation of holomorphic Euler characteristic of a bundle on an orbifold to the computation of a cohomological integral on the inertia orbifold.

In the process, we need to study more general cohomological Gromov-Witten invariants of an orbifold $\mathcal{X}$, which we call *twisted* invariants. These are obtained by capping the virtual fundamental classes of the moduli spaces $\mathcal{X}_{g,n,d}$ with certain multiplicative characteristic classes. We twist the Gromov-Witten potential by three types of twisting classes and we allow several twistings of each type. We use a Mumford's Grothendieck-Riemann-Roch computation on the universal curve to give closed formulae which show the effect of each type of twist on their generating series (the *twisted* potential). This generalizes earlier results of [CG] and [TS].

To my parents.

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Chapter 1

Introduction

In this chapter we give a self-contained presentation of the main theorems proved in later chapters.

1.1 K-theoretic Gromov-Witten invariants

We work over the field of complex numbers $\mathbb{C}$. Let X be a nonsingular complex projective variety. Let $X_{g,n,d}$ be the moduli spaces of stable maps of [K] : they parametrize data $(C, x_1, \ldots, x_n, f)$ where C is an n-pointed genus g Riemann surface and $f : C \to X$ is a degree $d \in H_2(X, \mathbb{Z})$ holomorphic map. There are natural maps:

$$ev_i : X_{g,n,d} \to X, \quad i = 1, \ldots, n$$

given by evaluation at the ith marked point. There are line bundles

$$L_i \to X_{g,n,d}, \quad i = 1, \ldots, n$$

called universal cotangent line bundles. The fiber of L_i over the point $(C, x_1, \ldots, x_n, f)$ is the cotangent line to C at the point x_i.
Let $a_1, \ldots a_n \in K^0(X, \mathbb{C})$. K-theoretic Gromov-Witten invariants are holomorphic Euler characteristics over $X_{g,n,d}$ of the sheaves:

$$ev_1^*(a_1)L_1^{k_1} \cdot \ldots ev_n^*(a_n)L_n^{k_n} \otimes \mathcal{O}^{vir}.$$

We will often use correlator notation for these numbers:

$$\left\langle a_1 L^{k_1}, \ldots, a_n L^{k_n} \right\rangle_{g,n,d}^X := \chi\left(X_{g,n,d}; ev_1^*(a_1)L_1^{k_1} \cdot \ldots ev_n^*(a_n)L_n^{k_n} \otimes \mathcal{O}^{vir}\right).$$

The sheaf $\mathcal{O}^{vir}$ was introduced by Y.-P. Lee in [L2] and plays a role in the K-theoretic version of the GW theory of X analogue to the role played by the virtual fundamental class $[X_{g,n,d}]$ in the cohomological GW theory of X.

The following generating series:

$$\mathcal{F}_X^0 := \sum_{d,n} \frac{Q^d}{n!} \langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \rangle_{0,n,d}^X$$

is called the genus 0 K-theoretic Gromov-Witten descendant potential. Here Q^d is the monomial representing the degree $d \in H_2(X, \mathbf{Z})$ in the Novikov ring $\mathbb{C}[[Q]]$, which is a completion of the semigroup ring of degrees of holomorphic curves in X and $\mathbf{t}(L)$ is a Laurent polynomial in L with vector coefficients $t_i \in K^0(X)$. Thus F_X^0 is a formal series in $\mathbf{t}(L)$ with coefficients in the Novikov ring. The summation is taken after all stable pairs (d, n) i.e. if $d = 0$ then $n \geq 3$.

Our aim is to express K-theoretic invariants in terms of cohomological Gromov-Witten invariants. The main tool is the Kawasaki-Riemann-Roch theorem of Section 1.3, which computes holomorphic Euler characteristics of orbifolds. We will use Kawasaki-Riemann-Roch to give a characterisation of the expansions of the J-function (which is introduced in Section 1.2) near each of its poles. This is the result of Theorem 1.6.2.

1.2 Symplectic loop space

The totality of genus 0 Gromov-Witten invariants can be encoded by a certain Lagrangian submanifold $\mathcal{L}$ living in a certain symplectic vector space $\mathcal{K}$, called the *loop space* and defined as:

$$\mathcal{K} := \left[K^0(X) \otimes \mathbb{C}(q, q^{-1})\right] \otimes \mathbb{C}[[Q]],$$

where $\mathbb{C}(q, q^{-1})$ is the ring of rational functions on the complex circle with coordinate q. Let $(\ ,\ )$ be the pairing on $K^0(X)$:

$$(a, b) := \chi(X, a \otimes b).$$

We endow $\mathcal{K}$ with the following symplectic form:

$$\mathbf{f}, \mathbf{g} \mapsto \Omega(\mathbf{f}, \mathbf{g}) := [Res_{q=0} + Res_{q=\infty}] \left(\mathbf{f}(q), \mathbf{g}(q^{-1})\right) \frac{dq}{q}.$$

The following two subspaces:

$$\mathcal{K}_+ = K^0(X)[q, q^{-1}] \otimes \mathbb{C}[[Q]], \qquad \mathcal{K}_- := \{\mathbf{f} \in \mathcal{K} \quad | \quad \mathbf{f}(0) \neq \infty, \mathbf{f}(\infty) = 0\}$$

form a Lagrangian polarisation of $\mathcal{K}$. $K^0(X)[q, q^{-1}]$ is the ring of Laurent polynomials in q with coefficients in $K^0(X)$.

We now introduce the big J function of X, which is the generating function:

$$\mathcal{J}(\mathbf{t}) = 1 - q + \mathbf{t}(q) + \sum_a \phi_a \sum_{d,n} \frac{Q^d}{n!} \left\langle \frac{\phi^a}{1-qL}, \mathbf{t}(L), \dots, \mathbf{t}(L) \right\rangle_{0,n+1,d}^X .$$

Here $\{\phi_a\}$ and $\{\phi^a\}$ are any Poincaré dual bases of $K^0(X)$. $\mathbf{t} \mapsto \mathcal{J}(\mathbf{t})$ is a map $\mathcal{K}_+ \to \mathcal{K}$ and is identified with the graph of the differential of the genus 0 potential, via the identification $T^*\mathcal{K}_+ = \mathcal{K}_+ \oplus \mathcal{K}_-$ and the dilaton shift $\mathbf{f} \mapsto \mathbf{f} + 1 - q$:

$$\mathcal{J}(\mathbf{t}) = 1 - q + \mathbf{t}(q) + d_{\mathbf{t}}\mathcal{F}_X^0.$$

The genus-0 general properties of K-theoretic Gromov-Witten invariants from [L2] are captured by the following:

Theorem 1.2.1. *The range of the $\mathcal{J}$ function is the formal germ of a Lagrangian cone $\mathcal{L}$ such that each tangent space T to $\mathcal{L}$ is tangent to $\mathcal{L}$ exactly along $(1-q)T$. In other words $T \cap \mathcal{L} = (1-q)T$ and the tangent space at all points of $(1-q)T$ is T.*

The theorem is a variant of results in [G1]. We'll sketch its proof in Section 3.5.

We call the submanifolds with the properties of the theorem *overruled Lagrangian cones.*

1.3 Kawasaki Riemann Roch theorem

For a complex manifold M, one can reduce the computation of the Euler characterstic of a holomorphic Euler bundle E to the computation of a cohomological integral via the Hirzebruch Riemann Roch theorem of [HR], which states that:

$$\chi(M,E) = \int_M ch(E)Td(T_M),$$

where Td is the Todd class. In [Ka] Kawasaki generalized this formula to the case when M is an orbifold. He reduces the computation of Euler characteristics on M to computation of certain cohomological integrals on *the inertia orbifold IM*.

$$\chi(M,E) = \sum_i \frac{1}{m_i} \int_{M_i} Td(T_{M_i}) ch \left(\frac{Tr(E)}{Tr(\Lambda^\bullet N_i^*)} \right).$$

We explain below the ingredients in Kawasaki's formula:
IM is defined as follows: around any point $p \in M$ there is a local chart $(\widetilde{U}_p, G_p)$ such that locally M is represented as the quotient of $\widetilde{U}_p$ by G_p. Consider the set of conjugacy classes $(1) = (h_p^1)$, (h_p^2), …, $(h_p^{n_p})$ in G_p. Define:

$$IM := \{(p, (h_p^i) \quad | \quad i = 1, 2, \dots, n_p\}.$$

Pick an element h_p^i in each conjugacy class. Then a local chart on IM is given by:

$$\coprod_{i=1}^{n_p} \widetilde{U}_p^{(h_p^i)} / Z_{G_p}(h_p^i),$$

where $Z_{G_p}(h_p^i)$ is the centralizer of h_p^i in G_p. Denote by M_i the connected components of the inertia orbifold (we'll often refer to them as Kawasaki strata). The multiplicity m_i associated to each M_i is given by:

$$m_i := \left| ker \left(Z_{G_p}(g) \to Aut(\widetilde{U}_p^g) \right) \right|.$$

The restriction of E to M_i decomposes in characters of the g action. Let $E_r^{(l)}$ be the subbundle of the restriction of E to M_i on which g acts with eigenvalue $e^{\frac{2\pi i l}{r}}$. Then the trace $Tr(E)$ is defined to be the orbibundle whose fiber over the point $(p,(g))$ of M_i is :

$$Tr(E) := \sum_l e^{\frac{2\pi i l}{r}} E_r^{(l)}.$$

Finally, $\Lambda^\bullet N_i^*$ is the K-theoretic Euler class of the normal bundle N_i of M_i in M. $Tr(\Lambda^\bullet N_i^*)$ is invertible because the symmetry g acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. We call the terms corresponding to the identity component in the formula *fake Euler characteristics*:

$$\chi^f(M,E) = \int_M ch(E) Td(T_M).$$

Notice that all the terms in Kawasaki's formula are fake Euler characteristics of certain bundles.

1.4 Fake quantum K-theory

The fake K-theoretic Gromov-Witten invariants are defined as :

$$\left\langle a_1 L^{k_1}, \ldots, a_n L^{k_n} \right\rangle_{0,n,d}^f := \int_{[X_{0,n,d}]} ch\left(ev_1^*(a_1) L_1^{k_1} \cdot \ldots \cdot ev_n^*(a_n) L_n^{k_n} \right) \cdot Td(\mathcal{T}_{0,n,d}^{vir})$$

where $\mathcal{T}_{0,n,d}^{vir}$ is the virtual tangent bundle to $X_{0,n,d}$. In general they are rational numbers. We define the big J function as:

$$\mathcal{J}_f(\mathbf{t}) = 1 - q + \mathbf{t}(q) + \sum_a \phi_a \sum_{d,n} \frac{Q^d}{n!} \left\langle \frac{\phi^a}{1 - qL_1}, \mathbf{t}(L_2), \ldots, \mathbf{t}(L_{n+1}) \right\rangle_{0,n+1,d}^f.$$

The loop space of the fake theory is defined as:

$$\mathcal{K}^f = \left[K^0(X) \otimes \mathbb{C}(((q-1)^{-1}))\right] \otimes \mathbb{C}[[Q]].$$

The symplectic structure is:

$$\mathbf{f}, \mathbf{g} \mapsto \Omega^f(\mathbf{f}, \mathbf{g}) = -Res_{q=1}\left(\mathbf{f}(q), \mathbf{g}(q^{-1})\right)\frac{dq}{q}.$$

A Lagrangian polarisation for $\mathcal{K}^f$ is given by:

$$\mathcal{K}^f_+ := K^0(X)[[(q-1)]] \otimes \mathbb{C}[[Q]],$$
$$\mathcal{K}^f_- := \frac{1}{1-q}K^0(X)[[\frac{1}{1-q}]] \otimes \mathbb{C}[[Q]].$$

In fact, if we expand

$$\frac{1}{1-qL} = \sum_{k\geq 1}(L-1)^k \frac{q^k}{(1-q)^{k+1}},$$

then a Darboux basis of $\mathcal{K}^f$ is given by $\{\phi^a(q-1)^k, \phi_a\frac{q^k}{(1-q)^{k+1}}\}$. It is a result of [G1] that, just like in the case of the genuine theory, the range of the J function of the genus 0 invariants is a formal germ of an overruled Lagrangian cone, which we call $\mathcal{L}^f$.
We call symplectic transformations on $\mathcal{K}^f$ which commute with multiplication by q loop group elements. They are series in $q-1$ with $End(K^0(X))$ coefficients.

1.5 Cohomological Gromov-Witten theory

The relation between the fake K-theoretic invariants of X and the cohomological ones has been studied in [C] and described in terms of the symplectic geometry of the loop space. Before stating the result, we need to briefly recall the setup of the cohomological theory. Let

$$\mathcal{H} := \mathbb{C}[[Q]] \otimes H^*(X, \mathbb{C})((z))$$

be the cohomological loop space. We endow $\mathcal{H}$ with the symplectic form:

$$\Omega(\mathbf{f}, \mathbf{g}) := \oint_{z=0} (\mathbf{f}(z), \mathbf{g}(-z))\, dz,$$

where $(\ ,\ )$ is the Poincaré pairing on $H^*(X)$. Consider the following polarisation of $\mathcal{H}$:

$$\mathcal{H}_+ := H^*(X, \mathbb{C})[[z]] \quad and \quad \mathcal{H}_- := z^{-1}H^*(X, \mathbb{C})[z^{-1}].$$

Let $\psi_i = c_1(L_i)$. We define the genus 0 potential as:

$$\mathcal{F}_H^0(\mathbf{t}) := \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{0,n,d}.$$

Let $\mathbf{q}(z) = \mathbf{t}(z) - z$. Consider the graph of the genus 0 potential, regarded as a function of $\mathbf{q}$:

$$\mathcal{L}^H := \{(\mathbf{p}, \mathbf{q}) \quad | \quad \mathbf{p} = d_{\mathbf{q}} \mathcal{F}_H^0\} \subset T^*\mathcal{H}_+ \simeq \mathcal{H}.$$

Then according to [G1], $\mathcal{L}^H$ is the formal germ of an overruled cone with vertex at the shifted origin $-\mathbf{z}$. Overruled means that the tangent spaces T to $\mathcal{L}^H$ are tangent to $\mathcal{L}^H$ exactly along zT.

Given a function $x \to s(x)$ the Euler-Maclaurin asymptoics of $\prod_{r=1}^\infty e^{s(x-rz)}$ is obtained as follows:

$$\sum_{r=1}^\infty s(x - rz) = \left(\sum_{r=1}^\infty e^{-rz\partial_x} \right) s(x) = \frac{z\partial_x}{e^{z\partial_x}} (z\partial_x)^{-1} s(x) =$$
$$= \frac{s^{(-1)}(x)}{z} - \frac{s(x)}{2} + \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!} s^{(2k-1)}(x) z^{2k-1},$$

where $s^{(k)} = d^k s/dx^k$, $s^{(-1)}$ is the antiderivative $\int_0^x s(t)dt$, and B_k are the Bernoulli numbers. Let x_i be the Chern roots of T_X, and let Δ be the Euler-Maclaurin asymptotics of the infinite product:

$$\Delta \sim \prod_i \prod_{r=1}^\infty \frac{x_i - rz}{1 - e^{-x_i + rz}}.$$

We identify $\mathcal{K}^f$ with $\mathcal{H}$ extending the Chern character isomorphism $ch : K^0(X, \mathbb{C}) \to H^*(X, \mathbb{C})$:

$$ch : \mathcal{K}^f \to \mathcal{H},$$
$$q \mapsto e^z.$$

This maps $\mathcal{K}_+^f$ to $\mathcal{H}_+$, but it doesn't map $\mathcal{K}_-^f$ to $\mathcal{H}_-$.

Theorem([C]) 1.5.1. *$\mathcal{L}^f$ is obtained from $\mathcal{L}^H$ by pointwise multiplication by Δ:*

$$\mathcal{L}^f = ch^{-1}(\Delta \mathcal{L}^H).$$

1.6 The main theorem

We will use Kawasaki's formula to express genus 0 K-theoretic GW invariants in terms of cohomological ones. We will first identify the Kawasaki strata in the moduli spaces $X_{0,n,d}$. Points with nontrivial symmetries in $X_{0,n,d}$ are those for which the domain contains a distinguished connected component, call it C, such that the map $f_{|C} : C \to X$ (of degree say md') factors through an m cover $C \to C'$ given in local coordinates $z \mapsto z^m$. The set of special points of C is fixed by the symmetry. Hence the only possible marked points lying on C are $0, \infty$. We also encounter the following situation: there are m-tuples of curves $C_1, \ldots, C_m$, isomorphic as stable maps, which intersect C and are permuted by the $\mathbb{Z}_m$ action. Notice that this prevents them from carrying marked points , which in turn, by the stability condition forces the maps $f_{|C_i} \to X$ to have positive degrees. Assume there are l tuples of such curves, denote the nodes by $\{x_1, \ldots, x_{lm}\}$. We call the moduli spaces parametrizing such objects $(C, 0, x_1, \ldots, x_{lm}, \infty, f)$ *stem spaces*. It is tempting to identify the stem spaces with the moduli spaces $X_{0,l+2,d'}$. This is not true, because the orbifold structure near the nodes is different.

Proposition 1.6.1. *The stem spaces are identified with the moduli spaces denoted* $(X \times B\mathbb{Z}_m)_{0,l+2,d',(g,0,\ldots,g^{-1})}$ *of orbimaps to the orbifold* $X/\mathbf{Z}_m$.

We explain the notation $(X \times B\mathbb{Z}_m)_{0,l+2,d',(g,0,\ldots,g^{-1})}$ in Section 1.7.

We refer to the (K-theoretic, cohomological) class in a certain seat in the correlators as the "input" at the corresponding marked point. Notice that the first marked point has distinguished input $\frac{\phi^a}{1-qL}$. Assume the first marked point lies on a stem space with $\mathbb{Z}_m$ symmetries, where the generator g of $\mathbb{Z}_m$ acts on the cotangent line with eigenvalue ζ. Denote by $\widetilde{\mathbf{t}}$ the sum of contributions in $\mathcal{J}$ coming from integrals on stem spaces corresponding to $\zeta \neq 1$. If on the contrary the first marked point lies on a component of the inertia orbifold indexed by the identity, the contributions to $\mathcal{J}$ are fake Euler characteristics on strata where the other special points are marked points or nodes. But the rational tails at these nodes must have nontrivial symmetries, otherwise we can regard the whole curve as a degeneration within a stratum without symmetries. When we sum after these possibilities the input in the correlators at each of these points is $\mathbf{t} + \widetilde{\mathbf{t}}$. This shows that:

$$\mathcal{J}(\mathbf{t}(q)) = \mathcal{J}_f(\mathbf{t} + \widetilde{\mathbf{t}}).$$

So we see that the expansion near $q = 1$ of $\mathcal{J}$ lies on $\mathcal{L}^f$. But $\mathcal{J}$ has poles at all roots of unity ζ. After making the change of variable $q \mapsto q\zeta^{-1}$ we can regard $\mathcal{J}$ as an element of $\mathcal{K}^f$. It turns out that this element lies in the tangent space to a cone obtained from $\mathcal{L}^f$ by an explicit procedure. For $\mathbf{f} \in \mathcal{K}$, denote by $\mathbf{f}_\zeta$ the expansion of $\mathbf{f}$ as a Laurent polynomial in $(1 - q\zeta)$. The main result of the thesis is the following theorem:

Theorem 1.6.2. *Let* $\mathcal{L} \in \mathcal{K}$ *the overruled Lagrangian cone of quantum K theory on* X. *Then* $\mathbf{f} \in \mathcal{L}$ *iff the following hold :*

1. $\mathbf{f}_\zeta$ *doesn't have poles unless* $\zeta \neq 0, \infty$ *is a root of unity.*

2. $\mathbf{f}_1$ *lies on* $\mathcal{L}^f$.

3. *In particular* $\mathcal{J}_1(0) \in \mathcal{L}^f$. *The tangent space to* $\mathcal{L}^f$ *at* $\mathcal{J}_1(0)$ *is given as* $S^{-1}(\mathcal{K}_+^f)$, *where* S^{-1} *is the matrix of a linear transformation, whose entries are Laurent series in* $q-1$ *with coefficients in* $\mathbb{C}[[Q]]$. *Denote by* $\widetilde{S}$ *the matrix obtained from* S *via the change of variable* $q \mapsto q^m$, $Q^d \mapsto Q^{md}$, *and denote by* $\mathcal{T} := \widetilde{S}^{-1}(\mathcal{K}_+^f)$. *Let* P_i *be the K-theoretic Chern roots of* T_X^* *and let* ∇_ζ *denote the Euler-Maclaurin asymptotics as* $q\zeta \to 1$ *of the infinite product:*

$$\nabla_\zeta \sim_{q\zeta\to 1} \prod_i \frac{\prod_{r=1}^{\infty}(1-q^{mr}P_i)}{\prod_{r=1}^{\infty}(1-q^r P_i)}.$$

Then if $\zeta \neq 1$ *is a primitive* m *root of* 1, $(\nabla_\zeta^{-1}\mathbf{f}_\zeta)(q/\zeta) \in \mathcal{T}$.

Conversely, every point that satisfies conditions 1-3 above lies on $\mathcal{L}$.

These conditions allow one to compute the values $\mathcal{J}(\mathbf{t})$ for all $\mathbf{t}$, assuming the cone $\mathcal{L}^f$ is known. Since we know how $\mathcal{L}^f$ is related to $\mathcal{L}^H$, the theorem expresses the K-theoretic invariants in terms of cohomological ones.

1.7 Orbifold Gromov-Witten invariants

Let $\mathcal{X}$ be a compact orbifold. Moduli spaces of orbimaps to orbifolds have been constructed by [CR1] in the setup of symplectic orbifolds and by [AGV2] in the context of Deligne-Mumford stacks. Informally, the domain curve is allowed to have nontrivial orbifold structure at the marked points and nodes.

Definition 1.7.1. *A nodal* n*-pointed orbicurve is a nodal marked curve* $(\mathcal{C}, x_1, \ldots, x_n)$, *such that*

- $\mathcal{C}$ has trivial orbifold structure on the complement of the marked points and nodes.

- In an analytic neighborhood of a marked point, $\mathcal{C}$ is isomorphic to the quotient [Spec $\mathbb{C}[z]/\mathbb{Z}_r$], for some r, and the generator of $\mathbb{Z}_r$ acts by $z \mapsto \zeta z$, $\zeta^r = 1$.

- In an analytic neighborhood of a node, $\mathcal{C}$ is isomorphic to [Spec $(\mathbb{C}[z,w]/(zw))/\mathbb{Z}_r$], and the generator of $\mathbb{Z}_r$ acts by $z \mapsto \zeta z$, $w \mapsto \zeta^{-1}w$.

Just like in the case of manifold target spaces, there are evaluation maps $\overline{ev}_i$ at the marked points. Although it is clear how these maps are defined on geometric points, it turns out that, to have well-defined morphisms of Deligne-Mumford stacks the target of the evaluation

maps is the rigidified inertia stack of $\mathcal{X}$. The rigidified inertia stack, which we denote $\overline{I\mathcal{X}}$, is defined by taking the quotient at $(x,(g))$ of the automorphism group by the cyclic subgroup generated by g. So, whereas a local chart at $(x,(g))$ on $I\mathcal{X}$ is given by $\widetilde{U_g}/Z_{G_x}(g)$, on $\overline{I\mathcal{X}}$ a local chart is $\widetilde{U_g}/[Z_{G_x}(g)/\langle g\rangle]$. We write $\overline{I\mathcal{X}} := \coprod_\mu \overline{\mathcal{X}_\mu}$.
We denote by:

$$\mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)} := \mathcal{X}_{g,n,d} \cap (\overline{ev}_1)^{-1}(\overline{\mathcal{X}}_{\mu_1}) \cap \ldots \cap (\overline{ev}_n)^{-1}(\overline{\mathcal{X}}_{\mu_n}).$$

$I\mathcal{X}$ and $\overline{I\mathcal{X}}$ have the same geometric points (coarse spaces), hence we can identify the rings $H^*(I\mathcal{X},\mathbb{C})$ and $H^*(\overline{I\mathcal{X}},\mathbb{C})$. We consider the cohomological pullbacks by the maps ev_i having domain $H^*(I\mathcal{X},\mathbb{C})$. More precisely, if r_i is the order of the automorphism group of x_i, then define:

$$\begin{aligned} ev_i^* : H^*(I\mathcal{X},\mathbb{C}) &\to H^*(\mathcal{X}_{g,n,d},\mathbb{C}),\\ a &\mapsto r_i^{-1}(\overline{ev}_i)^*(p_*a). \end{aligned}$$

This accounts for the difference of degree of fundamental classes of $\overline{I\mathcal{X}}$ and $I\mathcal{X}$.
For each i, there are line bundles $L_i, \overline{L}_i$ whose fiber over each point $(\mathcal{C}, x_1,\ldots,x_n,f)$ are the cotangent line to $\mathcal{C}$ at x_i, respectively to the coarse space C at x_i. We denote by $\psi_i = c_1(L_i)$ and $\overline{\psi}_i = c_1(\overline{L}_i)$. If x_i has an automorphism group of order r_i than $\overline{\psi}_i = r_i\psi_i$.

We denote the universal family by $\pi : \mathcal{U}_{g,n,d} \to \mathcal{X}_{g,n,d}$. $\mathcal{U}_{g,n,d}$ can be identified with $\cup_{(\mu_1,\ldots,\mu_n)}\mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)}$. The moduli spaces $\mathcal{X}_{g,n,d}$ are equipped with virtual fundamental classes $[\mathcal{X}_{g,n,d}] \in H_*(\mathcal{X}_{g,n,d},\mathbb{Q})$. Orbifold Gromov-witten invariants are obtained by integrating $\overline{\psi}_i$ and evaluation classes on these cycles. We use our favourite correlator notation:

$$\left\langle a_1\overline{\psi}^{k_1},\ldots,a_n\overline{\psi}^{k_n}\right\rangle_{g,n,d} := \int_{[\mathcal{X}_{g,n,d}]}\prod_{i=1}^n ev_i^* a_i \overline{\psi}_i^{k_i}.$$

The following generating series are called the genus g potential, respectively the total potential:

$$\mathcal{F}^g_{\mathcal{X}}(\mathbf{t}) = \sum_{d,n}\frac{Q^d}{n!}\langle \mathbf{t},\ldots,\mathbf{t}\rangle_{g,n,d},$$

$$\mathcal{D}_{\mathcal{X}}(\mathbf{t}) = exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}^g(\mathbf{t})\right).$$

They are functions on a suitable infinite dimensional vector space, which we describe below.

We denote by $\iota : I\mathcal{X} \to I\mathcal{X}$ the involution which maps $(x,(g))$ to $(x,(g^{-1}))$. It descends to an involution on $\overline{I\mathcal{X}}$, which we also denote ι. Let $\mathcal{X}_{\mu^I} = \iota(\mathcal{X}_\mu)$.

The orbifold Poincaré pairing on $I\mathcal{X}$ is defined for $a \in H^*(\mathcal{X}_\mu, \mathbb{C})$, $b \in H^*(\mathcal{X}_{\mu^I}, \mathbb{C})$ as:

$$(a, b)_{orb} := \int_{\mathcal{X}_\mu} a \cup \iota^* b.$$

Let

$$\mathcal{H} := \mathbb{C}[[Q]] \otimes H^*(I\mathcal{X}, \mathbb{C})((z)).$$

We equip $\mathcal{H}$ with the symplectic form:

$$\Omega(\mathbf{f}, \mathbf{g}) := \oint_{z=0} (\mathbf{f}(z), \mathbf{g}(-z))_{orb}\, dz.$$

Consider the following polarisation of $\mathcal{H}$:

$$\mathcal{H}_+ := H^*(I\mathcal{X}, \mathbb{C})[[z]] \quad and \quad \mathcal{H}_- := z^{-1} H^*(I\mathcal{X}, \mathbb{C})[z^{-1}].$$

Let Λ be the completion of the semigroup of the Mori cone of $\mathcal{X}$. Then $\mathcal{D}_\mathcal{X}$ is a well defined formal function on $\mathcal{H}_+$ taking values in $\Lambda \otimes \mathbb{C}[[\hbar, \hbar^{-1}]]$.

1.8 Twisted Gromov-Witten invariants

"Twisted Gromov-Witten invariants" are obtained from the usual ones by systematically inserting in the correlators multiplicative classes of certain bundles. We first describe the result of [TS] on twisted GW invariants, and then explain our generalizations. Let $E \in K^0(X)$, let a general multiplicative class be

$$\mathcal{A}(E) = exp\left(\sum_k s_k ch_k E\right).$$

More precisely let $E_{g,n,d} := \pi_*(ev_{n+1}^* E) \in K^0(\mathcal{X}_{g,n,d})$ and let the twisted genus 0 potential be:

$$\mathcal{F}^0_\mathcal{A} := \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(z), \ldots, \mathbf{t}(z); \mathcal{A}(E_{g,n,d})\rangle^{\mathcal{X}}_{0,n,d},$$

where the insert of the multiplicative class in the correlators means we cup it with the integrand. Let

$$\mathcal{H}^\mathcal{A} := \mathcal{H} \otimes \mathbb{C}[[s_0, s_1, \ldots]].$$

The "twisted" Poincaré pairing on $\mathcal{H}^{\mathcal{A}}$ is defined for $a \in H^*(I\mathcal{X}_\mu)$, $b \in H^*(I\mathcal{X}_{\mu^I})$ as:

$$(a,b)_{\mathcal{A}} = \int_{\mathcal{X}_\mu} a \cdot \iota^* b \cdot \mathcal{A}((q^*E)_{inv}),$$

where $q : I\mathcal{X} \to \mathcal{X}$ is the map $(x,(g)) \mapsto x$ and $(q^*E)_{inv}$ is the invariant part of q^*E under the g action.
We equip $\mathcal{H}^{\mathcal{A}}$ with the symplectic form:

$$\mathbf{f}, \mathbf{g} \mapsto \oint_{z=0} (\mathbf{f}(z), \mathbf{g}(-z))_{\mathcal{A}}.$$

The polarisation:

$$\mathcal{H}^{\mathcal{A}}_+ := H^*(X,\Lambda)[[z]], \quad \mathcal{H}^{\mathcal{A}}_- := z^{-1}H^*(X,\Lambda)[z^{-1}]$$

realizes $\mathcal{H}^{\mathcal{A}}$ as $T^*\mathcal{H}^{\mathcal{A}}_+$. Then $\mathcal{F}^0_{\mathcal{A}}$ is a function on $\mathcal{H}^{\mathcal{A}}_+$ of $\mathbf{q}(z) = \mathbf{t}(z) - z$ and its graph:

$$\mathcal{L}_{\mathcal{A}} := \left\{ (\mathbf{p}, \mathbf{q}) \quad | \quad \mathbf{p} = d_{\mathbf{q}} \mathcal{F}^0_{\mathcal{A}} \right\}$$

is an overruled Lagrangian cone. We identify $\mathcal{H}^{\mathcal{A}}$ with $\mathcal{H}$ via the symplectomorphism:

$$\begin{aligned} \mathcal{H}^{\mathcal{A}} &\to \mathcal{H} \\ x &\mapsto x\sqrt{\mathcal{A}((q^*E)_{inv})}. \end{aligned}$$

Using this, we can view $\mathcal{F}^0_{\mathcal{A}}$ as a function on $\mathcal{H}_+$, and $\mathcal{L}_{\mathcal{A}}$ as a section of $T^*\mathcal{H}_+$.

Let Δ be defined as follows:

$$\Delta := \sum_{k \geq 0} s_k \left(\sum_{m \geq 0} \frac{(A_m)_{k+1-m} z^{m-1}}{m!} + \frac{ch_k(q^*E)_{inv}}{2} \right),$$

where at its turn A_m are defined as operators of ordinary multiplication by certain elements $A_m \in H^*(I\mathcal{X})$. To define A_m we introduce more notation: let r_μ be the order of each element in the conjugacy class which is labeled by $\mathcal{X}_\mu$. The restriction of the bundle E to $\mathcal{X}_\mu$ decomposes into characters : let $E^{(l)}_\mu$ be the subbundle on which every element of the conjugacy class acts with eigen value $e^{2\pi i l / r_\mu}$. Then:

$$(A_m)_{|\mathcal{X}_\mu} := \sum_{l=0}^{l=r-1} B_m(\frac{l}{r_\mu}) ch(E^{(l)}_\mu).$$

Remark 1.8.1. The decomposition:

$$H^*(I\mathcal{X}, \mathbb{C})((z^{-1})) = \oplus H^*(\mathcal{X}_\mu, \mathbb{C})((z^{-1}))$$

is preserved by the action of this loop group element. A_m acts by cup product multiplication on each $H^*(\mathcal{X}_\mu)$.

Theorem([TS]) 1.8.2.

$$\mathcal{L}_{\mathcal{A}} = \Delta \mathcal{L}^{\mathcal{H}}.$$

1.9 Twisted GW invariants - Motivation

In this section we explain why we look at "twistings" more general than the ones already present in the literature.
Let ζ be an mth root of unity and denote by $X_{0,n+2,d}(\zeta)$ the stem space on which the generator g of $\mathbb{Z}_m$ acts by ζ on the cotangent line at the first marked point. It is a Kawasaki stratum in $X_{0,mn+2,md}$. Assume for symplicity the point ∞ is a marked point. Contributions coming from integration on $X_{0,n+2,d}(\zeta)$ in Kawasaki's formula are of the form:

$$\int_{[X_{0,n+2,d}(\zeta)]} td(T)ch\left(\frac{ev_1^*\phi\prod_{i=1}^{n+2}ev_i^*\mathbf{t}(L_i)}{(1-q\zeta L_1^{1/m})Tr(\Lambda^\bullet N^*)}\right),$$

where $[X_{0,n+2,d}(\zeta)]$ is the virtual fundamental class to the stratum and T, N are the (virtual) tangent, respectively normal bundles to $X_{0,n+2,d}(\zeta)$. The virtual tangent bundle $\mathcal{T}^{vir}_{0,mn+2,dm} \in K^0(X_{0,nm+2,dm})$ equals:

$$\mathcal{T}^{vir}_{0,mn+2,dm} = \widetilde{\pi}_*(ev^*_{mn+3}(T_X-1)) - \widetilde{\pi}_*(L^{-1}_{mn+3}-1) - (\widetilde{\pi}_*i_*(\mathcal{O}_{\widetilde{\mathcal{Z}}}))^\vee,$$

where $\widetilde{\mathcal{Z}}$ is the codimension two locus of nodes in the universal family.

As we've said, $X_{0,n+2,d}(\zeta)$ is identified with $(X\times B\mathbb{Z}_m)_{0,n+2,d,(g,0,\ldots,g^{-1})}$. We want to express contributions from $\mathcal{T}^{vir}_{0,mn+2,dm}$ to T and N in terms of the universal family on the moduli space $(X\times B\mathbb{Z}_m)_{0,n+2,d,(g,0,\ldots,g^{-1})}$, which we denote π.

Let $\mathbb{C}_{\zeta^k}$ be the $\mathbb{Z}_m$ module $\mathbb{C}$ where g acts by multiplication ζ^k. Then the eigenspace of g in $\pi_*(ev^*_{mn+3}(T_X-1))$ with eigenvalue ζ^{-k} is :

$$\pi_*\left(ev^*_{n+3}(T_X\otimes\mathbb{C}_{\zeta^k})\right).$$

Taking this into account, as well as the description of the contribution to T, N coming from $\widetilde{\pi}_*(L^{-1}_{mn+3}-1)$ and $(\widetilde{\pi}_*i_*(\mathcal{O}_{\widetilde{\mathcal{Z}}}))^\vee$ we are led to consider three types of twistings:

- twistings by a finite number of multiplicative index classes $\mathcal{A}_\alpha$ as in Tseng's theorem.
- twistings by classes $\mathcal{B}_\beta$ of the form:

$$\mathcal{B}_{g,n,d} = \prod_{\beta=1}^{i_B}\mathcal{B}_\beta\left(\pi_*(f_\beta(L^{-1}_{n+1})-f_\beta(1))\right),$$

  where f_β are polynomials with coefficients in $K^0(X)$.

- twistings by nodal classes $\mathcal{C}_\delta$ of the form:

$$\mathcal{C}_{g,n,d} = \prod_\mu\prod_{\delta=1}^{i_\mu}\mathcal{C}^\mu_\delta\left(\pi_*(ev^*_{n+1}F_{\delta\mu}\otimes i_{\mu *}\mathcal{O}_{\mathcal{Z}_\mu}\right),$$

where $F_{\delta\mu} \in K^0(\mathcal{X})$. The extra index μ keeps track of the orbifold type of the node. More precisely, we denote by $\mathcal{Z}_\mu$ the node where the map $\overline{ev}_+$ lands in $\overline{\mathcal{X}}_\mu$ and by i_μ the corresponding inclusion $\mathcal{Z}_\mu \to \mathcal{U}_{g,n,d}$. Hence we allow different types of twistings localised near the loci $\mathcal{Z}_\mu$.

We will refer to these as type $\mathcal{A}, \mathcal{B}, \mathcal{C}$ twistings respectively.

1.10 Twisted GW invariants - statement of results

One can define the twisted potentials $\mathcal{F}^g_{\mathcal{A},\mathcal{B},\mathcal{C}}$, $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ in a way very similar with the definitions of Sections 1.7. and 1.8. We describe the effect of these twistings on the potential in terms of the geometry of the corresponding cones $\mathcal{L}_{\mathcal{A},\mathcal{B},\mathcal{C}}$.
The first generalisation we propose is rather straightforward - we allow type $\mathcal{A}$ twistings by an arbitrary finite number of different multiplicative classes $\mathcal{A}_\alpha$. Slightly abusively, we keep notation from Section 1.8. i.e. $\mathcal{L}_\mathcal{A}$, $\mathcal{H}^\mathcal{A}$ etc. Denote by Δ_α the loop group transformation obtained from A_α via the procedure described in Section 1.8. Then after a suitable identification of $\mathcal{H}^\mathcal{A}$ with $\mathcal{H}$:

Theorem 1.10.1. *The cone*

$$\mathcal{L}_\mathcal{A} = \left(\prod_\alpha \Delta_\alpha \right) \mathcal{L}.$$

Let $\mathbf{L}_z$ be a line bundle with first chern class z.

Theorem 1.10.2. *The twisting by the classes $\mathcal{B}_{g,n,d}$ has the same effect as a translation on the Fock space:*

$$\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}(\mathbf{t}) = \mathcal{D}_{\mathcal{A},\mathcal{C}} \left(\mathbf{t} + z - z \prod_{i=1}^{i_B} \mathcal{B}_\beta \left(-\frac{f_\beta(\mathbf{L}_z^{-1}) - f_\beta(1)}{\mathbf{L}_z - 1} \right) \right). \tag{1.1}$$

The type $\mathcal{C}$ twisting doesn't move the cone, but changes the polarisation of $\mathcal{H}$. For each μ let the series $u_\mu(z)$ be defined by:

$$\frac{z}{u_\mu(z)} = \prod_{\delta=1}^{i_\mu} \mathcal{C}^\mu_\delta \left((q^* F_{\delta\mu})_\mu \otimes (-\mathbf{L}_{-z}) \right).$$

Moreover define Laurent series $v_{k,\mu}$, $k = 0, 1, 2, \ldots$ by:

$$\frac{1}{u_\mu(-x-y)} = \sum_{k \geq 0} (u_\mu(x))^k v_{k,\mu}(u(y)) \quad ,$$

where we expand the left hand side in the region where $|x| < |y|$.

Theorem 1.10.3. *The type $\mathcal{C}$ twisting is tantamount to considering the potential $\mathcal{D}_{\mathcal{A},\mathcal{B}}$ as a generating function with respect to the new polarisation $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_{-,\mathcal{C}}$, where $\mathcal{H}_+$ is the same, but $\mathcal{H}_{-,\mathcal{C}} = \oplus_\mu \mathcal{H}^\mu_{-,\mathcal{C}}$ and each $\mathcal{H}^\mu_{-,\mathcal{C}}$ is spanned by $\{\varphi_{\alpha,\mu} v_{k,\mu}(u(z))\}$ where $\{\varphi_{\alpha,\mu}\}$ runs over a basis of $H^*(\mathcal{X}_\mu, \mathbb{C})$ and k runs from 0 to ∞.*

1.11 $\mathcal{D}_q$ module structure

It is known that the tangent spaces to the cone $\mathcal{L}^H$ are preserved by the action of certain differential operators. A corresponding statement in quantum K-theory was missing in the absence of the divisor equation. While it was noticed in examples (see [GL]) that $\mathcal{J}(0)$ satisfies some finite difference equations, the reasons for that were unknown. The characterisation of Theorem 1.6.2 allows one to deduce the tangent spaces to $\mathcal{L}$ carry a structure of $\mathcal{D}_q$ module, where $\mathcal{D}_q$ is a certain algebra of differential operators.

Theorem 1.11.1. *Let $\mathcal{D}_q$ be the finite-difference operators, generated by integer powers of P_a and Q^d. Define a representation f $\mathcal{D}_q$ on $\mathcal{K}$ using the operators $P_a q^{Q_a \partial_{Q_a}}$. Then tangent spaces to $\mathcal{L}$ are $\mathcal{D}_q$ invariant.*

Chapter 2

Twisted Gromov-Witten invariants

2.1 Introduction

Twisted Gromov-Witten invariants have been introduced in [CG] for manifold target spaces and extended by [TS] to the case of orbifolds. The original motivation was to express Gromov-Witten invariants of complete intersections (the "twisted" ones) in terms of the GW invariants of the ambient space (the untwisted ones). In addition they were used in [C] to express Gromov-Witten invariants with values in cobordism in terms of cohomological Gromov-Witten invariants.

The results of this chapter incorporate and generalise all of the above: we consider three types of twisting classes. These are multiplicative cohomological classes of bundles of the form $\pi_* E$, where π is the universal family of the moduli space of stable maps to an orbifold $\mathcal{X}$. The main tool in the computations is the Grothendieck-Riemann-Roch theorem for stacks of [TN], applied to the morphism π: this gives differential equations satisfied by the generating functions of the twisted GW invariants. To the Gromov-Witten potential of an orbifold $\mathcal{X}$ one can associate an overruled Lagrangian cone in a symplectic space $\mathcal{H}$ - as explained in Section 1.5. Solving the differential equations for each type of twisting has an interpretation in terms of the geometry of the cone: change its position by a symplectic transformation, translation of the origin and a change of polarisation of $\mathcal{H}$.

In [TL], Teleman studies a group action on 2 dimensional quantum field theories. Our results match his, if the field theories come from Gromov-Witten theory. Our main motivation comes from studying the quantum K-theory of a manifold X, detailed in the next chapter. However it is very likely that they have other applications - for instance extending the work of [C] on quantum extraordinary cohomology to orbifold target spaces.

The material of the chapter is structured as follows. Sections 2.2-2.4 describe the main objects of study and introduce notation used throughout the rest of the chapter: in Section 2.2 we introduce the inertia orbifold $I\mathcal{X}$ and define the orbifold product, in Section 2.3 we define the moduli spaces $\mathcal{X}_{g,n,d}$, the symplectic space $\mathcal{H}$ and the Gromov-Witten potential and in Section 2.4 we define the twisted Gromov-Witten invariants and the twisted potential. In Section 2.5 we state Toën's Grothendieck-Riemann Roch theorem for stacks. Section 2.6 contains the technical results which are the core of the computations - mainly how the twisting cohomological classes pullback on the universal family, the locus of nodes and the divisors of marked points. We are now ready to prove the Theorems 1.10.1, 1.10.2 and 1.10.3 - which we do in Section 2.7. Finally, in Section 2.8 we extract the corollaries which we'll use in the next chapter on quantum K-theory.

2.2 Orbifold Cohomology

Let $\mathcal{X}$ be a compact Kähler orbifold over $\mathbb{C}$.

Definition 2.2.1. We define the inertia orbifold $I\mathcal{X}$ by specifying local charts. Around any point $p \in \mathcal{X}$ there is a local chart $(\widetilde{U}_p, G_p)$ such that locally $\mathcal{X}$ is represented as the quotient of $\widetilde{U}_p$ by G_p. Consider the set of conjugacy classes $(1) = (h_p^1)$, (h_p^2), …, $(h_p^{n_p})$ in G_p. Then:

$$I\mathcal{X} := \{(p, (h_p^i) \quad | \quad i = 1, 2, \dots, n_p\}.$$

Pick an element h_p^i in each conjugacy class. Then a local chart on $I\mathcal{X}$ is given by:

$$\coprod_{i=1}^{n_p} \widetilde{U}_p^{(h_p^i)} / Z_{G_p}(h_p^i),$$

where $Z_{G_p}(h_p^i)$ is the centralizer of h_p^i in G_p.

Remark 2.2.2. For the reader more confortable with the language of stacks, $I\mathcal{X}$ can be defined as the fiber product

$$I\mathcal{X} := \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X},$$

where both maps are the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$.

For foundational material on stacks, see [F] and [LM].

Remark 2.2.3. There are natural maps: $q : I\mathcal{X} \to \mathcal{X}$ which sends the pair (x, g) to x and $\iota : I\mathcal{X} \to I\mathcal{X}$ which maps (x, g) to (x, g^{-1}). We denote by $\mathcal{X}_{\mu^I} := \iota(\mathcal{X}_\mu)$.

In general $I\mathcal{X}$ is disconnected, even if $\mathcal{X}$ is connected. We write:

$$I\mathcal{X} := \coprod_{\mu} \mathcal{X}_\mu.$$

There is a distinguished stratum $\mathcal{X}_0$ of $I\mathcal{X}$ which is isomorphic to $\mathcal{X}$.
For a bundle $E \in K^0(\mathcal{X})$, we denote by $(q^*E)_{inv}$ the bundle whose restriction on each $\mathcal{X}_\mu$ is the invariant part under the action of the (conjugacy class of the) symmetry associated to $\mathcal{X}_\mu$.

Example 2.2.4. If $\mathcal{X}$ is a global quotient of the form Y/G then the strata of $I\mathcal{X}$ are in bijection with the conjugacy classes in G. More precisely to each conjugacy class (g) we associate the stratum $\mathcal{X}_{(g)} := Y^g/C_G(g)$. If $g = e$, then $\mathcal{X}_e = \mathcal{X}_0 = Y/G$.

Definition([CR2]) 2.2.5. The cohomology $H^*(I\mathcal{X}, \mathbb{C})$ is called the orbifold cohomology.

The orbifold Poincaré pairing on $I\mathcal{X}$ is defined for $a \in H^*(\mathcal{X}_\mu, \mathbb{C})$, $b \in H^*(\mathcal{X}_{\mu^I}, \mathbb{C})$ as:

$$(a, b)_{orb} := \int_{\mathcal{X}_\mu} a \cup \iota^* b.$$

We extend this by linearity to $H^*(I\mathcal{X}, \mathbb{C})$. The orbifold product is different from the usual cup product on $H^*(I\mathcal{X}, \mathbb{C})$:

Definition 2.2.6. For $a, b \in H^*(I\mathcal{X}, \mathbb{C})$ their orbifold product $a \cdot_{orb} b$ is the class whose pairing with any $c \in H^*(I\mathcal{X}, \mathbb{C})$ is given by

$$(a \cdot_{orb} b, c) = \langle a, b, c \rangle_{0,3,0},$$

where the right hand side is defined in Section 2.3.

2.3 Moduli of orbifold stable maps

In this section we recall the definition of the moduli spaces of orbifold stable maps of [CR1] and [AGV2]. The idea to extend the definition of a stable map to an orbifold target space is quite natural. One then notices that in order to obtain compact moduli spaces parametrizing these objects one has to allow orbifold structure on the domain curve at the nodes and marked points (see e.g. [A]).

Definition 2.3.1. *A nodal n-pointed orbicurve is a nodal marked curve $(\mathcal{C}, x_1, \ldots, x_n)$, such that*

- $\mathcal{C}$ has trivial orbifold structure on the complement of the marked points and nodes.
- Locally near a marked point, $\mathcal{C}$ is isomorphic to the quotient [Spec $\mathbb{C}[z]/\mathbb{Z}_r$], for some r, and the generator of $\mathbb{Z}_r$ acts by $z \mapsto \zeta z$, $\zeta^r = 1$.
- Locally near a node, $\mathcal{C}$ is isomorphic to [Spec $(\mathbb{C}[z, w]/(zw))/\mathbb{Z}_r$], and the generator of $\mathbb{Z}_r$ acts by $z \mapsto \zeta z$, $w \mapsto \zeta^{-1} w$. We call this action *balanced* at the node.

We now define twisted stable maps:

Definition 2.3.2. *An n-pointed, genus g, degree d orbifold stable map is a representable morphism $f : \mathcal{C} \to \mathcal{X}$, whose domain is an n-pointed, genus g orbicurve $\mathcal{C}$ such that the induced morphism of the coarse moduli spaces $C \to X$ is a stable map of degree d.*

Remark 2.3.3. The word "representable" in the Definition 2.3.2 means that for every point $x \in \mathcal{C}$ the associated morphism $G_x \to G_{f(x)}$ is injective. So the orbifold structure on $\mathcal{C}$ does not include "unnecessary" automorphisms.

We denote the moduli space parametrizing n-pointed, genus g, degree d orbifold stable maps by $\mathcal{X}_{g,n,d}$. It is proved in [AV] that $\mathcal{X}_{g,n,d}$ is a proper Deligne-Mumford stack. Just like the case of stable maps to manifolds, there are evaluation maps at the marked points, but these land naturally in the *rigidified* inertia orbifold of $\mathcal{X}$, which we denote $\overline{I\mathcal{X}}$. To explain this, notice that a marking (say x_1) is not a point but a stack $B\mathbb{Z}_r$. Consider a family of stable maps $(\mathcal{U}, x_1, \dots x_n, f)$ to $\mathcal{X}$ parametrized by a base scheme S. Let $\mathcal{C}$ be the fiber over $s \in S$. Then part of the data of $f : \mathcal{C} \to \mathcal{X}$ is an injective morphism $\mathbb{Z}_r = G_{x_1} \to G_{f(x_1)}$ (slightly abusively we denote by $f(x_1)$ the image of the geometric point $x_1 \in \mathcal{C}$). Call g the image of the fixed generator of $1 \in \mathbb{Z}_r$. Going around a nontrivial loop based at s induces an automorphism $\mathbb{Z}_r \to \mathbb{Z}_r$, which is not necessarily identity. So g is defined only up to composition $\mathbb{Z}_r \to \mathbb{Z}_r \to G_{f(x_1)}$. For any such composition the image of $1 \in \mathbb{Z}_r$ lands in the cyclic subgroup generated by g. To get a well-defined evaluation map, we need to factor by this subgroup.

$\overline{I\mathcal{X}}$ is a version of $I\mathcal{X}$ defined by changing the local stabilizer groups in Definition 2.2.1. Keeping notation from Definition 2.2.1, local charts on $\overline{I\mathcal{X}}$ are :

$$\coprod_{i=1}^{n_p} \widetilde{U}_p^{(h_p^i)} / [Z_{G_p}(h_p^i)/\langle h_p^i \rangle].$$

Example 2.3.4. If $\mathcal{X}$ is a global quotient Y/G then the strata of $\overline{I\mathcal{X}}$ are $\overline{\mathcal{X}}_{(g)} := Y^g/\overline{C_G(g)}$, where $\overline{C_G(g)} = C_G(g)/\langle g \rangle$ for each conjugacy class $(g) \in G$.

See [AGV1] and [AGV2] for the definition of $\overline{I\mathcal{X}}$ in the category of stacks. In general there is no map $\overline{I\mathcal{X}} \to \mathcal{X}$. The involution ι descends to an involution of $\overline{I\mathcal{X}}$, which we also denote by ι.

The connected components of $\mathcal{X}_{g,n,d}$ are $\mathcal{X}_{g,n,d,(\mu_1,\dots,\mu_n)}$, where:

$$\mathcal{X}_{g,n,d,(\mu_1,\dots,\mu_n)} := \mathcal{X}_{g,n,d} \cap (\overline{ev}_1)^{-1}(\overline{\mathcal{X}}_{\mu_1}) \cap \dots \cap (\overline{ev}_n)^{-1}(\overline{\mathcal{X}}_{\mu_n}).$$

Since we work with cohomology with complex coefficients we consider the cohomological pullbacks by the maps ev_i having domain $H^*(I\mathcal{X}, \mathbb{C})$. $I\mathcal{X}$ and $\overline{I\mathcal{X}}$ have the same coarse spaces (i.e. the same geometric points, only the stabilizer groups differ), which implies that both spaces have the same cohomolgy rings with rational coefficients. In fact there is a map $\Pi : I\mathcal{X} \to \overline{I\mathcal{X}}$, which maps a point $(x, (g))$ to $(x, (\overline{g}))$. If r_i is the order of the automorphism group of x_i, then define:

$$\begin{aligned} &ev_i^* : H^*(I\mathcal{X}, \mathbb{C}) \to H^*(\mathcal{X}_{g,n,d}, \mathbb{C}), \\ &a \mapsto r_i^{-1} (\overline{ev}_i)^* (\Pi_* a). \end{aligned}$$

Remark 2.3.5. The moduli spaces $\mathcal{X}_{g,n,d}$, as well as the evaluation maps, differ from those considered in [TS]. However the Gromov-Witten invariants agree, since integration in [TS] is done over a weighted virtual fundamental class.

Notice that if a marked point x_i has trivial orbifold structure, ev_i lands in the distinguished component $\mathcal{X}_0$ of $\overline{I\mathcal{X}}$. The universal family can be therefore identified with the diagram:

$$\begin{array}{ccc} \mathcal{U}_{g,n,d} := \cup_{(\mu_1,\ldots,\mu_n)}\mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)} & \xrightarrow{ev_{n+1}} & \mathcal{X} \\ \pi\downarrow & & \\ \mathcal{X}_{g,n,d} & & . \end{array}$$

In the universal family $\mathcal{U}_{g,n,d}$ lies the divisor of the i-th marked point $\mathcal{D}_i$: its points parametrize maps whose domain has a distinguished node separating two orbicurves $\mathcal{C}_0$ and $\mathcal{C}_1$. $\mathcal{C}_1$ is isomorphic to $\mathbb{CP}^1$ and carries only three special points: the node, the i-th marked point and the $(n+1)$-st marked point and is mapped with degree 0 to $\mathcal{X}$. We write:

$$\mathcal{D}_{i,(\mu_1,\ldots,\mu_n)} := \mathcal{D}_i \cap \mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)}.$$

Let $\mathcal{Z}$ be the locus of nodes in the universal family. It has codimension two in $\mathcal{U}_{g,n,d}$. Denote by $p : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ the double cover over $\mathcal{Z}$ given by a choice of $+,-$ at the node. For the inclusion of a stratum:

$$\mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathcal{Z} \hookrightarrow \mathcal{X}_{g,n+1,d}$$

we will denote by by p_i $(i = 1, 2)$ the projections:

$$p_i : \mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathcal{X}_{g_i,n_i+1,d_i}.$$

We denote $\mathcal{Z}^{irr}, \mathcal{Z}^{red}$ the loci of nonseparating nodes, respectively separating nodes and i^{irr}, i^{red} for the inclusion maps. Moreover we will need to keep track of the orbifold structure of the node. We denote by $\mathcal{Z}_\mu$ the locus of nodes where the evaluation map at one branch lands in $\overline{\mathcal{X}}_\mu$. We denote by i_μ the corresponding inclusions.

The moduli spaces $\mathcal{X}_{g,n,d}$ have perfect obstruction theory (see [BF]). This yields virtual fundamental classes:

$$[\mathcal{X}_{g,n,d}] \in H_*(\mathcal{X}_{g,n,d}, \mathbb{Q}).$$

We define $\overline{\psi}_i, \psi_i$ to be the first Chern classes of line bundles whose fibers over each point $(\mathcal{C}, x_1, \ldots, x_n, f)$ are the cotangent spaces at x_i to the *coarse curve* C, respectively to $\mathcal{C}$. If r_i is the order of the automorphism group of x_i then $\overline{\psi}_i = r_i\psi_i$.

We denote by:

$$\langle a_1\overline{\psi}^{k_1}, \ldots, a_n\overline{\psi}^{k_n}\rangle_{g,n,d} := \int_{[\mathcal{X}_{g,n,d}]} \prod_{i=1}^{n} ev_i^*(a_i)\overline{\psi}_i^{k_i}.$$

Let $\mathbb{C}[[Q]]$ be the Novikov ring which is the formal power series completion of the semi-group ring of degrees of holomorphic curves in X. For more on Novikov rings see [MS]. We introduce one more Novikov variable λ. We define the ground ring $\Lambda := \mathbb{C}[[Q, \lambda]]$ and:

$$\mathcal{H} := H^*(I\mathcal{X}, \Lambda)((z)).$$

We equip $\mathcal{H}$ with the Q, λ-adic topology. This means that when we say "elements of $\mathcal{H}$ have property P" we mean "elements of $\mathcal{H}$ have property P modulo any power in the variables Q, λ". $\mathcal{H}$ is a slight modification of the Fock space (from e.g. [C]). We do this because we need $\mathcal{H}$ to include expressions of the form $e^{x\lambda/z}$ and e^z.

Convention 2.3.6. Throughout this thesis we will refer to the "usual" Novikov variables Q by just writing "Novikov variables" without explicit mention of λ.

Let:

$$\mathbf{t}(z) := t_0 + t_1 z + \cdots \in H^*(I\mathcal{X}, \mathbb{C})[[z]][\lambda].$$

where the coefficient of $\mathbf{1} \cdot z^0$ in $\mathbf{t}(z)$ is proportional to λ. Then the genus g, respectively total potential are defined to be:

$$\mathcal{F}^g(\mathbf{t}) = \sum_{d,n} \frac{Q^d}{n!} \langle \mathbf{t}, \ldots, \mathbf{t} \rangle_{g,n,d},$$
$$\mathcal{D}(\mathbf{t}) = exp\left(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g(\mathbf{t}) \right).$$

We endow $\mathcal{H}$ with the symplectic form:

$$\Omega(\mathbf{f}, \mathbf{g}) := \oint_{z=0} (\mathbf{f}(z), \mathbf{g}(-z))_{orb}\, dz.$$

Consider the following polarisation of $\mathcal{H}$:

$$\mathcal{H}_+ := H^*(I\mathcal{X}, \mathbb{C})[[z]] \quad and \quad \mathcal{H}_- := z^{-1} H^*(I\mathcal{X}, \mathbb{C})[z^{-1}].$$

This identifies $\mathcal{H}$ with $T^*\mathcal{H}_+$. We introduce Darboux coordinates $\{p_a^\alpha, q_b^\beta\}$ on $\mathcal{H}$ and we write:

$$\mathbf{p}(z) = \sum_{a,\alpha} p_a^\alpha \varphi_\alpha (-z)^{-a-1} \in \mathcal{H}_-$$
$$\mathbf{q}(z) = \sum_{b,\beta} q_b^\beta \varphi_\beta z^b \in \mathcal{H}_+.$$

For $\mathbf{t}(z) \in \mathcal{H}_+$ we call the translation $\mathbf{q}(z) := \mathbf{t}(z) - \mathbf{1}z$ the *dilaton shift*. We regard the total descendant potential as a formal function on $\mathcal{H}_+$ taking values in $\mathbb{C}[[Q, \lambda, \hbar, \hbar^{-1}]]$.

The graph of the differential of $\mathcal{F}^0$ defines a formal germ of a Lagrangian submanifold of $\mathcal{H}$:

$$\mathcal{L}^H := \{(\mathbf{p}, \mathbf{q}), \mathbf{p} = d_{\mathbf{q}}\mathcal{F}^0\} \in \mathcal{H}.$$

Theorem 2.3.7. ([G1])$\mathcal{L}^H$ *is (the formal germ of) a Lagrangian cone with vertex at the shifted origin* $-\mathbf{1}z$ *such that each tangent space* T *is tangent to* $\mathcal{L}^H$ *exactly along* zT.

The class of cones satisfying properties of Theorem 2.3.7 is preserved under the action of symplectic transformations on $\mathcal{H}$ which commute with multiplication by z. We call these symplectomorphisms *loop group elements*. They are matrix valued Laurent series in z:

$$S(z) = \sum_{i \in \mathbb{Z}} S_i z^i,$$

where $S_i \in End\,(H^*(I\mathcal{X}) \otimes \Lambda)$. Being a symplectomorphism amounts to:

$$S(z)S^*(-z) = I,$$

where I is the identity matrix and S^* is the adjoint transpose of S. Differentiating the relation above at the identity, we see that infinitesimal loop group elements R satisfy:

$$R(z) + R^*(-z) = 0.$$

2.4 Twisted Gromov-Witten invariants

In this section we introduce more general Gromov-Witten twisted potentials than the ones of [TS].

For a bundle E we will denote by $\mathcal{A}(E)$, $\mathcal{B}(E)$, $\mathcal{C}(E)$ general multiplicative classes of E. These are of the form:

$$exp\left(\sum_{k \geq 0} s_k ch_k(E)\right).$$

We then define the classes $\mathcal{A}_{g,n,d}, \mathcal{B}_{g,n,d}, \mathcal{C}_{g,n,d} \in H^*(\mathcal{X}_{g,n,d})$ as products of *different* multiplicative classes of bundles:

$$\mathcal{A}_{g,n,d} = \prod_{\alpha=1}^{i_A} \mathcal{A}_\alpha(\pi_*(ev^* E_\alpha)),$$

$$\mathcal{B}_{g,n,d} = \prod_{\beta=1}^{i_B} \mathcal{B}_\beta\left(\pi_*(f_\beta(L_{n+1}^{-1}) - f_\beta(1))\right),$$

$$\mathcal{C}_{g,n,d} = \prod_{\mu}\prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu\left(\pi_*(ev_{n+1}^* F_{\delta\mu} \otimes i_{\mu *}\mathcal{O}_{\mathcal{Z}_\mu}\right).$$

Here f_i are polynomials with coefficients in $K^0(\mathcal{X})$, the bundles $E_\alpha, F_{\delta\mu}$ are elements of $K^0(\mathcal{X})$. If we denote by:

$$\Theta_{g,n,d} := \mathcal{A}_{g,n,d} \cdot \mathcal{B}_{g,n,d} \cdot \mathcal{C}_{g,n,d}$$

these "twisted" Gromov-Witten invariants are:

$$\langle a_1\overline{\psi}^{k_1}, \ldots, a_n\overline{\psi}^{k_n}; \Theta\rangle_{g,n,d} := \int_{[\mathcal{X}_{g,n,d}]} \prod_{i=1}^n ev_i^*(a_i)\overline{\psi}_i^{k_i} \cdot \Theta_{g,n,d}.$$

We now define the twisted potential $\mathcal{D}_{A,B,C}$:

$$\mathcal{F}^g_{\mathcal{A},\mathcal{B},\mathcal{C}}(\mathbf{t}) := \sum_{d,n} \frac{Q^d}{n!}\langle \mathbf{t}(\overline{\psi}), \ldots, \mathbf{t}(\overline{\psi}); \Theta\rangle_{g,n,d}$$

$$\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}} := exp(\sum_g \hbar^{g-1}\mathcal{F}^g_{A,B,C}).$$

We view $\mathcal{D}_{A,B,C}$ as a formal function on $\mathcal{H}_+^{\mathcal{A},\mathcal{B},\mathcal{C}}$.

The symplectic vector space $(\mathcal{H}^{\mathcal{A},\mathcal{B},\mathcal{C}}, \Omega_{\mathcal{A},\mathcal{B},\mathcal{C}})$ is defined as $\mathcal{H}^{\mathcal{A},\mathcal{B},\mathcal{C}} = \mathcal{H}$, but with a different symplectic form :

$$\Omega_{\mathcal{A},\mathcal{B},\mathcal{C}}(\mathbf{f},\mathbf{g}) := \oint_{z=0} (\mathbf{f}(z), \mathbf{g}(-z))_{\mathcal{A}} dz$$

where $(\ ,\ )_{\mathcal{A}}$ is the *twisted* pairing given for $a, b \in H^*(I\mathcal{X})$ by:

$$(a,b)_{\mathcal{A}} := \langle a, b, 1; \Theta\rangle_{0,3,0}.$$

Remark 2.4.1. We briefly discuss the case $(g,n,d) = (0,3,0)$. According to [AGV1] in this case the evaluation maps lift to $ev_i : \mathcal{X}_{0,3,0} \to I\mathcal{X}$. The spaces $\mathcal{X}_{0,3,0,(\mu_1,\mu_2,0)}$ are empty unless $\mu_2 = \mu_1^I$, in which case they can be identified with $\mathcal{X}_{\mu_1}$, with the evaluation maps being $ev_1 = id : \mathcal{X}_{\mu_1} \to \mathcal{X}_{\mu_1}$, $ev_2 = \iota : \mathcal{X}_{\mu_1} \to \mathcal{X}_{\mu_1^I}$ and ev_3 is the inclusion of $\mathcal{X}_{\mu_1}$ in $\mathcal{X}$.

Remark 2.4.2. Notice that on $\mathcal{X}_{0,3,0}$ there are no twistings of type $\mathcal{B}$ (the corresponding push-forwards are trivial for dimensional reasons) and of type $\mathcal{C}$ (there are no nodal curves). That's why the twisted pairing only depends on the $\mathcal{A}$ classes.

According to the previous two remarks we can rewrite the pairing as:

$$(a,b)_{\mathcal{A}} := \int_{I\mathcal{X}} a \cdot \iota^* b \cdot \prod_{\alpha} \mathcal{A}_\alpha \left((q^* E_\alpha)_{inv} \right).$$

There is a rescaling map:

$$(\mathcal{H}_{A,B,C}, \Omega_{A,B,C}) \to (\mathcal{H}, \Omega)$$
$$a \mapsto a \sqrt{\prod_{\alpha} A_\alpha((q^* E_\alpha)_{inv})}$$

which identifies the symplectic spaces. We denote by $\mathcal{D}_{A,B}$ the potential twisted only by classes of type $\mathcal{A}, \mathcal{B}$ etc. and by:

$$[\mathcal{X}_{g,n,d}]^{tw} := [\mathcal{X}_{g,n,d}] \cap \Theta_{g,n,d}.$$

2.5 Grothendieck-Riemann-Roch for stacks

The main tool for proving Theorems 1.10.1, 1.10.2 and 1.10.3 is a theorem of Grothendieck-Riemann Roch for stacks due to B.Toën ([TN]). Before stating it we will introduce more notation:

Definition 2.5.1. Define $Tr : K^0(\mathcal{X}) \to K^0(I\mathcal{X})$ to be the map:

$$F \mapsto \oplus \lambda_i(g) F_i$$

on each component $(g, \mathcal{X}_\mu)$ of the inertia stack, where F_i is the decomposition of the g action and $\lambda_i(g)$ is the eigenvalue of g on F_i.

Definition 2.5.2. Define $\widetilde{ch} : K^0(\mathcal{X}) \to H^*(I\mathcal{X})$ to be the map $ch \circ Tr$.

Now each vector bundle E on $\mathcal{X}$ restricts on each connected component $(g, \mathcal{X}_\mu)$ of the inertia stack as the direct sum $E_{inv} \oplus E_{mov}$.

Definition 2.5.3. *Define $\widetilde{Td}(E) : K^0(\mathcal{X}) \to H^*(I\mathcal{X})$ to be the class:*

$$\widetilde{Td} := \frac{Td(E_{inv})}{ch(Tr \circ \lambda_{-1}(E_{mov})^\vee)}$$

where λ_{-1} is the operation in K-theory defined as $\lambda_{-1}(V) := \sum_{a \geq 0} (-1)^a \Lambda^a V$. We can now state:

Theorem 2.5.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of smooth Deligne Mumford stacks. This induces a morphism $If : I\mathcal{X} \to I\mathcal{Y}$. Then under some nice technical assumptions we have:*

$$\widetilde{ch}(f_*E) = If_*\left(\widetilde{ch}(E)\widetilde{Td}(T_f)\right). \tag{2.1}$$

Restricting to the identity component $\mathcal{Y}$ of $I\mathcal{Y}$ we get:

$$ch(f_*E) = If_*\left(\widetilde{ch}(E)\widetilde{Td}(T_f)_{|If^{-1}\mathcal{Y}}\right). \tag{2.2}$$

The universal curve π is not necessarily a local complete intersection, so following [TS] we proceed as follows. The construction in [AGOT] provides a family of orbicurves

$$\widetilde{\pi} : \mathcal{U} \to \mathcal{M} \tag{2.3}$$

and an embedding $\mathcal{X}_{g,n,d} \to \mathcal{M}$ satisfying the following properties:

- The family $\mathcal{U} \to \mathcal{M}$ pulls back to the universal family over $\mathcal{X}_{g,n,d}$.
- A vector bundle of the form $ev_{n+1}^*(E)$ extends to a vector bundle over $\mathcal{U}$.
- The Kodaira-Spencer map $T_m\mathcal{M} \to Ext^1(\mathcal{O}_{\mathcal{U}_m}, \mathcal{O}_{\mathcal{U}_m})$ is surjective for all $m \in \mathcal{M}$.
- The locus $\mathcal{Z} \subset \mathcal{U}$ of the nodes of $\widetilde{\pi}$ is smooth and $\widetilde{\pi}(\mathcal{Z})$ is a divisor with normal crossings.
- The pull-back of the normal bundle $N_{\mathcal{Z}/\mathcal{U}}$ to the double cover $\widetilde{Z}$ given by choice of marked points at the node is isomorphic to the direct sum of the cotangent line bundles at the two marked points.

So technically we apply Grothendieck-Riemann Roch to $\widetilde{\pi}$ and then cap with the virtual fundamental classes $[\mathcal{X}_{g,n,d}]^{tw}$. Therefore for the rest of the chapter we assume the universal family π satisfies the above properties.

In our situation there are three strata on the universal curve which get mapped to $\mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)}$:

- The total space $\mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)}$.
- The locus of marked points $\mathcal{D}_{j,(\mu_1,\ldots,\mu_n)}$.
- The nodal loci $\mathcal{Z}_\mu$ where $\mu \neq 0$, i.e. the node is an orbifold point.

2.6 Prerequisites

We need to know how the classes $\Theta_{g,n,d}$ pullback on the universal orbicurve, on the divisor of marked points and on the locus of nodes. We state below the main result of this section, which we'll use in the proofs of the theorems:

Proposition 2.6.1. The following equalities hold:

1.
$$\pi^*[\mathcal{X}_{g,n,d}]^{tw} = [\mathcal{X}_{g,n+1,d}]^{tw} \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta \left(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1} \right) +$$
$$+ \sum_{j=1}^{n} [\mathcal{X}_{g,n+1,d}]^{tw} \cdot \left(\prod_{\delta=1}^{i_{\mu_j}} \mathcal{C}_\delta^{\mu_j} \left(-ev_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j} \right) - 1 \right) +$$
$$+ [\mathcal{X}_{g,n+1,d}]^{tw} \cdot \left(\prod_{\mu} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^{\mu} \left(-ev_{n+1}^*(F_{\delta\mu}) \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu} \right) - 1 \right). \tag{2.4}$$

2.
$$\sigma_j^*[\mathcal{X}_{g,n,d}]^{tw} = [\mathcal{X}_{g,n,d}]^{tw} \cdot \prod_{\delta=1}^{i_{\mu_j}} \mathcal{C}_\delta^{\mu_j} \left(ev_j^*(q^*F_{\delta\mu_j})_{\mu_j} \otimes (1 - L_j) \right). \tag{2.5}$$

3.
$$(\pi \circ i_\mu^{red} \circ p)^*[\mathcal{X}_{g,n,d}]^{tw} =$$
$$= \frac{p_1^*([\mathcal{X}_{g_1,n_1+1,d_1}]^{tw}) \cdot p_2^*([\mathcal{X}_{g_2,n_2+1,d_2}]^{tw})}{(ev_+^* \times ev_-^*)\Delta_{\mu*} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu \left((q^*F_{\delta\mu})_\mu) \otimes (L_+L_- - 1) \right)}. \tag{2.6}$$

4.
$$(\pi \circ i_\mu^{irr} \circ p)^*[\mathcal{X}_{g,n,d}]^{tw} =$$
$$= \frac{[\mathcal{X}_{g-1,n+2,d}]^{tw}}{(ev_+^* \times ev_-^*)\Delta_{\mu*} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu \left((q^*F_{\delta\mu})_\mu) \otimes (L_+L_- - 1) \right)}. \tag{2.7}$$

Proof: all the equalities follow from the corresponding statements about the classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ separatedly, which we'll prove below. Formula (2.4) follows from (2.9), (2.13), (2.38) combined with some more cancelation: namely the terms in (2.38) supported on $\mathcal{D}_j$ and $\mathcal{Z}$ are killed by the correction factor in (2.13) which is of the form $1 + \psi_{n+1} \cdot \ldots$. The untwisted virtual fundamental classes satisfy $\pi^*[\mathcal{X}_{g,n,d}] = [\mathcal{X}_{g,n+1,d}]$.
(2.6) and (2.7) follow from the corresponding Lemmata 2.6.2, 2.6.3 and 2.6.7 for each of the classes $\mathcal{A}_{g,n,d}$, $\mathcal{B}_{g,n,d}$ and $\mathcal{C}_{g,n,d}$ combined with the splitting axiom in orbifold Gromov-Witten theory for the untwisted fundamental classes $[\mathcal{X}_{g,n,d}]$, which we briefly review below. Let $\mathfrak{M}_{g,n}^{tw}$ be the stack of genus g twisted curves with n marked points. There is a natural map:

$$\mathrm{gl} : \mathfrak{D}^{tw}(g_1; n_1 | g_2, n_2) \to \mathfrak{M}_{g,n}^{tw}$$

induced by gluing two family of twisted curves into a reducible curve with a distinguished node. Here $\mathfrak{D}^{tw}(g_1; n_1 | g_2, n_2)$ is defined as in Section 5.1 of [AGV2]. This induces a cartesian diagram:

$$\begin{array}{ccc} \mathfrak{D}^{tw}_{g,n}(\mathcal{X}) & \longrightarrow & \mathcal{X}_{g,n,d} \\ \downarrow & & \downarrow \\ \mathfrak{D}^{tw}(g_1;n_1|g_2,n_2) & \xrightarrow{\text{gl}} & \mathfrak{M}^{tw}_{g,n}. \end{array}$$

There is a natural map:

$$\mathfrak{g} : \bigcup_{d_1+d_2=d} \mathcal{X}_{g_1,n_1+1,d_1} \times_{\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathfrak{D}^{tw}_{g,n}(\mathcal{X}).$$

Then the diagram:

$$\begin{array}{ccc} \mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2} \subset \mathcal{Z} & \longrightarrow & I\mathcal{X} \\ \downarrow & & \Delta\downarrow \\ \mathcal{X}_{g_1,n_1+1,d_1} \times \mathcal{X}_{g_2,n_2+1,d_2} & \xrightarrow{ev_+ \times \check{ev}_-} & I\mathcal{X} \times I\mathcal{X} \end{array}$$

gives:

$$\sum_{d_1+d_2=d} \Delta^!([\mathcal{X}_{g_1,n_1+1,d_1}] \times [\mathcal{X}_{g_2,n_2+1,d_2}]) = \mathfrak{g}^*(\text{gl}^!([\mathcal{X}_{g,n,d}])). \tag{2.8}$$

For details and proofs of the statements we refer the reader to the paper of [AGV2] (Prop. 5.3.1.) . The only modification we have made is - we consider the class of the diagonal with respect to the *twisted* pairing on $I\mathcal{X} = \mathcal{X}_{0,3,0,(\mu_1,\mu_2,0)}$. This cancels the factor $ev^*_\Delta(\mathcal{A}_{0,3,0})$ in (2.6) and (2.7).

Roughly speaking relation (2.8) says that the restriction of the virtual fundamental class of $\mathcal{X}_{g,n,d}$ to $\mathcal{Z}$ coincides with the push forward of the product of virtual fundamental classes under the gluing morphisms. Hence integration on $\mathcal{Z}$ factors "nicely" as products of integrals on the two separate moduli spaces.

The rest of the section is devoted to proving pullback results about each type of twisting class separately.

Lemma 2.6.1. *Consider the following diagram:*

$$\begin{array}{ccc} \mathcal{X}_{g,n+\circ+\bullet,d,(\mu_1,\ldots,\mu_n,0,0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n+\bullet,d,(\mu_1,\ldots,\mu_n,0)} \\ \pi_2\downarrow & & \pi_2\downarrow \\ \mathcal{X}_{g,n+\circ,d,(mu_1,\ldots,\mu_n,0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n,d},(\mu_1,\ldots,\mu_n) \end{array}$$

where π_1 forgets the $(n+1)$-st marked point (which I denoted $\circ$) and π_2 forgets the $(n+2)$-nd marked point (denoted $\bullet$) and let $\alpha \in K^0(\mathcal{X}_{g,n+\circ,d,(\mu_1,\ldots,\mu_n,0)})$. Then $\pi_2^*\pi_{1*}\alpha = \pi_{1*}\pi_2^*\alpha$.

Proof: for symplicity of notation we suppress the labeling $(\mu_1, \ldots, \mu_n)$ in the proof. Consider the fiber product:

$$\mathcal{F} := \mathcal{X}_{g,n+\circ,d} \times_{\mathcal{X}_{g,n,d}} \mathcal{X}_{g,n+\bullet,d}$$

and denote by p_1, p_2 the projections from $\mathcal{F}$ to the factors and by $\varphi : \mathcal{X}_{g,n+\circ+\bullet,d} \to \mathcal{F}$ the morphism induced by π_1, π_2. φ is a birational map: it has positive dimensional fibers along the locus where the two extra marked points hit another marked point or a node. We'll prove that

$$\varphi_*(\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}}) = \mathcal{O}_{\mathcal{F}}.$$

By definition of K-theoretic push-forward

$$\varphi_* \mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}} = R^0 \varphi_* \mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}} - R^1 \varphi_* \mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}}.$$

It is easy to see that $\varphi_*(\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}}) = \mathcal{O}_{\mathcal{F}}$ as quasicoherent sheaves (this is true for every proper birational map with normal target). We only have to prove that $R^1 = 0$, which we do by looking at the stalks:

$$(R^1 \varphi_* \mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}})_x = H^1(\varphi^{-1}(x), \mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d} | \varphi^{-1}(x)}).$$

If the fiber over x is a point, there's nothing to prove. If x is in the blowup locus the fiber is a (possibly weighted) $\mathbb{P}^1$. A calculation in [AGV2] shows that :

$$\chi(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 1 - g,$$

where g is the arithmetic genus of the coarse curve C. Therefore $H^1(\varphi^{-1}(x), \mathcal{O}) = 0$.
We have $p_{1*} p_2^* \alpha = \pi_2^* \pi_{1*} \alpha$ because the diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{p_1} & \mathcal{X}_{g,n+\bullet,d,(\mu_1,\ldots,\mu_n,0)} \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{X}_{g,n+\circ,d,(\mu_1,\ldots,\mu_n,0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)} \end{array}$$

is a fiber square. Hence:

$$\pi_{1*} \pi_2^* \alpha = p_{1*} \varphi_* \left(\varphi^* p_2^* \alpha \right) = p_{1*} p_2^* \alpha \varphi_*(\mathcal{O}) = p_{1*} p_2^* \alpha = \pi_2^* \pi_{1*} \alpha.$$

We need to know how the classes $\mathcal{A}_{g,n,d}, \mathcal{B}_{g,n,d}, \mathcal{C}_{g,n,d}$ behave under pullback by the morphisms π, σ and $\pi \circ i \circ p$.

Proposition 2.6.2. *The following identities hold:*

$$\begin{aligned} &a. \quad \pi^*\mathcal{A}_{g,n,d} = \mathcal{A}_{g,n+1,d}. && (2.9)\\ &b. \quad \sigma_i^*\mathcal{A}_{g,n,d} = \mathcal{A}_{g,n,d}. && (2.10)\\ &c. \quad (\pi \circ i^{red} \circ p)^*\mathcal{A}_{g,n,d} = \frac{p_1^*\mathcal{A}_{g_1,n_1+1,d_1} \cdot p_2^*\mathcal{A}_{g_2,n_2+1,d_2}}{ev_\Delta^*\mathcal{A}_{0,3,0}}. && (2.11)\\ &d. \quad (\pi \circ i^{irr} \circ p)^*\mathcal{A}_{g,n,d} = \frac{\mathcal{A}_{g-1,n+2,d}}{ev_\Delta^*\mathcal{A}_{0,3,0}}. && (2.12) \end{aligned}$$

These are proved in [TS]. Denote by $E_{g,n,d} := \pi_*(ev_{n+1}^*E)$. Then he shows that:

$$\begin{aligned} &a. \quad \pi^*E_{g,n,d} = E_{g,n+1,d},\\ &b. \quad (\pi \circ i^{red} \circ p)^*E_{g,n,d} = p_1^*(E_{g_1,n_1+1,d_1}) + p_2^*(E_{g_2,n_2+1,d_2}) - ev_\Delta^*(q^*E_{inv}),\\ &c. \quad (\pi \circ i^{irr} \circ p)^*E_{g,n,d} = E_{g-1,n+2,d} - ev_\Delta^*(q^*E_{inv}). \end{aligned}$$

The identities then follow by multiplicativity of the classes $\mathcal{A}_\alpha$. Since $\mathcal{X}_{0,3,0,(i_1,i_2,0)} \simeq I\mathcal{X}$, the class $\mathcal{A}_{0,3,0}$ is an element of $H^*(I\mathcal{X}, \mathbb{Q})$. We can pull it back by the diagonal evaluation morphism ev_Δ at the node.

Proposition 2.6.3. The following hold:

$$\begin{aligned} &a. \quad \pi^*\mathcal{B}_{g,n,d} = \mathcal{B}_{g,n+1,d} \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta\left(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}\right). && (2.13)\\ &b. \quad \sigma_i^*\mathcal{B}_{g,n,d} = \mathcal{B}_{g,n,d}. && (2.14)\\ &c. \quad (\pi \circ i^{red})^*\mathcal{B}_{g,n,d} = p_1^*\mathcal{B}_{g_1,n_1+1,d_1} \cdot p_2^*\mathcal{B}_{g_2,n_2+1,d_2}. && (2.15)\\ &d. \quad (\pi \circ i^{irr})^*\mathcal{B}_{g,n,d} = \mathcal{B}_{g-1,n+2,d}. && (2.16) \end{aligned}$$

Proof: The first identity is a consequence of Lemma 2.6.1 . More precisely we apply the lemma to the class $\alpha = ev_{n+1}^*(E)(L_{n+1} - 1)^{k+1}$. This gives:

$$\begin{aligned} \pi_2^*\pi_{1*}\left(ev_{n+1}^*(E)(L_{n+1} - 1)^{k+1}\right) &= \pi_{1*}\pi_2^*\left(ev_{n+1}^*(E)(L_{n+1} - 1)^{k+1}\right) =\\ &= \pi_{1*}\left(ev_{n+1}^*(E)(L_{n+1} - 1)^{k+1} - (\sigma_\bullet)_*\left[ev_{n+1}^*(E)(L_{n+1} - 1)^k\right]\right) =\\ &= \pi_{1*}\left(ev_{n+1}^*(E)(L_{n+1} - 1)^{k+1}\right) - ev_{n+1}^*(E)(L_{n+1} - 1)^k. \end{aligned}$$

The last equality follows because $\pi_1 \circ \sigma_\bullet = Id$ and the second equality uses the comparison identity for cotangent line bundles L_i:

$$\pi^*((L_i - 1)^{k+1}) = (L_i - 1)^{k+1} - \sigma_{i*}\left[(L_i - 1)^k\right].$$

But both morphisms π_1, π_2 can be identified with the universal orbicurve π. Hence we deduce:

$$\begin{aligned}\pi^*\pi_*\left(ev_{n+1}^*(E)(L_{n+1}-1)^{k+1}\right) = \pi_*\left(ev_{n+2}^*(E)(L_{n+2}-1)^{k+1}\right) - \\ - ev_{n+1}^*(E)(L_{n+1}-1)^k,\end{aligned} \tag{2.17}$$

or more generally if we expand

$$f_\beta(L_{n+1}^{-1}) - f_\beta(1) = \sum_{k\geq 0} a_k (L_{n+1}-1)^{k+1},$$

then:

$$\pi^*\pi_*(f_\beta(L_{n+1}^{-1}) - f_i(1)) = \pi_*(f_\beta(L_{n+2}^{-1}) - f_i(1)) - \frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1}-1}. \tag{2.18}$$

Then (2.13) follows because $\mathcal{B}_\beta$ are multiplicative classes:

$$\begin{aligned}\pi^*\mathcal{B}_\beta\left(\pi_*(f_\beta(L_{n+1}^{-1}) - f_\beta(1))\right) &= \mathcal{B}_\beta\left(\pi^*\pi_*(f_\beta(L_{n+1}^{-1}) - f_\beta(1))\right) = \\ &= \mathcal{B}_\beta\left(\pi_*(f_\beta(L_{n+2}^{-1}) - f_\beta(1)) - \frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1}-1}\right) = \\ &= \mathcal{B}_\beta\left(\pi_*(f_\beta(L_{n+2}^{-1}) - f_\beta(1))\right) \cdot \mathcal{B}_\beta\left(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1}-1}\right).\end{aligned}$$

Example 2.6.4. In the case $f_\beta = ev_{n+1}^*(E_\beta) \otimes L_{n+1}^{-1}$ (which is the only one we'll need) we have:

$$\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1}-1} = -E_\beta L_{n+1}^{-1}$$

and relation (2.13) reads:

$$\pi^*\mathcal{B}_{g,n,d} = \mathcal{B}_{g,n+1,d} \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta(E_\beta \otimes L_{n+1}^{-1}). \tag{2.19}$$

Relation (2.15) follows from the identity:

$$\begin{aligned}(\pi \circ i_{red})^*[\pi_*(f(L_{n+1}^{-1}) - f(1))] = \\ = p_1^*[\pi_*(f(L_{n_1+2}^{-1}) - f(1))] + p_2^*[\pi_*(f(L_{n_2+2}^{-1}) - f(1))],\end{aligned}$$

which we prove below. By linearity is enough to prove the result for $f = (L_{n+1}-1)^{k+1}$ for $k \geq 0$. Relation (2.17) gives:

$$\pi^*\pi_*(L_{n+1}-1)^{k+1}) = \pi_*(L_{n+2}-1)^{k+1} - (L_{n+1}-1)^k. \tag{2.20}$$

Assume for now that $k \geq 1$. When we apply i^*_{red} to this relation the second summand in the RHS of (2.20) vanishes because L_{n+1} is trivial on $\mathcal{Z}$. Therefore

$$i^*_{red}\pi^*\pi_*(L_{n+1}-1)^{k+1} = (i\circ p)^*\pi_*(L_{n+2}-1)^{k+1}.$$

Let $\mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2}$ be a stratum of $\mathcal{Z}$. If we denote by $\pi : \mathcal{U}'_{g,n,d} \to \mathcal{U}_{g,n,d}$ the universal curve then we have a fiber diagram:

$$\begin{array}{ccc} \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 & \xrightarrow{i} & \mathcal{U}'_{g,n,d} \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2} & \xrightarrow{i} & \mathcal{U}_{g,n,d}. \end{array}$$

Here $\mathcal{Z}_1$ and $\mathcal{Z}_3$ are the universal curves over the factors $\mathcal{X}_{g_1,n_1+1,d_1}$ and $\mathcal{X}_{g_2,n_2+1,d_2}$. So using

$$i^*_{red}\pi_*(L_{n+2}-1)^{k+1} = \pi_* i^*_{red}(L_{n+2}-1)^{k+1}, \tag{2.21}$$

we see that the contribution of the strata $\mathcal{Z}_1$ and $\mathcal{Z}_3$ above is:

$$p_1^*[\pi_*(f(L^{-1}_{n_1+2}) - f(1))] + p_2^*[\pi_*(f(L^{-1}_{n_2+2}) - f(1))]. \tag{2.22}$$

So if we show that the contribution from $\mathcal{Z}_2$ is 0 we are done. Notice that $\mathcal{Z}_2$ is the universal curve over the factor $\mathcal{X}_{0,3,0}$, hence it is a fiberproduct $\mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,4,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2}$. The fibers of the map $\mathcal{Z}_2 \to \mathcal{Z}$ are (weighted) $\mathbb{P}^1$. However the class L is a cotangent line at a point with trivial orbifold structure, so we can use Y.P.Lee's formula in [L1] which in this particular case reads:

$$\chi(\overline{M}_{0,4}, L_i^k) = k+1. \tag{2.23}$$

Hence the Euler characteristics of $(L_{n+2}-1)^{k+1}$ is:

$$\begin{aligned} \chi\left(\overline{M}_{0,4}, (L_{n+2}-1)^{k+1}\right) &= \sum_{i=0}^{k+1} (i+1)(-1)^{k+1-i}\binom{k+1}{i} = \\ = \sum_{i=0}^{k+1}(-1)^{k+1-i}\binom{k+1}{i} &+ (k+1)\sum_{i=1}^{k+1}(-1)^{k+1-i}\binom{k}{i-1} = 0+0 = 0. \end{aligned}$$

This almost proves the statement. We are left with the case $k=0$, which is slightly different: the sum above equals 1, but this is cancelled by the -1 in the second term of (2.20). Relation (2.15) follows then from the multiplicativity of the classes $\mathcal{B}_\beta$. A similar computation shows relations (2.14) and (2.16).

Lemma 2.6.5. *Let $F \in K^0(\mathcal{X})$. Then:*

$$\begin{aligned} a. \quad & \pi^*\pi_* i_{\mu *}(ev_{n+1}^*(F)\otimes \mathcal{O}_{\mathcal{Z}_\mu}) = \pi_* i_{\mu *}(ev_{n+1}^*(F)\otimes \mathcal{O}_{\mathcal{Z}_\mu}) - \\ & \sum_{j,\mu_j=\mu} ev_{n+1}^*(F)\otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j} - i_{\mu *}(ev_{n+1}^*(F)\otimes \mathcal{O}_{\mathcal{Z}_\mu}). \end{aligned} \tag{2.24}$$

$$\begin{aligned} b. \quad & (\pi\circ i)^*(\pi_* i_{\mu *}(ev_{n+1}^*(F)\otimes \mathcal{O}_{\mathcal{Z}_\mu}) = p_1^*(\pi_* i_{\mu *}(ev_{n+1}^*(F)\otimes \mathcal{O}_{\mathcal{Z}_\mu})) + \\ & p_2^*(\pi_* i_{\mu *}(ev_{n+1}^*(F)\otimes \mathcal{O}_{\mathcal{Z}_\mu})) + \left(ev_{n+1}^* F \otimes (1 - L_+ L_-)\right). \end{aligned} \tag{2.25}$$

Remark 2.6.2. Before delving in the technicalities of the proof, we try a heuristic explanation of why the formulae, which look rather ugly, "should" be true:

- Assume for now that F is the trivial bundle $\mathbb{C}$. The nodal locus $\mathcal{Z}$ "separates nodes" in the following sense: above a point of $\mathcal{X}_{g,n,d}$ representing a nodal curve with k nodes lie exactly k points of $\mathcal{Z}$. This is very similar with the way the normalisation of a nodal curve $\widetilde{C} \to C$ separates the nodes. But the structure sheaves of $\widetilde{C}$ and C differ (in K-theory) by skyscraper sheaves at the preimages of nodes. That's pretty much what the first formula expresses: the pull-back of the structure sheaf of the codimension one stratum of nodal curves in $\mathcal{X}_{g,n,d}$ equals the structure sheaf of the nodal locus in the universal family, minus a copy of the structure sheaf of $\mathcal{Z}$ (which has codimension two in the universal family) itself. The terms supported on the divisors $\mathcal{D}_j$ are substracted because they are nodes in the universal family, but they lie over the whole space $\mathcal{X}_{g,n,d}$. We'll see that the presence of the class $ev_{n+1}^*(F)$ doesn't complicate things too much.

- For the second formula, think of $\pi_* i_{\mu *}\alpha$ as a class supported on a codimension one subvariety. We pull it back along the map (πi) , which is like restricting to another codimension one subvariety. If these subvariesties intersect along a codimension two cycle (represented by curves with two nodes), then they contribute $p_i^*(\pi_* i_{\mu *}\alpha)$ to (2.25). If they are the same subvariety, then α gets multiplied with the Euler class of the normal bundle of it in the ambient space, which is $1 - L_+ L_-$.

Proof of Lemma 2.6.5: Denote by $\mathcal{Z}_\bullet$, $\mathcal{Z}_\circ$, respectively $\mathcal{Z}_{\bullet\circ}$ the nodal loci living inside the corresponding moduli spaces (and by $\mathcal{Z}_{\circ,\mu}$ etc. the ones with nodes of specific orbifold type) in the following diagram:

$$\begin{array}{ccccc} \pi_2^{-1}(\mathcal{Z}_{\circ,\mu}) & \xrightarrow{i_\mu} & \cup_{(i_1,\dots,i_n)}\mathcal{X}_{g,n+\circ+\bullet,d,(i_1,\dots,i_n,0,0)} & \xrightarrow{\pi_1} & \cup_{(i_1,\dots,i_n)}\mathcal{X}_{g,n+\bullet,d,(i_1,\dots,i_n,0)} \\ \pi_2 \downarrow & & \pi_2 \downarrow & & \pi_2 \downarrow \\ \mathcal{Z}_{\circ,\mu} & \xrightarrow{i_\mu} & \cup_{(i_1,\dots,i_n)}\mathcal{X}_{g,n+\circ,d,(i_1,\dots,i_n,0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n,d}. \end{array}$$

Remember that $\mathcal{Z}_{\circ,\mu}$ is defined as the total range of the gluing map:

$$\mathcal{X}_{g-1,n+2,d} \times_{\mathcal{X}_\mu \times \mathcal{X}_{\mu^I}} \mathcal{X}_{0,3,0} \coprod \mathcal{X}_{g_1,n_1+1,d_1} \times_{\mathcal{X}_\mu \times \mathcal{X}_{\mu^I}} \mathcal{X}_{0,3,0} \times_{\mathcal{X}_\mu \times \mathcal{X}_{\mu^I}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathcal{Z}_\circ \hookrightarrow \mathcal{X}_{g,n+\circ,d}.$$

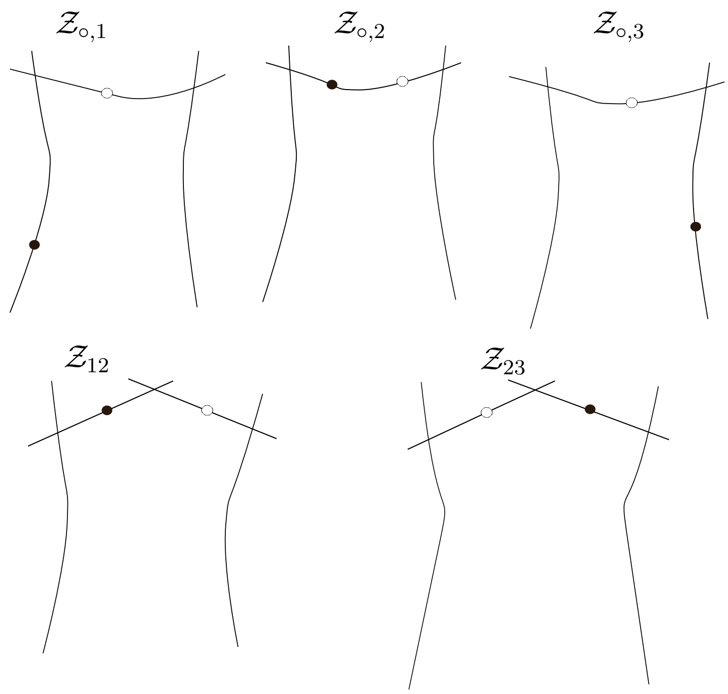

Figure 2.1: Strata of $\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})$.

We will compute $\pi_2^*(\pi_{1*}i_{\mu*}(ev_\circ^*(F)\otimes\mathcal{O}_{\mathcal{Z}_{\circ,\mu}}))$.

The square on the left is a fiber diagram, hence $i_*\pi_2^* = \pi_2^* i_*$. For the one on the right we have proved that $\pi_2^*\pi_{1*} = \pi_{1*}\pi_2^*$. Therefore:

$$\pi_2^*(\pi_{1*}i_{\mu*}(ev_{n+1}^*(F)\otimes\mathcal{O}_{\mathcal{Z}_{\circ,\mu}})) = \pi_{1*}i_{\mu*}\pi_2^*(ev_\circ^*(F)\otimes\mathcal{O}_{\mathcal{Z}_{\circ,\mu}})). \tag{2.26}$$

But:

$$\pi_2^*(ev_\circ^*F\otimes\mathcal{O}_{\mathcal{Z}_{\circ,\mu}}) = ev_\circ^*F\otimes\mathcal{O}_{\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})}.$$

The space $\pi_2^{-1}(\mathcal{Z}_{\circ,\mu}) := \mathcal{Z}_{\circ,1}\cup\mathcal{Z}_{\circ,2}\cup\mathcal{Z}_{\circ,3}$ is a singular space, where each codimension two stratum is the universal curve over one factor of $\mathcal{Z}_{\circ,\mu}$ and they intersect along two codimension three strata, call them $\mathcal{Z}_{12}$ and $\mathcal{Z}_{23}$:

$$\mathcal{Z}_{12} = \mathcal{X}_{g_1,n_1+1,d_1}\times\mathcal{X}_{0,3,0}\times\mathcal{X}_{0,3,0}\times\mathcal{X}_{g_2,n_2+1,d_2}$$

where the two rational components carry the points $\bullet$, $\circ$ and two nodes. Figure 2.1 above schematically represents each of these five strata. We can write the structure sheaf of $\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})$ as:

$$\mathcal{O}_{\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})} = \mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathcal{O}_{\mathcal{Z}_{\circ,3}} + \mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}.$$

We tensor this with the class $ev_\circ^* F$, keeping in mind that on the strata $\mathcal{Z}_{\circ,2}, \mathcal{Z}_{12}, \mathcal{Z}_{23}$ $ev_\circ = ev_\bullet$:

$$ev_\circ^* F \mathcal{O}_{\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})} = ev_\circ^* F \otimes \left[\mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathcal{O}_{\mathcal{Z}_{\circ,3}}\right] + ev_\bullet^* F \otimes \left[\mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}\right]. \tag{2.27}$$

We plug (2.27) in (2.26) and we get:

$$\pi_2^*(\pi_{1*} i_{\mu*}(ev_\circ^*(F) \otimes \mathcal{O}_{\mathcal{Z}_{\circ,\mu}})) = \pi_{1*} i_{\mu*} \left[ev_\circ^* F \left(\mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathcal{O}_{\mathcal{Z}_{\circ,3}}\right) + ev_\bullet^* F \left(\mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}\right)\right]. \tag{2.28}$$

We now notice that the union of $\mathcal{Z}_{\circ,1}$ and $\mathcal{Z}_{\circ,3}$ is almost $\mathcal{Z}_{\bullet\circ,\mu}$ but not quite. There are strata:

$$\mathcal{X}_{g,n,d} \times_{\mathcal{X}_\mu} \mathcal{X}_{0,3,0} \times_{\mathcal{X}_\mu} \mathcal{X}_{0,3,0}$$

which are in $\mathcal{Z}_{\bullet\circ,\mu}$, but they are missing from $\mathcal{Z}_{\circ,1} \cup \mathcal{Z}_{\circ,3}$ because the map $\pi_2 \circ i_\mu$ contracts one rational tail. These are mapped by $\pi_1 \circ i_\mu$ isomorphically to divisors $\mathcal{D}_j \in \mathcal{X}_{g,n+\bullet,d}$. There is one such stratum for each j such that $\mu_j = \mu$. Hence we can write:

$$\pi_{1*} i_{\mu*} \left[ev_\circ^* F \mathcal{O}_{\mathcal{Z}_{\circ,1}} + ev_\circ^* F \mathcal{O}_{\mathcal{Z}_{\circ,3}}\right] == \pi_{1*} i_{\mu*}(ev_\circ^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) - \sum_{j,\mu_j=\mu} ev_\circ^*(F) \otimes \sigma_{j*} \mathcal{O}_{\mathcal{D}_j}. \tag{2.29}$$

The codimension three strata $\mathcal{Z}_{12}$ and $\mathcal{Z}_{23}$ are mapped by $\pi_1 i_\mu$ isomorphically to $\mathcal{Z}_{\bullet,\mu}$. As for $\mathcal{Z}_{\circ,2}$, this is a $\mathbb{P}^1$ fibration over $\mathcal{Z}_{\bullet,\mu}$. When we push forward, we integrate the structure sheaf of (weighted) $\mathbb{P}^1$. This equals 1, as already explained. At the end of the day we see that the last three terms in (2.28) contribute:

$$\pi_{1*} i_{\mu*} \left[ev_\bullet^* F \left(\mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}\right)\right] = -ev_\bullet^* F \otimes i_{\mu_*} \mathcal{O}_{\mathcal{Z}_{\bullet,\mu}}. \tag{2.30}$$

Adding up (2.29) with (2.30) and identifying $\pi_1 = \pi_2 = \pi$ and $ev_\circ = ev_{n+1}$ proves the first equality in the lemma.

For the second equality, we first prove:

Lemma 2.6.6. *Let $j : \mathcal{Z} \hookrightarrow \mathcal{U}_{g,n,d}$ be the codimension two nodal locus. Then:*

$$\begin{aligned} j^* \pi_* i_{\mu*} \left(ev_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}\right) &= p_1^* \pi_* i_{\mu*} \left(ev_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}\right) + \\ &+ p_2^* \pi_* i_{\mu*} \left(ev_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}\right) + (2 - L_+ - L_-) ev_{n+1}^*(F). \end{aligned} \tag{2.31}$$

Proof of the Lemma 2.6.6: let $\mathcal{U}'_{g,n,d}$ be the universal curve over $\mathcal{U}_{g,n,d}$. The universal curve over $\mathcal{Z}$ is a union of three types of strata, depending on which component the extra marked point on $\mathcal{U}'_{g,n,d}$ - which we denote $\bullet$ - lies on (see also Figure 2.1):

$$\begin{aligned} \mathcal{Z}_1 &= \mathcal{X}_{g_1,n_1+1+\bullet,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2}, \\ \mathcal{Z}_2 &= \mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3+\bullet,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2}, \\ \mathcal{Z}_3 &= \mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1+\bullet,d_2}. \end{aligned}$$

The diagram below is a fiber square:

$$\begin{array}{ccc} \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 & \xrightarrow{j} & \mathcal{U}'_{g,n,d} \\ \pi\downarrow & & \pi\downarrow \\ \mathcal{Z} & \xrightarrow{j} & \mathcal{U}_{g,n,d}. \end{array}$$

Hence : $j^*\pi_* i_{\mu*}\alpha = \pi_* j^* i_{\mu*}\alpha$. To compute $j^* i_{\mu*}\alpha$ we form the following fiber diagram:

$$\begin{array}{ccc} \overline{\mathcal{Z}} & \xrightarrow{j} & \mathcal{Z}_{\bullet,\mu} \\ \pi\downarrow & & \pi\downarrow \\ \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 & \xrightarrow{j} & \mathcal{U}'_{g,n,d}. \end{array}$$

The space $\overline{\mathcal{Z}}$ is simply the intersection of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3$ with $\mathcal{Z}_{\bullet,\mu}$. Where the intersection is transversal one can simply write $j^* i_{\mu*}\alpha = i_{\mu*} j^*\alpha$. On components where the intersection is not transversal, there is some excess bundle N and $j^* i_{\mu*}\alpha = i_{\mu*} e(N) j^*\alpha$. The strata $\mathcal{Z}_1$ and $\mathcal{Z}_3$ intersect the nodal locus $\mathcal{Z}_{\bullet,\mu}$ in $\mathcal{U}'_{g,n,d}$ transversely along codimension four strata which can be seen as the nodal locus in $\mathcal{X}_{g_1,n_1+1+\bullet,d_1}$ and $\mathcal{X}_{g_2,n_2+1+\bullet,d_2}$ respectively. Hence the contribution to (2.31) is:

$$p_1^*\pi_* i_{\mu*}\left(ev_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}\right) + p_2^*\pi_* i_{\mu*}\left(ev_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}\right).$$

On the other hand $\mathcal{Z}_2$ intersects $\mathcal{Z}_{\bullet,\mu}$ along two codimension three strata of the form:

$$\mathcal{Z}_1 = \mathcal{X}_{g_1,n_1+1,d_1} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{0,3,0} \times_{I\mathcal{X}} \mathcal{X}_{g_2,n_2+1,d_2}.$$

Each gives a one dimensional excess normal bundle with Euler classes $1 - L_+$ and $1 - L_-$ respectively. They project isomorphically to $\mathcal{Z}$ downstairs. Hence they contribute:

$$(2 - L_+ - L_-) ev_{n+1}^*(F).$$

Adding up, we get (2.31).

We now prove formula 2.25 in Lemma 2.6.5. It falls out easily by combining (2.24) with Lemma 2.6.6. More precisely we take i^* of formula (2.24): the first term is computed in Lemma 2.6.6, the part supported on $\mathcal{D}_j$ vanishes and:

$$i_\mu^* i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu} = e(N) = (1 - L_-)(1 - L_+) \tag{2.32}$$

where N is the normal bundle of $\mathcal{Z}_\mu$ in the ambient space. When we add this with (2.31) we get:

$$\begin{aligned} &(\pi \circ i)^*(\pi_* i_{\mu*}(ev_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu}) = p_1^*(\pi_* i_{\mu*}(ev_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu})) + \\ &p_2^*(\pi_* i_{\mu*}(ev_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu})) + \left(ev_{n+1}^* F \otimes (1 - L_+ L_-)\right), \end{aligned} \tag{2.33}$$

as stated.

Proposition 2.6.7. The following hold:

$$a.\quad \pi^*\mathcal{C}_{g,n,d} = \mathcal{C}_{g,n+1,d} \cdot \prod_{j=1}^{n} \prod_{\delta=1}^{i_{\mu_j}} \mathcal{C}_\delta^{\mu_j}\left(-ev_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}\right)$$

$$\cdot \prod_{\mu} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^{\mu}\left(-ev_{n+1}^*(F_{\delta\mu}) \otimes (i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu}\right). \tag{2.34}$$

$$b.\quad \sigma_j^*\mathcal{C}_{g,n,d} = \mathcal{C}_{g,n,d} \cdot \prod_{\delta=1}^{i_{\mu_j}} \mathcal{C}_\delta^{\mu_j}\left(ev_{n+1}^*(F_{\delta\mu_j}) \otimes (1 - L_j)\right). \tag{2.35}$$

$$c.\quad p^*(i_\mu^{red})^*\pi^*\mathcal{C}_{g,n,d} = \left(p_1^*\mathcal{C}^{\mu}_{g_1,n_1+1,d_1} \cdot p_2^*\mathcal{C}^{\mu}_{g_2,n_2+1,d_2}\right) \cdot$$

$$\cdot (ev_+^* \times ev_-^*)\Delta_{\mu*}\left(\prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^{\mu}((q^*F_{\delta\mu})_\mu) \otimes (1 - L_+L_-))\right). \tag{2.36}$$

$$d.\quad p^*(i_\mu^{irr})^*\pi^*\mathcal{C}_{g,n,d} =$$

$$= \mathcal{C}^{\mu}_{g-1,n+2,d} \cdot (ev_+^* \times ev_-^*)\Delta_{\mu*}\left(\prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^{\mu}((q^*F_{\delta\mu})_\mu) \otimes (1 - L_+L_-))\right). \tag{2.37}$$

Proof: the equalities (2.34) and (2.36) are immediate consequences of (2.24) and (2.25) and of the multiplicativity of the classes $\mathcal{C}_{g,n,d}$. As for (2.35) we can view it as a particular case of (2.36) in the following way: the divisor $\mathcal{D}_i$ can be identified with the stratum $\mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)} \otimes_{I\mathcal{X}} \mathcal{X}_{0,3,0} \otimes_{I\mathcal{X}} \mathcal{X}_{0,3,0}$ in $\mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0,0)}$. On that stratum $L_+ = L_i$ and $L_- = 1$ - hence the formula.

We will use (2.34) in a different form, using the same trick as for the Todd class of $\Omega_\pi^\vee$ to transform the product into a sum:

$$\pi^*\mathcal{C}_{g,n,d} = \mathcal{C}_{g,n+1,d} \cdot \prod_{j=1}^{n} \prod_{\delta=1}^{i_C} \left(1 + \mathcal{C}_\delta^{\mu_j}\left(-ev_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}\right) - 1\right)$$

$$\prod_{\mu}\left(1 + \prod_{\delta=1}^{i_{C_\mu}} \mathcal{C}_\delta^{\mu}\left(-ev_{n+1}^*(F_{\delta\mu}) \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu}\right) - 1\right) =$$

$$\mathcal{C}_{g,n+1,d} + \sum_j \mathcal{C}_{g,n+1,d} \cdot \prod_{\delta=1}^{i_{C_{\mu_j}}} (\mathcal{C}_\delta^{\mu_j}\left(-ev_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j})\right) - 1) +$$

$$+ \sum_{\mu} \mathcal{C}_{g,n+1,d} \cdot \left(\prod_{\delta=1}^{i_{C_\mu}} \mathcal{C}_\delta^{\mu}\left(-ev_{n+1}^*(F_{\delta\mu}) \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu}\right) - 1\right). \tag{2.38}$$

This happens because the classes $\mathcal{C}_\delta^{\mu}(..) - 1$ are supported on $\mathcal{D}_i$ and $\mathcal{Z}$ and $\mathcal{D}_i \cdot \mathcal{D}_j = \mathcal{D}_i \cdot \mathcal{Z}_\mu = 0$ if $i \neq j$.

We conclude the section by doing a short Grothendieck-Riemann-Roch computation which will turn out useful in the next section:

Lemma 2.6.8. *Let $F \in K^0(\mathcal{X})$. Then*

$$ch\left(\pi_* i_{\mu *}(ev_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu})\right) = \pi_* i_{\mu *}\left(ch(ev_{n+1}^* F) \cdot Td^\vee(-L_+ \otimes L_-)\right). \tag{2.39}$$

Proof: recall that $r(\mu)$ is the order of the distinguished node on $\mathcal{Z}_\mu$. We'll simply write r throughout the proof.

We apply Toen's GRR to the map $f = \pi \circ i$. The map π is given in local coordinates near $\mathcal{Z}_\mu$ by:

$$(z, x, y)/\mathbb{Z}_r \times \mathbb{Z}_r \mapsto (z, xy)/\mathbb{Z}_r$$

where z is a vector coordinate along $\mathcal{Z}_\mu$ and $\mathcal{Z}_\mu$ is given by $x = y = 0$. The generator of $\mathbb{Z}_r \times \mathbb{Z}_r$ acts on the (x, y) plane as follows: $(x, y) \mapsto (\zeta^a x, \zeta^b y)$ and necessarily by multiplication by ζ^{a+b} on the base. So in this local description If maps r copies of the point $(z, 0, 0)$ to $(z, 0)$ on the base. Each copy has weight $1/r$ from Kawasaki's formula. The relativ tangent bundle is $-L_+^{-1} - L_-^{-1}$ because the coordinate on the base is $\varepsilon = xy$ and is invariant with respect to the $\mathbb{Z}_r$ action. This proves the statement.

2.7 Proofs of Theorems

Proof of Theorem 1.10.1: this is an easy consequence of Tseng's result and of the commutativity of the operators Δ_α.

Proof of Theorem 1.10.2: Remember that $\mathcal{B}_{g,n,d}$ is a product of i_B multiplicative characteristic classes. We'll prove the statement using induction on i_B. The case $i_B = 0$ is trivial. Assuming the statement holds for $i_B - 1$, we'll prove the infinitesimal version of the proposition for i_B. Namely assume the twisting class $\mathcal{B}_{i_B}$ to be:

$$\mathcal{B}_{i_B} = exp\left(\sum_{l \geq 1} v_l ch_l \pi_* \left(f(L_{n+1}^{-1}) - f(1)\right)\right).$$

We compute:

$$\begin{aligned}&\frac{\partial \mathcal{D}_{A,B,C}}{\partial v_l}\mathcal{D}_{A,B,C}^{-1} = \\ &= \sum_{d,n} \frac{Q^d \hbar^{g-1}}{n!}\left\langle \prod_{i=1}^n \mathbf{t}(\overline{\psi}_i) \cdot ch_l \pi_* \left(f(L_{n+1}^{-1}) - f(1)\right) \cdot \Theta_{g,n,d}\right\rangle_{g,n,d}.\end{aligned} \tag{2.40}$$

To compute $ch_l\pi_*\left(f(L_{n+1}^{-1})-f(1)\right)$ above we apply Toen's GRR to the morphism π to get:

$$ch\left(\pi_*(f(L_{n+1}^{-1})-f(1))\right)=I\pi_*\left(\widetilde{ch}(f(L_{n+1}^{-1})-f(1))Td^\vee(\Omega_\pi)\right). \tag{2.41}$$

Notice that $\widetilde{ch}=ch$ because the last marked point is not an orbifold point. We have:

$$\widetilde{ch}(f(L_{n+1}^{-1})-f(1))=f(e^{-\psi_{n+1}})-f(1). \tag{2.42}$$

There are three strata in the (relative) inertia stack that map to $\mathcal{X}_{g,n,d}$. But the expression on the RHS in (2.42) above is a multiple of ψ_{n+1} and ψ_{n+1} vanishes on the locus of marked points $\mathcal{D}_j$ and on the locus of nodes $\mathcal{Z}$. Hence only the total space contributes to GRR. As usually we write the sheaf of relative differentials:

$$\Omega_\pi=L_{n+1}-\oplus_{i=1}^n\sigma_{j*}\mathcal{O}_{\mathcal{D}_{j,(\mu_1,\ldots,\mu_n)}}-i_*\mathcal{O}_\mathcal{Z} \tag{2.43}$$

which for Todd classes gives

$$Td^\vee(\Omega_\pi)=Td^\vee(L_{n+1})\prod_{i=1}^n Td^\vee(-\sigma_{j*}\mathcal{O}_{\mathcal{D}_{j,(\mu_1,\ldots,\mu_n)}})Td^\vee(-i_*\mathcal{O}_\mathcal{Z}) \tag{2.44}$$

and then using the fact that L_{n+1} is trivial when restricted to D_i and $\mathcal{Z}$ we can rewrite the product as a sum:

$$Td^\vee(\Omega_\pi)=Td^\vee(L_{n+1})+\sum_{i=1}^n(Td^\vee(-\sigma_{j*}\mathcal{O}_{\mathcal{D}_{j,(\mu_1,\ldots,\mu_n)}})-1)+Td^\vee(-i_*\mathcal{O}_\mathcal{Z})-1. \tag{2.45}$$

The last $n+1$ summands are classes supported on $\mathcal{D}_j$ and $\mathcal{Z}$, so they are killed by the presence of ψ in $f(e^{-\psi_{n+1}})-f(1)$. After all these cancelations we see that:

$$ch\left(\pi_*(f(L_{n+1}^{-1})-f(1))\right)=\pi_*\left((f(e^{-\psi_{n+1}})-f(1))\cdot Td^\vee(L_{n+1})\right). \tag{2.46}$$

(2.46) is a linear combination of kappa classes $K_{aj}=\pi_*(ev_{n+1}^*\varphi_a\psi_{n+1}^{j+1})$. Now we pull the correlators back on the universal orbicurve. It is essential here that the corrections in the $\mathcal{C}_{g,n,d}$ classes are also supported on $\mathcal{D}_j$ and $\mathcal{Z}$ (as we can see from (2.38)) and the presence of ψ_{n+1} kills them. Therefore (we denote by$[f]_l$ the homogeneous part of degree l of f):

$$\begin{aligned}\mathcal{D}_{A,B,C}^{-1}\frac{\partial\mathcal{D}_{A,B,C}}{\partial v_l}&=\sum_{d,n,g}\frac{Q^d\hbar^{g-1}}{n!}\int_{\mathcal{X}_{g,n+1,d}}\prod_{i=1}^n\left(\sum_{k_i\geq 0}\left(ev_i^*(t_{k_i})\cdot\overline{\psi}_i^{k_i}\right)\right)\cdot\\&\cdot\left[(f(e^{-\psi_{n+1}})-f(1))\cdot Td^\vee(L_{n+1})\right]_{l+1}\cdot\Theta_{g,n+1,d}\cdot\prod_{\beta=1}^{i_B}\mathcal{B}_\beta\left(-\frac{f_\beta(L_{n+1}^{-1})-f_\beta(1)}{L_{n+1}-1})\right)\\&-\int_{\mathcal{X}_{0,3,0}}\varphi_a\psi_3^{m+1}(\cdots)-\int_{\mathcal{X}_{1,1,0}}\varphi_a\psi_1^{m+1}(\cdots).\end{aligned} \tag{2.47}$$

The correction terms occur because the spaces $\mathcal{X}_{0,3,0}$ and $\mathcal{X}_{1,1,0}$ are not universal families. Notice that the first correction is always 0 for dimensional reasons, and the second is $\neq 0$ only for $m=0$ (again for dimensional reasons), in which case equals $\frac{1}{24}\int_{\mathcal{X}} e(T_{\mathcal{X}})$. So the "new" twisting by the class $\mathcal{B}_{i_B}$ has the same effect as the translation:

$$\mathbf{t}_B(z) = \mathbf{t}(z) + z - z\prod_{\gamma=1}^{i_B}\mathcal{B}_\beta\left(-\frac{f_\beta(\mathbf{L}_z^{-1}) - f_\beta(1)}{\mathbf{L}_z - 1}\right),$$

because both potentials satisfy the same differential equation. To see this differentiate the potential $\mathcal{D}_{A,B}(\mathbf{t}_B(z))$ in v_l:

$$\frac{\partial\mathcal{D}_{A,B}(\mathbf{t}_B(z))}{\partial v_l}\mathcal{D}_{A,B}^{-1} = \sum_{d,n,g}\frac{Q^d\hbar^{g-1}}{n!}\int_{\mathcal{X}_{g,n+1,d}}\prod_{i=1}^{n}\left(\sum_{k_i\geq 0}\left(ev_i^*(t_{k_i})\cdot\overline{\psi}_i^{k_i}\right)\right)\cdot$$

$$\cdot\,\psi_{n+1}ch_l\left(\frac{f(L_{n+1}^{-1}) - f(1)}{L_{n+1}-1}\right)\cdot\Theta_{g,n+1,d}\cdot\prod_{\beta=1}^{i_B}\mathcal{B}_\beta\left(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1}-1}\right). \tag{2.48}$$

But:

$$\psi_{n+1}ch_l\left(\frac{f(L_{n+1}^{-1}) - f(1)}{L_{n+1}-1}\right) = \psi_{n+1}\left[\frac{f(e^{-\psi_{n+1}}) - f(1)}{e^{\psi}-1}\right]_l =$$

$$\left[\psi_{n+1}\frac{f(e^{-\psi_{n+1}}) - f(1)}{e^{\psi}-1}\right]_{l+1} = \left[(f(e^{-\psi_{n+1}}) - f(1))\cdot Td^\vee(L_{n+1})\right]_{l+1} \tag{2.49}$$

because

$$Td^\vee(L_{n+1}) = \frac{\psi_{n+1}}{e^{\psi_{n+1}}-1}.$$

Plugging (2.49) in (2.48) we see that (2.48) and (2.47) are of exactly the same form. The potentials also satisfy the same initial condition at $\mathbf{v}=0$ by the induction hypothesis.

Proof of Theorem 1.10.3: we'll prove that

$$\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}} = exp\left(\frac{\hbar}{2}\sum_{a,b,\alpha,\beta,\mu}A^\mu_{a,\alpha;b;\beta}\partial_a^{\alpha,\mu}\partial_b^{\beta,\mu^I}\right)\mathcal{D}_{\mathcal{A},\mathcal{B}} \tag{2.50}$$

where $A^\mu_{a,\alpha;b,\beta}$ are the coefficients of the expansion:

$$\sum_{a,b}A^\mu_{a,\alpha;b,\beta}\varphi_{\alpha,\mu}\overline{\psi}_+^a\otimes\varphi_{\beta,\mu^I}\overline{\psi}_-^b = -\frac{\Delta_{\mu*}\left(\prod_{\delta=1}^{i_\mu}\mathcal{C}^\mu_\delta\left((q^*F_{\delta\mu})_\mu\otimes(1-\mathbb{L}_z)\right)-1\right)}{\psi_+ + \psi_-}\in$$

$$\in H^*(\mathcal{X}_\mu,\mathbb{Q})[\overline{\psi}_+]\otimes H^*(\mathcal{X}_{\mu^I},\mathbb{Q})[\overline{\psi}_-]. \tag{2.51}$$

Here $\psi_+ = c_1(L_+)$, $\psi_- = c_1(L_-)$ and $\Delta_\mu : \mathcal{X}_\mu \to \mathcal{X}_\mu \otimes \mathcal{X}_{\mu^I}$ is the composition $(Id \times \iota) \circ \Delta$. The map:

$$\Delta_{\mu_*} : H^*(\mathcal{X}_\mu, \mathbb{Q}) \to H^*(\mathcal{X}_\mu, \mathbb{Q}) \otimes H^*(\mathcal{X}_{\mu^I}, \mathbb{Q})$$

extends naturally to a map, which we abusively also call $\Delta_{\mu*}$:

$$\Delta_{\mu_*} : H^*(\mathcal{X}_\mu, \mathbb{Q})[z] \to H^*(\mathcal{X}_\mu, \mathbb{Q})[\overline{\psi}_+] \otimes H^*(\mathcal{X}_{\mu^I}, \mathbb{Q})[\overline{\psi}_-],$$

by mapping $z \mapsto \psi_+ \otimes 1 + 1 \otimes \psi_-$ and the RHS of (2.51) should be understood in this way.

Accordind to [C], relation (2.50) is equivalent to the statement of 1.10.3.

We'll prove (2.50) using induction on the total number $\sum_\mu i_\mu$ of twisting classes $\mathcal{C}^\mu_\delta$. If $\sum i_\mu = 0$ then the equality is trivial. Let now $\sum i_\mu \geq 1$. Assuming (2.50) to be true for $\sum i_\mu - 1$, we'll prove the infinitesimal version of the theorem for $\sum i_\mu$. More precisely fix an μ_0 and let the multiplicative class $\mathcal{C}^{\mu_0}$ (we omit the lower index) be of the form :

$$\mathcal{C}^{\mu_0}(E) = exp\left(\sum_l w_l ch_l(E)\right). \tag{2.52}$$

As we vary the coefficients w_l we obtain a family of elements in the Fock space. We prove (2.50) by showing that both sides satisfy the same differential equations with the same initial condition. Notice that the induction hypothesis ensures that both sides of (2.50) satisfy the same initial condition at $\mathbf{w} = 0$. Moreover $\partial \mathcal{D}_{A,B}/\partial w_l = 0$ so on the RHS only the coefficients $A^{\mu_0}_{a,\alpha;b,\beta}$ depend on w_l. So if denote the RHS by $\mathcal{G}$ and differentiate it we get:

$$\frac{\hbar}{2}\sum_{a,b}\frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l}\partial^{\alpha,\mu_0}_a \partial^{\beta,\mu_0^I}_b \mathcal{G} = \frac{\partial}{\partial w_l}\mathcal{G}. \tag{2.53}$$

To compute $\partial A^{\mu_0}_{a,\alpha;b,\beta}/\partial w_l$ we differentiate in w_l relation (2.51) to get:

$$\sum_{a,\alpha;b,\beta}\frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l}\varphi_{\alpha,\mu_0}\overline{\psi}^a_+ \otimes \varphi_{\beta,\mu_0^I}\overline{\psi}^b_- =$$

$$= \frac{-1}{\psi_+ + \psi_-}\cdot \Delta_{\mu_0 *}\left(ch_l\left((q^*F)_{\mu_0}(1 - L_+L_-)\right)\prod_{\delta=1}^{i_{\mu_0}}\mathcal{C}^{\mu_0}_\delta\left((q^*F)_{\mu_0}(1 - L_+L_-)\right)\right). \tag{2.54}$$

But:

$$ch_l((q^*F)_{\mu_0}(1 - L_+L_-)) = \left[ch(q^*F)_{\mu_0}(1 - e^{\psi_+ + \psi_-})\right]_l, \tag{2.55}$$

hence

$$\sum_{a,\alpha;b,\beta} \frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l} \varphi_{\alpha,\mu_0}\overline{\psi}^a_+ \otimes \varphi_{\beta,\mu_0^I}\overline{\psi}^b_- =$$

$$= \frac{-1}{\psi_+ + \psi_-} \cdot \Delta_{\mu_0 *}\left(\left[ch(q^*F)_{\mu_0}(1-e^{\psi_+ + \psi_-})\right]_l \prod_{\gamma=1}^{i_{\mu_0}} \mathcal{C}^{\mu_0}_\delta\left((q^*F)_{\mu_0}(1-L_+L_-)\right)\right). \quad (2.56)$$

Below we prove that $\mathcal{D}_{A,B,C}$ satisfies the same second order differential equation. The partial derivative of $\mathcal{D}_{A,B,C}$ with respect to w_l equals:

$$\mathcal{D}^{-1}_{A,B,C}\frac{\partial \mathcal{D}_{A,B,C}}{\partial w_l} =$$

$$= \sum_{d,n} \frac{Q^d \hbar^{g-1}}{n!} \left\langle \mathbf{t}(\overline{\psi}_1), \ldots, \mathbf{t}(\overline{\psi}_n)\right) ; ch_l(\pi_*(ev^*_{n+1}(F) \otimes i_{\mu_0 *}\mathcal{O}_{\mathcal{Z}_\mu}) \cdot \Theta_{g,n,d}\rangle_{g,n,d}. \quad (2.57)$$

Lemma 2.6.8 shows that:

$$ch_l(\pi_*(ev^*_{n+1}(F) \otimes i_{\mu_0 *}\mathcal{O}_{\mathcal{Z}}) = \pi_* i_{\mu_0 *}\left[ev^*_{n+1} ch(F) \cdot \frac{e^{\psi_+ + \psi_-} - 1}{\psi_+ + \psi_-}\right]_{l-1}. \quad (2.58)$$

Using (2.58) and the formula :

$$\int_{[\mathcal{X}_{g,n,d}]} (\pi_* i_* a) \cdot b = \int_{[\mathcal{Z}]} a \cdot (\pi \circ i)^* b$$

we pullback the RHS of (2.57) on $\mathcal{Z}$. Moreover we use Proposition 2.6.1 to pullback the correlators on the factors $\mathcal{X}_{g_1,n_1+1,d_1} \times \mathcal{X}_{g_2,n_2+1,d_2}$.

The classes $[\mathcal{X}_{g,n,d}]^{tw}$ pullback as in formulae (2.6), (2.7) which we copy below:

$$(\pi \circ i^{red}_{\mu_0} \circ p)^*[\mathcal{X}_{g,n,d}]^{tw} = \frac{p_1^*([\mathcal{X}_{g_1,n_1+1,d_1}]^{tw}) \cdot p_2^*([\mathcal{X}_{g_2,n_2+1,d_2}]^{tw})}{(ev^*_+ \times ev^*_-)\Delta_{\mu_0 *}\left(\prod_{\delta=1}^{i_{\mu_0}} \mathcal{C}^{\mu_0}_\delta\left((q^*F)_{\mu_0}) \otimes (L_+L_- - 1)\right)\right)}. \quad (2.59)$$

$$(\pi \circ i^{irr}_{\mu_0} \circ p)^*[\mathcal{X}_{g,n,d}]^{tw} = \frac{[\mathcal{X}_{g-1,n+2,d}]^{tw}}{(ev^*_+ \times ev^*_-)\Delta_{\mu_0 *}\left(\prod_{\delta=1}^{i_{\mu_0}} \mathcal{C}^{\mu_0}_\delta\left(((q^*F)_{\mu_0}) \otimes (L_+L_- - 1)\right)\right)}. \quad (2.60)$$

As a consequence we see that if we define the coefficients $A^{\mu_0,l}_{a,\alpha;b,\beta}$ by:

$$\sum_{a,b,\alpha,\beta} A^{\mu_0,l}_{a,\alpha;b,\beta}\varphi_{\alpha,\mu_0}\overline{\psi}^a_+ \otimes \varphi_{\beta,\mu_0^I}\overline{\psi}^b_- =$$

$$= r(\mu_0)\Delta_{\mu_0 *}\left(\left[ch(q^*F)_{\mu_0} \cdot \frac{e^{\psi_+ + \psi_-} - 1}{\psi_+ + \psi_-}\right]_{l-1}\left(\prod_{\delta=1}^{i_{\mu_0}} \mathcal{C}_\delta((q^*F)_{\mu_0} \otimes (1 - L_+L_-)\right)\right), \quad (2.61)$$

we can express (2.57) as:

$$\begin{aligned}
&\mathcal{D}_{A,B,C}^{-1}\frac{\partial \mathcal{D}_{A,B,C}}{\partial w_l} = \\
&= \sum_{g_1,g_2,n_1,n_2,d_1,d_2} \frac{Q^{d_1+d_2}\hbar^{g_1+g_2-1}}{n_1!n_2!} \sum_{a,b,\alpha,\beta} \frac{1}{2}\left\langle \mathbf{t},\dots,\mathbf{t}, A^{\mu_0,l}_{a,\alpha;b,\beta}\varphi_{\alpha,\mu_0}\overline{\psi}^a_+, \Theta_{g_1,n_1+1,d_1}\right\rangle_{g_1,n_1+1,d_1} \times \\
&\times \left\langle \mathbf{t},\dots,\mathbf{t}, \varphi_{\beta,\mu_0^I}\overline{\psi}^b_-, \Theta_{g_2,n_2+1,d_2}\right\rangle_{g_2,n_2+1,d_2} + \\
&+ \sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\sum_{a,b,\alpha,\beta}\frac{1}{2}\left\langle \mathbf{t},\dots,\mathbf{t}, A^{\mu_0,l}_{a,\alpha;b,\beta}\varphi_{\alpha,\mu_0}\overline{\psi}^a_+, \varphi_{\beta,\mu_0^I}\overline{\psi}^b_-, \Theta_{g-1,n+2,d}\right\rangle_{g-1,n+2,d}. \qquad (2.62)
\end{aligned}$$

Hence the generating function $\mathcal{D}_{A,B,C}$ satisfies the equation:

$$\frac{\partial \mathcal{D}_{A,B,C}}{\partial w_l} = \frac{\hbar}{2}\sum_{a,b} A^{\mu_0,l}_{a,\alpha;b,\beta}\partial_a^{\alpha,\mu_0}\partial_b^{\beta,\mu_0^I}\mathcal{D}_{A,B,C}. \qquad (2.63)$$

Comparing (2.56) with (2.61) we see that

$$\frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l} = A^{\mu_0,l}_{a,\alpha;b,\beta}. \qquad (2.64)$$

Therefore both sides of (2.50) satisfy the same PDE. The theorem follows.

Remark 2.7.1. According to [C] (pages 91 – 95) this change of generating function corresponds to a change of polarisation, namely we regard the potential $\mathcal{D}_{A,B,C}$ as an element of the Fock space $\mathcal{H}_C = \mathcal{H}_+ \oplus \mathcal{H}_{-,C}$. The corresponding element in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with the usual polarisation is $\mathcal{G}$. If $\{q_a^{\alpha,\mu}, p_b^{\beta,\mu}\}$, $\{\overline{q}_a^{\alpha,\mu}, \overline{p}_b^{\beta,\mu}\}$ are Darboux coordinates systems on $\mathcal{H}$, respectively $\mathcal{H}_C$ then this change of polarisation is given in coordinates by:

$$\begin{aligned}
p_b^{\beta,\mu} &= \overline{p}_b^{\beta,\mu}, \\
\overline{q}_a^{\alpha,\mu} &= q_a^{\alpha,\mu} - \sum_{a,b} A^{\mu}_{a,\alpha;b,\beta}p_b^{\beta,\mu}. \qquad (2.65)
\end{aligned}$$

Example 2.7.2. Let $\mathcal{X}$ be a manifold and let $\mathcal{C}(\pi_* i_* \mathcal{O}_{\mathcal{Z}}) = Td(-\pi_* i_* \mathcal{O}_{\mathcal{Z}})^\vee$. Then $A_{a,\alpha;b,\beta}$ don't depend on α or β and we have:

$$\mathcal{C}(1 - L_+L_-) = Td^\vee(L_+L_-) = \frac{-\psi_+ - \psi_-}{1 - e^{\psi_+ + \psi_-}}.$$

This gives:

$$\sum_{a,b} A_{a,\alpha,b,\beta}\psi^a\psi^b = \frac{1}{\psi_+ + \psi_-} - \frac{1}{e^{\psi_+ + \psi_-} - 1}.$$

According to [C] the expansion of :

$$\frac{1}{1 - e^{\psi_+ + \psi_-}} = \sum_{k \geq 0} \frac{e^{k\psi_+}}{(1 - e^{\psi_+})^{k+1}}(e^{\psi_-} - 1)^k$$

gives a Darboux basis on $\mathcal{H}_{\mathcal{C}}$.

2.8 Examples and applications

Let $\mathcal{X} = X \times BG$, the stack theoretic quotient. Notation: $[\gamma_i]$ the conjugacy class of $\gamma_i \in G$, $C(\gamma)$ is the centralizer of γ. The inertia stack of X/G is the disjoint union $\coprod_i([\gamma_i], X/C(\gamma_i))$. Therefore :

$$H^*(I(X/G), \mathbb{C}) = \oplus_{[\gamma_i]} H^*(X, \mathbb{C}).$$

Denote by $e_{[\gamma_i]} := 1 \in H^*(([\gamma_i], pt/C([\gamma_i])))$. Then a basis of $H^*(([\gamma_i], X/C(\gamma_i)))$ is given by $\varphi_a \times e_{[\gamma_i]}$, where $\{\varphi_a\}$ is a basis of $H^*(X, \mathbb{C})$. The Poincaré pairing is given by :

$$(\varphi_a \times e_{[\gamma_i]}, \varphi_b \times e_{[\gamma_j]}) = \frac{\delta_{[\gamma_i][\gamma_j^{-1}]}}{|C(\gamma_i)|} \int_X \varphi_a \smile \varphi_b.$$

The J function is defined as:

$$J_{\mathcal{X}}(t, z) = -z + t(z) + \sum_{n,d} \frac{Q^d}{n!} \phi_a \langle \frac{\widetilde{\phi}^a}{-z - \overline{\psi}_1}, t(\overline{\psi}_2), \ldots, t(\overline{\psi}_n) \rangle_{n,d}^{X/G}. \tag{2.66}$$

where $\{\phi_a\}, \{\widetilde{\phi}^a\}$ are dual basis. We use results of [JK] to express the correlators in terms of correlators on $X_{0,n,d}$. In fact there is a finite degree map: $(X \times B\mathbb{Z}_m)_{0,n,d,([\gamma_1],\ldots,[\gamma_n])} \to X_{0,n,d}$. In [JK] it is shown the degree equals

$$\frac{|\chi_0^G(\gamma)|}{|G|},$$

where

$$\chi_0^G(\gamma) := \{(\sigma_1, \ldots, \sigma_n) | 1 = \prod_{j=1}^n \sigma_j, \sigma_j \in [\gamma_j] \quad for \quad all \quad j\}.$$

Since the $\overline{\psi}$ classes in the correlators are pullbacks of ψ classes from the coarse curve it follows that:

$$\langle\prod_i \overline{\psi}_i^{k_i}(ev_i^*(t_i \times e_{[\gamma_i]}))\rangle_{0,n,d}^{X/G} = \frac{|\chi_0^G(\gamma)|}{|G|}\langle\prod_i \psi_i^{k_i} ev_i^*(t_i)\rangle_{0,n,d}^{X} \tag{2.67}$$

where $t_i \in H^*(X)$.

From now on, let $G = \mathbb{Z}_m$. Denote by td_ζ the multiplicative class defined for line bundles L by:

$$td_\zeta(L) := \frac{1}{1-\zeta e^{-c_1(L)}}.$$

We twist the cohomological potential of $\mathcal{X}$ with 3 types of twisting classes as follows:

- the type $\mathcal{A}$ classes we take to be:

$$td(\pi_* ev^*(T_X)) \prod_{k=1}^{m-1} td_{\zeta^k}(\pi_* ev^*(T_X \otimes \mathbb{C}_{\zeta^k})).$$

  The effect of the type $\mathcal{A}$ twisting is:

  Corollary 2.8.1. *The cone rotates by the loop group element:*

$$\mathcal{L}^{tw} = \prod_{j=0}^{m-1}(\Box_j)\mathcal{L}_{\mathcal{X}},$$

  where we think of $\mathcal{L}_{\mathcal{X}}$ as a product of m copies of $\mathcal{L}_X$ and each operator $\Box_j$ acts on the copy corresponding to the sector labeled by g^j. Let $[kj/m]$ denote the greatest integer less than kj/m. The operators are Euler-MacLaurin expansions of the products:

$$\Box_0 = \prod_i \prod_{r=1}^{\infty} \frac{x_i - rz}{1-e^{-mx_i+mrz}},$$

$$\Box_j = \prod_{k=0}^{m-1}\prod_i\prod_{r=1}^{\infty} \frac{x_i - rz}{1-\zeta^k e^{-x_i+rz-(kj/m-[kj/m])z}}.$$

- the type $\mathcal{B}$ classes :

$$td(\pi_*(1-L_{n+1}^{-1})) \prod_{k=1}^{m-1} td_{\zeta^k}(\pi_*((1-L_{n+1}^{-1} \otimes ev^*\mathbb{C}_{\zeta^k})).$$

Corollary 2.8.2. *The dilaton shift changes from* $q(z) = \mathbf{t}(z) - z$ *to* $q(z) = \mathbf{t}(z) - (1 - e^{mz})$.

Proof: We apply Theorem 1.10.2 to the potential $\mathcal{F}$.

In our case $f_\beta = -ev_{n+1}^*(\mathbb{C}_\zeta) \otimes L_{n+1}^{-1}$ we have:

$$\frac{f_\beta(L_{n+3}^{-1}) - f_\beta(1)}{L_{n+3} - 1} = \mathbb{C}_\zeta L_{n+3}^{-1}.$$

So according to Theorem 1.10.2 (fix ζ primitive m root of unity) the translation is:

$$\begin{aligned}\mathbf{t}(z) &:= \mathbf{t}(z) + z - z\prod_{k=0}^{m-1} Td_{\zeta^k}(-\mathbb{C}_{\zeta^k}\mathbf{L}_z^{-1}) = \\ &:= \mathbf{t}(z) + z - z\frac{1-e^z}{z}\prod_{k=1}^{m-1}(1-\zeta^k e^z) = \mathbf{t}(z) + z - (1 - e^{mz}).\end{aligned} \tag{2.68}$$

- the type $\mathcal{C}$ classes we take to be: we twist by the class $Td^\vee(-\pi_* i_{g*}\mathcal{O}_{\mathcal{Z}_g})$ the nodal locus $\mathcal{Z}_g$; the locus $\mathcal{Z}_0$ of nonstacky nodes by:

$$td^\vee(-\pi_*(i_*\mathcal{O}_{\mathcal{Z}_0}))\prod_{k=1}^{m-1} td^\vee_{\zeta^k}(-\pi_*(i_*\mathcal{O}_{\mathcal{Z}_0} \otimes ev^*\mathbb{C}_{\zeta^k})).$$

We don't twist the other nodal loci.

Corollary 2.8.3. *The nodal twisting changes the polarisation in the sectors* $(\mathcal{X}, 1)$ *and* $(\mathcal{X}, g)$ *of* $I\mathcal{X}$*. The new Darboux basis are given by expansions of*

$$\frac{1}{1 - e^{m\psi_+ + m\psi_-}}$$

for $(\mathcal{X}, 1)$ and

$$\frac{1}{1 - e^{\frac{\overline{\psi}_+ + \overline{\psi}_-}{m}}} = \frac{1}{1 - e^{\psi_+ + \psi_-}}$$

for $(\mathcal{X}, g)$.

Proof.: according to Theorem 1.10.3, the coefficients $A^0_{a,\alpha,b,\beta}$ in the untwisted sector are given by:

$$-\frac{\prod_{i=0}^{m-1}\mathcal{C}^0_k(1-L_+L_-)-1}{\psi_++\psi_-} = -\frac{1}{\psi_++\psi_-}\left(\frac{\psi_++\psi_-}{\prod_{k=0}^{m-1}(1-\zeta^k e^{\psi_++\psi_-})}-1\right)$$
$$= \frac{1}{\psi_++\psi_-} - \frac{1}{e^{m\psi_++m\psi_-}-1}.$$

Then (see Example 2.7.2 and [C]) the Darboux basis is given by the expansion of $\frac{1}{1-e^{m\psi_++m\psi_-}}$. In the same way the coefficients $A^g_{a,\alpha,b,\beta}$ are given by expansion of:

$$-\frac{(Td^\vee(L_+L_--1)-1)}{\psi_++\psi_-} = \frac{1}{\psi_++\psi_-} - \frac{1}{e^{(\psi_++\psi_-)}-1}$$

and hence the polarisation is given by the expansion $\frac{1}{1-e^{\overline{\psi}_+/m+\overline{\psi}_-/m}}$.

Chapter 3

Quantum K-theory

3.1 Introduction

K-theoretic Gromov-Witten invariants have been introduced in [L2], as a tool for a better understanding of the geometry of the moduli spaces of stable maps. They are K-theoretic pushforwards to the point of some natural bundles on the moduli spaces of stable maps $X_{g,n,d}$. Most of the structure present in cohomological Gromov-Witten theory and quantum cohomology is present in its K-theoretic analogue, but there are also some essential pieces missing: the grading axiom and the divisor equation. Moreover, K-theory on orbifolds is "harder" than intersection theory, which makes the invariants harder to compute.

In this chapter we prove a Hirzebruch-Riemann-Roch type theorem which allows one to compute all genus 0 K-theoretic Gromov-Witten invariants in terms of cohomological ones. We apply Kawasaki's formula to the moduli spaces of stable maps $X_{0,n,d}$. The reason why we have to restrict ourselves to genus 0 maps is that the automorphisms of points in $X_{0,n,d}$ come only from multiple covers of the map, i.e. the domain curve has trivial automorphism group. This makes things considerably simpler. The main result is Theorem 1.6.2 and is stated in terms of the geometry of the uniruled Lagrangian cone $\mathcal{L}$ in the symplectic loop space $\mathcal{K}$. As a consequence we deduce a $\mathcal{D}$-module structure in quantum K-theory from the corresponding statement in quantum cohomology.

The material of this chapter is joint work with A. Givental. The chapter is arranged as follows. In Sections 3.2 and 3.3 be briefly recall the definitions of the moduli spaces of stable maps and of K-theoretic Gromov-Witten invariants. In Sections 3.4 and 3.5 we introduce the K-theoretic symplectic loop space $\mathcal{K}$, the K-theoretic Gromov-Witten potential and the J-function. The main tool for computing holomorphic Euler characteristics on orbifolds - Kawasaki's formula - is explained in Section 3.6. In Section 3.7 we define the "fake" K-theoretic Gromov-Witten invariants and recall their relation with cohomological ones. In Sections 3.8 and 3.9 we describe the strata of maps with symmetries in $X_{0,n,d}$ and the tangent and normal bundles to these strata. Section 3.10 contains a reformulation of Theorem 1.6.2, introducing a new object - the adelic cone $\widehat{\mathcal{K}}$. The proof of Theorem 1.6.2 is given in Section 3.11. Finally, the $\mathcal{D}$-module structure is proved in Section 3.12.

3.2 Moduli spaces of stable maps

Let X be a nonsingular complex projective variety. For $d \in H_2(X, \mathbb{Z})$ let $X_{g,n,d}$ be the moduli spaces of stable maps of degree d from n-pointed, genus g curves to X. This is a compact complex orbifold. In the case when X is a point, it coincides with the Deligne-Mumford space of stable curves $\overline{M}_{g,n}$. There are natural maps:

$$ev_i : X_{g,n,d} \to X, \quad i = 1, \ldots, n$$

given by evaluation at the ith marked point. There are line bundles

$$L_i \to X_{g,n,d}, \quad i = 1, \ldots, n$$

called universal cotangent line bundles. The fiber of L_i over the point $(C, x_1, \ldots, x_n, f)$ is the cotangent line to C at the point x_i.
There are also maps:

$$ct : X_{g,n,d} \to \overline{M}_{g,n}$$

given by forgetting the map and contracting the unstable components of the curve. The universal family can be identified with the diagram:

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{ev_{n+1}} & X \\ \pi \big\downarrow & & \\ X_{g,n,d} & & \end{array}$$

where the morphism π forgets the last marked point.

3.3 K-theoretic Gromov-Witten invariants

In [L2], Y.-P. Lee introduced the sheaf $\mathcal{O}^{vir}$ and used it to define K-theoretic Gromov-Witten invariants of X. These are holomorphic Euler charactersitics of sheaves of the form:

$$ev_1^*(a_1)L_1^{k_1} \cdot \ldots ev_n^*(a_n)L_n^{k_n} \otimes \mathcal{O}^{vir}$$

where $a_i \in K^0(X)$. Unlike the cohomological invariants, they are *integers*. We will use the notation:

$$\left\langle a_1 L^{k_1}, \ldots, a_n L^{k_n} \right\rangle_{g,n,d}^X := \chi\left(X_{g,n,d}; ev_1^*(a_1)L_1^{k_1} \cdot \ldots ev_n^*(a_n)L_n^{k_n} \otimes \mathcal{O}^{vir}\right).$$

According to [L2], the virtual structure sheaves on the spaces $X_{g,n,d}$ satisfy axioms analogue to Kontsevich-Manin's axioms in [KM] for cohomological theories. This leads to relations among K-theoretic Gromov-Witten invariants.

Example 3.3.1. The K-theoretic *string equation* can be deduced from the equality $\pi_* 1 = 1$:

$$\langle a_1, \ldots, a_n, 1 \rangle^X_{g,n+1,d} = \langle a_1, \ldots, a_n \rangle^X_{g,n,d}.$$

Example 3.3.2. The K-theoretic *dilaton equation* follows from $\pi_*(1 - L_{n+1}) = 2 - n$. This leads to:

$$\left\langle a_1 L^{k_1}, \ldots, a_n L^{k_n}, 1 - L \right\rangle^X_{0,n+1,d} = (2-n) \left\langle a_1 L^{k_1}, \ldots, a_n L^{k_n} \right\rangle^X_{0,n,d}.$$

There are however pieces of structure missing, most blatantly an analogue of the divisor equation.

3.4 The K-theoretic genus 0 potential

We define:

$$\mathcal{K} := \left[K^0(X) \otimes \mathbb{C}(q, q^{-1}) \right] \otimes \Lambda$$

where $\mathbb{C}(q, q^{-1})$ is the ring of rational functions on the complex circle with coordinate q. Elements of $\mathcal{K}$ are rational functions of q with coefficients in $K^0(X) \otimes \Lambda$ in the Q, λ-adic sense, i.e. modulo any power of the maximal ideal in the Novikov ring Λ. Let $(\ , \ )$ be the pairing on $K^0(X)$:

$$(a, b) := \chi(X, a \otimes b).$$

We endow $\mathcal{K}$ with the symplectic form:

$$\mathbf{f}, \mathbf{g} \mapsto \Omega(\mathbf{f}, \mathbf{g}) := [Res_{q=0} + Res_{q=\infty}] \left(\mathbf{f}(q), \mathbf{g}(q^{-1}) \right) \frac{dq}{q}.$$

Denote by $K^0(X)[q, q^{-1}]$ the ring of Laurent polynomials in q with coefficients in $K^0(X)$. The following two subspaces:

$$\mathcal{K}_+ = K^0(X)[q, q^{-1}] \otimes \Lambda, \qquad \mathcal{K}_- := \{ \mathbf{f} \in \mathcal{K} \quad | \quad \mathbf{f}(0) \neq \infty, \mathbf{f}(\infty) = 0 \}$$

form a Lagrangian polarisation of $\mathcal{K}$. This will allow us to identify $\mathcal{K}$ with $T\mathcal{K}_+^*$.

The subgroup of the symplectomorphisms of $\mathcal{K}$ which commute with multiplication by q are called loop group elements. They are of the form:

$$S(q) = \sum_{i \in \mathbb{Z}} S_i q^i$$

where $S_i \in End\,(K^0(X) \otimes \Lambda)$. Being a symplectomorphism amounts to:

$$S(q)S^*(q^{-1}) = I$$

where I is the identity matrix and S^* is the adjoint transpose of S. Differentiating the relation above at the identity, we see that infinitesimal loop group elements R satisfy:

$$R(q) + R^*(q^{-1}) = 0.$$

Example 3.4.1. The operator of multiplication by

$$\frac{\lambda}{1-q} - \frac{\lambda}{2} = \lambda \frac{1+q}{1-q}$$

is an infinitesimal loop group element.

Let $\mathbf{t}(q) \in \mathcal{K}_+$. The K-theoretic genus 0 potential is the following generating series:

$$\mathcal{F}_X^0 := \sum_{d,n} \frac{Q^d}{n!} \langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \rangle_{0,n,d}^X .$$

It is a formal series of $\mathbf{t}$ with coefficients in Λ. It is well defined because for any d there are finitely many monomials in coordinates t_k^a on $\mathcal{K}_+$ with nonzero coefficients.

3.5 The big J function of X

Let $\{\phi_a\}$ and $\{\phi^a\}$ be any dual basis of $K^0(X)$. Define:

$$\mathcal{J} : \mathcal{K}_+ \to \mathcal{K} \qquad , \quad \mathcal{J}(\mathbf{t}) = 1 - q + \mathbf{t}(q) + \sum_a \phi_a \sum_{d,n} \frac{Q^d}{n!} \left\langle \frac{\phi^a}{1-qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle_{0,n+1,d}^X .$$

Lemma 3.5.1. Each correlator in $\mathcal{J}$ is a reduced rational function in q without poles at $q = 0$, hence an element of $\mathcal{K}_-$.

Proof: The spaces $X_{0,n+1,d}$ are finite dimensional (virtual) orbifolds, hence their K-rings are finitely generated. This implies that there exists a minimal polynomial P' such that $P'(L^{-1}) = 0$. We can write correlators in this form:

$$\left\langle -\frac{G(L^{-1})}{L}, \ldots \right\rangle_{0,n+1,d} = \left\langle \frac{1}{2\pi i} \oint \frac{G(q)dq}{1-qL}, \ldots \right\rangle_{0,n+1,d} .$$

If G is a multiple of P' the LHS of the above equality is 0. This shows that the RHS is a rational function with denominator $P'(q^{-1})$. But P' can be written as the quotient of the form $P(q)/q^m$, where $m = deg(P') \geq 1$ and P is a polynomial in q. Hence the correlators are rational functions in q with denominator $P(q)$. Since each line bundle has an inverse, 0 is not a root of P, or in other words the correlators are different from ∞ at $q = 0$. Moreover they are 0 at $q = \infty$ q.e.d.

Notice that the first two terms in $\mathcal{J}$, $1-q$ and $\mathbf{t}(q)$ are elements of $\mathcal{K}_+$. We refer to them as *the dilaton shift* and *the input* respectively.

Proposition 3.5.2. The J function coincides with the graph of the differential of the genus 0 potential, considered as a section of the cotangent bundle of $T^*\mathcal{K}_+$, identified with $\mathcal{K}$ via the polarisation $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ and the dilaton shift $\mathbf{f} \mapsto \mathbf{f} + 1 - q$:

$$\mathcal{J}(\mathbf{t}) = 1 - q + \mathbf{t}(q) + d_{\mathbf{t}}\mathcal{F}^0_X.$$

Proof: We have already seen that that the $\mathcal{K}_+$ part of $\mathcal{J}$ coincides with the input shifted by $1-q$. Pick a variation $\delta\mathbf{t} \in \mathcal{K}_+$. By the definition of the canonical symplectic structure on $T^*\mathcal{K}_+$ it is enough to prove that the symplectic inner product of the $\mathcal{K}_-$ part of $\mathcal{J}$ is the same as $d_{\mathbf{t}}\mathcal{F}^0_X(\delta\mathbf{t})$. This follows from the following calculation (notice that $\delta\mathbf{t}$ has poles only at 0 and ∞) :

$$\Omega\left(\sum_a \phi_a \otimes \frac{\phi^a}{1-qL}, \delta\mathbf{t}\right) = -\Omega\left(\delta\mathbf{t}, \sum_a \phi_a \otimes \frac{\phi^a}{1-qL}\right) =$$
$$= -\left[Res_{q=0} + Res_{q=\infty}\right] \frac{\sum_a \delta\mathbf{t}_a(q)\phi^a}{1-q^{-1}L}\frac{dq}{q} = Res_{q=L}\frac{\delta\mathbf{t}(q)}{q-L} = \delta\mathbf{t}(L).$$

This shows that:

$$\Omega\left(\sum_a \phi_a \sum_{d,n} \frac{Q^d}{n!}\left\langle \frac{\phi^a}{1-qL}, \mathbf{t}(L), \dots, \mathbf{t}(L)\right\rangle^X_{0,n+1,d}, \delta\mathbf{t}\right) =$$
$$= \sum_{d,n}\frac{Q^d}{n!}\langle \delta\mathbf{t}, \mathbf{t}(L), \dots, \mathbf{t}(L)\rangle^X_{0,n+1,d} = d_{\mathbf{t}}\mathcal{F}^0_X(\delta\mathbf{t}).$$

Let $\mathcal{L} \subset \mathcal{K}$ be the range of the J function.

Theorem 3.5.3. *$\mathcal{L}$ is the formal germ of a Lagrangian cone such that each tangent space T to $\mathcal{L}$ is tangent to $\mathcal{L}$ exactly along $(1-q)T$. In other words $T \cap \mathcal{L} = (1-q)T$ and the tangent space at all points of $(1-q)T$ is T.*

The proof of the theorem is very similar with that of the correponding statement in [G1]. It relies on the comparison between descendant and ancestor potentials. More precisely, let $\overline{L}_i := ct^*L_i$ be pullbacks of cotangent line classes along the maps $ct : X_{0,n+l,d} \to \overline{M}_{0,n}$. Then for any $\tau \in K^0(X)$ we define the ancestor potential as:

$$\overline{\mathcal{F}}^0_\tau := \sum_{d,n}\frac{Q^d}{n!}\left\langle \mathbf{t}(\overline{L}), \dots, \mathbf{t}(\overline{L})\right\rangle^X_{0,n,d}(\tau)$$

where

$$\left\langle \mathbf{t}(\overline{L}), \ldots, \mathbf{t}(\overline{L}) \right\rangle_{0,n,d}^{X}(\tau) := \sum_{l=0}^{\infty} \frac{1}{l!} \left\langle \mathbf{t}(\overline{L}), \ldots, \mathbf{t}(\overline{L}), \tau, \ldots, \tau \right\rangle_{0,n+l,d}^{X}.$$

One then defines the ancestor J function as:

$$\overline{\mathcal{J}}_\tau := 1 - q + \mathbf{t}(q) + \sum_{a,b} \phi_a G^{ab}(\tau) \sum_{d,n} \frac{Q^d}{n!} \left\langle \frac{\phi_b}{1 - qL}, \mathbf{t}(L), \ldots, \mathbf{t}(L) \right\rangle_{0,n+1,d}^{X}(\tau).$$

Here $(G^{ab}) = (G_{ab})^{-1}$ and

$$G_{ab}(\tau) := (\phi_a, \phi_b) + \sum_{d,n} \frac{Q^d}{n!} \left\langle \phi_a, \tau, \ldots, \tau, \phi_b \right\rangle_{0,n,d}^{X}.$$

The reason of the occurence of this new tensor G_{ab} lies in the form of the WDVV equation for K-theoretic GW invariants. See [G3] for more details. Let $\overline{\mathcal{L}}_\tau \in \mathcal{K}$ be the range of $\overline{\mathcal{J}}_\tau$. We view $\overline{\mathcal{L}}_\tau$ as a Lagrangian submanifold of $(\mathcal{K}, \Omega_\tau)$ where Ω_τ is defined in the same way as Ω, replacing the pairing $(\, , \,)$ by $(\phi_a, \phi_b)_\tau = G_{ab}(\tau)$.

It turns out that $\overline{\mathcal{L}}_\tau$ is obtained from $\mathcal{L}$ by a loop group transformation. Define S_τ as a matrix with entries:

$$S_b^a = \delta_b^a + \sum_{d,n} \frac{Q^d}{n!} \sum_c G^{ac}(\tau) \left\langle \phi_c, \tau, \ldots, \tau, \frac{\phi_b}{1 - qL} \right\rangle_{0,n+2,d}^{X}.$$

Theorem 3.5.4. S_τ is a symplectomorphism $S_\tau : (\mathcal{K}, \Omega) \to (\mathcal{K}, \Omega_\tau)$ and

$$\overline{\mathcal{L}}_\tau = S_\tau \mathcal{L}.$$

The proof of the theorem is the same as of the corresponding cohomological theorem in [C], keeping in mind to replace at all times the string, dilaton, and WDVV equations with their K-theoretic counterparts.

Then we can deduce the properties of $\mathcal{L}$ described in Theorem 3.5.3 from the corresponding ones of $\overline{\mathcal{L}}_\tau$. It is essential here that the spaces $\overline{M}_{0,n}$ are manifolds of dimension $n-3$, which implies that any monomial in $(L_i - 1)$ of degree $n-3$ or more is 0. Consequently, at points $\mathbf{t} \in \mathcal{K}_+$ such that $\mathbf{t}(1) = 0$ the partial derivatives of order at most two of $\overline{\mathcal{F}}_\tau$ are 0. One can show that for each $\mathbf{f} \in \mathcal{K}_+$ there exists τ such that $S_\tau \cdot \mathbf{f} := \overline{\mathbf{f}} \in (1-q)\mathcal{K}_+$. By what we said above $(\overline{\mathbf{f}}, 0) \in \overline{\mathcal{L}}_\tau$ and $T_{(\overline{\mathbf{f}},0)} = \mathcal{K}_+$. One then sees that $\mathcal{K}_+ \cap \overline{\mathcal{L}}_\tau = (1-q)\mathcal{K}_+$. Hence $\mathcal{L}$ is ruled by the finite dimensional family (indexed by $\tau \in K^0(X)$) of subspaces $(1-q)S_\tau^{-1}\overline{\mathcal{L}}_\tau$.

The conical property of $\mathcal{L}$ is a consequence of the dilaton equation 3.3.2. More precisely, if we pick coordinates $\{t_k^a\}$ on $\mathcal{K}_+$ and denote by $\partial_{a,k}$ the corresponding partial derivatives then:

$$\sum_{d,n} \frac{Q^d}{n!} \langle \mathbf{t}(L), \ldots, \mathbf{t}(L), 1 - L \rangle_{0,n+1,d}^X = 2\mathcal{F}_X^0 - \sum t_k^a \partial_{a,k} \mathcal{F}_X^0.$$

This is equivalent to the degree two homogeneity of $\mathcal{F}_X^0$ after the dilaton shift.

3.6 Kawasaki's formula

Assume $M = \widetilde{M}/G$ is a global quotient orbifold of the manifold $\widetilde{M}$ by the finite group G. Then Lefschetz' holomorphic fixed point formula asserts that for a G-equivariant bundle E on $\widetilde{M}$ (which induces an orbibundle on M) we have:

$$\chi(M, E) = \sum_i (-1)^i dim H^i(\widetilde{M}, E)^G = \frac{1}{|G|} \sum_{g \in G} \sum_i (-1)^i tr \left(g | H^i(\widetilde{M}, E) \right).$$

Kawasaki generalized this formula to the case of orbifolds which are not global quotients, by reducing the computation of Euler characteristics on M to computation of certain cohomological integrals on *the inertia orbifold* IM of Definition 2.2.1.
Denote by M_i the connected components of the inertia orbifold (we'll often refer to them as Kawasaki strata). The multiplicity m_i associated to each M_i is given by:

$$m_i := \left| ker \left(Z_{G_p}(g) \to Aut(\widetilde{U}_p^g) \right) \right|.$$

The restriction of E to M_i decomposes in characters of the g action. Let $E_r^{(l)}$ be the subbundle of the restriction of E to M_i on which g acts with eigenvalue $e^{\frac{2\pi i l}{r}}$. Recall that the trace $Tr(E)$ is defined to be the orbibundle whose fiber over the point $(p, (g))$ of M_i is :

$$Tr(E) := \sum_l e^{\frac{2\pi i l}{r}} E_r^{(l)}.$$

$\Lambda^\bullet N_i^*$ is the K theoretic Euler class of the normal bundle N_i of M_i in M. For a line bundle L it is defined as $1 - L^\vee$. $Tr(\Lambda^\bullet N_i^*)$ is invertible because the symmetry g acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. Finally let Td be the Todd class, defined for a line bundle L as:

$$Td(L) = \frac{c_1(L)}{1 - e^{-c_1(L)}}.$$

We can now state Kawasaki's formula:

Theorem([Ka]) 3.6.1. *For any orbibundle E on M we have:*

$$\chi(M,E)=\sum_i \frac{1}{m_i}\int_{M_i} Td(T_{M_i})ch\left(\frac{Tr(E)}{Tr(\Lambda^\bullet N_i^*)}\right).$$

We call the terms corresponding to the identity component in the formula *fake Euler characteristics*:

$$\chi^f(M,E)=\int_M ch(E)Td(T_M).$$

Notice that one can rewrite Kawasaki's formula as:

$$\chi(M,E)=\sum_i \frac{1}{m_i}\chi^f\left(M_i,\frac{Tr(E)}{Tr(\Lambda^\bullet N_i^*)}\right).$$

Hence all the terms in the formula are fake Euler characteristics of certain bundles.

3.7 Fake quantum K-theory

According to what we said in the previous section, the fake Gromov-Witten invariants are in some sense an intermediate step in between the true K-theoretic ones and the cohomological ones. They are defined by :

$$\left\langle a_1L^{k_1},\dots,a_nL^{k_n}\right\rangle^f_{0,n,d}:=\int_{[X_{0,n,d}]} ch\left(ev_1^*(a_1)L_1^{k_1}\cdot\ldots\cdot ev_n^*(a_n)L_n^{k_n}\right)\cdot Td(\mathcal{T}^{vir}_{0,n,d})$$

where $\mathcal{T}^{vir}_{0,n,d}$ is the virtual tangent bundle to $X_{0,n,d}$. They coincide with the true invariants only if the spaces $X_{0,n,d}$ are virtual manifolds. In general they are rational numbers. We define the big J function as:

$$\mathcal{J}_f(\mathbf{t})=1-q+\mathbf{t}(q)+\sum_a \phi_a\sum_{d,n}\frac{Q^d}{n!}\left\langle\frac{\phi^a}{1-qL_1},\mathbf{t}(L_2),\dots,\mathbf{t}(L_{n+1})\right\rangle^f_{0,n+1,d}.$$

The loop space of the fake theory is defined as:

$$\mathcal{K}^f=\left[K^0(X)\otimes\mathbb{C}(((q-1)^{-1}))\right]\otimes\Lambda.$$

The symplectic structure is:

$$\mathbf{f},\mathbf{g}\mapsto\Omega^f(\mathbf{f},\mathbf{g})=-Res_{q=1}\left(\mathbf{f}(q),\mathbf{g}(q^{-1})\right)\frac{dq}{q}.$$

A Lagrangian polarisation for $\mathcal{K}^f$ is given by:

$$\mathcal{K}^f_+ := K^0(X)[[(q-1)]] \otimes \Lambda,$$
$$\mathcal{K}^f_- := \frac{1}{1-q} K^0(X)[[\frac{1}{1-q}]] \otimes \Lambda.$$

In fact, if we expand

$$\frac{1}{1-qL} = \sum_{k\geq 1} (L-1)^k \frac{q^k}{(1-q)^{k+1}}$$

then a Darboux basis of $\mathcal{K}^f$ is given by $\{\phi^a(q-1)^k, \, \phi_a \frac{q^k}{(1-q)^{k+1}}\}$. Just like in the case of the genuine theory, the range of the J function of the genus 0 invariants is a formal germ of an overruled Lagrangian cone, which we call $\mathcal{L}^f$.

The relation between the fake K-theoretic invariants of X and the cohomological ones has been studied in [C] and described in terms of the symplectic geometry of the loop space. Roughly speaking, the theorem says that the cones $\mathcal{L}^f$ and $\mathcal{L}^H$ (to be defined below) are related by a loop group transformation, after a suitable identification of the corresponding loop spaces. We now recall the setup of the cohomological theory: let

$$\mathcal{H} := \otimes H^*(X, \Lambda)((z))$$

be the cohomological loop space. We endow $\mathcal{H}$ with the symplectic form:

$$\Omega(\mathbf{f}, \mathbf{g}) := \oint_{z=0} (\mathbf{f}(z), \mathbf{g}(-z)) \, dz$$

where $(\, , \,)$ is the Poincaré pairing on $H^*(X)$. Consider the following polarisation of $\mathcal{H}$:

$$\mathcal{H}_+ := H^*(I\mathcal{X}, \mathbb{C})[[z]] \quad \textit{and} \quad \mathcal{H}_- := z^{-1} H^*(I\mathcal{X}, \mathbb{C})[z^{-1}].$$

Let $\psi_i = c_1(L_i)$. We define the genus 0 potential as:

$$\mathcal{F}^0_H(\mathbf{t}) := \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle_{0,n,d} \, .$$

Let $\mathbf{q}(z) = \mathbf{t}(z) - z$. Consider the graph of the genus 0 potential, regarded as a function of $\mathbf{q}$:

$$\mathcal{L}^H := \{(\mathbf{p}, \mathbf{q}) \quad | \quad \mathbf{p} = d_{\mathbf{q}} \mathcal{F}^0_H\} \subset T^*\mathcal{H}_+ \simeq \mathcal{H}.$$

Then according to [G1], $\mathcal{L}^H$ is the formal germ of an overruled cone with vertex at the shifted origin $-\mathbf{z}$. Overruled means that the tangent spaces T to $\mathcal{L}^H$ are tangent to $\mathcal{L}^H$ exactly along zT.

Let x_i be the Chern roots of T_X, and let Δ be the Euler-Maclaurin asymptotics of the infinite product:

$$\Delta \sim \prod_i \prod_{r=1}^{\infty} \frac{x_i - rz}{1 - e^{-x_i + rz}}.$$

We identify $\mathcal{K}^f$ with $\mathcal{H}$ extending the Chern character isomorphism $ch : K^0(X, \mathbb{C}) \to H^*(X, \mathbb{C})$:

$$\begin{aligned} ch : \mathcal{K}^f &\to \mathcal{H} \\ q &\mapsto e^z. \end{aligned}$$

This maps $\mathcal{K}^f_+$ to $\mathcal{H}_+$, but it doesn't map $\mathcal{K}^f_-$ to $\mathcal{H}_-$.

Theorem([C]) 3.7.1. *$\mathcal{L}^f$ is obtained from $\mathcal{L}^H$ by pointwise multiplication by Δ:*

$$\mathcal{L}^f = ch^{-1}(\Delta \mathcal{L}^H).$$

Remarks:

1. In our case

$$e^{s(x)} = \frac{x}{1 - e^{-x}}$$

which gives:

$$\Delta = \frac{1}{\sqrt{td(T_X)}} exp \left\{ \sum_{k \geq 0} \sum_{l \geq 0} s_{2k-1+l} \frac{B_2 k}{(2k)!} ch_l(T_X) z^{2k-1} \right\},$$

where the coefficients s_l and the Bernoulli numbers B_l are given by:

$$exp \left(\sum_{l \geq 0} s_l \frac{x^l}{l!} \right) = \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \sum_{l \geq 1} \frac{B_{2l}}{(2l)!} x^{2l}.$$

2. The result extends nicely to a statement about the total potentials (which we won't need), using the quantization formalism of [G2].

3. The transformations $ch : \mathcal{K}^f \to \mathcal{H}$ and $\Delta : \mathcal{H} \to \mathcal{H}$ are not symplectic, but their composition is.

4. The results of the previous chapter generalize the initial proof of [C], offering on the way more conceptual explanations for the change of dilaton shift and of polarisation.

3.8 Kawasaki strata in $X_{0,n+1,d}$

It is easily seen that points with non-trivial symmetries in $X_{0,n+1,d}$ come from maps which can be realised as multiple covers.

Example 3.8.1. Consider a point $(C, x_1, x_2, f) \in X_{0,2(d_0+d_1),2}$ such that:

- the domain C has three irreducible components, C_0, C_1, C_2 such that the nodes are $1, -1 \in C_0$ and the marked points are $0, \infty \in C_0$.
- the maps $f_{|C_1} : C_1 \to X$ and $f_{|C_2} : C_2 \to X$ are isomorphic stable maps of degree d_1.
- the map $f_{|C_0} : C_0 \to X$ factors as $C_0 \to C_0' \to X$, where the first map is given in local coordinates as $z \mapsto z^2$ and the second one has degree d_0.

Then this point has a $\mathbb{Z}_2$ symmetry given in local coordinates on C_0 as $z \mapsto -z$.

This example shows that an irreducible component of the domain is not necessarily fixed by a symmetry.

We now introduce a dictionary to help us keep track of these Kawasaki strata and of their contributions to the J function. We will use Figure 3.1 as book-keeping device for such strata. Pick C a generic domain curve in a Kawasaki stratum and denote the symmetry associated with it by g. We call the distinguished first marked point of C *the horn*. g acts with eigenvalue ζ on the cotangent line at the horn. If $\zeta = 1$ the symmetry is trivial on the irreducible component of the curve that carries the horn. We call the maximal connected connected component of the curve that contains the horn on which the symmetry is trivial *the head*. Notice that the head can be a nodal curve. Heads are parametrized by moduli spaces $X_{0,n'+1,d'}$ for some n', d'. In addition, there might be nodes connecting the head with strata of maps with nontrivial symmtries. We call these *the arms*.

Assume now that $\zeta \neq 1$, in which case it is an mth root of unity for some $m \geq 2$. Identifying the horn with 0, as in the example, we see that the other fixed point by the $\mathbb{Z}_m$ symmetry can be either a regular point, a marked point or a node. We call the maximal connected component of the curve on which g^m acts trivially and on which g acts with inverse eigenvalues on the cotangent line at each node *the stem*. The reason why we allow nodes subject to this constraint is because each such node can be smoothened while staying in the same Kawasaki stratum. So stems are chains of $\mathbb{P}^1$'s. In the last $\mathbb{P}^1$ in this chain lies the distinguished point ∞, fixed by the symmetry g. If it is a node, we call the rest of the curve connected to the stem at that node *the tail*. In addition we encounter the situation in the example above , i.e. there are m-tuples of curves $(C_1, \dots C_m)$ isomorphic as stable maps, which are permuted by the symmetry g. We call these *the legs*.

Notice that by Kawasaki's formula the input at the horn is $\frac{1}{1-q\zeta ch L_1^{1/m}}$, where L_1 is the cotangent line at the horn on the moduli spaces of stems. The contributions in the J function corresponding to a given ζ give the polar part of $\mathcal{J}$ at the pole $q = \zeta^{-1}$.

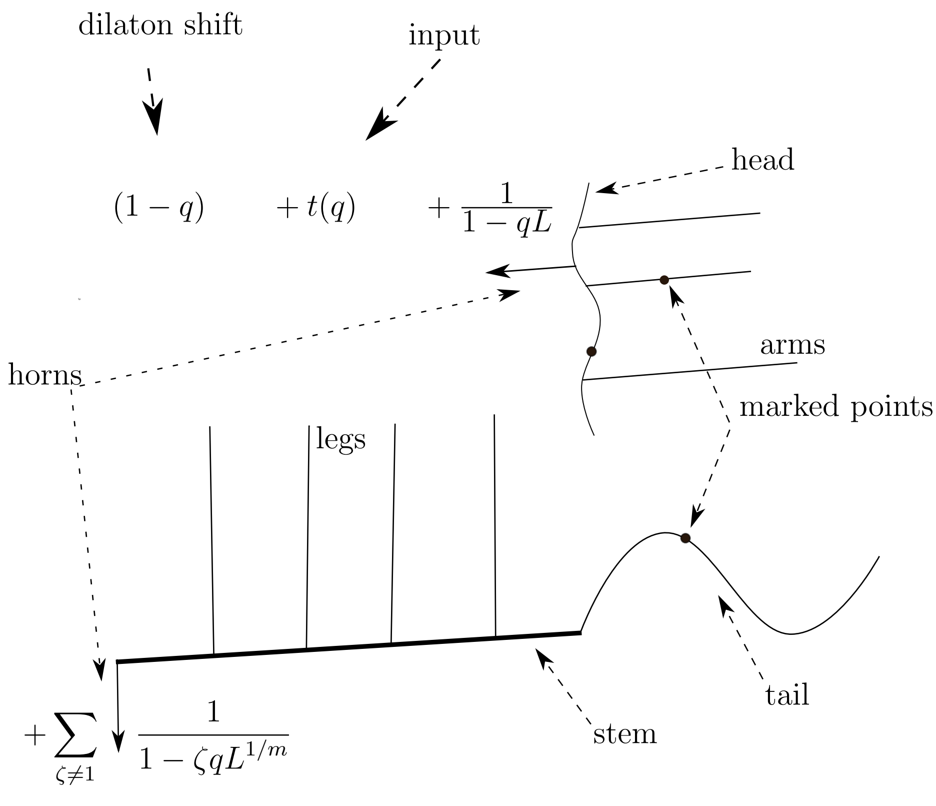

Figure 3.1: Contributions from various Kawasaki strata in the J function of X.

3.9 Stems as maps to $X/\mathbb{Z}_m$

Let $B\mathbb{Z}_m$ be the stack quotient $[pt/\mathbb{Z}_m]$. In this section we identify the stem spaces with moduli spaces of maps to the orbifold $X \times B\mathbb{Z}_m$. We use the description of maps to $\mathbb{Z}_m$ given in [JK]. Moduli spaces of orbimaps to an orbifold $\mathcal{X}$ in full generality will be introduced in the next chapter.
Let ζ be a primitive mth root of 1, and let $X_{0,n+2,d}(\zeta)$ be the stem space in $X_{0,nm+2,dm}$ which parametrizes maps $(C, x_0, \ldots, x_{nm+1}, f)$ which factor as $C \to C' \to X$ where the first map is given in coordinates as $z \mapsto z^m$, $x_0 = 0 \in C$, $x_{nm+1} = \infty \in C$ and each m-tuple $(x_{mk+1}, \ldots, x_{mk+m})$ is mapped to the same point in C'. Here ζ is the eigenvalue of the action of the generator $g \in \mathbb{Z}_m$ on the cotangent line at x_0.

Proposition 3.9.1. *The stem spaces are identified with the moduli spaces of orbimaps to the orbifold $X \times B\mathbf{Z}_m$, denoted $(X \times B\mathbb{Z}_m)_{0,n+2,d,(g,0,\ldots,0,g^{-1})}$.*

Proof: We describe stable maps to $X \times B\mathbb{Z}_m$ in a way very similar with the paper [JK], where this is done only for the case $X = point$. More precisely a map $(\mathcal{C}, x_0, x_1, \ldots, x_{n+1}, f)$ to $X \times B\mathbb{Z}_m$ of degree d is equivalent to the following data:

- a map $C \to X$ of degree d from the coarse space C of $\mathcal{C}$ to X ,
- a principal $\mathbb{Z}_m$ bundle on the complement to the set of special points of $\mathcal{C}$, possibly ramified over the nodes in a *balanced* way, i.e. such that the holonomies around the node of the two branches of the curve are inverse to each other.

The notation $(g, 0, \ldots, 0, g^{-1})$ keeps track of the holonomies of the principal bundle around the marked points, which determine it.

Remark 3.9.2. As we've seen in the previous chapter that the evaluation maps land in the rigidified inertia stack of $X \times B\mathbb{Z}_m$, whose connected components are indexed by elements of $\mathbb{Z}_m$. The sequence $(g, 0, \ldots, 0, g^{-1})$ designates the sectors picked up by the evaluation maps.

Remark 3.9.3. The geometric points of the stem space $X_{0,n+2,d}(\zeta)$ are the same as those of the moduli space $X_{0,n+2,d}$, however they are not identified as (virtual) orbifolds for the following reason: near a nodal curve of $X_{0,n+2,d}$, if we realize the branches as quotient curves then there are two copies of $\mathbb{Z}_m$ acting independently on each branch. In our moduli spaces there is one $\mathbb{Z}_m$ acting and the action is balanced as explained above.

We now describe the tangent and normal bundle to $\overline{\mathcal{M}} := X_{0,n+2,d}(\zeta)$ in $X_{0,nm+2,dm}$ in terms of the universal family $\pi : \mathcal{U} \to \overline{\mathcal{M}}$. Denote by $\widetilde{\pi} : X_{0,mn+3,md} \to X_{0,mn+2,md}$ the universal family over $X_{0,mn+2,md}$. Denote by $\widetilde{\mathcal{U}} := \widetilde{\pi}^{-1}(\overline{\mathcal{M}})$. Then the map $\widetilde{\pi} : \widetilde{\mathcal{U}} \to \overline{\mathcal{M}}$ is a $\mathbb{Z}_m$ equivariant lift of π, i.e. each fiber of $\widetilde{\pi}$ is a ramified $\mathbb{Z}_m$ cover of the corresponding fiber of π. There are also evaluation maps at the last marked point (we omit the index)

$ev : \mathcal{U} \to X/\mathbb{Z}_m$ and its $\mathbb{Z}_m$ lift $\widetilde{ev} : \widetilde{\mathcal{U}} \to X$. According to [C], the virtual tangent bundle to $X_{0,mn+2,md}$ is described as an element of the K-ring by:

$$\mathcal{T}^{vir}_{0,mn+2,dm} = \widetilde{\pi}_*(\widetilde{ev}^*(T_X - 1)) - \widetilde{\pi}_*(L^{-1}_{mn+3} - 1) - (\widetilde{\pi}_* i_*(\mathcal{O}_{\widetilde{\mathcal{Z}}}))^{\vee}.$$

We need to compute the trace of $g \in \mathbb{Z}_m$ on each piece of this bundle. Denote by $\mathbb{C}_{\zeta^k}$ the $\mathbb{Z}_m$ representation $\mathbb{C}$ where g acts by multiplication by ζ^k. K-theoretic push-forwards on orbifolds considered as global quotients extract invariants, so the piece of $\widetilde{\pi}_*(\widetilde{ev}^* T_X)$ on which g acts by ζ^{-k} can be expressed as $\pi_* ev^*(T_X \otimes \mathbb{C}_{\zeta^k})$. Therefore the trace is given by:

$$Tr\left(\widetilde{\pi}_*(\widetilde{ev}^* T_X)\right) = \sum_{k=0}^{m-1} \zeta^{-k} \pi_* ev^*(T_X \otimes \mathbb{C}_{\zeta^k}). \tag{3.1}$$

Of course the term $k = 0$ corresponds to the tangent bundle and the others to the normal bundle. Similarly:

$$Tr\left(\widetilde{\pi}_*(L^{-1}_{mn+3})\right) = \sum_{k=0}^{m-1} \zeta^{-k} \pi_*(L^{-1}_{n+3} ev^* \mathbb{C}_{\zeta^k}). \tag{3.2}$$

We denote the nodal locus in $\widetilde{\mathcal{U}}$ by $\widetilde{\mathcal{Z}}$ and in $\mathcal{U}$ by $\mathcal{Z}$. We distinguish two types of nodes. When the node is a balanced ramification point of order m then the tangent bundle is one dimensional and it is invariant (K theoretic Euler class class is $1 - L_+^{1/m} L_-^{1/m}$). If we denote by $\mathcal{Z}_g$ this nodal locus, downstairs this corresponds to twisting by the class $Td(-\pi_* i_{g*} \mathcal{O}_{\mathcal{Z}_g})^{\vee}$. If on the other hand the node is unramified then the covering curve has a $\mathbb{Z}_m$ symmetric m-tuple of nodes. The smoothing bundle has dimension m; it contains a one dimensional subspace which is tangent to the stratum and a $m - 1$ dimensional subspace normal to it. We denote by $\mathcal{Z}_0, \widetilde{\mathcal{Z}}_0$ the corresponding nodal loci and we claim that:

$$\left((\widetilde{\pi}_* i_{0*} \mathcal{O}_{\widetilde{\mathcal{Z}}_0}) \otimes \mathbb{C}_{\zeta^{-1}}\right)^{\mathbb{Z}_m} = \pi_*\left(\varphi^* \mathbb{C}_{\zeta^{-1}} \otimes i_{0*} \mathcal{O}_{\mathcal{Z}_0}\right).$$

Proof of the claim: we think of the sheaf $i_{0*}\mathcal{O}_{\widetilde{\mathcal{Z}_0}}$ as the trivial bundle on $\widetilde{\mathcal{Z}}_0$. The map $p : \widetilde{\mathcal{Z}_0} \to \mathcal{Z}_0$ is an m cover. Pushforwards of a vector bundle E along this map is the vector bundle $E \otimes \mathbb{C}^m$, where the transition matrices map $v \otimes e_i \mapsto v \otimes e_{i+1}$, or equivalently it is the regular $\mathbb{Z}_m$ representation acting on the direct sum of m copies of E. For each ζ the subbundle on which the generator of $\mathbb{Z}_m$ acts with eigen value ζ is isomorphic to E. Applying this to the trivial bundle proves the claim because:

$$\left((\widetilde{\pi}_* i_{0*} \mathcal{O}_{\widetilde{\mathcal{Z}}_0}) \otimes \mathbb{C}_{\zeta^{-1}}\right)^{\mathbb{Z}_m} = \left(\widetilde{\pi}_*(i_{0*} \mathcal{O}_{\widetilde{\mathcal{Z}}_0} \otimes \mathbb{C}_{\zeta^{-1}})\right)^{\mathbb{Z}_m} =$$
$$\pi_*\left(p_*(i_{0*} \mathcal{O}_{\widetilde{\mathcal{Z}}_0} \otimes \widetilde{\varphi}^* \mathbb{C}_{\zeta^{-1}})\right)^{\mathbb{Z}_m} = \pi_*\left(\varphi^* \mathbb{C}_{\zeta^{-1}} \otimes i_{0*} \mathcal{O}_{\mathcal{Z}_0}\right).$$

3.10 The adelic cone

In this section we give an alternative formulation of Theorem 1.6.2. For each $\zeta \neq 0, \infty$, let $\mathcal{K}^\zeta$ be the space of Laurent series in $1 - q\zeta$ with coefficients in $\mathbb{C}[[Q]] \otimes K^0(X)$. We endow it with the symplectic form:

$$\Omega^\zeta(\mathbf{f}, \mathbf{g}) = -Res_{q=\zeta^{-1}} \left(\mathbf{f}(q), \mathbf{g}(q^{-1})\right) \frac{dq}{q}.$$

Let $\mathcal{K}_+^\zeta := K^0(X)[[1 - q\zeta]] \otimes \Lambda$. Notice that we have symplectomorphisms:

$$\begin{aligned} &\varphi_\zeta : \mathcal{K}^f \to \mathcal{K}^\zeta \\ &q \mapsto q\zeta. \end{aligned}$$

The *adele space* $\widehat{\mathcal{K}}$ is defined as the subset in the cartesian product:

$$\prod_{\zeta \neq 0, \infty} \mathcal{K}^\zeta$$

consisting of collections $\{\mathbf{f}_\zeta\}$ such that, modulo any monomial Q^d, $\mathbf{f}_\zeta \in \mathcal{K}_+^\zeta$ for all but finitely many ζ. We endow it with the symplectic form:

$$\widehat{\Omega}(\mathbf{f}, \mathbf{g}) = -\sum Res_{q=\zeta^{-1}} \left(\mathbf{f}_\zeta(q), \mathbf{g}_\zeta(q^{-1})\right) \frac{dq}{q}.$$

Recall that for every $\mathbf{f} \in \mathcal{K}$ we denote by $\mathbf{f}_\zeta$ its expansion in $1 - q\zeta$ and we call it the localization at $q = \zeta^{-1}$. There is a map:

$$\widehat{\ } : \mathcal{K} \to \widehat{\mathcal{K}}, \qquad \mathbf{f} \mapsto \widehat{\mathbf{f}} := \{\mathbf{f}_\zeta\}$$

where we take all localizations at $\zeta^{-1} \neq 0, \infty$. The map is symplectic:

$$\Omega(\mathbf{f}, \mathbf{g}) = \widehat{\Omega}(\widehat{\mathbf{f}}, \widehat{\mathbf{g}}),$$

as can be easily seen from the definitions of Ω and $\widehat{\Omega}$.
Assume now we have a collection of overruled Lagrangian cones $\mathcal{L}^\zeta \subset (\mathcal{K}^\zeta, \Omega^\zeta)$ such that modulo any power of Novikov variables $\mathcal{L}^\zeta = \mathcal{K}_+^\zeta$ for all but finitely many values of ζ. Then their product $\prod_{\zeta \neq 0, \infty} \mathcal{L}^\zeta \subset \widehat{\mathcal{K}}$ is an adelic overruled Lagrangian cone.
In fact, one of the properties of being overruled is invariance with respect to multiplication by $1 - q$. Since $1 - q$ is an invertible element of $\mathcal{K}^\zeta$ for all $\zeta \neq 1$, we see that each such $\mathcal{L}^\zeta$ coincides with its tangent space at each point, i.e. it is linear subspace. We can now restate Theorem 1.6.2:

Theorem 3.10.1. *The image $\widehat{\mathcal{L}} \subset \widehat{\mathcal{K}}$ of $\mathcal{L} \subset \mathcal{K}$ under the map $\widehat{}$ followed by pointwise completion is an adelic overruled Lagrangian cone*

$$\widehat{\mathcal{L}} = \prod_{\zeta \neq 0,\infty} \mathcal{L}^\zeta$$

such that $\mathcal{L}^\zeta = \mathcal{K}_+^\zeta$ unless ζ is a root of unity, $\mathcal{L}^\zeta = \mathcal{L}^f$ if $\zeta = 1$ and $\mathcal{L}^\zeta = \nabla_\zeta \varphi_\zeta(\mathcal{T})$ if $\zeta \neq 1$ is a root of unity, where ∇_ζ and $\mathcal{T}$ are as described in Theorem 1.6.2.

3.11 The proof of Theorem 1.6.2

Proposition 3.11.1. *The localisation $\mathcal{J}_1$ lies on the cone $\mathcal{L}^f$.*

Proof: according to the discusion in Section 3.8, the $\mathcal{K}_-^f$ part of $\mathcal{J}_1$ is the sum of correlators for which the horn lies on heads. Denote by $\widetilde{\mathbf{t}}$ the sum of correlators for which $\zeta \neq 1$. Then we have:

$$\mathcal{J}_1 = 1 - q + \mathbf{t} + \widetilde{\mathbf{t}} + \sum_{n,m,d} \frac{Q^d}{n!}\frac{1}{m!} \sum_a \phi_a \left\langle \frac{\phi^a}{1-qL}, \mathbf{t}(L), \dots \mathbf{t}(L), \widetilde{\mathbf{t}}(L), \dots, \widetilde{\mathbf{t}}(L) \right\rangle^f_{0,n+m+1,d}. \tag{3.3}$$

where there are n occurences of $\mathbf{t}$ and m of $\widetilde{\mathbf{t}}$ in the correlators.
The reason why this is true is because each of the special points on the irreducible component that carries the horn is either a marked point or a node connecting it to an arm. If it is a marked point the input in the correlator is $\mathbf{t}(L)$. If it a node, it is known that the Euler class of the normal direction to the stratum which smoothens the node is $1 - L_+L_-$ where L_-, L_+ are the cotangent lines to the head and arm respectively. The input is therefore:

$$\sum_a \frac{\phi^a \otimes \phi_a}{1 - L_- Tr(L'_+)}.$$

The node becomes the horn for the integral on the arm. When we sum after all such possibilities, the contribution is $\widetilde{\mathbf{t}}(q)$ at the point $q = L_-$. The factor $\frac{1}{n!m!}$ in front of the correlators is combinatorial, it accounts for choosing which are the marked points and occurs because $\frac{1}{(n+m)!}\binom{n+m}{m} = \frac{1}{n!m!}$. But we can rewrite (3.3) as:

$$\mathcal{J}_1 = 1 - q + \mathbf{t} + \widetilde{\mathbf{t}} + \sum_{a,n',d} \phi_a \frac{Q^d}{n'!} \left\langle \frac{\phi^a}{1-qL}, \mathbf{t}(L) + \widetilde{\mathbf{t}}(L), \dots, \mathbf{t}(L) + \widetilde{\mathbf{t}}(L) \right\rangle^f_{0,n'+1,d} = \mathcal{J}_f(\mathbf{t} + \widetilde{\mathbf{t}}). \tag{3.4}$$

This proves the proposition.

We now explain the leg contrbiutions, which we denote $\mathbf{T}$. Recall that ψ^m are Adams' operations which are ring morphisms $\psi^m : K^0(X) \to K^0(X)$ which map line bundles E to $E^{\otimes m}$. We extend ψ^m on $\mathcal{K}$ by seting $\psi^m(q) = q^m$, $\psi^m(Q^d) = Q^{md}$.

Lemma 3.11.2. *Let* $\widetilde{\mathbf{T}}$ *be the arm contributions computed at the input* $\mathbf{t}(q) = 0$*. Then:*

$$\mathbf{T}(L) = \psi^m \left(\widetilde{\mathbf{T}}(L) \right).$$

Proof: The symmetry g^m acts nontrivially at the cotangent line to each copy of the leg because otherwise they are degenerations inside a higher degree stem space. When we sum after all possible contributions each copy of the leg we get the arm contributions. As noted, the legs are not allowed to ontain marked points, hence the input is $\mathbf{t} = 0$. Since we have m copies of each leg the contribution is $Tr(g|\widetilde{\mathbf{T}}(L)^{\otimes m})$. The proposition then follows from:

Lemma 3.11.3. Let V be a vector bundle. Then

$$Tr\left(g|V^{\otimes m}\right) = \psi^m(V).$$

Let $N = rk(V)$. It is enough to prove the lemma for the case when V is the universal $U(N)$ bundle on $BU(N)$ as every vector bundle is induced by pullback from this one. As $BU(N)$ is the homotopy quotient $pt/U(N)$, its K-theory ring is identified with the ring of representations of $U(N)$. The universal bundle corresponds to the standard representation of $U(N)$ on $\mathbb{C}^N$. Let $h \in U(N)$ and let $\vec{e_i}$ be eigenvectors of h with eigenvalues x_i. Then we compute the character of h on $Tr(g|V^{\otimes m})$, regarded as a $U(N)$ representation. This is equal to the $Tr(gh^{\otimes m})$ on $V^{\otimes m}$, because g and $h^{\otimes m}$ commute. But the matrix $gh^{\otimes m}$, written with respect to the basis $\vec{e}i_1 \otimes \cdots \otimes \vec{e}_{i_m}$, has zero diagonal entries unless $i_1 = \ldots = i_m$, in which case the entry is $x_{i_1}^m$. Thus:

$$Tr(gh^{\otimes m}) = x_1^m + \cdots + x_N^m.$$

This equals the trace of h on $\psi^m(V)$, q.e.d.

Proposition 3.11.4. *Let* ζ *be a root of unity. The localisation* $\mathcal{J}_\zeta$ *near* $q = \zeta^{-1}$ *is a tangent vector to the cone of some "twisted" fake theory (after identifying the loop spaces using the Chern isomorphism). The application point is the leg* $\mathbf{T}$*.*

Proof: Denote by $\delta\mathbf{t}(q)$ the sum of terms in $\mathcal{J}$ which don't have a pole at $q = \zeta^{-1}$. Then we can write:

$$\mathcal{J}_\zeta(\mathbf{t}) = \delta\mathbf{t}(q) + \sum_{a,n,d} \phi_a \frac{Q^{dm}}{n!} \left\langle \frac{\phi^a}{1 - q\zeta L^{1/m}}, \mathbf{T}(L), \ldots, \mathbf{T}(L), \delta\mathbf{t}(\zeta^{-1} L^{\frac{1}{m}}); Tr(\Lambda^* N_{g,n,d}) \right\rangle_{0,n+2,d}^{X/\mathbb{Z}_m, f} \tag{3.5}$$

where $N_{g,n,d}$ is the trace of the Euler class of the normal bundle to each stem space. Remember that g acts by ζ on the cotangent line at the first marked point, which explains the denominator $1 - q\zeta L^{1/m}$ of the input at that point in the correlators. We now explain the input $\delta\mathbf{t}(\zeta^{-1} L^{\frac{1}{m}})$ at the second branch point ∞. If ∞ is a marked point, then the input is $\mathbf{t}(\zeta^{-1}L^{1/m})$. If it is a nonspecial point of the original curve, than I claim the input is $1-\zeta^{-1}L^{1/m}$. For this look at the diagram (assume $n=0$ for symplicity, since the presence of legs doesn't change the following argument):

$$\begin{array}{ccc} X_{0,2,d}(\zeta) & \xrightarrow{i} & X_{0,2,dm} \\ {\scriptstyle ft_2}\big\downarrow & & {\scriptstyle ft_2}\big\downarrow \\ X_{0,2,d}(\zeta) & \xrightarrow{i} & X_{0,1,dm}. \end{array}$$

The restriction of ft_2 to the Kawasaki stratum $X_{0,2,d}(\zeta)$ is an isomorphism so the conormal bundle of $X_{0,2,d}(\zeta)$ in $X_{0,2,dm}$ (denote it $\overline{N}^*$) is the direct sum of the conormal bundle of $X_{0,2,d}(\zeta)$ in $D_2 := \sigma_2(X_{0,1,dm})$ (which is the same as the conormal bundle of $X_{0,2,d}(\zeta)$ in $X_{0,1,dm}$ - call it N^*) and the conormal bundle of D_2 in $X_{0,2,dm}$. Taking equivariant Euler classes we get:

$$\Lambda^*(\overline{N}^*) = \Lambda^*(N^*)(1-\zeta^{-1}L_2^{1/m}).$$

Hence integrals on $X_{0,2,d}(\zeta)$ viewed as a Kawasaki stratum in $X_{0,1,dm}$ can be expressed as integrals on the stem space with the input $1-\eta^{-1}L^{1/m}$ at ∞. Finally when ∞ is a node, then the input is the polar part of $\delta\mathbf{t}(\zeta^{-1}L^{1/m})$.
The reason why we view as $\mathcal{J}_\zeta(\mathbf{t})$ as a tangent vector to a Lagrangian cone is that we can identify tangent spaces to cones of theories with first order derivatives of their J function. Taking the derivative of J in the direction of $\vec{v}(q)$ replaces the input by $\vec{v}(q)$ and one seat in the correlators by $\vec{v}(L)$.
Although the correlators are on $X/\mathbb{Z}_m$, we will see soon that we can identify this generating series with a tangent space to the cone of a twisted theory on X.

Notice that we already proved conditions 1 and 2 in Theorem 1.6.2. Before attacking part 3, we prove the following:

Proposition 3.11.5. *$ch^{-1}(\Box_0\mathcal{L}^H) = \psi^m(\mathcal{L}^f)$, where $\Box_0$ is the operator in Corollary* 2.8.1 *and the Adams operation $\psi^m : \mathcal{K}^f \to \mathcal{K}^f$ acts on q by $\psi^m(q) = q^m$.*

Proof: we first show that:

$$\Box_0 = m^{1/2dim_{\mathbb{C}}X}\psi^m(\triangle)e^{-(logm)c_1(T_X)/z}. \tag{3.6}$$

Let x_i be the Chern roots of T_X. Note that $\triangle$ and $\Box_0$ are Euler-Maclaurin asymptotics of infinite products:

$$\triangle = \prod_{x_i}\prod_{r=1}^{\infty} s_1(x_i - rz),$$

$$\Box_0 = \prod_{x_i}\prod_{r=1}^{\infty} s_2(x - rz),$$

where

$$s_1(x) = \frac{x}{1-e^{-x}} \quad and \quad s_2(x) = \frac{x}{1-e^{-mx}} = \frac{1}{m}\psi^m(s_1(x)).$$

It follows that

$$lns_2(x) = -log(m) + \psi^m lns_1(x).$$

But from the definition of Euler-Maclaurin expansion we see $-logm$ influences only the terms

$$\sum_i \left[\int_0^{x_i} (-logm)dt/z + logm/2\right] = \frac{(-logm)c_1(T_X)}{z} + dim(X)\frac{logm}{2}.$$

since the sum is taken after Chern roots of T_X. Formula (3.6) follows.
Going back to the proof of 3.11.5, we know that:

$$\triangle^{-1}ch(\mathcal{J}_f^X) = J_X^H. \tag{3.7}$$

We use the chern character to define the Adams operation in cohomology:

$$\psi^m(a) := ch\left(\psi^m(ch^{-1}a)\right).$$

Notice that if a is homogeneous then $\psi^m(a) = m^{deg(a)/2}a$.
The J function J_X^H has degree two with respect to the grading $deg(z) = 2$, $deg(Q^d) = 2\int_d c_1(T_X)$, and the usual grading in cohomology. Therefore if we write $J_X^H = -z\sum_d J_d Q^d$ then $deg(J_d) = -deg(Q^d)$, hence:

$$\psi^m(J_X^H) = \sum_d m^{1-\int_d c_1(T_X)}(-z)J_d Q^d. \tag{3.8}$$

We can rewrite this as:

$$m^{-1}\psi^m(J_X^H) = \sum_d e^{-log(m)\int_d c_1(T_X)}(-z)J_d Q^d. \tag{3.9}$$

We now use the divisor equation (see [C]), to write the RHS above as:

$$\sum_d e^{-log(m)c_1(T_X)/z}(-z)J_d Q^d = e^{-log(m)c_1(T_X)/z}J_X^H. \tag{3.10}$$

Now combine (3.6) and (3.10) to write:

$$\begin{aligned}\Box_0 J_X^H &= m^{1/2dim_{\mathbb{C}}X}\psi^m(\triangle)e^{-(logm)c_1(T_X)/z}J_X^H = \\ &= m^{1/2dim_{\mathbb{C}}X}\psi^m(\triangle)m^{-1}\psi^m(J_X^H) = m^{-1+1/2dim_{\mathbb{C}}X}\psi^m(\triangle J_X^H).\end{aligned}$$

This proves the proposition because the range of $c \triangle J_X^H$ is the cone $ch(\mathcal{L}^f)$, for any scalar $c \in \mathbb{C}$.

We now complete the proof of Theorem 1.6.2. Before that we introdue more notation: we index the components of $I(X/\mathbb{Z}_m)$ by elements $g \in \mathbb{Z}_m$. We write:

$$H^*(IX/\mathbb{Z}_m, \mathbb{C}) := \oplus_{g \in \mathbb{Z}_m} H^*(X/\mathbb{Z}_m, \mathbb{C}) e_g.$$

For cohomology classes in the identiy sector we will drop the element e_1 from the notation. We now introduce the following generating series:

$$J_{X/\mathbb{Z}_m} := -z + t(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\widetilde{\phi}^a}{-z - \overline{\psi}_1}, \mathbf{t}(\psi_2), \ldots, \mathbf{t}(\psi_{n+1}) \right\rangle_{0,n+1,d}^{X/\mathbb{Z}_m}$$

$$\delta J_{X/\mathbb{Z}_m} := \delta t(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\widetilde{\phi}^a e_g}{-z - \overline{\psi}_1}, \mathbf{t}(\psi_2), \ldots, \delta\mathbf{t}(\psi_{n+2}) e_{g^{-1}} \right\rangle_{0,n+2,d}^{X/\mathbb{Z}_m}$$

where $\{\phi_a\}$ and $\{\widetilde{\phi}^a\}$ are dual basis with respect to the Poincaré pairing on $I(X/\mathbb{Z}_m)$. It follows from formula 2.67 that

$$J_{X/\mathbb{Z}_m} = J_X, \qquad \textit{where} \qquad J_X(z, t(z)) \in H^*((1, X))((z^{-1})) \simeq \mathcal{H}_X,$$

$$\delta J_{X/\mathbb{Z}_m} = \delta \mathbf{t}(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\widetilde{\phi}^a}{-z - \psi_1}, \mathbf{t}(\psi_2), \ldots, \delta\mathbf{t}(\psi_{n+2}) \right\rangle_{0,n+2,d}^{X}.$$

We now define their twisted counterparts :

$$J^{tw}_{X/\mathbb{Z}_m} := -z + t(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\widetilde{\phi}^a}{-z - \overline{\psi}_1}, \mathbf{t}(\psi_2), \ldots, \mathbf{t}(\psi_{n+1}); \Theta_{g,n,d} \right\rangle_{0,n+1,d}^{X/\mathbb{Z}_m},$$

$$\delta J^{tw}_{X/\mathbb{Z}_m} := \delta t(z) + \sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!} \left\langle \frac{\widetilde{\phi}^a e_g}{-z - \overline{\psi}_1}, \mathbf{t}(\psi_2), \ldots, \delta\mathbf{t}(\psi_{n+2}) e_{g^{-1}}; \Theta_{g,n,d} \right\rangle_{0,n+2,d}^{X/\mathbb{Z}_m},$$

where $\Theta_{g,n,d}$ is the twisting of Section 2.8 by the following classes:

$$\Theta_{g,n,d} := td(\pi_* ev^*(T_X)) \prod_{k=1}^{m-1} td_{\zeta^k}(\pi_* ev^*(T_X \otimes \mathbb{C}_{\zeta^k})).$$

.

Proposition 3.11.6. *The series $J^{tw}_{X/\mathbb{Z}_m}$ lies in the overruled Lagrangian cone $\Box_0 \mathcal{L}_X^H$ and $\delta J^{tw}_{X/\mathbb{Z}_m}$ lies in the tangent space $\Box_1 \mathcal{T}_{\Box_0^{-1} J^{tw}_{X/\mathbb{Z}_m}} \mathcal{L}_X^H$, where $\Box_0, \Box_1$ are the operators introduced in Section 2.8.*

Proof: This follows from Corollary 2.8.1: the range of $J^{tw}_{X/\mathbb{Z}_m}$ is the part of the untwisted sector of the cone $\mathcal{L}^{tw}$, which gets rotated by $\square_0$. The tangent vector $\delta J^{tw}_{X/\mathbb{Z}_m}$ on the other hand pertains to the sector indexed by g.

We introduce one more generating series:

$$\delta\mathcal{J}^{st}(\delta\mathbf{t},\mathbf{T}) = \delta\mathbf{t}(q)+$$
$$+\sum_a \phi_a \sum_{n,d} \frac{Q^d}{n!}\left\langle \frac{\phi^a}{1-q^{1/m}L^{1/m}}, \mathbf{T}(L),\ldots,\mathbf{T}(L), \delta\mathbf{t}(\zeta^{-1}L^{\frac{1}{m}}); Tr(\Lambda^* N_{g,n,d})\right\rangle^{X/\mathbb{Z}_m,f}_{0,n+2,d}. \quad (3.11)$$

Notice that if we do the change of variables $q^{1/m} \mapsto q\zeta$ and $Q^d \mapsto Q^{md}$ we obtain the localisation $\mathcal{J}_\zeta$.

Proposition 3.11.7. *$ch\,(\delta\mathcal{J}^{st}(\delta\mathbf{t},\mathbf{T}))$ lies in the subspace $\square_1\square_0^{-1}\mathcal{T}_{J^{tw}_{X/\mathbb{Z}_m}}\square_0\mathcal{L}^H$ where the input $\mathbf{T}$ is related to the application point $J^{tw}_{X/\mathbb{Z}_m}$ by the projection $[\ldots]_+$ along the polarisation pertaining to the identity sector in Corollary* 2.8.3*:*

$$ch[1-q^m+\mathbf{T}(q)] = [J^{tw}_{X/\mathbb{Z}_m}]_+.$$

Proof: According to the description of $N_{g,n,d}$ of Section 3.9, $ch(\delta\mathcal{J}^{st})$ is obtained from $\delta J^{tw}_{X/\mathbb{Z}_m}$ by twisting by the type $\mathcal{B}$ and $\mathcal{C}$ classes of Corollaries 2.8.2 and 2.8.3. Therefore it lies in the same space as $\delta J^{tw}_{X/\mathbb{Z}_m}$, which according to the Proposition 3.11.6 is $\square_1\square_0^{-1}\mathcal{T}_{J^{tw}_{X/\mathbb{Z}_m}}\square_0\mathcal{L}^H$.

However, the dilaton shift (see Corollary 2.8.2) changes from $-z$ to $1-e^{mz}$, and so does the space $\mathcal{H}_-$ of the polarisation. Changing the input at the first marked point from $\widetilde{\phi^a}/-z-\psi = \phi^a/(-z/m-\psi/m)$ to $\phi^a/(1-e^{(z+\psi)/m})$ is equivalent to applying to the same space the polarisation pertaining to the sector $g\in\mathbb{Z}_m$. The input $\mathbf{T}$ is related to $J^{tw}_{X/\mathbb{Z}_m}$ by:

$$ch[\mathbf{T}(q)]_+ = [J^{tw}_{X/\mathbb{Z}_m}]_+ - 1 + e^{mz} \quad (3.12)$$

due to the new polarisation and dilaton shift.

Proposition 3.11.8. *$\delta\mathcal{J}^{st}(\delta\mathbf{t},\mathbf{T})$ lies in the space $\square_1\triangle^{-1}\mathcal{T}_{\mathcal{J}_f(\widetilde{\mathbf{T}})}\mathcal{L}^f$, where $\mathbf{T}=\psi^m(\widetilde{\mathbf{T}})$.*

Proof: $ch\,(\delta\mathcal{J}^{st}(\delta\mathbf{t},\mathbf{T}))$ lies in the space $\square_1\square_0^{-1}\mathcal{T}_{J^{tw}_{X/\mathbb{Z}_m}}\square_0\mathcal{L}^H$ according to Proposition 3.11.7. But $J^{tw}_{X/\mathbb{Z}_m}=\square_0 J^H_X$, and (up to a scalar) we also have from 3.11.5 that $ch(J^{tw}_{X/\mathbb{Z}_m}) = \psi^m(\mathcal{J}_f)$. The input of $\mathcal{J}_f$ is determined as:

$$\widetilde{\mathbf{T}} = (\mathcal{J}_f)_+ - (1-q),$$

where $(\ldots)_+$ means projection along the space $\mathcal{K}^f_-$. Comparing with (3.12) and recalling from Corollary 2.8.3 that the negative space of the identity sector is $ch^{-1}(\psi^m(\mathcal{K}^f_-))$, and that $\psi^m(1-q)=1-q^m$ we see that we must have $\mathbf{T}=\psi^m(\widetilde{\mathbf{T}})$. Replacing:

$$\square_0 J^H_X = \square_0\triangle^{-1} ch(\mathcal{J}_f)$$

we see that $\delta\mathcal{J}^{st}(\delta\mathbf{t},\mathbf{T})$ lies in the space $\Box_1\,\triangle^{-1}\,\mathcal{T}_{\mathcal{J}_f(\widetilde{\mathbf{T}})}\mathcal{L}^f$ q.e.d.

Now notice that

$$\Box_0\triangle^{-1}=\prod_{x_i}\prod_{r=1}^{\infty}\frac{1-q^re^{-x}}{1-\zeta^{-r}q^{r/m}e^{-x}}.$$

The operator ∇_ζ of Theorem 1.6.2 is obtained by the change of variable $q^{1/m}\mapsto q\zeta$ in the formula above and computing the Euler-Maclaurin asymptotics as $q\zeta\to 1$. But the same change of variable and $Q^d\mapsto Q^{md}$ transforms $\delta\mathcal{J}^{st}(\delta\mathbf{t},\mathbf{T})$ to the localisation $\mathcal{J}_\zeta(\mathbf{t})$. Combining this with Proposition 3.11.8 we see that $\nabla_\zeta^{-1}\mathcal{J}_\zeta(\mathbf{t})$, after the change $q\mapsto q\zeta^{-1}$, lies in the subspace $\mathcal{T}$ of $\mathcal{L}^f$ obtained from the tangent space $\mathcal{T}_{\mathcal{J}_1(0)}\mathcal{L}^f$ by the change of variable $q^{1/m}\mapsto q$ and $Q^d\mapsto Q^{md}$.

This concludes the proof of Theorem 1.6.2.

Proposition 3.11.9. The Theorem 1.6.2 determines $\mathcal{J}$ in terms of head and stem correlators.

Proof: we use induction on the degree d of Novikov's monomials Q^d. The Deligne-Mumford spaces $\overline{M}_{0,n}$ are manifolds, hence in degree $d=0$ there are only head contributions to $\mathcal{J}(\mathbf{t})$. Assume now we have computed $\mathcal{J}(\mathbf{t})$ for all $d<d_0$. We can compute $\mathcal{J}(0)$ up to degree d_0: if the head has degree 0, then it suffices to know the arms up to degree $<d$, since there are two arms attached and since the arms have positive degrees. Also, when the stem and the tail have degree 0, and there is only one leg attached, we can recover the information about the leg from that of the arm up to degree $d_0/m<d_0$.

We can now project $\mathcal{J}_1(0)$ and $\mathcal{J}_\zeta(0)$ to $\mathcal{K}_+^f$ and $\mathcal{K}_+^\zeta$ respectively to reconstruct the arm $\widetilde{\mathbf{T}}(q)$ and the tail $\delta\mathbf{t}(q)$ up to degree d_0. But if we know $\widetilde{\mathbf{T}}$ we can reconstruct the leg $\mathbf{T}$ up to degree d_0. Hence we know all contributions and we can recover $\mathcal{J}(\mathbf{t})$ up to degree d_0.

Corollary 3.11.10. The Theorem 1.6.2 expresses genuine K-theoretic invariants of X in terms of cohomological invariants.

Proof: this follows from the previous corollary, combined with the twisting formulae which express head correlators in terms of cohomological ones, and stem correlators in terms of the cohomological ones of $X/\mathbb{Z}_m$. But it is known ([JK]) how cohomological GW invariants of $X/\mathbb{Z}_m$ are related to those of X, q.e.d.

Corollary 3.11.11. Two points $\mathbf{f},\mathbf{g}\in\mathcal{L}$ lie in the same ruling space of $\mathcal{L}$ if and only if their expansions $\mathbf{f}_1,\mathbf{g}_1$ near $q=1$ lie in the same ruling space of $\mathcal{L}^f$.

Proof: If $\mathbf{f}_1$ and $\mathbf{g}_1$ lie in the same ruling space of $\mathcal{L}^f$, then $\varepsilon\widehat{\mathbf{f}}+(1-\varepsilon)\widehat{\mathbf{g}}\in\widehat{\mathcal{L}}$ for each value of ε and therefore, by the theorem, the whole line $\varepsilon\widehat{\mathbf{f}}+(1-\varepsilon)\widehat{\mathbf{g}}$ lies in $\mathcal{L}$. The converse is also true: if the line through $\mathbf{f},\mathbf{g}$ lies in $\mathcal{L}$ then the line through $\mathbf{f}_1$, $\mathbf{g}_1$ lies in $\mathcal{L}^f$. It remains to notice that ruling subspaces of $\mathcal{L}$ and $\mathcal{L}^f$ are *maximal* linear subspaces of these cones - because this is true modulo Novikov's variables, i.e. in classical K-theory.

3.12 Floer's S^1 equivariant K-theory and $\mathcal{D}_q$ modules

In this section we show that tangent spaces to the overruled Lagrangian cone $\mathcal{L}$ carry a natural structure of modules over a certain algebra $\mathcal{D}_q$ of finite difference operators with respect to Novikov's variables. This structure, although manifest in examples ([GL]) and predictable on heuristic grounds of S^1- equivariant Floer theory ([G4], [G5]), has been missing so far in the realm of K-theoretic Gromov-Witten invariants. We first recall the heuristics and then derive the $\mathcal{D}_q$ invariance of tangent spaces to $\mathcal{L}$ from the divisor equation in quantum cohomology theory and the HRR Theorem 1.6.2.

Let X be a compact symplectic (or Kähler) target space, which is assumed simply-connected in this discussion, so that $\pi_2(X) = H_2(X)$. Let $k = rkH_2(X)$, let $d = (d_1, \dots, d_k)$ be integer coordinates on $H_2(X, \mathbb{Q})$, and let $\omega_1, \dots, \omega_k$ be closed two forms on X with integer periods, representing the corresponding basis in $H^2(X, \mathbb{R})$.

On the space $L_0(X)$ of parametrized loops $S^1 \to X$, as well as on its universal cover $\widetilde{L_0X}$, one defines closed two forms Ω_a, that to two vector fields ξ and η along a given loop associates the value:

$$\Omega_a(\xi, \eta) = \oint \omega_a\left(\xi(t), \eta(t)\right) dt.$$

A point $\gamma \in \widetilde{L_0X}$ is a loop in X together with a homotopy type of disk $u : D^2 \to X$ attached to it. One defines the *actionn functionals* $H_a : \widetilde{L_0X} \to X$ by evaluating the 2-forms ω_a on such disks:

$$H_a(\gamma) := \int_{D^2} u^* \omega_a.$$

Consider the action of S^1 on $\widetilde{L_0X}$ defined by the rotation of loops and let V be the velocity vector field of this action. It is well known that V is Ω_a - Hamiltonian with Hamilton function H_a, i.e.:

$$i_V(\Omega_a) + dH_a = 0, \quad a = 1, \dots, k.$$

Denote by z the generator of the coefficient ring $H^*(BS^1)$ of S^1-equivariant cohomology theory. The S^1-equivariant De Rham complex (of $\widetilde{L_0X}$ in our case) consists of S^1 equivariant differential forms with coefficients in $\mathbb{R}[z]$, and is equipped with the differential $D := d + zi_V$. Then:

$$p_a := \Omega_a + zH_a, \dots a = 1, \dots, k,$$

are degree 2 S^1-equivariantly closed elements of the complex: $Dp_a = 0$. This is a standard fact that usually accompanies the formula of Duistermaat-Heckman.

Furthermore, the lattice $\pi_2(X)$ acts by deck transformations on the universal cover $\widetilde{L_0X} \to L_0X$. Namely, an element $d \in \pi_2(X)$ acts on $\gamma \in \widetilde{L_0X}$ by replacing the homotopy type $[u]$ of the disk with $[u]+d$. We denote by $Q^d = Q_1^{d_1} \cdots Q_k^{d_k}$ the operation of pulling-back differential forms by this deck transformation. It is an observation from [G4], [G5] that the operations Q_a and the operations of exterior multiplication by p_a do not commute:

$$p_a Q_b - p_b Q_a = -\delta_{ab} z Q_a.$$

These are commutation relations between generators of the algebra of differential operators on the k-dimensional torus:

$$[-z\partial_{\tau_a}, e^{\tau_b}] = -\delta_{ab} z e^{\tau_a}.$$

Likewise, if P_a denotes the S^1-equivariant line bundle on $\widetilde{L_0X}$ whose Chern character is e^{-p_a}, then tensoring vector bundles by P_a and pulling back vector bundles by Q_a do not commute:

$$P_a Q_b = \delta_{ab} q Q_a P_b.$$

These are commutation relations in the algebra of finite-difference operators, generated by multiplications and translations:

$$Q_a \mapsto e^{\tau_a}, \quad P_a \mapsto e^{z\partial_{\tau_a}} = q^{\partial_{\tau_a}}, \quad where \quad q = e^z.$$

Thinking of these operations acting on S^1-equivariant Floer theory of the loop space, one arrives at the conclusion that S^1-equivariant Floer cohomology (K-theory) should carry the structure of a module over the algebra of differential (respectively finite-difference) operators. Here is how this heuristic prediction materializes in Gromov-Witten theory:

Proposition 3.12.1. *Let $\mathcal{D}$ denote the algebra of differential operators generated by $p_a, a = 1, \ldots, k$, and Q^d, with d lying in the Mori cone of X. Define a representation of $\mathcal{D}$ on the symplectic loop space $\mathcal{H}$ using the operators $p_a - zQ_a\partial_{Q_a}$ where p_a acts by multiplication in the classical cohomology algebra of X) and Q^d acts by multiplication in the Novikov ring. Then tangent spaces in the overruled Lagrangian cone $\mathcal{L}^H \subset \mathcal{H}$ of cohomological GW-theory of X are $\mathcal{D}$-invariant.*

Proof: invariance with respect to multiplication by Q^d is tautological since the Novikov ring $\mathbb{C}[[Q, \lambda]]$ is considered as the ground ring of scalars. To prove invariance with respect to operators $p_a - zQ_a\partial_{Q_a}$, recall from [G1] that tangent spaces to $\mathcal{L}^H$ have the form $S_\tau^{-1}\mathcal{H}_+$, where $H \ni \tau \mapsto S_\tau(z)$ is a matrix power series in $1/z$ whose matrix entries are:

$$S_a^b = \delta_a^b + \sum_{n,d} \frac{Q^d}{n!} \sum_\mu \left\langle \phi^\mu, \tau, \ldots, \tau, \frac{\phi_b}{z - \psi} \right\rangle_{0,n+2,d}^X .$$

The matrix S_τ lies in the twisted loop groop, i.e. $S_\tau{-}1(z) = S_\tau^*(-z)$. Let ∂_{τ_a} denote differentiation in τ in the direction of the degree two cohomology class p_a. According to the divisor equation:

$$zQ_a\partial_{Q_a}S_\tau(z) + S_\tau(z)p_a = z\partial_{\tau_a}S_\tau(z). \tag{3.13}$$

In fact the property of $\mathcal{L}^H$ to be uniruled implies that $z\partial_{\tau_a}S = p_a \bullet S$, where $\bullet$ stands for quantum cup product. Transposing, we get:

$$(p_a - zQ_a\partial_{Q_a})S_\tau^{-1}(z) = -z\partial_{\tau_a}S_\tau^{-1}(z) = S_\tau^{-1}(z)(p_a\bullet).$$

Also, if $\tau = \sum_\alpha \tau_\alpha(Q)\phi_\alpha$ and $\mathbf{h} \in \mathcal{H}_+$, so that $\mathbf{f}(z,Q) = S_\tau^{-1}(z)\mathbf{h}(z,Q) \in \mathcal{T}_\tau$, then:

$$(p_a - zQ_a\partial_{Q_a})\mathbf{f} = S_\tau^{-1}(z)\left[(p_a\bullet) - zQ_a\partial_{Q_a} + z\sum Q_a\partial_{Q_a}\tau_g a(\phi_\alpha\bullet)\right]\mathbf{h}.$$

Since $\mathcal{H}_+$ is invariant under the operators in brackets, the result follows.

Remark 3.12.2. Each ruling space $z\mathcal{T}_\tau$, and therefore the whole cone $\mathcal{L}^H$, is $\mathcal{D}$-invariant.

Corollary 3.12.3. Tangent and ruling spaces of $\mathcal{L}^f$ are $\mathcal{D}$-invariant.

Proof: In the QHRR formula $ch(\mathcal{L}^f) = \triangle\mathcal{L}^H$ of Section 3.7, the operator $\triangle$ commutes with $\mathcal{D}$, since it does not involve Novikov's variables, and since the operators (which do occur in $\triangle$) of multiplication in the classical cohomological ring of X commute with p_a.

Lemma 3.12.4. The subspace $\mathcal{T} \subset \mathcal{K}^f$ obtained from $\mathcal{T}_{\mathcal{J}(0)_1}\mathcal{L}^f$ by the change $z \mapsto mz, Q \mapsto Q^m$, is $\mathcal{D}$-invariant.

Proof: The tangent space in question is $\triangle(z)S^{-1}_{\overline{\tau}(Q)}(z,Q)\mathcal{H}_+$ for some $\overline{\tau} = \sum_\alpha \overline{\tau}_\alpha\phi_\alpha \in H$. (Recall that $H = H^*(X,\mathbb{C}[[Q]])$). The space $\mathcal{T}$ is therefore $\triangle(mz)S^{-1}_{\overline{\tau}(Q^m)}(mz,Q^m)\mathcal{H}_+$, where $\mathcal{H}_+$ is $\mathcal{D}$-invariant, and $\triangle$ commutes with $\mathcal{D}$. Since $zQ_a\partial_{Q_a} = mzQ^m\partial_{Q_a^m}$, we find that the divisor equation still holds in the form:

$$(p_a - zQ_a\partial_{Q_a})S_\tau^{-1}(mz,Q^m) = S_\tau^{-1}(mz,Q^m)\left(p_a\bullet_{(\tau,Q^m)}\right),$$

where the last subscript indicates that the matrix elements of $p_a\bullet$ depend on τ and Q^m. The result now follows as in Proposition 3.12.1.

Corollary 3.12.5. Let ζ be a primitive m-th root of unity. Then the factor $\mathcal{L}^\zeta = \nabla_\zeta\mathcal{T}^\zeta$ of the adelic cone $\widehat{\mathcal{L}}$ is $\mathcal{D}$-invariant.

Proof: recall that $\mathcal{T}^\zeta$ is related to $\mathcal{T}$ by the change $q = \zeta e^z$ and the action of z in the operator $p_a - zQ_a\partial_{Q_a}$ should be understood in the sense of this identification. The result follows from Lemma 3.12.4 since ∇_ζ commutes with $\mathcal{D}$.

Theorem 3.12.6. *Let $\mathcal{D}_q$ denote the algebra of finite-difference operators, generated by integer powers of P_a, $a = 1, \dots, k$, and Q^d, with d lying in the Mori cone of X. Define a representation of $\mathcal{D}_q$ on the symplectic loop space $\mathcal{K}$, using the operators $P_a q^{Q_a \partial_{Q_a}}$ and Q^d. Here P_a acts by multiplication in $K^0(X)$ by the line bundle with Chern character e^{-p_a} and Q^d acts by multplication in the Novikov ring. Then tangent spaces to the overruled Lagrangian cone $\mathcal{L} \subset \mathcal{K}$ of true quantum-K theory on X are $\mathcal{D}_q$ invariant.*

Proof: thanks to the adelic characterization of the cone $\mathcal{L}$ and its ruling spaces given by Theorem 1.6.2 and Corollary 3.11.11, this is an immediate consequence of the following:

Lemma 3.12.7. The adelic cone $\widehat{\mathcal{L}}$ is $\mathcal{D}_q$ invariant.

Proof: If ζ is not a root of unity it is obvious that $\mathcal{L}^\zeta$ are $\mathcal{D}_q$-invariant because $\mathcal{L}^\zeta = \mathcal{K}_+^f$. For $\zeta = 1$, it follows from Corollary 3.12.3 that the family of operators $e^{\varepsilon(zQ_a\partial_{Q_a} - p_a)}$ preserves $\mathcal{L}^f$, and so does the operator with $\varepsilon = 1$, which is $P_a q^{Q_a\partial_{Q_a}}$. When $\zeta \neq 1$ is a primitive m-th root of unity, the family of operators $e^{\varepsilon(zQ_a\partial_{Q_a} - p_a)}$ preserves $\mathcal{L}^\zeta$ by Corollary 3.12.5. However at $\varepsilon = 1$ the operator differs $P_a q^{Q_a\partial_{Q_a}}$ by the factor $\zeta^{Q_a\partial_{Q_a}}$ because $q = \zeta e^z$. The operator $\zeta^{Q_a\partial_{Q_a}}$ acts by $Q_a \mapsto \zeta Q_a$. It is essential here that this extra factor commutes with $S^{-1}_{\overline{\tau}(Q^m)}(mz, Q^m)$ because $\zeta^m = 1$. Since it also preserves $\mathcal{H}_+$ the result follows.

Example 3.12.8. It is known from [GL] that for $X = \mathbb{CP}^{n-1}$,

$$\mathcal{J}(0) = (1-q) \sum_{d=0}^{\infty} \frac{Q^d}{(1-Pq)^n \cdots (1-Pq^d)^n} \quad ,$$

where $P \in K^0(\mathbb{CP}^{n-1})$ is the Hopf bundle. It follows from the string equation that $\mathcal{J}(0)/(1-q)$ lies in the tangent space $\mathcal{T}_{\mathcal{J}(0)}\mathcal{L}$. Applying powers T^r of the translation operator $T := Pq^{Q_a\partial_{Q_a}}$, we conclude that, for all integers r, the same tangent space contains

$$P^r \sum_{d=0}^{\infty} \frac{Q^d q^{rd}}{(1-Pq)^n \cdots (1-Pq^d)^n}.$$

In fact $\mathcal{J}(0)$ satisfies the second order finite-difference equation $D^n\mathcal{J}(0) = Q\mathcal{J}(0)$, where $D := 1-T$. Therefore the $\mathcal{D}_q$-module structure generated by $\mathcal{J}(0)/(1-q)$ is spanned over the Novikov ring by $T^r\mathcal{J}(0)/(1-q)$, with $r = 0, 1, \dots, n-1$. The projections of these elements to $\mathcal{K}_+$ are P^r, $r = 0, \dots, n-1$, which generate the ring $K^0(\mathbb{CP})^{n-1} = \mathbb{Z}[P, P^{-1}]/(1-P)^n$. The K-theoretic Poincaré pairing on this ring is given by the residue formula:

$$(\phi(P), \phi'(P)) = -Res_{P=1} \frac{\phi(P)\phi'(P)}{(1-P)^n} \frac{dP}{P}.$$

By computing the pairings with the above series we actually evaluate K-theoretic Gromov-Witten invariants:

$$\left(\phi(P), \frac{T^r \mathcal{J}(0)}{1-q}\right) = \sum_d Q^d \left\langle \frac{\phi(P)}{1-qL}, P^r \right\rangle_{0,2,d}^X, \quad r = 0, \ldots, n-1.$$

Thus, using the $\mathcal{D}_q$ structure alone we can compute all values $\langle \phi L^k, \phi' \rangle_{0,2,d}^X$ from $\langle \phi L^k, 1 \rangle_{0,2,d}^X$. By virtue of general properties of the invariants we can then compute all $\langle \phi L^k, \phi' L^l \rangle_{0,2,d}^X$.

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Appendix A

Virtual Kawasaki formula

In this appendix we prove that Kawasaki's formula "behaves well" with working with virtual structure sheaves in the following sense: if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts then Kawasaki's formula stays true.

Let $\mathcal{X}$ be a compact, complex orbispace (Deligne-Mumford stack) with a perfect obstruction theory $E^{-1} \to E^0$. This gives rise to the intrinsic normal cone, which is embedded in E_1 - the dual bundle to E^{-1} (see [LT], also [BF]). The virtual structure sheaf $\mathcal{O}_{\mathcal{X}}^{vir}$ was defined in [L2] as the K-theoretic pull-back by the zero section of the structure sheaf of this cone. Let $I\mathcal{X} = \coprod_\mu \mathcal{X}_\mu$ be the inertia orbifold of $\mathcal{X}$. We denote by i_μ the inclusion of a stratum $\mathcal{X}_\mu$ in $\mathcal{X}$. For a bundle V on $\mathcal{X}$ we write $i_\mu^* V = V_\mu^f \oplus V_\mu^m$ for its decomposition as the direct sum of the fixed part and the moving part under the action of the symmetry associated to $\mathcal{X}_\mu$. To avoid ugly notation , we will often not write the lower index μ in the notation and simply write V^m, V^f. The virtual normal bundle to $\mathcal{X}_\mu$ in $\mathcal{X}$ is defined as $[E_0^m] - [E_1^m]$. We will in addition assume that $\mathcal{X}$ admits an embedding j in a smooth compact orbifold $\mathcal{Y}$. This is always true for the moduli spaces of stable maps $X_{0,n,d}$ because an embedding $X \hookrightarrow \mathbb{P}^N$ induces an embedding $X_{0,n,d} \hookrightarrow (\mathbb{P}^N)_{0,n,d}$.

Proposition A.0.1. Denote by N_μ^{vir} the virtual normal bundle of $\mathcal{X}_\mu$ in $\mathcal{X}$. Then in the notation of Section 3.6:

$$\chi\left(\mathcal{X}, j^*(V) \otimes \mathcal{O}_{\mathcal{X}}^{vir}\right) = \sum_\mu \frac{1}{m_\mu} \chi^f \left(\mathcal{X}_\mu, \frac{Tr(V_\mu \otimes \mathcal{O}_{\mathcal{X}_\mu}^{vir})}{Tr\left(\Lambda^\bullet (N_\mu^{vir})^*\right)} \right). \tag{A.1}$$

Remark A.0.2. A perfect obstruction theory $E^{-1} \to E^0$ on $\mathcal{X}$ induces canonically a perfect obstruction theory on $\mathcal{X}_\mu$ by taking the fixed part of the complex $E_\mu^{-1,f} \to E_\mu^{0,f}$. The proof is the same as that of Proposition 1 in [GP]. This is then used to define the sheaf $\mathcal{O}_{\mathcal{X}_\mu}^{vir}$.

Remark A.0.3. It is proved in [FG] that the Grothendieck-Riemann-Roch theorem, which gives the fake invariants, is compatible with virtual fundamental classes and virtual fundamental sheaves i.e.:

$$\chi^f(\mathcal{X}, V \otimes \mathcal{O}_{\mathcal{X}}^{vir}) = \int_{[\mathcal{X}]} ch(V \otimes \mathcal{O}_{\mathcal{X}}^{vir}) \cdot T^{vir}$$

where $[\mathcal{X}]$ is the virtual fundamental class of $\mathcal{X}$ and T^{vir} is its virtual tangent bundle.

Remark A.0.4. The bundles V to which we apply the proposition in Section 3 are (sums and products of) cotangent line bundles L_i and evaluation classes $ev_i^*(a_i)$. They are pull-backs of the corresponding bundles on $(\mathbb{P}^N)_{0,n,d}$.

Example A.0.5. We first look at the toy-case when there is a bundle E on $\mathcal{Y}$ with a section $s : \mathcal{Y} \to E$ such that $\mathcal{X} = s^{-1}(0) \subset \mathcal{Y}$. In this case the sheaf $\mathcal{O}_{\mathcal{X}}^{vir}$ is the K-theoretic Euler class $\Lambda^\bullet E^*$ and the obstruction theory is the differential $ds : T\mathcal{Y} \to E$. If we denote by i_μ the inclusion of the Kawasaki stratum $\mathcal{Y}_\mu$ in $\mathcal{Y}$ then by Kawasaki formula applied to $\mathcal{Y}$ we have:

$$\chi(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{vir}) = \chi(\mathcal{Y}, j_*\Lambda^\bullet E^*) = \sum_\mu \frac{1}{m_\mu} \chi^f \left(\mathcal{Y}_\mu, \frac{Tr(i_\mu^* j_* \Lambda^\bullet E^*)}{Tr(\Lambda^\bullet N_\mu^*)} \right). \tag{A.2}$$

The following diagram:

$$\begin{array}{ccc} \mathcal{X}_\mu & \xrightarrow{i'_\mu} & \mathcal{X} \\ {\scriptstyle j'}\downarrow & & {\scriptstyle j}\downarrow \\ \mathcal{Y}_\mu & \xrightarrow{i_\mu} & \mathcal{Y} \end{array}$$

is cartesian hence $i_\mu^* j_* \Lambda^\bullet E^* = j'_*(i'_\mu)^* \Lambda^\bullet E^*$. By multiplicativity of Euler classes:

$$i'^*_\mu \Lambda^\bullet E^* = \Lambda^\bullet (E^f)^* \Lambda^\bullet (E^m)^*$$

and the sheaf $\Lambda^\bullet(E^f)^* = \mathcal{O}_{\mathcal{X}_\mu}^{vir}$ by definition. Moreover $N_\mu^{vir} = N_\mu - E^m$. This gives:

$$\chi^f \left(\mathcal{Y}_\mu, \frac{Tr(i_\mu^* j_* \Lambda^\bullet E^*)}{Tr(\Lambda^\bullet N_\mu^*)} \right) = \chi^f \left(\mathcal{Y}_\mu, j'_* \frac{Tr(\mathcal{O}_{\mathcal{X}_\mu}^{vir})}{Tr(\Lambda^\bullet N_\mu^{vir})} \right).$$

Plugging the above expression in $(A.2)$ and pulling back to $\mathcal{X}_\mu$ proves the proposition in this case.

Before proving Proposition $A.0.1$ in the general case we recall a lemma of [L2] which we will use. For any fiber square:

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ B' & \xrightarrow{i} & B \end{array}$$

with i a regular embedding one can define K-theoretic refined Gysin homomorphisms $i^! : K_0(V) \to K_0(V')$ (see [L2]). Consider now the diagram:

$$\begin{array}{ccc} \iota^* C_{X/Y} & \longrightarrow & C_{X/Y} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow j \\ Y' & \xrightarrow{i} & Y \end{array}$$

with i a regular embedding and j an embedding, $C_{X/Y}$ is the normal cone of X in Y and both squares are fiber diagrams. Then Lemma 2 of [L2] states that:

$$i^![\mathcal{O}_{C_{X/Y}}] = [\mathcal{O}_{C_{X'/Y'}}] \in K_0(\iota^* C_{X/Y}). \tag{A.3}$$

Proof of Proposition $A.0.1$: we have:

$$\chi(\mathcal{X}, j^* V \otimes \mathcal{O}^{vir}_{\mathcal{X}}) = \chi(\mathcal{Y}, V \otimes j_* \mathcal{O}^{vir}_{\mathcal{X}}).$$

We now apply Kawasaki's formula to the sheaf $V \otimes j_* \mathcal{O}^{vir}_{\mathcal{X}}$ on $\mathcal{Y}$. This gives:

$$\chi(\mathcal{Y}, V \otimes j_* \mathcal{O}^{vir}_{\mathcal{X}}) = \sum_{\mu} \frac{1}{m_\mu} \chi^f \left(\mathcal{Y}_\mu, \frac{Tr(V_\mu \otimes i^*_\mu j_* \mathcal{O}^{vir}_{\mathcal{X}})}{Tr(\Lambda^\bullet N^*_\mu)} \right). \tag{A.4}$$

From the fiber diagram:

$$\begin{array}{ccc} \mathcal{X}_\mu & \xrightarrow{i'_\mu} & \mathcal{X} \\ j' \downarrow & & \downarrow j \\ \mathcal{Y}_\mu & \xrightarrow{i_\mu} & \mathcal{Y} \end{array}$$

and Theorem **6.2** in [FL] we have $i^*_\mu j_* \mathcal{O}^{vir}_{\mathcal{X}} = j'_* i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}}$. Plugging this in $(A.4)$ gives:

$$\chi^f \left(\mathcal{Y}_\mu, \frac{Tr\left(V_\mu \otimes i^*_\mu j_* \mathcal{O}^{vir}_{\mathcal{X}}\right)}{Tr(\Lambda^\bullet N^*_\mu)} \right) = \chi^f \left(\mathcal{Y}_\mu, \frac{Tr\left(V_\mu \otimes j'_* i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}}\right)}{Tr(\Lambda^\bullet N^*_\mu)} \right). \tag{A.5}$$

Let G_μ be the cyclic group generated by one element of the conjugacy class associated to $\mathcal{X}_\mu$. Then we will show that:

$$Tr\left(\frac{i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}}}{\Lambda^\bullet(N^*_\mu)} \right) = Tr\left(\frac{\mathcal{O}^{vir}_{\mathcal{X}_\mu}}{\Lambda^\bullet(N^{vir}_\mu)^*} \right) \tag{A.6}$$

in the G_μ-equivariant K-ring of $\mathcal{X}_\mu$. This is essentially the computation of Section 3 in [GP] carried out in $\mathbb{C}^*$-equivariant K-theory. Relation $(A.6)$ then follows by embedding the group G_μ in the torus and specializing the value of the variable t in the ground ring of $\mathbb{C}^*$-equivariant K-theory to a $|G_\mu|$-root of unity.

If we define a cone $D := C_{\mathcal{X}/\mathcal{Y}} \times_{\mathcal{X}} E_0$, then this is a $T\mathcal{Y}$ cone (see [BF]). The virtual normal cone D^{vir} is defined as $D/T\mathcal{Y}$ and $\mathcal{O}^{vir}_{\mathcal{X}}$ is the pull-back by the zero section of the structure sheaf of D^{vir}. Alternatively there is a fiber diagram:

$$\begin{array}{ccc} T\mathcal{Y} & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ \mathcal{X} & \xrightarrow{0_{E_1}} & E_1 \end{array}$$

whre the bottom map is the zero section of E_1. Then one can define $\mathcal{O}^{vir}_{\mathcal{X}}$ as $0^*_{T\mathcal{Y}} 0^!_{E_1}[\mathcal{O}_D]$. We'll prove formula $(A.6)$ following closely the calculation in [GP]. First by definition of $\mathcal{O}^{vir}_{\mathcal{X}}$ and by commutativity of Gysin maps we have :

$$i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}} = i^!_\mu 0^*_{T\mathcal{Y}} 0^!_{E_1}[\mathcal{O}_D] = 0^*_{T\mathcal{Y}} 0^!_{E_1} i^!_\mu[\mathcal{O}_D]. \tag{A.7}$$

We pull-back relation $(A.3)$ to $(i'_\mu)^* D = (i'_\mu)^*(C_{\mathcal{X}/\mathcal{Y}} \times E_0)$ to get:

$$i^!_\mu[\mathcal{O}_D] = [\mathcal{O}_{D_\mu} \times (E_0^m)^*]. \tag{A.8}$$

In the equality above we have used the fact that $D_\mu = C_{\mathcal{X}_\mu/\mathcal{Y}_\mu} \times E_0^f$ and we identified the sheaf of sections of the bundle E_0^m with the dual bundle $(E_0^m)^*$. Plugging $(A.8)$ in $(A.7)$ we get:

$$i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}} = 0^*_{T\mathcal{Y}} 0^!_{E_1}[\mathcal{O}_{D_\mu} \times (E_0^m)^*]. \tag{A.9}$$

Notice that the action of $T\mathcal{Y}_\mu$ leaves $D_\mu \times (E_0^m)^*$ invariant (it acts trivially on $(E_0^m)^*$). Now we can write $0^*_{T\mathcal{Y}} = 0^*_{T\mathcal{Y}^f_\mu} \times 0^*_{T\mathcal{Y}^m_\mu}$ and since $D^{vir}_\mu = D_\mu / T\mathcal{Y}_\mu$ we rewrite $(A.9)$ as:

$$i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}} = 0^*_{T\mathcal{Y}^m_\mu} 0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*]. \tag{A.10}$$

The proof of Lemma 1 in [GP] works in our set-up as well: it uses excess intersection formula which holds in K-theory. It shows that the following relation holds in the $\mathbb{C}^*$-equivariant K-ring of $\mathcal{X}_\mu$:

$$0^*_{T\mathcal{Y}^m_\mu} 0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*] = 0^*_{E_0^m}\left(0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*]\right) \cdot \frac{\Lambda^\bullet (T\mathcal{Y}^m)^*}{\Lambda^\bullet (E_0^m)^*}. \tag{A.11}$$

The class $0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times E_0^m]$ lives in the $\mathbb{C}^*$-equivariant K-ring of E_0^m. The class doesn't depend on the bundle map $E_0^m \to E_1^m$ so we can assume this map to be 0. Then by excess intersection formula and the definition of $\mathcal{O}^{vir}_{\mathcal{X}_\mu}$ we get :

$$0^*_{E_0^m}\left(0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*]\right) = \mathcal{O}^{vir}_{\mathcal{X}_\mu} \cdot \Lambda^\bullet (E_1^m)^*. \tag{A.12}$$

Formula $(A.12)$ holds because $D_\mu^{vir} \times (E_0^m) \subset E_1^f \times E_0^m$ and $0^!_{E_1}$ acts as $0^!_{E_1^f} \times 0^!_{E_1^m}$ on factors. $0^!_{E_1^f}[\mathcal{O}_{D_\mu^{vir}}] = \mathcal{O}^{vir}_{\mathcal{X}_\mu}$ by definition of $\mathcal{O}^{vir}_{\mathcal{X}_\mu}$. By excess intersection formula applied to the fiber square:

$$\begin{array}{ccc} E_0^m & \longrightarrow & E_0^m \\ \pi \downarrow & & \downarrow \\ \mathcal{X}_\mu & \xrightarrow{0_{E_1^m}} & E_1^m \end{array}$$

we have $0^*_{E_0^m} 0^!_{E_1^m}[(E_0^m)^*] = 0^*_{E_0^m} \pi^* \Lambda^\bullet (E_1^m)^* = \Lambda^\bullet (E_1^m)^*$. Plugging formula $(A.12)$ in $(A.11)$ (note that $N_\mu = T\mathcal{Y}_\mu^m$ and $N_\mu^{vir} = [E_0^m] - [E_1^m]$) and taking traces proves $(A.6)$. We now plug $(A.6)$ in $(A.5)$ and then pull-back to $\mathcal{X}_\mu$ to get:

$$\chi^f \left(\mathcal{Y}_\mu, \frac{Tr(V_\mu \otimes j_* i_\mu^* \mathcal{O}^{vir}_{\mathcal{X}})}{Tr(\Lambda^\bullet N_\mu^*)} \right) = \chi^f \left(\mathcal{Y}_\mu, Tr(V_\mu) \otimes j'_* \frac{Tr(\mathcal{O}^{vir}_{\mathcal{X}_\mu})}{Tr(\Lambda^\bullet (N_\mu^{vir})^*)} \right) =$$
$$= \chi^f \left(\mathcal{X}_\mu, \frac{Tr(V_\mu \otimes \mathcal{O}^{vir}_{\mathcal{X}_\mu})}{Tr(\Lambda^\bullet (N_\mu^{vir})^*)} \right). \qquad (A.13)$$

This concludes the proof of the proposition.