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Block patterns in permutations and words and generalized clusters

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Block patterns in permutations and words and generalized clusters

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by

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Professor Jeffrey Remmel, Chair
Professor Ronald Graham
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The dissertation of Ran Pan is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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DEDICATION

To my grandfather.

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ABSTRACT OF THE DISSERTATION

Block patterns in permutations and words and generalized clusters

by

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Doctor of Philosophy in Mathematics

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Professor Jeffrey Remmel, Chair

Goulden and Jackson introduced a very powerful method to study the distributions of certain consecutive patterns in permutations, words, and other combinatorial objects which is now called the cluster method. There are a number of natural classes of combinatorial objects which start with either permutations or words and add additional restrictions. These include up-down permutations, generalized Euler permutations, words without consecutive repeats, colored permutations without consecutive repeated colors, Carlitz integer compositions, Young tableaux, non-backtracking random walks, ordered set partitions, cycle structures in permutations and so on. We develop an extension of the cluster method which we call the generalized cluster method to study the distribution of certain consecutive patterns in such restricted combinatorial objects. The generalized cluster method enables us to express the generating function for distribution of a pattern in such restricted combinatorial objects in terms of so-called generalized cluster polynomials. Compared to the original problem, computing generalized cluster polynomials is usually more tractable. We also generalize a multi-variate version of both cluster method and generalized cluster method which is used to study joint distribution of multiple patterns. We use combinatorial objects mentioned above as concrete examples to illustrate our methods.

Chapter 1

Introduction

In enumerative combinatorics, the study of pattern enumeration has mainly focused on the following question.

> *Given some pattern and a class of objects, how many objects in this class have exactly k many matches of such pattern?*

The study of patterns in permutations and words is now a very active area of combinatorics. There are two types of patterns that have been extensively studied. For example, for classical patterns, one looks for subsequences in a permutation $\sigma = \sigma_1 \dots \sigma_n$ which are order isomorphic to a given permutation $\tau = \tau_1 \dots \tau_j$. For consecutive pattern matchings, one looks for a consecutive sequence in σ which is order isomorphic to a given permutation τ. Research on patterns in permutations and words started over a century ago. See for example, MacMahon's work in [39]. More recently, researchers have defined a number extensions of pattern matching conditions. For example, in a generalized pattern introduced in [61], one can force only some elements of a pattern to occur consecutively. For example, an occurrence of a 1-23 pattern in σ would mean that we are looking for a subsequence $1 \leq a < b \leq n-1$ such that $\sigma_a < \sigma_b < \sigma_{b+1}$. Thus the dash between 1 and 2 means that we allow the "1" of the subsequence to occur anywhere before the "2" and "3" of the subsequence but the lack of dash between 2 and 3 means that the "2" and the "3" of the subsequence must occur consecutively. There are now a number of extensions of pattern matching conditions included barred patterns, mesh patterns, and marked mesh patterns. See Kitaev's book [33].

In the last twenty years, there have been numerous articles on finding generating functions for the number of consecutive occurrences of a given pattern in various sets of combinatorial objects such as permutations, words, colored permutations, set partitions, ordered set partitions, lattice paths, and various types of arrays. Goulden and Jackson [24] developed the so-called cluster method which has been used by many authors to find such generating functions. Remmel and his students have shown that many such generating functions arise by applying a ring homomorphism on the ring of symmetric functions over infinitely many variable $x_1, x_2, \ldots$ to simple symmetric function identities. This method is now called the homomorphism method and is explained in the recent book by Mendes and Remmel [44].

The main focus of this thesis is how one can extend the cluster method to find generating functions for patterns in such objects where there are additional restrictions on the objects. For example, in permutations, one can ask to find generating functions for the number of consecutive patterns in up-down permutations. In words, one can ask to find generating functions for the number of consecutive patterns in words that have no consecutive repeated letters, no consecutive repeated letters which are even, or no consecutive repeated letters which are odd. In rectangular arrays, one can ask to find generating functions for the number of consecutive patterns in standard tableaux, column strict tableaux, or row-strict tableaux.

In this thesis, we develop a new method called the *generalized cluster method* to handle such questions. Our work was inspired by the work of Remmel [55] who defined generalized maximum packings as a way to find generating functions for the number of consecutive patterns in up-down permutations. Generalized maximum packings are a special case of what we call generalized clusters and we show that our desired generating functions can be expressed in terms of certain polynomials associated with generalized cluster.

1.1 Pattern avoidance and matching

In this section, we shall briefly review various concepts and definitions from the theory of pattern matching. We let $\mathcal{S}_n$ to denote the symmetric group. That is, $\mathcal{S}_n$ is the set of bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ under composition. We shall use the one-line notation for permutations. That is, we shall write $\sigma = \sigma_1 \ldots \sigma_n$ if $\sigma(i) = \sigma_i$ for $i = 1, \ldots, n$.

1.1.1 Classical permutation patterns

Given any sequence $\tau = \tau_1 \ldots \tau_n$ of positive integers which are pairwise distinct, the **reduction** of τ is a permutation that results from replacing the i-th smallest number in τ by i. We denote the reduction of τ by red(τ). For example, assume $\tau = 2\ 6\ 5$ and $\pi = 3\ 4\ 9$, then red$(\tau) = 1\ 3\ 2$ and red$(\pi) = 1\ 2\ 3$. If two sequences have the same reduction, we say they are **order-isomorphic**.

Definition 1.1. *Given a permutation $\tau = \tau_1\tau_2 \cdots \tau_j \in \mathcal{S}_j$ and a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathcal{S}_n$, we say*

1. *τ **occurs** in σ if there exist indices $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ such that*

$$red(\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_j}) = \tau,$$

2. *σ **avoids** τ if there is no occurrence of τ in σ.*

This type of pattern matching condition is often called classical pattern matching. Note that the indices are not required to be contiguous. For example, assume there are patterns $\tau = 1\ 3\ 2$ and pattern $\pi = 3\ 2\ 1$, and the permutation is $\sigma = 1\ 3\ 4\ 2$. Then σ has two occurrences of τ because red$(1\ 3\ 2)$ =red$(1\ 4\ 2) = \tau$, but σ avoids π.

Clearly, a classical permutation pattern could be regarded as reduction of subsequence in a permutation. Classical pattern avoidance and matching in permutations have been studied for over a century. Some instances and work about this topic haven already be recorded in MacMahon's classical book *Combinatory Analysis* [39] in 1915. After 1965, Knuth's work [35] on sorting permutations using various data structures inspired more and more researchers to focus their attention to patterns in permutations. He showed that, for any $\tau \in \mathcal{S}_3$, the number of permutations in $\mathcal{S}_n$ avoid classical pattern τ is always given by n-th Catalan number, that is, $\frac{1}{n+1}\binom{2n}{n}$. Simon and Schmidt [57] were the first to systematically study the problem of enumerating the number of permutations in $\mathcal{S}_n$ which avoid certain patterns. Kitaev's book [33] provides a good modern reference to the theory of patterns in permutations and words.

Two permutations τ and π in $\mathcal{S}_j$ are said to be **Wilf-equivalent** if for any n, the number of permutations in $\mathcal{S}_n$ avoiding τ is equal to the number of permutations in $\mathcal{S}_n$ avoiding π. They are said to be **strongly Wilf-equivalent** if for any n and k, the number of permutations in $\mathcal{S}_n$ having exactly k occurrences of τ is equal to the number of permutations in $\mathcal{S}_n$ having exactly k occurrences of π. Clearly, strong Wilf-equivalence implies Wilf-equivalence. It is not true that Wilf-equivalence implies strong Wilf equivalence. That is, the patterns 123 and 132 are Wilf-equivalent, but in, the permutation $\sigma = 1234$ has 4 occurrences of the pattern 123, but there is no permutation in $\mathcal{S}_4$ which as 4 occurrences of 132.

One can easily extend our definitions to consider sets of permutations Γ.

Definition 1.2. *Given a set of permutation* Γ *and a permutation* $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$*, we say*

1. Γ ***occurs*** *in* σ *if there exist indices* $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ *such that*

$$red(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_j}) \in \Gamma,$$

2. σ ***avoids*** Γ *if there is no occurrence of* Γ *in* σ.

1.1.2 Consecutive patterns in permutations and words

Generalized permutation patterns were introduced in [61]. In a generalized permutation pattern σ, some elements are required to be consecutive and some are not. A dash is used to connect two elements that are not required to be consecutive. For example although $\sigma = 1\ 3\ 4\ 2$ contains classical pattern 1 3 2 and generalized pattern 1 3-2 but does not contain generalized pattern 1-3 2.

Consecutive patterns are a special class of generalized patterns. In contrast to classical patterns, all elements are required to be consecutive. A consecutive occurrence of a permutation τ in a permutation σ is called a τ-match of σ.

Definition 1.3. *Given a permutation* $\tau = \tau_1\tau_2\cdots\tau_j \in \mathcal{S}_j$ *and a permutation* $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$*, we say*

1. *there is a* τ***-match*** *starting at position* i *in* σ*, if there exists an integer* i *such that*
$$red(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}) = \tau,$$
2. σ ***consecutively avoids*** τ *if* σ *does not have a* τ*-match.*

We let τ*-mch*(σ) *denote the number of* τ*-matches in* σ*.*

For example, if $\tau = 1\ 2\ 3$ and $\sigma = 2\ 6\ 7\ 1\ 3\ 4\ 5$, then there is τ-matches starting at positions 1, 4 and 5 and therefore, $\tau\text{-mch}(\sigma) = 3$. Naturally the definition above can be extended to a set of patterns.

Definition 1.4. *Given a set of permutations* Γ *and a permutation* $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$*, we say*

1. *there is a* Γ***-match*** *at starting at position* i *in* σ*, if there exists an integer* i *such that*
$$red(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j-1}) \in \Gamma,$$
2. σ ***consecutive avoids*** Γ *if* σ *does not have a* Γ*-match.*

For a set of permutations Γ, we let

$$\Gamma\text{-mch}(\sigma) = \sum_{\tau\in\Gamma} \tau\text{-mch}(\sigma)$$

denote the number of Γ-matches in σ.

To study pattern enumeration in permutations, we usually study the following exponential generating function $A_{\tau,\mathcal{S}}(x,t)$ for a given pattern τ and try to find an expression or explicit formula for $A_{\tau,\mathcal{S}}(x,t)$.

$$A_{\tau,\mathcal{S}}(x,t) := 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{S}_n} x^{\tau\text{-mch}(\sigma)}. \tag{1.1}$$

Although systematic study on consecutive permutation patterns was not started until 2003 [19], some basic statistics over permutations have been studied for a long time. Suppose $\sigma \in \mathcal{S}_n$,

$$\begin{aligned} \mathrm{Des}(\sigma) &= \{i : \sigma_i > \sigma_{i+1}\} & \mathrm{des}(\sigma) &= |\mathrm{Des}(\sigma)|, \\ \mathrm{Ris}(\sigma) &= \{i : \sigma_i < \sigma_{i+1}\} \ \text{ and } & \mathrm{ris}(\sigma) &= |\mathrm{Ris}(\sigma)|. \end{aligned}$$

Apparently, for $\sigma \in \mathcal{S}_n$, $\mathrm{des}(\sigma) = 21\text{-mch}(\sigma)$ and $\mathrm{ris}(\sigma) = 12\text{-mch}(\sigma)$. Euler first gave the explicit generating function for number of descents in permutations,

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{des}(\sigma)} = \frac{x-1}{x - e^{(x-1)t}}.$$

Let $i, j, k,$ and n be non-negative integers satisfying $k \geq 2$, $i, j \geq 0$. Let $\mathcal{C}^{i,j,k}_{i+kn+j}$ denote the set of permutations $\sigma = \sigma_1 \ldots \sigma_{i+kn+j}$ in $\mathcal{S}_{i+kn+j}$ with $\mathrm{Des}(\sigma) \subseteq \{i, i+k, \ldots, i+nk\}$ and $C^{i,j,k}_{i+kn+j} = |\mathcal{C}^{i,j,k}_{i+kn+j}|$. Thus permutations in $\mathcal{C}^{i,j,k}_{i+kn+j}$ start with an increasing block of size i followed by n increasing blocks of size k and ending with an increasing block of size j. Given $\sigma \in \mathcal{C}^{i,j,k}_{i+kn+j}$, we let $\mathrm{Ris}_{i,k}(\sigma) = \{i + sk : \sigma_{i+sk} < \sigma_{i+sk+1}\}$ and $\mathrm{ris}_{i,k}(\sigma) = |\mathrm{Ris}_{i,k}(\sigma)|$.

We let $\mathcal{E}^{i,j,k}_{i+kn+j}$ denote the set of permutations $\sigma \in \mathcal{S}_{i+kn+j}$ with $\mathrm{Des}(\sigma) = \{i, i+k, \ldots, i+nk\}$ and $E^{i,j,k}_{i+kn+j} = |\mathcal{E}^{i,j,k}_{i+kn+j}|$. For any $\sigma \in \mathcal{S}_{i+kn+j}$, let

$$\mathrm{Ris}_{i,k}(\sigma) = \{s : 0 \leq s \leq n \text{ and } \sigma_{i+sk} < \sigma_{i+sk+1}\}$$

and $\mathrm{ris}_{i,k}(\sigma) = |Ris_{i,k}(\sigma)|$. Then $E^{i,j,k}_{i+kn+j}$ is the number of $\sigma \in \mathcal{C}^{i,j,k}_{i+kn+j}$ such that $\mathrm{Ris}_{i,k}(\sigma) = \varnothing$. Thus permutations σ in $\mathcal{E}^{i,j,k}_{i+kn+j}$ have the same block structure as permutations in $\mathcal{C}^{i,j,k}_{i+kn+j}$, but we require the additional restriction that for any two consecutive blocks B and C, the last element of block B must be larger than the first element of block C.

In the special case where $k = 2$, $i = 0$, and $j = 2$, $E^{0,2,2}_{2n+2}$ is the number of permutations in $\mathcal{S}_{2n+2}$ with descent set $\{2, 4, \ldots, 2n\}$. These permutations with alternating descents and ascents are classical up-down permutations (or alternating permutations). André [2, 3] proved that

$$1 + \sum_{n \geq 0} \frac{E^{0,2,2}_{2n+2}}{(2n+2)!} t^{2n+2} = \sec t.$$

Similarly, $E_{2n+1}^{0,1,2}$ counts the number of odd up-down permutations and André proved that

$$\sum_{n\geq 0} \frac{E_{2n+1}^{0,1,2}}{(2n+1)!} t^{2n+1} = \tan t.$$

These numbers are also called the Euler numbers. When $k \geq 2$, $E_{kn+j}^{0,j,k}$ are called generalized Euler numbers. Mendes and Remmel [44] showed that

$$\sum_{n\geq 0} \left(\frac{E_{3n}^{0,0,3}}{(3n)!} t^{3n} + \frac{E_{3n+1}^{0,1,3}}{(3n)!} t^{3n+1} + \frac{E_{3n}^{0,2,3}}{(3n+2)!} t^{3n+2} \right) = \frac{3 + 2\sqrt{3} e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}t}{2}\right)}{e^{-t} + 2e^{\frac{t}{2}} \cos\left(\frac{\sqrt{3}t}{2}\right)}.$$

One can all study generating functions for the number of consecutive occurrence of a pattern in words. Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ denote the set of positive integers and for any $k \in \mathbb{P}$, let $[k] = \{1, 2, \ldots, k\}$. A **word** of length n over alphabet $[k]$ is a sequence consisting of letters from the set $\{1, 2, 3, \cdots, k\}$. The set of all such words of length n is denoted by $[k]^n$. The set of all words over alphabet $[k]$ is denoted by $[k]^*$, that is, $[k]^* = \bigcup_{n\geq 0} [k]^n$.One can also define a natural notion of reduction for words. That is, given a word $u_1 \ldots u_j$ in $[k]^*$ for some $k \in \mathbb{P}$, we let $\text{red}(u)$ denote the word which results by replacing the i-th smallest letter in u by i. For example, if $u = 543364$, then $\text{red}(u)$ equals 321142. There are two types of consecutive patterns in words, one uses exact matches and the other uses reduced matches.

Definition 1.5. *Given a word $u = u_1 u_2 \cdots u_j \in \mathbb{P}^j$ and a word $w = w_1 w_2 \cdots w_n \in \mathbb{P}^n$, we say*

1. *there is an **exact** u**-match** starting at position i in w if there exists an integer i such that*

$$w_i w_{i+1} \cdots w_{i+j-1} = u,$$

2. *w **exactly consecutively avoids** u if w does not have an exact u-match.*

The number of exact u-matches in w is denoted by u-Emch(w).

For example, suppose we have word $w = 4\ 3\ 1\ 3\ 2\ 5\ 2\ 3\ 1 \in [5]^9$ and the pattern $u = 3\ 1$, then u-Emch$(w) = 2$.

Definition 1.6. *Given a word $u = u_1 \dots u_j$ such that red$(u) = u$ and a word $w = w_1 w_2 \cdots w_n \in \mathbb{P}^n$, we say*

1. *there is a u-**match** starting at position i if there exists an integer i such that*

$$\text{red}(w_i w_{i+1} \cdots w_{i+j-1}) = u,$$

2. *w **consecutively avoids** u if w does not have an exact τ-match.*

We let u-mch(w) denote the number of u-matches in w.

Given $w = w_1 \dots w_n \in \mathbb{P}^*$, we let

$$\begin{aligned} \text{Des}(w) &= \{i : w_i > w_{i+1}\}, & \text{des}(w) &= |\text{Des}(w)|, \\ \text{Lev}(w) &= \{i : w_i = w_{i+1}\}, & \text{lev}(w) &= |\text{Lev}(w)|, \\ \text{Ris}(w) &= \{i : w_i < w_{i+1}\} \text{ and } & \text{ris}(w) &= |\text{Ris}(w)|. \end{aligned}$$

1.1.3 Consecutive patterns in arrays

For our purposes, we shall picture a σ in $\mathcal{C}^{i,j,k}_{i+kn+j}$ as an array $F(\sigma)$ starting with a column of size i, followed by n columns of size k, and ending with a column of size j filled with the permutation σ so that one recovers σ by reading the elements in each column from bottom to top and reading the columns from left to right. This means that in each column, the numbers are increasing when read from bottom to top. For example, the array associated with the permutation

$$\sigma = 2\ 5\ 6\ 8\ 9\ 1\ 7\ 10\ 4\ 11\ 12\ 3$$

in $\mathcal{C}^{2,1,3}_{12}$ is pictured in Figure 1.1. Elements of $\mathcal{E}^{i,j,k}_{i+kn+j}$ can be viewed as restricted arrays where the top element of each column has to be bigger than the bottom element in the column immediately to its right.

More generally, let $D^{i,j,k}_{i+kn+j}$ denote the diagram which consists of a column of height i, followed by n columns of height k, and ending with a column of height j. We let (s,t) denote the cell which is in the s-th column reading from left to right, and the t-th row reading from bottom to top. For example, for the filling of $D^{2,1,3}_{12}$ is pictured in Figure 1.1, the number 7 is in cell $(3,2)$.

	9	10	12	
5	8	7	11	
2	6	1	4	3

Figure 1.1: The array for an element of $\mathcal{C}_{12}^{2,1,3}$.

Given an alphabet $A \subseteq \mathbb{P}$, $F_{i+kn+j,A}^{i,j,k}$ denote the set of all fillings of $D_{i+kn+j}^{i,j,k}$ with elements from A. We let $\mathcal{WI}_{i+kn+j,A}^{i,j,k}$ ($\mathcal{SI}_{i+kn+j,A}^{i,j,k}$) be the set of all fillings of F of $D_{i+kn+j}^{i,j,k}$ with elements of A such that the elements are weakly increasing (strictly increasing) reading from bottom to top. The word $w(F)$ of any filling $F \in F_{i+kn+j,A}^{i,j,k}$ is the word obtained by reading the columns from bottom to top and the columns from left to right. For example, the words of the element $F_1 \in \mathcal{WI}_{12,[5]}^{1,2,3}$ and the element $F_2 \in \mathcal{SI}_{12,[5]}^{1,2,3}$ are pictured in Figure 1.2. We let $\mathcal{P}_{i+kn+j}^{i,j,k}$ denote the set of all fillings of $D_{i+kn+j}^{i,j,k}$ with the elements of $1, \ldots, i+kn+j$ such that the elements are increasing reading from bottom to top in each column. Thus for any $F \in \mathcal{P}_{n}^{i,j,k}$, $w(F) \in \mathcal{C}_{i+kn+j}^{i,j,k}$.

F_1

	4	5	3	
	4	4	1	3
2	1	3	1	2

$w(F_1) = 2\ 1\ 4\ 4\ 3\ 4\ 5\ 1\ 1\ 3\ 2\ 3$

F_2

	4	5	3	
	3	4	2	3
2	1	3	1	2

$w(F_2) = 2\ 1\ 3\ 4\ 3\ 4\ 5\ 1\ 2\ 3\ 2\ 3$

Figure 1.2: The words of elements of $\mathcal{WI}_{12}^{1,2,3}$ and $\mathcal{SI}_{12}^{1,2,3}$.

In this paper, we will be mostly interested in patterns that occur between columns of height k for elements of $\mathcal{WI}_{i+kn+j,\mathbb{P}}^{i,j,k}$, $\mathcal{SI}_{i+kn+j,\mathbb{P}}^{i,j,k}$, $\mathcal{WI}_{i+kn+j,[s]}^{i,j,k}$, $\mathcal{SI}_{i+kn+j,[s]}^{i,j,k}$, and $\mathcal{P}_{i+kn+j}^{i,j,k}$. These types of patterns were first studied by Harmse and Remmel [27] for elements in $\mathcal{P}_{nk}^{0,0,k}$.

If F is any filling of a $k \times n$-rectangle with positive integers, then we let $\mathrm{red}(F)$ denote the filling which results from F by replacing the i-th smallest element of

F by i. For example, Figure 1.3 demonstrates a filling, F, with its corresponding reduced filling, $\mathrm{red}(F)$.

$$\begin{array}{|c|c|}\hline 5 & 9 \\ \hline 2 & 5 \\ \hline 2 & 3 \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

Figure 1.3: An example of an $F \in \mathcal{WI}_6^{0,0,3}$ and $\mathrm{red}(F)$.

If $F \in \mathcal{F}_{i+kn+j,\mathbb{P}}^{i,j,k}$ and $2 \leq c_1 < \cdots < c_s \leq n+1$, then we let $F[c_1, \ldots, c_s]$ be the filling of the $k \times s$ rectangle where the elements in column a of $F[c_1, \ldots, c_s]$ equal the elements in column c_a in F for $a = 1, 2, \ldots, s$. We can then extend the usual pattern matching definitions from permutations to elements of $\mathcal{F}_{i+kn+j,\mathbb{P}}^{i,j,k}$ as follows.

Definition 1.7. *Let P be an element of $\mathcal{F}_{kr,\mathbb{P}}^{0,0,k}$ and $F \in \mathcal{F}_{i+kn+j,\mathbb{P}}^{i,j,k}$ where $r \leq n$. Then we say*

1. *P **occurs** in F if there are $2 \leq i_1 < i_2 < \cdots < i_r \leq n+1$ such that $\mathrm{red}(F[i_1, \ldots, i_r]) = P$,*
2. *F **avoids** P if there is no occurrence of P in F,*
3. *there is a P**-match in** F **starting at position** i if $\mathrm{red}(F[i, i+1, \ldots, i+r-1]) = P$, and*
4. *F **consecutively avoids** P if F does not have P-matches.*

Clearly, consecutive patterns for an array F can be regarded as block patterns for $w(F)$. We note that P-matches are often referred to as consecutive pattern matches of P. When $i = j = 0$ and $k = 1$, then $\mathcal{P}_n^{0,0,1} = \mathcal{S}_n$, where $\mathcal{S}_n$ is the symmetric group, and our definitions reduce to the standard definitions that have appeared in the pattern matching literature. We note that Kitaev, Mansour, and Vella [34] have studied pattern matching in matrices which is a more general setting than the one we are considering for $i = j = 0$ in this paper.

1.1.4 c-Wilf equivalence

c-Wilf equivalence is consecutive version of Wilf equivalence. We take column-strict array patterns as an example. Given a pattern $P \in \mathcal{P}_{kj}^{0,0,k}$, we define generating functions as follows,

$$A_{P,\mathcal{P}}(x) := 1 + \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} \sum_{F\in\mathcal{P}_{kn}^{0,0,k}} x^{P\text{-mch}(F)}.$$

Definition 1.8. *Given two patterns $P, Q \in \mathcal{P}_{kj}^{0,0,k}$, we say P and Q are c-Wilf equivalent if*

$$A_{P,\mathcal{P}}(0,t) = A_{Q,\mathcal{P}}(0,t).$$

In the other words, two patterns $P, Q \in \mathcal{P}_{kj}^{0,0,k}$ are c-Wilf equivalent if for any n, the number of elements in $\mathcal{P}_{nj}^{0,0,k}$ avoiding P consecutively is always equal to the number of elements in $\mathcal{P}_{kn}^{0,0,k}$ avoiding Q consecutively. A stronger equivalence is called strong c-Wilf equivalence.

Definition 1.9. *Given two patterns $P, Q \in \mathcal{P}_{kj}^{0,0,k}$, we say P and Q are strongly c-Wilf equivalent if*

$$A_{P,\mathcal{P}}(x,t) = A_{Q,\mathcal{P}}(x,t).$$

Clearly, strong c-Wilf equivalence implies c-Wilf equivalence and researchers doubt c-Wilf equivalence also implies strong c-Wilf equivalence. In [47], Nakamura conjectured that if two permutations are c-Wilf equivalent then they are also strongly c-Wilf equivalent. Harmse and Remmel gave a similar conjecture in [27] when $k \geq 2$ which generalized the conjecture for permutation patterns.

Conjecture 1.10. *$P, Q \in \mathcal{P}_{kj}^{0,0,k}$ are c-Wilf equivalent if and only if P and Q are strongly c-Wilf equivalent.*

The conjecture is still open and in fact, even to find the c-Wilf equivalence classes for $\mathcal{S}_n$ is a difficult task. The number of equivalence classes in $\mathcal{S}_n$ is currently known up to $k = 6$, and they are 1, 1, 2, 7, 25, 92 [47]. For instance, there are two classes in $\mathcal{S}_3$, namely, $\{1\ 2\ 3,\ 3\ 2\ 1\}$ and $\{1\ 3\ 2,\ 3\ 1\ 2,\ 2\ 1\ 3,\ 2\ 3\ 1\}$. The number of equivalence classes in $\mathcal{S}_n$ is proved to be less than $1, 1, 2, 8, 32,\ 192, 1272, 10176,$ $90816, \cdots$ (see [51]). The problem of finding the c-Wilf equivalence classes for $\mathcal{P}_{kj}^{0,0,k}$ is completely open.

1.2 Ring homomorphism method

In this section, we review a remarkable approach, ring homomorphism method, to obtain generating function for consecutive pattern enumeration.

There is a long line of research that uses certain homomorphisms from the ring of symmetric functions Λ in infinitely many variable $x_1, x_2, \ldots$ to obtain results about generating functions for permutation statistics. This line of research started with the work of Brenti [8] who introduced a homomorphism ξ mapping Λ to the polynomial ring $\mathbb{Q}[x]$ over the rationals $\mathbb{Q}$ that demonstrated a remarkable connection between permutation enumeration and symmetric functions. Let the elementary symmetric function e_n and the homogeneous symmetric function h_n be defined by

$$\begin{aligned} \sum_{n\geq 0} h_n t^n &= \prod_{i\geq 1} \frac{1}{1-x_i t} \text{ and} \\ \sum_{n\geq 0} e_n t^n &= \prod_{i\geq 1} 1 + x_i t. \end{aligned}$$

Then Brenti defined a ring homomorphism $\xi : \Lambda \to Q[x]$ by setting for $k \geq 1$,

$$\xi(e_k) = \frac{(x-1)^{k-1}}{k!}$$

and setting $\xi(e_0) = 1$. Let $p_k = \sum_{i\geq 1} x_i^k$, denote the k-th power symmetric function. Also, for a permutation σ in the symmetric group $\mathcal{S}_n$, let $\text{des}(\sigma)$ and $\text{exc}(\sigma)$ denote the number of descents and excedences of σ, respectively. Then Brenti proved

$$\begin{aligned} n!\xi(h_n) &= \sum_{\sigma\in\mathcal{S}_n} x^{\text{des}(\sigma)} \text{ and} \\ \frac{n!}{z_\lambda}\xi(p_\lambda) &= \sum_{\sigma\in\mathcal{S}_n(\lambda)} x^{\text{exc}(\sigma)} \end{aligned} \qquad (1.2)$$

where if $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ is a partition of n, then $\mathcal{S}_n(\lambda)$ is the set of permutations in $\mathcal{S}_n$ with cycle type λ and $z_\lambda = \prod_{i=1}^n i^{m_i} m_i!$.

Brenti's proofs were mainly algebraic. However in [6], Beck and Remmel gave combinatorial proofs of Brenti's results that used the combinatorial interpretations of the entries of the connection matrices between various bases of symmetric function introduced by Eğecioğlu and Remmel [15]. These combinatorial proofs suggested natural modifications of Brenti's original homomorphism ξ that could be used to obtain q-analogues of Brenti's results [8] or to obtain similar permutation enumeration results for other groups such as the hyperoctahedral group B_n [5] or wreath products, $\mathcal{C}_k \wr \mathcal{S}_n$, of cyclic groups $\mathcal{C}_k$ with the symmetric group $\mathcal{S}_n$ [63]. For example, Beck and Remmel [6] defined a homomorphism $\xi_q : \Lambda \to Q(q)[x]$ by

$$\xi_q(e_k) = \frac{(x-1)^{k-1}q^{\binom{k}{2}}}{[k]_q!}$$

where for a positive integer k, $[k]_q = 1 + q + \cdots + q^{k-1}$ and
$[k]_q! = [k]_q[k-1]_q \cdots [1]_q$. They proved that

$$\begin{aligned} [n]_q!\xi_q(h_n) &= \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)} \\ [n]_q!\xi_q(p_n) &= \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{rise}(\sigma)-f(\sigma)+1} q^{\mathrm{coinv}(\sigma)} \left(x^{f(\sigma)} - (x-1)^{f(\sigma)}\right) \end{aligned} \tag{1.3}$$

where rise(σ) and coinv(σ) are the number of rises and coinversions of σ, respectively, and $f(\sigma)$ is the length of the last increasing sequence of σ when σ is written in one-line notation.

Later, Harmse and Remmel applied this method to study pattern matching in column-strict arrays [27], Jones and Remmel used this method to obtain left-to-right minima and distribution of descents in permutations avoiding certain patterns [29], and Duane and Remmel also studied minimal overlapping patterns in colored permutations using this method [14]. More details about the homomorphism method and its history can be found in the book by Mendes and Remmel [44].

Of particular interest to us is the work of Duane and Remmel [14] who introduced the notion of minimally overlapping patterns in permutations, words, and colored permutations. We will discuss these types of patterns in the next section.

1.2.1 Minimally overlapping patterns

One special class of patterns are called minimally overlapping patterns (sometimes also called non-overlapping patterns [7] [17]). For a pattern P of length n, if there does not exist an integer $1 < k < n$ such that the subpattern consisting of the first k positions in P matches the subpattern consisting of the last k positions in P, we say P is **minimally overlapping** or has **minimal overlapping** property.

Take permutation pattern as an example, suppose $\tau \in \mathcal{S}_j$, we say that τ has the minimal overlapping property or τ is minimally overlapping if for any integer m, $1 < m < j$, we have

$$\mathrm{red}(\tau_1\tau_2\cdots\tau_m) \neq \mathrm{red}(\tau_{j-m+1}\cdots\tau_{j-1}\tau_j).$$

Alternatively, $\tau \in \mathcal{S}_j$ is minimally overlapping if the smallest n such that there exists $\sigma \in \mathcal{S}_n$ such that $\tau\text{-mch}(\sigma) = 2$ is $2j - 1$. This means in any two consecutive τ-matches in a permutation σ can share at most one position which must necessarily be at the end of the first τ-match and the start of the second τ-match. It follows that if $\tau \in \mathcal{S}_j$ is minimally overlapping, then the smallest n such that there exists a $\sigma \in \mathcal{S}_n$ such that $\tau\text{-mch}(\sigma) = k$ is $k(j-1)+1$. A $\sigma \in \mathcal{S}_{k(j-1)+1}$ such that $\tau\text{-mch}(\sigma) = k$ is called a **maximum packing** for τ.

Similarly, for column-strict arrays, pattern $P \in \mathcal{P}^{0,0,k}_{kj}$ is minimally overlapping if and only if for any integer m, $1 < m < j$, the reduction of first m columns of P is different from the reduction of last m columns of P, that is, $\mathrm{red}(P[1, 2, \cdots, m]) \neq \mathrm{red}(P[j-m+1, j-m+2, \cdots, j])$.

Minimally overlapping patterns are nice because we are then able to define maximum packings for minimally overlapping patterns in that we have a nice expression for generating functions in terms of maximum packings. The definition of maximum packings are discussed in next subsection.

Actually determining the percentage of minimally overlapping pattern among all patterns of given length is itself is also a research topic. For example, Bóna [7] found that the lower bound of percentage of minimally overlapping permutation patterns in $\mathcal{S}_n$ is $3-e \approx 0.2817$ and showed that the percentage is convergent as the length of permutations increases. The author and Remmel [52] extended Bóna's result to column strict arrays, generalized Euler permutations and standard Young tableaux of rectangular shapes. Moreover, for arrays of height k where $k \geq 2$, regardless of number of columns, proportion of minimally overlapping patterns converges to 1 very fast as k increases.

Another fact worth mentioning fact is that it has been proved by Duane and Remmel in the case where $k = 1$ and Harmse and Remmel in the case where $k \geq 2$ that Conjecture 1.10 holds for minimally overlapping patterns in column-strict arrays $\mathcal{P}_{kn}^{0,0,k}$. That is, the following result holds.

Theorem 1.11. *Suppose $P, Q \in \mathcal{P}_{kj}^{0,0,k}$ are minimally overlapping patterns, then P and Q are strongly c-Wilf equivalent if and only if P and Q are c-Wilf equivalent.*

1.2.2 Maximum packings

For minimally overlapping patterns, we are able to define its maximum packings. Take a minimally overlapping permutation pattern $\tau \in \mathcal{S}_j$ as an example. A permutation in $\mathcal{S}_{k(j-1)+1}$ that has exactly k τ-matches is called a **maximum packings** for τ and we let $\mathcal{MPK}_{\tau,k(j-1)+1} = \{\sigma \in \mathcal{S}_{k(j-1)+1} : \tau\text{-mch}(\sigma) = k\}$. We let $\text{mp}_{\tau,k(j-1)+1} = |\mathcal{MPK}_{\tau,k(j-1)+1}|$.

For example, $\tau = 1\ 3\ 2$ is a minimally overlapping pattern in $\mathcal{S}_3$, then $\text{mp}_{\tau,5} = 3$ because $\mathcal{MPK}_{\tau,5} = \{1\ 3\ 2\ 5\ 4,\ 1\ 4\ 2\ 5\ 3,\ 1\ 5\ 2\ 4\ 3\}$.

In [14], for a minimally overlapping pattern τ, Duane and Remmel showed that the generating function $A_{\tau,\mathcal{S}}(x,t)$ defined in Equation 1.1 can be expressed in terms of $\text{mp}_{\tau,n}$.

Theorem 1.12. *If $\tau \in \mathcal{S}_j$ has the minimal overlapping property, then*

$$A_{\tau,\mathcal{S}}(x,t) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{S}_n} x^{\tau\text{-}mch(\sigma)} = \frac{1}{1 - \left(t + \sum_{n\geq 1} \frac{t^{n(j-1)+1}}{(n(j-1)+1)!}(x-1)^n \text{mp}_{\tau,n(j-1)+1}\right)}. \quad (1.4)$$

According to Theorem 1.12, we see that the number of maximum packings $\text{mp}_{\tau,n(j-1)+1}$ determines $A_{\tau,\mathcal{S}}(0,t)$ and also $A_{\tau,\mathcal{S}}(x,t)$ is completely determined by $\text{mp}_{\tau,n(j-1)+1}$, which indicates that Conjecture 1.10 holds for minimally overlapping permutation patterns.

In [27], Harmse and Remmel extended Theorem 1.12 for column-strict arrays. It has been shown that Conjecture 1.10 holds for minimally overlapping patterns and the first and the last column of a minimally overlapping pattern determines which c-Wilf equivalence class it belongs to (see [12], [14], [16], and [27]). The key to proving such results is to prove an analogue to Theorem 1.12. This was done by Harmse and Remmel [27] who proved the following theorem for minimally overlapping patterns in $\mathcal{P}_{kj}^{0,0,k}$.

Theorem 1.13. *Suppose that* $k \geq 2$, $j \geq 2$, *and* $P \in \mathcal{P}_{kj}^{0,0,k}$ *has the minimal overlapping property. Then*

$$A_{P,\mathcal{P}}(x,t) = 1 + \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} \sum_{F \in \mathcal{P}_{kn}^{0,0,k}} x^{P\text{-}mch(F)} = \frac{1}{1 - \left(\frac{t^k}{k!} + \sum_{n\geq 1} \frac{t^{n(j-1)+1}}{(k(n(j-1)+1))!}(x-1)^n \mathrm{mp}_{P,n(j-1)+1}\right)}. \tag{1.5}$$

Therefore, to obtain the formula for generating functions we only need to compute the number of maximum packings.

1.3 Cluster method

The cluster method, which is based on inclusion-exclusion principle, was first introduced by Goulden and Jackson [23] [24] when they studied pattern matching in words in 1979. It didn't gain much attention at the beginning until Noonan and Zeilberger re-emphasized importance of the cluster method in 1999 in [48]. Now it has been widely utilized to solve pattern matching in permutations. Rawlings [54] used cluster method to find enumeration formulas for permutation patterns in form of $1\ 2\ \cdots\ m$, $1\ 2\ \cdots\ (m-2)\ m\ (m-1)$ and $1\ m\ (m-1)\ \cdots\ 2$. Elizalde and Noy gave a more general discussion about permutation pattern matching using cluster method in [19].

When we use ring homomorphism method for pattern matching, it's required that the pattern is minimally overlapping. However, the cluster method works generally for arbitrary patterns although it might be very difficult in computation.

1.3.1 Clusters

Given a permutation pattern $\tau \in \mathcal{S}_j$ and a permutation $\sigma \in \mathcal{S}_n$ satisfying τ-mch$(\sigma) > 0$, we mark some of the τ-matches in σ by putting 'x' on the top of the first element in the match. We could even mark all of the τ-matches or none of them. It's apparent that for given a pattern $\tau \in \mathcal{S}_j$ and a permutation $\sigma \in \mathcal{S}_n$, there are $2^{\tau\text{-mch}(\sigma)}$ distinct τ-marked σ's. We let $\mathcal{MS}_{n,\tau}$ denote the set of all the τ-marked permutations in $\mathcal{S}_n$. For example, for pattern $\tau = 1\ 3\ 2$ and $\sigma = 1\ 4\ 2\ 5\ 3$, there are four distinct τ-marked σ's, shown in Figure 1.4.

$$\begin{array}{ccccc} & & & & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{ccccc} x & & & & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{ccccc} & & x & & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{ccccc} x & & x & & \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Figure 1.4: Four τ-marked σ's

A **τ-cluster** of length n for pattern $\tau \in \mathcal{S}_j$ is a permutation σ in $\mathcal{S}_n$ such that

1. each element of σ is contained in some marked τ-matches and
2. any two consecutive marked τ-matches share at least one element.

We let $m_\tau(\sigma)$ denote the number of marked τ-matches in σ. The set of all the τ-clusters of size n is denoted by $\mathcal{C}_{n,\tau}$.

Accordingly we could extend the notion of clusters to arrays, such as $\mathcal{P}_{kn}^{0,0,k}$, $\mathcal{WI}_{kn,A}^{0,0,k}$, $\mathcal{SI}_{kn,A}^{0,0,k}$ and even $\mathcal{P}_{i+kn+j}^{i,j,k}$. Details about these clusters will be discussed in following chapters.

1.3.2 Cluster polynomials

Based on the cluster method, the exponential generating function $A_{\tau,\mathcal{S}}(x,t)$ for a pattern $\tau \in \mathcal{S}_j$ can be expressed as (see [19])

$$A_{\tau,\mathcal{S}}(x,t) = \frac{1}{1 - \left(t + \sum_{n\geq 2} C_{\tau,n}(x-1)\frac{t^n}{n!}\right)}, \tag{1.6}$$

where $C_{\tau,n}(x)$ is so-called cluster polynomial. It was shown in [44] that (1.6) can also be derived via the homomorphism method.

A **cluster polynomial** for pattern τ of size n is defined as

$$C_{\tau,n}(x) := \sum_{\sigma \in \mathcal{C}_{\tau,n}} x^{m_\tau(\sigma)}.$$

Hence, the problem of find generating functions is now converted to finding formulas for corresponding cluster polynomials which is typically easier than to find generating functions directly.

For example, assume $\tau = 1\ 2\ 3$, then the cluster polynomials are as follows,

$$\begin{array}{rcl} C_{123,3}(x) & = & x \\ C_{123,4}(x) & = & x^2 \\ C_{123,5}(x) & = & x^3 + x^2 \\ & \cdots & \\ C_{123,n}(x) & = & x(C_{123,n-2}(x) + C_{123,n-1}(x)). \end{array}$$

In the next section (1.6), we shall extend the concept of cluster polynomials to arrays.

1.4 Patterns in restricted arrays

From the results above, we can see that we could use either the homomorphism method or the cluster method to find generating functions for the number of consecutive occurrence of a pattern in permutations, arrays and various other combinatorial objects. The main focus of this thesis is to extend the cluster method to find generating functions for the number of consecutive occurrences of patterns in various subclasses of permutations, words, and arrays. In the next subsection, we shall discuss various natural examples of such subclasses.

1.4.1 Restricted arrays

Moreover, we could add various restrictions to arrays in $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s]}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s]}$, and $\mathcal{P}^{i,j,k}_{i+kn+j}$. For example, suppose that $i = j = 0$ and $k \geq 2$. In this case, we label the columns of $D^{0,0,k}_n$ with $1, \ldots, n$, reading from left to right, and the rows with $1, \ldots, k$, reading from bottom to top.

1. Elements of $\mathcal{E}^{0,0,k}_{kn}$ are the elements $F \in \mathcal{P}^{0,0,k}_{kn}$ such that satisfy the additional restriction that $F(s,k) > F(s+1,1)$ for $s = 1,\ldots,n-1$.
2. Standard tableaux of shape n^k are the elements $F \in \mathcal{P}^{0,0,k}_{kn}$ such that satisfy the additional restriction that $F(s,r) < F(s+1,r)$ for $s = 1,\ldots,n-1$ and $r = 1,\ldots,k$.
3. Column strict tableaux of shape n^k are the elements $F \in \mathcal{SI}^{0,0,k}_{kn.\mathbb{P}}$ such that satisfy the additional restriction that $F(s,r) \leq F(s+1,r)$ for $s = 1,\ldots,n-1$ and $r = 1,\ldots,k$.
4. Words with no consecutive repeated letters can be viewed at elements $\mathcal{WI}^{0,0,1}_{n,\mathbb{P}}$ with the restriction that $F(s,1) \neq F(s+1,1)$ for $s = 1,\ldots,n-1$.

In each of these cases, one can describe our collection of elements as the set of elements in $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s]}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s]}$, or $\mathcal{P}^{i,j,k}_{i+kn+j}$ whose consecutive columns satisfy a certain binary relation. That is, let $\mathscr{R}$ be some binary relation $\mathscr{R}$ between pairs of columns of integers. Then we let $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P},\mathscr{R}}$ ($\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P},\mathscr{R}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s],\mathscr{R}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s],\mathscr{R}}$, $\mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}$) denote the set of all elements F in $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P}}$ ($\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s]}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s]}$, $\mathcal{P}^{i,j,k}_{i+kn+j}$) such that for all $1 \leq i < n$, $(F[i], F[i+1]) \in \mathscr{R}$. For example, consider the following relations $\mathscr{R}$.

1. Let $\mathscr{R}$ is the relation that holds between a pair if columns of integers (C,D) if and only if the top element of C is greater than the bottom element of D. Then it is easy to see that $\mathcal{E}^{i,j,k}_{kn}$ equals $\mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}$.
2. Let $\mathscr{R}$ is the relation that holds between a pair of columns of integers (C,D) if and only if in the array CD, the rows are strictly increasing. Then it is easy to see that set of standard tableaux of shape n^k equals $\mathcal{P}^{0,0,k}_{kn,\mathscr{R}}$ and the set of row tableaux of shape n^k equals $\mathcal{WI}^{0,0,k}_{kn,\mathbb{P},\mathscr{R}}$.
3. Let $\mathscr{R}$ is the relation that holds between a pair of columns of integers (C,D) of integers if and only if in the array CD, the rows are weakly increasing. Then it is easy to see that the set of column strict tableaux of shape n^k equals $\mathcal{SI}^{0,0,k}_{kn,\mathbb{P},\mathscr{R}}$.

4. Let $\mathscr{R}$ is the relation that holds between a pair of integers (a, b) if and only if $a \neq b$. Then it is easy to see that the set of words with no consecutive repeated letters equals $\mathcal{WI}^{0,0,1}_{n,\mathbb{P},\mathscr{R}}$.

1.4.2 Generalized clusters

Although ring homomorphism method and cluster method can solve pattern enumeration in permutations, words or arrays, they fail to deal with such objects with restrictions.

Remmel [55] extended maximum packings to generalized maximum packings in order to solve length-4 pattern matching in up-down permutations. As mentioned in Chapter 1.4.1, up-down permutations of even lengths could be treated as a class of two-row columns strict arrays with some special restrictions. In scenario of $\mathcal{S}_4$, an up-down permutation of length 4 is always overlapping while if we think of an up-down permutation of length 4 as an element in $\mathcal{P}^{0,0,2}_4$, it is always a minimally overlapping pattern.

Generalized maximum packing gives us an inspiration to solve pattern matching in either permutations, words or arrays with customized restrictions. In this dissertation, we develop a so-called **generalized cluster method** which enables us to express the generating functions for pattern matching in restricted objects in terms of so-called generalized cluster polynomials.

The main result of this paper will allow to find generating functions of P-matches in $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P},\mathscr{R}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P},\mathscr{R}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s],\mathscr{R}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s],\mathscr{R}}$, and $\mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}$. We note that by varying the binary relation $\mathscr{R}$, we can study the distribution of P-matches in fillings with other types of restrictions such as elements of $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s]}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s]}$, or $\mathcal{P}^{i,j,k}_{i+kn+j}$ whose first row is strictly increasing, reading from left to right, or elements $\mathcal{WI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,\mathbb{P}}$, $\mathcal{WI}^{i,j,k}_{i+kn+j,[s]}$, $\mathcal{SI}^{i,j,k}_{i+kn+j,[s]}$, or $\mathcal{P}^{i,j,k}_{i+kn+j}$ whose elements in even rows are strictly increasing, reading from left to right, and whose elements in odd rows are strictly decreasing, reading from left to right.

Furthermore, we are also able to obtain a multi-variate generating function for multiple pattern matchings by constructing generalized joint clusters. For instance, suppose we have m patterns, $P_1, P_2, \cdots, P_m$, a multi-variate version of a generating function is in form of

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{F \in \mathcal{R}_n} \prod_{i=1}^{m} x_i^{P_i\text{-mch}(F)}, \tag{1.7}$$

where $\mathcal{R}_n$ stands for a set of restricted combinatorial objects of size n.

1.5 Outline

In Chapter 2, we shall discuss clusters and generalized clusters for fillings of diagrams of rectangular shapes, that is, $D_{kn}^{0,0,k}$. Three examples are given and they are patterns in words with no consecutive repeats, patterns in Young tableaux of rectangular shapes and shortest loops in non-backtracking walks.

In Chapter 3, we discuss clusters and generalized clusters for fillings of $D_{kn+j}^{0,j,k}$, $D_{i+kn+j}^{i,0,k}$ and $D_{i+kn+j}^{i,j,k}$ respectively. We take up-down patterns in down-up permutations as an example to further illustrate clusters and generalized clusters for $D_{kn+j}^{i,j,k}$.

In Chapter 4, we extend clusters and generalized clusters to a multi-variate version and then we can study joint distribution of multiple patterns in restricted combinatorial objects. For examples, we consider co-runs in restricted colored permutations and multiple patterns of Carlitz integer compositions.

In Chapter 5, we extend the discussion to arrays of undetermined shapes. We proved that the cluster and generalized cluster method still work for patterns in such situation. For examples, we consider patterns in ordered set partitions and patterns in cycle structures of permutations. It is also pointed out that clusters and generalized clusters can be extended to arrays of undetermined shapes with some partial restrictions.

Although generalized cluster method is a powerful method of find distributions of patterns in various restricted combinatorial objects, we discuss its limitation in the final chapter. We also discuss its connection to joint clusters and directions of further research.

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mal overlapping patterns for generalized Euler permutations, standard tableaux of rectangular shapes, and column strict arrays, Discret. Math. and Theoretical Computer Science, **18–2** (2016), # 6.

Chapter 2

Cluster and Generalized Clusters for fillings of $D_{kn}^{0,0,k}$.

In this chapter, we shall describe cluster method and generalized cluster method in a fundamental situation where only fillings of rectangular shapes are considered. A few examples are given with details to explain how cluster and generalized cluster method work for block patterns in fillings of $D_{kn}^{0,0,k}$. Recall that $D_{kn}^{0,0,k}$ is a $k \times n$ rectangular diagram. $D_{18}^{0,0,3}$ is pictured in Figure 2.1 as an example.

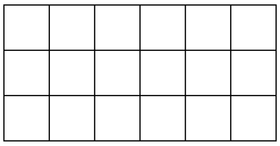


Figure 2.1: $D_{18}^{0,0,3}$.

2.1 Main theorems

In this section, we shall describe the cluster method and our generalized cluster method for fillings of $D_{kn}^{0,0,k}$. We shall start with the special case of elements in $\mathcal{P}_{kn}^{0,0,k}$.

We start by recalling the definition of clusters for permutations. Let $\tau \in \mathcal{S}_j$ be a permutation. Then for any $n \geq 1$, we let $\mathcal{MS}_{n,\tau}$ denote the set of all permutations in $\sigma \in \mathcal{S}_n$ where we have marked some of the τ-matches in σ by placing an x at the start of τ-match in σ. For example, suppose that $\tau = 1\ 3\ 2$ and $\sigma = 1\ 5\ 4\ 7\ 8\ 2\ 6\ 3$. Then there are two τ-matches in σ, one starting at position 1 and one starting at position 6. Thus σ gives rises to four elements of $\mathcal{MS}_{8,\tau}$.

$$\begin{array}{ll} 1\ 5\ 4\ 7\ 8\ 2\ 6\ 3 & \overset{x}{1}\ 5\ 4\ 7\ 8\ 2\ 6\ 3 \\ 1\ 5\ 4\ 7\ 8\ \overset{x}{2}\ 6\ 3 & \overset{x}{1}\ 5\ 4\ 7\ 8\ \overset{x}{2}\ 6\ 3 \end{array}$$

A τ-cluster is an element of $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{MS}_{n,\tau}$ such that

1. every σ_i is an element of a marked τ-match in σ and
2. any two consecutive marked τ-matches share at least one element.

We let $\mathcal{CMS}_{n,\tau}$ denote the set of all τ-clusters in $\mathcal{MS}_{n,\tau}$. Given a τ-cluster $\sigma \in \mathcal{MS}_{n,\tau}$, we let $m_\tau(\sigma)$ be the number of marked τ-matches in σ. For each $n \geq 1$, we define the cluster polynomial

$$C_{n,\tau}(x) = \sum_{\sigma \in \mathcal{CMS}_{n,\tau}} x^{m_\tau(\sigma)}.$$

For example, we say that a permutation $\tau \in \mathcal{S}_j$ is *minimal overlapping* if the smallest n such that there exists a $\sigma \in \mathcal{S}_n$ where $\tau\text{-mch}(\sigma) = 2$ is $2j-1$. This means that two consecutive τ-matches in a permutation σ can share at most one element which must be the element at the end of the first τ-match and the element which is at the start of the second τ-match. In such a situation, the smallest m such that there exists a $\sigma \in \mathcal{S}_m$ such that $\tau\text{-mch}(\sigma) = n$ is $n(j-1)+1$. We call elements of $\sigma \in \mathcal{S}_{n(j-1)+1}$ such that $\tau\text{-mch}(\sigma) = n$ *maximum packings* of τ. We let $\mathcal{MP}_{n(j-1)+1}$ denote the set of maximum packings for τ in $\mathcal{S}_{n(j-1)+1}$ and $mp_{n(j-1)+1,\tau} = |\mathcal{MP}_{n(j-1)+1}|$. It is easy to see that if $\tau \in \mathcal{S}_j$ is minimal overlapping, then the only τ-clusters are maximum packings for τ where the start of each τ-match is marked with an x. For example, $\tau = 132$ is a minimal overlapping permutation and it is easy to compute the number of maximum packings of size $2n+1$ for any $n \geq 1$. That is, if $\sigma = \sigma_1 \ldots \sigma_{2n+1}$ is in $\mathcal{MP}_{2n+1,132}$, then there must be 132-matches starting at positions $1, 3, 5, \ldots, 2n-1$. It easily follows that for each $i = 0, \dots, n-1$, σ_{2i+1} is smaller than σ_j for all $j > 2i+1$. Hence $\sigma_1 = 1$ and $\sigma_3 = 2$. We then have $2n-1$ choices for σ_3. Hence, it follows that

$$mp_{2n+1,132} = (2n-1)mp_{2n-1,132} = \prod_{i=0}^{n-1}(2i+1).$$

Thus $C_{2n+1,132}(x) = x^n \prod_{i=0}^{n-1}(2i+1)$ for all $n \geq 1$ and $C_{2n,132}(x) = 0$ for all $n \geq 1$.

On the other hand, suppose that $\tau = 1234$. It is easy to see that for any τ-cluster in $\mathcal{CMS}_{n,\tau}$ where $n \geq 4$, the underlying permutation must be the identity permutation. Moreover, if $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{CMS}_{n,\tau}$, then σ_1 must be marked with an x because σ_1 must be an element in a marked τ-match and σ_{n-3} must be marked since σ_n must be an element of a marked τ-match. Thus for $n = 7$, we are forced to mark 1 and 4,

$$\overset{x}{1}23\overset{x}{4}567.$$

However we free to mark either 2 or 3 with an x. Hence

$$C_{7,1234}(x) = x^2(1+x)^2.$$

Then Goulden and Jackson's [24] proved the following theorem.

Theorem 2.1. *Let $\tau \in \mathcal{S}_j$ where $j \geq 2$. Then*

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{S}_n} x^{\tau\text{-mch}(\sigma)} = \frac{1}{1 - t - \sum_{n \geq 2} \frac{t^n}{n!} C_{n,\tau}(x-1)}.$$

It is easy to generalize this result to deal with elements of $\mathcal{P}_{nk}^{0,0,k}$. Suppose that we are given a filling $P \in \mathcal{P}_{kr}^{0,0,k}$. For any $n \geq 1$, we let $\mathcal{MP}_{kn,P}^{0,0,k}$ denote the set of all fillings $F \in \mathcal{P}_{nk}^{0,0,k}$ where we have marked some of the P-matches in F by placing an x on top of the column that start a P-match in σ. For example, suppose that $P = \begin{array}{|c|c|c|}\hline 4 & 6 & 5 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ and $F \in \mathcal{P}_{12}^{0,0,2}$ pictured in Figure 2.2. Then there are two P-matches in F, one starting at column 1 and one starting at column 4. Thus F gives rise to four elements of $\mathcal{MP}_{12}^{0,0,2}$. Given a $F \in \mathcal{MP}_{kn,P}^{0,0,k}$, we let $m_P(F)$ be the number of marked P-matches in F.

We can also extend the reduction operation to P-marked fillings. That is, suppose $P \in \mathcal{P}_{jk}^{0,0,k}$ and F is a filling of the $k \times n$ array with integers which strictly

8	11	10	7	12	9
3	4	5	1	2	6

x					
8	11	10	7	12	9
3	4	5	1	2	6

			x		
8	11	10	7	12	9
3	4	5	1	2	6

x			x		
8	11	10	7	12	9
3	4	5	1	2	6

Figure 2.2: P-marked fillings.

increasing in columns, reading from bottom to top, where we have marked some of the P-matches by placing an x at the top of the column that starts a marked P-match. Then by red(F), we mean the element of $\mathcal{MP}_{kn}^{0,0,k}$ that results by replacing the i^{th} smallest element in F by i and marking a column in red(F) if and only if it is marked in F.

A P-cluster is a filling of $F \in \mathcal{MP}_{kn,P}^{0,0,k}$ such that

1. every column of F is contained in a marked P-match of F and
2. any two consecutive marked P-matches share at least one column.

P

4	5	6
2	1	3

Q

x		x		
5	6	8	9	10
3	2	4	1	7

T

		x		
5	6	8	9	10
3	2	4	1	7

Figure 2.3: Q is a P-cluster but T is not.

In Figure 2.3, Q is a P-cluster while T is not a P-cluster.

We let $\mathcal{CM}_{kn,P}^{0,0,k}$ denote the set of all P-clusters in $\mathcal{MP}_{kn,P}^{0,0,k}$. For each $n \geq 2$, we define the cluster polynomial

$$C_{kn,P}^{0,0,k}(x) = \sum_{F \in \mathcal{CM}_{kn,P}^{0,0,k}} x^{m_P(F)}$$

where $m_P(F)$ is the number of marked P-matches in F. By convention, we let $C_{k,P}^{0,0,k}(x) = 1$.

Theorem 2.2. *Let $P \in \mathcal{P}^{0,0,k}_{jk}$ where $j \geq 2$. Then*

$$1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{P}^{0,0,k}_{kn}} x^{P\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,P}(x-1)}. \tag{2.1}$$

Proof. Replace x by $x+1$ in (2.1). Then the left-hand side of (2.1) is the generating function of $m_P(F)$ over all $F \in \mathcal{MP}^{0,0,k}_{kn,P}$. That is, it easy to see that

$$1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{P}^{0,0,k}_{kn}} (x+1)^{P\text{-mch}(F)} = 1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{MP}^{0,0,k,P}_{kn}} x^{m_P(F)}. \tag{2.2}$$

Thus we must show that

$$1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{MP}^{0,0,k}_{kn}} x^{m_P(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,P}(x)}. \tag{2.3}$$

Now

$$\frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,P}(x)} = 1+\sum_{m\geq 1}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,P}(x)\right)^m. \tag{2.4}$$

Taking the coefficient of $\frac{t^{ks}}{(ks)!}$ on both sides of (2.3) where $n \geq 1$, we see that we must show that

$$\begin{aligned}
\sum_{F\in\mathcal{MP}^{0,0,k}_{sn}} x^{m_P(F)} &= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,P}(x)\right)^m \Big|_{\frac{t^{ks}}{(ks)!}} \\
&= \sum_{m=1}^{s}\left(\sum_{n=1}^{s}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,P}(x)\right)^m \Big|_{\frac{t^{ks}}{(ks)!}} \\
&= \sum_{m=1}^{s}\sum_{\substack{a_1+a_2+\cdots+a_m=s\\ a_i\geq 1}}\binom{kn}{ka_1,\ldots,ka_m}\prod_{j=1}^{m}C^{0,0,k}_{ka_j,P}(x). \qquad (2.5)
\end{aligned}$$

The right-hand side of (2.5) is now easy to interpret. First we pick an m such that $1 \leq m \leq s$. Then we pick $a_1,\ldots,a_m \geq 1$ such that $a_1+a_2+\cdots+a_m = s$. Next the binomial coefficient $\binom{kn}{ka_1,\ldots,ka_m}$ allows us to pick sets $S_1,\ldots,S_m$ which partition $\{1,\ldots,ks\}$ such that $|S_i| = ka_i$ for $i = 1,\ldots,m$. Finally the product $\prod_{j=1}^{m} C^{0,0,k}_{ka_j,P}(x)$ allows us to pick clusters $C_i \in \mathcal{CM}^{0,0,k}_{ka_i,P}$ for $i = 1,\ldots,m$ with weight $\prod_{j=1}^{m} x^{m_P(C_i)}$. Note that in the cases where $a_i = 1$, we will interpret C_i as just a column of height k filled with the numbers $1,\ldots,k$ which is increasing, reading from bottom to top.

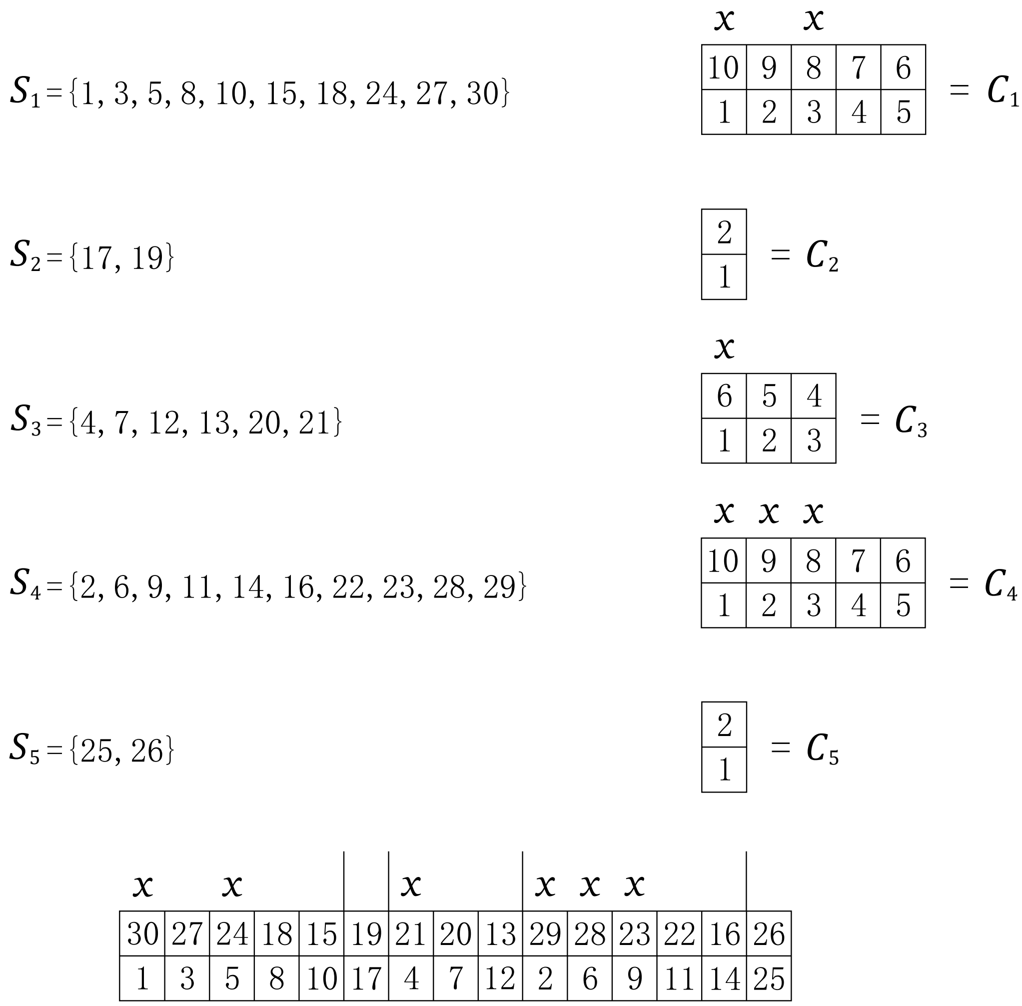


Figure 2.4: Construction for the right-hand side of (2.5).

For example, suppose that $k = 2$ and $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Then in Figure 2.4, we have pictured S_1, S_2, S_3, S_4, S_5 which partition $\{1, \ldots, 30\}$ and corresponding clusters $C_1, \ldots, C_5$. Then for each $i = 1, \ldots, m$, we create a cluster D_i which results by replacing the j in C_i by the j^{th} element of S_i. If we concatenate $D_1 \ldots D_m$ together, then we will obtain an element of $Q \in \mathcal{MP}^{0,0,k}_{kn,P}$. It is easy to see that one can recover $D_1, \ldots, D_5$ from Q. That is, given an element $F \in \mathcal{MP}^{0,0,k}_{kn,P}$, we say that a marked subsequence $F[i, i+1, \ldots, j]$ is a *maximal P-subcluster* of F if $\text{red}(F[i, i+1, \ldots, j])$ is a P-cluster and $F[i, i+1, \ldots, j]$ is not properly contained in a marked subsequence $F[a, a+1, \ldots, b]$ such that $\text{red}(F[a, a+1, \ldots, b])$ is a

P-cluster. In the special case where $i = j$ and the column $F[i]$ is not marked, then we say that $F[i]$ is maximal P-subcluster if $F[i]$ is not properly contained in a marked subsequence $F[a, a+1, \ldots, b]$ such that $\mathrm{red}(F[a, a+1, \ldots, b])$ is a P-cluster. Thus $D_1, \ldots, D_5$ are the maximal P-subclusters of Q. Of course, once we have recovered $D_1, \ldots, D_5$, we can recover the sets $S_1, \ldots, S_5$ and the p-clusters $C_1, \ldots, C_5$.

In this manner, we can see that the right-hand side of (2.5) just classifies the elements of $\mathcal{MP}^{0,0,k}_{kn,P}$ by its maximal P-subclusters which proves our theorem.

$\square$

Next suppose that we are given a binary relation $\mathscr{R}$ between $k \times 1$ arrays of integers and a pattern $P \in \mathcal{P}^{0,0,k}_{kj}$.

Definition 2.3. *We say that* $Q \in \mathcal{MP}^{0,0,k}_{kn,P}$ *is a* **generalized** $P, \mathscr{R}$**-cluster** *if we can write* $Q = B_1 B_2 \cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

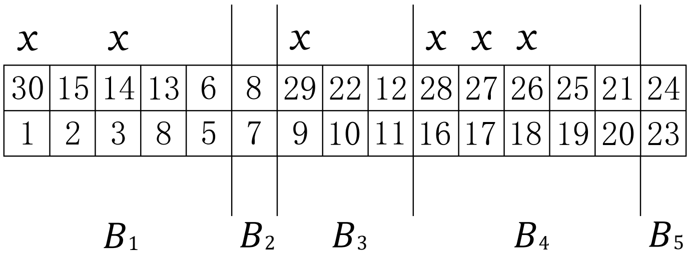
1. *either* B_i *is a single column or* B_i *consists of* r*-columns where* $r \geq 2$, $\mathrm{red}(B_i)$ *is a* P*-cluster in* $\mathcal{MP}_{kr,P}$*, and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*

2. *for* $1 \leq i \leq m-1$*, the pair* $(last(B_i), first(B_{i+1}))$ *is not in* $\mathscr{R}$ *where for any* i*,* $last(B_i)$ *is the right-most column of* B_i *and* $first(B_i)$ *is the left-most column of* B_j*.*

Let $\mathcal{GC}^{0,0,k}_{kn,P,\mathscr{R}}$ denote the set of all generalized $P, \mathscr{R}$-clusters which have n columns of height k. For example, suppose that $\mathscr{R}$ is the relation that holds for a pair of columns (C, D) if and only if the top element of column C is greater than the bottom element of column D and $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline\end{array}$.

Then in Figure 2.5, we have pictured a generalized $P, \mathscr{R}$-cluster with 5 blocks B_1, B_2, B_3, B_4, B_5.

Given $Q = B_1 B_2 \ldots B_m \in \mathcal{GC}^{0,0,k}_{kn,P,\mathscr{R}}$, we define the weight of B_i, $\omega_{P,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{m_P(\mathrm{red}(B_i))}$ if B_i is order isomorphic to a P-cluster.



x		x				x			x	x	x			
30	15	14	13	6	8	29	22	12	28	27	26	25	21	24
1	2	3	8	5	7	9	10	11	16	17	18	19	20	23

Figure 2.5: A generalized $P,\mathscr{R}$-cluster.

Then we define the weight of Q, $\omega_{P,\mathscr{R}}(Q)$, to be $(-1)^{m-1}\prod_{i=1}^{m}\omega_{P,\mathscr{R}}(B_i)$. We let

$$GC^{0,0,k}_{kn,P,\mathscr{R}}(x) = \sum_{Q\in\mathcal{GC}^{0,0,k}_{kn,P,\mathscr{R}}} \omega_{P,\mathscr{R}}(Q). \tag{2.6}$$

Then we have the following theorem.

Theorem 2.4. *Let $\mathscr{R}$ be a binary relation on pairs of columns (C,D) of height k which are filled with integers which are increasing from bottom to top. Let $P\in\mathcal{P}^{0,0,k}_{jk}$ where $j\geq 2$. Then*

$$1+\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}\sum_{F\in\mathcal{P}^{0,0,k}_{kn,\mathscr{R}}} x^{P\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,P,\mathscr{R}}(x-1)}. \tag{2.7}$$

Proof. Replace x by $x+1$ in (2.7). Then the left-hand side of (2.7) is the generating function of $m_P(F)$ over all $F\in\mathcal{MP}^{0,0,k}_{kn,P,\mathscr{R}}$. That is, it easy to see that

$$1+\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}\sum_{F\in\mathcal{P}^{0,0,k}_{kn,\mathscr{R}}}(x+1)^{P\text{-mch}(F)} = 1+\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}\sum_{F\in\mathcal{MP}^{0,0,k}_{kn,P,\mathscr{R}}} x^{m_P(F)}. \tag{2.8}$$

Thus we must show that

$$1+\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}\sum_{F\in\mathcal{MP}^{0,0,k}_{kn,P,\mathscr{R}}} x^{m_P(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,P,\mathscr{R}}(x)}. \tag{2.9}$$

Now

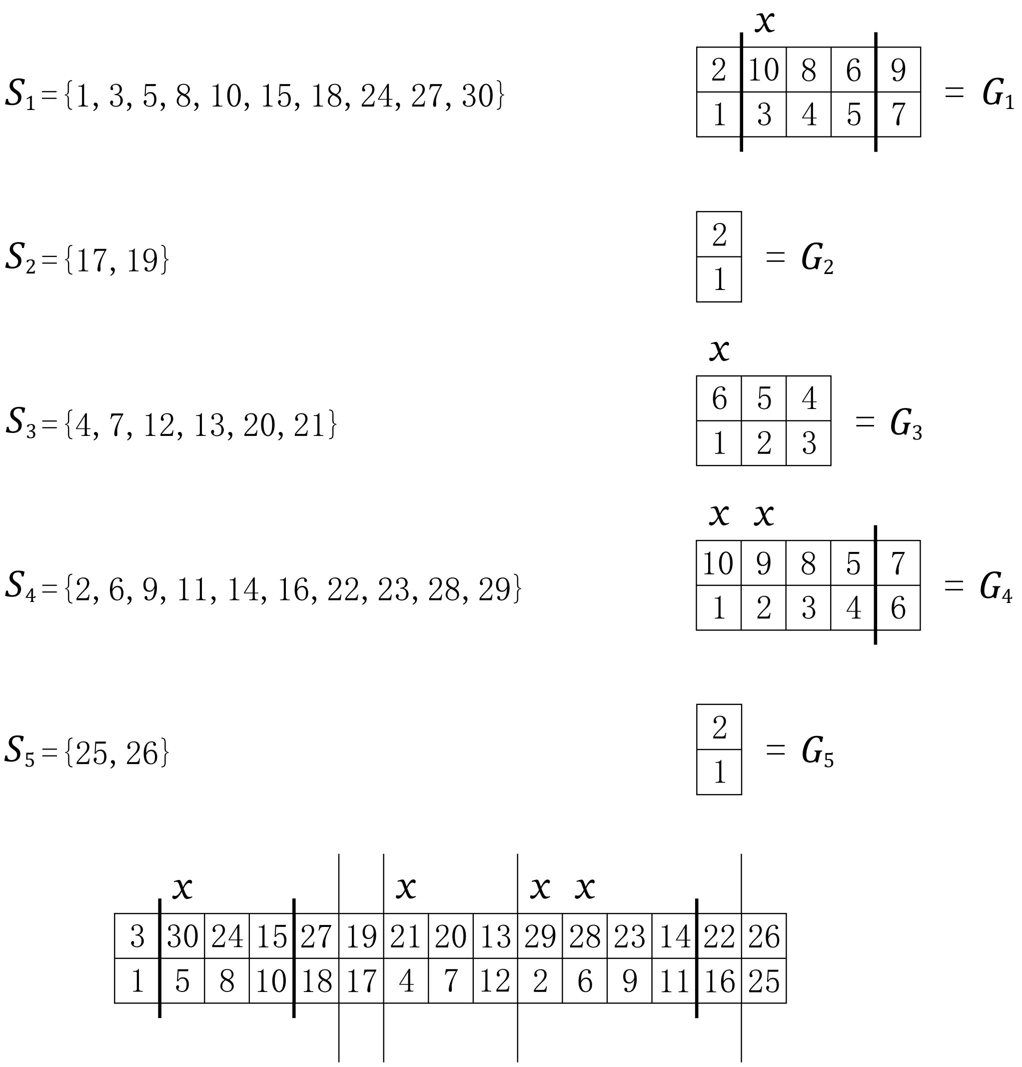
$$\frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,P,\mathscr{R}}(x)} = 1+\sum_{m\geq 1}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,P,\mathscr{R}}(x)\right)^{m}. \tag{2.10}$$

Taking the coefficient of $\frac{t^{ks}}{(ks)!}$ on both sides of (2.3) where $n \geq 1$, we see that we must show that

$$\begin{aligned}\sum_{F \in \mathcal{MP}^{0,0,k}_{sn}} x^{m_P(F)} &= \sum_{m=1}^{\infty} \left(\sum_{n \geq 1} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,P,\mathscr{R}}(x) \right)^m |_{\frac{t^{ks}}{(ks)!}} \\ &= \sum_{m=1}^{s} \left(\sum_{n=1}^{s} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,P,\mathscr{R}}(x) \right)^m |_{\frac{t^{ks}}{(ks)!}} \\ &= \sum_{m=1}^{s} \sum_{\substack{a_1+a_2+\cdots+a_m=s \\ a_i \geq 1}} \binom{ks}{ka_1, \ldots, ka_m} \prod_{j=1}^{m} GC^{0,0,k}_{ka_j,P,r}(x).(2.11)\end{aligned}$$

The right-hand side of (2.11) is now easy to interpret. First we pick an m such that $1 \leq m \leq s$. Then we pick $a_1, \ldots, a_m \geq 1$ such that $a_1 + a_2 + \cdots + a_m = s$. Next the binomial coefficient $\binom{ks}{ka_1,\ldots,ka_m}$ allows us to pick sets $S_1, \ldots, S_m$ which partition $\{1, \ldots, ks\}$ such that $|S_i| = ka_i$ for $i = 1, \ldots, m$. Finally the product $\prod_{j=1}^{m} GC^{0,0,k}_{ka_j,P,\mathscr{R}}(x)$ allows us to pick generalized $P,\mathscr{R}$-clusters $G_i \in \mathcal{GC}^{0,0,k}_{ka_i,P,\mathscr{R}}$ for $i = 1, \ldots, m$ with weight $\prod_{j=1}^{m} \omega_{P,\mathscr{R}}(G_i)$. Note that in the cases where $a_i = 1$, our definitions imply that C_i is just a column of height k filled with the numbers $1, \ldots, k$ which is increasing, reading from bottom to top.

For example, suppose that $k = 2$ and $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Suppose that $\mathscr{R}$ is relation where, for any two columns C and D which filled with integers and are strictly increasing in columns, $(C, D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom elements of D. Then in Figure 2.6, we have pictured S_1, S_2, S_3, S_4, S_5 which partition $\{1, \ldots, 30\}$ and corresponding generalized $P,\mathscr{R}$-clusters $G_1, \ldots, G_5$. For each i, we have indicated the separation between the blocks of G_i by dark black lines. Then for each $i = 1, \ldots, m$, we create a cluster E_i which results by replacing the j in G_i by the j^{th} element of S_i. If we concatenate $E_1 \ldots E_5$ together, then we will obtain an element of $Q \in \mathcal{MP}^{0,0,k}_{kn,P}$. The weight of



$S_1 = \{1, 3, 5, 8, 10, 15, 18, 24, 27, 30\}$

	x			
2	10	8	6	9
1	3	4	5	7

$= G_1$

$S_2 = \{17, 19\}$

2
1

$= G_2$

$S_3 = \{4, 7, 12, 13, 20, 21\}$

x		
6	5	4
1	2	3

$= G_3$

$S_4 = \{2, 6, 9, 11, 14, 16, 22, 23, 28, 29\}$

x	x			
10	9	8	5	7
1	2	3	4	6

$= G_4$

$S_5 = \{25, 26\}$

2
1

$= G_5$

	x					x			x	x				
3	30	24	15	27	19	21	20	13	29	28	23	14	22	26
1	5	8	10	18	17	4	7	12	2	6	9	11	16	25

Figure 2.6: Construction for the right-hand side of (2.11).

Q equals $\prod_{j=1}^{5} \omega_{P,\mathscr{R}}(G_i)$ where

$$\begin{aligned} \omega_{P,\mathscr{R}}(G_1) &= (-1)^2 x, \\ \omega_{P,\mathscr{R}}(G_2) &= 1, \\ \omega_{P,\mathscr{R}}(G_3) &= x, \\ \omega_{P,\mathscr{R}}(G_4) &= (-1)^1 x^2, \text{ and} \\ \omega_{P,\mathscr{R}}(G_5) &= 1. \end{aligned}$$

In Figure 2.6, we have indicated the boundaries between the G_is by light lines.

We let $\mathcal{HGC}_{ks,P,\mathscr{R}}$ denote the set of all elements that can be constructed in this way. Thus $Q = E_1 \dots E_m$ is an element of $\mathcal{HGC}_{ks,P,\mathscr{R}}$ if and only if for each $i = 1, \dots, m$, red(E_i) is a generalized $P, \mathscr{R}$-cluster. Next we define a sign reversing involution $\theta : \mathcal{HGC}_{ks,P,\mathscr{R}} \to \mathcal{HGC}_{ks,P,\mathscr{R}}$. Given $Q = E_1 \dots E_m \in \mathcal{HGC}_{ks,P,\mathscr{R}}$, look for the first i such that either

1. the block structure of red$(E_i) = B_1^{(i)} B_2^{(i)} \dots B_{k_i}^{(i)}$ consists of more than one block or
2. E_i consists of a single block $B_1^{(i)}$ and $(last(B_i), first(E_i))$ is not in $\mathscr{R}$.

In case (1), we let $\theta(E_1 \dots E_m)$ be the result of replacing E_i by two generalized $P, \mathscr{R}$-clusters, E_i^* and E_i^{**} where E_i* consists just of $B_1^{(i)}$ and E_i^{**} consists of $B_2^{(i)} \dots B_{k_i}^{(i)}$. Note that in this case

$$\omega_{P,\mathscr{R}}(E_i) = (-1)^{k_i - 1} \prod_{j=1}^{k_i} \omega_{P,\mathscr{R}}(B_j^{(i)})$$

while

$$\omega_{P,\mathscr{R}}(E_i^*) \omega_{P,\mathscr{R}}(E_i)^{**} = (-1)^{k_i - 2} \prod_{j=1}^{k_i} \omega_{P,\mathscr{R}}(B_j^{(i)}).$$

In case (2), we let $\theta(E_1 \dots E_m)$ be the result of replacing E_i and E_{i+1} by the single generalized $P, \mathscr{R}$-cluster $E = B_1^{(i)} B_1^{(i+1)} \dots B_{k_{i+1}}^{(i+1)}$. Since $(last(B_i), first(E_i))$ is not in $\mathscr{R}$, $B_1^{(i)} B_1^{(i+1)} \dots B_{k_{i+1}}^{(i+1)}$ reduces to a generalized $P, \mathscr{R}$-cluster. In this case,

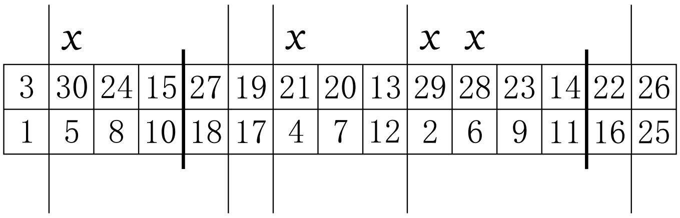
$$\omega_{P,\mathscr{R}}(E_i) \omega_{P,\mathscr{R}}(E_{i+1}) = (-1)^{k_{i+1} - 1} \omega_{P,\mathscr{R}}(B_1^{(i)}) \prod_{j=1}^{k_{i+1}} \omega_{P,\mathscr{R}}(B_j^{(i+1)})$$

while

$$\omega_{P,\mathscr{R}}(E) = (-1)^{k_{i+1}} \omega_{P,\mathscr{R}}(B_1^{(i)}) \prod_{j=1}^{k_{i+1}} \omega_{P,\mathscr{R}}(B_j^{(i+1)}).$$

If neither case (1) or case (2) applies, then we let $\theta(E_1 \dots E_m) = E_1 \dots E_m$. For example, suppose that $\mathscr{R}$ is the binary relation where, for any two columns C and D, which filled with integers and are strictly increasing in columns, $(C, D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom elements of D and $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Then if $Q = E_1 \dots E_5$ is the generalized $P, \mathscr{R}$-cluster pictured in

Figure 2.6, then we are in case (1) with $i = 1$ since E_1 consists of more than one block. Thus $\theta(Q)$ results by breaking that generalized $P, \mathscr{R}$-cluster into to two clusters E_1^* of size 1 and E_1^{**} of size 4. $\theta(Q)$ is pictured in Figure 2.7.



	x					x			x	x				
3	30	24	15	27	19	21	20	13	29	28	23	14	22	26
1	5	8	10	18	17	4	7	12	2	6	9	11	16	25

Figure 2.7: The involution θ.

It is easy to see that θ is an involution. That is, if $Q = E_1 \dots E_m$ is in case (1) using E_i, then $\theta(Q)$ will be in case (2) using E_i^* and E_i^{**}. Similarly if $Q = E_1 \dots E_m$ is in case (1) using E_i and E_{i+1}, then $\theta(Q)$ will be in case (2) using $E = E_i E_{i+1}$. It follows that if $\theta(E_1 \dots E_m) \neq E_1 \dots E_M$, then $\omega_{P,\mathscr{R}}(E_1 \dots E_m) = -\omega_{P,\mathscr{R}}(\theta(E_1 \dots E_m))$ so that the right-hand side of (2.11) equals

$$\sum_{Q=E_1 \dots E_m \in \mathcal{HGC}_{ks,P,\mathscr{R}}, \theta(Q)=Q} \prod_{i=1}^{m} \omega_{P,\mathscr{R}}(E_i).$$

Thus we must examine the fixed points of θ.

If $Q = E_1 \dots E_m \in \mathcal{HGC}_{ks,P,\mathscr{R}}$ and $\theta(Q) = Q$, then it must be the case that each E_i consists of single column of weight 1 or it reduces to generalized $P, \mathscr{R}$-cluster $\overline{E}_i$ consisting of a single block $B_1^{(i)}$ whose weight is the weight of red$(B_1^{(i)})$ as a P-cluster. Moreover, it must be the case that for all $i = 1, \dots m-1$, $(last(E_i), first(E_{i+1}))$ is in $\mathscr{R}$. But this means for all $j = 1, \dots, n-1$, $(Q[j], Q[j+1])$ is in $\mathscr{R}$. That is, either $Q[j]$ equals $last(E_i)$ for some i or column j is contained in one of the P-clusters E_i in which case $(Q[j], Q[j+1])$ is in $\mathscr{R}$ by our definition of generalized $P, \mathscr{R}$-clusters. Thus any fixed point Q of θ is an element $\mathcal{MP}_{ks,\mathscr{R}}^{0,0,k}$. Then just like our proof Theorem 2.2, it follows that $E_1, \dots, E_m$ are just the maximal P-subclusters of an element in $\mathcal{P}_{ks,\mathscr{R}}^{0,0,k}$. Vice versa, if $T = F_1 \dots F_r$ is an element of $\mathcal{P}_{ks,\mathscr{R}}^{0,0,k}$ where $F_1, \dots, F_r$ are the maximal P-subclusters of T, then $T = F_1 \dots F_r$

is a fixed point of θ. Thus we have proved that the right-hand side of (2.11) equals

$$\sum_{F \in \mathcal{MP}_{ks}^{0,0,k}} x^{m_P(F)}$$

which is what we wanted to prove. $\square$

It is quite easy to extend our results to consider pattern matching relative to sets of elements in $\mathcal{P}_{kr}^{0,0,k}$. That is, let $\Gamma \subseteq \mathcal{P}_{kr}^{0,0,k}$ and $F \in \mathcal{F}_{i+kn+j,\mathbb{P}}^{i,j,k}$ where $r \leq n$. Then we say

1. Γ **occurs** in F if there are $2 \leq i_1 < i_2 < \cdots < i_r \leq n+1$ such that $\mathrm{red}(F[i_1, \ldots, i_r]) \in \Gamma$,
2. F **avoids** Γ if there is no occurrence of Γ in F, and
3. there is a **Γ-match in F starting at position** i if $\mathrm{red}(F[i, i+1, \ldots, i+r-1]) \in \Gamma$.

We let Γ-mch(F) denote the number of Γ-matches in F.

We let $\mathcal{MP}_{kn,\Gamma}^{0,0,k}$ denote the set of elements that arise by starting with an element F of $\mathcal{P}_{kn}^{0,0,k}$ and marking some of the Γ-matches in F by placing on x on the column which starts the Γ-match. A Γ-cluster is a filling of $F \in \mathcal{MP}_{kn,\Gamma}^{0,0,k}$ such that

1. every column of F is contained in a marked Γ-match of F and
2. any two consecutive marked Γ-matches share at least one column.

We let $\mathcal{CM}_{kn,\Gamma}^{0,0,k}$ denote the set of all Γ-clusters in $\mathcal{MP}_{kn,\Gamma}^{0,0,k}$. For each $n \geq 2$, we define the cluster polynomial

$$C_{kn,\Gamma}^{0,0,k}(x) = \sum_{F \in \mathcal{CM}_{kn,\Gamma}^{0,0,k}} x^{m_\Gamma(F)}$$

where $m_\Gamma(F)$ is the number of marked Γ-matches in F. By convention, we let $C_{k,\Gamma}^{0,0,k}(x) = 1$. Then it is easy to modify the proof of Theorem 2.2 to prove the following theorem.

Theorem 2.5. *Let* $\Gamma \subseteq \mathcal{P}_{kr}^{0,0,k}$ *where* $r \geq 2$. *Then*

$$1 + \sum_{n\geq 1} \frac{t^n}{kn!} \sum_{F \in \mathcal{P}_{kn}^{0,0,k}} x^{\Gamma\text{-mch}(F)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} C_{kn,\Gamma}^{0,0,k}(x-1)}. \tag{2.12}$$

Next suppose that we are given a binary relation $\mathscr{R}$ between $k \times 1$ arrays of integers and a set of patterns $\Gamma \subseteq \mathcal{P}_{kr}^{0,0,k)}$.

Definition 2.6. *We say that* $Q \in \mathcal{MP}_{kn,\Gamma}^{0,0,k}$ *is a* generalized $\Gamma, \mathscr{R}$-cluster *if we can write* $Q = B_1 B_2 \cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

1. *either* B_i *is a single column or* B_i *consists of* r*-columns where* $r \geq 2$, $\text{red}(B_i)$ *is a* Γ*-cluster in* $\mathcal{MP}_{kr,\Gamma}$, *and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*
2. *for* $1 \leq i \leq m-1$, *the pair* $(last(B_i), first(B_{i+1}))$ *is not in* $\mathscr{R}$ *where for any* i, $last(B_i)$ *is the right-most column of* B_i *and* $first(B_i)$ *is the left-most column of* B_i.

Let $\mathcal{GC}_{kn,\Gamma,\mathscr{R}}^{0,0,k}$ denote the set of all generalized $\Gamma, \mathscr{R}$-clusters which have n columns of height k. Given $Q = B_1 B_2 \ldots B_m \in \mathcal{GC}_{kn,\Gamma,\mathscr{R}}^{0,0,k}$, we define the weight of B_i, $\omega_{\Gamma,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{m_\Gamma(\text{red}(B_i))}$ if B_i is order isomorphic to a Γ-cluster. Then we define the weight of Q, $\omega_{\Gamma,\mathscr{R}}(Q)$, to be $(-1)^{m-1} \prod_{i=1}^{m} \omega_{\Gamma,\mathscr{R}}(B_i)$. We let

$$GC_{kn,\Gamma,\mathscr{R}}^{0,0,k}(x) = \sum_{Q \in \mathcal{GC}_{kn,\Gamma,\mathscr{R}}^{0,0,k}} \omega_{\Gamma,\mathscr{R}}(Q). \tag{2.13}$$

Then we have the following theorem.

Theorem 2.7. *Let* $\mathscr{R}$ *be a binary relation on pairs of columns* (C, D) *of height* k *which are filled with integers which are increasing from bottom to top. Let* $\Gamma \subseteq \mathcal{P}_{kr}^{0,0,k}$ *where* $r \geq 2$. *Then*

$$1 + \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} \sum_{F \in \mathcal{P}_{kn,\mathscr{R}}^{0,0,k}} x^{\Gamma\text{-mch}(F)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC_{kn,\Gamma,\mathscr{R}}^{0,0,k}(x-1)}. \tag{2.14}$$

It is also easy to extend our results to find generating functions for consecutive patterns in $\mathcal{F}_{kn,A,\mathscr{R}}^{0,0,k}$, where $A \subseteq \mathbb{P}$ is some given alphabet.

Recall that there are two types of pattern matching when we consider filling where a given letter can occur more than once. Let Γ be a finite set of elements of $\mathcal{F}^{0,0,k}_{kn,A,\mathscr{R}}$ such that $\text{red}(P) = P$ for all P in Γ and and let Δ be an arbitrary set of elements of $\mathcal{F}^{0,0,k}_{kn,A,\mathscr{R}}$. Recall that $P \in \mathcal{F}^{0,0,k}_{kn,A,\mathscr{R}}$ has a Γ-match starting at column i if there is a $j > i$ such that $\text{red}F([i, i+1, \ldots, j]) \in \Gamma$ and has an exact Δ-match starting at column i if there is a $j > i$ such that $F([i, i+1, \ldots, j]) \in \Delta$. We let $\Gamma\text{-mch}(F)$ denote the number of Γ-matches in F and $\Delta\text{-Emch}(F)$ denote the number of exact Δ-matches in F.

We let $\mathcal{MF}^{0,0,k}_{kn,A,\Gamma}$ denote the set of elements with an element F of $\mathcal{F}^{0,0,k}_{kn,A}$ where we have marked some of the Γ-matches by placing an x on top of the column that starts a Γ-match and let $\mathcal{EMF}^{0,0,k}_{kn,A,\Delta}$ denote the set of elements with an element F of $\mathcal{F}^{0,0,k}_{kn,A}$ where we have marked some of the exact Δ-matches by placing an x on top of the column that starts a Δ-match. If $P \in \mathcal{MF}^{0,0,k}_{kn,A,\Gamma}$, we let $m_\Gamma(P)$ denote the number of marked Γ-matches in P and if $Q \in \mathcal{EMF}^{0,0,k}_{kn,A,\Delta}$, we let $em_\Delta(Q)$ denote the number of marked exact Δ-matches in Q.

A Γ-cluster is a filling $F \in \mathcal{MF}^{0,0,k}_{kn,A,\Gamma}$ such that

1. every column of F is contained in a marked Γ-match of F and
2. any two consecutive marked Γ-matches share at least one column.

An exact Δ-cluster is a filling $F \in \mathcal{EMF}^{0,0,k}_{kn,A,\Delta}$ such that

1. every column of F is contained in a marked exact Δ-match of F and
2. any two consecutive marked exact Δ-matches share at least one column.

Let $C^{0,0,k}_{n,\Gamma}(x)$ denote the sum of $x^{m_\Gamma(C)}$ over all Γ-clusters C of length n and $EC^{0,0,k}_{n,\Delta}(x)$ denote the sum of $x^{em_\Gamma(C)}$ over all exact Δ-clusters C of length n. Then we have the following analogues of Theorem 2.2

Theorem 2.8. *Let* $\Gamma, \Delta \subseteq \mathcal{F}^{0,0,k}_{kr,A}$ *where* $r \geq 2$, $A \subseteq \mathbb{P}$ *is some given alphabet, and* red$(P) = P$ *for all* $P \in \Gamma$. *Then*

$$1 + \sum_{n\geq 1} t^{kn} \sum_{F \in \mathcal{F}^{0,0,k}_{kn,A}} x^{\Gamma\text{-}mch(F)} = \frac{1}{1 - |A|^k t - \sum_{n\geq r} t^{kn} C^{0,0,k}_{kn,\Gamma}(x-1)}, \tag{2.15}$$

and

$$1+\sum_{n\geq 1} t^{kn} \sum_{F\in \mathcal{F}_{kn,A}^{0,0,k}} x^{\Delta\text{-}Emch(F)} = \frac{1}{1-|A|^k t - \sum_{n\geq r} t^{kn} EC_{kn,\Delta}^{0,0,k}(x-1)}, \tag{2.16}$$

where $|A|$ *is the cardinality of alphabet* A*.*

The proof of Theorem 2.8 is very similar to the proof of Theorem 2.2 so that we will leave the details to the reader.

Next suppose that we are given a binary relation $\mathscr{R}$ between $k\times 1$ arrays of integers

Definition 2.9. *We say that* $Q\in \mathcal{MF}_{kn,A,\Gamma}^{0,0,k}$ *is a* generalized $\Gamma,\mathscr{R}$-cluster *if we can write* $Q = B_1B_2\cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

1. *either* B_i *is a single column or* B_i *consists of* r*-columns where* $r\geq 2$*,* B_i *is a* Γ*-cluster in* $\mathcal{MF}_{kr,A,\Gamma}^{0,0,k}$*, and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*

2. *for* $1\leq i\leq m-1$*, the pair* $(last(B_i), first(B_{i+1}))$ *is not in* $\mathscr{R}$ *where for any* i*,* $last(B_i)$ *is the right-most column of* B_i *and* $first(B_i)$ *is the left-most column of* B_i*.*

We say that $Q\in \mathcal{MF}_{kn,A,\Gamma}^{0,0,k}$ *is a* generalized exact $\Delta,\mathscr{R}$-cluster *if we can write* $Q = B_1B_2\cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

1. *either* B_i *is a single column or* B_i *consists of* r*-columns where* $r\geq 2$*,* B_i *is an exact* Δ*-cluster in* $\mathcal{EMF}_{kr,A,\Delta}^{0,0,k}$*, and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*

2. *for* $1\leq i\leq m-1$*, the pair* $(last(B_i), first(B_{i+1}))$ *is not in* $\mathscr{R}$ *where for any* i*,* $last(B_i)$ *is the right-most column of* B_i *and* $first(B_i)$ *is the left-most column of* B_i*.*

Let $\mathcal{GC}_{kn,A,\Gamma,\mathscr{R}}^{0,0,k}$ denote the set of all generalized $\Gamma,\mathscr{R}$-clusters which have n columns of height k. Given $Q = B_1B_2\ldots B_m \in \mathcal{GC}_{kn,A,\Gamma,\mathscr{R}}^{0,0,k}$, we define the weight of

B_i, $\omega_{\Gamma,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{m_\Gamma(B_i)}$ if B_i is order isomorphic to a Γ-cluster. Then we define the weight of Q, $\omega_{\Gamma,\mathscr{R}}(Q)$, to be $(-1)^{m-1}\prod_{i=1}^{m}\omega_{\Gamma,\mathscr{R}}(B_i)$. We let

$$GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x) = \sum_{Q\in\mathcal{GC}^{0,0,k}_{kn,A,\Gamma,\mathscr{R}}} \omega_{\Gamma,\mathscr{R}}(Q). \tag{2.17}$$

Similarly, let $\mathcal{EGC}^{0,0,k}_{kn,A,\Delta,\mathscr{R}}$ denote the set of all exact generalized $\Delta,\mathscr{R}$-clusters which have n columns of height k. Given $Q = B_1B_2\ldots B_m \in \mathcal{EGC}^{0,0,k}_{kn,A,\Delta,\mathscr{R}}$, we define the weight of B_i, $e\omega_{\Delta,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{em_\Gamma(B_i)}$ if B_i is order isomorphic to an exact Δ-cluster. Then we define the weight of Q, $e\omega_{\Delta,\mathscr{R}}(Q)$, to be $(-1)^{m-1}\prod_{i=1}^{m} 3\omega_{\Delta,mthscrR}(B_i)$. We let

$$EGC^{0,0,k}_{kn,A,\Delta,\mathscr{R}}(x) = \sum_{Q\in\mathcal{EGC}^{0,0,k}_{kn,A,\Delta,\mathscr{R}}} e\omega_{\Delta,\mathscr{R}}(Q). \tag{2.18}$$

Then we the following theorem.

Theorem 2.10. *Let $\mathscr{R}$ be a binary relation on pairs of columns (C,D) of height k. Let $\Gamma,\Delta \subseteq \mathcal{F}^{0,0,k}_{kr,A,\mathscr{R}}$ where $r \geq 2$, $A \subseteq \mathbb{P}$ is some given alphabet, and red$(P) = P$ for all $P \in \Gamma$. Then*

$$1+\sum_{n\geq 1} t^{kn} \sum_{F\in\mathcal{F}^{0,0,k}_{kn,A,\mathscr{R}}} x^{\Gamma\text{-}mch(F)} = \frac{1}{1-\sum_{n\geq 1} t^{kn} GC^{0,0,k}_{kn,A,\Gamma,\mathscr{R}}(x-1)}. \tag{2.19}$$

$$1+\sum_{n\geq 1} t^{kn} \sum_{F\in\mathcal{F}^{0,0,k}_{kn,A,\mathscr{R}}} x^{\Delta\text{-}Emch(F)} = \frac{1}{1-\sum_{n\geq 1} t^{kn} EGC^{0,0,k}_{kn,A,\Delta,\mathscr{R}}(x-1)}. \tag{2.20}$$

The proof of Theorem 2.10 is very similar to the proof of Theorem 2.4 so that we will leave the details to the reader.

2.2 Examples

The key to using either Theorems 2.4, 2.7, or 2.10 to compute explicit formulas for our generating functions is to be able to compute generalized cluster polynomials. In this section, we shall give several examples where we can compute the required generalized cluster polynomials.

2.2.1 Words with no consecutive repeats

We use words with no consecutive repeats as our first and fundamental example. Suppose the alphabet is $[k] = \{1, 2, 3, \cdots, k\}$, $w = w_1 w_2 \cdots w_n$ is a word over $[k]$. According to our previous notations, the set of words of length n over $[k]$ is denoted by $\mathcal{WI}^{0,0,1}_{n,[k]}$.

We say w has no consecutive repeated letters if $w_i \neq w_{i+1}$ for any $1 \leq i \leq n-1$. Clearly, if we let $\mathscr{R}$ be the relation which holds on pairs of two elements (a, b) if and only if $a \neq b$, then the set of words without consecutive repeated letters of length n over $[k]$ equals $\mathcal{WI}^{0,0,1,\mathscr{R}}_{n,[k]}$. It is easy to see that

$$\left|\mathcal{WI}^{0,0,1,\mathscr{R}}_{n,[k]}\right| = k(k-1)^{n-1}.$$

We start by considering some simple patterns first. First, we study the distribution of descents in $\mathcal{WI}^{0,0,1}_{n,[k]}$ and $\mathcal{WI}^{0,0,1,\mathscr{R}}_{n,[k]}$. That is, we consider the following two generating functions

$$A_{\mathrm{des},[k]}(x,t) := 1 + \sum_{n\geq 1} t^n \sum_{w \in \mathcal{WI}^{0,0,1}_{n,[k]}} x^{\mathrm{des}(w)}, \tag{2.21}$$

and

$$A_{\mathrm{des},[k],\mathscr{R}}(x,t) := 1 + \sum_{n\geq 1} t^n \sum_{w \in \mathcal{WI}^{0,0,1}_{n,[k],\mathscr{R}}} x^{\mathrm{des}(w)}. \tag{2.22}$$

According to Goulden and Jackson' cluster method for words, we have

$$A_{\mathrm{des},[k]}(x,t) = \frac{1}{1 - kt - \sum_{n\geq 2} t^n C_{n,\mathrm{des}}(x-1)}, \tag{2.23}$$

where $C_{n,\mathrm{des}}(x)$ is the cluster polynomial for the pattern u=21. First, we need to figure out the structures of u-clusters. It is easy to see for any $F \in \mathcal{C}_{n,\mathrm{des}}$, F is a u-cluster if and only if F is a monotonically decreasing word over $[k]$ and $m_{\mathrm{des}}(F) = n-1$. If follows that n, $|\mathcal{C}_{n,21}| = \binom{k}{n}$. Then

$$C_{n,21}(x) = \binom{k}{n} x^{n-1}$$

and hence

$$\begin{aligned} A_{\mathrm{des},[k]}(x,t) &= \frac{1}{1 - kt - \sum_{n\geq 2} t^n \binom{k}{n} (x-1)^{n-1}} \\ &= \frac{x-1}{x - (t(x-1)+1)^k} \end{aligned}$$

As for $A_{\mathrm{des},[k],\mathscr{R}}(x,t)$, according to Theorem 2.10,

$$A_{\mathrm{des},[k],\mathscr{R}}(x,t) = \frac{1}{1-kt-\sum_{n\geq 2} t^n GC_{n,21}(x-1)}, \tag{2.24}$$

where $GC_{n,21}(x) =$ is the generalized cluster polynomial for the pattern $u = 21$.

Suppose $z = B_1B_2\cdots B_m$ is a generalized $u,\mathscr{R}$-cluster. Since the restriction here is that no adjacent elements are the same, we have levels between blocks in z. Therefore, z is a weakly decreasing sequence and there are $m-1$ levels in z. An example of a generalized $u,\mathscr{R}$-cluster is pictured in Figure 2.8.

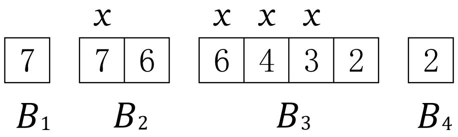


Figure 2.8: A generalized u-cluster of size 8 with 4 blocks.

Similar to Equation 2.30, we have

$$GC_{n,u}(x) = \sum_{1\leq m\leq n} (-1)^{m-1}|\mathcal{GC}^m_{n,u}|x^{n-m}, \tag{2.25}$$

where $|\mathcal{GC}^m_{n,u}|$ is the number of generalized u-clusters of size n with m blocks. Since the size of the alphabet is k, $m \geq \max\{1, n+1-k\}$. $|\mathcal{GC}^m_{n,u}|$ can be counted easily

$$|\mathcal{GC}^m_{n,u}| = \binom{n-1}{m-1}\binom{k}{n-m+1}$$

and then we have the generalized cluster polynomials as follows,

$$\begin{aligned}
GC_{1,u}(x) &= k\\
GC_{2,u}(x) &= \binom{k}{2}x - k\\
GC_{3,u}(x) &= \binom{k}{3}x^2 - 2\binom{k}{2}x + k\\
GC_{4,u}(x) &= \binom{k}{4}x^3 - 3\binom{k}{3}x^2 + 3\binom{k}{2}x - k\\
GC_{5,u}(x) &= \binom{k}{5}x^4 - 4\binom{k}{4}x^3 + 6\binom{k}{3}x^2 - 4\binom{k}{2}x + k\\
GC_{6,u}(x) &= \binom{k}{6}x^5 - 5\binom{k}{5}x^4 + 10\binom{k}{4}x^3 - 10\binom{k}{3}x^2 + 5\binom{k}{2}x - k\\
&\cdots
\end{aligned}$$

Then we just need to plug polynomials above into the generating function and then we can get the result.

For example, suppose that the alphabet is [3], i.e., $k = 3$, then we can get generalized cluster polynomials as follows,

$$GC_{1,u}(x-1) = 3$$

$$GC_{2,u}(x-1) = 3x-6$$

$$GC_{3,u}(x-1) = x^2-8x+10$$

$$GC_{4,u}(x-1) = -3x^2+15x-15$$

$$GC_{5,u}(x-1) = 6x^2-24x+21$$

$$GC_{6,u}(x-1) = -10x^2+35x-28$$

$\cdots$

For $n \geq 3$,

$$GC_{n,u}(x-1) = (-1)^{n-1}\left(\binom{n-1}{2}x^2+(n^2-1)x+\binom{n+2}{2}\right).$$

It is straightforward to check that

$$\begin{aligned}\sum_{n\geq 3}(-1)^{n-1}\binom{n-1}{2}x^2t^n &= \frac{x^2t^3}{(t+1)^3},\\ \sum_{n\geq 2}(-1)^{n-1}(n-1)(n+1)xt^n &= \frac{xt^2(t+3)}{(t+1)^3},\\ \sum_{n\geq 1}(-1)^{n-1}\binom{n-1}{2}t^n &= 1-\frac{1}{(t+1)^3}.\end{aligned}$$

Then we obtained the following explicit formula of the generating function for descents in words over [3] with no repeated letters:

$$\begin{aligned}A_{\text{des},[3],\mathscr{R}}(x,t) &= -\frac{(t+1)^3}{t^2x(tx+t+3)-1}\\ &= 1+3t+(3x+3)t^2+(x^2+10x+1)t^3+(12x^2+12x)t^4\\ &\quad +(6x^3+36x^2+6x)t^5+(x^4+47x^3+47x^2+x)t^6+\cdots.\end{aligned}$$

We end up this subsection with another example. Suppose the alphabet is $[k+1]$ and the pattern $u = 1\ 3\ 4\ \cdots\ k\ 2$. Then we are interested in the following generating functions for distribution of reduced matches of u,

$$A_{u,[k+1]}(x,t) := 1 + \sum_{n \geq 1} t^n \sum_{w \in \mathcal{WI}^{0,0,1}_{n,[k+1]}} x^{u\text{-mch}(w)}. \tag{2.26}$$

and

$$A_{u,[k+1],\mathscr{R}}(x,t) := 1 + \sum_{n \geq 1} t^n \sum_{w \in \mathcal{WI}^{0,0,1}_{n,[k+1],\mathscr{R}}} x^{u\text{-mch}(w)}, \tag{2.27}$$

where $\mathscr{R}$ holds for a pair (a,b) in $[k+1]^2$ if and only if $a \neq b$.

Clearly, u is a minimally overlapping pattern and the length of a u-cluster can only be either k or $2k-1$. That suppose that $a_1 \dots a_{2k-1}$ is a u-cluster. Then it is easy to see that $a_1 < a_k$. But if $a_k \geq 3$, then we do not have a enough room in $[k+1]$ to have an increasing sequence of length $k-1$ starting at 3. Thus it must be the case that $a_1 = 1$ and $a_k \dots a_{2k-1} = 24 \dots (k+1)3$. Because u is minimally overlapping, all the u-matches in a u-cluster have to be marked. Clearly there are $k+1$ ways to fill a u-cluster of length k using elements from $[k+1]$ and $\binom{k}{k-1} = k-1$ ways to fill a u-cluster of length $2k-1$ using elements from $[k+1]$. Hence

$$C_{k,u}(x) = (k+1)x \quad \text{and} \quad C_{2k-1,u}(x) = (k-1)x^2.$$

Then

$$A_{u,[k+1]}(x,t) = \frac{1}{1 - kt - (k+1)(x-1)t^k - (k-1)(x-1)^2 t^{2k-1}}. \tag{2.28}$$

To compute generalized cluster polynomials in (2.27), we first need to figure out structures of generalized $u,\mathscr{R}$-clusters. It is obvious to see that for $z \in \mathcal{GC}_{n,u,\mathscr{R}}$, the length of any block in z has to be 1, k or $2k-1$. From previous discussion, there are $k+1$ different u-clusters of length k and $k-1$ u-cluster of length $2k-1$. If $B_1 \dots B_m$ is a generalized $u,\mathscr{R}$-cluster, then we must have $last(B_i) = first(B_{i+1})$ for all $1 \leq ileqm - 1$. If follows that in an arbitrary generalized u-cluster of any length, there can at most two u-matches. That is, if there exists a generalized u-cluster having three u-matches, the first number in the third u-match is at least 3 and then the largest number in the third u-match is at least $k+2$ which is out range of the alphabet $[k+1]$.

Based on above discussion, we split generalized u-clusters of size n into 4 cases. Case 1 is that there are only singleton blocks, Case 2 is that there is only one block of length k, Case 3 is that there are two blocks of length k and Case 4 is that there is there is one block of length $2k-1$.

Assume the size of the generalized cluster is n. In Case 1, there are only singleton blocks which contributes $(-1)^{n-1}(k+1)$ to the generalized cluster polynomial $GC_{n,u}(x)$. In Case 2, $n \geq k$ and there are $n-k$ singleton blocks and a block of size k which is actually a u-cluster of length k. There are $(n-k+1)$ ways to choose the position of the block of length k and there are $(k+1)$ different fillings of the cluster. Therefore, the generalized clusters in Case 2 contribute $(-1)^{n-k}(n-k+1)(k+1)x$ to the generalized cluster polynomial $GC_{n,u}(x)$ if $n \geq k$. In Case 3, $n \geq 2k$, there are are $n-2k$ singleton blocks and two blocks of size k. Clearly, there are $\binom{n-2k+2}{2}$ to choose positions of these two clusters. We have $k-1$ different fillings of the first u-cluster because the cluster must begin with 1 and end with 2. For the second cluster, there is only one legal filling. Therefore, the generalized clusters in Case 3 contribute $(-1)^{n-2k+1}\binom{n-2k+2}{2}(k-1)x^2$ to the generalized cluster polynomial $GC_{n,u}(x)$ if $n \geq 2k$ In Case 4, there is a block of size $2k-1$ and $n-2k+1$ singleton blocks which contributes $(-1)^{n-2k+1}(n-2k+2)(k-1)x^2$ to the generalized cluster polynomial $GC_{n,u}(x)$ if $n \geq 2k-1$.

Taking the sum of the polynomials obtained in each case, we have

$$
\begin{aligned}
& 1 \leq n \leq k-1, \\
& \qquad GC_{n,u}(x) = (-1)^{n-1}(k+1). \\
& k \leq n \leq 2k-2, \\
& \qquad GC_{n,u}(x) = (-1)^{n-k}(n-k+1)(k+1)x + (-1)^{n-1}(k+1). \\
& n \geq 2k-1, \\
& \qquad GC_{n,u}(x) = (n-2k+2)(k-1)(-1)^{n-2k+1}x^2 \\
& \qquad\qquad + \binom{n-2k+2}{2}(k-1)(-1)^{n-2k+1}x^2 \\
& \qquad\qquad + (n-k+1)(-1)^{n-k}(k+1)x + (-1)^{n-1}(k+1).
\end{aligned}
$$

Then we can compute that

$$\sum_{n=1}^{k-1} t^n GC_{n,u}(x) = \frac{(k+1)\left((-1)^k t^k + t\right)}{t+1},$$

$$\sum_{n=k}^{2k-2} t^n GC_{n,u}(x) = \\ \left((-t)^k\left(kx-(-1)^k\right)-(-1)^k t^2+t\left(x-(-1)^k\right)+(-t)^{k+1}\left((1-k)x+(-1)^k\right)\right) \\ \times \frac{(k+1)t^{k-1}}{(t+1)^2},$$

$$\sum_{n=2k-1}^{\infty} t^n GC_{n,u}(x) = \\ \left(-(-1)^k k^2(t+1)^2 x + k\left((-1)^{2k}t^2+(-1)^k t\left(2(-1)^k - x\right)-(-1)^k x \right.\right. \\ \left.\left. +(-1)^{2k}+x^2\right)+(-1)^k t^2\left((-1)^k+x\right)+(-1)^k t\left(2(-1)^k+x\right)+(-1)^{2k}-x^2\right) \\ \times \frac{t^{2k-1}}{(t+1)^3}.$$

Therefore,

$$\begin{aligned} & A_{u,[k+1],\mathscr{R}}(x,t) \\ = \; & \frac{1}{1-\sum_{n\geq 1} t^n GC_{n,u}(x-1)} \\ = \; & \frac{1}{1-\sum\limits_{n=1}^{k-1} t^n GC_{n,u}(x-1) - \sum\limits_{n=k}^{2k-2} t^n GC_{n,u}(x-1) - \sum\limits_{n=2k-1}^{\infty} t^n GC_{n,u}(x-1)} \\ = \; & \frac{-t(t+1)^3}{(k-1)(x-1)^2t^{2k}+(k+1)(t+1)(x-1)t^{k+1}+(t+1)^2t(kt-1)}. \end{aligned} \tag{2.29}$$

Now we assume the alphabet is [4] and then the pattern $u = 132$. Then letting $k = 3$ in (2.29), we have

$$A_{u,[4],\mathscr{R}}(x,t) = -\frac{(t+1)^3}{t(t(t(2t(x-1)(t(x-1)+2)+4x-1)+5)+1)-1}.$$

A few initial terms are

$$\begin{aligned} & 1+4t+12t^2+(32+4x)t^3+(84+24x)t^4+(218+104x+2x^2)t^5 \\ & +(566+380x+26x^2)t^6+(1468+1276x+172x^2)t^7 \\ & +(3808+4064x+860x^2+16x^3)t^8+\cdots. \end{aligned}$$

Letting $x = 0$ and $k = 3$ in (2.29), we have the generating function for the number of elements in $\mathcal{WI}_{n,\mathscr{R}}^{0,0,1}$ avoiding 132 consecutively,

$$\begin{aligned}
A_{u,[4],\mathscr{R}}(0,t) &= -\frac{(t+1)^3}{2t^5-4t^4-t^3+5t^2+t-1} \\
&= 1+4t+12t^2+32t^3+84t^4+218t^5+566t^6+1468t^7+3808t^8 \\
&\quad +9878t^9+25622t^{10}+66464t^{11}+172400t^{12}+\cdots .
\end{aligned}$$

Taking partial derivative of (2.29) with respect to x and letting $x = 0, k = 3$, we have

$$\begin{aligned}
\left.\frac{\partial A_{u,[4],\mathscr{R}}(x,t)}{\partial x}\right|_{x=0} &= \frac{4t^3(t+1)^3\left(-t^2+t+1\right)}{\left(-2t^5+4t^4+t^3-5t^2-t+1\right)^2} \\
&= 4t^3+24t^4+104t^5+380t^6+1276t^7+4064t^8+12496t^9 \\
&\quad +37492t^{10}+110404t^{11}+320536t^{12}+919976t^{13}+\cdots ,
\end{aligned}$$

which is the generating function for the number of elements in $\mathcal{WI}_{n,\mathscr{R}}^{0,0,1}$ having exactly one u-match.

2.2.2 'N'-shaped pattern in standard Young tableaux of shape n^3

A standard Young tableau of shape n^k is a filling of a $k \times n$ rectangular array with integers $1, 2, 3, \cdots, kn$ such that the elements increase from bottom to top in each column and increase from left to right in each row. Here we adopt the French notation of standard Young tableaux.

The set of all standard Young tableaux of shape n^k is denoted by $\mathcal{SYT}(n^k)$. An element in $\mathcal{SYT}(2^3)$ is given in Figure 2.9. $\mathcal{SYT}(n^k)$ can also be regarded as a special subset of $\mathcal{P}_{kn}^{0,0,k}$ equipped with some relation $\mathscr{R}$. Here $\mathscr{R}$ is a binary relation on a pair of two columns (C, D) of height k such that the i-th element in C is less than the i-th element in D, for all $1 \le i \le k$.

For a given pattern $P \in \mathcal{SYT}(j^k)$, our goal is to compute the following generating function,

$$A_{P,\mathcal{SYT}}(x,t) := 1 + \sum_{n\ge 1} \frac{t^{kn}}{(kn)!} \sum_{F\in\mathcal{SYT}(n^k)} x^{P\text{-mch}(F)},$$

	3	6
P	2	5
	1	4

Figure 2.9: An element in $\mathcal{SYT}(2^3)$.

which according to Theorem 2.4, equals

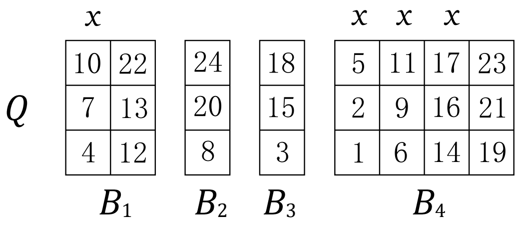
$$A_{P,\mathcal{SYT}}(x,t) = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC_{kn,P}(x-1)},$$

where $GC_{kn,P}(x)$ is generalized P-cluster polynomial. To compute $GC_{kn,P}(x)$, we need to figure out the structure of generalized clusters.

Since the structures of P-clusters and generalized P-clusters depends on the pattern P itself, we can use our methods to compute $A_{P,\mathcal{SYT}}(x,t)$ for a simple pattern P. Suppose we let $P \in \mathcal{SYT}(2^3)$ be the standard tableau pictured in Figure 2.9. Before we can compute a formula of $GC_{3n,P}(x)$, we need to understand the structure of P-clusters and generalized P-clusters.

Here the pattern P only consists of two columns, which implies that in an arbitrary P-cluster, all P-matches must be marked. Otherwise two consecutive marked P-matches don't share a column, which violates the definition of P-clusters. The number of columns in P-clusters could be any integer greater than or equal to 2. If we use Hasse diagram to represent the partial ordering of pattern P, as pictured in Figure 2.11, we see that it looks like capital letter 'N'. It is easy to see that there is only one P-cluster Q with n columns, namely, the i-th column of Q must consists of the numbers $3i-2$, $3i-1$ and $3i$, reading from bottom to top. For example, a P-cluster of size 4 is pictured in Figure 2.11 whose first three columns are marked 'x'.

Next we consider the structures of generalized clusters. First we know each block in a generalized cluster is either a single increasing column or a P-cluster. Between blocks, the row increasing condition must be violated, that is, we fail to observe row increasing condition in at least one row. For example, in Figure 2.10, $Q = B_1B_2B_3B_4$ is a generalized cluster of 8 columns, where B_1 is a cluster that has two columns and B_4 is a cluster having four columns. Between B_1 and B_2, in the base row, $12 \longleftrightarrow 8$ is not increasing, between B_2 and B_3, no rows are increasing, and between B_3 and B_4, no rows are increasing. Thus Q satisfies the requirement of being a generalized cluster.



	x				x	x	x	
	10	22	24	18	5	11	17	23
Q	7	13	20	15	2	9	16	21
	4	12	8	3	1	6	14	19
	B_1		B_2	B_3	B_4			

Figure 2.10: $Q = B_1B_2B_3B_4$ is generalized cluster consisting of 4 blocks.

In general, suppose $Q \in \mathcal{GC}_{3n,P}$ is a generalized cluster of size $3n$ which has m blocks, and l_i is the number of columns in the i-th block. Since a block is either a single column or a cluster and it's known that a P-cluster can be any size greater or equal to 6, clearly $(l_1, l_2, \cdots, l_m)$ can be thought as an integer composition of n. Since all P-matches have to be marked, in a block having l_i columns, there are $l_i - 1$ many marked P-matches and then hence

$$m_P(Q) = \sum_{1 \le i \le m} (l_i - 1) = n - m,$$

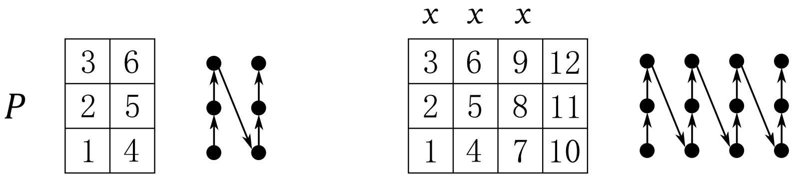
which implies the weight of Q, $\omega(Q) = (-1)^{\mathrm{B}(Q)-1} x^{n-\mathrm{B}(Q)}$, where $\mathrm{B}(Q)$ is the number of blocks in Q. Based on these facts, we have

$$GC_{3n}(x) = \sum_{Q \in \mathcal{GC}_{3n,P}} \omega_P(Q) = \sum_{L \models n} (-1)^{\ell(L)-1} |\mathcal{GC}_{n,P}^{(L)}| x^{n-\ell(L)(Q)}, \tag{2.30}$$

where L is composition of n, $\ell(L)$ is number of parts in l and $\mathcal{GC}_{3n}^{(L)}$ is the set of all the generalized clusters whose blocks have sizes $L = (l_1, l_2, \cdots, l_m)$. Therefore, we only need to count the cardinality of the set $\mathcal{GC}_{3n}^{(L)}$.

To count the number of generalized clusters, we use the so-called graphic method which is based on counting linear extensions of posets. It has been a widely utilized technique in many papers such as [19] [14] [27].

We use Hasse diagrams to represent the pattern P and P-clusters. For example, two corresponding Hasse diagrams are drawn in Figure 2.11. According to the



3	6
2	5
1	4

3	6	9	12
2	5	8	11
1	4	7	10

Figure 2.11: Pattern P, a P-cluster of length 4 and their corresponding Hasse diagrams.

definition of generalized clusters, the restriction should be violated between blocks, that is, arrows between blocks should be in the set C, as shown in Figure 2.12. Then we can represent generalized clusters as Hasse diagrams. For instance, all the generalized clusters with 8 columns whose such that the number of columns in the blocks induce the composition $L(2,4,1,1)$ corresponding to the Hasse diagram D_l in Figure 2.13. We let $\mathrm{LE}(D_l)$ denote the number of linear extensions of the diagram D_l. Then we have

$$\mathrm{LE}(D_l) = |\mathcal{GC}^{(l)}_{3n,P}| \tag{2.31}$$

and then according to Equation (2.30), we have

$$GC_n(x) = \sum_{l \vDash n} (-1)^{m-1} \mathrm{LE}(D_l) x^{n-m}, \tag{2.32}$$

where $l = (l_1, \cdots, l_m)$ is a composition of n.

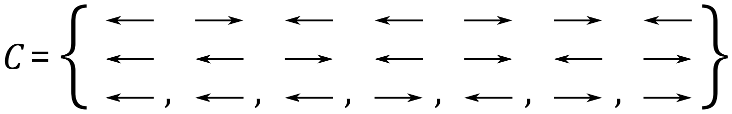


Figure 2.12: The set of arrows between blocks.

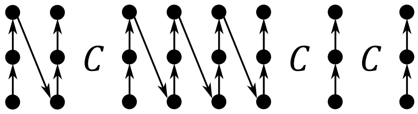


Figure 2.13: Hasse diagram $D_{(2,4,1,1)}$

The strategy of counting $\mathrm{LE}(D_l)$ is based on recursion and Inclusion-Exclusion principle. We take $D_{(2,4,1,1)}$ in Figure 2.13 to demonstrate the computation. First we remove the rightmost C and then subtract the row-increasing case. We keep this process until there is no 'C' in the diagrams. Figure 2.15 shows the results after we remove 2 C's from right to left. That is, the second line means that the set of elements where C holds between the last two columns can be expressed as the set of elements where there are no conditions imposed on the relative order of the rows between the last two columns minus set of elements where the last two column form a column strict tableau. For the first element in row 2, we can choose the elements in the last column in $\binom{24}{3}$ ways and we are reduced to considering a diagram with 7 columns. The next row then shows how we can eliminate the second C in each case. Eventually, we will reduce ourselves to a case where there are no C and the blocks consists of patterns of the form pictured in Figure 2.14. In such a case, we can represent the structure by a sequence of pairs $(a_1, b_1, a_2, b_2, \ldots, a_r, b_r)$ where we start out with a_1 columns whose top element is bigger than the bottom element of the next column, followed by b_1 columns whose Hasse diagram is that of standard tableaux of shape b_1^3, then the top element of part corresponding to b_1 is larger than the bottom element of the next column which starts a sequence of a_2 columns whose top element is bigger than the bottom element of the next column, followed by b_2 columns whose Hasse diagram is that of standard tableaux of shape b_2^3, etc. For the diagram pictured at the top of Figure 2.14 is associated with the sequence (3,4,4,4) and the diagram picutured at the bottom of Figure 2.14 is associated with the sequence (0,4,4,5). Given such a diagram corresponding to $(a_1, b_1, a_2, b_2, \ldots, a_r, b_r)$, we will say that a_is correspond to the linear part of the diagram and b_is correspond to the standard part of the diagram. It is easy to see that the number of linear extension of the diagram if just $\prod_{i=1}^{r} f_{b_i}$ where f_b is the number of standard tableaux of shape b^3. In this case, one can apply the hook length formula of Frame, Robinson, and Thrall [20] for the number of standard tableaux of shape λ to compute that

$$f_b = \frac{(3b)!}{\prod_{i=1}^{b} i(i+1)(i+2)}.$$

For example, consider the Hasse diagram at the top of Figure 2.14. It is easy to see that we must use the number $1, \ldots, 9$ to label the first linear part of 3 columns which can be done in only one way. Then we must use the numbers $10, \ldots, 21$

to label the next four columns because the label of the right top-most element of that block must be the largest element of the block and it is smaller than all the remaining elements. Thus f_4 ways to arrange the numbers $10, \ldots, 21$. Next we must use the numbers $22, \ldots, 33$, to label the second linear part which can be done in only one way. Finally we can use the numbers $34, \ldots 45$ to label the second standard block which can be done in f_4 ways.

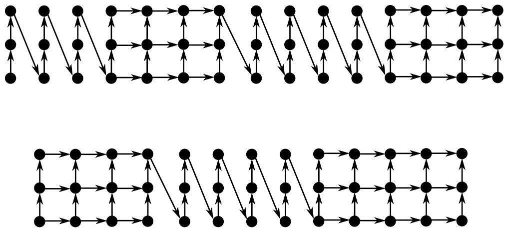


Figure 2.14: The final structure of blocks in the inclusion-exclusion process.

Continuing on in this way, we can obtain a formula of $\mathrm{LE}(D_{(2,4,1,1)})$ which is given by

$$\begin{aligned}
\mathrm{LE}(D_l) \;&=\; \left(\binom{24}{3,3,6} - \binom{24}{3,3}f_2\right) - \left(\binom{24}{3,6}f_2 - \binom{24}{3}f_2f_2\right) \qquad (2.33)\\
&\quad - \left(\binom{24}{6,6}f_2 - \binom{24}{6}f_2f_2\right) + \left(\binom{24}{6}f_3 - f_2f_3\right), \qquad (2.34)
\end{aligned}$$

where f_j is number of standard Young tableaux of shape j^3 which can be computed by hook-length formula.

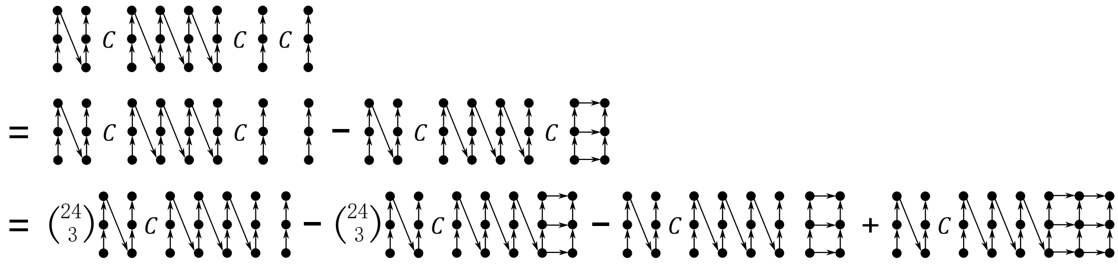


Figure 2.15: Computing $\mathrm{LE}(D_{(2,4,1,1)})$

We can embed the Hasse diagrams whose linear extensions that we want to compute in a larger class of Hasse diagrams where there are simple recursions to compute the number of linear extensions. To this end, we define a new class of Hasse diagrams $\Gamma(b_1, b_2, b_3, \cdots, b_r; c_1, d_1, c_2, d_2, c_3, d_3, \cdots)$, where the semicolon separates the diagram into two parts at the rightmost C. That is, $b_1, b_2, \ldots, b_r$ correspond to linear blocks which are separated by C, block b_r is followed by a C which is turned is followed by a single block consisting of alternating linear and standard parts. Here c_i is the length of the i-th linear part to the right of the rightmost C and d_i is the length of the i-th standard part to the right of the rightmost C.

Then we set

$$\Gamma(b_1, b_2, b_3, \cdots, b_m; 0, 0) = \Gamma(b_1, b_2, b_3, \cdots, b_{m-1}; b_m, 0) = D_{(b_1, b_2, \cdots, b_m)}. \tag{2.35}$$

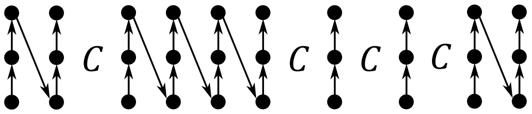


Figure 2.16: $D_{(2,4,1,1,2)} = \Gamma(2, 4, 1, 1, 2; 0, 0) = \Gamma(2, 4, 1, 1; 2, 0)$

Then there are natural recursions which split into two cases.

Case (i) If $c_1 = 0$, then

$$\begin{aligned} & \mathrm{LE}(\Gamma(b_1, b_2, \cdots, b_m; 0, d_1, c_2, d_2, \cdots, c_t, d_t)) \\ = & \binom{3\sum_{i=1}^m b_i + 3\sum_{j=1}^t (c_t + d_t)}{3\sum_{i=1}^m b_i} \mathrm{LE}(\Gamma(b_1, b_2, \cdots, b_{m-1}; b_m, 0)) \prod_{j=1}^t f_{d_j} \\ & -\mathrm{LE}(\Gamma(b_1, b_2, \cdots, b_{m-1}; b_m - 1, d_1 + 1, c_2, d_2, \cdots, c_t, d_t)). \end{aligned}$$

That is, we can express the desired number of linear extensions as the number of linear extensions where there is no relation imposed between the last block of b_m and the diagram of corresponding the pairs $(0, d_1, c_2, d_2, \ldots c_t, d_t)$, in which case we have $\binom{3\sum_{i=1}^m b_i + 3\sum_{j=1}^t (c_t + d_t)}{3\sum_{i=1}^m b_i}$ ways to choose the set of labels for the last block and $\prod_{j=1}^t f_{d_j}$ ways to that set of labels to label the last block, minus the number of linear extensions where the row increasing condition holds between the last column of b_m and the first column of d_1. In the latter case, the last block corresponds to the sequence $(b_m - 1, d_1 + 1, c_2, d_2, \ldots, c_t, d_t)$.

Figure 2.17 is an example of case (i).

$$\mathrm{LE}(\Gamma(2, 4, 1; 0, 2, 1, 0)) = \binom{30}{9} f_2 \mathrm{LE}(\Gamma(2, 4; 1, 0)) - \mathrm{LE}(\Gamma(2, 4; 0, 3, 1, 0)),$$

where f_2 is the number of standard Young tableuax of shape 2^3.

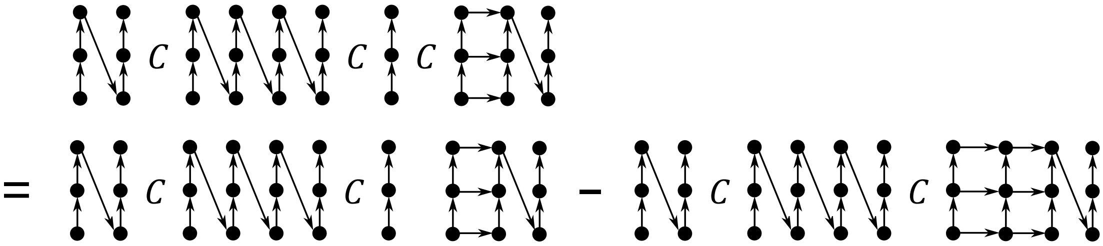


Figure 2.17: Recursion for $\Gamma(2,4,1;0,2,1,0)$

Case (ii) If $c_1 \neq 0$, then

$$\begin{aligned}
&\mathrm{LE}(\Gamma(b_1, b_2, \cdots, b_m; c_1, d_1, c_2, d_2, \cdots, c_t, d_t)) \\
= &\binom{3\sum_{i=1}^{m} b_i + 3\sum_{j=1}^{t}(c_t + d_t)}{3\sum_{i=1}^{m} b_i} \mathrm{LE}(\Gamma(b_1, b_2, \cdots, b_{m-1}; b_m, 0)) \prod_{j=1}^{t} f_{d_j} \\
&- \mathrm{LE}(\Gamma(b_1, b_2, \cdots, b_{m-1}; b_m - 1, 2, c_1 - 1, d_1, c_2, d_2, \cdots, c_t, d_t)).
\end{aligned}$$

Figure 2.18 is an example of case (ii).

$$\mathrm{LE}(\Gamma(2,4;1,2,1,0)) = \binom{30}{12} f_2 \mathrm{LE}(\Gamma(2;4,0)) - \mathrm{LE}(\Gamma(2;3,2,0,2,1,0)).$$

That is, in this case, we can express the desired number of linear extensions as the number of linear extensions where there is no relation imposed between the last block of b_m and the diagram of corresponding the pairs $(c_1, d_1, c_2, d_2, \ldots c_t, d_t)$, in which case we have $\binom{3\sum_{i=1}^{m} b_i + 3\sum_{j=1}^{t}(c_t + d_t)}{3\sum_{i=1}^{m} b_i}$ ways to choose the set of labels for the last block and $\prod_{j=1}^{t} f_{d_j}$ ways to that set of labels to label the last block, minus the number of linear extensions where the row increasing condition holds between the last column of b_m and the first column of c_1. In the latter case, the last block corresponds to the sequence $(b_m - 1, 2, c_1 - 1, d_1, c_2, d_2, \ldots, c_t, d_t)$.

For the base case, $\mathrm{LE}(\Gamma(\emptyset; c_1, d_1, c_2, d_2, \cdots, c_t, d_t)) = \prod_{j=1}^{t} f_{d_j}$, where f_i is the number of standard Young tableaux of shape i^3. Based on the recursion, we can compute the number of linear extensions of D_L for any composition of n. Then using these recursions and dynamic programming, we can compute the generalized cluster polynomials. The initial generalized cluster polynomials in this case are as

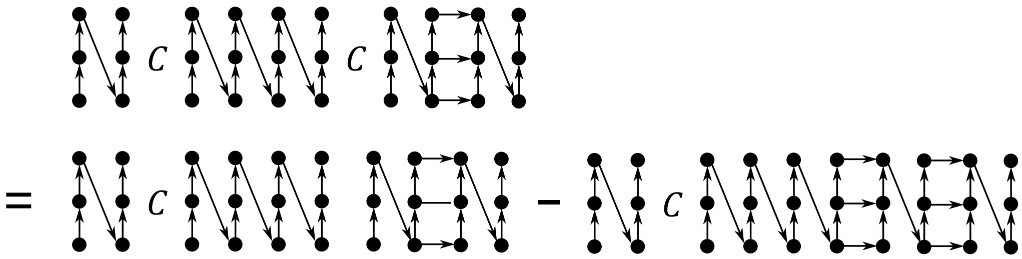


Figure 2.18: Recursion for $\Gamma(2,4;1,2,1,0)$

follows.

$$
\begin{aligned}
&GC_3(x) = 1\\
&GC_6(x) = x - 15\\
&GC_9(x) = x^2 - 158x + 882\\
&GC_{12}(x) = x^3 - 1349x^2 + 41909x - 133518\\
&GC_{15}(x) = x^4 - 10900x^3 + 1397961x^2 - 19036766x + 41627586\\
&GC_{18}(x) = x^5 - 87355x^4 + 41024174x^3 - 1759633773x^2 +\\
&\qquad\qquad 14037147012x - 23252213556\\
&\cdots
\end{aligned}
$$

Using these values in Theorem 2.4, we can compute the initial terms of $A_{P,\mathcal{SYT}}(x,t)$. In this case, we computed that

$$
\begin{aligned}
A_{P,\mathcal{SYT}}(x,t) &= 1 + \frac{t^3}{3!} + \frac{(x+4)t^6}{6!}\\
&+\frac{(x^2+8x+33)t^9}{9!} + \frac{(x^3+12x^2+82x+367)t^{12}}{12!}\\
&+\frac{(x^4+16x^3+147x^2+998x+4844)t^{15}}{15!}\\
&+\frac{(x^5+20x^4+228x^3+1957x^2+13713x+71597)t^{12}}{18!} + \cdots .
\end{aligned}
$$

In this example, although we are not able to obtain an explicit formula for the generating function $A_{P,\mathcal{SYT}}(x,t)$, we convert the original problem to a more tractable one which can be thought as a recursive problem over the set of all the integer compositions.

2.2.3 Shortest loops in non-backtracking random walks

A random walk on $\mathbb{Z}^2$ of length n is a path consisting of n random steps of the form $(1,0)$, $(-1,0)$, $(0,1)$ or $(0,-1)$. Clearly, a random walk can be represented as word over alphabet $\{1,\bar{1},2,\bar{2}\}$ if we denote $(1,0)$ by 1, $(-1,0)$ by $\bar{1}$, $(0,1)$ by 2 and $(0,-1)$ by $\bar{2}$. For example, the random walk on $\mathbb{Z}^2$ of length 10 pictured in Figure 2.19 corresponds to the word $1\ 2\ 1\ 2\ \bar{1}\ \bar{1}\ \bar{2}\ \bar{1}\ \bar{2}\ \bar{2}$. For convenience, let $\mathcal{W}_{n,2}$ denote the set of words of length n over the alphabet $\{1,\bar{1},2,\bar{2}\}$.

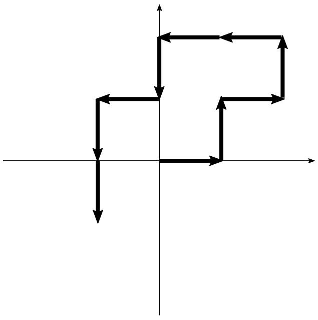


Figure 2.19: The path corresponding to $1\ 2\ 1\ 2\ \bar{1}\ \bar{1}\ \bar{2}\ \bar{1}\ \bar{2}\ \bar{2}$.

More generally, a random walk of length n on a k-dimensional lattice $\mathbb{Z}^d$ can be regarded as a word of length n over the alphabet $\{1,\bar{1},2,\bar{2},\cdots,d,\bar{d}\}$, where i represents a forward unit step along the i-th axis and $\bar{i}$ represents a backward unit step along the i-th axis and . For convenience, let $\mathcal{W}_{n,d}$ denote the set of words of length n over the alphabet $\{1,\bar{1},2,\bar{2},\cdots,d,\bar{d}\}$.

A non-backtracking random walk is a walk which cannot visit the previous vertex immediately. In other words, in the word corresponding to a non-backtracking random walk on $\mathbb{Z}^d$, i and $\bar{i}$ can not be adjacent in non-backtracking walks, for $1 \leq i \leq d$. Clearly, non-backtracking walks are corresponding to a restricted class of $\mathcal{W}_{n,d}$. For convenience, let $\overline{\mathcal{W}_{n,d}}$ denote the set of words of length n over the alphabet $\{1,\bar{1},2,\bar{2},\cdots,d,\bar{d}\}$ where i and $\bar{i}$ cannot be adjacent .

Non-backtracking walks and regular walks are different in several prospectives such as non-backtracking walks mix faster. Note that the only situation where a non-backtracking could revisit a lattice point is that it has a loop that ends at this point.

Here we apply generalized cluster method to enumeration of loops in non-backtracking random walks. Although most problems about words could be handled in a recursive manner, the generalized cluster method provides an easier alternative approach to solve this problem.

Here we focus the discussion on shortest loops. Clearly, shortest loops on $\mathbb{Z}^d$ have length 4. For example, on $\mathbb{Z}^2$, there are 8 loops of length 4, namely, $12\bar{1}\bar{2}$, $2\bar{1}\bar{2}1$, $\bar{1}\bar{2}12$, $\bar{2}12\bar{1}$, $\bar{2}\bar{1}21$, $\bar{1}21\bar{2}$, $21\bar{2}\bar{1}$ and $1\bar{2}\bar{1}2$. For $\mathbb{Z}^d$, there are $8\binom{d}{2}$ loops of length 4.

Suppose that we want to keep track of the number of loops of the form $12\bar{1}\bar{2}$ of all non-backtracking walks of length n. Obviously, the is equivalent to finding the number of exact matches of $12\bar{1}\bar{2}$ over all words of length n over the alpahbet $\{1, \bar{1}, 2, \bar{2}$ where 1 and $\bar{1}$ can never be adjacent and 2 and $\bar{2}$ can never be adjacent By Theorem 2.10, the generating function has following formula

$$A_{u,\overline{\mathcal{W}_d}}(x,t) = 1 + \sum_{n\geq 1} t^n \sum_{w\in\overline{\mathcal{W}_{n,d}}} x^{u\text{-Emch}(w)} = \frac{1}{1 - \sum_{n\geq 1} EGC_n(x-1)t^n}.$$

where

$$EGC_n(x) = \sum_{w\in\mathcal{EGC}_n} (-1)^{\mathrm{B}} x^{em_u(w)}$$

and $\mathcal{EGC}_n$ is the set of exact generalized clusters for u of size n.

Thus we must study the structure of exact generalized clusters. We can partitions the exact generalized u-clusters into two groups in this case.

1. **Type 1: All blocks are singleton cells**

   In this case, once the element in the first block is chosen and then the generalized cluster is uniquely determined. It is because if the element in previous block is i then the element in next block is forced to be $\bar{i}$, and vice versa, if the element in previous block is $\bar{i}$ then the element in next block is forced to be i. An example is given in Figure 2.20. Note that it is also possible that a Type 1 generalized cluster starts with $\bar{i}$, for $1 \leq i \leq d$. The set of all Type 1 exact generalized clusters of size n is denoted by $\mathcal{EGC}_n^1$. If the size of alphabet is $2d$, then $|\mathcal{EGC}_n^1| = 2d$.

$$\boxed{i}\ \boxed{\bar{i}}\ \boxed{i}\ \boxed{\bar{i}}\ \cdots$$

Figure 2.20: An example of Type 1 exact generalized u-cluster, where $1 \leq i \leq d$

2. **Type 2: There is a cluster of size 4**.

   First observe that there is only one exact u-cluster which consist of the sequence $12\bar{1}\bar{2}$ where 1 is marked with an x. An exact generalized u-cluster can not have more than one u-clusters because between blocks we can only have either i followed by $\bar{i}$ or $\bar{i}$ followed by i. The set of all Type 1 exact generalized u-clusters of size n is denoted by $\mathcal{EGC}_n^2$. An example in $\mathcal{EGC}_9^2$ is drawn in Figure 2.21. It is not hard to see that the cardinality of $\mathcal{GC}_n^2$ is $n-3$ for $n \geq 4$

$$\boxed{1}\ \boxed{\bar{1}}\ \overset{x}{\boxed{1}}\!\boxed{2}\!\boxed{\bar{1}}\!\boxed{\bar{2}}\ \boxed{2}\ \boxed{\bar{2}}\ \boxed{2}$$

Figure 2.21: An example in $\mathcal{EGC}_9^2$

Then the corresponding exact generalized u-clusters polynomials are defined as follows,

$$EGC_n^{\mathrm{J}}(x) := \sum_{w \in \mathcal{GC}_n^{\mathrm{J}}} (-1)^{\mathrm{B}(w)-1}\ x^{em_u(w)},$$
$$EGC^{\mathrm{J}}(x,t) := \sum_{n \geq 1} t^n EGC_n^{\mathrm{J}}(x),$$

where $\mathrm{J} \in \{1,\ 2\}$. Clearly, $\mathcal{EGC}_n^1$ and $\mathcal{EGC}_n^2$ are disjoint so that

$$\mathcal{EGC}_n = \biguplus_{\mathrm{J} \in \{1,2\}} \mathcal{EGC}_n^{\mathrm{J}}$$

As discussed above, in a Type 1 generalized cluster of length n, there are n singleton blocks and since $|\mathcal{GC}_n^1| = 2d$, we have

$$EGC_n^1(x) = (-1)^{n-1} \cdot 2d,$$

and then hence

$$EGC^1(x,t) = \sum_{n \geq 1} t^n GC_n^1(x) = \sum_{n \geq 1} t^n (-1)^{n-1} \cdot 2d = -2d \sum_{n \geq 1} (-t)^n = \frac{2dt}{1+t}.$$

As for Type 2 generalized clusters of length n, there are $n-4$ singleton blocks and exactly one block of size 4 and since $|\mathcal{GC}_n^2| = n-3$, we have

$$EGC_n^2(x) = (-1)^{n-1}(n-3)x, \quad \text{for } \ n \geq 4$$

and then hence

$$\begin{aligned} EGC^2(x,t) &= \sum_{n\geq 4} t^n GC_n^2(x) \\ &= \sum_{n\geq 4} t^n (-1)^{n-1}(n-3)x \\ &= -xt^3 \sum_{n\geq 1} n(-t)^n \\ &= \frac{xt^4}{(1+t)^2} \end{aligned}$$

Finally, we could get

$$\begin{aligned} A_{u,\overline{\mathcal{W}_d}}(x,t) &= \frac{1}{1 - GC^1(x-1,t) - GC^2(x-1,t)} && (2.36) \\ &= \frac{1}{1 - \frac{2dt}{1+t} - \frac{(x-1)t^4}{(t+1)^2}} && (2.37) \end{aligned}$$

A few initial terms of expansion are

$$\begin{aligned} &1 + 2dt + \left(-2d + 4d^2\right)t^2 + \left(2d - 8d^2 + 8d^3\right)t^3 + \\ &\left(-1 - 2d + 12d^2 - 24d^3 + 16d^4 + x\right)t^4 + \\ &\left(2 - 2d - 16d^2 + 48d^3 - 64d^4 + 32d^5 - 2x + 4dx\right)t^5 + \\ &\left(-3 + 10d + 8d^2 - 80d^3 + 160d^4 - 160d^5 + 64d^6 + 3x - 12dx + 12d^2x\right)t^6 \\ &+ \cdots . \end{aligned}$$

Based on the formula for $A_{u,\overline{\mathcal{W}_d}}(x,t)$ in Equation (2.36), we are able answer some further question from an enumerative point of view, such as, how many shortest loops in a random walk on $\mathbb{Z}^d$ of length n do we expect to see in average?

Taking partial derivative of the generating function with respect to x and then let $x=1$, we have

$$\begin{aligned} \left.\frac{\partial A_{u,\overline{\mathcal{W}_d}}(x,t)}{\partial x}\right|_{x=1} &= 8\binom{d}{2}\frac{t^4}{(1-(2d-1)t)^2} \\ &= \sum_{n\geq 4}(n-3)(2d-1)^{n-4}t^n, \end{aligned}$$

where coefficients of t^n is the total number of $12\bar{1}\bar{2}$ in $\overline{W_{n,d}}$. To answer the question, we also need to use a fact that in a random walk, the number of any given shortest loop are identically distributed. In other words, $A_{u,\overline{\mathcal{W}_d}}(x,t) \equiv A_{v,\overline{\mathcal{W}_d}}(x,t)$ as long as u and v are both loops of length 4. According to previous discussion, there are $2d(2d-1)^{n-1}$ random walks in $\overline{W_{n,d}}$ and $8\binom{d}{2}$ different shortest loops. Then the average number of shortest loops in a non-backtracking random walk of length n is given by

$$\frac{8\binom{d}{2}(n-3)(2d-1)^{n-4}}{2d(2d-1)^n} = \frac{2(d-1)(n-3)}{(2d-1)^3},$$

for $n \geq 3$. For example, when $d = 2$, the average number of loops of length 4 in a non-backtracking random walk on $\mathbb{Z}^2$ of length $n \geq 4$ is $\frac{2(n-3)}{27}$, that is, each step after the third step contributes $\frac{2}{27}$ loop in average. When $d = 3$, the average number of loops of length 4 in a non-backtracking random walk on $\mathbb{Z}^3$ of length $n \geq 4$ is $\frac{4(n-3)}{125}$.

The contents of Chapter 2 are currently under preparation for submission. Some portion is co-authored with J. B. Remmel. The dissertation author is the author of this material.

Chapter 3

Clusters and Generalized Clusters for fillings of $D_{i+kn+j}^{i,j,k}$.

In Chapter 2, we discussed clusters and generalized clusters for fillings of $D_{kn}^{0,0,0}$ which are of rectangular shapes. In this chapter, we extend the methods to $D_{i+kn+j}^{i,j,k}$ which are almost rectangular shapes but we allow the first and the last columns have different heights. In following sections of this chapter, $D_{kn+j}^{0,j,k}$, $D_{i+kn}^{i,0,k}$ and $D_{i+kn+j}^{i,j,k}$ will be discussed respectively. For example, $D_{20}^{0,0,4}$, $D_{18}^{2,0,4}$, $D_{19}^{0,3,4}$ and $D_{21}^{2,3,4}$ are pictured in Figure 3.1.

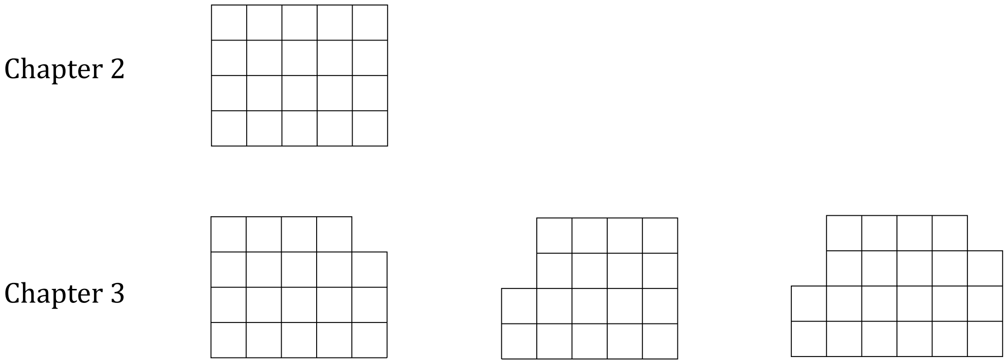


Figure 3.1: Focus of Chapter 3.

3.1 Clusters and Generalized Clusters for fillings of $D^{0,j,k}_{kn+j}$

In this section, we shall extend the generalized cluster method to deal with various types of fillings of $D^{0,j,k}_{kn+j}$, $j \neq k$. Assume a set of patterns $\Gamma \subseteq \mathcal{P}^{0,0,k}_{kr}$, we let $\mathcal{MP}^{0,j,k}_{kn+j,\Gamma}$ denote the set of elements that arise by starting with an element F of $\mathcal{P}^{0,j,k}_{kn+j}$ and marking some of the Γ-matches in F by placing an 'x' on the column which starts the Γ-match. Given an element $F \in \mathcal{MP}^{0,j,k}_{kn+j,\Gamma}$, we let $m_\Gamma(F)$ denote the number of marked Γ-matches in F. Clearly, the last column of F cannot be contained in any Γ-match because the heights do not agree.

To find an extension of Theorem 2.7 for these types of arrays, we need to define a special type of generalized cluster which we call a generalized end-cluster. That is, suppose that we are given a binary relation $\mathscr{R}$ on columns of integers and a set of patterns $\Gamma \subseteq \mathcal{P}^{0,0,k}_{kr}$.

Definition 3.1. *We say that* $Q \in \mathcal{MP}^{0,j,k}_{kn+j,\Gamma}$ *is a* generalized $\Gamma, \mathscr{R}$-end-cluster *if we can write* $Q = B_1 B_2 \cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

1. B_m *is a column of height* j*,*
2. *for* $i < m$*, either* B_i *is a single column or* B_i *consists of* r*-columns where* $r \geq 2$*,* $\mathrm{red}(B_i)$ *is a* Γ*-cluster in* $\mathcal{MP}_{kr,\Gamma}$*, and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*
3. *for* $1 \leq i \leq m-1$*, the pair* $(last(B_i), first(B_{i+1}))$ *is not in* $\mathscr{R}$ *where for any* i*,* $last(B_i)$ *is the right-most column of* B_i *and* $first(B_i)$ *is the left-most column of* B_i*.*

Let $\mathcal{GEC}^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}$ denote the set of all generalized $\Gamma, \mathscr{R}$-end-clusters which have n columns of height k followed by a column of height j. Given $Q = B_1 B_2 \ldots B_m \in \mathcal{GEC}^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}$, we define the weight of B_i, $\omega_{\Gamma,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{m_\Gamma(\mathrm{red}(B_i))}$ if B_i is order isomorphic to a Γ-cluster. Then we define the weight of Q, $\omega_{\Gamma,\mathscr{R}}(Q)$, to be $(-1)^{m-1}\prod_{i=1}^{m}\omega_{\Gamma,\mathscr{R}}(B_i)$. We let

$$GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x) = \sum_{Q\in\mathcal{GEC}^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}} \omega_{\Gamma,\mathscr{R}}(Q) \tag{3.1}$$

Let $\mathcal{P}^{0,j,k}_{kn+j,\mathscr{R}}$ denote the set of all elements $F \in \mathcal{P}^{0,j,k}_{kn+j}$ such that the relation $\mathscr{R}$ holds for any pair of consecutive columns in F. We let $\mathcal{MP}^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}$ denote the set of elements that arise by starting with an element F of $\mathcal{P}^{0,j,k}_{kn+j,\mathscr{R}}$ and marking some of the Γ-matches in F by placing on x on the column which starts the Γ-match.

Then we have the following theorem.

Theorem 3.2. *Let $\mathscr{R}$ be a binary relation on pairs of columns (C,D) which are filled with integers which are increasing from bottom to top. Let $\Gamma \subseteq \mathcal{P}^{0,0,k}_{kr}$ where $r \geq 2$. Then*

$$\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}\sum_{F\in\mathcal{P}^{0,j,k}_{kn+j,\mathscr{R}}} x^{\Gamma\text{-mch}(F)} = \frac{\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x-1)}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x-1)}. \tag{3.2}$$

Proof. Replace x by $x+1$ in (3.2). Then the left-hand side of (3.2) is the generating function of $m_\Gamma(F)$ over all $F \in \mathcal{MP}^{0,j,k}_{kn,+j,\Gamma,\mathscr{R}}$. That is, it easy to see that

$$\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}\sum_{F\in\mathcal{P}^{0,j,k}_{kn+j,\mathscr{R}}}(x+1)^{\Gamma\text{-mch}(F)} = \sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}\sum_{F\in\mathcal{MP}^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}} x^{m_\Gamma(F)}. \tag{3.3}$$

Thus we must show that

$$\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}\sum_{F\in\mathcal{MP}^{0,0,k}_{kn,\Gamma,\mathscr{R}}} x^{m_\Gamma(F)} = \frac{\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x)}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)}. \tag{3.4}$$

Now

$$\frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)} = 1+\sum_{m\geq 1}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^m. \tag{3.5}$$

Taking the coefficient of $\frac{t^{ks+j}}{(ks+j)!}$ on both sides of (3.4) where $s \geq 0$, we see that we must show that

$$
\begin{aligned}
&\sum_{F \in \mathcal{MP}^{0,j,k}_{ks+j,\mathscr{R}}} x^{m_\Gamma(F)} \qquad\qquad (3.6)\\
&= \left(\left(\sum_{m=1}^{\infty}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^m\right) \times \right.\\
&\qquad \left.\left(\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x)\right)\right|_{\frac{t^{ks+j}}{(ks+j)!}}\\
&= \sum_{a+b=s,a,b\geq 0}\binom{ks+j}{ka,kb+j}\left.\left(\sum_{m=1}^{\infty}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^m\right)\right|_{\frac{t^{ka}}{(ka)!}} \times\\
&\qquad \left.\left(\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x)\right)\right|_{\frac{t^{kb+j}}{(kb+j)!}}\\
&= \sum_{a+b=s,a,b\geq 0}\binom{ks+j}{ka,kb+j} \times\\
&\qquad \left(\sum_{m=1}^{a}\sum_{\substack{a_1+a_2+\cdots+a_m=a\\ a_i\geq 1}}\binom{ka}{ka_1,\ldots,ka_m}\prod_{i=1}^{m}GC^{0,0,k}_{ka_i,\Gamma,r}(x)\right)GEC^{0,j,k}_{kb+j,\Gamma,\mathscr{R}}(x)\\
&= \sum_{a+b=s,a,b\geq 0}\sum_{m=1}^{a}\sum_{\substack{a_1+a_2+\cdots+a_m=a\\ a_i\geq 1}}\binom{ks+j}{ka_1,\ldots,ka_m,kb+j} \times\\
&\qquad\qquad GEC^{0,j,k}_{kb+j,\Gamma,\mathscr{R}}(x)\prod_{j=1}^{m}GC^{0,0,k}_{ka_j,\Gamma,r}(x). \qquad\qquad (3.7)
\end{aligned}
$$

The right-hand side of (3.6) is now easy to interpret. First we pick non-negative integers a and b such that $a+b=s$. Then we pick an m such that $1 \leq m \leq a$. Next we pick $a_1,\ldots,a_m \geq 1$ such that $a_1+a_2+\cdots+a_m = a$. Next the binomial coefficient $\binom{ks+j}{ka_1,\ldots,ka_m,kb+j}$ allows us to pick sets $S_1,\ldots,S_m,S_{m+1}$ which partition $\{1,\ldots,ks+j\}$ such that $|S_i| = ka_i$ for $i=1,\ldots,m$ and $|S_{m+1}| = kb+j$. The factor $GEC^{0,j,k}_{kb+j,\Gamma,\mathscr{R}}(x)$ allows us to pick an a a generalized $\Gamma,\mathscr{R}$-end-cluster G_{m+1} of size $kb+j$ with weight $\omega_{\Gamma,\mathscr{R}}(G_{m+1})$. Note that in the cases where $b=0$, our definitions imply that G_{m+1} is just a column of height j filled with the numbers $1,\ldots,j$ which is increasing, reading from bottom to top. Finally the product $\prod_{j=1}^{m}GC^{0,0,k}_{ka_j,\Gamma,\mathscr{R}}(x)$ allows us to pick generalized $\Gamma,\mathscr{R}$-clusters $G_i \in \mathcal{GC}^{0,0,k}_{ka_i,\Gamma,\mathscr{R}}$ for $i=1,\ldots,m$ with weight $\prod_{i=1}^{m} \omega_{\Gamma,\mathscr{R}}(G_i)$. Note that in the cases where $a_i = 1$, our definitions imply that G_i is just a column of height k filled with the numbers $1, \ldots, k$ which is increasing, reading from bottom to top.

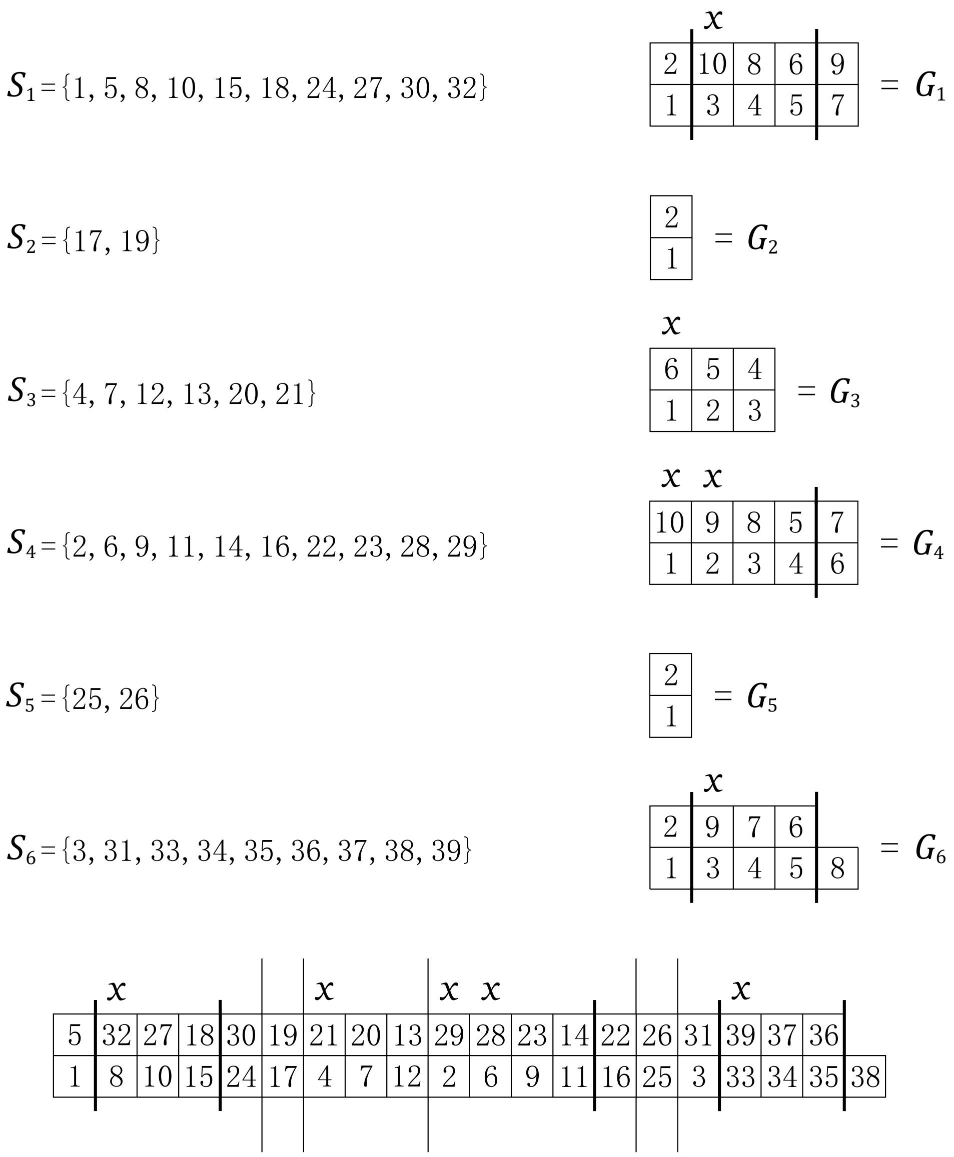


Figure 3.2: Construction for the right-hand side of (3.6).

For example, suppose that $k = 2$ and $j = 1$ and $\Gamma = \{P\}$ where $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Suppose that $\mathscr{R}$ is relation where for any two columns C and D which filled with integers and are strictly increasing in columns, $(C, D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom element of D. Then in Figure 3.2, we have pictured $S_1, S_2, S_3, S_4, S_5, S_6$ which partition $\{1, \ldots, 39\}$ and corresponding generalized $\Gamma, \mathscr{R}$-clusters $G_1, \ldots, G_5$ and a general $\Gamma, \mathscr{R}$-end-cluster G_6. For each i, we have indicated the separation between the blocks of G_i by dark black lines. Then for each $i = 1, \ldots, m+1$, we create a cluster E_i which results by replacing the j in G_i by the j^{th} element of S_i. If we concatenate $E_1 \ldots E_6$ together, then we will obtain an element of $Q \in \mathcal{MP}^{0,j,k}_{ks+j,\Gamma}$. The weight of Q equals $\prod_{j=1}^{6} \omega_{\Gamma,\mathscr{R}}(G_i)$ where

$$\begin{array}{rcl}
\omega_{\Gamma,\mathscr{R}}(G_1) & = & (-1)^2 x, \\
\omega_{\Gamma,\mathscr{R}}(G_2) & = & 1, \\
\omega_{\Gamma,\mathscr{R}}(G_3) & = & x, \\
\omega_{\Gamma,\mathscr{R}}(G_4) & = & (-1)^1 x^2, \\
\omega_{\Gamma,\mathscr{R}}(G_5) & = & 1, \text{ and} \\
\omega_{\Gamma,\mathscr{R}}(G_6) & = & x.
\end{array}$$

In Figure 3.2, we have indicated the boundaries between the G_is by light lines.

We let $\mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}}$ denote the set of all elements that can be constructed in this way. Thus $Q = E_1 \ldots E_m E_{m+1}$ is an element of $\mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}}$ if and only if for each $i = 1, \ldots, m$, $\text{red}(E_i)$ is a generalized $\Gamma, \mathscr{R}$-cluster and $\text{red}(E_{m+1})$ is a generalized $\Gamma, \mathscr{R}$-end-cluster.

Next we define a sign reversing involution $\theta : \mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}} \to \mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}}$. Given $Q = E_1 \ldots E_m E_{m+1} \in \mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}}$, look for the first i such that either

1. the block structure of $\text{red}(E_i) = B_1^{(i)} \ldots B_{k_i}^{(i)}$ consists of more than one block or
2. E_i consists of a single block $B_1^{(i)}$ and $(last(B_i), first(E_i))$ is not in $\mathscr{R}$.

In case (1), if $i \leq m$, we let $\theta(E_1 \ldots E_{m+1})$ be the result of replacing E_i by two generalized $\Gamma, \mathscr{R}$-clusters, E_i^* and E_i^{**} where E_i* consists just of $B_1^{(i)}$ and E_i^{**} consists of $B_2^{(i)} \ldots B_{k_i}^{(i)}$. If $i = m+1$, we let $\theta(E_1 \ldots E_{m+1})$ be the result of replacing E_{m+1} by a generalized $\Gamma, \mathscr{R}$-cluster, E^*_{m+1} and a generalized $\Gamma, \mathscr{R}$-end-cluster E^{**}_{m+1} where $E_{m+1}*$ consists just of $B_1^{(m+1)}$ and E^{**}_{m+1} consists of $B_2^{(m+1)} \dots B_{k_i}^{(m+1)}$. Note that in either case, $\omega_{\Gamma,\mathscr{R}}(E_i) = (-1)^{k_i-1} \prod_{j=1}^{k_i} \omega_{\Gamma,\mathscr{R}}(B_j^{(i)})$ while $\omega_{\Gamma,\mathscr{R}}(E_i^*)\omega_{\Gamma,\mathscr{R}}(E_i)^{**} = (-1)^{k_i-2} \prod_{j=1}^{k_i} \omega_{\Gamma,\mathscr{R}}(B_j^{(i)})$.

In case (2), if $i < m$, we let $\theta(E_1 \dots E_{m+1})$ be the result of replacing E_i and E_{i+1} by the single generalized $\Gamma, \mathscr{R}$-cluster $E = B_1^{(i)} B_1^{(i+1)} \dots B_{k_{i+1}}^{(i+1)}$. If $i = m,$, we let $\theta(E_1 \dots E_{m+1})$ be the result of replacing E_m and E_{m+1} by the single generalized $\Gamma, \mathscr{R}$-end-cluster $E = B_1^{(m)} B_1^{(m+1)} \dots B_{k_{i+1}}^{(m+1)}$. In either case,

$$\omega_{\Gamma,\mathscr{R}}(E_i)\omega_{\Gamma,\mathscr{R}}(E_{i+1}) = (-1)^{k_{i+1}-1}\omega_{\Gamma,\mathscr{R}}(B_1^{(i)}) \prod_{j=1}^{k_{i+1}} \omega_{\Gamma,\mathscr{R}}(B_j^{(i+1)})$$

while

$$\omega_{\Gamma,\mathscr{R}}(E) = (-1)^{k_{i+1}}\omega_{\Gamma,\mathscr{R}}(B_1^{(i)}) \prod_{j=1}^{k_{i+1}} \omega_{\Gamma,\mathscr{R}}(B_j^{(i+1)}).$$

If neither case (1) or case (2) applies, then we let $\theta(E_1 \dots E_{m+1}) = E_1 \dots E_{m+1}$. For example, suppose that $\mathscr{R}$ is the binary relation where for any two columns C and D, which filled with integers and are strictly increasing in columns, $(C, D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom elements of D and $\Gamma = \{P\}$ where $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Then if $Q = E_1 \dots E_6$ is the generalized $\Gamma, \mathscr{R}$-cluster pictured in Figure 3.2, then we are in case (1) with $i = 1$ since E_1 consists of more than one block. Thus $\theta(Q)$ results by breaking that generalized $\Gamma, \mathscr{R}$-cluster into to two clusters E_1^* of size 1 and E_1^{**} of size 4. $\theta(Q)$ is pictured in Figure 3.3.

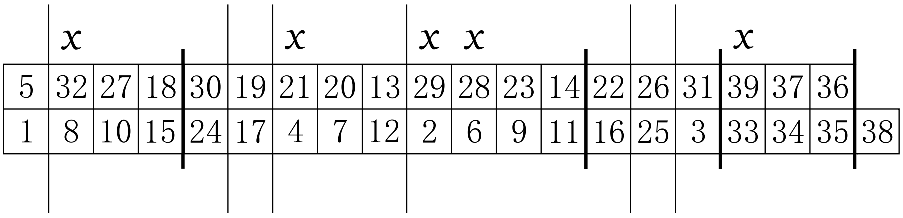


Figure 3.3: The involution θ.

It is easy to see that θ is an involution. That is, if $Q = E_1 \dots E_{m+1}$ is in case (1) using E_i, then $\theta(Q)$ will be in case (2) using E_i^* and E_i^{**}. Similarly if $Q =$

$E_1 \dots E_{m+1}$ is in case (1) using E_i and E_{i+1}, then $\theta(Q)$ will be in case (2) using $E = E_iE_{i+1}$. It follows that if $\theta(E_1 \dots E_{m+1}) \neq E_1 \dots E_{m+1}$, then $\omega_{\Gamma,\mathscr{R}}(E_1 \dots E_{m+1}) = -\omega_{\Gamma,\mathscr{R}}(\theta(E_1 \dots E_{m+1}))$ so that the right-hand side of (2.11) equals

$$\sum_{Q=E_1\dots E_{m+1}\in\mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}},\theta(Q)=Q} \prod_{i=1}^{m+1} \omega_{\Gamma,\mathscr{R}}(E_i).$$

Thus we must examine the fixed points of θ.

If $Q = E_1 \dots E_{m+1} \in \mathcal{HGEC}_{ks+j,\Gamma,\mathscr{R}}$ and $\theta(Q) = Q$, then it must be the case that for each $i \leq m$, E_i consists of single column of weight 1 or it reduces to generalized $\Gamma,\mathscr{R}$-cluster $\overline{E}_i$ consisting of a single block $B_1^{(i)}$ whose weight is the weight of $\text{red}(B_1^{(i)})$ as a Γ-cluster. Moreover, it must be the case that for all $i = 1, \dots m-1$, $(last(E_i), first(E_{i+1})$ is in $\mathscr{R}$. Similarly, E_{m+1} must consists of a single column of height j and $(last(E_m), E_{m+1})$ must be in $\mathscr{R}$. But this means for all $j = 1, \dots, s$, $(Q[j], Q[j+1])$ is in $\mathscr{R}$. That is, either $Q[j]$ equals $last(E_i)$ for some i or column j is contained in one of the Γ-clusters E_i in which case $(Q[j], Q[j+1])$ is in $\mathscr{R}$ by our definition of generalized $\Gamma,\mathscr{R}$-clusters. Thus any fixed point Q of θ is an element $\mathcal{MP}^{0,j,k}_{ks+j,\mathscr{R}}$. Then just like our proof Theorem 2.2, it follows that $E_1, \dots, E_m$ are just the maximal Γ-subclusters of an element in $\mathcal{P}^{0,j,k}_{kn,\mathscr{R}}$. Vice versa, if $T = F_1 \dots F_rF_{r+1}$ is an element of $\mathcal{P}^{0,0,k}_{ks+j,\mathscr{R}}$ where $F_1, \dots, F_r$ are the maximal Γ-subclusters of T and F_{r+1} is the last column of height j, then $T = F_1 \dots F_r$ is a fixed point of θ. Thus we have proved that the right-hand side of (3.6) equals

$$\sum_{F\in\mathcal{MP}^{0,j,k}_{ks+j},\mathscr{R}} x^{m_\Gamma(F)}$$

which is what we wanted to prove. □

3.2 Clusters and Generalized Clusters for fillings of $D^{i,0,k}_{i+kn}$

In this section, we shall extend the generalized cluster method to deal with various types of fillings of $D^{i,0,k}_{i+kn}$, for $i \neq k$. Let a set of patterns $\Gamma \subseteq \mathcal{P}^{0,0,k)}_{kr}$. We let $\mathcal{MP}^{i,0,}_{i+kn,\Gamma}$ denote the set of elements that arise by starting with an element F

of $\mathcal{P}^{i,0,k}_{i+kn}$ and marking some of the Γ-matches in F by placing on x on the column which starts the Γ-match. Given an element $F \in \mathcal{MP}^{i,0,k}_{i+kn,\Gamma}$, we let $m_\Gamma(F)$ denote the number of marked Γ-matches in F. Clearly, the first column of F cannot be contained in any Γ-match because the heights do not agree.

To find an extension of Theorem 2.7 for these types of arrays, we need to define a special type of generalized cluster which we call an start cluster. That is, suppose that we are given a binary relation $\mathscr{R}$ on columns of integers a set of patterns $\Gamma \subseteq \mathcal{P}^{0,0,k)}_{kr}$.

Definition 3.3. *We say that* $Q \in \mathcal{MP}^{i,0,k}_{i+kn,\Gamma}$ *is a* generalized $\Gamma,\mathscr{R}$-start-cluster *if we can write* $Q = B_1B_2\cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

1. B_1 *is a single column of height* i*,*

2. *for* $2 < a \le m$*, either* B_a *is a single column or* B_a *consists of* r*-columns where* $r \geq 2$*,* $\mathrm{red}(B_a)$ *is a* Γ*-cluster in* $\mathcal{MP}_{kr,\Gamma}$*, and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*

3. *for* $1 \le i \le m-1$*, the pair* $(last(B_i), first(B_{i+1})$ *is not in* $\mathscr{R}$ *where for any* j*,* $last(B_j)$ *is the right-most column of* B_j *and* $first(B_j)$ *is the left-most column of* B_j*.*

Let $\mathcal{GSC}^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}$ denote the set of all generalized $\Gamma,\mathscr{R}$-start-clusters which start with a column of height i and which is followed by n columns of height k. Given $Q = B_1B_2\ldots B_m \in \mathcal{GSC}^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}$, we define the weight of B_i, $\omega_{\Gamma,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{m_\Gamma(\mathrm{red}(B_i))}$ if B_i is order isomorphic to a Γ-cluster. Then we define the weight of Q, $\omega_{\Gamma,\mathscr{R}}(Q)$, to be $(-1)^{m-1}\prod_{i=1}^{m}\omega_{\Gamma,\mathscr{R}}(B_i)$. We let

$$GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x) = \sum_{Q \in \mathcal{GSC}^{i,0,,k}_{i+kn,\Gamma,\mathscr{R}}} \omega_{\Gamma,\mathscr{R}}(Q). \tag{3.8}$$

Let $\mathcal{P}^{i,0,k}_{i+kn,\mathscr{R}}$ denote the set of all elements $F \in \mathcal{P}^{i,0,k}_{i+kn}$ such that the relation $\mathscr{R}$ holds for any pair of consecutive columns in F. We let $\mathcal{MP}^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}$ denote the set of elements that arise by starting with an element F of $\mathcal{P}^{i,0,k}_{i+kn,\mathscr{R}}$ and marking some of the Γ-matches in F by placing on x on the column which starts the Γ-match.

Then we have the following theorem.

Theorem 3.4. *Let $\mathscr{R}$ be a binary relation on pairs of columns (C, D) which are filled with integers which are increasing from bottom to top. Let $\Gamma \subseteq \mathcal{P}^{0,0,k}_{kr}$ where $r \geq 2$. Then*

$$\sum_{n \geq 0} \frac{t^{i+kn}}{(i+kn)!} \sum_{F \in \mathcal{P}^{i,0,k}_{i+kn,\mathscr{R}}} x^{\Gamma\text{-mch}(F)} = \frac{\sum_{n \geq 0} \frac{t^{i+kn}}{(i+kn)!} GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x-1)}{1 - \sum_{n \geq 1} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x-1)}. \tag{3.9}$$

Proof. Replace x by $x+1$ in (3.9). Then as in our previous theorems, the left-hand side of (3.2) becomes of $m_\Gamma(F)$ over all $F \in \mathcal{MP}^{i,0,k}_{kn,+j,\Gamma,\mathscr{R}}$. As in our previous theorems, it easy to see that

$$\sum_{n \geq 0} \frac{t^{i+kn}}{(i+kn)!} \sum_{F \in \mathcal{MP}^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}} x^{m_\Gamma(F)}. \tag{3.10}$$

Thus we must show that

$$\sum_{n \geq 0} \frac{t^{i+kn}}{(i+kn)!} \sum_{F \in \mathcal{MP}^{i,0,k}_{kn,\Gamma,\mathscr{R}}} x^{m_\Gamma(F)} = \frac{\sum_{n \geq 0} \frac{t^{i+kn}}{(i+kn)!} GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x)}{1 - \sum_{n \geq 1} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)}. \tag{3.11}$$

Taking the coefficient of $\frac{t^{i+ks}}{(i+ks)!}$ on both sides of (3.11) where $s \geq 0$, we see that we must show that

$$
\begin{aligned}
&\sum_{F \in \mathcal{MP}^{i,0,k}_{i+ks,\mathscr{R}}} x^{m_\Gamma(F)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.12)\\
= &\left(\left(\sum_{m=1}^{\infty}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^{m}\right)\times\right.\\
&\qquad\left.\left.\left(\sum_{n\geq 0}\frac{t^{i+kn}}{(i+kn)!}GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x)\right)\right)\right|_{\frac{t^{i+ks}}{(i+ks)!}}\\
= &\sum_{a+b=s,a,b\geq 0}\binom{i+ks}{ka,i+kb}\left.\left(\sum_{m=1}^{\infty}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^{m}\right)\right|_{\frac{t^{ka}}{(ka)!}}\times\\
&\qquad\left.\left(\sum_{n\geq 0}\frac{t^{i+kn}}{(i+kn)!}GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x)\right)\right|_{\frac{t^{i+kb}}{(i+kb)!}}\\
= &\sum_{a+b=s,a,b\geq 0}\binom{i+ks}{ka,i+kb}\times\\
&\left(\sum_{m=1}^{a}\sum_{\substack{a_1+a_2+\cdots+a_m=a\\ a_i\geq 1}}\binom{ka}{ka_1,\ldots,ka_m}\prod_{j=1}^{m}GC^{0,0,k}_{ka_j,\Gamma,r}(x)\right)GSC^{i,0,k}_{i+kb,\Gamma,\mathscr{R}}(x)\\
= &\sum_{a+b=s,a,b\geq 0}\sum_{m=1}^{a}\sum_{\substack{a_1+a_2+\cdots+a_m=a\\ a_i\geq 1}}\binom{ks+i}{ka_1,\ldots,ka_m,i+kb}\times\\
&\qquad\qquad GSC^{i,0,k}_{i+kb,\Gamma,\mathscr{R}}(x)\prod_{j=1}^{m}GC^{0,0,k}_{ka_j,\Gamma,r}(x). \qquad\qquad\qquad\qquad\qquad (3.13)
\end{aligned}
$$

The right-hand side of (3.12) is now easy to interpret. First we pick non-negative intergers a and b such that $a+b=s$. Then we pick an m such that $1 \leq m \leq a$. Next we pick $a_1,\ldots,a_m \geq 1$ such that $a_1+a_2+\cdots+a_m = a$. Next the binomial coefficient $\binom{ks+j}{ka_1,\ldots,ka_m,kb+j}$ allows us to pick sets $S_1,\ldots,S_m,S_{m+1}$ which partition $\{1,\ldots,i+ks\}$ such that $|S_i| = ka_i$ for $i = 2,\ldots,m+1$ and $|S_1| = i+kb$. The factor $GSC^{i,0,k}_{i+kb,\Gamma,\mathscr{R}}(x)$ allows us to pick an a a generalized $\Gamma,\mathscr{R}$-start-cluster G_1 of size $i+kb$ with weight $\omega_{\Gamma,\mathscr{R}}(G_1)$. Note that in the cases where $b=0$, our definitions imply that G_{m+1} is just a column of height i filled with the numbers $1,\ldots,i$ which is increasing, reading from bottom to top. Finally the product $\prod_{j=1}^{m} GC^{0,0,k}_{ka_j,\Gamma,\mathscr{R}}(x)$ allows us to pick generalized $\Gamma,\mathscr{R}$-clusters $G_i \in \mathcal{GC}^{0,0,k}_{ka_i,\Gamma,\mathscr{R}}$ for

$p = 2, \ldots, m+1$ with weight $\prod_{p=2}^{m+1} \omega_{\Gamma,\mathscr{R}}(G_p)$. Note that in the cases where $a_i = 1$, our definitions imply that G_i is just a column of height k filled with the numbers $1, \ldots, k$ which is increasing, reading from bottom to top.

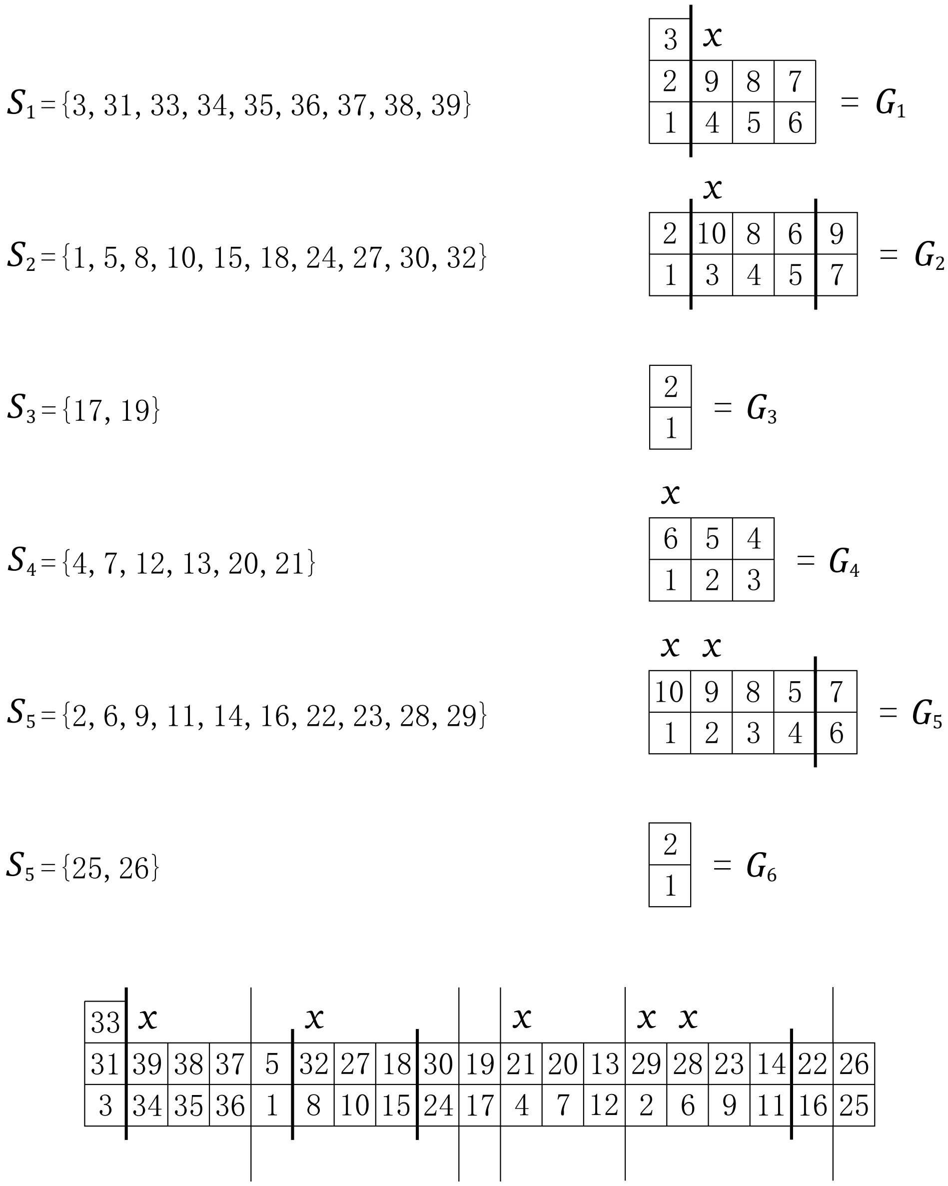


Figure 3.4: Construction for the right-hand side of (3.12).

For example, suppose that $k = 2$ and $i = 3$ and $\Gamma = \{P\}$ where $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Suppose that $\mathscr{R}$ is relation where for any two columns C and D which filled with integers and are strictly increasing in columns, $(C, D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom element of D. Then in Figure 3.4, we have pictured $S_1, S_2, S_3, S_4, S_5, S_6$ which partition $\{1, \ldots, 39\}$ and corresponding generalized $\Gamma, \mathscr{R}$-clusters $G_2, \ldots, G_6$ and a generalized $\Gamma, \mathscr{R}$-start-cluster G_1. For each i, we have indicated the separation between the blocks of G_i by dark black lines. Then for each $i = 1, \ldots, m+1$, we create a cluster E_i which results by replacing the j in G_i by the j^{th} element of S_i. If we concatenate $E_1 \ldots E_6$ together, then we will obtain an element of $Q \in \mathcal{MP}^{i,0,k}_{i+ks,\Gamma}$. The weight of Q equals $\prod_{j=1}^{6} \omega_{\Gamma,\mathscr{R}}(G_i)$ where

$$\begin{aligned}
\omega_{\Gamma,\mathscr{R}}(G_1) &= -x, \\
\omega_{\Gamma,\mathscr{R}}(G_2) &= (-1)^2 x, \\
\omega_{\Gamma,\mathscr{R}}(G_3) &= 1, \\
\omega_{\Gamma,\mathscr{R}}(G_4) &= x, \\
\omega_{\Gamma,\mathscr{R}}(G_5) &= (-1)^1 x^2, \text{ and} \\
\omega_{\Gamma,\mathscr{R}}(G_6) &= 1.
\end{aligned}$$

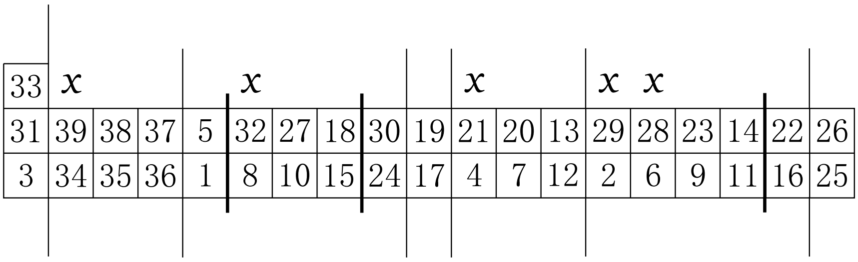
In Figure 3.4, we have indicated the boundaries between the G_is by light lines.

We let $\mathcal{HGSC}_{ks+j,\Gamma,\mathscr{R}}$ denote the set of all elements that can be constructed in this way. Thus $Q = E_1 \ldots E_m E_{m+1}$ is an element of $\mathcal{HGSC}_{ks+j,\Gamma,\mathscr{R}}$ if and only if for each $i = 1, \ldots, m$, $\text{red}(E_i + 1)$ is a generalized $\Gamma, \mathscr{R}$-cluster and $\text{red}(E_1)$ is a generalized $\Gamma, \mathscr{R}$-start-cluster. Next we define a sign reversing involution $\theta : \mathcal{HGSC}_{ks+j,\Gamma,\mathscr{R}} \to \mathcal{HGSC}_{ks+j,\Gamma,\mathscr{R}}$. Given $Q = E_1 \ldots E_m E_{m+1} \in \mathcal{HGAC}_{ks+j,\Gamma,\mathscr{R}}$, look for the first i such that either

1. the block structure of $\text{red}(E_i) = B_1^{(i)} \ldots B_{k_i}^{(i)}$ consists of more than one block or
2. E_i consists of a single block $B_1^{(i)}$ and $(last(B_i), first(E_i))$ is not in $\mathscr{R}$.

In case (1), if $2 \leq i \leq m+1$, we let $\theta(E_1 \ldots E_{m+1})$ be the result of replacing E_i by two generalized $\Gamma, \mathscr{R}$-clusters, E_i^* and E_i^{**} where E_i* consists just of $B_1^{(i)}$ and E_i^{**} consists of $B_2^{(i)} \ldots B_{k_i}^{(i)}$. If $i = 1$, we let $\theta(E_1 \ldots E_{m+1})$ be the result of replacing E_1 by a generalized $\Gamma, \mathscr{R}$-start-cluster, E_1^* and a generalized $\Gamma, \mathscr{R}$-cluster

E_1^{**} where E_1* consists just of $B_1^{(1)}$ and E_1^{**} consists of $B_2^{(1)} \dots B_{k_i}^{(1)}$. Note that in either case, $\omega_{\Gamma,\mathscr{R}}(E_i) = (-1)^{k_i-1} \prod_{j=1}^{k_i} \omega_{\Gamma,\mathscr{R}}(B_j^{(i)})$ while $\omega_{\Gamma,\mathscr{R}}(E_i^*)\omega_{\Gamma,\mathscr{R}}(E_i)^{**} = (-1)^{k_i-2} \prod_{j=1}^{k_i} \omega_{\Gamma,\mathscr{R}}(B_j^{(i)})$. In case (2), if $2 \leq i \leq m$, we let $\theta(E_1 \dots E_{m+1})$ be the result of replacing E_i and E_{i+1} by the single generalized $\Gamma,\mathscr{R}$-cluster $E = B_1^{(i)} B_1^{(i+1)} \dots B_{k_{i+1}}^{(i+1)}$. If $i = 1$,, we let $\theta(E_1 \dots E_{m+1})$ be the result of replacing E_1 and E_2 by the single generalized $\Gamma,\mathscr{R}$-start-cluster $E = B_1^{(1)} B_1^{(2)} \dots B_{k_{i+1}}^{(2)}$. In either case, $\omega_{\Gamma,\mathscr{R}}(E_i)\omega_{\Gamma,\mathscr{R}}(E_{i+1}) = (-1)^{k_{i+1}-1}\omega_{\Gamma,\mathscr{R}}(B_1^{(i)}) \prod_{j=1}^{k_{i+1}} \omega_{\Gamma,\mathscr{R}}(B_j^{(i+1)})$ while $\omega_{\Gamma,\mathscr{R}}(E) = (-1)^{k_{i+1}}\omega_{\Gamma,\mathscr{R}}(B_1^{(i)}) \prod_{j=1}^{k_{i+1}} \omega_{\Gamma,\mathscr{R}}(B_j^{(i+1)})$. If neither case (1) or case (2) applies, then we let $\theta(E_1 \dots E_{m+1}) = E_1 \dots E_{m+1}$. For example, suppose that $\mathscr{R}$ is the binary relation where for any two columns C and D, which filled with integers and are strictly increasing in columns, $(C, D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom elements of D and $\Gamma = \{P\}$ where $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Then if $Q = E_1 \dots E_6$ is the generalized $\Gamma,\mathscr{R}$-cluster pictured in Figure 3.4, then we are in case (1) with $i = 1$ since E_1 consists of more than one block. Thus $\theta(Q)$ results by breaking that generalized $\Gamma,\mathscr{R}$-start-cluster into to two clusters E_1^* of size 1 and E_1^{**} of size 2. $\theta(Q)$ is pictured in Figure 3.5.



33	x				x					x			x	x				
31	39	38	37	5	32	27	18	30	19	21	20	13	29	28	23	14	22	26
3	34	35	36	1	8	10	15	24	17	4	7	12	2	6	9	11	16	25

Figure 3.5: The involution θ.

As in our previous theorems, it is easy to verify that θ is an involution. Moreover, if $\theta(E_1 \dots E_{m+1}) \neq E_1 \dots E_{m+1}$, then

$$\omega_{\Gamma,\mathscr{R}}(E_1 \dots E_{m+1}) = -\omega_{\Gamma,\mathscr{R}}(\theta(E_1 \dots E_{m+1}))$$

so that the right-hand side of (3.12) equals

$$\sum_{Q=E_1 \dots E_{m+1} \in \mathcal{HGSC}_{i+ks,\Gamma,\mathscr{R}}, \theta(Q)=Q} \prod_{i=1}^{m+1} \omega_{\Gamma,\mathscr{R}}(E_i).$$

Thus we must examine the fixed points of θ.

If $Q = E_1 \dots E_{m+1} \in \mathcal{HGSC}_{i+ks,\Gamma,\mathscr{R}}$ and $\theta(Q) = Q$, then it must be the case that for each $2 \leq i \leq m+1$, E_i consists of single column of weight 1 or it reduces to generalized $\Gamma, \mathscr{R}$-cluster $\overline{E}_i$ consisting of a single block $B_1^{(i)}$ whose weight is the weight of red$(B_1^{(i)})$ as a Γ-cluster. Moreover, it must be the case that for all $i = 2, \dots m$, $(last(E_i), first(E_{i+1})$ is in $\mathscr{R}$. Similarly, E_1 must consists of a single column of height i and $(E_1, first(E_2))$ must be in $\mathscr{R}$. But this means for all $j = 1, \dots, s$, $(Q[j], Q[j+1])$ is in $\mathscr{R}$. That is, either $Q[j]$ equals $last(E_i)$ for some i or column j is contained in one of the Γ-clusters E_i in which case $(Q[j], Q[j+1])$ is in $\mathscr{R}$ by our definition of generalized $\Gamma, \mathscr{R}$-clusters. Thus any fixed point Q of θ is an element $\mathcal{MP}^{i,0,k}_{i+ks,\mathscr{R}}$. Then just like our proof Theorem 2.2, it follows that $E_1, \dots, E_m$ are just the maximal Γ-subclusters of an element in $\mathcal{P}^{0,0,k}_{i+ks,\mathscr{R}}$. Vice versa, if $T = F_1 \dots F_r F_{r+1}$ is an element of $\mathcal{P}^{0,0,k}_{kn,\mathscr{R}}$ where $F_2, \dots, F_r$ are the maximal Γ-subclusters of T and F_1 is the initial column of height i, then $T = F_1 \dots F_r$ is a fixed point of θ. Thus we have proved that the right-hand side of (3.12) equals

$$\sum_{F \in \mathcal{MP}^{i,0,k}_{i+ks,\mathscr{R}}} x^{m_\Gamma(F)}$$

which is what we wanted to prove. □

3.3 Clusters and Generalized Clusters for fillings of $D^{i,j,k}_{i+kn+j}$.

In this section, we shall combine Section 3.1 and 3.2 to extend the generalized cluster method to deal with various types of fillings of $D^{i,j,k}_{i+kn+j}$. Let a set of patterns $\Gamma \subseteq \mathcal{P}^{0,0,k}_{kr}$. We let $\mathcal{MP}^{i,j,k}_{i+kn+j,\Gamma}$ denote the set of elements that arise by starting with an element F of $\mathcal{P}^{i,j,k}_{i+kn+j}$ and marking some of the Γ-matches in F by placing on x on the column which starts the Γ-match. Given an element $F \in \mathcal{MP}^{i,j,k}_{i+kn+j,\Gamma}$, we let $m_\Gamma(F)$ denote the number of marked Γ-matches in F. Clearly, neither the first or the last column of F is contained in a Γ-match.

To find an extension of Theorem 2.7 for fillings of $D^{0,j,k}_{kn+j}$, we defined generalized end clusters and for fillings of $D^{i,0,k}_{kn+i}$, we defined generalized start clusters. However, they are not enough to extend Theorem 2.7 for fillings of $D^{i,j,k}_{i+kn+j}$. Next we define a special type of generalized cluster which we call a generalized start-end cluster. That is, suppose that we are given a binary relation $\mathscr{R}$ on columns of integers a set of patterns $\Gamma \subseteq \mathcal{P}^{0,0,k)}_{kr}$.

Definition 3.5. *We say that* $Q \in \mathcal{MP}^{i,j,k}_{i+kn+j,\Gamma}$ *is a* generalized $\Gamma, \mathscr{R}$-start-end-cluster *if we can write* $Q = B_1 B_2 \cdots B_m$ *where* B_i *are blocks of consecutive columns in* Q *such that*

1. $m \geq 2$,
2. B_1 *is a column of height* i,
3. B_m *is a column of height* j,
4. *for* $1 < i < m$, *either* B_i *is a single column or* B_i *consists of* r*-columns where* $r \geq 2$, $\mathrm{red}(B_i)$ *is a* Γ*-cluster in* $\mathcal{MP}_{kr,\Gamma}$, *and any pair of consecutive columns in* B_i *are in* $\mathscr{R}$ *and*
5. *for* $1 \leq i \leq m-1$, *the pair* $(last(B_i), first(B_{i+1}))$ *is not in* $\mathscr{R}$ *where for any* j, $last(B_j)$ *is the right-most column of* B_j *and* $first(B_j)$ *is the left-most column of* B_j.

Let $\mathcal{GSEC}^{i,j,k}_{i+kn+j,\Gamma,\mathscr{R}}$ denote the set of all generalized $\Gamma, \mathscr{R}$-start-end-clusters which have n columns of height k between a column of height i and a column of height j. Given $Q = B_1 B_2 \ldots B_m \in \mathcal{GSEC}^{i,j,k}_{i+kn+j,\Gamma,\mathscr{R}}$, we define the weight of B_i, $\omega_{\Gamma,\mathscr{R}}(B_i)$, to be 1 if B_i is a single column and $x^{m_\Gamma(\mathrm{red}(B_i))}$ if B_i is order isomorphic to a Γ-cluster. Then we define the weight of Q, $\omega_{\Gamma,\mathscr{R}}(Q)$, to be $(-1)^{m-1} \prod_{i=1}^m \omega_{\Gamma,\mathscr{R}}(B_i)$. We let

$$GSEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x) = \sum_{Q \in \mathcal{GSEC}^{i,j,k}_{i+kn+j,\Gamma,\mathscr{R}}} \omega_{\Gamma,\mathscr{R}}(Q). \tag{3.14}$$

Let $\mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}$ denote the set of all elements $F \in \mathcal{P}^{i,j,k}_{i+kn+j}$ such that the relation $\mathscr{R}$ holds for any pair of consecutive columns in F. We let $\mathcal{MP}^{i,j,k}_{i+kn+j,\Gamma,\mathscr{R}}$ denote the set of elements that arise by starting with an element F of $\mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}$ and marking some of the Γ-matches in F by placing on 'x' on the column which starts the Γ-match.

Then we have the following theorem.

Theorem 3.6. *Let $\mathscr{R}$ be a binary relation on pairs of columns (C, D) which are filled with integers which are increasing from bottom to top. Let $\Gamma \subseteq \mathcal{P}_{kr}^{0,0,k}$ where $r \geq 2$. Then*

$$\sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} \sum_{F \in \mathcal{P}_{i+kn+j,\mathscr{R}}^{i,j,k}} x^{\Gamma\text{-mch}(F)} = \frac{\left(\sum_{n\geq 0} \frac{t^{i+kn}}{(i+kn)!} GSC_{i+kn,\Gamma,\mathscr{R}}^{i,0,k}(x-1)\right)\left(\sum_{n\geq 0} \frac{t^{kn+j}}{(kn+j)!} GEC_{kn+j,\Gamma,\mathscr{R}}^{0,j,k}(x-1)\right)}{1 - \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC_{kn,\Gamma,\mathscr{R}}^{0,0,k}(x-1)} + \sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} GSEC_{i+kn+j,\Gamma,\mathscr{R}}^{i,j,k}(x-1). \quad (3.15)$$

Proof. Without causing any ambiguity, as a subscript of many variables, Γ is sometimes omitted for convenience. Replace x by $x+1$ in (3.15). Then the left-hand side of (3.2) is the generating function of $m_\Gamma(F)$ over all $F \in \mathcal{MP}_{i+kn,+j,\Gamma,\mathscr{R}}^{i,j,k}$. That is, it easy to see that

$$\sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} \sum_{F \in \mathcal{P}_{i+kn+j,\mathscr{R}}^{i,j,k}} (x+1)^{\Gamma\text{-mch}(F)} = \sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} \sum_{F \in \mathcal{MP}_{i+kn+j,\Gamma,\mathscr{R}}^{i,j,k}} x^{m_\Gamma(F)}. \quad (3.16)$$

Thus we must show that

$$\sum_{n\geq 0} \frac{t^{kn+j}}{(kn+j)!} \sum_{F \in \mathcal{MP}_{kn,\mathscr{R}}^{0,0,k}} x^{m_\Gamma(F)} = \frac{\left(\sum_{n\geq 0} \frac{t^{i+kn}}{(i+kn)!} GSC_{i+kn,\Gamma,\mathscr{R}}^{i,0,k}(x)\right)\left(\sum_{n\geq 0} \frac{t^{kn+j}}{(kn+j)!} GEC_{kn+j,\Gamma,\mathscr{R}}^{0,j,k}(x)\right)}{1 - \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC_{kn,\Gamma,\mathscr{R}}^{0,0,k}(x)} + \sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} GSEC_{i+kn+j,\Gamma,\mathscr{R}}^{i,j,k}(x). \quad (3.17)$$

Now we rewrite the right-hand side of (3.17) as

$$
\begin{aligned}
\Bigg(\left(\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x)\right)&\left(\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x)\right)\\
&\times\sum_{m\geq 0}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^{m}\Bigg)\\
&\qquad+\sum_{n\geq 0}\frac{t^{i+kn+j}}{(i+kn+j)!}GSEC^{i,j,k}_{i+kn+j,\Gamma,\mathscr{R}}(x). \qquad (3.18)
\end{aligned}
$$

Taking the coefficient of $\frac{t^{ks+j}}{(ks+j)!}$ on both sides of (3.18) where $s\geq 0$, we see that we must show that

$$
\begin{aligned}
&\sum_{F\in\mathcal{MP}^{i,j,k}_{i+ks+j,\mathscr{R}}} x^{m_\Gamma(F)}\\
=\;&\Bigg(\Bigg(\left(\sum_{n\geq 0}\frac{t^{i+kn}}{(i+kn)!}GSC^{i,0,k}_{i+kn,\Gamma,\mathscr{R}}(x)\right)\left(\sum_{n\geq 0}\frac{t^{kn+j}}{(kn+j)!}GEC^{0,j,k}_{kn+j,\Gamma,\mathscr{R}}(x)\right)\\
&\times\sum_{m\geq 0}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,\Gamma,\mathscr{R}}(x)\right)^{m}\Bigg)\\
&+\sum_{n\geq 0}\frac{t^{i+kn+j}}{(i+kn+j)!}GSEC^{i,j,k}_{i+kn+j,\Gamma,\mathscr{R}}(x)\Bigg)\Bigg|_{\frac{t^{i+ks+j}}{(i+ks+j)!}}\\
=\;&\Bigg(\sum_{a+b+c=s,a,b,c\geq 0}\;\sum_{m=1}^{b}\;\sum_{\substack{b_1+b_1+\cdots+b_m=b\\ b_i\geq 1}}\binom{i+ks+j}{i+ka,kb_1,\ldots,kb_m,kc+j}\times\\
&\qquad GSC^{i,0,k}_{i+ka,\mathscr{R}}(x)\;GEC^{0,j,k}_{kc+j,\mathscr{R}}(x)\;\prod_{j=1}^{m}GC^{0,0,k}_{kb_j,\mathscr{R}}(x)\Bigg)+GSEC^{i,j,k}_{i+ks+j,\mathscr{R}}(x)
\end{aligned}
$$

(3.19)

Now we interpret the right-hand side of (3.19). Different from interpretation of (3.6) or (3.12), the plus sign in (3.19) offers us two ways to construct an array in $\mathcal{MP}^{i,j,k}_{i+ks+j,\mathscr{R}}$.

One way is via $\mathcal{GSC}$, $\mathcal{GEC}$ and $\mathcal{GC}$. First we pick non-negative integers a, b and c such that $a+b+c=s$. Then we pick an m such that $1\leq m\leq b$. Next we pick $b_1,\ldots,b_m\geq 1$ such that $b_1+b_2+\cdots+b_m=b$. Next the multinomial coefficient

$\binom{i+ks+j}{i+ka,kb_1,\ldots,kb_m,kc+j}$ allows us to pick sets $S_1,\ldots,S_m,S_{m+1},S_{m+2}$ which partition $\{1,2,\ldots,i+ks+j\}$ such that $|S_1| = i+ka$, $|S_{h+1}| = kb_h$ for $h = 1,\ldots,m$ and $|S_{m+2}| = kc+j$. The factor $GSC^{i,0,k}_{i+ka,\mathscr{R}}(x)$ allows us to pick a generalized start-cluster G_1 of size $i+ka$ with weight $\omega_{\Gamma,\mathscr{R}}(G_1)$. Note that in the cases where $a = 0$, our definitions imply that G_1 is just a column of height i filled with the numbers $1,\ldots,i$ which is increasing, reading from bottom to top. The factor $GEC^{0,j,k}_{kc+j,\mathscr{R}}(x)$ allows us to pick an a a generalized end-cluster G_{m+2} of size $kc+j$ with weight $\omega_{\Gamma,\mathscr{R}}(G_{m+2})$. Note that in the cases where $c = 0$, our definitions imply that G_{m+2} is just a column of height j filled with the numbers $1,\ldots,j$ which is increasing, reading from bottom to top. Finally, the product $\prod_{j=1}^{m} GC^{0,0,k}_{kb_h,\mathscr{R}}(x)$ allows us to pick generalized clusters $G_h \in \mathcal{GC}^{0,0,k}_{kb_h,\mathscr{R}}$ for $h = 2,\ldots,m+1$ with weight $\prod_{h=1}^{m} \omega_{\Gamma,\mathscr{R}}(G_h)$. Note that in the cases where $b_h = 1$, our definitions imply that G_h is just a column of height k filled with the numbers $1,\ldots,k$ which is increasing, reading from bottom to top.

Another way is via $\mathcal{GSEC}$. The term $GSEC^{i,j,k}_{i+ks+j,\mathscr{R}}(x)$ allows us to have a generalized start-end-cluster G_* of size $i+ks+j$. Note that in the case where $s = 0$, our definition implies that G_* has two blocks and the first block is a single column of height i and the second block is a single column of height j.

For example, suppose that $i = 3$, $k = 2$ and $j = 1$ and $\Gamma = \{P\}$ where $P = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$. Suppose that $\mathscr{R}$ is relation where for any two columns C and D which filled with integers and are strictly increasing in columns, $(C,D) \in \mathscr{R}$ if and only if the top element of C is greater than the bottom element of D. Then in Figure 3.6, we have pictured S_1,S_2,S_3,S_4,S_5,S_6 which partition $\{1,\ldots,42\}$ and corresponding a general start-cluster G_1, generalize clusters $G_2,\ldots,G_5$ and a general end-cluster G_6. This is a construction for the right-hand side of (3.12) via $\mathcal{GSC}$, $\mathcal{GEC}$ and $\mathcal{GC}$. For each h, we have indicated the separation between the blocks of G_i by dark black lines. Then for each $h = 1,\ldots,m+1$, we create a cluster E_i which results by replacing the j in G_i by the j^{th} element of S_i. If we concatenate $E_1 \ldots E_6$ together, then we will obtain an element of $Q \in \mathcal{MP}^{i,j,k}_{i+ks+j,\Gamma}$.

$S_1 = \{3, 31, 33, 34, 35, 36, 37, 38, 39\}$

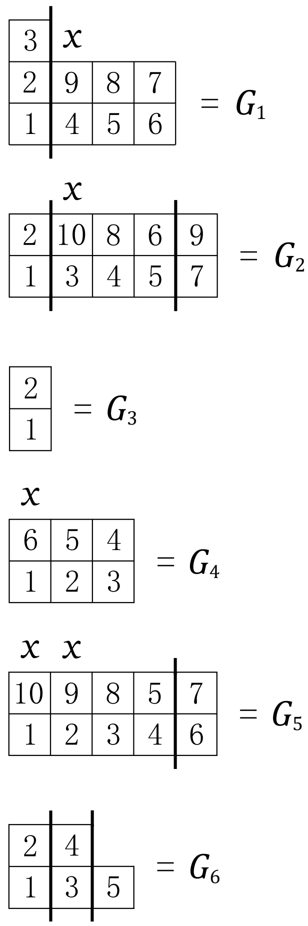
$S_2 = \{1, 5, 8, 10, 15, 18, 24, 27, 30, 32\}$

$S_3 = \{17, 19\}$

$S_4 = \{4, 7, 12, 13, 20, 21\}$

$S_5 = \{2, 6, 9, 11, 14, 16, 22, 23, 28, 29\}$

$S_6 = \{25, 26, 40, 41, 42\}$



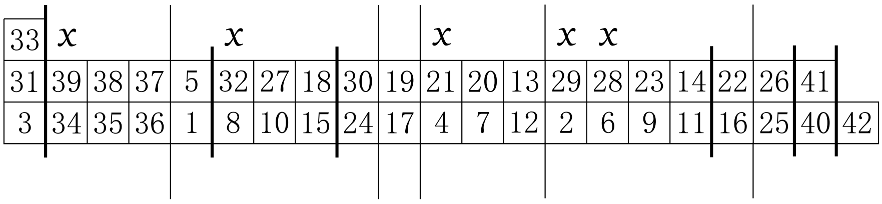


Figure 3.6: Construction for the right-hand side of (3.12) via $\mathcal{GSC}$, $\mathcal{GEC}$ and $\mathcal{GC}$.

$S_* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$

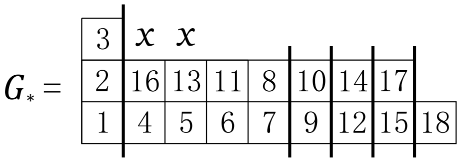


Figure 3.7: Construction for the right-hand side of (3.12) via $\mathcal{GSEC}$.

The weight of Q equals $\prod_{h=1}^{6} \omega_{\Gamma,\mathscr{R}}(G_h)$ where

$$\begin{aligned}
\omega_{\Gamma,\mathscr{R}}(G_1) &= (-1)^{2-1}x, \\
\omega_{\Gamma,\mathscr{R}}(G_2) &= (-1)^{3-1}x, \\
\omega_{\Gamma,\mathscr{R}}(G_3) &= (-1)^{1-1}, \\
\omega_{\Gamma,\mathscr{R}}(G_4) &= (-1)^{1-1}x, \\
\omega_{\Gamma,\mathscr{R}}(G_5) &= (-1)^{2-1}x^2, \text{ and} \\
\omega_{\Gamma,\mathscr{R}}(G_6) &= (-1)^{3-1}.
\end{aligned}$$

In Figure 3.6, we have indicated the boundaries between the G_h's by light lines.

In Figure 3.7, we have pictured $S_* = \{1, 2, \ldots, 18\}$ and corresponding a general start-end-cluster G_*. This is a construction for the right-hand side of (3.12) via $\mathcal{GSEC}$. Clearly, $G_* \in \mathcal{MP}^{i,j,k}_{i+ks+j,\Gamma}$. The weight of G_*

$$\omega_{\Gamma,\mathscr{R}}(G_*) = (-1)^{6-1}x^2.$$

We let $\mathcal{HGC}_{i+ks+j,\Gamma,\mathscr{R}}$ denote the set of all elements that can be constructed via $\mathcal{GSC}$, $\mathcal{GEC}$ and $\mathcal{GC}$, and let $\mathcal{HGSEC}_{i+ks+j,\Gamma,\mathscr{R}}$ denote the set of all elements that can be constructed via $\mathcal{GSEC}$. We let

$$\mathcal{H}_{i+ks+j,\Gamma,\mathscr{R}} := \mathcal{HGC}_{i+ks+j,\Gamma,\mathscr{R}} \ \cup \ \mathcal{HGSEC}_{i+ks+j,\Gamma,\mathscr{R}}.$$

For convenience, the subscripts Γ and $\mathscr{R}$ are omitted in following discussion.

Then $Q = E_1 \ldots E_{m+1}E_{m+2}$ is an element of $\mathcal{HGC}_{i+ks+j,\Gamma,\mathscr{R}}$ if and only if $m \geq 0$, $\text{red}(E_1)$ is a generalized start-cluster, for each $h = 1, \ldots, m$, $\text{red}(E_{h+1})$ is a generalized cluster and $\text{red}(E_{m+2})$ is a generalized end-cluster. On the other hand, $Q = E_1$ is an element of $\mathcal{HGSEC}_{i+ks+j}$ if and only if $\text{red}(E_1)$ is a generalized start-end-cluster.

Next we define a sign reversing involution $\theta : \mathcal{H}_{i+ks+j} \to \mathcal{HGEC}_{i+ks+j}$. We use $B_d^{(h)}$ to denote the d-th block in E_h.

If $Q \in \mathcal{HGC}_{i+ks+j}$, then we assume $Q = E_1 \ldots E_{m+1}E_{m+2}$, look for the first h such that either

Case (1). the block structure of $\mathrm{red}(E_h) = B_1^{(h)} B_2^{(h)} \dots B_{k_h}^{(h)}$ consists of more than one block or

Case (2). E_h consists of a single block $B_1^{(h)}$ and $(last(B_1^{(h)}), first(B_1^{(h+1)}))$ is not in $\mathscr{R}$.

If $Q \in \mathcal{HGSEC}_{i+ks+j}$, we assume $Q = E_1$ and then

Case (3). the block structure of $\mathrm{red}(E_1) = B_1^{(h)} B_2^{(h)} \dots B_{k_h}^{(h)}$ consists of more than one block.

In Case (1), if $h = 1$, we let $\theta(E_1 \dots E_{m+1} E_{m+2})$ be the result of replacing E_1 by a generalized cluster E_1^* and a generalized start-cluster E_1^{**} where E_1^* consists just of $B_1^{(1)}$ and E_1^{**} consists of $B_2^{(1)} \dots B_{k_1}^{(1)}$. If $2 \leq h \leq m+1$, we let $\theta(E_1 \dots E_{m+1} E_{m+2})$ be the result of replacing E_h by two generalized clusters E_h^* and E_h^{**} where E_i^* consists just of $B_1^{(h)}$ and E_i^{**} consists of $B_2^{(h)} \dots B_{k_h}^{(h)}$. If $h = m+2$, we let $\theta(E_1 \dots E_{m+1} E_{m+2})$ be the result of replacing E_{m+2} by a generalized cluster E_{m+1}^* and a generalized end-cluster E_{m+2}^{**} where E_{m+2}^* consists just of $B_1^{(m+2)}$ and E_{m+2}^{**} consists of $B_2^{(m+2)} \dots B_{k_{m+2}}^{(m+2)}$. Note that in any of these three situations above,

$$\omega_{\Gamma,\mathscr{R}}(E_h) = (-1)^{k_h - 1} \prod_{n=1}^{k_i} \omega_{\Gamma,\mathscr{R}}(B_n^{(h)})$$

while

$$\omega_{\Gamma,\mathscr{R}}(E_h^*) \omega_{\Gamma,\mathscr{R}}(E_h)^{**} = (-1)^{k_h - 2} \prod_{n=1}^{k_h} \omega_{\Gamma,\mathscr{R}}(B_n^{(h)}) = -\omega_{\Gamma,\mathscr{R}}(E_h).$$

In Case (1), it is obvious that $\theta(E_1 \dots E_{m+1} E_{m+2})$ is still an element in $\mathcal{HGC}_{i+ks+j}$.

In Case (2), if $h = 1$, we let $\theta(E_1 \dots E_{m+1} E_{m+2})$ be the result of replacing E_1 and E_2 by a generalized start-cluster $E = B_1^{(1)} B_1^{(2)} \dots B_{k_2}^{(2)}$. If $2 \leq h \leq m$, we let $\theta(E_1 \dots E_{m+1} E_{m+2})$ be the result of replacing E_h and E_{h+1} by a generalized cluster $E = B_1^{(h)} B_1^{(h+1)} \dots B_{k_{h+1}}^{(h+1)}$. If $h = m+1$, we let $\theta(E_1 \dots E_{m+1} E_{m+2})$ be the result of replacing E_{m+1} and E_{m+2} by a generalized end-cluster $E = B_1^{(m+1)} B_1^{(m+2)} \dots B_{k_{m+2}}^{(m+2)}$. In any of these three situations,

$$\omega_{\Gamma,\mathscr{R}}(E_h) \omega_{\Gamma,\mathscr{R}}(E_{h+1}) = (-1)^{k_{h+1} - 1} \omega_{\Gamma,\mathscr{R}}(B_1^{(h)}) \prod_{n=1}^{k_{h+1}} \omega_{\Gamma,\mathscr{R}}(B_n^{(h+1)})$$

while

$$\omega_{\Gamma,\mathscr{R}}(E) = (-1)^{k_{h+1}}\omega_{\Gamma,\mathscr{R}}(B_1^{(h)})\prod_{n=1}^{k_{h+1}}\omega_{\Gamma,\mathscr{R}}(B_n^{(h+1)}) = -\omega_{\Gamma,\mathscr{R}}(E_h)\omega_{\Gamma,\mathscr{R}}(E_{h+1}).$$

One particular situation in this case is that if $Q = E_1E_2$ and E_1 has only one block, then E_1 and E_2 will be combined, that is, $\theta(E_1E_2)$ is an element in $\mathcal{HSEGC}_{i+ks+j}$. Except for this situation, $\theta(E_1 \ldots E_{m+1}E_{m+2})$ is still an element in $\mathcal{HGC}_{i+ks+j}$.

In Case (3), note that for any $Q \in \mathcal{HSEGC}_{i+ks+j}$, Q itself is a generalized start-end cluster which must have at least two blocks. Suppose $Q = E_1$, then $B_1^{(1)}$ is a single increasing column of height i and $B_{k_1}^{(1)}$ is a single increasing column of height j. Then we let $\theta(E_1)$ be the result of replacing E_1 by a generalized start-cluster E_1^* and a generalized end-cluster E_1^{**} where E_1^* consists just of $B_1^{(1)}$ and E_1^{**} consists of $B_2^{(1)} \ldots B_{k_1}^{(1)}$. Clearly,

$$\omega_{\Gamma,\mathscr{R}}(E_1) = (-1)^{k_1-1}\prod_{n=1}^{k_1}\omega_{\Gamma,\mathscr{R}}(B_n^{(1)})$$

while

$$\omega_{\Gamma,\mathscr{R}}(E_1^*)\omega_{\Gamma,\mathscr{R}}(E_1)^{**} = (-1)^{k_1-2}\prod_{n=1}^{k_1}\omega_{\Gamma,\mathscr{R}}(B_n^{(1)}) = -\omega_{\Gamma,\mathscr{R}}(E_1).$$

Since $\theta(E_1)$ has a generalized start-cluster and a generalized end-cluster, $\theta(E_1)$ is an element in $\mathcal{HGC}_{i+ks+j}$.

If neither Case (1), Case (2), or Case (3) applies, then we let $\theta(E_1 \ldots E_{m+2}) = E_1 \ldots E_{m+2}$.

According to the discussion above, now it is easy to see that $\theta : \mathcal{H}_{i+ks+j} \mapsto \mathcal{H}_{i+ks+j}$ is a sign reversing involution. Then based on similar argument we used in proof of Theorem 3.2 and 3.4, if $Q = E_1 \ldots E_{m+2}$ and $\theta(Q) = Q$, Q must be an element in $\mathcal{MP}_{i+ks+j,\Gamma,\mathscr{R}}^{i,j,k}$, which finishes the proof. □

We can state analogues of Theorems 3.2, 3.4, and 3.6 in the case where are study P-matches and exact P-matches where we are studying fillings of $D_{i+kn+j}^{i,j,k}$ where we allow repeated entries. These analogues follow the same ideas that we used to state an analogue of Theorem 2.2 in Theorem 2.8 and the analogue of Theorem 2.7 in Theorem 2.10. Thus we shall not give the details here.

3.4 Example

3.4.1 Up-down patterns in down-up permutations

We say a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$ is an up-down permutation if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \cdots$. More precisely, $\sigma \in \mathcal{S}_n$ is up-down if and only if

$$\mathrm{Des}(\sigma) = \left\{2k : \forall \text{ integer } k, 0 < k < \frac{n}{2}\right\}.$$

We let $\mathcal{UD}_n$ denote the set of all the up-down permutations in $\mathcal{S}_n$. We say a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$ is an down-up permutation if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \cdots$. Similarly, a permutation $\sigma \in \mathcal{S}_n$ is down-up if and only if

$$\mathrm{Des}(\sigma) = \left\{2k+1 : \forall \text{ integer } k, 0 < k < \frac{n+1}{2}\right\}.$$

We let $\mathcal{DU}_n$ denote the set of all the up-down permutations in $\mathcal{S}_n$.

It is easy to see

$$|\mathcal{UD}_n| = |\mathcal{DU}_n|$$

because for any $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{UD}_n$, $(n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n) \in \mathcal{DU}_n$. For example, $\sigma = 2\ 6\ 1\ 4\ 3\ 5 \in \mathcal{UD}_n$, then we see $5\ 1\ 6\ 3\ 4\ 2 \in \mathcal{DU}_n$. The number of up-down permutations is counted by Euler numbers and André [2, 3] proved that

$$1 + \sum_{n\geq 1} |\mathcal{UD}_n| \frac{t^n}{n!} = \tan t + \sec t.$$

Classical permutation patterns (i.e., non-consecutive patterns) in up-down permutations have been studied in several papers. For example, [41] and [37] showed that the number of up-down permutations that don't contain τ as a classical pattern is a Catalan number, for any $\tau \in \mathcal{S}_3$. However, study on consecutive patterns for up-down (down-up) permutations is still a relatively new topic. Remmel studied consecutive up-down patterns of length 4, namely, 1324, 2314, 2413, 1432 and 3412, in up-down permutations in [55]. In this subsection, we mainly use Theorem 3.4 and Theorem 3.6 to consider up-down patterns in down-up permutations.

Alternatively, down-up permutations can be represented by arrays. Consider a binary relation $\mathscr{R}$ such that holds for a pair of columns (C, D) if and only if the top element of column C is greater than the bottom element of column D, then $\mathcal{P}^{1,1,2}_{2n+2,\mathscr{R}}$ is one-to-one corresponding to $\mathcal{DU}_{2n+2}$ and $\mathcal{P}^{1,0,2}_{2n+2,\mathscr{R}}$ is one-to-one corresponding to $\mathcal{DU}_{2n+1}$. For any $F \in \mathcal{P}^{i,j,k}_{i+kn+j}$, let $w(F)$ be the sequence obtained by reading the columns from bottom to top and then from left to right. Then it is easy to see that if F is a filling in $\mathcal{P}^{1,0,2}_{2n+1,\mathscr{R}}$ ($\mathcal{P}^{1,1,2}_{2n+2,\mathscr{R}}$), then $w(F)$ is an element in $\mathcal{DU}_{2n+1}$ ($\mathcal{DU}_{2n+2}$). For example, an $F \in \mathcal{P}^{1,1,2}_{10,\mathscr{R}}$ and $w(F)$ is given in Figure 3.8.

$$F \qquad \begin{array}{|c|c|c|c|c|c|} \cline{2-5} \multicolumn{1}{c|}{} & 8 & 10 & 5 & 9 & \multicolumn{1}{c}{} \\ \hline 4 & 2 & 7 & 1 & 3 & 6 \\ \hline \end{array}$$

$$w(F) = 4\ \ 2\ \ 8\ \ 7\ \ 10\ \ 1\ \ 5\ \ 3\ \ 9\ \ 6$$

Figure 3.8: $F \in \mathcal{P}^{1,1,2}_{10,\mathscr{R}}$ and $w(F) \in \mathcal{DU}_{10}$.

In this example, we shall consider an up-down pattern $\tau = 1\ 6\ 2\ 5\ 3\ 4 \in \mathcal{UD}_6$ and consider the following generating function

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{DU}_n} x^{\tau\text{-mch}(\sigma)}.$$

It is equivalent to consider the following two generating functions

$$A_P(x,t) := \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \sum_{F \in \mathcal{P}^{1,0,2}_{2n+1,\mathscr{R}}} x^{P\text{-mch}(F)}, \tag{3.20}$$

$$B_P(x,t) := \sum_{n\geq 0} \frac{t^{2n+2}}{(2n+2)!} \sum_{F \in \mathcal{P}^{1,1,2}_{2n+2,\mathscr{R}}} x^{P\text{-mch}(F)}, \tag{3.21}$$

where $P = \begin{array}{|c|c|c|} \hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \in \mathcal{P}^{0,0,2}_{0,0,2j}$. (3.20) and (3.21) are equivalent to generating functions for the distribution of τ in down-up permutations of even length and odd length respectively.

Then by Theorem 3.4,

$$A_P(x,t) = \frac{\sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} GSC^{1,0,2}_{2n+1,P,\mathscr{R}}(x-1)}{1 - \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} GC^{0,0,2}_{2n,P,\mathscr{R}}(x-1)}, \tag{3.22}$$

and by Theorem 3.6,

$$B_P(x,t) = \frac{\left(\sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} GSC^{1,0,2}_{2n+1,P,\mathscr{R}}(x-1)\right)\left(\sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} GEC^{0,1,2}_{2n+1,P,\mathscr{R}}(x-1)\right)}{1-\sum_{n\geq 1} \frac{t^{2n}}{(2n)!} GC^{0,0,2}_{2n,P,\mathscr{R}}(x-1)} + \sum_{n\geq 0} \frac{t^{2n+2}}{(2n+2)!} GSEC^{1,1,2}_{2+2,P,\mathscr{R}}(x-1). \quad (3.23)$$

Observing that both $A_P(x,t)$ and $B_P(x,t)$ need generalized start-cluster polynomials and generalized cluster polynomials, firstly we compute $GC^{0,0,2}_{2n,P,\mathscr{R}}(x)$.

We shall start by discussing structures of P-clusters. We let $\mathcal{C}_{2n,P}$ denote the set of P-clusters consisting of n columns. Clearly n could be any integer greater than 2. As a usual technique, we think of $\mathcal{C}_{2n,P}$ as posets, represented by Hasse diagrams. For example, the poset corresponding to P is pictured in the left-hand side of Figure 3.9. To obtain the poset corresponding to $\mathcal{C}_{8,P}$, we superimpose the diagram at the second column of itself, which is pictured in right-hand side of Figure 3.9. It is clear that the two P-matches in $\mathcal{C}_{8,P}$ have to be marked. It follows that if

$$C_{2n}(x) = \sum_{Q\in\mathcal{C}_{2n,P}} x^{m_P(Q)},$$

then $C_6(x) = x$ and $C_8(x) = x^2$.

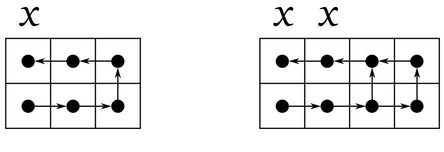


Figure 3.9: Poset for pattern P and $\mathcal{C}_{8,P}$.

In general, in a P-cluster C with n-columns, it easy to see that the cells in the bottom row must be filled with $1,\ldots,n$, reading from left to right, and the numbers in the top row must be filled with the numbers $n+1,\ldots,2n$, reading from right to left. That is, because any two consecutive marked P-matches in a P-cluster must share at least one column, it follows that the elements in the first row must be increasing, reading from left to right, and the numbers in the second row must be increasing, reading from right to left. Since every column must be in a marked P-matches, it follows that column $n-2$ must start a marked P-match so that column n is part of marked P-match. This means that the elements in the last column must be increasing, reading from bottom to top. Thus the underlying Hasse diagram must be of the form pictured in Figure 3.10.

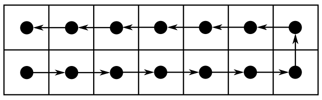


Figure 3.10: The Hasse diagram of a P-cluster.

Now if C is a P-cluster with $n \geq 5$ columns, then there are two possibilities that we have to consider. Because of the form of the Hasse diagram, there are P-matches which start at columns $1, \ldots, n-2$. We know that the P-match starting at column $n-2$ must be marked in C because the last column must be part of a marked P-match. However it could be that (a) column $n-3$ is start of marked P-match in C or (b) column $n-3$ is not the start of marked P-match in C in which case column $n-4$ must be the start of a marked P-match in C. This situation is pictured in Figure 3.11.where case (a) is pictured on the top left and case (b) is pictured on the top right. In case (a), we can remove the last column and the x on top of column $n-2$ to obtain a cluster with $n-1$ columns and in case (b) we can remove the last two columns and the x on top of column $n-2$ to obtain a P-cluster with $n-2$ columns. It follows that for $n \geq 5$,

$$C_{2n}(x) = \sum_{Q \in \mathcal{C}_{2n,P}} x^{m_P(Q)} = x\left(C_{2n-4}(x) + C_{2n-2}(x)\right).$$

However, there are multiple ways to mark P-matches in a cluster. Consider the cluster polynomial $C_{2n}(x)$,

$$C_{2n}(x) = \sum_{Q \in \mathcal{C}_{2n,P}} x^{m_P(Q)}.$$

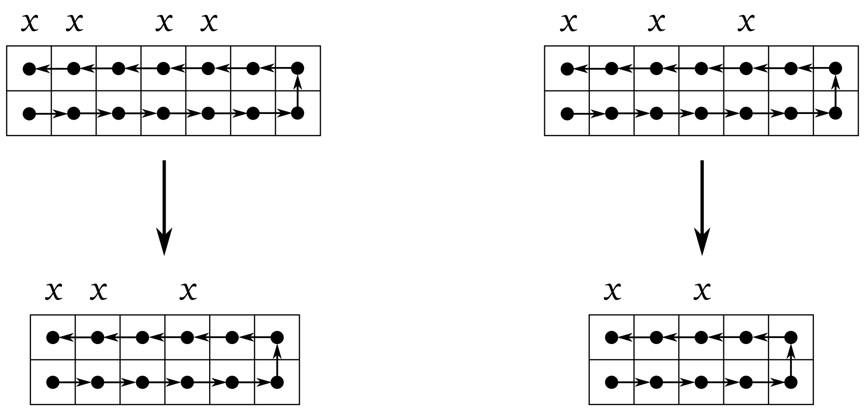


Figure 3.11: The recursions for P-clusters.

Thus we have proved that

$$C_6(x) = x \tag{3.24}$$

$$C_8(x) = x^2 \tag{3.25}$$

$$C_{2n}(x) = x\left(C_{2n-4}(x) + C_{2n-2}(x)\right), \quad \text{for } n \geq 5. \tag{3.26}$$

To compute $GC^{0,0,2}_{2n,P,\mathscr{R}}(x)$, we need to figure structures of $\mathcal{GC}^{0,0,2}_{2n,P,\mathscr{R}}$. Suppose $Q \in \mathcal{GC}^{0,0,2}_{2n,P,\mathscr{R}}$ has m blocks, i.e., $Q = B_1B_2\ldots B_m$. For an array F, we use $\mathrm{Col}(F)$ to denote the number of columns in $\mathrm{Col}(F)$. Each B_i is either order isomorphic to a P-cluster or a single column. We let $\mathcal{GC}_{\mathrm{Col}=(b_1,b_2,\ldots,b_m)}$ to denote the set of generalized clusters such that $\mathrm{Col}(B_i) = b_i$. Given $\mathrm{Col}(B_i)$ for each $1 \leq i \leq m$, we can represent the filling for the set of such generalized clusters uniquely by Hasse diagram. Since we are assuming that for any block $(last(B_i), first(B_{i+1})$ is not in $\mathscr{R}$, there must be an arrow directed from the top of the last column of B_i to the bottom of the first column of B_{i+1}. We let $\Gamma(b_1, b_2, \ldots, b_m)$ denote the Hasse diagram corresponding to $\mathcal{GC}_{\mathrm{Col}=(b_1,b_2,\ldots,b_m)}$. For example, $\Gamma(3, 1, 1, 5, 1)$ is pictured in Figure 3.12.

It is clear that

$$\sum_{Q \in \mathcal{GC}_{\mathrm{Col}=(b_1,b_2,\ldots,b_m)}} \omega_{P,\mathscr{R}}(Q) = (-1)^{m-1}\mathrm{LE}(\Gamma(b_1, b_2, \ldots, b_m)) \prod_{i=1}^{m} C_{2b_i}(x), \tag{3.27}$$

where $\mathrm{LE}(\Gamma(b_1, b_2, \ldots, b_m))$ is the number of linear extensions of $\Gamma(b_1, b_2, \ldots, b_m)$ and by convention, we let $C_2(x) = 1$. Therefore, to compute (3.27), we only

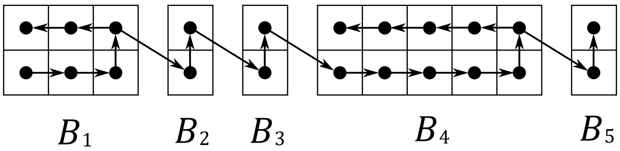


Figure 3.12: $\Gamma(3,1,1,5,1)$.

need to count linear extensions of $\Gamma(b_1, b_2, \ldots, b_m)$. Continue using $\Gamma(3,1,1,5,1)$ as example, the Hasse diagram in Figure 3.12 is actually a tree-like diagram, as drawn in Figure 3.13, and then we can easily compute that

$$\mathrm{LE}(\Gamma(3,1,1,5,1)) = \binom{6}{4}\binom{18}{2}.$$

That is, it is easy to see that the first four elements in the bottom row must be labeled with $1, \ldots, 4$, reading from left to right, since there is a directed path from these elements to any other elements in the poset. Once we remove these four elements, the Hasse diagram becomes disconnected so that $\binom{18}{2}$ ways to pick the labels for the two elements above the element labeled with 4 on only one way to label those two elements. Once we have picked those two elements a and b, the next 10 elements must be the smallest elements of $\{1, \ldots, 22\} - \{1, 2, 3, 4, , a, b\}$. Once we remove those 10 elements, the Hasse diagram again becomes disconnected so that there are $\binom{6}{4}$ ways to pick the lables of the vertical segment and only one way to order them.

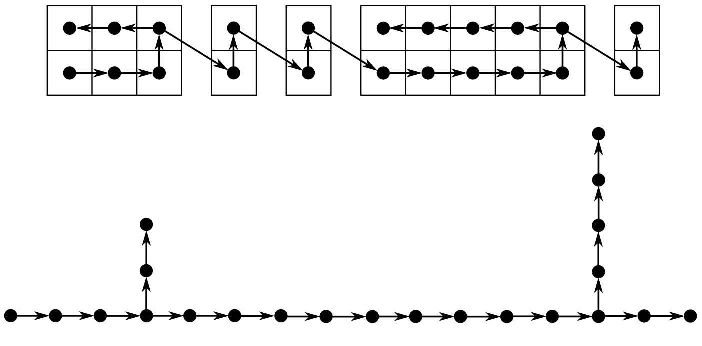


Figure 3.13: $\Gamma(3,1,1,5,1)$.

In general, we see that

$$\mathrm{LE}(\Gamma(b_1, b_2, \ldots, b_m)) = \prod_{i=1}^{m-1} \binom{b_{m-i} - 1 + 2 \sum_{j=m-i+1}^{n} b_i}{b_{m-i} - 1}$$

and then hence

$$\sum_{Q \in \mathcal{GC}_{\mathrm{Col}=(b_1, b_2, \ldots, b_m)}} \omega_{P,\mathscr{R}}(Q) =$$

$$(-1)^{m-1} C_{2b_m}(x) \prod_{i=1}^{m-1} \binom{b_{m-i} - 1 + 2 \sum_{j=m-i+1}^{n} b_i}{b_{m-i} - 1} C_{2b_i}(x). \quad (3.28)$$

Since a generalized P-cluster can not have a block of 2 columns,

$$GC_{2n,P,\mathscr{R}}^{0,0,2}(x) = \sum_{m=1}^{n} \sum_{\substack{b_1 + \cdots + b_m = n \\ b_i = 1 \text{ or } b_i \geq 3}} \sum_{Q \in \mathcal{GC}_{\mathrm{Col}=(b_1, \ldots, b_m)}} \omega_{P,\mathscr{R}}(Q) =$$

$$\sum_{m=1}^{n} \sum_{\substack{b_1 + \cdots + b_m = n \\ b_i = 1 \text{ or } b_i \geq 3}} (-1)^{m-1} C_{2b_m}(x) \prod_{i=1}^{m-1} \binom{b_{m-i} - 1 + 2 \sum_{j=m-i+1}^{m} b_i}{b_{m-i} - 1} C_{2b_i}(x).$$

Using computer programs, it is easy to obtain that

$$\begin{aligned}
GC_{2,P,\mathscr{R}}^{0,0,2}(x) &= 1 \\
GC_{4,P,\mathscr{R}}^{0,0,2}(x) &= -1 \\
GC_{6,P,\mathscr{R}}^{0,0,2}(x) &= 1 + x \\
GC_{8,P,\mathscr{R}}^{0,0,2}(x) &= -1 - 7x + x^2 \\
GC_{10,P,\mathscr{R}}^{0,0,2}(x) &= 1 + 22x - 10x^2 + x^3 \\
GC_{12,P,\mathscr{R}}^{0,0,2}(x) &= -1 - 50x + 2x^2 - 14x^3 + x^4 \\
GC_{14,P,\mathscr{R}}^{0,0,2}(x) &= 1 + 95x + 299x^2 - 86x^3 - 19x^4 + x^5 \\
GC_{16,P,\mathscr{R}}^{0,0,2}(x) &= -1 - 161x - 1796x^2 + 1705x^3 - 377x^4 - 25x^5 + x^6 \\
&\cdots \qquad (3.29)
\end{aligned}$$

Next we compute $GSC_{2n+1,P,\mathscr{R}}^{1,0,2}(x)$. Suppose $Q \in \mathcal{GSC}_{2n+1,P,\mathscr{R}}^{1,0,2}$ has m blocks, i.e., $Q = B_1 B_2 \ldots B_m$. B_1 has to be a block consisting of one element, and for

$2 \leq i \leq m$, each B_i is either order isomorphic to a P-cluster or a single column. We let $\mathcal{GSC}_{\mathrm{Col}=(1,b_2,\ldots,b_m)}$ to denote the set of generalized start-clusters such that $\mathrm{Col}(B_i) = b_i$. Given $\mathrm{Col}(B_i)$ for each $2 \leq i \leq m$, we can represent the filling for the set of such generalized start-clusters uniquely by Hasse diagram. Arrows between blocks should agree with $\mathscr{R}$. We let $\Gamma_S(1, b_2, \ldots, b_m)$ denote the Hasse diagram corresponding to $\mathcal{GSC}_{\mathrm{Col}=(1,b_2,\ldots,b_m)}$. For example, $\Gamma_S(1, 3, 1, 1, 5, 1)$ is pictured in Figure 3.14.

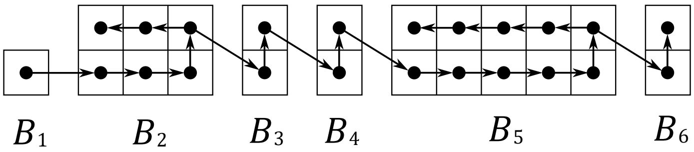


Figure 3.14: $\Gamma_S(1, 3, 1, 1, 5, 1)$.

It is clear that

$$\sum_{Q \in \mathcal{GSC}_{\mathrm{Col}=(1,b_2,\ldots,b_m)}} \omega_{P,\mathscr{R}}(Q) = (-1)^{m-1} \mathrm{LE}(\Gamma_S(1, b_2, \ldots, b_m)) \prod_{i=1}^{m} C_{2b_i}(x). \qquad (3.30)$$

By comparing Figure 3.12 and 3.14, we see that

$$\mathrm{LE}(\Gamma_S(1, b_2, \ldots, b_m)) = \mathrm{LE}(\Gamma(b_2, \ldots, b_m)).$$

Therefore, for $n \geq 1$,

$$\begin{aligned} GSC^{1,0,2}_{2n+1,P,\mathscr{R}}(x) &= \sum_{m=1}^{n+1} \sum_{\substack{1+b_2+\cdots+b_m=n+1 \\ b_i=1 \text{ or } b_i \geq 3}} \sum_{Q \in \mathcal{GSC}_{\mathrm{Col}=(1,b_2,\ldots,b_m)}} \omega_{P,\mathscr{R}}(Q) \\ &= -GC^{0,0,2}_{2n,P,\mathscr{R}}(x), \end{aligned}$$

and $GSC^{1,0,2}_{1,P,\mathscr{R}}(x) = 1$. Then by (3.29), we have

$$\begin{aligned}
GSC^{0,0,2}_{1,P,\mathscr{R}}(x) &= 1\\
GSC^{0,0,2}_{3,P,\mathscr{R}}(x) &= -1\\
GSC^{0,0,2}_{5,P,\mathscr{R}}(x) &= 1\\
GSC^{0,0,2}_{7,P,\mathscr{R}}(x) &= -1-x\\
GSC^{0,0,2}_{9,P,\mathscr{R}}(x) &= 1+7x-x^2\\
GSC^{0,0,2}_{11,P,\mathscr{R}}(x) &= -1-22x+10x^2-x^3\\
GSC^{0,0,2}_{13,P,\mathscr{R}}(x) &= 1+50x-2x^2+14x^3-x^4\\
GSC^{0,0,2}_{15,P,\mathscr{R}}(x) &= -1-95x-299x^2+86x^3+19x^4-x^5\\
GSC^{0,0,2}_{17,P,\mathscr{R}}(x) &= 1+161x+1796x^2-1705x^3+377x^4+25x^5-x^6\\
&\cdots
\end{aligned} \tag{3.31}$$

Then replacing $GC^{0,0,2}_{2n,P,\mathscr{R}}(x-1)$ and $GSC^{1,0,2}_{2n+1,P,\mathscr{R}}(x-1)$ in (3.20) by expressions in (3.29) and (3.31), we have

$$\begin{aligned}
A_P(x,t) &= t+\frac{2}{3!}t^3+\frac{16}{5!}t^5+\frac{266+6x}{7!}t^7+\frac{7623+303x+9x^2}{9!}t^9\\
&+\frac{333475+19557x+695x^2+10x^3}{11!}t^{11}+\cdots.
\end{aligned}$$

Next we compute $GEC^{0,1,2}_{2n+1,P,\mathscr{R}}(x)$. Suppose $Q \in \mathcal{GEC}^{0,1,2}_{2n+1,P,\mathscr{R}}$ has m blocks, i.e., $Q = B_1B_2\dots B_m$. B_m has to be a block consisting of one element, and for $1 \le i \le m-1$, each B_i is either order isomorphic to a P-cluster or a single column. We let $\mathcal{GEC}_{\mathrm{Col}=(b_1,\dots,b_{m-1},1)}$ to denote the set of generalized end-clusters such that $\mathrm{Col}(B_i) = b_i$. Given $\mathrm{Col}(B_i)$ for each $1 \le i \le m-1$, we can represent the filling for the set of such generalized end-clusters uniquely by a Hasse diagram. Since we are assuming that for any block $(last(B_i), first(B_{i+1}))$ is not in $\mathscr{R}$, there must be an arrow directed from the top of the last column of B_i to the bottom of the first column of B_{i+1}. We let $\Gamma_E(b_1,\dots,b_{m-1},1)$ denote the Hasse diagram corresponding to $\mathcal{GEC}_{\mathrm{Col}=(b_1,\dots,b_{m-1},1)}$. For example, $\Gamma_E(3,1,1,5,1,1)$ is pictured in Figure 3.15.

Similarly, we have

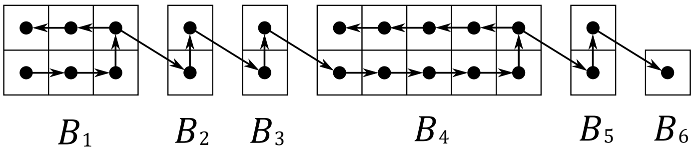


Figure 3.15: $\Gamma_E(3,1,1,5,1,1)$.

$$\mathrm{LE}(\Gamma_E(b_1,b_2,\ldots,1)) = \prod_{i=1}^{m-1} \binom{b_{m-i} + 2\sum_{j=m-i+1}^{m-1} b_i}{b_{m-i}-1}.$$

Then

$$GEC^{0,1,2}_{2n+1,P,\mathscr{R}}(x) = \sum_{m=1}^{n} \sum_{\substack{2b_1+\cdots+2b_{m-1}+1=2n+1 \\ b_i=1 \text{ or } b_i\geq 3}} \sum_{Q\in\mathcal{GEC}_{\mathrm{Col}=(b_1,\ldots,b_{m-1},1)}} \omega_{P,\mathscr{R}}(Q) =$$

$$\sum_{m=1}^{n} \sum_{\substack{2b_1+\cdots+2b_{m-1}+1=2n+1 \\ b_i=1 \text{ or } b_i\geq 3}} (-1)^{m-1} C_{2b_m}(x) \prod_{i=1}^{m-1} \binom{b_{m-i} + 2\sum\limits_{j=m-i+1}^{m-1} b_i}{b_{m-i}-1} C_{2b_i}(x).$$

Using computer programs, we have

$$\begin{aligned}
GEC^{0,1,2}_{1,P,\mathscr{R}}(x) &= 1\\
GEC^{0,1,2}_{3,P,\mathscr{R}}(x) &= -1\\
GEC^{0,1,2}_{5,P,\mathscr{R}}(x) &= 1\\
GEC^{0,1,2}_{7,P,\mathscr{R}}(x) &= -1-3x\\
GEC^{0,1,2}_{9,P,\mathscr{R}}(x) &= 1+13x-4x^2\\
GEC^{0,1,2}_{11,P,\mathscr{R}}(x) &= -1-34x+19x^2-5x^3\\
GEC^{0,1,2}_{13,P,\mathscr{R}}(x) &= 1+70x68x^2+28x^3-6x^4\\
GEC^{0,1,2}_{15,P,\mathscr{R}}(x) &= -1-125x-789x^2+531x^3+41x^4-7x^5\\
GEC^{0,1,2}_{17,P,\mathscr{R}}(x) &= 1+203x+3551x^2-3973x^3+2195x^4+59x^5-8x^6\\
&\cdots
\end{aligned} \tag{3.32}$$

Finally we shall compute $GSEC^{1,1,2}_{2n+2,P,\mathscr{R}}(x)$. Suppose $Q \in \mathcal{GSEC}^{1,1,2}_{2n+2,P,\mathscr{R}}$ has m blocks, i.e., $Q = B_1B_2\ldots B_m$. Because of the definition of generalized start-end clusters, $m \geq 2$. B_1 as well as B_m has to be a block consisting of one element, and for $2 \leq i \leq m-1$, each B_i is either order isomorphic to a P-cluster or a single column. We let $\mathcal{GEC}_{\text{Col}=(1,b_2,\ldots,b_{m-1},1)}$ to denote the set of generalized start-end-clusters such that $\text{Col}(B_i) = b_i$. Given $\text{Col}(B_i)$ for each $2 \leq i \leq m-1$, we can represent the filling for the set of such generalized start-end-clusters uniquely by a Hasse diagram. Arrows between blocks should agree with $\mathscr{R}$. We let $\Gamma_{SE}(1, b_2, \ldots, b_{m-1}, 1)$ denote the Hasse diagram corresponding to $\mathcal{GSEC}_{\text{Col}=(1,b_2,\ldots,b_{m-1},1)}$. For example, $\Gamma_{SE}(1, 3, 1, 1, 5, 1, 1)$ is pictured in Figure 3.16.

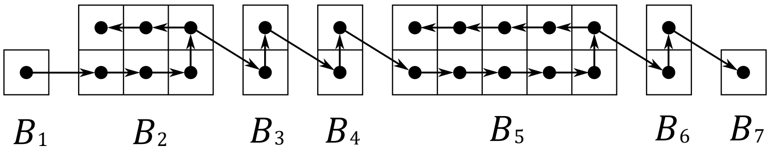


Figure 3.16: $\Gamma_{SE}(1, 3, 1, 1, 5, 1, 1)$.

Comparing Figure 3.15 and 3.16, it is easy to see that

$$\text{LE}(\Gamma_{SE}(1, b_2, \ldots, b_{m-1}, 1)) = \text{LE}(\Gamma_E(b_2, \ldots, b_{m-1}, 1)),$$

and then hence

$$GSEC^{1,1,2}_{2n+2,P,\mathscr{R}}(x) = -GEC^{0,1,2}_{2n+1,P,\mathscr{R}}(x).$$

Based on (3.32),

$$\begin{aligned}
GSEC^{1,1,2}_{2,P,\mathscr{R}}(x) &= -1 \\
GSEC^{1,1,2}_{4,P,\mathscr{R}}(x) &= 1 \\
GSEC^{1,1,2}_{6,P,\mathscr{R}}(x) &= -1 \\
GSEC^{1,1,2}_{8,P,\mathscr{R}}(x) &= 1 + 3x \\
GSEC^{1,1,2}_{10,P,\mathscr{R}}(x) &= -1 - 13x + 4x^2 \\
GSEC^{1,1,2}_{12,P,\mathscr{R}}(x) &= 1 + 34x - 19x^2 + 5x^3 \\
GSEC^{1,1,2}_{14,P,\mathscr{R}}(x) &= -1 - 70x - 68x^2 - 28x^3 + 6x^4 \\
GSEC^{1,1,2}_{16,P,\mathscr{R}}(x) &= 1 + 125x + 789x^2 - 531x^3 - 41x^4 + 7x^5 \\
GSEC^{1,1,2}_{18,P,\mathscr{R}}(x) &= -1 - 203x - 3551x^2 + 3973x^3 - 2195x^4 - 59x^5 + 8x^6 \\
&\cdots
\end{aligned} \tag{3.33}$$

Then substituting $GC^{0,0,2}_{2n,P,\mathscr{R}}(x-1)$, $GSC^{1,0,2}_{2n+1,P,\mathscr{R}}(x-1)$, $GEC^{0,1,2}_{2n+1,P,\mathscr{R}}(x-1)$ and $GSEC^{1,1,2}_{2n+2,P,\mathscr{R}}(x-1)$ in (3.20) by expressions obtained in (3.29), (3.31), (3.32) and (3.33), we have

$$\begin{aligned} B_P(x,t) &= \frac{1}{2!}t^2 + \frac{5}{4!}t^4 + \frac{61}{6!}t^6 + \frac{1358+27x}{8!}t^8 + \frac{48806+1611x+86x^23}{10!}t^{10} \\ &\quad + \frac{2561283+133803x+6734x^2+65x^3}{12!}t^{12} + \cdots . \end{aligned}$$

Adding $A_P(x,t)$ and $B_P(x,t)$, we can get distributions of pattern 123654 in down-up permutations,

$$\begin{aligned} & t + \frac{1}{2!}t^2 + \frac{2}{3!}t^3 + \frac{5}{4!}t^4 + \frac{16}{5!}t^5 + \frac{61}{6!}t^6 + \frac{266+6x}{7!}t^7 + \frac{1358+27x}{8!}t^8 \\ & + \frac{7623+303x+9x^2}{9!}t^9 + \frac{48806+1611x+86x^2}{10!}t^{10} + \cdots . \end{aligned}$$

The contents of Chapter 3 are currently under preparation for submission. Some portion is co-authored with J. B. Remmel. The dissertation author is the author of this material.

Chapter 4

Joint Clusters and Generalized Joint Clusters

In this chapter, we extend the notions P-clusters and generalized P-clusters for a single pattern P to joint cluster and generalized joint clusters for a sequence of patterns $P_1, \ldots, P_m$. Our goal here is different from finding generating functions for the number of Γ-matches where $\Gamma = \{P_1, \ldots, P_m\}$. We are interested in computing generating functions where the variable x_i keeps track of the number of P_i-matches for each i. For example, if we have patterns $P_1, P_2, \cdots, P_m$, then the multi-variate generating function for these patterns in $\mathcal{P}_{kn}^{0,0,k}$ is

$$1 + \sum_{n \geq 1} \frac{t^{kn}}{(kn)!} \sum_{F \in \mathcal{P}_{kn}^{0,0,k}} \prod_{i=1}^{m} x_i^{P_i\text{-mch}(F)}.$$

Similarly, if we are given some binary relation $\mathscr{R}$ on a pairs of columns, then the multi-variate generating function for these patterns in $\mathcal{P}_{kn,\mathscr{R}}^{0,0,k}$ is

$$1 + \sum_{n \geq 1} \frac{t^{kn}}{(kn)!} \sum_{F \in \mathcal{P}_{kn,\mathscr{R}}^{0,0,k}} \prod_{i=1}^{m} x_i^{P_i\text{-mch}(F)}.$$

In Section 4.1, we state and prove theorems for joint clusters and generalized joint clusters. In Section 4.2, several examples are given with details to show how joint and generalized joint clusters work. In Section 4.3, we discuss joint and generalized joint clusters for fillings of $D_{i+kn+j}^{i,j,k}$.

4.1 Main theorems

For convenience, we use pattern matching in $\mathcal{P}_{nk}^{0,0,k}$ and $\mathcal{P}_{nk,\mathscr{R}}^{0,0,k}$ to state and prove theorems. But obviously, the idea of joint and generalized joint clusters still hold for $\mathcal{WI}_{kn}^{0,0,k}$, $\mathcal{SI}_{kn}^{0,0,k}$ and other fillings.

We start by adapting the notion of marked matches for multiple patterns. Previously, we let $\mathcal{MS}_{n,\tau}$ denote the set of permutations in $\mathcal{S}_n$ where we have marked some of the τ-matches in the permutation by placing an 'x' at the start of τ-match. Similarly, we let $\mathcal{MS}_{n,(u_1,u_2,\cdots,u_m)}$ denote the set of permutations in $\mathcal{S}_n$ where we have marked some of the u_m-matches in the permutation by placing an 'x_m' at the start of u_m-match. For a permutation σ, the number of distinct $(u_1, u_2, \cdot, u_m)$-marked σ is $\prod_{i=1}^{m} 2^{u_i\text{-mch}(\sigma)}$. For example, suppose $u_1 = 1\ 2\ 3$ and $u_2 = 3\ 2\ 1$, for $\sigma = 6\ 1\ 3\ 7\ 2\ 8\ 5\ 4\ 9$, there is a u_1-match at position 2 and a u_2-match at position 6, there for there are total 4 marked permutations, pictured in Figure 4.1.

$$6\ 1\ 3\ 7\ 2\ 8\ 5\ 4\ 9 \qquad\qquad 6\ \overset{x_1}{1}\ 3\ 7\ 2\ 8\ 5\ 4\ 9$$

$$6\ 1\ 3\ 7\ 2\ \overset{x_2}{8}\ 5\ 4\ 9 \qquad\qquad 6\ \overset{x_1}{1}\ 3\ 7\ 2\ \overset{x_2}{8}\ 5\ 4\ 9$$

Figure 4.1: (u_1, u_2)-marked σ.

Note that it is also possible that a position has more than one label. Suppose $u_3 = 1\ 2$, two examples in $\mathcal{MS}_{9,(u_1,u_2,u_3)}$ are pictured in Figure 4.2.

$$6\ \overset{x_3}{\overset{x_1}{1}}\ 3\ 7\ \overset{x_3}{2}\ \overset{x_2}{8}\ 5\ 4\ 9 \qquad\qquad 6\ \overset{x_3}{\overset{x_1}{1}}\ \overset{x_3}{3}\ 7\ 2\ \overset{x_2}{8}\ 5\ 4\ 9$$

Figure 4.2: Two examples in $\mathcal{MS}_{9,(u_1,u_2,u_3)}$

A **joint** $(u_1, u_2, \cdots, u_m)$**-cluster** is an element $\sigma = \sigma_1 \ldots \sigma_n \in \mathcal{MS}_{n,(u_1,u_2,\cdots,u_m)}$ such that

1. every σ_i is an element contained in a marked u_j-match in σ, for some $1 \leq j \leq m$ and

2. any two consecutive marked matches share at least one element.

Since $\mathcal{S}_n$ is actually $\mathcal{P}_n^{0,0,1}$, it is easy to generalize this definition to deal with elements of $\mathcal{P}_{nk}^{0,0,k}$. Suppose that we are given patterns $P_1, P_2, \cdots, P_m$. For any $n \geq 1$, we let $\mathcal{MP}_{kn,(P_1,P_2,\cdots,P_m)}^{0,0,k}$ denote the set of all fillings $F \in \mathcal{P}_{nk}^{0,0,k}$ where we have marked some of the P_j-matches in F by placing an 'x_j' on top of the column that starts a P_j-match in F. We use $m_{P_j}(F)$ to denote the number of marked P_j-matches in F.

Then a **joint** $(P_1, P_2, \cdots, P_m)$**-cluster** is an element $F \in \mathcal{MP}_{kn,(P_1,P_2,\cdots,P_m)}^{0,0,k}$ such that

1. every column of F is contained in a marked P_j-match in F, for some j, $1 \leq j \leq m$ and

2. any two consecutive marked matches share at least one column.

For convenience, we shall refer to joint $(P_1, \cdots, P_m)$-clusters as just $(P_1, \cdots, P_m)$-clusters.

Let $\mathcal{CM}_{kn,(P_1,\cdots,P_m)}^{0,0,k}$ denote the set of all $(P_1, \cdots, P_m)$-clusters in $\mathcal{MP}_{kn,(P_1,\cdots,P_m)}^{0,0,k}$. For each $n \geq 2$, we define the multi-variate cluster polynomial

$$C_{kn,(P_1,\cdots,P_m)}^{0,0,k}(x_1, \cdots, x_m) := \sum_{F \in \mathcal{CM}_{kn,(P_1,\cdots,P_m)}^{0,0,k}} \prod_{i=1}^{m} x_i^{m_{P_i}(F)}, \tag{4.1}$$

where $m_P(F)$ is the number of marked P-matches in F. By convention, we let $C_{k,(P_1,\cdots,P_m)}^{0,0,k}(x_1, \cdots, x_m) = 1$.

Then we have the multi-variate version of Theorem 2.2.

Theorem 4.1. *Let $P_i \in \mathcal{P}_{j_i k}^{0,0,k}$ where $j_i \geq 2$ and $1 \leq i \leq m$. Then*

$$1 + \sum_{n \geq 1} \frac{t^{kn}}{kn!} \sum_{F \in \mathcal{P}_{kn}^{0,0,k}} \prod_{i=1}^{m} x_i^{P_i\text{-}mch(F)} = \frac{1}{1 - \sum_{n \geq 1} \frac{t^{kn}}{(kn)!} C_{kn,(P_1,P_2,\cdots,P_m)}^{0,0,k}(x_1 - 1, x_2 - 1, \cdots, x_m - 1)}. \tag{4.2}$$

Proof. The strategy is similar to the proof of Theorem 2.2. Replace x by $x+1$ in (4.2). Then the left-hand side of (4.2) is the generating function of $m_{P_i}(F)$ over all $F \in \mathcal{MP}^{0,0,k}_{kn,(P_1,\cdots,P_m)}$. That is, it easy to see that

$$1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{P}^{0,0,k}_{kn}}\prod_{i=1}^{m}(x_i+1)^{P_i\text{-mch}(F)} = \\ 1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{MP}^{0,0,k}_{kn,(P_1,\cdots,P_m)}}\prod_{i=1}^{m}x_i^{m_{P_i}(F)}. \tag{4.3}$$

Thus we must show that

$$1+\sum_{n\geq 1}\frac{t^n}{kn!}\sum_{F\in\mathcal{MP}^{0,0,k}_{kn,(P_1,\cdots,P_m)}}\prod_{i=1}^{m}x_i^{m_{P_i}(F)} = \\ \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,(P_1,P_2,\cdots,P_m)}(x_1,x_2,\cdots,x_m)}. \tag{4.4}$$

Rewriting the right-hand side of (4.4) as

$$\frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,(P_1,\cdots,P_m)}(x_1,\cdots,x_m)} = \\ 1+\sum_{h\geq 1}\left(\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,(P_1,\cdots,P_m)}(x_1,\cdots,x_m)\right)^h$$

and taking coefficients of $\frac{t^{ks}}{(ks)!}$ on both sides of (4.4) where $n \geq 1$, we see that we must show that

$$\begin{aligned}
&\sum_{F\in\mathcal{MP}^{0,0,k}_{sn,(P_1,\cdots,P_m)}}\prod_{i=1}^{m}x_i^{m_{P_i}(F)} \\
&= \sum_{h=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,(P_1,\cdots,P_m)}(x_1,\cdots,x_m)\right)^h\Bigg|_{\frac{t^{ks}}{(ks)!}} \\
&= \sum_{h=1}^{s}\left(\sum_{n=1}^{s}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,(P_1,\cdots,P_m)}(x_1,\cdots,x_m)\right)^h\Bigg|_{\frac{t^{ks}}{(ks)!}} \\
&= \sum_{h=1}^{s}\sum_{\substack{a_1+\cdots+a_h=s\\ a_i\geq 1}}\binom{kn}{ka_1,\ldots,ka_h}\prod_{j=1}^{h}C^{0,0,k}_{ka_j,(P_1,\cdots,P_m)}(x_1,\cdots,x_m).
\end{aligned} \tag{4.5}$$

The right-hand side of (4.5) is now easy to interpret. First we pick an h such that $1 \leq h \leq s$. Then we pick $a_1, \ldots, a_h \geq 1$ such that $a_1 + a_2 + \cdots + a_h = s$. Next the multinomial coefficient $\binom{kn}{ka_1,\ldots,ka_h}$ allows us to pick sets $S_1, \ldots, S_h$ which partition $\{1, \ldots, ks\}$ such that $|S_i| = ka_i$ for $i = 1, \ldots, h$. Finally the product $\prod_{i=1}^{h} C^{0,0,k}_{ka_i,(P_1,\cdots,P_m)}(x_1, \cdots, x_m)$ allows us to pick clusters $C_i \in \mathcal{CM}^{0,0,k}_{ka_i,(P_1,\cdots,P_m)}$ for $i = 1, \ldots, h$ with weight $\prod_{i=1}^{h} \prod_{j=1}^{m} x_j^{m_{P_j}(C_i)}$. Note that in the cases where $a_i = 1$, we will interpret C_i as just a column of height k filled with the numbers $1, \ldots, k$ which is increasing, reading from bottom to top.

$S_1 = \{1, 3, 5, 8, 10, 15, 18, 24, 27, 30\}$

x_1	x_2	x_1		
10	9	8	7	6
1	2	3	4	5

$= C_1$

$S_2 = \{17, 19\}$

2
1

$= C_2$

$S_3 = \{4, 7, 12, 13, 20, 21\}$

x_1		
6	5	4
1	2	3

$= C_3$

$S_4 = \{2, 6, 9, 11, 14, 16, 22, 23, 28, 29\}$

x_2	x_1 x_2			
10	9	8	7	6
1	2	3	4	5

$= C_4$

$S_5 = \{25, 26\}$

2
1

$= C_5$

x_1	x_2	x_1				x_1			x_2	x_1 x_2				
30	27	24	18	15	19	21	20	13	29	28	23	22	16	26
1	3	5	8	10	17	4	7	12	2	6	9	11	14	25

Figure 4.3: Construction for the right-hand side of (4.5).

For example, suppose that $k = 2$, $P_1 = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ and $P_2 = \begin{array}{|c|c|c|c|}\hline 8 & 7 & 6 & 5 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$. Then in Figure 4.3, we have pictured S_1, S_2, S_3, S_4, S_5 which partition $\{1, \ldots, 30\}$ and corresponding clusters $C_1, \ldots, C_5$. Then for each $i = 1, \ldots, m$, we create a cluster D_i which results by replacing the j in C_i by the j^{th} element of S_i. If we concatenate $D_1 \ldots D_m$ together, then we will obtain an element of $Q \in \mathcal{MP}^{0,0,k}_{kn,(P_1,P_2)}$. It is easy to see that one can recover $D_1, \ldots, D_5$ from Q. That is, given an element $F \in \mathcal{MP}^{0,0,k}_{kn,(P_1,P_2)}$, we say that a marked subsequence $F[i, i+1, \ldots, j]$ is a *maximal* (P_1, P_2)*-subcluster* of F if $\mathrm{red}(F[i, i+1, \ldots, j])$ is a (P_1, P_2)-cluster and $F[i, i+1, \ldots, j]$ is not properly contained in a marked subsequence $F[a, a+1, \ldots, b]$ such that $\mathrm{red}(F[a, a+1, \ldots, b])$ is a (P_1, P_2)-cluster. In the special case where $i = j$ and the column $F[i]$ is not marked, then we say that $F[i]$ is maximal (P_1, P_2)-subcluster if $F[i]$ is not properly contained in a marked subsequence $F[a, a+1, \ldots, b]$ such that $\mathrm{red}(F[a, a+1, \ldots, b])$ is a (P_1, P_2)-cluster. Thus $D_1, \ldots, D_5$ are the maximal (P_1, P_2)-subclusters of Q. Of course, once we have recovered $D_1, \ldots, D_5$, we can recover the sets $S_1, \ldots, S_5$ and the (P_1, P_2)-clusters $C_1, \ldots, C_5$.

In this manner, we can see that the right-hand side of (4.5) just classifies the elements of $\mathcal{MP}^{0,0,k}_{kn,(P_1,\cdots,P_m)}$ by its maximal $(P_1, \cdots, P_m)$-subclusters which proves our theorem. □

Next suppose that we are given a binary relation $\mathscr{R}$ between $k \times 1$ arrays of integers and patterns $P_1, P_2, \cdots, P_m$ where $P_i \in \mathcal{P}^{0,0,k}_{kj_i}$, $1 \leq i \leq m$. Given our definition of joint clusters, we can easily modify the definition of generalized P-clusters given in Definition 2.3 to generalized joint $(P_1, \ldots, P_m)$-clusters.

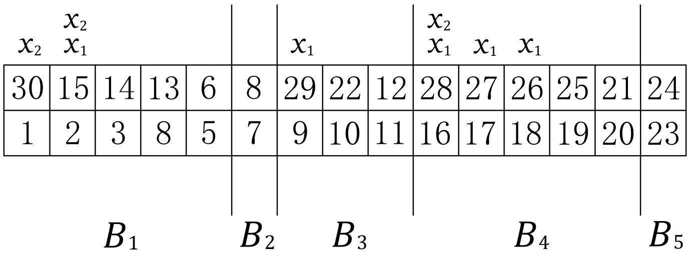
Definition 4.2. *Given patterns $P_1, P_2, \cdots, P_m$, we say that $Q \in \mathcal{MP}^{0,0,k}_{kn,(P_1,\cdots,P_m)}$ is a* ***generalized joint*** $(P_1, \cdots, P_m), \mathscr{R}$***-cluster*** *if we can write $Q = B_1 B_2 \cdots B_h$ where B_i are blocks of consecutive columns in Q such that*

1. *either B_i is a single column or B_i consists of r-columns where $r \geq 2$, $\mathrm{red}(B_i)$ is a $(P_1, \cdots, P_m)$-cluster in $\mathcal{MP}^{0,0,k}_{kn,(P_1,\cdots,P_m)}$, and any pair of consecutive columns in B_i are in $\mathscr{R}$ and*

2. *for $1 \leq i \leq h-1$, the pair $(last(B_i), first(B_{i+1}))$ is not in $\mathscr{R}$ where for any j, $last(B_j)$ is the right-most column of B_j and $first(B_j)$ is the left-most column of B_j.*

For convienence, a generalized joint $(P_1, \cdots, P_m), \mathscr{R}$-clusters will be refered to a just a generalized $(P_1, \cdots, P_m)$-cluster.

Let $\mathcal{GC}^{0,0,k}_{kn,(P_1,\cdots,P_m),\mathscr{R}}$ denote the set of all generalized $(P_1, \cdots, P_m), \mathscr{R}$-clusters which have n columns of height k. For example, suppose that $\mathscr{R}$ is the relation that holds for a pair of columns (C, D) if and only if the top element of column C is greater than the bottom element of column D, $P_1 = \begin{array}{|c|c|c|}\hline 6 & 5 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ and $P_2 = \begin{array}{|c|c|c|c|}\hline 8 & 7 & 6 & 5 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$. Then in Figure 4.4, we have pictured a generalized (P_1, P_2)-cluster with 5 blocks B_1, B_2, B_3, B_4, B_5.



30	15	14	13	6	8	29	22	12	28	27	26	25	21	24
1	2	3	8	5	7	9	10	11	16	17	18	19	20	23

Figure 4.4: A generalized (P_1, P_2)-cluster.

Given $Q = B_1 B_2 \dots B_h \in \mathcal{GC}^{0,0,k}_{kn,(P_1,\cdots,P_m),\mathscr{R}}$, we define the weight of B_i,

$$\omega_{(P_1,\cdots,P_m),\mathscr{R}}(B_i) := \begin{cases} 1, & \text{if } \mathrm{Col}(B_i) = 1, \\ \prod_{j=1}^{m} x_j^{m_{P_j}(\mathrm{red}(B_i))}, & \text{if } \mathrm{Col}(B_i) \geq 2, \end{cases}$$

where $\mathrm{Col}(B_i)$ is the number of columns in B_i. Then we define the weight of Q, $\omega_{(P_1,\cdots,P_m),\mathscr{R}}(Q)$, to be $(-1)^{h-1} \prod_{i=1}^{h} \omega_{(P_1,\cdots,P_m),\mathscr{R}}(B_i)$. We let

$$GC^{0,0,k}_{kn,(P_1,\cdots,P_m),\mathscr{R}}(x_1, \cdots, x_m) := \sum_{Q \in \mathcal{GC}^{0,0,k}_{kn,(P_1,\cdots,P_m),\mathscr{R}}} \omega_{(P_1,\cdots,P_m),\mathscr{R}}(Q), \tag{4.6}$$

which is called generalized joint cluster polynomial.

Then we have the following theorem which is the multi-variate version of Theorem 2.4.

Theorem 4.3. *Let $\mathscr{R}$ be a binary relation on pairs of columns (C, D) of height k which are filled with integers which are increasing from bottom to top. Given* *patterns* $P_1, P_2, \cdots, P_m$ *where* $P_i \in \mathcal{P}^{0,0,k}_{kj_i}$, $j_i \geq 2$, *for* $1 \leq i \leq m$, *then*

$$1 + \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} \sum_{F\in\mathcal{P}^{0,0,k}_{kn,\mathscr{R}}} \prod_{i=1}^{m} x_i^{P_i\text{-}mch(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,(P_1,\cdots,P_m),\mathscr{R}}(x_1-1,\cdots,x_m-1)}.$$

It is quite straightforward to modify the proof of Theorem 2.4 to prove theorem Theorem 4.3. Thus we shall not give the details.

Moreover, it is quite easy to extend Theorem 4.1 and Theorem 4.3 to consider multiple sets of patterns. In other words, suppose we have $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$ where Γ_i itself is a set of patterns, then

$$1 + \sum_{n\geq 1} \frac{t^{kn}}{kn!} \sum_{F\in\mathcal{P}^{0,0,k}_{kn}} \prod_{i=1}^{m} x_i^{\Gamma_i\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}C^{0,0,k}_{kn,(\Gamma_1,\cdots,\Gamma_m)}(x_1-1,\cdots,x_m-1)},$$

and given some binary relation $\mathscr{R}$, we have

$$1 + \sum_{n\geq 1} \frac{t^{kn}}{(kn)!} \sum_{F\in\mathcal{P}^{0,0,k}_{kn,\mathscr{R}}} \prod_{i=1}^{m} x_i^{\Gamma_i\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{kn}}{(kn)!}GC^{0,0,k}_{kn,(\Gamma_1,\cdots,\Gamma_m),\mathscr{R}}(x_1-1,\cdots,x_m-1)}.$$

By modifying Equation (4.1) and (4.6) accordingly, one can easily figure out the definitions of $C^{0,0,k}_{kn,(\Gamma_1,\cdots,\Gamma_m)}(x_1,\cdots,x_m)$ and $GC^{0,0,k}_{kn,(\Gamma_1,\cdots,\Gamma_m),\mathscr{R}}(x_1,\cdots,x_m)$.

It is worth mentioning that joint clusters and generalized joint clusters still hold for other fillings of $D^{0,0,k}_{kn}$ and even $D^{i,j,k}_{i+kn+j}$. Discussion about joint clusters and generalized joint clusters for $D^{i,j,k}_{i+kn+j}$ can be found in Section 4.3.

So far we only discussed joint and generalized joint clusters for reduced pattern matching. One can convince themselves that Theorem 4.1 and Theorem 4.3 still hold for joint exact patterns. With some modifications, the formulas in Theorem 4.1 and Theorem 4.3 still work even though some of the patterns are reduced patterns and some are exact patterns.

4.2 Examples

In this section, we take colored permutations and integer compositions as examples to further illustrate joint clusters and generalized joint clusters.

4.2.1 Bi-runs in restricted colored permutations

A k-colored permutation of length n can be thought as an element in wreath product $\mathcal{C}_k \wr \mathcal{S}_n$ of cyclic group $\mathcal{C}_k$ and symmetric group $\mathcal{S}_n$. We will use $\mathcal{C}_k \wr \mathcal{S}_n$ to denote the set of k-colored permutations of length n. For convenience, we use a two-row array to represent a colored permutation where the base row stands for the permutation and the top row stands for the corresponding colors. Thus a colored permutation in $\mathcal{C}_k \wr \mathcal{S}_n$ can be thought of as a special filling of $\mathcal{D}_{2n}^{0,0,2}$. In Figure 4.5, a 5-colored permutation of length 9 is pictured as an example.

1	2	1	5	2	4	1	3	4
2	5	9	1	3	7	4	6	8

Figure 4.5: An element in $\mathcal{C}_5 \wr \mathcal{S}_9$

Minimal overlapping patterns in colored permutations has been studied by Duane and Remmel in [14] using maximum packings which is based on ring homomorphism method. In this subsection, we would like to extend to multiple overlapping patterns for colored permutations and also a class of colored permutations with specific restrictions.

A run in a permutation or a word is just an consecutive increasing subsequence of length greater than or equal to 2. The distribution of runs in permutations and words have been the subject of many papers in the literature. A run of length l can be thought as a $1\ 2\ \cdots\ l$-match in a permuation or a word. In this subsection, we shall explore the enumeration of bi-runs in colored permutations and restricted colored permutations.

Definition 4.4. *For a colored permutation $(\sigma, w) \in \mathcal{C}_k \wr \mathcal{S}_n$, we say there is a bi-run of length l at position i if*

1. *red*$(\sigma_i\sigma_{i+1}\cdots\sigma_{i+l-1}) = 1\ 2\ 3\ \cdots\ l$ *and*

2. *red*$(w_iw_{i+1}\cdots w_{i+l-1}) = 1\ 2\ 3\ \cdots\ l$.

Throughout this subsection, for the sake of convenience, an l-run is short for a bi-run of length l. The number of l-runs in colored permutation (σ, w) is denoted by l-rn(σ, w). As for the 5-colored permutation in Figure 4.5, $\sigma = 2\ 5\ 9\ 1\ 3\ 7\ 4\ 6\ 8$ and $w = 1\ 2\ 1\ 5\ 2\ 4\ 1\ 3\ 4$. 2-rn$(\sigma, w) = 4$ because there are 2-runs at positions 1, 5, 7 and 8. 3-rn$(\sigma, w) = 1$ because there is a 3-run at position 7. Clearly, for any $(\sigma, w) \in \mathcal{C}_k \wr \mathcal{S}_n$, l-rn(σ, w) has to be zero if $l > \min(k, n)$.

For colored permutations (σ, w), there is a natural restricted subclass which consists of the all the colored permutations (σ, w) where w has no consecutive repeated letters. Thus we shall consider the binary relation $\mathscr{R}$ where $\mathscr{R}$ holds for consecutive pairs $(\sigma), w_i), (\sigma_{i+1}, w_{i+1}))$ if and only if $w_i \neq w_{i+1}$. The set of all such restricted colored permutations in $\mathcal{C}_k \wr \mathcal{S}_n$ is denoted by $\overline{\mathcal{C}_k \wr \mathcal{S}_n}$. Actually the example in Figure 4.5 is also an element in $\overline{\mathcal{C}_5 \wr \mathcal{S}_9}$

Our focus is on computing the following two generating functions which count colored permutations and restricted colored permutations by the number of bi-runs.

$$A_{\text{bi-run},\mathcal{C}_k\wr\mathcal{S}}(x_2, x_3, x_4, \cdots, x_k, t) := 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{(\sigma,w)\in\mathcal{C}_k\wr\mathcal{S}_n} \prod_{l=2}^{k} x_l^{l\text{-rn}(\sigma,w)}$$

$$A_{\text{bi-run},\overline{\mathcal{C}_k\wr\mathcal{S}}}(x_2, x_3, x_4, \cdots, x_k, t) := 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{(\sigma,w)\in\overline{\mathcal{C}_k\wr\mathcal{S}_n}} \prod_{l=2}^{k} x_l^{l\text{-rn}(\sigma,w)}$$

It is clear that, in above formulas, the subscript of variable x_l is up to k because for any $m > k$, m-rn$(\sigma, w) \equiv 0$.

Based on Theorem 4.1, we have

$$A_{\text{bi-run},\mathcal{C}_k\wr\mathcal{S}}(x_2, x_3, x_4, \cdots, x_k, t) = \frac{1}{1 - kt - \sum_{n\geq 2} \frac{t^n}{n!} C_n(x_2 - 1, x_3 - 1, \cdots, x_k - 1)}, \tag{4.7}$$

where $C_n(x_2, x_3, \cdots, x_k)$ is the (2-run, 3-run, $\cdots$, k-run)-cluster polynomial, and by Theorem 4.3,

$$A_{\text{bi-run},\overline{\mathcal{C}_k\wr\mathcal{S}}}(x_2, x_3, x_4, \cdots, x_k, t) = \frac{1}{1 - \sum_{n\geq 1} \frac{t^n}{n!} GC_n(x_2 - 1, x_3 - 1, \cdots, x_k - 1)}, \tag{4.8}$$

where $GC_n(x_2, x_3, \cdots, x_k)$ is the generalized (2-run, 3-run, $\cdots$, k-run)-cluster polynomial. Thus we only need to compute the cluster and generalized cluster polynomials in order to obtain our desired generating functions.

Throughout this subsection, we use $k = 4$ as our running example. That is, we want to study joint distributions of 2-runs, 3-runs and 4-runs in $\mathcal{C}_4 \wr \mathcal{S}_n$ and $\overline{\mathcal{C}_4 \wr \mathcal{S}}$. In this case, we shall refer to the corresponding clusters and generalized clusters as bi-run clusters and generalized bi-run clusters.

First let us work on bi-runs enumeration for unrestricted 4-colored permutations.

Before we could compute bi-run cluster polynomials, we shall figure out the structure of the bi-run clusters. Because each column in a bi-run cluster has to be contained in some marked run, the filling of a bi-run cluster of n columns itself has to be an n-run, but there could be multiple ways to label it. A labeling polynomial is the sum of weights for different labelings of an n-run. Let $L_n(x_2, x_3, x_4)$ be the labeling polynomial for an n-run cluster. Clearly, the base row of an n-run must be unique while there are $\binom{4}{n}$ ways to choose colors. This gives us

$$C_n(x_2, x_3, x_4) = \binom{4}{n} L_n(x_2, x_3, x_4).$$

In Figure 4.6, a 3-run $(\sigma, w) = (1\ 2\ 3,\ 1\ 3\ 4)$ is pictured and in order to make (σ, w) be a cluster, there are 5 ways to label marked runs. Then we can compute the labeling polynomial $L_3(x_2, x_3, x_4)$

$$L_3(x_2, x_3, x_4) = x_3 + x2_x3 + x_2^2 x_3 + x_2 x_3 + x_2^2 = x_2^2 + (x_2 + 1)^2 x_3$$

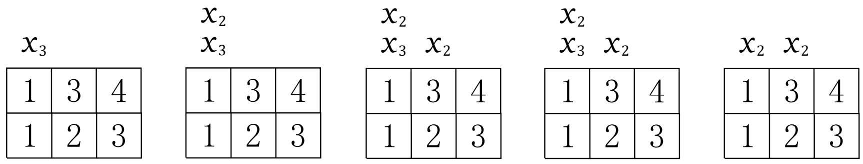


Figure 4.6: 5 different labelings of a 3-run-cluster in $\mathcal{C}_4 \wr \mathcal{S}_3$

The bi-run cluster with two columns can have only one label, namely, 'x_2'. Hence $L_2(x_1, x_2, x_3) = x_2$. For a cluster of 3 columns, we have shown that $L_3 =$

$x_2^2+(x_2+1)^2x_3$ according to Figure 4.6. For a cluster of size 4, we split discussion into three cases. Case 1 is the case where the first column is contained in a marked 4-run. The labeling polynomial for this case is $(x_2+1)^3(x_3+1)^2x_4$ because there are at most three marked 2-runs and two marked 3-runs. Case 2 is the case where the first column is not included in a marked 4-run and the second column is labeled. This means that the first column has label x_3, x_2, or both x_2 and x_3. Since the second column is labeled with either x_2 or x_3 or both, the marked second column will automatically ensure that the remaining marked columns meet the condition to be a bi-run cluster. Thus the bi-run clusters in case 2 contributes $(x_2+x_3+x_2x_3)L_3(x_2,x_3,x_4)$ to the labeling polynomial. Case 3 is the case where the first column is not included in a marked 4-run and the second column has no label. This means that the first column must be marked with x_3 but it could also be marked with x_2 or not. Moreover, the 3 column must be marked with x_2. Thus the bi-run clusters in case 3 contribute $x_2(x_2+1)x_3$ to the labeling polynomial. Thus,

$$\begin{aligned} L_4(x_2,x_3,x_4) &= (x_2+1)^3(x_3+1)^2x_4+(x_2+x_3+x_2x_3)L_3(x_2,x_3,x_4) \\ &\quad +x_2(x_2+1)x_3 \\ &= (x_2+x_2^2)x_3+(x_2+x_3+x_2x_3)\left(x_2^2+(1+x_2)^2x_3\right) \\ &\quad +(1+x_2)^3(1+x_3)^2x_4 \end{aligned} \tag{4.9}$$

Then we can compute $C_n(x_2,x_3,x_4)$ for $2\le n\le 4$ as

$$\begin{aligned} C_2(x_2,x_3,x_4) &= \binom{4}{2}L_2(x_2,x_3,x_4)=6x_2 \\ C_3(x_2,x_3,x_4) &= \binom{4}{3}L_3(x_2,x_3,x_4)=4x_2^2+4(1+x_2)^2x_3 \\ C_4(x_2,x_3,x_4) &= \binom{4}{4}L_4(x_2,x_3,x_4) \\ &= (x_2+x_2^2)x_3+(x_2+x_3+x_2x_3)\left(x_2^2+(1+x_2)^2x_3\right) \\ &\quad +(1+x_2)^3(1+x_3)^2x_4 \end{aligned}$$

Substituting $C_n(x_2, x_3, x_4)$ in Equation (4.7) by above expressions, we have

$$A_{\text{bi-run},\mathcal{C}_4\wr\mathcal{S}}(x_2,x_3,x_4,t) = \frac{1}{1-\sum_{n\geq 1}\frac{t^n}{n!}C_n(x_2-1,x_3-1,x_4-1)} =$$

$$24\left(24-96t-72(x_2-1)t^2-16(x_2^2x_3-2x_2+1)t^3 \right.$$
$$\left. -(3x_2+x_2^3x_3^2x_4-2x_2^2x_3-x_2^2-1)t^4\right)^{-1}.$$

A few initial terms of the expansion are

$$1+4t+\frac{1}{2!}(13+3x_2)t^2+\frac{1}{3!}(244+136x_2+4x_2^2x_3)t^3$$
$$+\frac{1}{24}(3031+2771x_2+215x_2^2+126x_2^2x_3+x_2^3x_3^2x_4)t^4+\cdots$$

By manipulating the explicit generating function above, we could derive various results, some of which are not known before. For example,

$$\begin{aligned}
& A_{\text{bi-run},\mathcal{C}_4\wr\mathcal{S}}(1,0,0,t) \\
= & -\frac{24}{t^4-16t^3+96t-24} \\
= & 1+4t+\frac{32}{2!}t^2+\frac{380}{3!}t^3+\frac{6017}{4!}t^4+\frac{119080}{5!}t^5+\frac{2828000}{6!}t^6+\frac{78354920}{7!}t^7 \\
& +\frac{2481104710}{8!}t^8+\frac{88384565640}{9!}t^9+\cdots,
\end{aligned}$$

which counts the number of 4-colored permutations that don't have a 3-run.

$$\begin{aligned}
& A_{\text{bi-run},\mathcal{C}_4\wr\mathcal{S}}(1,1,0,t)-A_{\text{bi-run},\mathcal{C}_4\wr\mathcal{S}}(1,0,0,t) \\
= & 24\left(\frac{1}{t^4-96t+24}+\frac{1}{t^4-16t^3+96t-24}\right) \\
= & \frac{4}{3!}t^3+\frac{126}{4!}t^4+\frac{3760}{5!}t^5+\frac{119680}{6!}t^6+\frac{4166680}{7!}t^7+\frac{159156480}{8!}t^8 \\
& +\frac{6649359360}{9!}t^9+\cdots,
\end{aligned}$$

which counts the number of 4-colored permutations that have at least one 3-run but don't have a 4-run.

$$\begin{aligned}
& \left.\frac{\partial A_{\text{bi-run},\mathcal{C}_4\wr\mathcal{S}}(x_2,1,1,t)}{\partial x_2}\right|_{x_2=1} \\
= & \frac{3t^2}{(1-4t)^2} \\
= & \frac{6}{2!}t^2+\frac{144}{3!}t^3+\frac{3456}{4!}t^4+\frac{92160}{5!}t^5+\frac{2764800}{6!}t^6+\frac{92897280}{7!}t^7 \\
& +\frac{3468165120}{8!}t^8+\cdots,
\end{aligned}$$

which counts the total number of 2-runs in all the 4-colored permutations of length n. It is not hard to see that the coefficient of $\frac{t^n}{n!}$ is equal to $3(n-1)\cdot 4^{n-2}\cdot n!$, which implies that the average number of 2-runs in a random 4-colored permutation is

$$\frac{3(n-1)\cdot 4^{n-2}\cdot n!}{|\mathcal{C}_4 \wr \mathcal{S}_n|} = \frac{3(n-1)\cdot 4^{n-2}\cdot n!}{4^n \cdot n!} = \frac{3n-3}{16}.$$

Next we shall work on joint distributions of bi-runs in restricted 4-colored permutations $\overline{\mathcal{C}_4 \wr \mathcal{S}_n}$. Here we restrict that adjacent colors has to be different, i.e., for any $(\sigma, w) \in \overline{\mathcal{C}_4 \wr \mathcal{S}_n}$, $w_i \neq w_{i+1}$ for $1 \leq i \leq n-1$.

According to Equation (4.8), we only need to compute $GC_n(x_2, x_3, x_4)$. Instead of computing $GC_n(x_2, x_3, x_4)$ for different n directly, we divide generalized bi-run clusters $\mathcal{GC}_n$ into several types based on different structures of bi-runs.

First let us review the definition for generalized bi-run clusters of $\overline{\mathcal{C}_4 \wr \mathcal{S}_n}$ for bi-runs. We say $(\sigma, w) \in \mathcal{C}_4 \wr \mathcal{S}_n$ is a generalized bi-run cluster of size n if (σ, w) can be written as $(\sigma, w) = B_1 B_2 \cdots B_l$ where

1. B_i is either a single column or a bi-run cluster and
2. the last element of the color row (top row) in block B_i is equal to the first color element in block B_{i+1} for $1 \leq i < l$.

From previous discussion with respect to bi-runs in $\mathcal{C}_4 \wr \mathcal{S}_n$, we know there are at most 3 2-runs in a generalized bi-run cluster. Then we have seven different types of generalized bi-run clusters based on choices on bi-run clusters.

1. Type 1 generalized bi-run clusters: In this type of generalized bi-run clusters, there are only singleton blocks, that is, each block contains only one column. The set of such generalized bi-run clusters of columns n is denoted by $\mathcal{GC}_n^1$, for $n \geq 1$. An example in $\mathcal{GC}_4^1$ is pictured in Figure 4.7, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4$ is an arbitrary element in $\mathcal{S}_4$ and $1 \leq i \leq 4$.
2. Type 2 generalized bi-run clusters: This type of generalized bi-run clusters contains exactly one block of size 2. The set of such generalized bi-run clusters of n columns is denoted by $\mathcal{GC}_n^2$, for $n \geq 2$. An example in $\mathcal{GC}_5^2$ is pictured in Figure 4.8, omitting labelings, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ is an arbitrary element in $\mathcal{S}_5$ satisfying $\sigma_3 < \sigma_4$, and $1 \leq i < j \leq 4$.

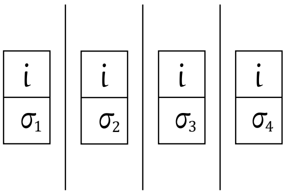


Figure 4.7: An example in $\mathcal{GC}_4^1$, where $1 \leq i \leq 4$.

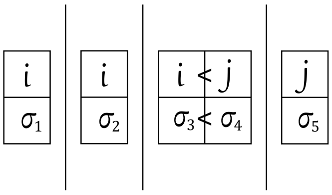


Figure 4.8: An example in $\mathcal{GC}_5^2$, omitting labelings.

3. Type 3 generalized bi-run clusters: this type of generalized bi-run clusters contains exactly one block of size 3. The set of such generalized bi-run clusters of n columns is denoted by $\mathcal{GC}_n^3$, for $n \geq 3$. An example in $\mathcal{GC}_6^3$ is pictured in Figure 4.9, omitting labelings, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6$ is an arbitrary element in $\mathcal{S}_6$ satisfying $\sigma_2 < \sigma_3 < \sigma_4$, and $1 \leq i < j < k \leq 4$.

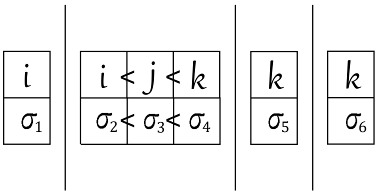


Figure 4.9: An example in $\mathcal{GC}_6^3$, omitting labelings.

4. Type 4 generalized bi-run clusters: this type of generalized bi-run clusters contains exactly two blocks of size 2. The set of such generalized bi-run clusters of n columns is denoted by $\mathcal{GC}_n^4$, for $n \geq 3$. An example in $\mathcal{GC}_7^4$ is pictured in Figure 4.10, omitting labelings, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7$ is an arbitrary element in $\mathcal{S}_7$ satisfying $\sigma_3 < \sigma_4$, $\sigma_6 < \sigma_7$, and $1 \leq i < j < k \leq 4$.

5. Type 5 generalized bi-run clusters: this type of generalized bi-run clusters contains exactly one block of size 4. The set of such generalized bi-run

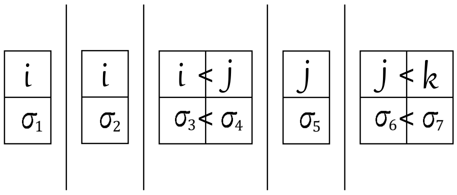


Figure 4.10: An example in $\mathcal{GC}_7^4$, omitting labelings.

clusters of n columns is denoted by $\mathcal{GC}_n^5$, for $n \geq 4$. An example in $\mathcal{GC}_7^5$ is pictured in Figure 4.12, omitting labelings, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7$ is an arbitrary element in $\mathcal{S}_7$ satisfying $\sigma_3 < \sigma_4 < \sigma_5 < \sigma_6$.

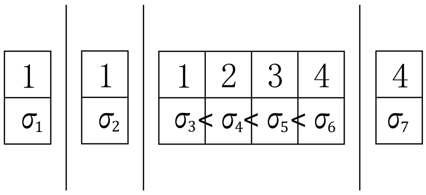


Figure 4.11: An example in $\mathcal{GC}_7^5$, omitting labelings.

6. Type 6 generalized bi-run clusters: this type of generalized bi-run clusters contains exactly three blocks of size 2. The set of such generalized bi-run clusters of n columns is denoted by $\mathcal{GC}_n^6$, for $n \geq 6$. An example in $\mathcal{GC}_{10}^6$ is pictured in Figure 4.12, omitting labelings, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}$ is an arbitrary element in $\mathcal{S}_{10}$ satisfying $\sigma_3 < \sigma_4$, $\sigma_6 < \sigma_7$ and $\sigma_8 < \sigma_9$.

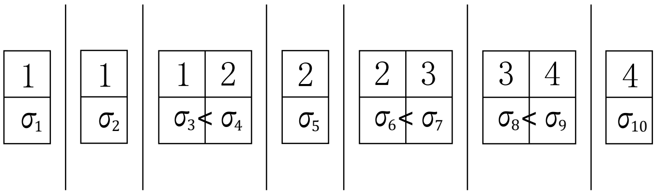


Figure 4.12: An example in $\mathcal{GC}_{10}^6$, omitting labelings.

7. Type 7 generalized bi-run clusters: this type of generalized bi-run clusters contains exactly one block of size 2 and exactly one block of size 3. Note that we didn't specify the order of the 2-column block and 3-column block,

which means we need take two possibilities into account where one is that 2-run is to left of the 3-run and the other possibility is that 2-run is to the right of the 3-run. The set of such generalized bi-run clusters of size n is denoted by $\mathcal{GC}_n^7$, for $n \geq 5$. An example in $\mathcal{GC}_9^7$ is pictured in Figure 4.13, omitting labelings, where $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9$ is an arbitrary element in $\mathcal{S}_9$ satisfying $\sigma_3 < \sigma_4$, $\sigma_6 < \sigma_7 < \sigma_8$.

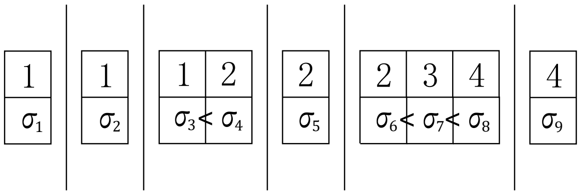


Figure 4.13: An example in $\mathcal{GC}_9^7$, omitting labelings.

Then we defined the corresponding generalized joint bi-run clusters polynomials are defined as follows,

$$GC_n^{\mathrm{J}}(x_2,x_3,x_4) := \sum_{(\sigma,w)\in\mathcal{GC}_n^{\mathrm{J}}} (-1)^{\mathrm{B}(\sigma,w)-1}\ x_2^{m_{2\text{-rn}}(\sigma,w)} x_3^{m_{3\text{-rn}}(\sigma,w)} x_4^{m_{4\text{-rn}}(\sigma,w)},$$

$$GC^{\mathrm{J}}(x_2,x_3,x_4,t) := \sum_{n\geq 1} \frac{t^n}{n!} GC_n^{\mathrm{J}}(x_2,x_3,x_4),$$

where $\mathrm{J} \in \{1,2,3,4,5,6,7\}$, $\mathrm{B}(\sigma,w)$ is the number of blocks in (σ,w) and $m_{l\text{-rn}}(\sigma,w)$ denotes the number of marked l-runs in (σ,w). Clearly, $\mathcal{GC}_n^{\mathrm{J}}$ are disjoint sets, i.e.,

$$\mathcal{GC}_n = \biguplus_{\mathrm{J}\in\{1,2,\cdots,7\}} \mathcal{GC}_n^{\mathrm{J}}$$

and then

$$GC_n(x_2,x_3,x_4) = \sum_{\mathrm{J}\in\{1,2,\cdots,7\}} GC_n^{\mathrm{J}}(x_2,x_3,x_4)$$

Then we shall discuss $\mathcal{GC}_n^{\mathrm{J}}$ and compute GC_n^{J} for each J.

For any arbitrary $(\sigma,w) \in \mathcal{GC}_n^1$, there are no bi-runs because $w_1 = w_2 = \cdots = w_n$ and σ could be any permutation in $\mathcal{S}_n$. There are $4n!$ Type 1 generalized bi-run clusters of size n and then we have

$$GC_n^1(x_2,x_3,x_4) = \sum_{(\sigma,w)\in\mathcal{GC}_n^{\mathrm{J}}} (-1)^{\mathrm{B}(\sigma,w)-1} x_2^0 x_3^0 x_4^0 = 4(-1)^{n-1} n!,$$

and

$$GC^1(x_2,x_3,x_4,t) = \sum_{n\geq 1} \frac{t^n}{n!} GC^1_n(x_2,x_3,x_4) = -4\sum_{n\geq 1}(-t)^n = \frac{4t}{1+t}.$$

For any arbitrary $(\sigma, w) \in \mathcal{GC}^2_n$, there are exactly one 2-run which has to be marked here, and no other bi-runs. Apparently, there are $n-1$ blocks in (σ, w), which means we have $n-1$ choices for the position of the 2-run. We have $\binom{4}{2}$ ways to choose colors. The permutation could be arbitrary except for the two elements in the 2-run where the two elements have to be increasing. Therefore, there are $(n-1)\binom{4}{2}\binom{n}{2}(n-2)!$ Type 2 generalized bi-run clusters of size n, which implies

$$GC^2_n(x_2,x_3,x_4) = \sum_{(\sigma,w)\in\mathcal{GC}^2_n} (-1)^{\mathrm{B}(\sigma,w)-1} x_2^1 x_3^0 x_4^0 = 3(-1)^{n-2}(n-1)n!x_2.$$

Then

$$\begin{aligned} GC^2(x_2,x_3,x_4,t) &= \sum_{n\geq 2}\frac{t^n}{n!}GC^2_n(x_2,x_3,x_4) \\ &= \sum_{n\geq 2}\frac{t^n}{n!}3(-1)^n(n-1)n!x_2 \\ &= 3x_2 t\sum_{n\geq 1} n(-t)^n \\ &= \frac{3x_2t^2}{(1+t)^2}. \end{aligned}$$

For any arbitrary $(\sigma, w) \in \mathcal{GC}^3_n$, there are exactly one 3-run and no other bi-runs. Apparently, there are $n-2$ blocks in (σ, w), which means we have $n-2$ choices for the position of the 3-run. We have $\binom{4}{3}$ ways to choose colors. The permutation could be arbitrary except for the three elements in the 3-run where the three numbers have to be increasing. Therefore, there are $(n-2)\binom{4}{3}\binom{n}{3}(n-3)!$ Type 3 generalized bi-run clusters of size n. Because there are multiple ways to mark bi-runs in a 3-run, which is given by $L_3(x_2,x_3,x_4)$, which implies

$$\begin{aligned} GC^3_n(x_2,x_3,x_4) &= 4(-1)^{n-3}(n-2)\binom{n}{3}(n-3)!L_3(x_2,x_3,x_4) \\ &= \frac{2}{3}(-1)^{n-3}(n-2)n!L_3(x_2,x_3,x_4). \end{aligned}$$

Then

$$\begin{aligned}
GC^3(x_2,x_3,x_4,t) &= \sum_{n\geq 3}\frac{t^n}{n!}GC^3_n(x_2,x_3,x_4)\\
&= L_3(x_2,x_3,x_4)\sum_{n\geq 3}\frac{t^n}{n!}\frac{2}{3}(-1)^{n-3}(n-2)n!\\
&= -\frac{2}{3}L_3(x_2,x_3,x_4)\sum_{n\geq 3}(n-2)(-t)^n\\
&= \frac{2\ L_3(x_2,x_3,x_4)\ t^3}{3(1+t)^2}\\
&= \frac{2\,(x_2^2+(x_2+1)^2x_3)\,t^3}{3(1+t)^2}.
\end{aligned}$$

For any arbitrary $(\sigma, w) \in \mathcal{GC}^4_n$, there are exactly two blocks of size 2 and no other bi-runs. Apparently, there are $n-2$ blocks in (σ, w), which means we have $\binom{n-2}{2}$ choices for the position of the 2-runs. We have $\binom{4}{3}$ ways to choose colors. The permutation could be arbitrary except for the elements in the w-run where the numbers have to be increasing. Therefore, there are $\binom{n-2}{2}\binom{4}{3}\binom{n}{2}\binom{n-2}{2}(n-4)!$ Type 4 generalized bi-run clusters of size n. Because the two 2-runs have to be labeled with 'x_2', which implies

$$\begin{aligned}
GC^4_n(x_2,x_3,x_4) &= 4(-1)^{n-3}\binom{n-2}{2}\binom{n}{2}\binom{n-2}{2}(n-4)!x_2^2\\
&= (-1)^{n-3}\binom{n-2}{2}n!\ x_2^2.
\end{aligned}$$

Then

$$\begin{aligned}
GC^4(x_2,x_3,x_4,t) &= \sum_{n\geq 4}\frac{t^n}{n!}GC^4_n(x_2,x_3,x_4)\\
&= x_2^2\sum_{n\geq 4}(-1)^{n-3}\binom{n-2}{2}t^n\\
&= -x_2^2\sum_{n\geq 4}\binom{n-2}{2}(-t)^n\\
&= -\frac{x_2^2t^4}{(1+t)^3}.
\end{aligned}$$

For any arbitrary $(\sigma, w) \in \mathcal{GC}^5_n$, there are exactly one block of 4 columns. Apparently, there are $n-3$ blocks in (σ, w), which means we have $n-3$ choices for the position of the 4-run. Since all colors must be used in Type 5 generalized bi-run clusters, we have only 1 ways to choose colors. The permutation could be arbitrary except for the four elements in the 4-run where the four numbers have to be increasing. Therefore, there are $(n-3)\binom{n}{4}(n-4)!$ Type 5 generalized bi-run clusters of size n. Because there are multiple ways to label marked bi-runs in a 4-run, which is given by $L_4(x_2, x_3, x_4)$, which implies

$$\begin{aligned} GC_n^5(x_2, x_3, x_4) &= 4(-1)^{n-4}(n-3)\binom{n}{4}(n-4)!\ L_4(x_2, x_3, x_4) \\ &= \frac{1}{24}(-1)^n(n-3)n!\ L_4(x_2, x_3, x_4). \end{aligned}$$

Then

$$\begin{aligned} GC^5(x_2, x_3, x_4, t) &= \sum_{n\geq 4} \frac{t^n}{n!} GC_n^5(x_2, x_3, x_4) \\ &= \frac{1}{24} L_4(x_2, x_3, x_4) \sum_{n\geq 4}(n-3)(-t)^n \\ &= -\frac{-t^3}{3} L_4(x_2, x_3, x_4) \sum_{n\geq 1} n(-t)^n \\ &= \frac{L_4(x_2, x_3, x_4)\ t^4}{24(1+t)^2}, \end{aligned}$$

where $L_4(x_2, x_3, x_4)$ is given by Equation (4.9).

For any arbitrary $(\sigma, w) \in \mathcal{GC}_n^6$, there are exactly three blocks of 2 columns and no other bi-runs. Apparently, there are $n-3$ blocks in (σ, w), which means we have $\binom{n-3}{3}$ choices for the positions of the three 2-runs. Since all colors must be used in Type 6 generalized bi-run clusters, we have only 1 ways to choose colors. The permutation could be arbitrary except for the four elements in the 2-runs where the numbers have to be increasing. Therefore, there are $\binom{n-3}{3}n!\frac{1}{2^3}$ Type 6 generalized bi-run clusters of size n. Because all 2-runs have to be labeled with 'x_2', which implies

$$\begin{aligned} GC_n^6(x_2, x_3, x_4) &= (-1)^{n-4}\binom{n-3}{3} n! \frac{1}{2^3} x_2^3 \\ &= \frac{1}{8}(-1)^n \binom{n-3}{3} n!\ x_2^3. \end{aligned}$$

Then

$$\begin{aligned} GC^6(x_2,x_3,x_4,t) &= \sum_{n\geq 6} \frac{t^n}{n!} GC_n^6(x_2,x_3,x_4) \\ &= \frac{x_2^3}{8} \sum_{n\geq 6} \binom{n-3}{3} (-t)^n \\ &= \frac{x_2^3\ t^6}{8(1+t)^4}. \end{aligned}$$

For any arbitrary $(\sigma, w) \in \mathcal{GC}_n^7$, there are exactly one block of 2 columns and one block of three columns and no other blocks containing bi-runs. Apparently, there are $n-3$ blocks in (σ, w), which means we have $(n-3)(n-4)$ choices for the positions of the 2-column block and 3-column block. Since all colors must be used in Type 7 generalized bi-run clusters, we have only 1 ways to choose colors. The permutation could be arbitrary except for the elements in the bi-runs where the numbers have to be increasing. Therefore, there are $\frac{1}{2!}\frac{1}{3!}(n-3)(n-4)n!$ Type 7 generalized bi-run clusters of size n. Because a 2-run has to be labeled with 'x_2' and a 3-run has labeling polynomial $L_3(x_2,x_3,x_4)$, which implies

$$\begin{aligned} GC_n^7(x_2,x_3,x_4) &= (-1)^{n-4} \frac{1}{2!}\frac{1}{3!}(n-3)(n-4)n! x_2 L_3(x_2,x_3,x_4) \\ &= \frac{1}{12}(-1)^n (n-3)(n-4)n!\ x_2 L_3(x_2,x_3,x_4). \end{aligned}$$

Then

$$\begin{aligned} GC^7(x_2,x_3,x_4,t) &= \sum_{n\geq 5} \frac{t^n}{n!} GC_n^7(x_2,x_3,x_4) \\ &= \frac{x_2 L_3(x_2,x_3,x_4)}{12} \sum_{n\geq 5} (n-3)(n-4)(-t)^n \\ &= -\frac{x_2 L_3(x_2,x_3,x_4)\ t^5}{6(1+t)^3} \\ &= -\frac{(x_2^3 + x_2(x_2+1)^2 x_3)\ t^5}{6(1+t)^3}. \end{aligned}$$

Finally, we obtain the generating function for bi-runs in $\overline{\mathcal{C}_4 \wr \mathcal{S}_n}$ as follows,

$$\begin{aligned}
& A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(x_2, x_3, x_4, t) \\
= & \frac{1}{1 - \sum_{n \geq 1} \frac{t^n}{n!} GC_n(x_2 - 1, x_3 - 1, x_4 - 1)} \\
= & \frac{1}{1 - \sum_{J \in \{1,2,\cdots,7\}} GC^{\mathrm{J}}(x_2 - 1, x_3 - 1, x_4 - 1, t)} \\
= & \frac{24(1+t)^4}{24(1+t)^4 - GC(x_2, x_3, x_4, t)},
\end{aligned}$$

where

$$\begin{aligned}
GC(x_2, x_3, x_4, t) \; = \; & 96t + (360 - 72x_2)\, t^2 + \left(416 - 112x_2 - 16x_2^2 x_3\right) t^3 \\
& + \left(161 - 59x_2 + 25x_2^2 - 30x_2^2 x_3 - x_2^3 x_3^2 x_4\right) t^4 \\
& + \left(6 - 10x_2 + 18x_2^2 - 16x_2^2 x_3 + 4x_2^3 x_3 - 2x_2^3 x_3^2 x_4\right) t^5 \\
& + \left(2x_2^2 - 3x_2^3 - 2x_2^2 x_3 + 4x_2^3 x_3 - x_2^3 x_3^2 x_4\right) t^6.
\end{aligned}$$

A few initial terms in expansion of $A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(x_2, x_3, x_4, t)$ are

$$\begin{aligned}
& 1 + 4t + \frac{1}{2!}(6x_2 + 18)t^2 + \frac{1}{3!}\left(4x_2^2 x_3 + 100x_2 + 112\right) t^3 \\
& + \frac{1}{4!}\left(x_2^3 x_3^2 x_4 + 94x_2^2 x_3 + 191x_2^2 + 1371x_2 + 935\right) t^4 \\
& + \frac{1}{5!}\left(30x_2^3 x_3^2 x_4 + 460x_2^3 x_3 + 1640x_2^2 x_3 + 7090x_2^2 + 19950x_2 + 9710\right) t^5 + \cdots .
\end{aligned}$$

By manipulating the generating function $A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(x_2, x_3, x_4, t)$ above, we could derive various results, some of which are not known before. For example,

$$\begin{aligned}
& A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(1, 0, 0, t) \\
= & -\frac{24(t+1)^2}{t^4 - 16t^3 + 72t^2 + 48t - 24} \\
= & 1 + 4t + \frac{24}{2!}t^2 + \frac{212}{3!}t^3 + \frac{2497}{4!}t^4 + \frac{36750}{5!}t^5 + \frac{649130}{6!}t^6 + \frac{13376160}{7!}t^7 \\
& + \frac{315015190}{8!}t^8 + \frac{8346046800}{9!}t^9 + \cdots ,
\end{aligned}$$

which counts the number of restricted 4-colored permutations that don't have a

3-run.

$$
\begin{aligned}
& A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(1,1,0,t) - A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(1,0,0,t) \\
= \; & \frac{48(t-8)t^3(t+1)^2}{(t^4-72t^2-48t+24)\,(t^4-16t^3+72t^2+48t-24)} \\
= \; & \frac{4}{3!}t^3 + \frac{94}{4!}t^4 + \frac{2100}{5!}t^5 + \frac{49900}{6!}t^6 + \frac{1297800}{7!}t^7 + \frac{37023840}{8!}t^8 \\
& + \frac{1155336000}{9!}t^9 + \cdots ,
\end{aligned}
$$

which counts the number of 4-colored permutations in $\overline{\mathcal{C}_4 \wr \mathcal{S}_n}$ that have at least one 3-run but don't have a 4-run.

$$
\begin{aligned}
& \left. \frac{\partial A_{\text{bi-run},\overline{\mathcal{C}_4 \wr \mathcal{S}}}(x_2,1,1,t)}{\partial x_2} \right|_{x_2=1} \\
= \; & \frac{3t^2}{(1-3t)^2} \\
= \; & \frac{6}{2!}t^2 + \frac{108}{3!}t^3 + \frac{1944}{4!}t^4 + \frac{38880}{5!}t^5 + \frac{874800}{6!}t^6 + \frac{22044960}{8!}t^8 \\
& + \frac{617258880}{8!}t^8 + \cdots ,
\end{aligned}
$$

which counts the total number of 2-runs in all the restricted 4-colored permutations of length n. It is easy to see that the average number of 2-runs in a random colored permutation in $\overline{\mathcal{C}_4 \wr \mathcal{S}_n}$ is

$$
\frac{(n-1)\cdot 3^{n-1}\cdot n!}{|\overline{\mathcal{C}_4 \wr \mathcal{S}}|} = \frac{(n-1)\cdot 3^{n-1}\cdot n!}{4\cdot 3^{n-1}\cdot n!} = \frac{n-1}{4}.
$$

Although there are many similar interesting results derived from our generating function for bi-runs, we will not expand the discussion here.

4.2.2 Exact patterns in Carlitz integer compositions

An integer composition w of n is a sequence of positive integers whose sum is equal to n, denoted by $w \vDash n$. The length of w is the number of parts in w, denoted by Len(w). For example, $w = 2\ 1\ 1\ 3 \vDash 7$ has four parts. It is well-known that the number of compositions of n into k parts is

$$
\binom{n-1}{k-1}
$$

and the total number of compositions of n is 2^{n-1}. We denote the set of all the integer compositions of n by $\mathcal{L}_n$.

Consecutive patterns in integer compositions, as a research topic, has been studied for a while (see [21] [22] [42]). However, not much work has been done on pattern enumeration in restricted compositions.

A well-known class of restricted of restricted composition is called Carlitz compositions, first introduced by Carlitz [10] and then further studied by Knopfmacher and Prodinger [36]. A Carlitz composition is a composition where adjacent parts have to be different. In other words, Carlitz compositions are integer compositions equipped with a binary relation $\mathscr{R}$ which requires adjacent elements to be different. For example, $w = 2\ 1\ 2$ is a Carlitz composition of 5 while $u = 2\ 2\ 1$ is not a Carlitz composition. We use $\mathcal{CL}_n$ to denote the set of all Carlitz compositions of n and

$$\mathcal{CL} := \biguplus_{n \geq 1} \mathcal{CL}_n.$$

For any $w \in \mathcal{CL}$, the number of parts of w is denoted by $\mathrm{Len}(w)$ and the sum of elements in w is denoted by $\mathrm{Sum}(w)$. For example, suppose $w = 2\ 3\ 1\ 2 \in \mathcal{CL}$, $\mathrm{Len}(w) = 4$ and $\mathrm{Sum}(w) = 2 + 3 + 1 + 2 = 8$.

This subsection is mainly focused on exact pattern matching in Carlitz compositions. Exact patterns are actually subwords in a integer composition. For example, suppose the pattern $u = 1\ 2$, then in a Carlitz composition $w = 2\ 1\ 2\ 3\ 4\ 3\ 1\ 2 \vDash 18$, there are two exact u-matches, i.e., $u\text{-Emch}(w) = 2$.

In general, we are interested in following generating function

$$A_{(u_1,\cdots,u_m),\mathcal{CL}}(x_1,\cdots,x_m,q,t) := \\ 1 + \sum_{w \in \mathcal{CL}} \left(\prod_{i \geq 1}^{m} x_i^{u_i\text{-Emch}(w)} \right) q^{\mathrm{Sum}(w)} t^{\mathrm{Len}(w)}, \quad (4.10)$$

where q is used to keep track of sum of the compositions and t is used to keep track of the number of parts in w.

One can easily modifiy the proof of Theorem 4.3 to prove that

$$A_{u_1,\cdots,u_m,\mathcal{P}}(x_1,\cdots,x_m,q,t) = \frac{1}{1-\sum_{n\geq 1} EGC_{n,u_1,\ldots,u_m}(x_1-1,\cdots,x_m-1,q)t^n}, \quad (4.11)$$

where $EGC_{n,u_1,\ldots,u_m}(x_1,x_2,\cdots,x_m,q)$ is exact generalized joint $(u_1,u_2,\cdots,u_m)$-clusters polynomial of size n in which q is an additional variable we use to keep track of sum of elements in the corresponding exact generalized $(u_1,u_2,\cdots,u_m)$-cluster.

Before being able to compute polynomial $EGC_{n,u_1,\ldots,u_m}(x_1,\cdots,x_m)$, we must study the structure of exact $(u_1,u_2,\cdots,u_m)$-clusters and exact joint generalized $(u_1,u_2,\cdots,u_m)$-clusters.

Given compositions $u_1,u_2,\cdots,u_m$, an exact joint $(u_1,u_2,\cdots,u_m)$-cluster of length n is a composition w consisting of n parts such that

1. each part of w is contained in some marked exact $(u_1,u_2,\cdots,u_m)$-matches and
2. any two consecutive marked exact $(u_1,u_2,\cdots,u_m)$-matches share at least one part.

For our running example in this subsection, we let $u_1 = 12$, $u_2 = 123$ and $u_3 = 3$. In Figure 4.14, v is a (u_1,u_2,u_3)-cluster of length 2, q is a (u_1,u_2,u_3)-cluster of length 3 while s is not a (u_1,u_2,u_3)-cluster because marked u_1-match and marked u_3-match do not share parts. More precisely, $m_{u_1}(v) = 1$, $m_{u_2}(v) = m_{u_3}(v) = 0$ and $m_{u_1}(q) = m_{u_2}(q) = m_{u_3}(q) = 1$.

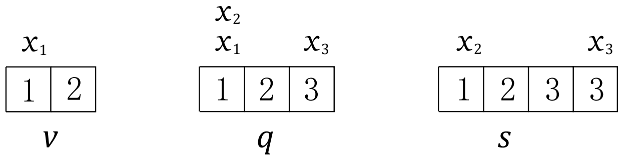


Figure 4.14: v and q are clusters while s is not.

Next we consider exact generalized joint $(u_1,u_2,\cdots,u_m)$-clusters. A composition w is called an exact generalized joint $(u_1,u_2,\cdots,u_m)$-cluster of length n for

Carlitz compositions if we can write $w = B_1 B_2 \cdots B_j$ where B_i are blocks such that

1. B_i is a joint $(u_1, u_2, \cdots, u_m)$-cluster or a single element
2. for $1 \leq i < j$, the last part of B_i is equal to the part of B_{i+1}. In other words, if we combine blocks B_i and B_{i+1}, we do not have a Carlitz composition.

As pictured in Figure 4.15, w and r are both exact generalized joint (u_1, u_2, u_3)-clusters of length 9, where $\text{Sum}(w) = 14$, $m_{u_1}(w) = 1$, and $m_{u_2}(w) = m_{u_3}(w) = 0$, and $\text{Sum}(r) = 20$, $m_{u_1}(r) = m_{u_2}(r) = 1$ and $m_{u_3}(r) = 3$. We let $\mathcal{GC}_n$ denote the set of all the exact generalized joint (u_1, u_2, u_3)-clusters of length n.

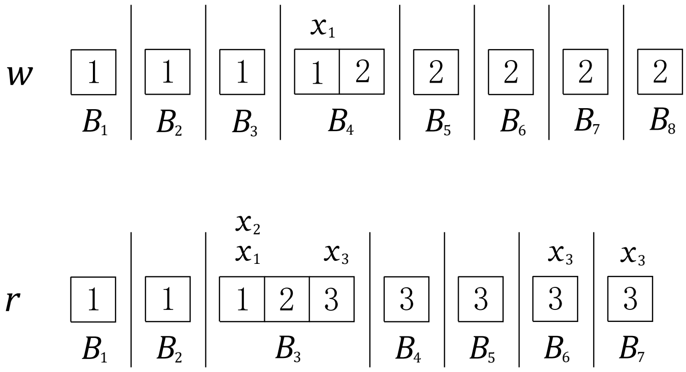


Figure 4.15: $w, r \in \mathcal{GC}_9$.

Next, we shall continue using $u_1 = 1\ 2$, $u_2 = 1\ 2\ 3$ and $u_3 = 3$ as the patterns to demonstrate how we compute the generalized joint cluster polynomials and then hence obtain the generating functions.

The generalized joint cluster polynomials that we are interested in is

$$GC_n(x_1, x_2, x_3, q) := \sum_{w \in \mathcal{GC}_n} (-1)^{\text{B}(w)-1}\, x_1^{m_{u_1}(w)} x_2^{m_{u_2}(w)} x_3^{m_{u_3}(w)} q^{\text{Sum}(w)},$$

where $\text{B}(w)$ is the number of blocks in w and $m_{u_i}(w)$ is the number of exact matches of u_i which are marked with x_i.

Regardless of labellings, there are only three types of exact (u_1, u_2, u_3)-clusters, namely, u_1-cluster, u_2-cluster and u_3-cluster, pictured in Figure 4.16. It is because

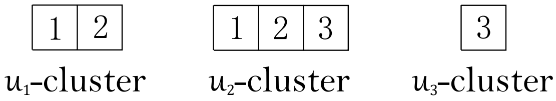


Figure 4.16: Three types of (u_1, u_2, u_3)-clusters

all these three patterns are impossible to be concatenated and then hence each type of cluster contains exactly one pattern.

Since an exact generalized joint (u_1, u_2, u_3)-cluster consists of singletons and exact joint (u_1, u_2, u_3)-clusters, there are only four types of exact generalized joint (u_1, u_2, u_3)-clusters. Note that the restriction of Carlitz compositions is that adjacent parts have to be different and therefore, to violate this restriction, we must last element of any block is equal to the first element of the next block.

- Type 1 exact generalized joint (u_1, u_2, u_3)-clusters are the ones which contain exactly one u_1-cluster, as pictured in Figure 4.17 (omitting labellings). The set of such exact generalized joint (u_1, u_2, u_3)-clusters of length n is denoted by $\mathcal{GC}_n^1$.

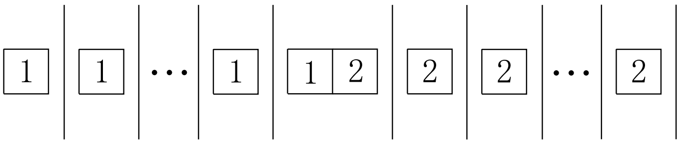


Figure 4.17: Type 1 generalized clusters.

- Type 2 exact generalized joint (u_1, u_2, u_3)-clusters are the ones that contain exactly one u_2-cluster, as pictured in Figure 4.18 (omitting labellings). Note that in Type 2 exact generalized joint (u_1, u_2, u_3)-clusters, there could be marked u_3-matches. The set of such exact generalized joint (u_1, u_2, u_3)-clusters of length n is denoted by $\mathcal{GC}_n^2$.

- Type 3 exact generalized joint (u_1, u_2, u_3)-clusters are the ones that contain at least one u_3-cluster but no u_1-clusters or u_2 clusters, as pictured in Figure 4.19. In other words, there is at least one '3' in this type of generalized

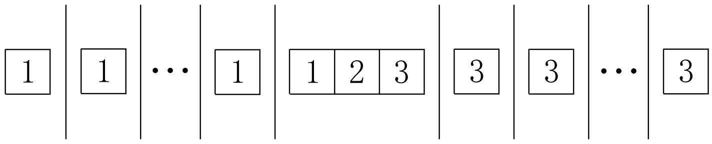


Figure 4.18: Type 2 generalized clusters.

clusters is marked. The set of such generalized clusters of length n is denoted by $\mathcal{GC}_n^3$.

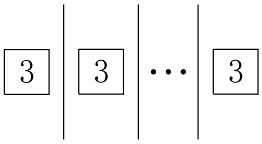


Figure 4.19: Type 3 generalized clusters.

- Type 4 exact generalized joint (u_1, u_2, u_3)-clusters are the ones that do not contain any clusters, as pictured in Figure 4.20 where i can be any positive integer, including 3. When $i = 3$, not marked '3' with x_3 is allowed. The set of such exact generalized joint (u_1, u_2, u_3)-clusters of length n is denoted by $\mathcal{GC}_n^4$.

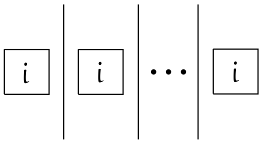


Figure 4.20: Type 4 generalized clusters.

We then set

$$GC_n^{\mathrm{J}}(x_1, x_2, x_3, q) := \sum_{w \in \mathcal{GC}_n^{\mathrm{J}}} (-1)^{\mathrm{B}(w)-1}\, x_1^{m_{u_1}(w)} x_2^{m_{u_2}(w)} x_3^{m_{u_3}(w)} q^{\mathrm{Sum}(w)}, \tag{4.12}$$

$$GC^{\mathrm{J}}(x_1, x_2, x_3, q, t) := \sum_{n \geq 1} t^n GC_n^{\mathrm{J}}(x_1, x_2, x_3, q), \tag{4.13}$$

where $\mathrm{J} \in \{1, 2, 3, 4\}$. Clearly, $\mathcal{GC}_n^{\mathrm{J}}$ are disjoint sets, i.e.,

$$\mathcal{GC}_n = \biguplus_{\mathrm{J} \in \{1,2,3,4\}} \mathcal{G}_n^{\mathrm{J}}$$

and then

$$\begin{aligned}\sum_{n\geq 1} t^n GC_n(x_1,x_2,x_3,q) &= \sum_{n\geq 1} t^n \sum_{\mathrm{J}\in\{1,2,3,4\}} GC_n^{\mathrm{J}}(x_1,x_2,x_3,q)\\ &= \sum_{\mathrm{J}\in\{1,2,3,4\}} GC^{\mathrm{J}}(x_1,x_2,x_3,q,t).\end{aligned}$$

Next we shall consider $\mathcal{GC}_n^{\mathrm{J}}$ and compute GC_n^{J} for each J.

For an arbitrary $w \in \mathcal{GC}_n^1$, w contains one '1 2', l_1 many singletons of '1' and l_2 many singletons of '2' such that

$$2 + l_1 + l_2 = n, \quad \text{for} \quad l_1 \geq 0, \; l_2 \geq 0,$$

and then

$$\mathrm{Sum}(w) = 1 + 2 + l_1 + 2l_2 = 3 + l_1 + 2l_2,$$

which gives us for $n \geq 2$,

$$\begin{aligned}GC_n^1(x_1,x_2,x_3,q) &= \sum_{l_1+l_2=n-2,l_1,l_2\geq 0} (-1)^{l_1+l_2}\; x_1 q^{3+l_1+2l_2}\\ &= (-1)^{n-2} x_1 q^{3+n-2} \sum_{l_2=0}^{n-2} q^{l_2}\\ &= (-1)^n x_1 \frac{q^{n+1}-q^{2n}}{1-q}.\end{aligned}$$

Thus

$$\begin{aligned}GC^1(x_1,x_2,x_3,q,t) &= \sum_{n\geq 2} t^n GC_n^1(x_1,x_2,x_3,q)\\ &= \sum_{n\geq 2} t^n (-1)^n x_1 \frac{q^{n+1}-q^{2n}}{1-q}\\ &= \frac{x_1q^3}{1-q}\left(t^2\sum_{n\geq 0}(-qt)^n - qt^2\sum_{n\geq 0}(-q^2t)^n\right)\\ &= \frac{x_1q^3}{1-q}\left(\frac{t^2}{1+qt} - \frac{qt^2}{1+q^2t}\right)\\ &= \frac{x_1q^3t^2}{(1+qt)\,(1+q^2t)}. \qquad (4.14)\end{aligned}$$

Now we move to computation of $GC^2(x_1, x_2, x_3, t)$. For an arbitrary $w \in \mathcal{GC}_n^2$, w contains one '1 2 3', l_1 many singletons of '1' and l_3 many singletons of '3' such that

$$3 + l_1 + l_3 = n, \quad \text{for} \;\; l_1 \geq 0, \;\; l_3 \geq 0,$$

and then

$$\text{Sum}(w) = 1 + 2 + 3 + l_1 + 3l_3 = 6 + l_1 + 3l_3.$$

Unlike the Type 1 exact generalized joint (u_1, u_2, u_3)-clusters, Typer 2 exact generalized joint (u_1, u_2, u_3)-clusters contains a block '1 2 3' which could be marked in several different ways. It is easy to see that 1 must be marked with x_2, but then we can mark 1 with either x_2 or not and mark 3 with either x_3 or not. THus the labeling polynomial for u_2 is $(x_1+1)x_2(x_3+1)$. For each singleton of '3', we could either mark it with 'x_3' or not.

Then for $n \geq 3$, we have

$$\begin{aligned}
& GC_n^2(x_1, x_2, x_3, q) \\
= & \sum_{l_1+l_3-n-3, l_1, l_3 \geq 0} (-1)^{l_1+l_3} \; (x_1+1)x_2(x_3+1)(x_3+1)^{l_3} q^{6+l_1+3l_3} \\
= & (-1)^{n-3} q^{6+n-3} (x_1+1)x_2(x_3+1) \sum_{l_3=0}^{n-3} ((x_3+1)q^2)^{l_3} \\
= & \frac{(x_1+1)x_2(x_3+1)(-q)^{n+3} \left(1 - ((x_3+1)q^2)^{n-2}\right)}{1-(x_3+1)q^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
& GC^2(x_1, x_2, x_3, q, t) \qquad (4.15) \\
= & \sum_{n \geq 3} t^n GC_n^2(x_1, x_2, x_3, q) \\
= & \sum_{n \geq 3} t^n \frac{(x_1+1)x_2(x_3+1)(-q)^{n+3} \left(1 - ((x_3+1)q^2)^{n-2}\right)}{1-(x_3+1)q^2} \\
= & \frac{(x_1+1)x_2(x_3+1)q^6t^3}{1-(x_3+1)q^2} \left(\sum_{n \geq 0} (-qt)^n - (x_3+1)q^2 \left(-(x_3+1)q^3t\right)^n \right) \\
= & \frac{(x_1+1)x_2(x_3+1)q^6t^3}{1-(x_3+1)q^2} \left(\frac{1}{1+qt} - \frac{(x_3+1)q^2}{1+(x_3+1)q^3t} \right) \\
= & \frac{(x_1+1)x_2(x_3+1)q^6t^3}{(1+qt)\left(1+(x_3+1)q^3t\right)} \qquad (4.16)
\end{aligned}$$

Next we shall compute $GC^3(x_1,x_2,x_3,q,t)$. For an arbitrary $w \in \mathcal{GC}_n^3$, w only contains n singleton '3' blocks. Each '3' block is either marked or not but at least one block has to be marked. Thus

$$GC_n^3(x_1,x_2,x_3,q) = (-1)^{n-1}\left((1+x_3)^n - 1\right) q^{3n}.$$

Hence,

$$\begin{aligned}
GC^3(x_1,x_2,x_3,q,t) &= \sum_{n\geq 1} t^n \; GC_n^3(x_1,x_2,x_3,q) \\
&= \sum_{n\geq 1} t^n (-1)^{n-1}\left((1+x_3)^n - 1\right) q^{3n} \\
&= -\sum_{n\geq 1} \left(-(x_3+1)q^3t\right)^n - (-q^3t)^n \\
&= -\left(\frac{-(x_3+1)q^3t}{1+(x_3+1)q^3t} - \frac{-q^3t}{1+q^3t}\right) \\
&= \frac{x_3q^3t}{(1+q^3t)\left(1+(x_3+1)q^3t\right)} \qquad (4.17)
\end{aligned}$$

Finally we shall compute $GC^4(x_1,x_2,x_3,q,t)$. For an arbitrary $w \in \mathcal{GC}_n^4$, w has no marked (u_1,u_2,u_3)-matches, which implies

$$x_1^{m_{u_1}(w)} x_2^{m_{u_2}(w)} x_3^{m_{u_3}(w)} = 1.$$

Since elements in each block in w are identical, say the element in the block is k, then $\mathrm{Sum}(w) = kn$, which gives

$$\begin{aligned}
GC_n^4(x_1,x_2,x_3,q) &= \sum_{w\in\mathcal{GC}_n^4} (-1)^{\mathrm{B}(w)-1} x_1^{m_{u_1}(w)} x_2^{m_{u_2}(w)} x_3^{m_{u_3}(w)} q^{\mathrm{Sum}(w)} \\
&= (-1)^{n-1} \sum_{k=1} q^{kn} \\
&= -\frac{(-q)^n}{1+q^n}.
\end{aligned}$$

Hence

$$\begin{aligned}
GC^4(x_1,x_2,x_3,q,t) &= \sum_{n\geq 1} t^n \; GC_n^4(x_1,x_2,x_3,q) \\
&= -\sum_{n\geq 1} t^n \frac{(-q)^n}{1+q^n} \\
&= -\sum_{n\geq 1} \frac{(-qt)^n}{1+q^n} \qquad (4.18)
\end{aligned}$$

Taking sum of Equation (4.14), (4.15), (4.17) and (4.18), plugging into the generating function $A_{12,123,3,\mathcal{CL}}(x_1, x_2, x_3, q, t)$, we have

$$\begin{aligned}
&A_{12,123,3,\mathcal{CL}}(x_1, x_2, x_3, q, t)\\
= \;& \frac{1}{1-\sum_{n\geq 1} t^n \cdot GC_n(x_1-1, x_2-1, x_3-1)}\\
= \;& \frac{1}{1-\sum_{\mathrm{J}\in\{1,2,3,4\}} GC^{\mathrm{J}}(x_1-1, x_2-1, x_3-1, q, t)}\\
= \;& \frac{1}{1-GC(x_1-1, x_2-1, x_3-1, q, t)},
\end{aligned}$$

where

$$\begin{aligned}
&G(x_1-1, x_2-1, x_3-1, q, t)\\
= \;& \sum_{\mathrm{J}\in\{1,2,3,4\}} GC^{\mathrm{J}}(x_1-1, x_2-1, x_3-1, q, t)\\
= \;& \frac{(x_3-1)q^3t}{(1+q^3t)\,(1+x_3q^3t)} + \frac{x_1(x_2-1)x_3q^6t^3}{(1+qt)\,(1+x_3q^3t)} + \frac{(x_1-1)q^3t^2}{(qt+1)\,(1+q^2t)} - \sum_{n\geq 1} \frac{(-qt)^n}{1+q^n}.
\end{aligned}$$

A few initial terms in the expansion are

$$\begin{aligned}
&A_{12,123,3,\mathcal{CL}}(x_1, x_2, x_3, q, t)\\
= \;& 1 + tq + tq^2 + (t^2 + x_1t^2 + x_3t)q^3 + (t + x_1t^3 + 2x_3t^2)q^4\\
&+(t + 2t^2 + x_1t^3 + 2x_3t^2 + x_3t^3)q^5\\
&+(t + 4t^2 + t^3 + x_1t^4 + x_1^2t^4 + 4x_3t^3 + x_1x_3t^3 + x_1x_2x_3t^3)q^6\\
&+\cdots.
\end{aligned}$$

Coefficients of q^n describe distribution of patterns in all the Carlitz compositions of n. For example, coefficient of q^4 is $(t + x_1t^3 + 2x_3t^2)$, which means in $\mathcal{P}_5$, there is one Carlitz composition having one part avoiding u_1, u_2 and u_3, namely 4, one Carlitz composition having three parts, exactly one u_1-match and no other matches, namely 121, and two Carlitz compositions having two parts, exactly one u_3-match and no other matches, namely 13 and 31.

The generating function $A_{12,123,3,\mathcal{CL}}(x_1, x_2, x_3, q, t)$ gives us many interesting results, some of which lead to integer sequences that haven't been recorded on

OEIS [49]. For example,

$$
\begin{aligned}
& A_{12,123,3,\mathcal{CL}}(1,1,0,q,1) \\
= & \frac{1}{1+\frac{q^3}{q^3+1}-\sum_{n\geq 1}\frac{q^n}{q^n+1}} \\
= & \frac{1}{1-\sum_{n\geq 1,\ n\neq 3}\frac{q^n}{q^n+1}} \\
= & 1+q+q^2+2q^3+2q^4+4q^5+8q^6+13q^7+20q^8+32q^9+51q^{10} \\
& +82q^{11}+137q^{12}+224q^{13}+362q^{14}+588q^{15}+\cdots,
\end{aligned}
$$

where the coefficients of q^n in $A_{12,123,3,\mathcal{P}}(1,1,0,q,1)$ is the the number of Carlitz compositions of n that do not contain '3' as a part, and they are 1, 1, 2, 2, 4, 8, 13, 20, 32, $\cdots$, for $n \geq 1$.

Another example is

$$
\begin{aligned}
& A_{12,123,3,\mathcal{CL}}(1,0,1,q,1)-A_{12,123,3,\mathcal{CL}}(0,0,0,q,1) \\
= & \frac{1}{1+\frac{q^6}{q^4+q^3+q+1}-\sum_{n=1}^{\infty}\frac{q^n}{q^n+1}}-\frac{1}{1+\frac{q^3}{q^3+1}+\frac{q^3}{q^3+q^2+q+1}-\sum_{n=1}^{\infty}\frac{q^n}{q^n+1}} \\
= & 2q^3+3q^4+4q^5+7q^6+12q^7+24q^8+48q^9+89q^{10}+152q^{11} \\
& +268q^{12}+478q^{13}+852q^{14}+1524q^{15}+2699q^{16}+\cdots,
\end{aligned}
$$

the coefficients of q^n in $A_{12,123,3,\mathcal{CL}}(1,0,1,q,1)-A_{12,123,3,\mathcal{CL}}(0,0,0,q,1)$ is the the number of Carlitz compositions of n that contain '1 2' or '3' but do not contain subword '1 2 3', and they are 2, 3, 4, 7, 12, 24, 48, 89, $\cdots$, for $n \geq 3$.

Furthermore, since t is used to keep track of number of parts in compositions, if we substitute t by 1 and -1 respectively and take difference of the two functions, we are able to get the generating function for the number of Carlitz compositions of n that have odd number of parts, contain '1 2' or '3' but do not contain subword

'1 2 3'.

$$\begin{aligned}
& \frac{1}{2}A_{12,123,3,\mathcal{CL}}(1,0,1,q,1) - \frac{1}{2}A_{12,123,3,\mathcal{CL}}(0,0,0,q,1) \\
& -\frac{1}{2}A_{12,123,3,\mathcal{CL}}(1,0,1,q,-1) + \frac{1}{2}A_{12,123,3,\mathcal{CL}}(0,0,0,q,-1) \\
= & \frac{1/2}{1+\frac{q^6}{q^4+q^3+t+1}-\sum_{n=1}^{\infty}\frac{q^n}{q^n+1}} - \frac{1/2}{1+\frac{q^3}{q^3+1}+\frac{q^3}{q^3+q^2+q+1}-\sum_{n=1}^{\infty}\frac{q^n}{q^n+1}} \\
& -\frac{1/2}{1-\frac{q^4(q^6+q^5+2q^4+q^2-q-1)}{q^7-q^4-q^3+1}-\sum_{n=1}^{\infty}\frac{q^n}{q^n+1}} + \frac{1/2}{1+\frac{q^3(q(2q-1)+2)}{q^5+q^3+q^2+1}-\sum_{n=1}^{\infty}\frac{q^n}{q^n+1}} \\
= & q^3+q^4+2q^5+5q^6+5q^7+12q^8+25q^9+41q^{10}+78q^{11}+137q^{12} \\
& +236q^{13}+426q^{14}+764q^{15}+\cdots,
\end{aligned}$$

where coefficient of q^n is the number of Carlitz compositions of n that have odd number of parts, contain '1 2' or '3' but do not contain subword '1 2 3'.

There are many other interesting integer sequences derived from this generating function, but we shall not pursue such results here.

4.3 Joint clusters and generalized joint clusters for $D^{i,j,k}_{i+kn+j}$.

We can also extend joint clusters and generalized joint clusters to fillings of $D^{i,j,k}_{i+kn+j}$, for $i,j \neq k$.

Suppose that we have m patterns $\{P_s \in \mathcal{P}^{i,j,k}_{i+kn_s+j}\}_{1\le s\le m}$ and a given binary relation $\mathscr{R}$, we consider the following generating function,

$$A_{(P_1,\ldots,P_m),\mathcal{P},\mathscr{R}}(x_1,\ldots,x_m,t) := 1+\sum_{n\ge 1}\frac{t^{kn}}{(kn)!}\sum_{F\in\mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}}\prod_{s=1}^{m}x_s^{P_s\text{-mch}(F)}. \tag{4.19}$$

Similar to Section 3.3, but we define joint versions of generalized start, end, and start-end clusters. First we let $\mathcal{MP}^{i,j,k}_{i+kn+j,(P_1,\ldots,P_m)}$ denote the set of fillings $F \in \mathcal{P}^{i,j,k}_{i+kn+j}$ where we have marked some of the P_s-match in F by placing an 'x_s' on the top of the column that starts a P_s-match in F, for $1 \le s \le m$.

Definition 4.5. *$Q \in \mathcal{MP}^{0,j,k}_{kn+j,(P_1,\ldots,P_m)}$ is a* ***generalized*** *$(P_1,\ldots,P_m),\mathscr{R}$-**end-cluster** if we can write $Q = B_1B_2\cdots B_h$ where B_s are blocks of consecutive* *columns in Q such that*

1. *B_h is a column of height j,*

2. *for $1 \leq s < h$, either B_s is a single column or B_s consists of r-columns where $r \geq 2$, $\mathrm{red}(B_s)$ is a $(P_1, \ldots, P_m)$-cluster in $\mathcal{MP}_{kr,(P_1,\ldots,P_m)}$, and any pair of consecutive columns in B_s are in $\mathscr{R}$ and*

3. *for $1 \leq s \leq h-1$, the pair $(last(B_s), first(B_{s+1}))$ is not in $\mathscr{R}$ where for any s, $last(B_s)$ is the right-most column of B_s and $first(B_s)$ is the left-most column of B_s.*

Let $\mathcal{GEC}^{0,j,k}_{kn+j,(P_1,\ldots,P_m),\mathscr{R}}$ denote the set of all generalized $(P_1, \ldots, P_m), \mathscr{R}$-end-clusters which have n columns of height k followed by a column of height j. We let

$$GEC^{0,j,k}_{kn+j,(P_1,\ldots,P_m),\mathscr{R}}(x_1, \ldots, x_m) := \sum_{Q \in \mathcal{GEC}^{0,j,k}_{kn+j,(P_1,\ldots,P_m),\mathscr{R}}} (-1)^{\mathrm{B}(Q)-1} \prod_{s=1}^{m} x_s^{m_{P_s}(Q)}, \tag{4.20}$$

where $\mathrm{B}(Q)$ the number of blocks in Q and $m_{P_s}(Q)$ is the number of P_s-matches in in Q which are marked with an x_s.

Definition 4.6. *$Q \in \mathcal{MP}^{i,0,k}_{i+kn,(P_1,\ldots,P_m)}$ is a **generalized** $(P_1, \ldots, P_m), \mathscr{R}$**-start-cluster** if we can write $Q = B_1 B_2 \cdots B_h$ where B_s are blocks of consecutive columns in Q such that*

1. *B_1 is a column of height i,*

2. *for $1 < s \leq h$, either B_s is a single column or B_s consists of r-columns where $r \geq 2$, $\mathrm{red}(B_s)$ is a $(P_1, \ldots, P_m)$-cluster in $\mathcal{MP}_{kr,(P_1,\ldots,P_m)}$, and any pair of consecutive columns in B_s are in $\mathscr{R}$ and*

3. *for $1 \leq s \leq h-1$, the pair $(last(B_s), first(B_{s+1}))$ is not in $\mathscr{R}$ where for any s, $last(B_s)$ is the right-most column of B_s and $first(B_s)$ is the left-most column of B_s.*

Let $\mathcal{GSC}^{i,0,k}_{i+kn,(P_1,\ldots,P_m),\mathscr{R}}$ denote the set of all generalized $(P_1,\ldots,P_m),\mathscr{R}$-start-clusters which have a column of height i followed by n columns of height k. We let

$$GSC^{i,0,k}_{i+kn,(P_1,\ldots,P_m),\mathscr{R}}(x_1,\ldots,x_m) := \sum_{Q\in\mathcal{GSC}^{i,0,k}_{i+kn,(P_1,\ldots,P_m),\mathscr{R}}} (-1)^{\mathrm{B}(Q)-1}\prod_{s=1}^{m} x_s^{m_{P_s}(Q)}, \tag{4.21}$$

where $\mathrm{B}(Q)$ the number of blocks in Q and $m_{P_s}(Q)$ is the number of P_s-matches in in Q which are marked with an x_s.

Definition 4.7. *$Q \in \mathcal{MP}^{i,j,k}_{i+kn+j,(P_1,\ldots,P_m)}$ is a* ***generalized*** *$(P_1,\ldots,P_m)$,**$\mathscr{R}$-start-end-cluster** if we can write $Q = B_1B_2\cdots B_h$ where $h \geq 2$ and B_s are blocks of consecutive columns in Q such that*

1. *B_1 is a column of height i,*
2. *B_h is a column of height j,*
3. *for $2 \leq s \leq h-1$, either B_s is a single column or B_s consists of r-columns where $r \geq 2$, $\mathrm{red}(B_s)$ is a $(P_1,\ldots,P_m)$-cluster in $\mathcal{MP}_{kr,(P_1,\ldots,P_m)}$, and any pair of consecutive columns in B_s are in $\mathscr{R}$ and*
4. *for $1 \leq s \leq h-1$, the pair $(last(B_s), first(B_{s+1}))$ is not in $\mathscr{R}$ where for any s, $last(B_s)$ is the right-most column of B_s and $first(B_s)$ is the left-most column of B_s.*

Let $\mathcal{GSEC}^{i,0,k}_{i+kn+j,(P_1,\ldots,P_m),\mathscr{R}}$ denote the set of all generalized $(P_1,\ldots,P_m),\mathscr{R}$-start-clusters which have a column of height i followed by n columns of height k and then followed by a column of height j. We let

$$GSEC^{i,j,k}_{i+kn+j,(P_1,\ldots,P_m),\mathscr{R}}(x_1,\ldots,x_m) := \sum_{Q\in\mathcal{GSEC}^{i,j,k}_{i+kn+j,(P_1,\ldots,P_m),\mathscr{R}}} (-1)^{\mathrm{B}(Q)-1}\prod_{s=1}^{m} x_s^{m_{P_s}(Q)}, \tag{4.22}$$

where $\mathrm{B}(Q)$ the number of blocks in Q and $m_{P_s}(Q)$ is the number of P_s-matches in in Q which are marked with an x_s.

Then we have following theorems.

Theorem 4.8. *Given a binary relation* $\mathscr{R}$ *and patterns* $P_1, \ldots, P_m$ *, then*

$$\sum_{n\geq 0} \frac{t^{kn+j}}{(kn+j)!} \sum_{F \in \mathcal{P}^{0,j,k}_{kn+j,\mathscr{R}}} \prod_{s=1}^{m} x_s^{P_s\text{-}mch(F)} = \frac{\left(\sum_{n\geq 0} \frac{t^{kn+j}}{(kn+j)!} GEC^{0,j,k}_{kn+j,(P_1,\ldots,P_m)\mathscr{R}}(x_1-1,\ldots,x_m-1)\right)}{1-\sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,(P_1,\ldots,P_m),\mathscr{R}}(x_1-1,\ldots,x_m-1)} \tag{4.23}$$

Theorem 4.9. *Given a binary relation* $\mathscr{R}$ *and patterns* $P_1, \ldots, P_m$ *, then*

$$\sum_{n\geq 0} \frac{t^{i+kn}}{(i+kn)!} \sum_{F \in \mathcal{P}^{i,0,k}_{i+kn,\mathscr{R}}} \prod_{s=1}^{m} x_s^{P_s\text{-}mch(F)} = \frac{\left(\sum_{n\geq 0} \frac{t^{i+kn}}{(i+kn)!} GSC^{i,0,k}_{i+kn,(P_1,\ldots,P_m)\mathscr{R}}(x_1-1,\ldots,x_m-1)\right)}{1-\sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,(P_1,\ldots,P_m),\mathscr{R}}(x_1-1,\ldots,x_m-1)} \tag{4.24}$$

Theorem 4.10. *Given a binary relation* $\mathscr{R}$ *and patterns* $P_1, \ldots, P_m$ *, then*

$$\begin{aligned}
\sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} \sum_{F \in \mathcal{P}^{i,j,k}_{i+kn+j,\mathscr{R}}} \prod_{s=1}^{m} x_s^{P_s\text{-}mch(F)} = \\
& \frac{1}{1-\sum_{n\geq 1} \frac{t^{kn}}{(kn)!} GC^{0,0,k}_{kn,(P_1,\ldots,P_m),\mathscr{R}}(x_1-1,\ldots,x_m-1)} \\
& \times \left(\sum_{n\geq 0} \frac{t^{i+kn}}{(i+kn)!} GSC^{i,0,k}_{i+kn,(P_1,\ldots,P_m),\mathscr{R}}(x_1-1,\ldots,x_m-1)\right) \\
& \times \left(\sum_{n\geq 0} \frac{t^{kn+j}}{(kn+j)!} GEC^{0,j,k}_{kn+j,(P_1,\ldots,P_m)\mathscr{R}}(x_1-1,\ldots,x_m-1)\right) \\
& + \sum_{n\geq 0} \frac{t^{i+kn+j}}{(i+kn+j)!} GSEC^{i,j,k}_{i+kn+j,(P_1,\ldots,P_m),\mathscr{R}}(x_1-1,\ldots,x_m-1).
\end{aligned} \tag{4.25}$$

The contents of Chapter 4 are currently under preparation for submission. Some portion is co-authored with J. B. Remmel. The dissertation author is the author of this material.

Chapter 5

Clusters and Generalized Clusters for undetermined shapes

Previous chapters are mainly focused on various fillings of rectangular shapes (i.e., $D_{kn}^{0,0,k}$) or almost rectangular shapes (i.e., $D_{i+kn+j}^{i,j,k}$). However, the idea of clusters and generalized clusters can be extended to irregular shapes or even unknown shapes naturally.

In next section, corresponding theorems for undetermined shapes shall be stated and proved. Then in Section 5.2, two examples will be given to illustrate how clusters and generalized clusters work for undetermined shapes. In Section 5.3, we discuss clusters and generalized clusters for undetermined shapes with partial restrictions.

5.1 Main theorem

In this section, we let D_n denote the diagrams consisting n cells. Obviously, the number of columns is at least one and at most n. We use $D_{n,k}$ to denote the set of diagrams in D_n having k columns. Clearly, for $i, j \geq 1$, $D_{i+kn+j}^{i,j,k} \subseteq D_{i+kn+j,n+2} \subseteq D_{i+kn+j}$. As pictured in Figure 5.1, $T_1 \in D_{12,6} \subseteq D_{12}$ and $T_2 \in D_{11,4} \subseteq D_{11}$.

We let $\mathcal{P}_n$ ($\mathcal{P}_{n,k}$) denote the set of all fillings of D_n ($D_{n,k}$) with the elements of $1, 2, 3 \ldots, n$ such that the elements are increasing reading from bottom to top in each column. Thus for any $F \in \mathcal{P}_{n,k}$, $w(F)$ is a permutation in $\mathcal{S}_n$ satisfying

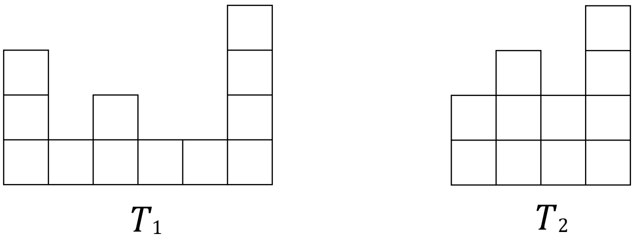
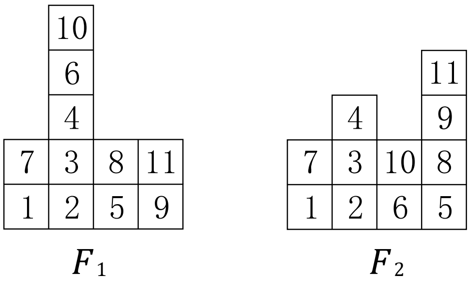


Figure 5.1: $T_1 \in D_{12,6} \subseteq D_{12}$ and $T_1 \in D_{11,4} \subseteq D_{11}$.

$\mathrm{des}(w(F)) < k$. Note that the correspondence here actually is not a bijection and counterexamples are given in Figure 5.2. Similarly, we let $\mathcal{F}_{n,A}$ ($\mathcal{F}_{n,k,A}$) denote the set of all fillings of D_n ($D_{n,k}$) with the elements from the alphabet A. Accordingly, let $\mathcal{WI}_{n,A}$ and $\mathcal{WI}_{n,k,A}$ ($\mathcal{SI}_{n,A}$ and $\mathcal{SI}_{n,k,A}$) denote the set of all fillings of D_n ($D_{n,k}$) respectively with the elements from the alphabet A such that elements are weakly (strictly) increasing in each column reading from bottom to top. In this chapter, for convenience of stating and proving theorems, major emphasis is on $\mathcal{P}_n$ but theorems and methods still hold for pattern matching in $\mathcal{F}_{n,A}$, $\mathcal{WI}_{n,A}$ and $\mathcal{SI}_{n,A}$.



$w(F_1) = w(F_2) = 1\ 7\ 2\ 3\ 4\ 6\ 10\ 5\ 8\ 9\ 11$

Figure 5.2: $F_1, F_2 \in \mathcal{P}_{11,4}$, $w(F_1) = w(F_2)$ but $F_1 \neq F_2$.

Due to the uncertainty of shapes, patterns are also allowed to be very flexible. Patterns could be associated to shapes of several consecutive columns or partial ordering of elements in consecutive columns. For instance, we might want to keep track of variety of quantities such as the number of columns of height 1 in $\mathcal{P}_n$ or $\mathcal{F}_{n,A}$, the number of pairs of adjacent columns satisfy the condition that the first element in the left column is greater than the second element in the right column, the number of pairs of adjacent columns that have the same height, the number of times we observe row-increasing condition between two columns, etc..

Generally speaking, a consecutive pattern can be understood as a condition occurring in columns or between consecutive columns. The number of pattern matches in a filling F of D_n is the number of times we observe this condition in F.

For some given pattern P and for $n \geq 1$, we let $\mathcal{MP}_{n,P}$ denote the set of all fillings $F \in \mathcal{P}_n$ where we have marked some of the P-matches in F by placing an 'x' on top of the column that start a P-match in F.

Then as in previous sections, a P-cluster of size n is a filling F in $\mathcal{MP}_{n,P}$ such that

1. every column of F is contained in a marked P-match of F and
2. any two consecutive marked P-matches share at least one column.

Due to flexibility of patterns, it is possible that a P-cluster has only one column or even one cell. We let $\mathcal{CM}_{n,P}$ denote the set of all P-clusters in $\mathcal{MP}_{n,P}$. We define the cluster polynomial for $n \geq 1$,

$$C_{n,P}(x) = \sum_{F \in \mathcal{CM}_{n,P}} x^{m_P(F)}$$

where $m_P(F)$ is the number of marked P-matches in F.

Theorem 5.1. *Given some pattern P, then*

$$\begin{aligned} 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{F \in \mathcal{P}_n} x^{P\text{-mch}(F)} &= \frac{1}{1 - \sum_{n \geq 1} \frac{t^n}{n!} \left(C_{n,P}(x-1) + |\mathcal{P}_{n,1}| \right)} && (5.1) \\ &= \frac{1}{2 - e^t - \sum_{n \geq 1} \frac{t^n}{n!} C_{n,P}(x-1)}, && (5.2) \end{aligned}$$

where $C_{n,P}(x)$ is P-cluster polynomial with respect to $\mathcal{P}_n$.

Proof. First, the equality between (5.1) and (5.2) is easy to understand. For any $F \in \mathcal{P}_{n,1}$, F has to be a single column of height n with fillings of $1, 2, \ldots, n$, reading from bottom to top. Then $|\mathcal{P}_{n,1}| = 1$ and hence,

$$\sum_{n \geq 1} \frac{t^n}{n!} |\mathcal{P}_{n,1}| = \sum_{n \geq 1} \frac{t^n}{n!} = e^t - 1.$$

If we replace x by $x+1$ in equation (5.1), we get

$$1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{F\in\mathcal{P}_n}(x+1)^{P\text{-mch}(F)}=\frac{1}{1-\sum_{n\geq 1}\frac{t^n}{n!}\left(C_{n,P}(x)+|\mathcal{P}_{n,1}|\right)} \tag{5.3}$$

As before, the left-hand side of (5.3) is the generating function of $m_P(F)$ over all $F\in\mathcal{MP}_{n,P}$.Tthat is,

$$1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{F\in\mathcal{P}_n}(x+1)^{P\text{-mch}(F)}=1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{F\in\mathcal{MP}_{n,P}}x^{m_P(F)}. \tag{5.4}$$

Thus we must show that

$$1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{F\in\mathcal{MP}_{n,P}}x^{m_P(F)}=\frac{1}{1-\sum_{n\geq 1}\frac{t^n}{n!}\left(C_{n,P}(x)+|\mathcal{P}_{n,1}|\right)}. \tag{5.5}$$

Next we rewrite the right-hand side of Equation (5.5) in form of power series as follows,

$$\frac{1}{1-\sum_{n\geq 1}\frac{t^n}{n!}\left(C_{n,P}(x)+|\mathcal{P}_{n,1}|\right)}=1+\sum_{m\geq 1}\left(\sum_{n\geq 1}\frac{t^n}{n!}\left(C_{n,P}(x)+|\mathcal{P}_{n,1}|\right)\right)^m. \tag{5.6}$$

Taking the coefficients of $\frac{t^s}{s!}$ on both sides of Equation (5.5) where $n\geq 1$, we see that we must show that

$$\begin{aligned}\sum_{F\in\mathcal{MP}_{s,P}}x^{m_P(F)} &= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{t^n}{n!}\left(C_{n,P}(x)+|\mathcal{P}_{n,1}|\right)\right)^m\Bigg|_{\frac{t^s}{s!}}\\ &= \sum_{m=1}^{s}\left(\sum_{n=1}^{s}\frac{t^n}{n!}\left(C_{n,P}(x)+|\mathcal{P}_{n,1}|\right)\right)^m\Bigg|_{\frac{t^s}{s!}}\\ &= \sum_{m=1}^{s}\sum_{\substack{a_1+\cdots+a_m=s\\ a_i\geq 1}}\binom{s}{a_1,\ldots,a_m}\prod_{j=1}^{m}\left(C_{a_j,P}(x)+|\mathcal{P}_{a_j,1}|\right).\end{aligned} \tag{5.7}$$

The right-hand side of (5.7) is now easy to interpret. We pick an m such that $1\leq m\leq s$. Then we pick $a_1,\ldots,a_m\geq 1$ such that $a_1+a_2+\cdots+a_m=s$. Next the multinomial coefficient $\binom{s}{a_1,a_2,\ldots,a_m}$ allows us to pick sets $S_1,S_2,\ldots,S_m$ which partition $\{1,2,\ldots,s\}$ such that $|S_i|=a_i$ for $i=1,2,\ldots,m$. Finally the product $\prod_{j=1}^{m}\left(C_{a_j,P}(x)+|\mathcal{P}_{a_j,1}|\right)$ allows us to pick either a P-cluster of size a_i or a single column of height a_i, for $1\leq i\leq m$. More precisely, $C_{a_i,P}(x)$ is P-cluster polynomial of size a_i, that is, weighted sum of all P-clusters of size a_i. A single column of height a_j are not allowed to have any marked matches (otherwise it would be a cluster), so weighted sum of them is just cardinality of single columns of height a_i.

Exploiting the same argument of maximal P-subclusters in proof of Theorem 2.2 finishes the proof here, which will not be elaborated again here. □

As for $\mathcal{F}_{n,A}$, $\mathcal{SI}_{n,A}$ and $\mathcal{WI}_{n,A}$, the idea and proof of Theorem 5.1 still hold and the difference is that now the generating function is ordinary instead of exponential because the fillings are words over A instead of permutations.

Theorem 5.2. *Given some pattern P, then*

$$1+\sum_{n\geq 1} t^n \sum_{F\in\mathcal{F}_{n,A}} x^{P\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1} t^n \left(C_{n,P}(x-1)+|\mathcal{F}_{n,1,A}|\right)} \tag{5.8}$$

where $C_{n,P}(x)$ is P-cluster polynomial with respect to $\mathcal{F}_{n,A}$.

Given some given pattern P, then

$$1+\sum_{n\geq 1} t^n \sum_{F\in\mathcal{WI}_{n,A}} x^{P\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1} t^n \left(C_{n,P}(x-1)+|\mathcal{WI}_{n,1,A}|\right)}, \tag{5.9}$$

where $C_{n,P}(x)$ is P-cluster polynomial with respect to $\mathcal{WI}_{n,A}$.

Given some pattern P, then

$$1+\sum_{n\geq 1} t^n \sum_{F\in\mathcal{SI}_{n,A}} x^{P\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1} t^n \left(C_{n,P}(x-1)+|\mathcal{SI}_{n,1,A}|\right)}, \tag{5.10}$$

where $C_{n,P}(x)$ is P-cluster polynomial with respect to $\mathcal{SI}_{n,A}$.

Next, suppose we want to study arrays which satify some extra restrictions between pairs of adjacent columns. Let $\mathscr{R}$ is a binary relation between two adjacent columns whose heights are not necessary the same. For example, we could restrict that two adjacent columns are forced to have different heights or the base element in a column must be greater than the top element in the next column. The set of fillings in $\mathcal{P}_n$ ($\mathcal{P}_{n,k}$) that satisfy the binary relation $\mathscr{R}$ is denoted by $\mathcal{P}_{n,\mathscr{R}}$ ($\mathcal{P}_{n,k,\mathscr{R}}$). As introduced in previous chapters, generalized clusters are suitable dealing with arrays with restrictions. Here we have the same definition for generalized clusters.

Let $\mathcal{MP}_{n,P}$ be the set of fillings in $\mathcal{P}_n$ such that we mark some of P-matches. We say $Q \in \mathcal{MP}_{n,P}$ is a generalized $P, \mathscr{R}$-cluster of size n if we can write $Q = B_1 B_2 \ldots B_m$ where B_i are blocks such that

1. B_i is either a single column or is order isomorphic to a P-cluster and $\mathscr{R}$ holds between any pair adjacent columns in B_i and
2. for $1 \leq i \leq m-1$, $(last(B_i), first(B_{i+1}))$ is not in $\mathscr{R}$.

The set of generalized clusters of size n is denoted by $\mathcal{GC}_{n,P,\mathscr{R}}$. Accordingly, generalized cluster polynomials $GC_{n,P,\mathscr{R}}(x)$ are defined in the same manner as previously defined. We let $\omega_{P,\mathscr{R}}(Q) = \prod_{i=1}^{m} x^{m_P(B_i)}$ and let

$$GC_{n,P,\mathscr{R}}(x) = \sum_{Q \in \mathcal{GC}_{n,P,\mathscr{R}}} (-1)^{\mathrm{B}(Q)-1} \omega_{P,\mathscr{R}}(Q),$$

where $\mathrm{B}(Q)$ is the number of blocks in Q.

Let $\mathcal{P}_{n,\mathscr{R}}$ denote the set of elements Q of $\mathcal{P}_n$ such that $\mathscr{R}$ holds for any two consecutive columns in Q. t Then we have following theorem.

Theorem 5.3. *Let $\mathscr{R}$ be a binary relation on all pairs of adjacent columns. For a given pattern P,*

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{F \in \mathcal{P}_{n,\mathscr{R}}} x^{P\text{-mch}(F)} = \frac{1}{1 - \sum_{n \geq 1} \frac{t^n}{n!} GC_{n,P,\mathscr{R}}(x-1)}. \tag{5.11}$$

The proof is essentially the same as proof of Theorem 2.4 so that we shall not give the details.

In fact, there are natural analogues of Theorems 5.1 and 5.3 in this setting. In Section 5.2, we shall provide and example that uses the multi-variate version of Theorem 5.3.

5.2 Examples

Examples in this section will show how we use Theorem 5.1 and 5.3 to find generating functions for arrays whose columns have unknown heights.

5.2.1 Singletons and patterns in ordered set partitions

Throughout this subsection, we still use $[n]$ to denote the set $\{1, 2, \cdots, n\}$ for convenience. A set partition of $[n]$ is a set of disjoint subsets of $[n]$ whose union is $[n]$. Usually, we call a subset in a set partition a block. Since in this paper blocks are an exclusive term in generalized clusters, we just call elements in set partition subsets to avoid confusion. The total number of set partitions of $[n]$ is given by the n-th Bell number, Bel_n, which has following recursion

$$Bel_n = \sum_{k=0}^{n} \binom{n}{k} Bel_k$$

and exponential generating function

$$\sum_{n\geq 0} \frac{Bel_n}{n!} t^n = e^{e^t - 1}.$$

Note that we don't distinguish the order of subsets in a set partition. For example, there are five partitions of [3], namely,

$$\{\{1,2,3\}\}, \quad \{\{1,2\},\{3\}\}, \quad \{\{1,3\},\{3\}\},$$

$$\{\{1\},\{2,3\}\}, \quad \{\{1\},\{2\},\{3\}\}.$$

Usually, we list elements in each subset in an increasing order. The first (least) element in a subset is called **opener** and the last (largest) element in a subset is called **closer**. In a set partition, we list subsets, by their openers, also in an increasing order.

Also, it is well-known that the number of set partitions of $[n]$ having k subsets is given Sterling number of the second kind, denoted by $\left\{ n \atop k \right\}$. For $n \geq 1$

$$\sum_{k=1}^{n} \left\{ n \atop k \right\} = Bel_n.$$

In contrast to set partitions of $[n]$, if orders of subsets are taken into account, we get a list of ordered disjoint subsets whose union is $[n]$, which is called an ordered set partition of $[n]$. We use $\mathcal{OSP}_n$ to denote the set of all the ordered set partitions of $[n]$. The total number of ordered set partitions of $[n]$ is given by the n-th Fubini number, Fub_n, which has following recursion,

$$Fub_n = \sum_{k=1}^{n} \binom{n}{k} Fub_{n-k}$$

and exponential generating function

$$\sum_{n\geq 0} \frac{Fub_n}{n!} t^n = \frac{1}{2-e^t}.$$

It can be also expressed in terms of Stirling numbers of the second kind,

$$\sum_{k=1}^{n} k! \left\{ {n \atop k} \right\} = Fub_n.$$

People sometimes think ordered set partitions as permutations with some prescribed descent set. As mentioned in Section 1.1.3, given information on descent sets, permutations can be represented in form of column-strict arrays whose shapes are not necessary to be rectangular. More precisely, for any $F \in \mathcal{OSP}_n$, we can represent F as an element in $\mathcal{P}_n$. An example in $\mathcal{OSP}_9$ is pictured as an element in $\mathcal{P}_9$ in Figure 5.3.

$$\{\{2\},\, \{1,4\},\, \{5\},\, \{7,8,9\},\, \{3\}\}$$

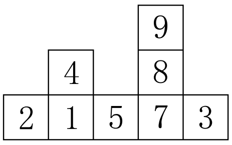


Figure 5.3: An ordered set partition of [9] and its array representation.

In [56], Remmel and Wilson used stars to indicate connectives of elements in ordered set partitions, that is, elements in a subset are connected by stars. Then the example in Figure 5.3 can be expressed as

$$2\ 1_*4\ 5\ 7_*8_*9\ 3.$$

We call such permutations **starred ascent permutations**. More precisely, $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a starred ascent permutation if we mark some of ascents in σ by stars. We let $\mathcal{S}_n^*$ denote the set of starred ascent permutations.

Statistics on ordered set partitions has been a popular topic especially in recent years, such as [28, 31, 62, 56]. Throughout this subsection, different from patterns in other examples in previous sections, we shall consider more abstract patterns. For example, we could consider a singleton as our pattern, that is, we are interested in enumerating the number of singletons in ordered set partitions, and equivalently, how many columns have only one row. For $p \in \mathcal{OSP}_n$, we use $\mathrm{Sgt}(p)$ to denote the number of singletons in p. Assume $p = \{\{2\}, \{1, 4\}, \{5\}, \{7, 8, 9\}, \{3\}\} \in \mathcal{OSP}_9$, then $\mathrm{Sgt}(p) = 3$. Actually, to some extent, to enumerate singletons or subsets of any giving size is similar to subword pattern in integer compositions. However, we could also consider other permutation patterns such as ascents.

We compute distribution of singletons in $\mathcal{OSP}_n$ and consider following generating function

$$A_{\mathrm{Sgt},\mathcal{OSP}}(x,t) := 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{p\in\mathcal{OSP}_n} x^{\mathrm{Sgt}(p)},$$

and according to Theorem 5.1,

$$A_{\mathrm{Sgt},\mathcal{OSP}}(x,t) = \frac{1}{2 - e^t - \sum_{n\geq 1} \frac{t^n}{n!} C_{n,\mathrm{Sgt}}(x-1)}, \tag{5.12}$$

where $C_{n,\mathrm{Sgt}}(x)$ is cluster polynomial of size n.

Based on the definition of cluster polynomial, a cluster polynomial is sum of weights for all clusters. Clearly, here the only feasible cluster is a single column of height 1. Therefore,

$$C_{n,\mathrm{Sgt}}(x) = \begin{cases} x, & \text{for } n = 1 \\ 0, & \text{for } n \geq 2. \end{cases}$$

Then

$$A_{\mathrm{Sgt},\mathcal{OSP}}(x,t) = \frac{1}{2 - e^t - C_{1,\mathrm{Sgt}}(x-1)} = \frac{1}{2 - e^t - (x-1)t}.$$

As mentioned, since $\mathcal{OSP}_n$ can be regarded as starred ascent permutations $\mathcal{S}_n^*$, it also allows to us to consider permutation patterns. For $\sigma \in \mathcal{S}_n^*$, elements in σ could be starred or non-starred and therefore, here we could consider permutation patterns with specification of starred positions.

For example, assume $\sigma \in \mathcal{S}_n^*$, we say there is a 1_*2_*-match in σ at position i if $\sigma_i < \sigma_{i+1}$ and both σ_i and σ_{i+1} are starred. For convenience, we use underscore

‘$\_$’ to indicate gaps that don’t have stars. We say there is a $1_*2_{\_}$-match in σ at position i if $\sigma_i < \sigma_{i+1}$ and σ_i is starred while σ_{i+1} is non-starred. We say there is a $1_{\_}2_{\_}$-match in σ at position i if $\sigma_i < \sigma_{i+1}$ and both σ_i and σ_{i+1} are not starred. $12_{\_}$-match can be recognized as a set of patterns, that is, $12_{\_} = \{1_{\_}2_{\_}, 1_*2_{\_}\}$. A $_{\_}12_*$-match in σ at position i means $\sigma_i < \sigma_{i+1}$, σ_{i+1} has a star and σ_{i-1} has no star if σ_{i-1} exists.

For the remainder of this subsection, we shall consider the pattern $1_{\_}2_{\_}$. Assume $\sigma = 2\ 1_*4\ 5\ 7_*8_*9\ 3$, then $1_{\_}2_{\_}\text{-mch}(\sigma) = 1$ because there is a $1_{\_}2_{\_}$-match at position 3. To clarify connections between $\mathcal{OSP}_n$, $\mathcal{P}_n$ and $\mathcal{S}_n^*$, we interpret $1_{\_}2_{\_}$ for $\mathcal{OSP}_n$ and $\mathcal{P}_n$. We see that for $\sigma \in \mathcal{S}_n^*$, a $1_{\_}2_{\_}$ pattern in σ means that in the ordered set partition corresponding to σ, we observe an ascent between a closer and the adjacent opener and the opener has to be a singleton element, which is also equivalent to that for $\mathcal{P}_n$, we observe the top element in a column is less than the element in next column which is forced to be a singleton column. For convenience, we denote such a pattern for $\mathcal{P}_n$ by P. Singleton column pattern is, in fact, $_{\_}1_{\_}$ in sense of starred patterns, denoted by Q. Then we consider following joint generating function,

$$A_{P,Q,\mathcal{P}}(x,y,t) := 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{F\in\mathcal{P}_n} x^{P\text{-mch}(F)} y^{Q\text{-mch}(F)}.$$

Then by the multi-variate analogue of Theorem 5.1,

$$A_{P,Q,\mathcal{P}}(x,y,t) = \frac{1}{2 - e^t - \sum_{n\geq 1} \frac{t^n}{n!} C_{n,(P,Q)}(x-1,y-1)}.$$

Before computing $C_{n,(P,Q)}(x,y)$, we need to figure out the structures of (P,Q)-clusters. For convenience, we let $\mathcal{C}_{n,k,(P,Q)}$ denote the set of (P,Q)-clusters of size n having k columns and let $C_{n,k,(P,Q)}(x,y)$ denote the corresponding cluster polynomial.

Suppose $F \in \mathcal{C}_{n,1,(P,Q)}$, F has to be a single column of height 1 and has to be labeled with ‘y’. Then

$$C_{n,1,(P,Q)}(x,y) = \begin{cases} y, & \text{for } n = 1, \\ 0, & \text{for } n \geq 2, \end{cases}$$

and thus

$$\sum_{n\geq 1}\frac{t^n}{n!}C_{n,1,(P,Q)}(x,y) = yt. \tag{5.13}$$

Suppose $F \in \mathcal{C}_{n,2,(P,Q)}$, then F has a marked P-match, which implies the second column is a singleton while the first column can be of any height including 1. Clearly, the filling of F is unique. Singleton column can be either marked with 'x' or not. Then

$$C_{n,2,(P,Q)}(x,y) = \begin{cases} 0, & \text{for } n = 1, \\ x(y+1)^2, & \text{for } n = 2, \\ x(y+1), & \text{for } n \geq 3, \end{cases}$$

and thus

$$\begin{aligned}\sum_{n\geq 1}\frac{t^n}{n!}C_{n,2,(P,Q)}(x,y) &= \frac{t^2}{2}x(y+1)^2 + \sum_{n\geq 3}\frac{t^n}{n!}x(y+1) \\ &= \frac{t^2}{2}x(y+1)^2 + x(y+1)\left(e^t - 1 - t - \frac{t^2}{2}\right).\end{aligned}$$

In general, for $k \geq 2$, suppose $F \in \mathcal{C}_{n,k,(P,Q)}$, then F is forced to have exactly $k-1$ P-matches and all columns are of height 1 except the first column can be of any height including 1. Clearly, the filling of F is unique. Singleton column can be either marked with 'x' or not. Then

$$C_{n,k,(P,Q)}(x,y) = \begin{cases} 0, & \text{for } n \leq k-1, \\ x^{k-1}(y+1)^k, & \text{for } n = k, \\ x^{k-1}(y+1)^{k-1}, & \text{for } n \geq k+1, \end{cases}$$

and thus

$$\sum_{n\geq 1}\frac{t^n}{n!}C_{n,k,(P,Q)}(x,y) = \frac{t^k}{k!}x^{k-1}(y+1)^k + \sum_{n\geq k+1}\frac{t^n}{n!}x^{k-1}(y+1)^{k-1}. \tag{5.14}$$

Combining (5.13) and (5.14) and letting $z = x(y+1)$ for convenience, we have

$$\begin{aligned}
&\sum_{n\geq 1} \frac{t^n}{n!} C_{n,(P,Q)}(x,y) \\
= \;& \sum_{n\geq 1} \frac{t^n}{n!} C_{n,1,(P,Q)}(x,y) + \sum_{k\geq 2}\sum_{n\geq 1} \frac{t^n}{n!} C_{n,k,(P,Q)}(x,y) \\
= \;& yt + \sum_{k\geq 2} \frac{t^k}{k!} x^{k-1}(y+1)^k + \sum_{k\geq 2}\sum_{n\geq k+1} \frac{t^n}{n!} x^{k-1}(y+1)^{k-1} \\
= \;& yt + \frac{1}{x}\sum_{k\geq 2} \frac{(x(y+1)t)^k}{k!} k + \sum_{n\geq 3} \frac{t^n}{n!} \sum_{k=1}^{n-2} (x(y+1))^k \\
= \;& yt + \frac{e^{zt}-zt-1}{x} + \sum_{n\geq 3} \frac{t^n}{n!} \frac{z-(z)^{n-1}}{1-z} \\
= \;& yt + \frac{e^{zt}-zt-1}{x} + \frac{z}{1-z}\sum_{n\geq 3} \frac{t^n}{n!} - \frac{1}{z-z^2}\sum_{n\geq 3} \frac{(zt)^n}{n!} \\
= \;& yt + \frac{e^{zt}-zt-1}{x} + \frac{z\left(e^t-1-t-\frac{t^2}{2}\right)}{1-z} - \frac{e^{zt}-1-zt-\frac{z^2t^2}{2}}{z-z^2}.
\end{aligned}$$

Therefore,

$$A_{(P,Q),\mathcal{P}}(x,y,t) = \frac{(x-1)y((x-1)y-1)}{e^t(x-1)y + (y(-xy+y+1)-1)e^{t(x-1)y} + xy((x-1)y-2) + y + 1}. \tag{5.15}$$

A few initial terms of $A_{(P,Q),\mathcal{P}}(x,y,t)$ are

$$\begin{aligned}
&1 + yt + \frac{t^2}{2!}\left(xy^2+y^2+1\right) + \frac{t^3}{3!}\left(x^2y^3+4xy^3+xy+y^3+5y+1\right) \\
&+\frac{t^4}{4!}\left(x^3y^4+11x^2y^4+x^2y^2+11xy^4+18xy^2+xy+y^4+17y^2+7y+7\right) \\
&+\cdots.
\end{aligned}$$

By manipulating the formula in (5.15), we could derive various results. For example,

$$\left.\frac{\partial A_{(P,Q),\mathcal{P}}(0,y,t)}{\partial y}\right|_{y=1} = \frac{(1-t)\sinh(t)+(t-2)\cosh(t)+2}{2(\sinh(t)-1)^2},$$

which is the exponential generating function for the total number of singleton columns in arrays in $\mathcal{P}_n$ that don't have pattern P. For $n \geq 1$, the numbers are $1, 2, 8, 45, 293, 2254, 20024, 200891, 2246471, \ldots$.

If we set $x = 0, y = 1$ in (5.15), we have

$$A_{(P,Q),\mathcal{P}}(x, y, t) = \frac{2}{e^{-t} - e^{t} + 2} = \frac{1}{1 - \sinh t},$$

which is the exponential generating function for the number of arrays in $\mathcal{P}_n$ avoiding P, also the number of ordered set partitions such that we don't see an ascent between a subset and a singleton subset immediately following it and it also the number of starred ascent permutations avoiding 1_2_.

It turns out that $\frac{2}{e^{-t}-e^{t}+2} = \frac{1}{1-\sinh t}$ is also the exponential generating function of ordered set partitions that don't have subsets of even sizes. Because of the trivial bijection between $\mathcal{OSP}_n$ and $\mathcal{P}_n$, it is natural to ask if we can prove this fact bijectively. Clearly we can partitions the $\mathcal{P}_n$ into three sets, $NE\mathcal{P}_n$ which is the set elements of $\mathcal{P}_n$ that have no even columns, $B\mathcal{P}_n$ which is the set of elements of $\mathcal{P}_n$ which have both even column and an occurrence of the pattern P, and $\hat{\mathcal{P}}_n$ which is the set of elements of elements of $\mathcal{P}_n$ which have no occurrence of pattern P but has columns of even height. Similarly, we can partition $\mathcal{P}_n$ into three sets, $NP\mathcal{P}_n$ which is the set elements of $\mathcal{P}_n$ no occurrence of pattern P, $B\mathcal{P}_n$ which is the set of elements of $\mathcal{P}_n$ which have both and even column and an occurrence of the pattern P, and $\bar{\mathcal{P}}_n$ which is the set of elements of elements of $\mathcal{P}_n$ which have no columns of even height but has at least an occurrence of the pattern P. Clearly to show that $|NE\mathcal{P}_n| = |NP\mathcal{P}_n|$, we need only show that $|\bar{\mathcal{P}}_n| = |\hat{\mathcal{P}}_n|$.

We can define a bijection $\Phi : \bar{\mathcal{P}}_n \mapsto \hat{\mathcal{P}}_n$ as follows, for $F \in \bar{\mathcal{P}}_n$, we scan columns in F from right to left, if j-th column has an even height, we remove the closer (i.e., top element) from the column and insert in between j-th column and $(j + 1)$-th. Obviously, in j-th column of the new array. This will create at P-match. Note that if $j+1$-st column has height 1, then since F has no P-matches it must be that the element in column $j + 1$ is less than top element of column j so that we can not create a new P between the two columns of height 1 that are now the $j + 1$-st and $j + 2$-nd elements of $\Phi(F)$. We keep doing this until there are no columns of even heights. It is possible that we can create some new P-matches in the process in the case where the height of the j-th column is 2 and the height of the $j - 1$-st column is 1 and the element in the $j - 1$-column is less than the bottom element of j-th column. However, one sees that the only way that we can create consecutive sequence of P-matches consisting of columns of height 1 in $\Phi(F)$ is if we started with consecutive sequence of r columns of height 2 such that top element of any column is less than the bottom of the next column. If the column preceding this column is a column of height 1 whose element is less than bottom element of the next column, we will produce a sequence of $2r+1$ columns of height 1 such that there is a P-match between any two consecutive columns and we will produce a sequence of $2r$ columns of height 1 such that there is a P-match between any two consecutive columns, otherwise. Nevertheless, the array we finally we obtain is an array in $\hat{\mathcal{P}}_n$. An example is pictured in Figure 5.4. When we apply the inverse Φ^{-1} to the new array we just got in $\hat{\mathcal{P}}_n$, we scan columns of the array from left to right, if we see a P-match at j-th column, we put the $(j+1)$-th column, actually a singleton, on the top of the j-th column. We know there were no columns of even heights. However, after one element was put on the top of the j-th column, now the j-th column has an even number of rows. We keep doing this until there are no P-matches. It is easy to check by our remarks above that we recover the original array F in this process.

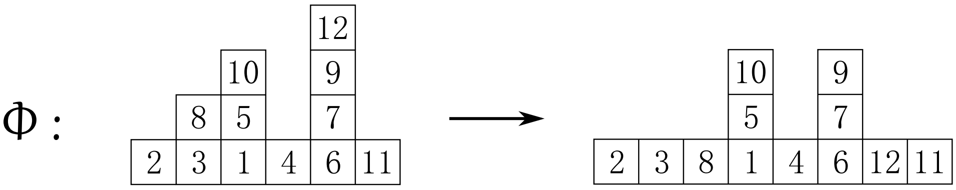


Figure 5.4: Φ maps an array in $\bar{\mathcal{P}}_n$ to an array in $\hat{\mathcal{P}}_n$.

However, in general, the number of arrays in $\mathcal{P}_n$ having k P-matches is not equal to the number of arrays in $\mathcal{P}_n$ having k columns of even heights. Take $\mathcal{P}_3$ as a counterexample, there is one array that has two P-matches, namely, $\{\{3\},\{2\},\{1\}\}$ but there does not exist any array that has two subsets of even sizes.

Besides regular ordered set partitions, we also consider the distribution of pattern P in restricted ordered set partitions. Thinking of order set partitions at elements of $\mathcal{P}_n$, $\mathscr{R}$ be the relation that holds between two columns C and D of integers, which increase from bottom to top, if either C or D has height greater than 1 or C and D both have height 1 and the element in C is bigger than the element of D. Thus $\mathscr{R}$ requires that we have to see a descent between two adjacent singleton subsets in an ordered set partition if there exist adjacent singletons, which is equivalent to that we see a descent between two adjacent singleton columns in an array in $\mathcal{P}_n$, and also equivalent to avoiding $\_1\_2\_$ in a starred ascent permutation. We denote the subset of $\mathcal{OSP}_n$ ($\mathcal{P}_n$, $\mathcal{S}_n^*$) equipped with relation $\mathscr{R}$ by $\mathcal{OSP}_{n,\mathscr{R}}$ ($\mathcal{P}_{n,\mathscr{R}}$, $\mathcal{S}_{n,\mathscr{R}}^*$).

We still consider the previous pattern P and Q which are $1\_2\_$ and $\_1\_$, respectively, in sense of starred starred permutations. Next we shall study

$$A_{(P,Q),\mathcal{P},\mathscr{R}}(x,y,t) := 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{F\in\mathcal{P}_{n,\mathscr{R}}} x^{P\text{-mch}(F)} y^{Q\text{-mch}(F)}.$$

By the multi-variate analogue of Theorem 5.3, we have

$$A_{(P,Q),\mathcal{P},\mathscr{R}}(x,y,t) = \frac{1}{1-\sum_{n\geq 1}\frac{t^n}{n!}GC_{n,(P,Q),\mathscr{R}}(x-1,y-1)}. \tag{5.16}$$

We will partition the generalized joint (P,Q)-$\mathscr{R}$-clusters into three types. Type 1, Type 2, and Type 3 are described below. A key fact to observe is that this scenario is that (P,Q)-clusters with more than 2 columns cannot appear in a generalized joint (P,Q)-$\mathscr{R}$-clusters because in a (P,Q)-clusters with k columns where $k \geq 3$, we must end with a sequence of $k-1$ columns of height 1 and the numbers in these columns must be increasing, reading from left to right. However, to be part of a generalized joint (P,Q)-$\mathscr{R}$-cluster, the number must decrease in consecutive singletons to meet the condition $\mathscr{R}$. For convenience, we let $\mathcal{GC}^{\mathrm{J}}_{n,(P,Q),\mathscr{R}}$ denote the Type J generalized joint (P,Q)-$\mathscr{R}$-clusters and let $GC^{\mathrm{J}}_{n,(P,Q),\mathscr{R}}(x,y)$ denote the corresponding generalized joint (P,Q)-$\mathscr{R}$-cluster polynomial. Also, let

$$GC^{\mathrm{J}}(x,y,t) := \sum_{n\geq 1}\frac{t^n}{n!}GC^{\mathrm{J}}_{n,(P,Q),\mathscr{R}}(x,y).$$

1. Type 1 generalized joint (P,Q)-$\mathscr{R}$-clusters are those clusters that have only one column. If the height of the column is 1, it can be either marked with 'y' or not. Then

$$GC^1_{n,(P,Q),\mathscr{R}}(x,y) = \begin{cases} y+1, & \text{for } n=1,\\ 1, & \text{for } n\geq 2,\end{cases}$$

and

$$GC^1(x, y, t) = yt + e^t - 1.$$

2. Type 2 generalized joint (P, Q)-$\mathscr{R}$-clusters are those cluster that have more than one block, but where each block is a column of height 1. To violate $\mathscr{R}$, the numbers in the blocks must be increasing, reading from left to right. Then for $n \geq 2$

$$GC^2_{n,(P,Q),\mathscr{R}}(x, y) = (-1)^{n-1}(y + 1)^n$$

and

$$\begin{aligned} GC^2(x, y, t) &= \sum_{n\geq 2} \frac{t^n}{n!}(-1)^{n-1}(y + 1)^n \\ &= -\sum_{n\geq 2} \frac{(-(y + 1)t)^n}{n!} \\ &= 1 - (y + 1)t - e^{-(y+1)t} \end{aligned}$$

3. Type 3 generalized joint (P, Q)-$\mathscr{R}$-clusters are those generalized joint (P, Q)-$\mathscr{R}$-clusters which have more than one column and at least one column of height ≥ 2. Suppose that $B_1 \dots B_m$ is such a generalized joint (P, Q)-$\mathscr{R}$-cluster. Then for each $1 \leq i < m$, $\mathscr{R}$ does hold for the pair consisting of the last column of B_i and the first column of B_{i+1}. We claim that no B_i can consist of a single of height greater than 1 since then either if $i \geq 2$, then the last column of B_i and first column of B_i automatically satisfy $\mathscr{R}$. Similarly, if $i = 1$, then B_1 and the first column of B_2 automatically satisfy $\mathscr{R}$. Thus the only columns of height ≥ 2 must be part of (P, Q)-clusters and this means the the cluster must have the form pictured on the left in Figure 5.5. That is, the (P, Q)-cluster must start with a column C_1 of height ≥ 2 and a column C_2 of height 1 where the top element of C_1 is bigger than the element in column C_2. Moreover this (P, Q) must be B_1 since if it equal to B_i for $i \geq 2$, then the last column of B_{i-1} and the first column of B_i would satisfy $\mathscr{R}$. It is then easy to see that the only possibilities for $B_2, \dots, B_m$ are that they consist of a single column and the elements in $C_2, B_2, \dots, B_m$ must be increasing reading from left to right. This means that the filling of

$B_1 \ldots, B_m$ is unique. That is, the Hasse diagram of $B_1, \ldots, B_m$ are of the form pictured in Figure 5.5. The first column must be marked with an x, but there remaining columns could be marked with a y or not. It follows that

$$\begin{aligned} GC^3_{n,(P,Q),\mathscr{R}}(x,y) &= \sum_{k=2}^{n-1} x(y+1)^{n-k}(-1)^{n-k-1} \\ &= x(y+1)\sum_{k=2}^{n-1}(-y-1)^{n-k-1} \\ &= x(y+1)\frac{1-(-y-1)^{n-2}}{1-(-y-1)} \\ &= \frac{x(y+1)}{2+y} - \frac{x}{(2+y)(1+y)}(-y-1)^n. \end{aligned}$$

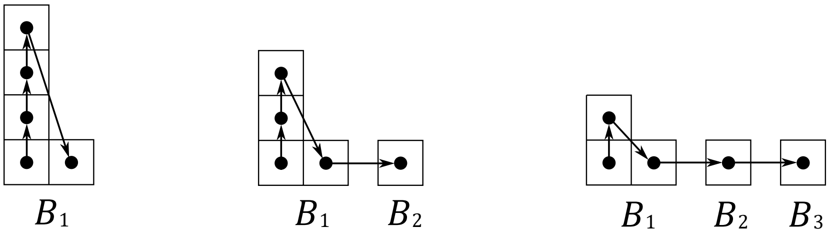


Figure 5.5: Examples of Type 3 generalized clusters.

Then

$$\begin{aligned} GC^3(x,y,t) &= \sum_{n\geq 3} \frac{t^n}{n!} GC^3_{n,(P,Q),\mathscr{R}}(x,y) \\ &= \frac{x(y+1)}{2+y}\sum_{n\geq 3}\frac{t^n}{n!} - \frac{x}{(2+y)(1+y)}\sum_{n\geq 3}\frac{(-y-1)^n t^n}{n!} \\ &= \frac{xe^{-t(y+1)}\left((y+1)^2 e^{t(y+2)} - (y+2)e^{t(y+1)}(ty+t+y)-1\right)}{(y+1)(y+2)}. \end{aligned}$$

Finally we are able to compute the generating function

$$\begin{aligned} & A_{(P,Q),\mathcal{P},\mathscr{R}}(x,y,t) \\ = \ & \frac{1}{1-GC^1(x-1,y-1,t)-GC^2(x-1,y-1,t)-GC^3(x-1,y-1,t)} \\ = \ & \frac{y(y+1)e^{ty}}{-ye^{ty+t}(xy+1)+(y+1)e^{ty}(x(ty+y-1)+1)+x+y^2+y-1}. \end{aligned} \tag{5.17}$$

The first few terms of this series are

$$\begin{aligned}&1+yt+\frac{1}{2}\left(1+y^2\right)t^2+\frac{1}{6}\left(1+5y+xy+y^3\right)t^3+\\&\frac{1}{24}\left(7+7y+xy+17y^2+7xy^2+y^4\right)t^4+\\&\frac{1}{120}\left(21+79y+21xy+31y^2+9xy^2+49y^3+31xy^3+y^5\right)t^5+\\&\frac{1}{720}\left(141+301y+71xy+549y^2+301xy^2+20x^2y^2+\right.\\&\qquad\left.111y^3+49xy^3+129y^4+111xy^4+y^6\right)t^6+\cdots.\end{aligned}$$

Setting $x=0$ and $y=0$ in (5.17) gives

$$A_{(P,Q),\mathcal{P},\mathscr{R}}(0,1,t)=\frac{2e^t}{1+2e^t-e^{2t}}.$$

This is the generating function ordered set partitions with no P-matches. The first few terms of this series is

$$1,1,2,7,32,181,1232,9787,88832,907081,\ldots$$

which is sequence A0006154 in the OEIS [49]. This all counts the number of ordered set partitions of $\{1,\ldots,n\}$ into only odd parts.

We could also compare the number of P-matches and Q-matches.

$$\begin{aligned}&\lim_{y=1}A_{(P,Q),\mathcal{P},\mathscr{R}}\left(\frac{1}{y},y,t\right)-\lim_{y=0}A_{(P,Q),\mathcal{P},\mathscr{R}}\left(\frac{1}{y},y,t\right)\\=\;&\frac{t(t+2)-4\cosh(t)+4}{(t(t+4)-4e^t+6)\left(t-2\sinh(t)+1\right)},\end{aligned}$$

which is generating function for the number of ordered set partitions in $\mathcal{OSP}_{n,\mathscr{R}}$ having more Q-matches than P-matches. For $n\geq 1$, they are $1,1,6,32,200,1552,13748,138406,1558488,\ldots$.

If we take the partial derivative of $A_{(P,Q),\mathcal{P},\mathscr{R}}(x,y,t)$ and then set $x=0$ and $y=1$, we will get the number of ordered set partitions in $\mathcal{P}_{n,\mathscr{R}}$ which has exactly one P-match. This give the generating function

$$\frac{2e^t(1-e^{2t}-2te^t}{(1-2e^t-e^{2t})^2}.$$

There are many other new sequences derived from (5.17) but we would not discuss here.

5.2.2 Patterns in cycle structures of permutations

Suppose that σ is a permutation in $\mathcal{S}_n$ with k cycles $L_1L_2\ldots L_k$, we shall always write cycle L_i in form of $L_i = (c_{1,i}, c_{2,i}, \ldots, c_{p_i,i})$ where $c_{1,i}$ is the smallest element in L_i and p_i is the length of L_i. We arrange cycles of σ by increasing the smallest element in each cycle. For example, the two-line notation and the cyclic notation for $\sigma = 3\ 6\ 1\ 4\ 2\ 5\ 7\ 9\ 8\ 10 \in \mathcal{S}_{10}$ are as follows,

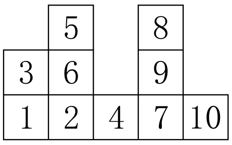
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 6 & 1 & 4 & 2 & 5 & 9 & 7 & 8 & 10 \end{pmatrix} = (1\ 3)(2\ 6\ 5)(4)(7\ 9\ 8)(10).$$

In [30], Jones and Remmel studied the joint distribution of number of cycles and the number of cycle descents for permutations that avoid certain cycle patterns. The number of cycles in σ is denoted by $\mathrm{Cyc}(\sigma)$. The number of cycle descents in a cycle L_i, denoted by $\mathrm{Cdes}(L_i)$, is defined as $\mathrm{Cdes}(L_i) = \mathrm{des}(L_i) + 1$. The number of cycle descents in a permutation is sum of cycle descents in all the cycles. In the example above, $\mathrm{Cdes}(\sigma) = \mathrm{Cdes}(L_1) + \mathrm{Cdes}(L_2) + \mathrm{Cdes}(L_3) + \mathrm{Cdes}(L_4) + \mathrm{Cdes}(L_5) = 1 + 2 + 1 + 2 + 1 = 7$.

Clearly, cycle descents can be regarded as descents within each cycle. In this subsection, we would use generalized cluster method to compute joint distribution of number of cycles and simple patterns between adjacent cycles. We say that there is a **cycle rise** at i-th cycle if $\max(L_i) < \min(L_{i+1})$. We let $\mathrm{Crise}(\sigma)$ denote the number of cycle rises in σ. For previous example, $\mathrm{Cyc}(\sigma) = 5$ and $\mathrm{Crise}(\sigma) = 2$ because $\max(L_3) = 4 < 7 = \min(L_4)$ and $\max(L_4) = 9 < 10 = \min(L_5)$. Our goal in this subsection is to compute the following generating function:

$$1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{S}_n} x^{\mathrm{Crise}(\sigma)} y^{\mathrm{Cyc}(\sigma)}. \tag{5.18}$$

Similar to treatment used for ordered set partitions, to apply generalized cluster method, we have to represent cycle structures in form of an array. A straightforward way to do this is to put elements of L_i in i-th column of an array from bottom to top. An example is given in Figure 5.6. Note that different from previous theorems and examples, here arrays are even not necessary to be column-strict increasing.



$$(1\ \ 3)\,(2\ \ 6\ \ 5)\,(4)\,(7\ \ 9\ \ 8)\,(10)$$

Figure 5.6: σ and its array representation.

We let $\mathcal{K}_n$ ($\mathcal{K}_{n,k}$) denote the set of fillings of D_n ($D_{n,k}$) with the elements of $1, 2, 3, \ldots, n$ such that in each column, the bottom element is the smallest.

For $F \in \mathcal{K}_n$, we use $\mathrm{Col}(F)$ to denote the number of columns in F, $F[j]$ to denote the j-th column and $F(i, j)$ to denote the element in i-th row, reading from bottom to top, and j-th column, reading from left to right. Next we let relation $\mathscr{R}$ be base-row increasing, that is, $F(1, i) < F(1, i + 1)$ for $1 \leq i < \mathrm{Col}(F)$. Clearly, $\mathcal{K}_{n,\mathscr{R}}$ is bijective to $\mathcal{S}_n$ and $\mathcal{K}_{n,k,\mathscr{R}}$ is bijective to $\mathcal{S}_n$ with k cycles.

Next we define pattern P, that is, cycle rise, for arrays. For $F \in \mathcal{K}_{n,\mathscr{R}}$, we say there is a P-match at position j if $\max(F[j]) < \min(F[j + 1])$. Equivalently, there is a P-match at position j if and only if $\max(F[j]) < F(1, j+1)$. Clearly, P-matches in $F \in \mathcal{K}_{n,\mathscr{R}}$ are equivalent to cycle rises in $\mathcal{S}_n$. The number of P-matches is denoted by $P\text{-mch}(F)$. We also define a trivial pattern Q which is used to keep track of the number of columns. In a (P, Q)-marked arrays, each column is either marked Q-match or not.

Then we can rewrite the generating function (5.18) as follows,

$$1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{Crise}(\sigma)} y^{\mathrm{Cyc}(\sigma)} = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{F \in \mathcal{K}_{n,\mathscr{R}}} x^{P\text{-mch}(F)} y^{Q\text{-mch}(F)}. \tag{5.19}$$

Although $\mathcal{K}_n$ are not column-strict arrays, Theorem 5.3 still holds. Then the right-hand side generating function in (5.19) has following formula

$$1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{F \in \mathcal{K}_{n,\mathscr{R}}} x^{P\text{-mch}(F)} y^{\mathrm{Col}(F)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^n}{n!} GC_{n,(P,Q),\mathscr{R}}(x - 1, y - 1)}. \tag{5.20}$$

Before computing $GC_{n,(P,Q),\mathscr{R}}(x, y)$, let us figure out the structures of (P, Q)-clusters. First, we see an array F in $\mathcal{K}_{n,k,\mathscr{R}}$ is a P-cluster if $\max(F[j]) < F(1, j+1)$ for all $1 \leq j < k$ where all columns have to be labeled with 'x' except the last column. Then, a (P, Q)-cluster is

1. either a single column marked with 'y'
2. or a P-cluster where each column is free to be marked with 'y' or not.

Next we consider generalized joint (P,Q)-$\mathscr{R}$-clusters. An array $H \in \mathcal{K}_n$ is a generalized joint (P,Q)-$\mathscr{R}$-cluster of size n if we can write H as $H = B_1 B_2 \cdots B_{\mathrm{B}(F)}$ where B_i are blocks such that

1. for $1 \le i \le \mathrm{B}(H)$, B_i is either a single column without any labeling or order-isomorphic to (P,Q)-cluster and
2. the base element in the last column of B_i is greater than the base element in the first column of B_{i+1}, for for $1 \le i \le \mathrm{B}(H)-1$,

where $\mathrm{B}(H)$ is the number of blocks in H. We denote the set of generalized (P,Q)-clusters of size n by $\mathcal{GC}_{n,(P,Q),\mathscr{R}}$. By definition of generalized cluster polynomial,

$$GC_{n,(P,Q),\mathscr{R}}(x,y) = \sum_{H \in \mathcal{GC}_{n,(P,Q),\mathscr{R}}} (-1)^{\mathrm{B}(H)-1} x^{m_P(H)} y^{m_Q(H)}.$$

In the remainer of the this subsection, we will focus on how to compute $GC_{n,(P,Q),\mathscr{R}}(x,y)$. As one may see from previous examples, for partial ordering sets, we usually represent them in form of Hasse diagrams and then count the linear extensions of these diagrams.

A column of height p_i in a generalized cluster can be represented as height 1 directed $(p_i - 1)$-ary tree whose number of linear extensions is $(p_i - 1)!$ because the base element must be the smallest element while there are no conditions for the other elements. A column of height 5 is pictured in Figure 5.7

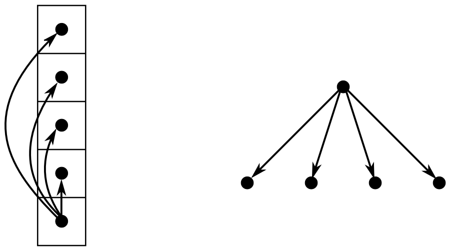


Figure 5.7: A column of height 5 and its Hasse diagram.

In a P-cluster F, all elements in i-th column are less than the base element in $(i+1)$-th column, and hence in the corresponding Hasse diagram, all the leaves in i-th tree are pointing to the root of $(i+1)$-th tree. For a P-cluster with k columns whose heights are $(p_1, p_2, \ldots, p_k)$, the number of linear extensions of the corresponding Hasse diagram is $\prod_{i=1}^{k}(p_i - 1)!$. The Hasse diagram corresponding to a P-cluster with three columns whose heights are $5, 2, 4$ is pictured in Figure 5.8, and the number of linear extensions is $4!1!3! = 144$.

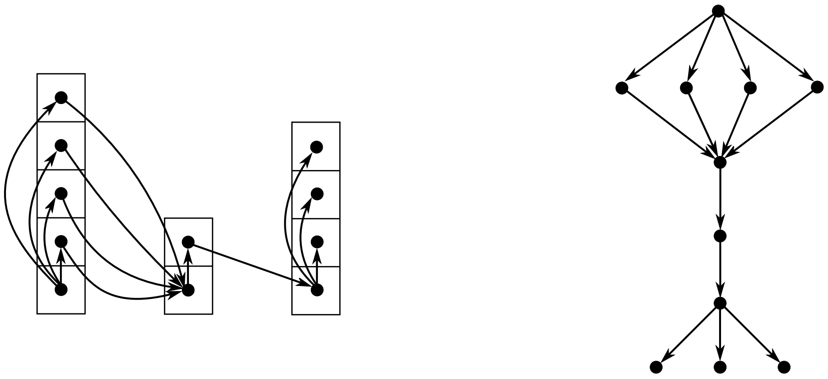


Figure 5.8: A P-cluster whose column heights are $5, 2, 4$.

Now assume a generalized cluster H has m blocks and each block has k_i columns, for $1 \leq i \leq m$ and we let $p_{i,j}$ denote the height of the i-th column in j-th block, and let $\mathcal{GC}([p_{1,1}, p_{2,1}, \ldots, p_{k_1,1}], [p_{1,2}, \ldots, p_{k_2,2}], \ldots, [p_{1,m}, \ldots, p_{k_m,m}])$ denote the set of such generalized clusters. It is clear that the sum of this nested list is just the size of the generalized cluster. Then there are $\prod_{j=1}^{m}\prod_{i=1}^{k_j}(p_{i,j} - 1)!$ ways to extend all the tree-like Hasse diagrams in linear orderings. After single columns and clusters are straightened out, Hasse diagrams of generalized clusters become easier to handle which will allows us to prove some simple recursions. Keep in mind that the arrows between blocks should from right to left. We use $\Gamma([s_1, p_{k_1,1}], [s_2, p_{k_2,2}], \ldots, [s_m, p_{k_m,m}])$, where $s_j = \sum_{i=1}^{k_j-1} p_{i,j}$ to denote the straightened Hasse diagram corresponding to $\mathcal{GC}([p_{1,1}, \ldots, p_{k_1,1}], \ldots, [p_{1,m}, \ldots, p_{k_m,m}])$.

Clearly,

$$|\mathcal{GC}([p_{1,1}, p_{2,1}, \ldots, p_{k_1,1}], [p_{1,2}, \ldots, p_{k_2,2}], \ldots, [p_{1,m}, \ldots, p_{k_m,m}])| = \left(\prod_{j=1}^{m}\prod_{i=1}^{k_j}(p_{i,j}-1)!\right) \mathrm{LE}(\Gamma([s_1, p_{k_1,1}], [s_2, p_{k_2,2}], \ldots, [s_m, p_{k_m,m}])). \quad (5.21)$$

For example, the set of generalized clusters $\mathcal{GC}([3, 2], [4], [1, 2, 1], [2, 2])$ and its straightened diagram $\Gamma([3, 2], [0, 4], [3, 1], [2, 2])$ are drawn in Figure 5.9.

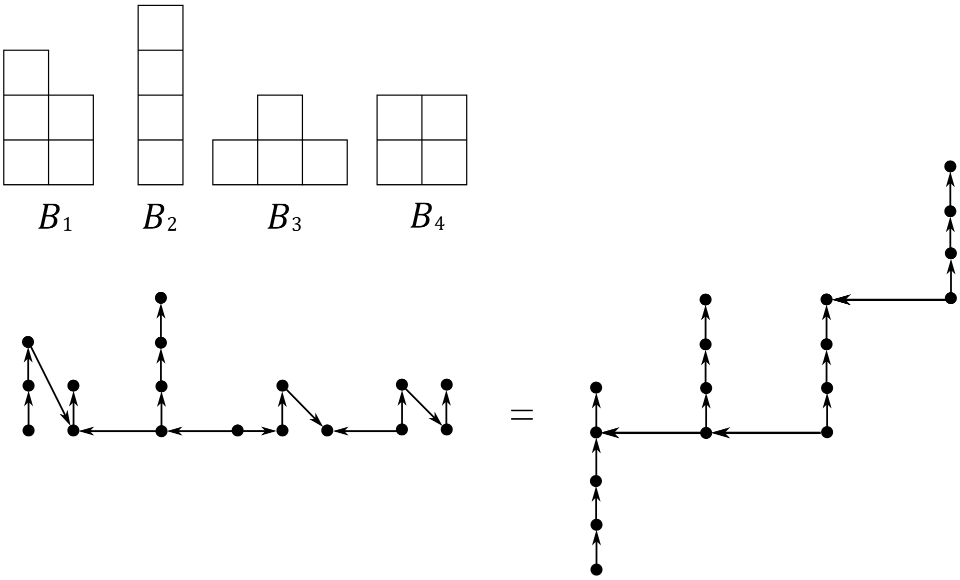


Figure 5.9: $\mathcal{GC}([3, 2], [4], [1, 2, 1], [2, 2])$ and $\Gamma([3, 2], [0, 4], [3, 1], [2, 2])$.

Applying Inclusion-Exclusion to the rightmost right-to-left arrow in the straightened Hasse diagram $\Gamma([s_1, p_{k_1,1}], \ldots, [s_m, p_{k_m,m}]$, we can compute the number of its linear extensions recursively. One Inclusion-Exclusion step is pictured in Figure 5.10 as an example, and for this example,

$$\mathrm{LE}(\Gamma([3, 2], [0, 4], [3, 1], [2, 2])) = \binom{17}{4}\mathrm{LE}(\Gamma([3, 2], [0, 4], [3, 1])) - \binom{4}{4}\mathrm{LE}(\Gamma([3, 2], [0, 4], [3, 5])). \quad (5.22)$$

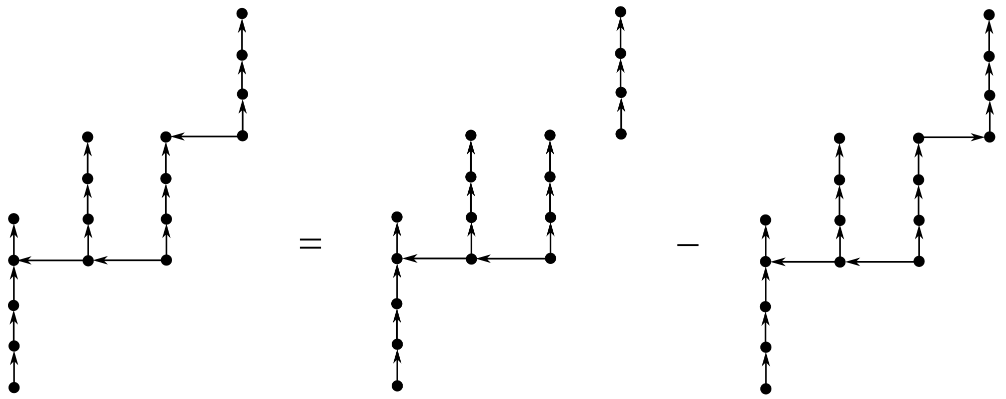


Figure 5.10: Recursion of LE(Γ([3, 2], [0, 4], [3, 1], [2, 2])).

In general,

$$\begin{aligned}
&\mathrm{LE}(\Gamma([s_1,p_{k_1,1}],[s_2,p_{k_2,2}],\ldots,[s_m,p_{k_m,m}])) = \\
&\qquad\binom{\sum_{j=1}^{n}(s_j+p_{k_j,j})}{s_m+p_{k_m,m}} \times \mathrm{LE}(\Gamma([s_1,p_{k_1,1}],\ldots,[s_{m-1},p_{k_{m-1},m-1}])) \\
&-\binom{p_{k_m,m}-1+s_m+p_{k_m,m}}{s_m+p_{k_m,m}} \times \mathrm{LE}(\Gamma([s_1,p_{k_1,1}],\ldots,[s_{m-1},p_{k_{m-1},m-1}+s_m+p_{k_m,m}]))
\end{aligned} \tag{5.23}$$

Although the formula seems long, the computation is indeed fairly tractable. To compute $GC_{n,(P,Q),\mathscr{R}}(x,y)$, we first need to generate all subsets

$$\mathcal{GC}([p_{1,1},p_{2,1},\ldots,p_{k_1,1}],[p_{1,2},\ldots,p_{k_2,2}],\ldots,[p_{1,m},\ldots,p_{k_m,m}])$$

satisfying $\sum_{j=1}^{m}\sum_{i=1}^{k_j} p_{i,j} = n$. Clearly, there are 3^{n-1} such subsets of $\mathcal{GC}_n$ in total, because we have three ways to extend a generalized cluster of size n to another of size $n+1$:

1. put the new element on the top of the last column in the last block
2. put the new element as a new column inserted in end of the last block
3. put the element as a new column in a new block.

Therefore, we could generate subsets of $\mathcal{GC}_n$ easily in manner of breath first search and then run the Inclusion-Exclusion recursion dynamically. For the weights of generalized clusters, keep in mind that, it is actually equal to

$$(-1)^{\mathrm{B}(H)}x^{\mathrm{Col}(H)-\mathrm{B}(H)}(y+1)^{\mathrm{Col}(H)}$$

because each column is marked with 'x' except the last column in each block and each column can be either marked with 'y' or not.

Generalized polynomial $GC_{n,(P,Q),\mathscr{R}}(x,y)$ for $n = 1, 2, \ldots, 7$ are listed as follows,

$$GC_1 = (y+1)$$

$$GC_2 = \begin{pmatrix} 1 \\ x \end{pmatrix}^T \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y+1 \\ (y+1)^2 \end{pmatrix}$$

$$GC_3 = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}^T \begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y+1 \\ (y+1)^2 \\ (y+1)^3 \end{pmatrix}$$

$$GC_4 = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}^T \begin{pmatrix} 6 & -11 & 6 & -1 \\ 0 & 5 & -19 & 11 \\ 0 & 0 & 3 & -11 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y+1 \\ (y+1)^2 \\ (y+1)^3 \\ (y+1)^4 \end{pmatrix}$$

$$GC_5 = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}^T \begin{pmatrix} 24 & -50 & 35 & -10 & 1 \\ 0 & 16 & -89 & 94 & -26 \\ 0 & 0 & 9 & -72 & 66 \\ 0 & 0 & 0 & 4 & -26 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y+1 \\ (y+1)^2 \\ (y+1)^3 \\ (y+1)^4 \\ (y+1)^5 \end{pmatrix}$$

$$GC_6 = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}^T \begin{pmatrix} 120 & -274 & 225 & -85 & 15 & -1 \\ 0 & 64 & -468 & 687 & -348 & 57 \\ 0 & 0 & 31 & -410 & 734 & -302 \\ 0 & 0 & 0 & 14 & -218 & 302 \\ 0 & 0 & 0 & 0 & 5 & -57 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y+1 \\ (y+1)^2 \\ (y+1)^3 \\ (y+1)^4 \\ (y+1)^5 \\ (y+1)^6 \end{pmatrix}$$

$GC_7 =$

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{pmatrix}^T \begin{pmatrix} 720 & -1764 & 1624 & -735 & 175 & -21 & 1 \\ 0 & 312 & -2818 & 5154 & -3630 & 1098 & -120 \\ 0 & 0 & 126 & -2444 & 6431 & -5058 & 1191 \\ 0 & 0 & 0 & 52 & -1462 & 4152 & -2416 \\ 0 & 0 & 0 & 0 & 20 & -585 & 1191 \\ 0 & 0 & 0 & 0 & 0 & 6 & -120 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y+1 \\ (y+1)^2 \\ (y+1)^3 \\ (y+1)^4 \\ (y+1)^5 \\ (y+1)^6 \\ (y+1)^7 \end{pmatrix}$$

We observe that the first row in each coefficient matrix of $GC_{n,(P,Q),\mathscr{R}}$ is signed Stirling number of the first kind. It is because that coefficients in each first row counts the the number generalized clusters having no marked P-matches, that is, each block is a cycle. Therefore, the first row counts permutations by the number of cycles which is exactly the interpretation of Stirling number of the first kind.

The first row in coefficient matrix of GC_n						
GC_1	1					
GC_2	1	-1				
GC_3	2	-3	1			
GC_4	6	-11	6	-1		
GC_5	24	-50	35	-10	1	
GC_6	120	-274	225	-85	15	-1

We observe that the last column in each coefficient matrix of $GC_{n,(P,Q),\mathscr{R}}$ is signed Eulerian number. It is because that coefficients in each first row counts the number generalized clusters where all columns are of height one. We know, inside a given block, there is forced to be an ascent between columns and for adjacent blocks, there is forced to be a descent between blocks. Therefore, it counts permutations by the number of descents which is exactly the interpretation of Eulerian numbers.

The last column in coefficient matrix of GC_n						
GC_1	1					
GC_2	-1	1				
GC_3	1	-4	1			
GC_4	-1	11	-11	1		
GC_5	1	-26	66	-26	1	
GC_6	-1	57	-302	302	-57	1

Elements on the main diagonal of each matrix count permutations where elements each cycle forms an interval by the number of cycles. This triangle is recorded in OEIS ([49]) as A084938.

The main diagonal in coefficient matrix of GC_n						
GC_1	1					
GC_2	1	1				
GC_3	2	2	1			
GC_4	6	5	3	1		
GC_5	24	16	9	4	1	
GC_6	120	64	31	14	5	1

It is easy to see that if we can compute $GC_{n,(P,Q),\mathscr{R}}(x-1,y-1)$ for $n \leq k$, we can plug those values into our formula for $1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{S}_n} x^{\text{Crise}(\sigma)}y^{\text{Cyc}(\sigma)}$ to compute the initial values of these series. In this case, we have computed that the first few initial terms of this series are

$$\begin{aligned}
& 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{S}_n} x^{\text{Crise}(\sigma)}y^{\text{Cyc}(\sigma)} \\
= \;& 1+yt+\frac{t^2}{2!}(y+xy^2)+\frac{t^3}{3!}(2y+y^2+2xy^2+x^2y^3) \\
& +\frac{t^4}{4!}(6y+6y^2+5xy^2+3xy^3+3x^2y^3+x^3y^4)t^4 \\
& +\frac{t^5}{5!}(24y+34y^2+16xy^2+3y^3+23xy^3+9x^2y^3+6x^2y^4+4x^3y^4+x^4y^5) \\
& +\cdots .
\end{aligned}$$

Setting $x=0$ and $y=1$, the coefficients of $\frac{t^n}{n!}$ are number of permutations avoiding cycle rises,

$$\begin{aligned}
& 1+t+\frac{t^2}{2!}+\frac{3t^3}{3!}+\frac{12t^4}{4!}+\frac{61t^5}{5!}+\frac{372t^6}{6!}+\frac{2639t^7}{7!}+\frac{21328t^8}{8!}+\frac{193403t^9}{9!} \\
& +\frac{1944730t^{10}}{10!}+\frac{21478849t^{11}}{11!}+\frac{258520960t^{12}}{12!}+\cdots ,
\end{aligned}$$

which is approximately equal to

$$\begin{aligned}
& 1+1.000\ t+0.500\ t^2+0.500\ t^3+0.500\ t^4+0.508\ t^5+0.517\ t^6+0.524\ t^7 \\
& +0.529\ t^8+0.533\ t^9+0.536\ t^{10}+0.538\ t^{11}+0.540\ t^{12}+\cdots+0.542\ t^{15} \\
& +\cdots+0.543\ t^{20}+\cdots+0.546\ t^{40}+\cdots+0.546\ t^{60}+\cdots+0.546\ t^{100}+\cdots
\end{aligned}$$

By observing coefficients above, we could ask, does the percentage of permutations avoiding cycle rises in $\mathcal{S}_n$ converge to some value between 54% and 55%? We shall leave this as on open problem.

5.3 Clusters and generalized clusters for undetermined shapes with restrictions

Now suppose shapes are not totally undetermined but are partially restricted. Similar to Chapter 3, we assume it is forced that the first column has height i and the last column has height j. For $n \geq 0$, We let $D_{i+n+j}^{i,j}$ denote the set of diagrams such that the first column has i rows and the last column has j columns, and the total number of cells is $i+n+j$. Except for the first and the last column, heights of the other columns are undetermined. For example, two elements in $D_{12}^{2,3}$ are pictured in Figure 5.11. $i=0$ means the first column has no restriction and $j=0$ means the last column has no restriction.

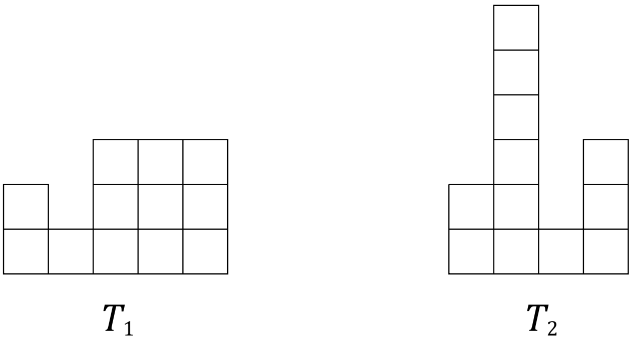


Figure 5.11: Two elements in $D_{12}^{2,3}$.

Similar to Theorem 3.6, it is necessary to define so-called start-clusters, end-clusters and start-end-clusters. Suppose $\mathcal{F}_{i+n+j,A}^{i,j}$ is the set of filling of $D_{i+n+j}^{i,j}$ with elements from the alphabet A. Given some pattern P, we let $\mathcal{MF}_{i+n+j,A,P}^{i,j}$ be the set of elements in $\mathcal{F}_{i+n+j,A}^{i,j}$ such that we mark some of P-matches. Note that since the pattern could be very flexible, even the first or the last column could be contained in some marked P-match. Then we define various types of clusters.

1. We say $F \in \mathcal{MF}^{0,0}_{n,A,P}$ is a P-**cluster** of size n if every column in F is contained in some marked P-match and any consecutive marked P-matches share at least one column. We let $\mathcal{C}^{0,0}_{n,A,P}$ denote the set of P-cluster of size n.
2. We say $F \in \mathcal{MF}^{i,0}_{i+n,A,P}$ is a P-**start-cluster** of size $i+n$ if every column in F is contained in some marked P-match and any consecutive marked P-match share at least one column. We let $\mathcal{SC}^{i,0}_{i+n,A,P}$ denote the set of P-start-cluster of size $i+n$.
3. We say $F \in \mathcal{MF}^{0,j}_{n+j,A,P}$ is a P-**end-cluster** of size $n+j$ if every column in F is contained in some marked P-match and any consecutive marked P-match share at least one column. We let $\mathcal{EC}^{0,j}_{n+j,A,P}$ denote the set of P-end-cluster of size $n+j$.
4. We say $F \in \mathcal{MF}^{i,j}_{i+n+j,A,P}$ is a P-**start-end-cluster** of size $i+n+j$ if every column in F is contained in some marked P-match and any consecutive marked P-match share at least one column. We let $\mathcal{SEC}^{i,j}_{i+n+j,A,P}$ denote the set of P-start-cluster of size $i+n+j$.

Then we define cluster polynomials as follows,

$$\begin{aligned}
C^{0,0}_{n,A,P}(x) &:= \sum_{F \in \mathcal{SC}^{i,0}_{n,A,P}} x^{P\text{-mch}(F)} \\
SC^{i,0}_{i+n,A,P}(x) &:= \sum_{F \in \mathcal{SC}^{i,0}_{i+n,A,P}} x^{P\text{-mch}(F)} \\
EC^{0,j}_{n+j,A,P}(x) &:= \sum_{F \in \mathcal{EC}^{0,j}_{n+j,A,P}} x^{P\text{-mch}(F)} \\
SEC^{i,j}_{i+n+j,A,P}(x) &:= \sum_{F \in \mathcal{SEC}^{i,j}_{i+n+j,A,P}} x^{P\text{-mch}(F)}
\end{aligned}$$

Then we have following theorems.

Theorem 5.4. *Given some pattern $P \subseteq \mathcal{F}^{0,0}_{r,A}$ and $i \geq 1$,*

$$\sum_{n\geq 1} t^n \sum_{F \in \mathcal{F}^{i,0}_{i+n,A}} x^{P\text{-}mch(F)} = \frac{\sum_{n\geq 0} t^{i+n} SC^{i,0}_{i+n,A,P}(x-1)}{1 - \sum_{n\geq 1} t^n \left(C^{0,0}_{n,A,P}(x-1) + |\mathcal{F}^{n,0}_{n,A}|\right)}.$$

Theorem 5.5. *Given some pattern $P \subseteq \mathcal{F}_{r,A}^{0,0}$ and $j \geq 1$,*

$$\sum_{n\geq 1} t^n \sum_{F\in\mathcal{F}_{n+j,A}^{0,j}} x^{P\text{-}mch(F)} = \frac{\sum_{n\geq 0} t^{n+j} EC_{n+j,A,P}^{0,j}(x-1)}{1-\sum_{n\geq 1} t^n \left(C_{n,A,P}^{0,0}(x-1)+|\mathcal{F}_{n,A}^{n,0}|\right)}.$$

Theorem 5.6. *Given some pattern $P \subseteq \mathcal{F}_{r,A}^{0,0}$ and $i, j \geq 1$,*

$$\sum_{n\geq 1} t^n \sum_{F\in\mathcal{F}_{i+n+j,A}^{i,j}} x^{P\text{-}mch(F)} = \frac{\left(\sum_{n\geq 0} t^{i+n} SC_{i+n,A,P}^{i,0}(x-1)\right)\left(\sum_{n\geq 0} t^{n+j} EC_{n+j,A,P}^{0,j}(x-1)\right)}{1-\sum_{n\geq 1} t^n \left(C_{n,A,P}^{0,0}(x-1)+|\mathcal{F}_{n,A}^{n,0}|\right)} + \sum_{n\geq 0} t^{i+n+j} SEC_{i+n+j,A,P}^{i,j}(x-1). \quad (5.24)$$

Now if we are given some binary relation $\mathscr{R}$, to study the distribution of patterns in arrays with $\mathscr{R}$, we need to define generalized $\mathscr{R}$-start-clusters, generalized $\mathscr{R}$-end-clusters and generalized $\mathscr{R}$-start-end-clusters.

Suppose $\mathcal{F}_{i+n+j,A,\mathscr{R}}^{i,j}$ is the subset of $\mathcal{F}_{i+n+j,A,\mathscr{R}}^{i,j}$ satisfying $\mathscr{R}$. Then we define various types of generalized clusters as follows.

We say $Q \in \mathcal{MF}_{n,A,P}^{0,0}$ is a **generalized $P,\mathscr{R}$-cluster** of size n if we can write $Q = B_1B_2\ldots B_m$ where for $1 \leq h \leq m$, B_h are blocks such that

1. B_h is either a single column or P-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$, and
2. for $1 \leq h \leq m-1$, the pair $(last(B_h), first(B_{h+1}))$ is not in $\mathscr{R}$ where for any h, $last(B_h)$ is the right-most column of B_h and $first(B_h)$ is the left-most column of B_h.

Let $\mathcal{GC}_{n,A,P,\mathscr{R}}^{0,0}$ denote the set of all generalized $P,\mathscr{R}$-clusters of size n. We define the weight of Q, to be

$$\omega_{P,\mathscr{R}}(Q) = (-1)^{m-1}x^{m_P(Q)}.$$

We let

$$GC_{n,A,P,\mathscr{R}}^{0,0}(x) := \sum_{Q\in\mathcal{GC}_{n,A,P,\mathscr{R}}^{0,0}} \omega_{P,\mathscr{R}}(Q).$$

We say $Q \in \mathcal{MF}_{i+n,A,P}^{i,0}$ is a **generalized $P,\mathscr{R}$-start-cluster** of size $i+n$ if we can write $Q = B_1B_2\ldots B_m$ where for $1 \leq h \leq m$, B_h are blocks such that

1. B_1 is a P-start-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$,
2. for $2 \leq h \leq m$, B_h is either a single column or P-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$, and
3. for $1 \leq h \leq m-1$, the pair $(last(B_h), first(B_{h+1}))$ is not in $\mathscr{R}$ where for any h, $last(B_h)$ is the right-most column of B_h and $first(B_h)$ is the left-most column of B_h.

Let $\mathcal{GSC}^{i,0}_{i+n,A,P,\mathscr{R}}$ denote the set of all generalized $P,\mathscr{R}$-start-clusters of size $i+n$. We define the weight of Q, to be

$$\omega_{P,\mathscr{R}}(Q) = (-1)^{m-1} x^{m_P(Q)}.$$

We let

$$GSC^{i,0}_{i+n,A,P,\mathscr{R}}(x) := \sum_{Q \in \mathcal{GSC}^{i,0}_{i+n,A,P,\mathscr{R}}} \omega_{P,\mathscr{R}}(Q).$$

We say $Q \in \mathcal{MF}^{0,j}_{n+j,A,P}$ is a **generalized $P,\mathscr{R}$-end-cluster** of size $n+j$ if we can write $Q = B_1 B_2 \ldots B_m$ where for $1 \leq h \leq m$, B_h are blocks such that

1. B_m is a P-end-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$,
2. for $1 \leq h \leq m-1$, B_h is either a single column or P-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$, and
3. for $1 \leq h \leq m-1$, the pair $(last(B_h), first(B_{h+1}))$ is not in $\mathscr{R}$ where for any h, $last(B_h)$ is the right-most column of B_h and $first(B_h)$ is the left-most column of B_h.

Let $\mathcal{GEC}^{0,j}_{n+j,A,P,\mathscr{R}}$ denote the set of all generalized $P,\mathscr{R}$-end-clusters of size $n+j$. We define the weight of Q, to be

$$\omega_{P,\mathscr{R}}(Q) = (-1)^{m-1} x^{m_P(Q)}.$$

We let

$$GEC^{0,j}_{n+j,A,P,\mathscr{R}}(x) := \sum_{Q \in \mathcal{GEC}^{0,j}_{n+j,A,P,\mathscr{R}}} \omega_{P,\mathscr{R}}(Q).$$

We say $Q \in \mathcal{MF}^{i,0}_{i+n,A,P}$ is a **generalized** $P,\mathscr{R}$**-start-end-cluster** of size $i+n+j$ if Q has at least two blocks, that is, we can write $Q = B_1B_2 \dots B_m$, $m \geq 2$ where for $1 \leq h \leq m$, B_h are blocks such that

1. B_1 is a P-start-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$,
2. B_m is a P-end-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$,
3. for $2 \leq h \leq m-1$, B_h is either a single column or P-cluster in which any pair of consecutive columns satisfies $\mathscr{R}$, and
4. for $1 \leq h \leq m-1$, the pair $(last(B_h), first(B_{h+1}))$ is not in $\mathscr{R}$ where for any h, $last(B_h)$ is the right-most column of B_h and $first(B_h)$ is the left-most column of B_h,

or if Q only has one block, that is, we can write $Q = B_1$, and B_1 is a P-start-end-cluster of size $i+n+j$.

Let $\mathcal{GSEC}^{i,j}_{i+n+j,A,P,\mathscr{R}}$ denote the set of all generalized $P,\mathscr{R}$-start-end-clusters of size $i+n+j$. We define the weight of Q, to be

$$\omega_{P,\mathscr{R}}(Q) = (-1)^{m-1} x^{m_P(Q)}.$$

We let

$$GSEC^{i,j}_{i+n+j,A,P,\mathscr{R}}(x) := \sum_{Q \in \mathcal{GSEC}^{i,j}_{i+n+j,A,P,\mathscr{R}}} \omega_{P,\mathscr{R}}(Q).$$

Then we have following theorems.

Theorem 5.7. *Given some pattern $P \subseteq \mathcal{F}^{0,0}_{r,A}$, a binary relation $\mathscr{R}$ and $i \geq 1$,*

$$\sum_{n\geq 1} t^n \sum_{F \in \mathcal{F}^{i,0}_{i+n,A,\mathscr{R}}} x^{P\text{-}mch(F)} = \frac{\sum_{n\geq 0} t^{i+n} GSC^{i,0}_{i+n,A,P,\mathscr{R}}(x-1)}{1 - \sum_{n\geq 1} t^n \left(GC^{0,0}_{n,A,P,\mathscr{R}}(x-1)\right)}.$$

Theorem 5.8. *Given some pattern $P \subseteq \mathcal{F}^{0,0}_{r,A}$, a binary relation $\mathscr{R}$ and $j \geq 1$,*

$$\sum_{n\geq 1} t^n \sum_{F \in \mathcal{F}^{0,j}_{n+j,A,\mathscr{R}}} x^{P\text{-}mch(F)} = \frac{\sum_{n\geq 0} t^{n+j} GEC^{0,j}_{n+j,A,P,\mathscr{R}}(x-1)}{1 - \sum_{n\geq 1} t^n \left(GC^{0,0}_{n,A,P,\mathscr{R}}(x-1)\right)}.$$

Theorem 5.9. *Given some pattern $P \subseteq \mathcal{F}^{0,0}_{r,A}$, a binary relation $\mathscr{R}$ and $i, j \geq 1$,*

$$\sum_{n\geq 1} t^n \sum_{F\in\mathcal{F}^{i,j}_{i+n+j,A,\mathscr{R}}} x^{P\text{-}mch(F)} = \\ \frac{\left(\sum_{n\geq 0} t^{i+n} GSC^{i,0}_{i+n,A,P,\mathscr{R}}(x-1)\right)\left(\sum_{n\geq 0} t^{n+j} GEC^{0,j}_{n+j,A,P,\mathscr{R}}(x-1)\right)}{1-\sum_{n\geq 1} t^n \left(GC^{0,0}_{n,A,P,\mathscr{R}}(x-1)\right)} \\ + \sum_{n\geq 0} t^{i+n+j} GSEC^{i,j}_{i+n+j,A,P,\mathscr{R}}(x-1). \quad (5.25)$$

Essentially we can prove the theorems above by combining the proof of Theorem 3.6 and proof of Theorem 5.3.

We can also prove multi-variate analogues of all these theorems. Also we can extend theorems above to consider patterns in $\mathcal{P}^{i,j}_{i+n+k,\mathscr{R}}$, where $\mathcal{P}^{i,j}_{i+n+j,\mathscr{R}}$ is the set of fillings of $D^{i,j}_{i+n+j}$ with elements from $\{1, 2, \ldots, i+n+j\}$ such that elements in each columns are increasing from bottom to top and also satisfies $\mathscr{R}$. In this situation, the resulting generating functions are exponential generating functions rather than ordinary generating functions.

The contents of Chapter 5 are currently under preparation for submission. Some portion is co-authored with J. B. Remmel. The dissertation author is the author of this material.

Chapter 6

Conclusion and further research

In this final chapter, we will discuss some of the limitations of generalized cluster method, the connection between generalized clusters and joint clusters, and some questions for research in the future.

6.1 Limitation of generalized cluster method

In this thesis, we developed a powerful method, which we called the generalized cluster method, to find generating functions for the number of consecutive occurrence of a pattern or sequence of patterns in various classes of combinatorial objects. This method is quite general and can be adapted to handle a large range of combinatorial objects beyond the ones that we considered in this thesis.

However, the generalized cluster method also inherits the limitations of the cluster method. The main limitation is due to the fact that in many cases, finding explicit formulas or recursions for the cluster polynomials or the generalized cluster polynomials is extremely difficult. In many examples, the computation of cluster polynomials or generalized cluster polynomials require that we find the number of linear extensions of a class of posets. However, the general problem of counting the number of linear extensions of a poset is known to be a very difficult problem. Indeed, in [9], Brightwell and Winkle showed that in general, it is #P-complete to count linear extensions.

Thus while the generalized cluster method is a powerful tool, it may not be the best tool to compute generating function of the number of consecutive occurrence of a given pattern is every situation. For example, suppose we let $\mathcal{F}_{2n}^{0,0,2}$ denote the set of fillings of $D_{2n}^{0,0,2}$ with elements in $\{1,2,\ldots,2n\}$. Let $\mathscr{R}$ be the relation that holds for a pair of column (C,D) of height 2 if and only if the base element in C is less than the base element in D and the top element in C is also less than the top element in D. Thus the elements $F \in \mathcal{F}_{2n}^{0,0,2}$ in which every consecutive pair of columns in F satisfies $\mathscr{R}$ are the ones that are increasing in rows, reading from left to right. Clearly,

$$|\mathcal{F}_{2n,\mathscr{R}}^{0,0,2}| = \binom{2n}{n}.$$

For example, the six elements in $\mathcal{F}_{4,\mathscr{R}}^{0,0,2}$ are pictured in Figure 6.1.

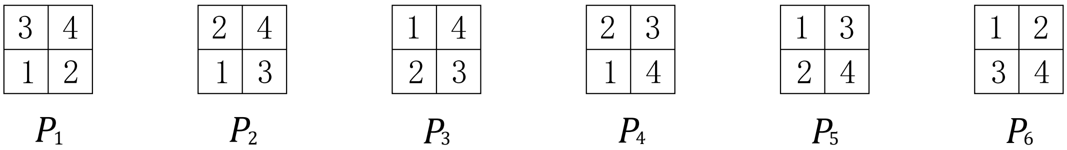


Figure 6.1: The six elements in $\mathcal{F}_{4,\mathscr{R}}^{0,0,2}$.

Suppose we want to compute the generating function

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{F\in\mathcal{F}_{2n,\mathscr{R}}^{0,0,2}} x^{P_j\text{-mch}(F)}.$$

By Theorem 2.4, we have

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{F\in\mathcal{F}_{2n,\mathscr{R}}^{0,0,2}} x^{P_j\text{-mch}(F)} = \frac{1}{1-\sum_{n\geq 1}\frac{t^{2n}}{(2n)!}GC_{2n,P_j,\mathscr{R}}^{0,0,2}(x-1)}.$$

We can calculate $GC_{2n,P_j,\mathscr{R}}^{0,0,2}(x)$ by counting linear extensions of certain Hasse diagrams. Details of computation will not be given here, but the kinds of computations that one needs are similar to kinds of computations that we carried out in Section 2.2.2. We were unable to find a recursive formula for the required generalized joint cluster polynomials in the case where we wanted to compute the multi-variate generating function

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{F\in\mathcal{F}_{2n,\mathscr{R}}^{0,0,2}} \prod_{j=1}^{6} x_j^{P_j\text{-mch}(F)}.$$

In fact, in this case, counting the number of linear extensions seemed no easier than directly counting the number pattern matches in $\mathcal{F}^{0,0,2}_{2n,\mathscr{R}}$.

However, the author and Jeff Remmel found an alternative way to find the distributions of patterns $\{P_j\}_{1\leq j\leq 6}$ without using the generalized cluster method. In [53], we defined so-called paired patterns for lattice paths. There is a trivial bijection between $\mathcal{F}^{0,0,2}_{2n,\mathscr{R}}$ and the set of all grand Dyck paths from $(0,0)$ to (n,n). The bijection is quite simple. We simply require that if k is in the base row of F, then k-th step in the corresponding path is an east step, and otherwise, it is a north step. Via this bijection, the six patterns $P_1, P_2, \ldots, P_6$ can be represented by six grand Dyck paths from $(0,0)$ to $(2,2)$, as drawn in Figure 6.2.

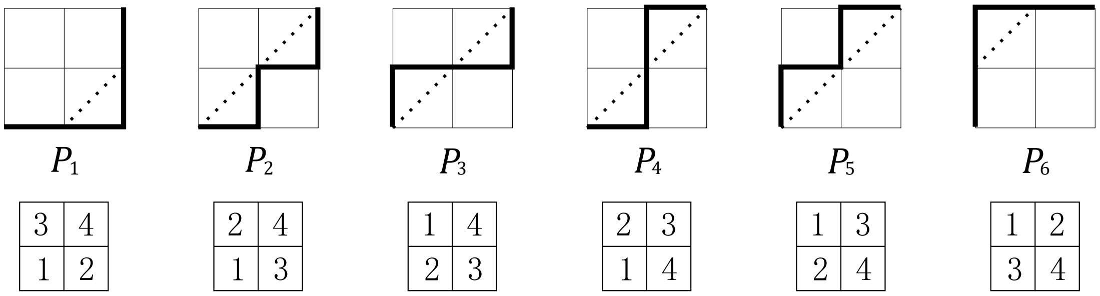


Figure 6.2: The six elements in $\mathcal{F}^{0,0,2}_{4,\mathscr{R}}$ and their corresponding paths.

In [53], we proved that

1. the number of P_1-match in $F \in \mathcal{F}^{0,0,2}_{2n,\mathscr{R}}$ is the number of east steps below the sub-diagonal $y = x - 1$ in the corresponding path

2. the number of P_2-match in $F \in \mathcal{F}^{0,0,2}_{2n,\mathscr{R}}$ is the number of times the corresponding path bounce off the diagonal $y = x$ to the right

3. the number of P_3-match in $F \in \mathcal{F}^{0,0,2}_{2n,\mathscr{R}}$ is the number of times the corresponding path cross the diagonal $y = x$ horizontally.

By symmetry, there are similar interpretations for P_4, P_5 and P_6. Based on these facts, we were able to obtain direct recursions for the number of occurrences of the patterns P_i which allowed us to compute the ordinary generating function for the number of occurrences of these patterns in grand Dyck paths. For example, we proved that the ordinary generating function for the number of occurrences of the patterns P_2 and P_4 has following explicit formula

$$
\begin{aligned}
&1+\sum_{n\geq 1} t^n \sum_{F\in\mathcal{F}^{0,0,2}_{2n,\mathscr{R}}} x_2^{P_2\text{-mch}(F)} x_4^{P_4\text{-mch}(F)} \\
= &\frac{(x_2-2)\left(-1+\sqrt{1-4t}\right)+2(x_2-1)t}{x_4\left(-1+\sqrt{1-4t}\right)+x_2\left(2+\left(-1+\sqrt{1-4t}\right)+3x_4-x_4\sqrt{1-4t}\right)t}.
\end{aligned}
$$

However, in this case, we could not directly compute the corresponding generalized joint cluster polynomials to find an exponential generating function for the number of occurrences of the patterns P_2 and P_4.

6.2 Connections between joint clusters and generalized clusters

In Chapter 4, we defined natural analogues of clusters and generalized clusters in the case where we want to keep track of the occurrence of several consecutive patterns at the same time. This lead us to define joint clusters and generalized joint clusters. The main point that we want to make in this subsection is that one can use joint clusters to compute the same type of generating functions that we computed by the generalized cluster method. That is, in the setting of generalized clusters, we considered binary relations $\mathscr{R}$ between pairs of consecutive columns and we wanted to consider only those fillings in which any two consecutive columns satisfied $\mathscr{R}$. Another way to obtain the same set of fillings, is to consider complement of $\mathscr{R}$, $\neg\mathscr{R}$. $\neg\mathscr{R}$ can always be translated to some 2-column consecutive pattern or some set of 2-column consecutive patterns. Then we just need to find the multi-variate generating function of the number of consecutive occurrences of the pattern $P = P_1$ that we are interested in plus the number of of occurrences of the patterns $P_2, \ldots, P_k$ that correspond to $\neg\mathscr{R}$ and, then set all the variables x_i for $i = 2, \ldots, k$ equal to 0.

For example, suppose the pattern is $P = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$ and $\mathscr{R}$ is a relation that holds between a pair of columns (C, D) of height 2 which are increasing, reading from bottom to top, if and only if the top element in C is greater than the base element in D. Suppose we are interested in distribution of pattern P in $\mathcal{P}^{0,0,2}_{2n,\mathscr{R}}$. In this case, there is a single pattern corresponding $\neg\mathscr{R}$, namely, $R = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$. By Theorem 2.4,

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{F \in \mathcal{P}^{0,0,2}_{2n,\mathscr{R}}} x^{P\text{-mch}(F)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} GC_{2n,P,\mathscr{R}}(x-1)}.$$

And by Theorem 4.1,

$$1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \sum_{F \in \mathcal{P}^{0,0,2}_{2n}} x^{P\text{-mch}(F)} y^{R\text{-mch}(F)} = \frac{1}{1 - \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} C_{2n,(P,R)}(x-1, y-1)}.$$

Thus it follows that

$$\sum_{F \in \mathcal{P}^{0,0,2}_{2n,\mathscr{R}}} x^{P\text{-mch}(F)} = \sum_{F \in \mathcal{P}^{0,0,2}_{2n}} x^{P\text{-mch}(F)} 0^{R\text{-mch}(F)},$$

which implies

$$GC_{2n,P,\mathscr{R}}(x-1) = C_{2n,(P,R)}(x-1, 0-1) = C_{2n,(P,R)}(x-1,-1).$$

A more straightforward way to understand this fact is that when we compute $C_{2n,(P,R)}(x,-1)$, we are considering joint (P,R)-clusters where we place 'x' on marked P-matches and '-1' instead of 'y' on marked R-matches. It is not hard to see that this labeling will lead to generalized $P,\mathscr{R}$-clusters.

6.3 Directions for further research

In this thesis, we only explored restrictions on combinatorial objects which corresponding to simple binary relations $\mathscr{R}$ between pairs of consecutive columns. However, we could consider more general restrictions such as the ones that arise by insisting that any k consecutive columns in a filling satisfies a k-ary relation where $k \geq 3$. One cannot compute the generating function for the number of consecutive occurrences of a pattern P in the set of fillings which meet such a restriction by a direct application of generalized cluster method. However, in principle, we could modify joint cluster, as described in our discussion in Section 6.2, to compute such generating functions. That is, we could represent $\neg\mathscr{R}$ as some pattern or a set of patterns, denoted by N. Then we could joint (P, N)-cluster polynomials where marked N-matches have the label '-1'. We call joint (P, N)-clusters with such labeling by negative $P, \mathscr{R}$-clusters, denoted by $\mathcal{NC}_{n,P,\mathscr{R}}$. Then we define

$$NC_{n,P,\mathscr{R}}(x) := \sum_{F \in \mathcal{NC}_{n,P,\mathscr{R}}} \omega_{P,\mathscr{R}}(x),$$

where $\omega_{P,\mathscr{R}}(x)$ is product of all the labels of F.

For example, suppose that we wanted to compute the generating function of the distribution of ascents in $\mathcal{S}_n$ where the ternary relation $\mathscr{R}$ holds on three consecutive elements (a, b, c) such that a, b, c is not a monotonically increasing sequence. Then $N = 123$ and $P = 12$. An example of negative $P, \mathscr{R}$-cluster of length 7 is given in Figure 6.3.

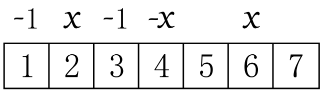


Figure 6.3: An example of negative $P, \mathscr{R}$-cluster of length 7.

In this case, negative cluster polynomials are easy to compute,

$$\begin{aligned} NC_{1,(P,R)}(x) &= 1, \\ NC_{2,(P,R)}(x) &= x, \\ &\text{and} \\ NC_{n,(P,R)}(x) &= -(x+1)N_{n-2,(P,R)}(x) - N_{n-1,(P,R)}(x), \quad \text{for } n \geq 3. \end{aligned}$$

In general, computing negative cluster polynomials with respect to relation $\mathscr{R}$ which involves more than two columns is much more difficult than computing generalized clusters. Thus a natural problem for further research is to develop methods to handle restrictions involving k-ary relations $\mathscr{R}$ where $k \geq 2$ by breaking the problem into various sub-problems which are easier to compute. Another direction of further research is to compute generating functions where we keep track of more information such as keeping track of the number of consective occurrences of a pattern P plus keeping track of inversions, co-inversions or the major index in $w(F)$. This should lead to natural q-analogues of many of the results of this thesis.

A portion of Chapter 6 is has been submitted to a special volume on Lattice Path Combinatorics and Applications in the Springer "Developments in Mathematics Series". R. Pan and J. B. Remmel, Paired patterns in lattice paths, submitted.

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