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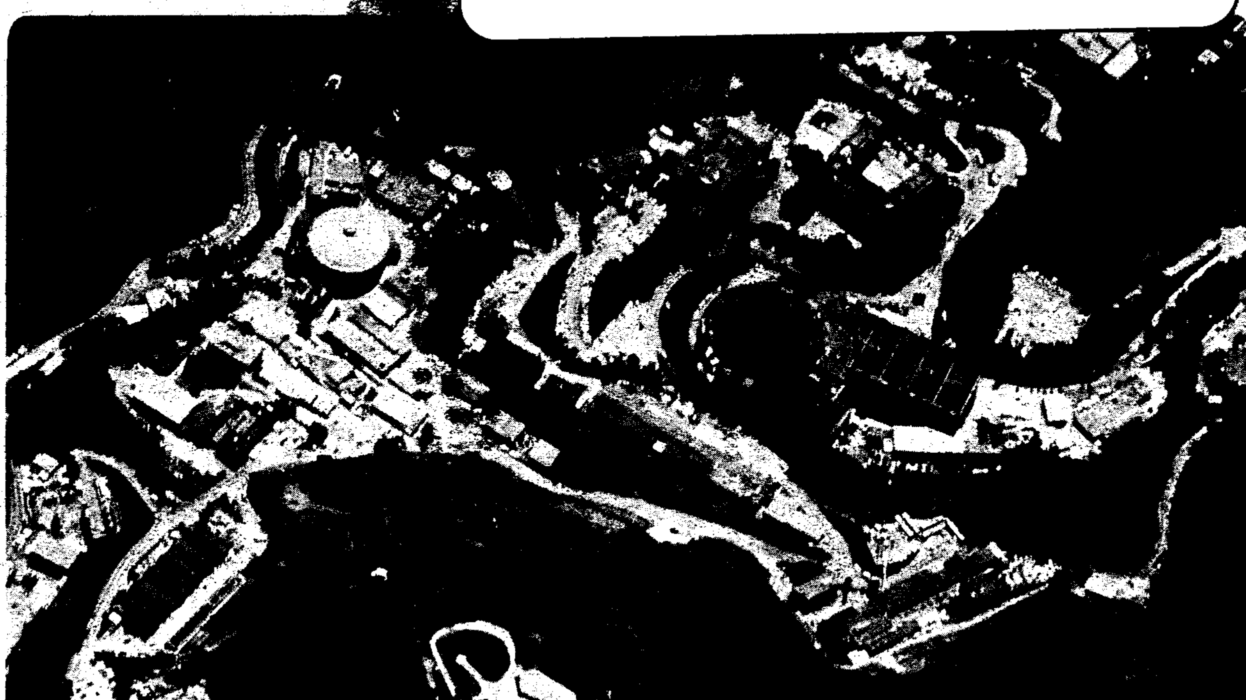

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On the accuracy of the vortex method[1]

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On the accuracy of the vortex method

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Introduction

The vortex method is a grid free method that simulates fluid flow by approximating the vorticity by blobs of vorticity and computing their evolution. We briefly describe the vortex method for an inviscid, incompressible fluid in the absence of boundaries. A detailed description of the method can be found in [3],[4],[5],[9],[10].

Consider Euler's equations

$$\omega_t + (u.\nabla)\omega = 0,$$
$$\Delta\Psi = -\omega,$$
$$u_1 = \Psi_y, \qquad u_2 = -\Psi_x,$$

where $u = (u_1, u_2)$ is the velocity vector, $z = (x, y)$ is the position vector, ω is the vorticity and Ψ is the stream function.

We write Ψ as a convolution of the Green's function of the Laplace operator with ω; the velocity u is then given by the Biot-Savart integral

$$u(z,t) = K * \omega \equiv \int K(z - z')\, \omega(z')\, dz',$$

where

$$K(z) = -\frac{1}{2\pi}\begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} \log|z| = -\frac{1}{2\pi|z|^2}\begin{pmatrix} y \\ -x \end{pmatrix}. \tag{1}$$

Assume the vorticity ω has bounded support in the ball of radius R centered at the origin. Introduce a grid with squares B_j of side h centered at $jh = (j_1, j_2)h$ and approximate the initial vorticity distribution by

$$\omega^h(z) = \sum_j \psi_\delta(z - jh)\, c_j, \tag{2}$$

where the c_j's have one of the following two forms

$$c_j = \int_{B_j} \omega(z)\, dz, \tag{2.a}$$

$$c_j = \omega(jh)\, h^2, \tag{2.b}$$

and ψ_δ is a smooth approximation to the Dirac delta function, defined by $\psi_\delta(z) = \frac{1}{\delta^2}\psi(\frac{z}{\delta})$, where ψ satisfies the following conditions:

(i) $\psi \,\varepsilon\, C^2(R^2)$

(ii) *Moment condition:*

$$\int \psi(z)\, dz = 1$$

$$\int z^\gamma \psi(z)\, dz = 0 \qquad \gamma = (\gamma_1, \gamma_2) \qquad 1 \le |\gamma| \le p - 1$$

(iii) *For some $L > 0$, and for any multi-index β the Fourier transform $\hat{\psi}(\xi)$ satisfies*

$$\sup_{\xi \varepsilon R^2} |D_\xi^\beta \hat{\psi}(\xi)| \le C_\beta (1 + |\xi|)^{-L-\beta}$$

ψ_δ is said to be of order p if (ii) holds.

The vorticity approximation (2) by a sum of vortex blobs results in the velocity approximation $\tilde{u}^h$:

$$\tilde{u}^h(z,t) = \sum_j K_\delta(z - \tilde{z}_j(t))\, c_j, \tag{3}$$

where K_δ is defined by

$$K_\delta(z) = K * \psi_\delta = \int K(z-z')\, \psi_\delta(z')\, dz',$$

the corresponding vorticity is

$$\tilde{\omega}^h(z,t) = \sum_j \psi_\delta(z - \tilde{z}_j(t))\, c_j, \tag{4}$$

where $\tilde{z}_j(t)$ are the approximate particle paths which can be found by solving the system of ordinary differential equations

$$\frac{d\tilde{z}_j}{dt} = \tilde{u}_j(\tilde{z}_j, t), \qquad \tilde{z}_j(0) = jh.$$

The accuracy of the vortex method depends both on the approximation of the initial vorticity distribution and the choice of cutoff functions ψ_δ.

Hald and Del Prete [8] and Hald [7] using a special class of cutoff functions and the vorticity approximation (2.a) proved that the vortex method converges to the solution of Euler's equations in the absence of boundaries.

Recently Beale and Majda [1] using the vorticity approximation (2.b) and a more general class of cutoff functions proved that the vortex method can be made to converge with arbitrarily high accuracy, under the same restrictions. In their proof they used stability and consistency estimates to establish convergence.

Cottet [6] proved that using the vorticity approximation (2.a), the vortex method converges only with second order accuracy, for any cutoff function satisfying $(i)-(iii)$ with $p \geq 4$.

Let $z_j(t)$ denote the exact particle paths, $z_j(0) = jh$, i.e. the particle paths determined by the exact solution of Euler's equations. Let $u^h(z,t)$ and $\omega^h(z,t)$ be the discrete approximations to the velocity and the vorticity determined by the z_j's , i.e. the velocity and vorticity fields obtained by using $z_j(t)$ rather than

$\tilde{z}_j(t)$ in (3) and (4),

$$u^h(z,t) = \sum_j K_\delta(z - z_j(t))\, \omega_j h^2, \tag{5}$$

$$\omega^h(z,t) = \sum_j \psi_\delta(z - z_j(t))\, \omega_j h^2. \tag{6}$$

The consistency error is defined by

$$E_u = \| u - u^h \| \tag{7}$$

for the velocity, and

$$E_\omega = \| \omega - \omega^h \| \tag{8}$$

for the vorticity, where E_u and E_ω depend on the mesh length h and the time t.

The stability error measures the difference between u^h and the discrete velocity approximation $\tilde{u}^h$ due to a collection of vortex blobs moving under the influence of the computed particle paths $\tilde{z}_j$.

Beale and Majda estimated the consistency error as the sum of two terms. The first term, the smoothing error, is due to the fact that the singular kernel K in (1) is replaced by the smooth kernel $K_\delta = K * \psi_\delta$ resulting in the velocity approximation:

$$u^\delta(z,t) = \int K_\delta(z - z')\, \omega(z')\, dz',$$

which can also be viewed as approximating the vorticity ω by $\omega^\delta = \psi_\delta * \omega$

$$\omega^\delta(z,t) = \int \psi_\delta(z - z')\, \omega(z')\, dz'.$$

The smoothing error depends on the parameter δ and on the time t and is defined by:

$$E_u^S = \| u - u^\delta \| \qquad E_\omega^S = \| \omega - \omega^\delta \| \tag{9}$$

The second term, the discretization error, is due to the fact that we approximate u^δ and ω^δ by their discrete analogues u^h and ω^h defined in (5), (6). We denote this error term by E^D. It depends upon the mesh length h, the parameter δ and the time t.

$$E_u^D = \| u^\delta - u^h \| \qquad E_\omega^D = \| \omega^\delta - \omega^h \| . \tag{10}$$

Beale and Majda have shown that provided the flow is smooth, the first error term E^S is of order δ^p, where p is related to the number of moments of the cutoff function ψ that vanish, while the discretization error E^D is of order $\delta^{-L} h^{-L-1-\varepsilon}$. Here h is the initial distance between the vortices, and L measures the decay of the Fourier transform of ψ. The best error estimates are attained when the two errors E^S and E^D are in balance. Choosing $\delta = h^q$ with $q = \dfrac{L-1-\varepsilon}{L+p}$, we balance the errors and obtain a total error of order h^{pq}. Cutoff functions ψ with L arbitrarily large, (for example Gaussian cutoff functions) allow us to choose $\delta = h^{1-\varepsilon}$, ε small, and obtain essentially a p^{th} order method.

Note that the Hald and Beale-Majda proofs do not establish that cutoff functions ψ which fail to satisfy conditions $(i)-(iii)$ above cannot lead to convergence. Chorin [3], [4], [5], used a different cutoff function which does not satisfy these conditions, but has been shown experimentally to be of second order accuracy [5],[8].

To test the accuracy of the vortex method in practice, we carried out a number of numerical experiments with several choices of cutoff functions and different values of h and δ. We measured the consistency errors E_u and E_ω as well as their components, the smoothing and discretization errors. These results are presented in the next section.

In the numerical experiments we used cutoff functions ψ which are linear combinations of gaussian functions as suggested in [2]. Since both ψ and its Fourier transform decay rapidly, L is arbitrarily large allowing us in principle to choose $\delta = h^{1-\varepsilon}$ with ε small.

The numerical experiments show that if $p \geq 4$ and δ close to h, then the error develops in an unexpected fashion both as a function of h and as a function of t. Looking at the two components of the error we find that this behavior is due to the discretization error, which grows sharply in time and for $t > 0$ does not decrease as $h \to 0$. This behavior of the discretization error is present for all $\delta = h^q$ with $0.5 < q < 1$, but the error decreases as δ increases.

The decrease of the discretization error as δ increases, and the fact that the smoothing error increases with δ, allow us to eliminate the 'odd' behavior of the consistency error by choosing $\delta = h^q$ substantially larger than h, i.e., $q \leq 0.65$ for $p = 4$ and $q \leq .0.60$ for $p = 6, 8$. Thus the accuracy provided by this new class of cutoff functions is reduced, i.e., instead of p^{th} order accuracy for a p^{th} order cutoff $(p \geq 4)$, we obtain pq order accuracy, $q \leq .65$.

Numerical results

In this section we present the numerical experiments carried out to test the accuracy of the vortex method. As initial vorticity distribution we choose a radially symmetric function:

$$\omega(z) = \begin{cases} (1 - |z|^2)^7 & |z| \leq 1 \\ 0 & |z| > 1 \end{cases}$$

The corresponding solution of Euler's equations is:

$$u(z) = \begin{cases} -\dfrac{1}{16|z|^2}(1 - (1 - |z|^2)^8)(y, -x) & |z| \leq 1 \\ -\dfrac{1}{16|z|^2}(y, -x) & |z| > 1 \end{cases}$$

We measure the consistency errors E_u and E_ω defined in (7), (8) in the discrete L^2 norm

$$E_u = \left(\sum_j |u(z_j,t) - u^h(z_j,t)|^2 h^2\right)^{½},$$

$$E_\omega = \left(\sum_j |\omega(z_j,t) - \omega^h(z_j,t)|^2 h^2\right)^{½},$$

where z_j, u, u^h and ω^h are as defined in the previous section.

We also estimate the rate of convergence by :

$$\textit{rate of convergence} = \frac{\log\left(E_{h_1} / E_{h_2}\right)}{\log\left(h_1 / h_2\right)}.$$

In our calculations we use gaussian cutoff functions of various orders (suggested by Beale and Majda, [2]):

$$(i) \quad p = 2 \qquad \psi_\delta = \frac{1}{2\pi\delta^2} e^{-\frac{r^2}{2\delta^2}}$$

$$(ii) \quad p = 4 \qquad \psi_\delta = \frac{1}{\pi\delta^2}\left(2 e^{-\frac{r^2}{\delta^2}} - \frac{1}{2} e^{-\frac{r^2}{2\delta^2}} \right)$$

$$(iii) \quad p = 6 \qquad \psi_\delta = \frac{1}{\pi\delta^2}\left(\frac{8}{3} e^{-\frac{r^2}{\delta^2}} - e^{-\frac{r^2}{2\delta^2}} + \frac{1}{12} e^{-\frac{r^2}{4\delta^2}} \right)$$

$$(iv) \quad p = 8 \qquad \psi_\delta = \frac{1}{\pi\delta^2}\left(\frac{64}{21} e^{-\frac{r^2}{\delta^2}} - \frac{4}{3} e^{-\frac{r^2}{2\delta^2}} + \frac{1}{6} e^{-\frac{r^2}{4\delta^2}} - \frac{1}{168} e^{-\frac{r^2}{8\delta^2}} \right)$$

The runs were made for $.05 \le h \le 0.2$, where h is the initial spacing between the particles. This corresponds to 60-950 vortices. We let $\delta = h^q$ with $0.5 < q < 1$ and assume that $0 \le t \le 20$. At time $t = 20$ the particles near zero travel 10 radians while those on $|z| = 1$ travel 1.25 radians.

We find that the errors using the cutoff functions $(ii)-(iv)$ are qualitatively similar. Hence we group them together as higher order cutoff functions as opposed to the second order cutoff function (i) for which the results are different.

The errors E_u and E_ω depend on three parameters: h, the initial distance between the particles, δ, the core size which depends on h, and the time t. For any of the cutoff functions $(ii)-(iv)$, we find that as a function of each one of these three parameters the errors E_u and E_ω develop in an unexpected fashion.

As a function of t ,for $\delta = h^q$ with $.75 \leq q < 1$, E_u and E_ω increase sharply reaching a maximum and then oscillating. Let T_c the time at which the maximum occurs. Although $T_c > 12$ (one rotation of the inner particles), T_c does not necessarily increase as $h \to 0$ (Figures 1a-b).

As a function of h, and with δ as above, neither E_u nor E_ω decrease uniformly as $h \to 0$, this is clearly seen when we compute the rate of convergence, (see figures 2a-b). The rate of convergence stays constant for a short time interval and then decreases sharply. The time interval becomes shorter as $h \to 0$. We also find that after $T > 0$ the errors do not decrease with h. This effect is more pronounced for the consistency error in the vorticity than in the velocity.

Consider now a fixed h and let $\delta = h^q$ with $0.5 < q < 1$. Beale and Majda's estimates [1] show that if $\delta = h^q$ with $q < 1$ then the error is of order h^{pq} , where p is the order of the cutoff function. Hence the error should increase as q decreases. We find that this holds for a short time interval $[0, T_*]$. This time interval becomes shorter as $h \to 0$ and as p increases. For $t > T_*$ and $p = 4$ the error decreases for $0.75 < q < 1$ and increases for $q < 0.75$, while for $t > T_*$ and $p = 6, 8$ the error decreases for $0.65 < q < 1$ and increases for $q < 0.65$, (Figures 3a-b).

As q decreases the sharp increase of the error in time is gradually attenuated and we observe a more uniform decrease of the error as a $h \to 0$. If $q \leq 0.65$ for a fourth order cutoff function and $q \leq 0.60$ for the 6^{th} and 8^{th} order cutoff functions then the accuracy is asymptotically pq throughout the interval

[0, 20]. For $h = 0.05$, which corresponds to 925 vortices, and $\delta = h^{.65}$ the errors are less than 1% for the velocity and between 2 and 3% for the vorticity, with the cutoff function (ii). For the same h but $\delta = h^{.60}$ the errors are 0.6% for the velocity and 1.4 - 2% for the vorticity, with the cutoff function (iii). Finally for the cutoff function (iv) we obtain an error of 0.3% in the velocity and 0.8 - 1.8% in the vorticity (Tables 1a-b).

In contrast to the higher order cutoffs, we obtain good results for $\delta = h^q$, $q \leq 0.9$, with the second order cutoff function (i). For example with $\delta = h^{.9}$, $h = 0.05$, the errors are 3% and 5% for the velocity and vorticity (Table 2).

To further understand the error behavior and following the spirit of the proof in [1], we measured the two components of the error, namely the smoothing error and the discretization error, in the discrete L^2 norm:

$$E_\omega^S = \left(\sum_j | \omega(z_j) - \omega^\delta(z_j,t) |^2 h^2 \right)^{\frac{1}{2}}$$

$$E_\omega^D = \left(\sum_j | \omega^\delta(z_j,t) - \omega^h(z_j,t) |^2 h^2 \right)^{\frac{1}{2}}$$

where ω^δ and ω^h are as defined in (9), (10).

We computed ω^δ by numerical integration using the routine D01DAF of the NAG library, with an error tolerance of 10^{-7}. Since ω^δ does not change in time and neither does ω, the smoothing error E_ω^S remains constant for all t. It is therefore enough to look at E_ω^S at time $t = 0$. From tables 3a-b we find that E_ω^S is asymptotically of order δ^p for a p^{th} order cutoff function.

The discretization error E_ω^D has the same qualitative behavior for all cutoff functions $(i)-(iv)$. We find from figures 4a-d that the discretization error E_ω^D increases sharply in time , reaching a maximum, oscillating later on. The time at which the maximum occurs changes with h. As a function of h the error decreases at $t = 0$, but for later times the error does not decrease as $h \to 0$. This behavior in h and t is present for all $\delta = h^q$ with $0.5 < q < 1$.

As a function of δ, keeping h fixed, the error E_ω^D decreases as δ increases. This indicates the presence of a negative power of δ in E_ω^D, (see [1], [7]).

If we keep h and δ fixed, and compare the discretization error E_ω^D for the different cutoff functions, we find that the error for the second order cutoff function (i) is substantially smaller than the errors for the higher order cutoff functions (ii)–(iv) . The latter are of comparable size, but increase slightly as p increases.

Having observed the behavior of the smoothing and discretization errors we can understand how the consistency error develops as a function of h , δ and t. Consider the second order cutoff function (ii), as we mentioned above the discretization error E_ω^D increases sharply in time and for $T > 0$, does not decrease as $h \to 0$, however it is small relative to the size of the smoothing error E_ω^S, which is of order δ^p. Thus the 'odd' behavior of E_ω^D is not felt in the total consistency error and we obtain an accuracy of order $2q$ with $q \le 0.9$.

For higher order cutoff function and $\delta = h^q, 0.75 < q < 1$ the sharp increase of the error in time and its behavior as $h \to 0$ is caused by its discretization component. We observe that the consistency error is almost equal to the discretization error. This indicates that except for a short initial time, there is no balance of the error components, but the dominant term in the total consistency error is the discretization component. Because E^S and E^D are of opposite character; i.e., E^S increases with δ while E^D decreases as δ increases, we are able to attenuate the sharp increase of the error as a function of t and to eliminate the uneven decrease as a function of h , by increasing δ so that E^S becomes the dominant term. We can clearly observe this in Table 4 which compares the consistency error E_ω to the smoothing and discretization errors for $\delta = h^{.65}$ and the cutoff function (ii).

Conclusion

Numerical experiments carried out to test the accuracy of the vortex method using cutoff functions suggested in [2] show that less than p^{th} order accuracy is obtained with a p^{th} order cutoff function. The rapid decrease of these cutoff functions and of their Fourier transform should allow us to choose $\delta = h^{1-\varepsilon}$, ε small. However, with this choice of δ, the consistency error grows sharply in time and decreases in an unexpected manner as $h \to 0$. We find that while the smoothing error is of order δ^p, the discretization error grows sharply in time and for $T>0$ does not decrease as $h \to 0$. This phenomena is present for all $\delta = h^q$ with $.5 < q < 1$, but as a function of δ, the discretization error decreases as δ increases. This decrease of the discretization error and the fact that the smoothing error increases with δ, allow us to eliminate this 'odd' behavior of the consistency error, by choosing δ substantially larger than h. In doing so we lose some of the increased accuracy provided by the higher order cutoff functions. Nevertheless, higher order cutoff functions improve the accuracy of the results. For example, with $p = 2$, $h = 0.05$ and $\delta = h^{.90}$, the consistency error for the velocity is 3%, while with $p = 8$, $h = 0.05$ and $\delta = h^{.60}$ the error is 0.8%.

We have not found an explanation for the behavior of the discretization error.

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References

[1] J. T. Beale and A. Majda, Vortex methods. II: Higher order accuracy in Two and three dimensions, Math. Comp. 39 (1982), p. 29.

[2] J. T. Beale and A. Majda, Explicit smooth velocity kernels for vortex methods, University of California, Berkeley, report PAM-140, 1983.

[3] A. J. Chorin, Numerical study of slightly viscous flow, J. Fluid Mech., 57 (1973), p. 785.

[4] A. J. Chorin, Vortex models and boundary layer instability, SIAM J. Sci. Stat. Comput., 1 (1980), p. 1.

[5] A. J. Chorin & P. S. Bernard, Discretization of a vortex sheet, with an example of roll-up, J. Computational Phys., 13, (1973) p.423.

[6] G. H. Cottet, Thèse de 3ème cycle, Université Pierre et Marie Curie, Paris 6, (1982).

[7] O. Hald, The convergence of vortex methods II, SIAM J. Numer. Anal., 16 (1979), p. 726.

[8] O. Hald and V. del Prete, The convergence of vortex methods, Math Comp. , 32 (1978), p. 791.

[9] A. Leonard, Vortex methods for flow simulations, J. Computational Phys., 37 (1980), p. 289.

[10] A. Majda and J. T. Beale, The design and analysis of vortex methods, Proc. Conf. on transonic, Shock and Multi-Dimensional Flows, Madison, WI, 1981.

Table 1-a. Consistency errors for different cutoff functions, $h = 0.05$

	E_u			E_ω		
	$t=0$	10	20	$t=0$	10	20
$\delta = h^{.65}$ $p=4$	1.336 -03	1.336 -03	1.445 -03	.9385 -02	.9399 -02	1.429 -02
$\delta = h^{.60}$ $p=6$	0.798 -03	0.798 -03	.8791 -03	.6262 -02	.6264 -02	.9348 -02
$\delta = h^{.60}$ $p=8$	0.421 -03	.4210 -03	.5765 -03	.3574 -02	.3577 -02	.8343 -02

Table 1-b. Relative errors for different cutoff functions, $h = 0.05$

	$E_u / \| u \|$			$E_\omega / \| \omega \|$		
	$t=0$	10	20	$t=0$	10	20
$\delta = h^{.65}$ $p=4$	.9118 -02	.9119 -02	.9866 -02	.2051 -01	.2054 -01	.3123 -01
$\delta = h^{.60}$ $p=6$	.5447 -02	.5447 -02	.6002 -02	.1358 -01	.1370 -01	.2043-01
$\delta = h^{.60}$ $p=8$	.2874 -02	.2874 -02	.3936 -02	.7810 -02	.7816 -02	.1823 -01

Table 2. Absolute and relative consistency errors for second order cutoff, $h = 0.05$, $\delta = h^{.90}$.

	$t=0$	10	20		$t=0$	10	20
E_u	.405 -02	.405 -02	.421 -02	E_ω	.220 -01	.223 -01	.319 -01
$\frac{E_u}{\| u \|}$	.276 -01	.276 -01	.287 -01	$\frac{E_\omega}{\| \omega \|}$	.482 -01	.486 -01	.697 -01

Table 3-a. Smoothing error E_ω^S

δ	$p=4$	$p=6$	$p=8$
0.2	.1414	.2801 -01	.1406 -01
0.14	.8345 -01	.9099 -02	.2977 -02
0.10	.4586 -01	.2636 -02	.5093 -03
0.07	.2411 -01	.7122 -03	.7642 -04
0.05	.1237 -01	.1853 -03	.1059 -04

Table 3-b. Order of accuracy of the approximation to the vorticity ω by ω^δ

δ	$p=4$	$p=6$	$p=8$
0.2	1.52	3.24	4.48
0.14	1.73	3.57	5.09
0.10	1.85	3.78	5.47
0.07	1.92	3.88	5.70

Table 4. Consistency error compared to its smoothing and discretization components, $h = 0.05$, $\delta = h^{.65}$, $p = 4$

	E_ω	E_ω^S	E_ω^D
$t = 0$	.9386 -02	.9385 -02	.5462 -05
$t = 10$	.9398 -02	.9385 -02	.1416 -03
$t = 20$	.1429 -01	.9385 -02	.9995 -02

Figure captions.

Figure 1a Consistency errors E_u and E_ω, $p=4$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

Figure 1b Consistency error E_u and E_ω, $p=6$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

Figure 2a Rate of convergence for the velocity and the vorticity approximations, $p=4$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

Figure 2b Rate of convergence for the velocity and the vorticity approximations, $p=6$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

Figure 3a Consistency errors E_u and E_ω, $p=4$, $h=0.1$, $\delta = h^q$, $0.75 \le q \le 0.95$.

Figure 3b Consistency errors E_u and E_ω, $p=4$, $h=.05$, $\delta = h^q$, $0.75 \le q \le 0.95$.

Figure 4a Discretization error E_ω^D, $p=2$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

Figure 4b Discretization error E_ω^D, $p=2$, $0.05 \le h \le 0.2$, $\delta = h^{.65}$.

Figure 4c Discretization error E_ω^D, $p=4$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

Figure 4d Discretization error E_ω^D, $p=4$, $0.65 \le h \le 0.2$, $\delta = h^{.65}$.

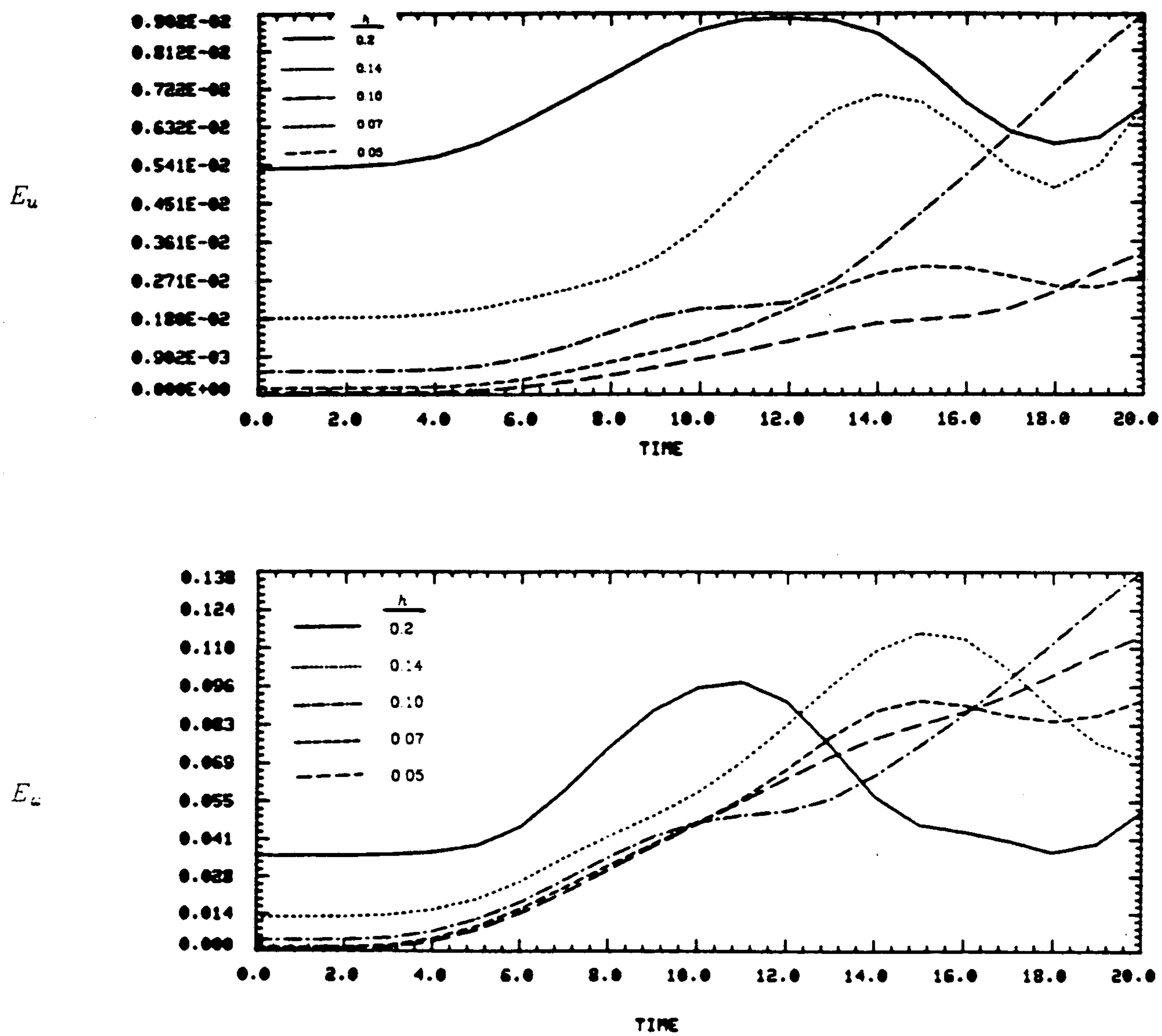
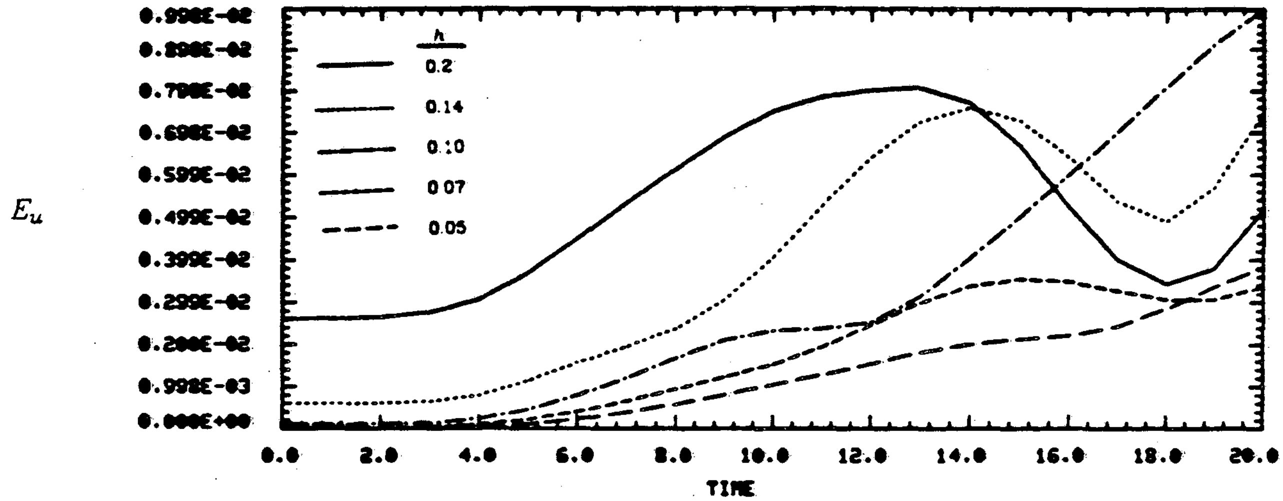


Figure 1a. Consistency errors for the velocity and the vorticity, $p = 4$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.



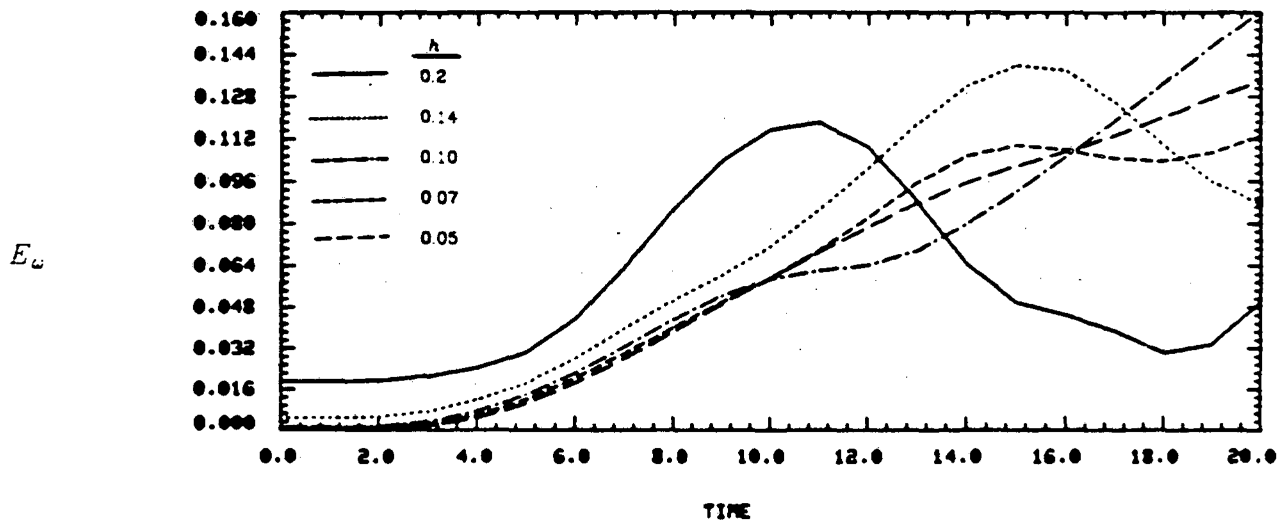


Figure 1b. Consistency errors for the velocity and the vorticity, $p = 6$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

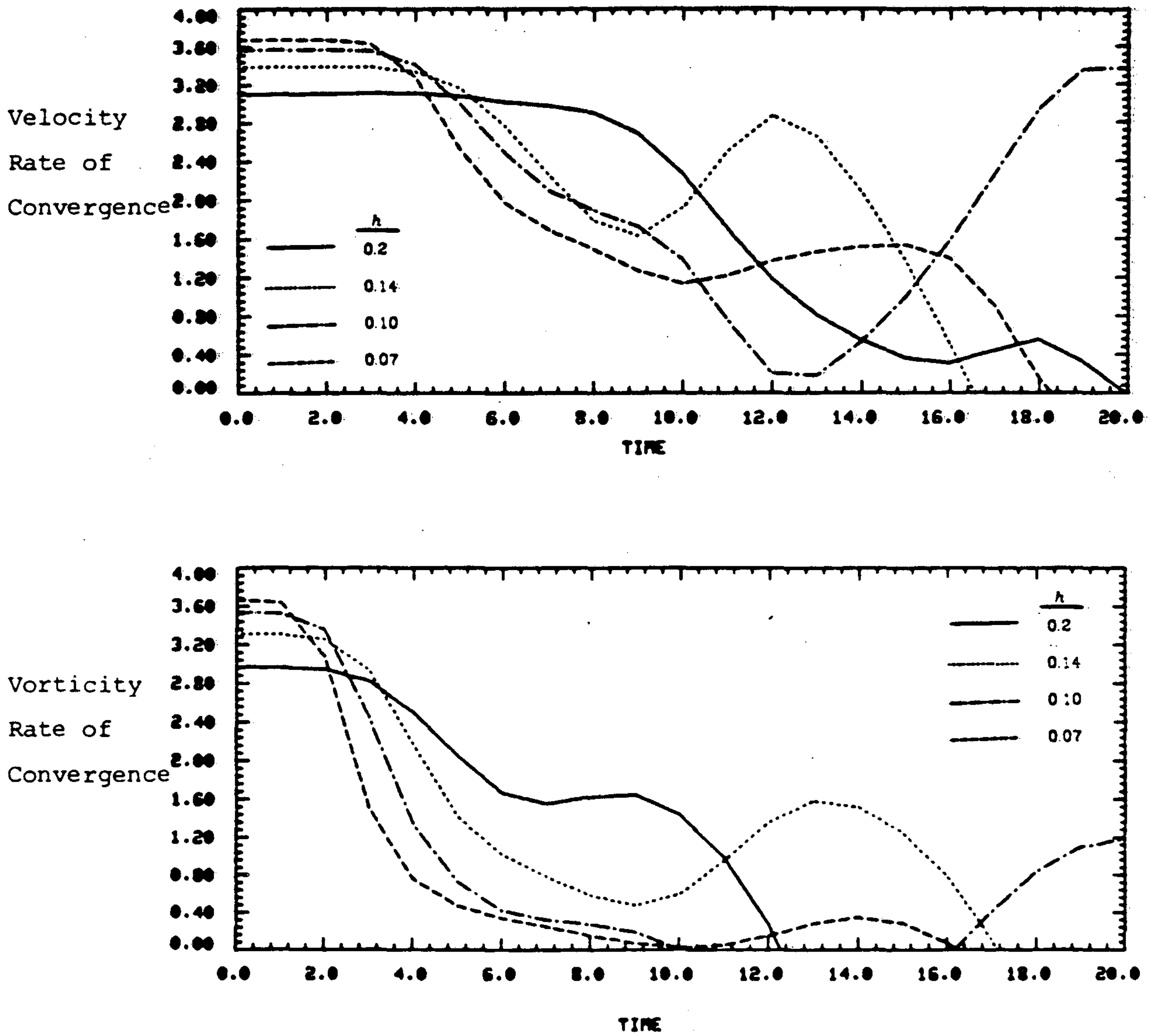
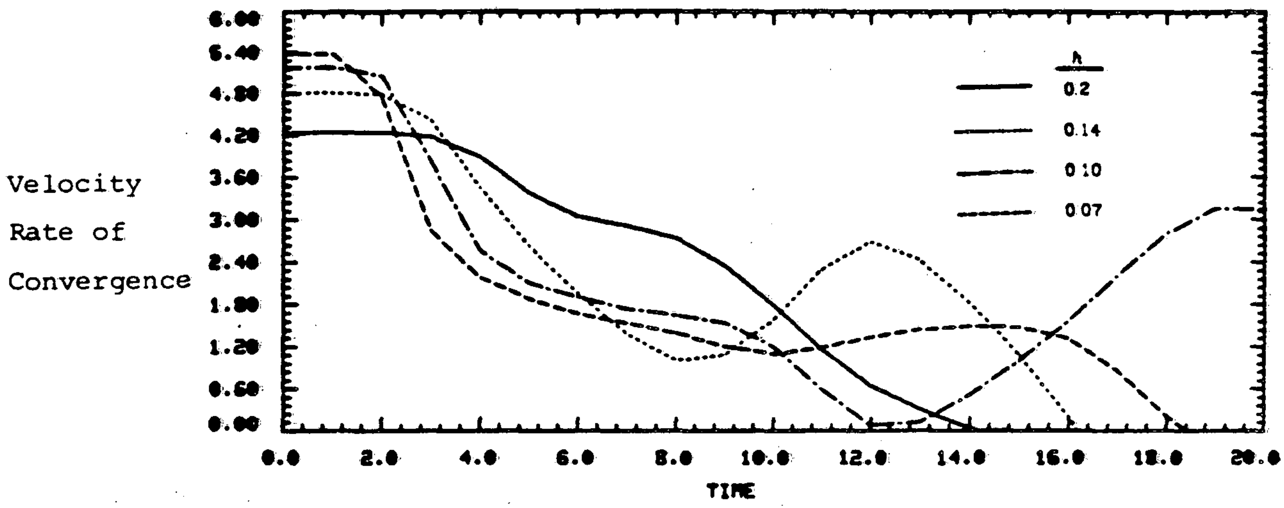


Figure 2a. Rate of convergence for velocity and vorticity, $p = 4$, $0.05 \leq h \leq 0.2$, $\delta = h^{.95}$.



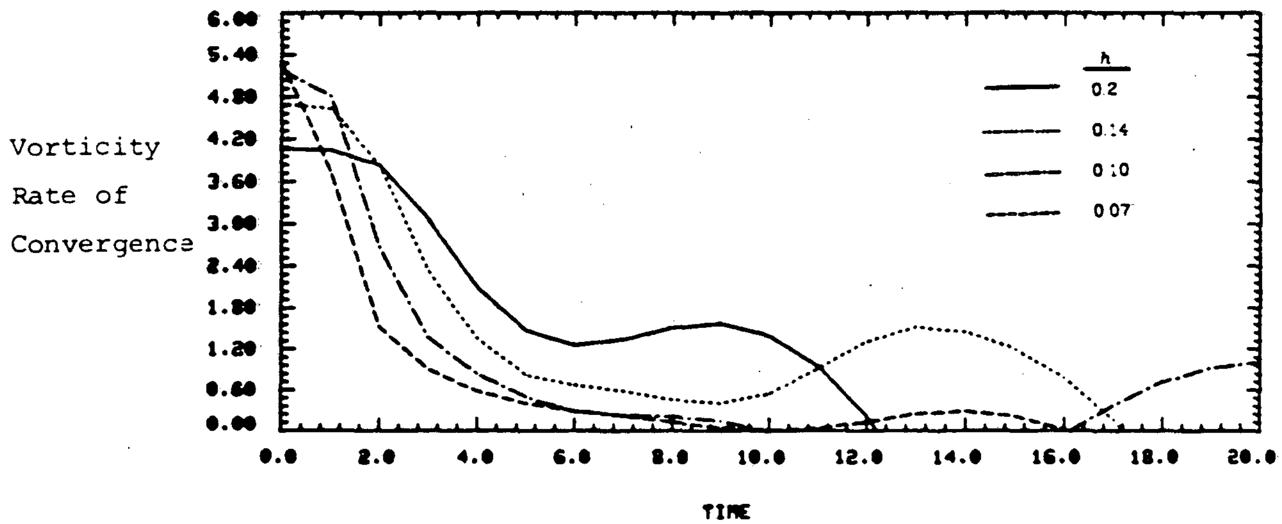
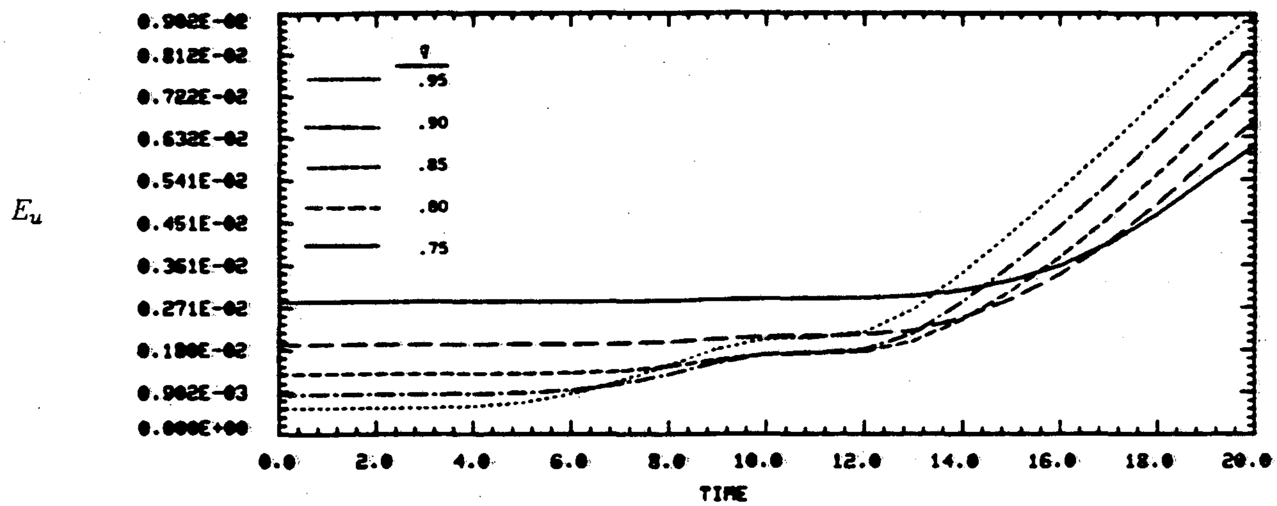


Figure 2b. Rate of convergence for velocity and vorticity, $p=6$, $0.05 \le h \le 0.2$, $\delta = h^{95}$.



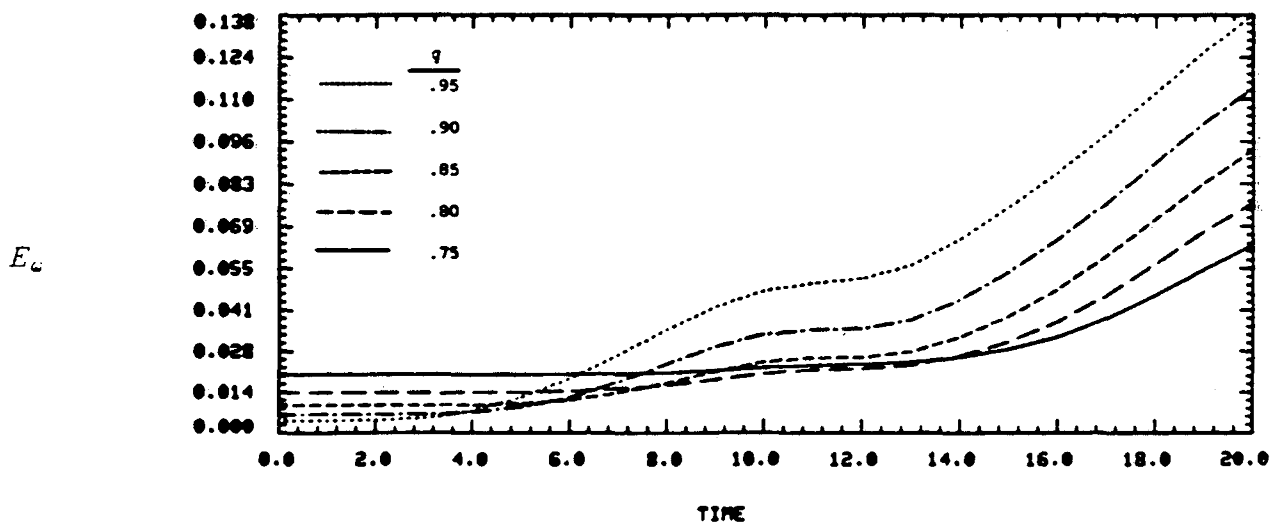


Figure 3a. Consistency error for velocity and vorticity, $p = 4$, $h = 0.1$, $\delta = h^q$, $0.75 \leq q \leq 0.95$.

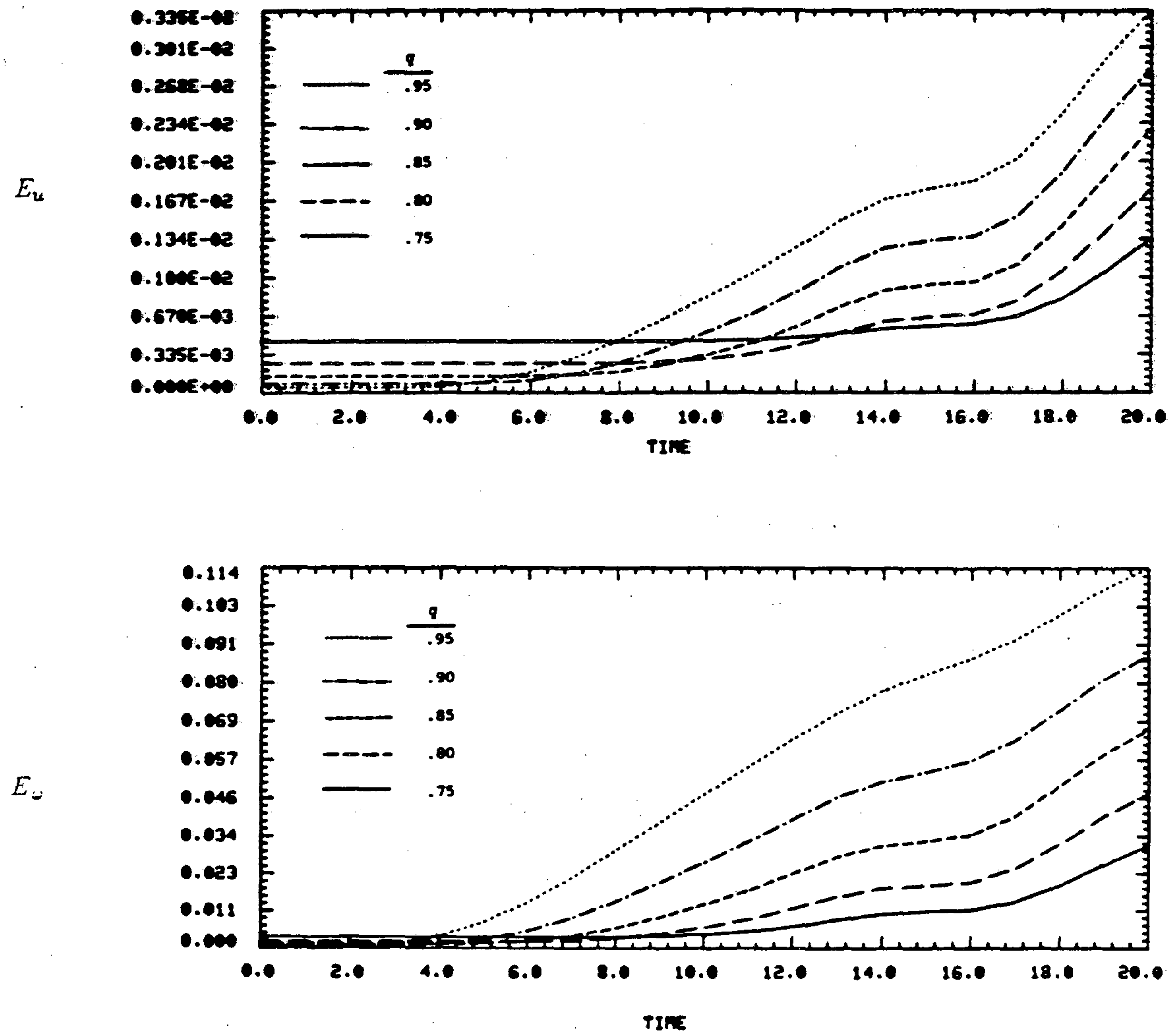


Figure 3b. Consistency error for velocity and vorticity, $p = 4$, $h = .05$, $\delta = h^q$, $0.75 \le q \le 0.95$.

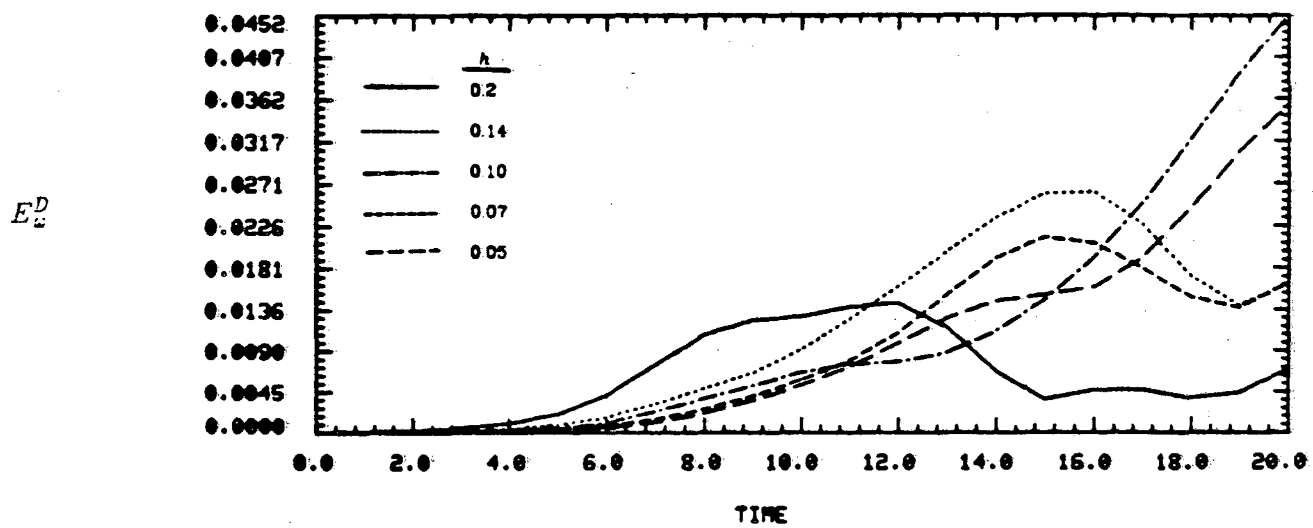


Figure 4a. Discretization error E_{ω}^{D} $p = 2,\ 0.05 \le h \le 0.2,\ \delta = h^{.95}$.

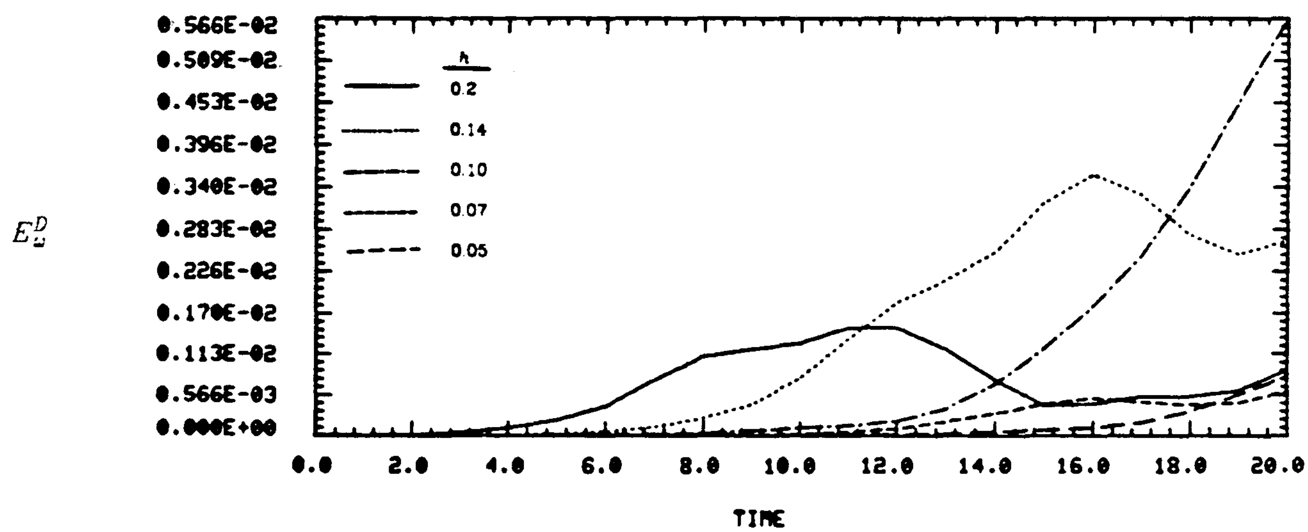


Figure 4b. Discretization error E_{ω}^{L} $p = 2,\ 0.05 \le h \le 0.2,\ \delta = h^{.85}$.

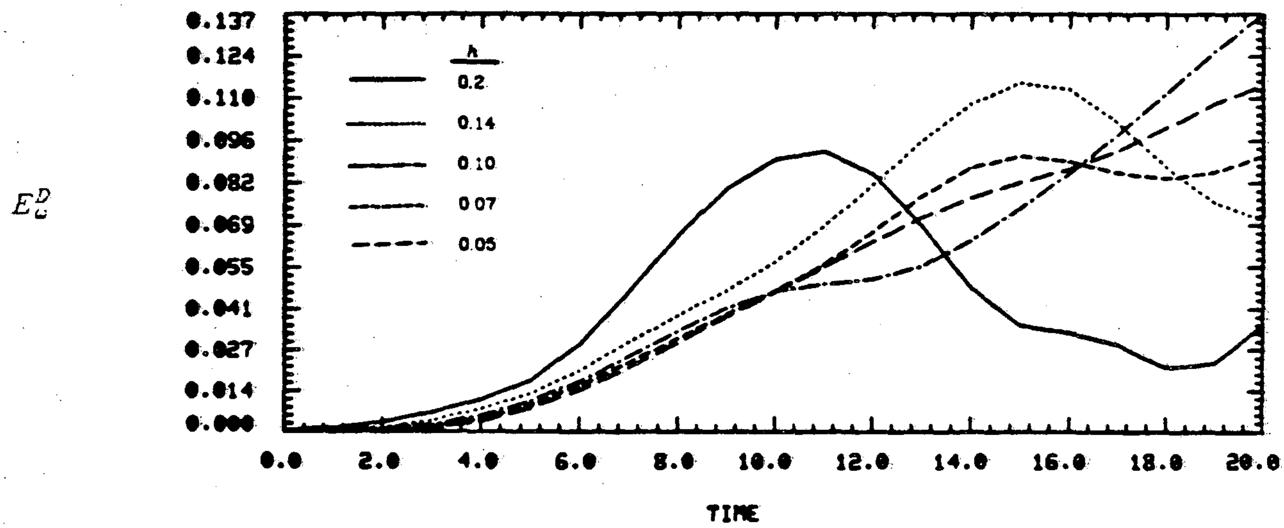


Figure 4c. Discretization error E_ω^D , $p = 4$, $0.05 \le h \le 0.2$, $\delta = h^{.95}$.

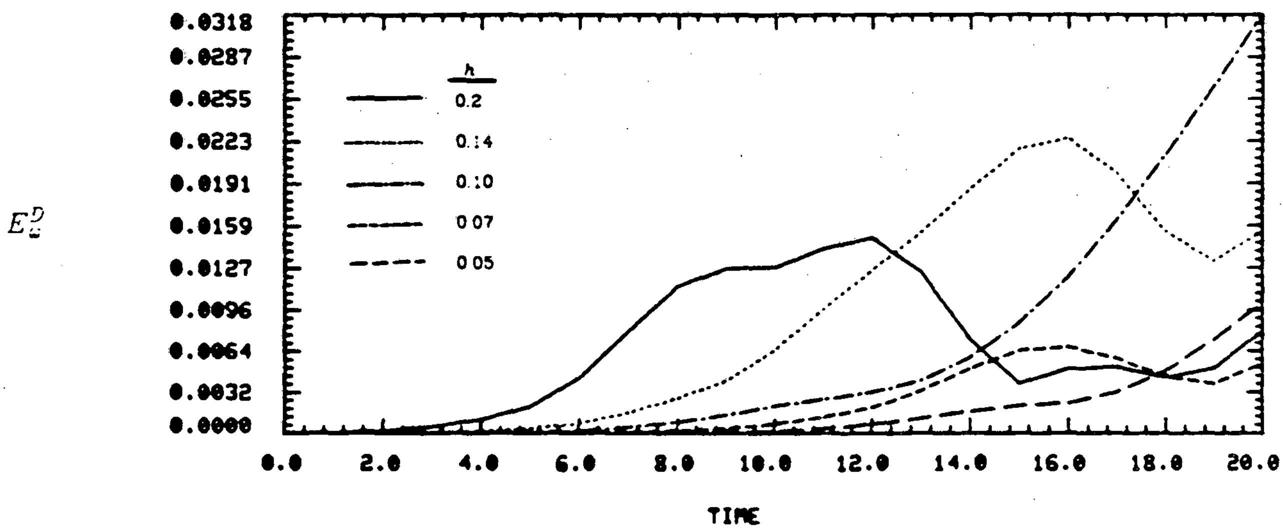


Figure 4d. Discretization error E_ω^D , $p = 4$, $0.05 \le h \le 0.2$, $\delta = h^{.85}$.

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