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Bayesian Linear State Estimation using Smart Meters and PMUs Measurements in Distribution Grids

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Abstract—In this work we address the problem of static state estimation (SE) in distribution grids by leveraging historical meter data (pseudo-measurements) with real-time measurements from synchrophasors (PMU data). We present a Bayesian linear estimator based on a linear approximation of the power flow equations for distribution networks, which is computationally more efficient than standard nonlinear weighted least squares (WLS) estimators. We show via numerical simulations that the proposed strategy performs similarly to the standard WLS estimator on a small distribution network. A key advantage of the proposed approach is that it provides explicit off-line computation of the estimation error confidence intervals, which we use to explore the trade-offs between number of PMUs, PMU placement and measurement uncertainty. Since the estimation error in distribution systems tends to be dominated by uncertainty in loads and scarcity of instrumented nodes, the linearized method along with the use of high-precision PMUs may be a suitable way to facilitate on-line state estimation where it was previously impractical.

Keywords—*Distribution Systems State estimation, Phasor Measurement Unit, Smart Grid, Power System Modeling, Smart Meters.*

I. INTRODUCTION

Optimal management of power networks depends on knowledge of the network state in real-time. Consequently, state estimation (SE) has long been recognized as a fundamental problem in power networks. Classic SE in transmission networks is based on real and reactive power injected at each bus, along with voltage and current measurements sampled every several seconds at substations and other suitable locations via Supervisory Control and Data Acquisition (SCADA) systems [1]. More recently, the introduction of phasor measurement units (PMUs) in transmission networks has increased the accuracy of SE algorithms, by measuring not only voltage magnitudes, but also voltage angle differences at different nodes [2]. However, since PMUs are still expensive, most of the recently proposed SE algorithms rely on measurements from a small number of PMUs along with conventional (magnitude only) measurements, as for example in [3], [4]. So the SE problem remains intrinsically nonlinear and hard to solve. Although there is a well-established tradition and an extensive literature on SE for transmission networks, only recently the SE problem has been taken into consideration at the distribution level. Primarily, this is because measurement data in distribution networks tend to be very scarce and often nonexistent beyond the substation. Also, distribution networks rely mostly on radial topologies and one-way power flows, which require evaluating just peak loads and fault currents,

rather than tracking the network operating state in real-time [5]. These circumstances have changed in recent years, due to the increasing penetration of distributed energy resources, which may introduce variability, uncertainty and even instabilities. Consequently, there is a growing interest in SE estimators based on pseudo-measurements [6] and measurements from relatively inexpensive PMUs tailored to the distribution context [5], [7]. However, not only distribution networks are radial, but the line reactance-resistance ratio (X/R) is substantially lower than the X/R ratio of transmission networks, thus making standard approximations such as the so-called DC power flow model inadequate [8]. Moreover, while the three phases are quite well balanced at the transmission level (owing to the statistics of aggregation), this is not true at the distribution level.To correctly capture these phenomena, some SE solutions rely explicitly on three-phase branch currents [9], [10], on linearized models for pseudo-measurements [11], or on unsynchronized phasor measurements [12]. In this paper, we address the problem of distribution SE based on PMU measurements and load pseudo-measurements for a small, balanced distribution network. While it remains to be extended to more general cases, the contribution of the present work is twofold. First, a Bayesian linear state estimator (BLSE) based on a fully linear approximation of the power flow equations is used [13], [14]. The second contribution is an integrated evaluation of the SE algorithm performance with respect to phasor measurement accuracy, pseudo-measurements uncertainty, number of PMUs, and PMU placement sequence. Specifically, we show via numerical Monte Carlo simulations on a small distribution grid that the BLSE offers the same performance as the traditional nonlinear weighted least squares (WLS) state estimator, but with the benefit of being numerically faster and more robust. A key feature of the proposed BLSE is that it allows the computation of estimation error confidence bounds off-line, since these do not depend on the actual measurements (in contrast to WLS, where estimation error confidence intervals have to be computed on-line since they depend on measurements). These confidence bounds can be used, for example, to decide where to optimally place the PMUs if only a limited number of them is available. Finally, some numerical simulations show that deploying few, but accurate, PMUs can provide better performance than using a larger number of less accurate instruments, thus shedding some light on possible trade-offs between number, accuracy and cost of PMUs.

II. POWER NETWORK MODEL

A grid is modelled as a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{V} = \{0, 1, \ldots, N\}$ are the ordered nodes or buses of the grid and

$\mathcal{E} = \{1, 2, \ldots, E\}$ are the ordered lines of the grid. Each edge ℓ is represented by the ℓ-th row of the incidence matrix $A \in \{-1, 0, 1\}^{E \times (N+1)}$ where all elements are zeros except for one entry set to -1 (source node) and another entry set to 1 (terminal node). The impedance matrix $Z \in \mathbb{C}^{E \times E}$ is a diagonal complex matrix given by $Z = \text{diag}\{z_1, \ldots, z_E\}$, where z_ℓ is the impedance of the ℓ-th line. Without loss of generality we assume that the 0-th node correspond to the point of common coupling (PCC), i.e. the substation or point where the distribution network under analysis is connected to the main grid. Let be $v_0, v_1, \ldots, v_N \in \mathbb{C}$ and $i_0, \ldots, i_N \in \mathbb{C}$ the vectors representing the voltage and current phasors at each node, respectively. The symbol i_k represents the phasor of the current injected into the k-th node. We also denote with $\xi_1, \ldots, \xi_E \in \mathbb{C}$ the phasors of the currents flowing across the ℓ-th line from the source node to the sink node defined by the ℓ-th row of the incidence matrix A. Let us define the complex column vectors $v^+ = [v_0 \ldots v_N]^T \in \mathbb{C}^{N+1}$, $i^+ = [i_0 \ldots i_N]^T \in \mathbb{C}^{N+1}$, and $\xi = [\xi_1 \ldots \xi_E]^T \in \mathbb{C}^E$ where $(\cdot)^T$ indicates the transpose operator. The Kirchhoff's laws at all nodes and lines of the grid can be compactly written as:

$$A^T \xi + i^+ = 0, \ (KCL), \quad Av^+ + Z\xi = 0, \ (KVL)$$

Since Z is invertible we can combine the previous two equations into a single matrix expression that relates voltage and current phasors at each node of the grid:

$$L^+ v^+ = i^+, \quad L^+ := A^T Z^{-1} A \tag{1}$$

where $L^+ \in \mathbb{C}^{(N+1) \times (N+1)}$ is referred to as the admittance matrix of the grid.

We assume that the node representing the PCC acts as an ideal voltage source generator, i.e.

$$v_0 = V_0 \tag{2}$$

where $V_0 \in \mathbb{R}$ is the nominal voltage at the PCC. The previous expression implies that $|v_0| = V_0$ and $\angle v_0 = 0$. We also assume that all the power loads (also referred to as PQ loads) are constant i.e.

$$s_k = v_k \bar{i}_k = p_k + j q_k, \quad k = 1, \ldots, N \tag{3}$$

where $s_i \in \mathbb{C}$ is the (complex) power, $\underline{p_k}$ is the active power, q_k is the reactive power, and the symbol $\overline{(\cdot)}$ indicates the complex conjugate. Note that when $p_k > 0$ then the active power is injected into the node according to the current direction defined above. If no load is attached to a specific node k, as, for example, in the case of nodes connecting a lateral line with the main line, we can simply assume $s_k = 0$, which indirectly implies $i_k = 0$ since under normal grid operating conditions $v_k \neq 0$.

III. Steady State Power Flow Computation

The previous section define the set of equations that voltage and current phasors need to satisfy in each node of the network. Let us define the complex column vectors $v = [v_1 \ldots v_N]^T \in \mathbb{C}^N$, $i = [i_1 \ldots i_N]^T \in \mathbb{C}^N$. Under assumption (2), then the constraints given by Eqn. (1) can be equivalently rewritten as:

$$\underbrace{\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L \end{bmatrix}}_{L^+} \underbrace{\begin{bmatrix} v_0 \\ v \end{bmatrix}}_{v^+} = \underbrace{\begin{bmatrix} i_0 \\ i \end{bmatrix}}_{i^+} \Leftrightarrow \begin{matrix} L_{10} V_0 + Lv = i \\ i_0 = -\sum_{k=1}^N i_k \end{matrix}$$

If the grid is connected, then $L \in \mathbb{C}^{N \times N}$ is invertible, therefore we have $v = L^{-1}(i - L_{10} V_0)$. Since $L\mathbf{1}_N = \mathbf{1}_N$ and $v_0 = V_0$, then $L^{-1} L_{10} = -\mathbf{1}_N$, therefore the previous equation can be written as

$$v = V_0 \mathbf{1}_N + L^{-1} i \tag{4}$$

If we define the complex column vector $s = [s_1 \ldots s_N]^T \in \mathbb{C}^N$, then equations (2), (3) and (4) can be equivalently written as:

$$v_0 = V_0 \tag{5}$$
$$v \circ \bar{i} = s \tag{6}$$
$$v - L^{-1} i = V_0 \mathbf{1}_N \tag{7}$$
$$i_0 + \sum_{k=1}^N i_k = 0 \tag{8}$$

where the symbol $\circ$ denotes the component-wise product and $\bar{i}$ is the component-wise complex conjugate of the current vector i. If the PCC voltage V_0 and the load vector s are known, then the previous equations represent a set of $2(N+1)$ (complex) nonlinear equations in $2(N+1)$ (complex) unknown variables $v_0, \ldots, v_N, i_0, \ldots, i_N$. The previous set of nonlinear equations may have no solutions or multiple solutions. Under the assumption that there exists at least one feasible solution, the solution of the previous set of equations reduces to the problem of solving Eqn. (7) and Eqn. (6). One possible numerical procedure to compute a feasible solution (which is also the most efficient in terms of power loss over the lines) is expressed by Algorithm 1. This algorithm coincides with

---

Algorithm 1 Power Flow computation

Require: $s \in \mathbb{C}^N, V_0 \in \mathbb{R}, r \in (0, 1), T \in \mathbb{N}$
1: $v[0] = V_0 \mathbf{1}_N$
2: $\tau = 0$
3: **repeat**
4: $\quad \tau = \tau + 1$
5: $\quad i[\tau] = \frac{\bar{s}}{v[\tau-1]}$ $\qquad \triangleright$ Component-wise division
6: $\quad v[\tau] = V_0 \mathbf{1}_N + L^{-1} i[\tau]$
7: $\quad s[\tau] = v[\tau] \circ i^*[\tau]$
8: **until** $\|s[\tau] - s\| < r\|s\|$ or $\tau > T$
9: **if** $\tau > T$ **then**
10: $\quad$ No solution found
11: **end if**

---

the forward-backward sweep method used to solve power flow equations [15] when the grid is radial, but it is somewhat more general since it can also be used for mesh networks. Alternative approaches for power flow analysis are the well-known Gauss-Seidel and Newton-Raphson methods [16]. In our simulations, we adopted the proposed algorithm, since it also provides a link with the linear model approximation described below.

IV. Measurement Model

Our goal is to estimate the state of the grid that can be uniquely determined by the voltage phasors of all nodes. We assume that two types of sources of information are available for estimating the state of the grid. The first source of information comes from historical time series of active and reactive power values collected at each node by smart-meters. The time-series can be used to predict the active and reactive

load power at a certain time in the future. Being a prediction, these estimates of active and reactive power at each load are affected by large uncertainty. As shown in [6], from historical time series it is possible to forecast day-ahead active and reactive power consumptions within $30\% - 50\%$ of the nominal value. Therefore, the active and reactive power values at node k at a specific time t can be written as

$$P_k(t) = p_k + w_k^p(t), \quad \mathbb{E}[(w_k^p(t))^2] = \sigma_L^2 p_k^2 \quad (9)$$

$$Q_k(t) = q_k + w_k^q(t), \quad \mathbb{E}[(w_k^q(t))^2] = \sigma_L^2 q_k^2 \quad (10)$$

where p_k and q_k are the nominal active and reactive power values (also used for prediction when pseudo-measurements are used), $w_k^p(t)$ and $w_k^q(t)$ are zero-mean random variables, and $\sigma_L \approx 0.3 - 0.5$ is the relative load (and prediction) uncertainty. In order to simplify the state estimation problem, we further assume that active and reactive power prediction error are uncorrelated, i.e. $\mathbb{E}[w_k^p(t)w_k^q(t)] = \mathbb{E}[w_k^p(t)]\mathbb{E}[w_k^q(t)] = 0$. As suggested in [6], it is also fair to assume that prediction errors at two different nodes k and h are uncorrelated, i.e. $\mathbb{E}[w_k^p(t)w_h^p(t)] = \mathbb{E}[w_k^q(t)]\mathbb{E}[w_h^q(t)] = 0$.

The second source of information comes from possible phasor measurement units (PMUs) which are able to measure both the magnitude and the phase of voltage phasor at some nodes. More formally, the PMU measurements at a node k at time t can be written as:

$$V_k(t) = |v_k(t)| + w_k^V(t), \quad \mathbb{E}[(w_k^V(t))^2] = b^2 V_0^2 \quad (11)$$

$$\theta_k(t) = \angle v_k(t) + w_k^\theta(t), \quad \mathbb{E}[(w_k^\theta(t))^2] = c^2 \quad (12)$$

where b and c represent the relative and absolute standard deviations of magnitude and phase measurement data, respectively. In the following, magnitude and angle measures will be assumed to be uncorrelated both at the same node and between nodes, i.e. $\mathbb{E}[w_k^V(t)w_k^\theta(t)] = 0$ and $\mathbb{E}[w_k^V(t)w_h^V(t)] = 0$ for $h \neq k$. Probably this assumption is expected to be relaxed in future. Finally, in order to simplify the analysis, we will suppose that

$$b = c = \sigma_{PMU} \quad (13)$$

where σ_{PMU} is a constant value depending on PMU technology.

V. LINEAR APPROXIMATION OF POWER FLOW

The power flow equations (5)-(8) are highly non-linear and this makes the problem of state estimation in the presence of noisy measurements and uncertainty very difficult and numerically intensive. Recently, a linear approximation has been proposed for modeling the power flow equations. This approximation can be obtained by stopping the power flow computation in Algorithm 1 after the first iteration, i.e.

$$i[1] = \frac{1}{V_0}\bar{s}, \quad v[1] = V_0 \mathbf{1}_N + \frac{1}{V_0}L^{-1}\bar{s} \quad (14)$$

The previous two equations show that the current and the voltage phasors are approximately linear in the power loads s. Obviously, the linearization error is not zero, i.e. $e[1] := v - v[1] \neq 0$, v being the actual voltage vector of the grid nodes. However, it has been shown that if the term $\frac{\|L^{-1}\|\|s\|}{V_0^2}$ is sufficiently smaller than unity, then the linearization error is small. As we will show later, this assumption is acceptable for distribution systems, where the voltage magnitude drop and

the phase angle difference between the PCC and any node in the network are smaller that $5 - 10\%$ and $2 - 5^o$, respectively [17].

VI. STATE ESTIMATION

The purpose of state estimation is to find the values of the voltage phasors at time t for a given amount of information, such as the active and reactive load predictions and the PMU measurements. In the following, we will assume that statistical information about the loads is available at each node of grid, i.e. $P_k(t)$ and $Q_k(t)$ are known for $k = 1, \ldots, N$. Moreover, we will assume that M PMUs are placed at the locations defined by the set $\mathcal{M} = \{m_1, \ldots, m_M\} \subseteq \{1, \ldots, N\}$. If no PMU is available, then $\mathcal{M} = \emptyset$. Therefore, the state estimation problem consists in finding a function $\widehat{v}(t) = f(P(t), Q(t), V(t), \theta(t)) \approx v(t)$ where $P = [P_1 \ldots P_N]^T, Q = [Q_1 \ldots Q_N]^T, V = [V_{m_1} \ldots V_{m_M}]^T, \theta = [\theta_{m_1} \ldots \theta_{m_M}]^T$.

We will present two state estimation approaches: the first approach is based on Bayesian estimation using the linear approximated power flow model described in Section V, while the second is based on the well-known nonlinear weighted least squares (WLS) approach [1]. We also restrict our analysis to static estimation, i.e. estimation based only on information available at time t. Thus, to simplify the notation we will drop the time dependence in all the variables, i.e. we will simply write $\widehat{v} = f(P, Q, V, \theta)$, where the variables are to be intended at some time instant t.

A. Bayesian Linear State Estimator (BLSE)

In this section, we adopt the linear approximation of the power flow equations described in (14) and we derive the best estimator in a Bayesian framework. More specifically, the prediction of active and reactive power of the loads can be used to derive a-priori information about the statistical distributions of voltage phasors. Such distributions can be computed off-line and, once PMU measurements are available, they can be used to improve the estimate of the true voltage phasors and to reduce the uncertainty given by the a-posteriori statistical distributions of the state estimate errors. We start by defining $S_k = P_k + jQ_k \in \mathbb{C}$ and $S = [S_1 \ldots S_N] \in \mathbb{C}^N$. The a-priori information about loads gives rise to the following a-priori information about voltage phasors:

$$v^0 := \mathbb{E}[v] = V_0 \mathbf{1}_N + \frac{1}{V_0}L^{-1}\overline{S} \quad (15)$$

$$\Sigma_0 : = \mathbb{E}[(v-v^0)(v-v^0)^*] = \sigma_L^2 \frac{1}{V_0^2}L^{-1}\Sigma_s(L^{-1})^* \quad (16)$$

$$\Sigma_s = diag\{|S_1|^2, \ldots, |S_N|^2\} \quad (17)$$

where the symbol $()^*$ indicates the transpose complex conjugate operator. The previous estimator is the optimal estimator among all possible linear estimators based on pseudo-measurements. The vector v^0 corresponds to the nominal voltage phasors according to the load prediction and Σ_0 is the corresponding error covariance. Note that, because pseudo-measurements are assumed to be available at all nodes, then the state of the grid is observable even without any PMU measurement. Adding PMU measurements will simply improve estimation performance. If a PMU measurement is available

at node k, we define the complex measurement voltage phasor as

$$u_k := V_k e^{j\theta_k}$$

The distribution of this phasor is not centered in the true voltage phasor v_k. However, since in distribution networks the voltage drop across a feeder is generally small, the measurement process can be approximated as

$$u_k = v_k + V_0 w_k^{rl} + j V_0 w_k^{im}, \quad \mathbb{E}[(w_k^{rl})^2] = \mathbb{E}[(w_k^{im})^2] = \sigma_{PMU}^2$$

where w_k^{rl} and w_k^{im} are independent and zero-mean uncertainty contributions. Observe that

$$\mathbb{E}[u_k] = v_k, \quad \mathbb{E}[|u_k - v_k|^2] = 2 V_0^2 \sigma_{PMU}^2$$

. Thus, the PMU measurement provides an unbiased estimate of the true voltage phasor with variance twice as large as $V_0^2 \sigma_{PMU}^2$. If we define $u_{\mathcal{M}} := [u_{m_1} \ldots u_{m_M}]^T \in \mathbb{C}^M$, then we can write

$$
\begin{aligned}
u_{\mathcal{M}} &= C_{\mathcal{M}} v + V_0 w_{\mathcal{M}}^{rl} + j V_0 w_{\mathcal{M}}^{im}, \\
\mathbb{E}[w_{\mathcal{M}}^{rl}(w_{\mathcal{M}}^{rl})^*] &= \mathbb{E}[w_{\mathcal{M}}^{im}(w_{\mathcal{M}}^{im})^*] = \sigma_{PMU}^2 I_M
\end{aligned}
$$

where $w_{\mathcal{M}}^{rl}$ and $w_{\mathcal{M}}^{im}$ are the measurement uncertainty terms associated with the PMUs belonging to set $\mathcal{M}$, I_M is the $M \times M$ identity matrix, and $C_{\mathcal{M}} \in \mathbb{R}^{M \times N}$ is a matrix whose ℓ-th row has all zeros except for the m_ℓ-th entry which is set to one. In practice $C_{\mathcal{M}}$ is a selection matrix that associates the PMU measurement u_{m_ℓ} with the corresponding voltage v_{m_ℓ}. In the context of Bayesian estimation, the optimal voltage estimate based on $u_{\mathcal{M}}$ and on the prior distribution v^0, is given by

$$\widehat{v} := \mathbb{E}[v \,|\, u_{\mathcal{M}}] = v^0 + K(u_{\mathcal{M}} - C_{\mathcal{M}} v^0) \tag{18}$$

$$\Sigma_{\mathcal{M}} = \Sigma_0 - \underbrace{\Sigma_0 C_{\mathcal{M}}^T (C_{\mathcal{M}} \Sigma_0 C_{\mathcal{M}}^T + 2 V_0^2 \sigma_{PMU}^2 I_M)^{-1}}_{K} C_{\mathcal{M}} \Sigma_0 \tag{19}$$

where $\Sigma_v = \mathbb{E}[(v - \widehat{v})(v - \widehat{v})^*]$. Note that the matrices $K, \Sigma_{\mathcal{M}}$ can be computed off-line, i.e. they do not depend on the actual PMU measurement vector $u_{\mathcal{M}}$. Therefore, the computation of (18) can be performed very rapidly on-line even for very large networks.

B. Non-linear Weighted Least Square Estimator (WLS)

Let $x = [\theta^T, V^T]^T$ be the state vector including the bus voltage magnitude and phases. If we assume to measure both the real and reactive power injected in each bus and a variable number of bus voltage phasors, the following measurement model holds, i.e.

$$z = h(x) + e = \begin{bmatrix} P(x) \\ Q(x) \\ V(x) \\ \theta(x) \end{bmatrix} + e \tag{20}$$

where e is the column vector including all measurement uncertainty contributions and $P = [P_1 \ldots P_N]^T, Q = [Q_1 \ldots Q_N]^T, V = [V_{m_1} \ldots V_{m_M}]^T, \theta = [\theta_{m_1} \ldots \theta_{m_M}]^T$, in accordance with the notation used in the previous Section. Note that in (20) all quantities are functions of the state vector. Evidently, the elements of $V(x)$ and $\theta(x)$ are identically equal to the respective state variables by definition, whereas the explicit expressions of $P(x), Q(x)$ as a function of the state are widely available in the literature (e.g. in [1]), so they are not reported for the sake of brevity.

Observe also that system (20) consists of M' equations, with $M' \leq 2N + 2M - 1$ (N denotes the number of nodes) depending on the number M of PMUs actually used to monitor the grid in excess of the pseudo-measurements. Assuming that all the measurement uncertainty contributions have zero-mean and are uncorrelated (i.e. $E\{e_i e_j\} = 0, \forall i \neq j = 1, \ldots, M'$), the classic Non-linear Weighted Least Square estimator (WLS) relies on the minimization of the cost function

$$J(x) = [z - h(x)]^T R^{-1} [z - h(x)] \tag{21}$$

where $R = \text{diag}(\sigma_1^2, \ldots, \sigma_{M'}^2)$ is built using the variances of the available true or pseudo-measurements. From the Taylor's series of the gradient of $J(x)$ truncated to the first order around x_k for $k \geq 0$, it can be shown that the state at iteration $k + 1$ can be obtained using the Gauss-Newton method as follows [1]

$$x_{k+1} = x_k + [H^T(x_k) R^{-1} H(x_k)]^{-1} H^T(x_k) R^{-1} (z - h(x_k)) \tag{22}$$

where $H(x)$ is the Jacobian of $h(x)$ computed at state x_k. The solution described above relies on the so-called normal equations (NE). This approach works correctly in most cases, but sometimes (e.g. when the elements of R are very small) it can suffer from numerical instabilities leading to poor accuracy or even to divergence. In order to tackle this problem, a preliminary QR Cholesky orthogonal factorization of $R^{-1/2} H(x_k)$ can be performed in (22). In this way, the Gauss-Newton method is applied to a better conditioned system [1].

VII. PERFORMANCE METRICS AND PMU PLACEMENT

The state estimation procedure described in Section VI-A also provides uncertainty bounds that can be used to evaluate the performance of the BLSE estimator. A possible performance metric is the mean square error averaged over the number of nodes and under square root, which is defined as

$$ARMSE(\mathcal{M}) := \sqrt{\frac{1}{N} \sum_{k=1}^{N} \mathbb{E}[|v_k - \widehat{v}_k|^2]} = \sqrt{\frac{1}{N} trace(\Sigma_{\mathcal{M}})} \tag{23}$$

and that corresponds to the average prediction error. If a limited number of PMUs is available for state estimation, a relevant problem is where to place them to achieve best accuracy. If the cost function is the ARMSE, then the desired optimization problem is given by:

$$\mathcal{M}^{op}(M) := arg \min_{\mathcal{M}} \quad ARMSE(\mathcal{M}) \tag{24}$$

$$s.t. \quad |\mathcal{M}| = M$$

$$ARMSE^{op}(M) := ARMSE(\mathcal{M}^{op}(M)) \tag{25}$$

where ARMSE is a function of the PMU placement set $\mathcal{M}$ according to (19), and $ARMSE^*(M)$ denotes the best accuracy when M PMUs are used. Obviously $ARMSE^{op}(M) \geq ARMSE^{op}(M + 1)$, i.e. it is monotonically non-increasing function of the number of PMUs. Moreover, after some simple manipulations, it is possible to verify that

$$ARMSE^{op}(0) = \sqrt{\frac{1}{N} trace(\Sigma_0)} \tag{26}$$

$$ARMSE^{op}(N) = \sqrt{\frac{1}{N} trace\left(\Sigma_0^{-1} + \frac{1}{2\sigma_{PMU}^2 V_0^2} I_N\right)^{-1}} \tag{27}$$
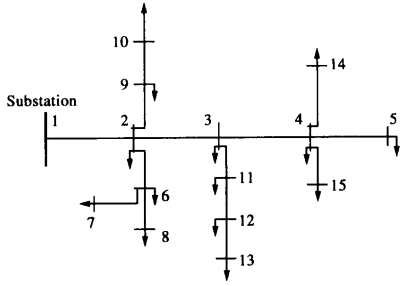
Fig. 1.   Small 15-node rural network [19].

Fig. 2.   ARMSE voltage curves corresponding to different values of load standard deviation, σ_L, for BLSE (red dashed line), WLS (blue solid line) and theoretical ARMSE value Eqn.(31) (black solid line with markers).

However, (24) is a combinatorial problem, so its solution could be unfeasible when large grids are considered. For this reason, we also propose a greedy PMU placement procedure based on the sequential addition of one PMU at a time able to provide the best local performance improvement. This procedure is described formally in Algorithm 2, and it is computationally more tractable than the original problem. In fact, $ARMSE(\mathcal{M})$ has to be computed for $\frac{N(N-1)}{2}$ different sets $\mathcal{M}$ to obtain the greedy PMU position set $\mathcal{M}^{gr}(N)$. The greedy algorithm is in general suboptimal, i.e. $ARMSE^{gr}(M) \geq ARMSE^{op}(M)$, $\forall M = 2, \ldots, N-1$, while equality is guaranteed only for $M = 0, 1, N$.
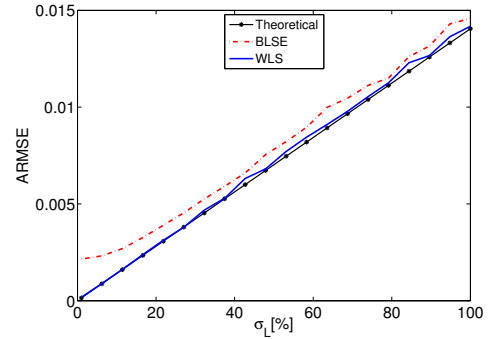

Algorithm 2 Greedy PMU placement

Require: $\Sigma_0, V_0, \sigma_{PMU}$
1: $\mathcal{M}^{gr}(0) := \emptyset, \overline{\mathcal{M}}(0) := \{1, \ldots, N\}$
2: **for** $\tau = 1$ to N **do**
3:      $m^*(\tau) = arg\min_{m \in \overline{\mathcal{M}}(\tau-1)} ARMSE(\mathcal{M}^{gr}(\tau-1) \cup m)$
4:      $\mathcal{M}^{gr}(\tau) = \mathcal{M}^{gr}(\tau - 1) \cup m^*(\tau)$
5:      $\overline{\mathcal{M}}(\tau) = \overline{\mathcal{M}}(\tau - 1) \setminus m^*(\tau)$
6:      $ARMSE^{gr}(\tau) = ARMSE(\mathcal{M}^{gr}(\tau))$
7: **end for**

VIII.   SIMULATION RESULTS

In this section the proposed BLSE state estimator is compared with the classic WLS-based approach, whose behavior has been extensively analyzed in [18]. In the present example, the small 15-node distribution rural network shown in Fig. 1 and introduced in [19] is considered. The parameters of the network and the nominal load values are available in [20]. The performance of BLSE and WLS estimators has been evaluated in Matlab by computing the parameter $\widehat{ARMSE}(\mathcal{M}) := \sqrt{\frac{1}{NT} \sum_{t=1}^{T} \|v^{[t]} - \widehat{v}^{[t]}\|^2}$ over $T = 2000$ Monte Carlo runs, where $v^{[t]}$ indicates the state vector of the t-th Monte Carlo run. All ARMSE values are expressed in per-unit (p.u.).

In the first test case the accuracy of BLSE and WLS is evaluated as a function of σ_L assuming that active and reactive power pseudo-measurements only are considered. Fig. 2 reports the empirical performance $\widehat{ARMSE}$ for both BLSE and WLS methods. At the beginning, the BSLE estimation accuracy is worse than WLS. However, the difference between the results of such techniques becomes small for $\sigma_L > 0.3$ (30%), which is typical when pseudo-measurements are considered. The figure also reports the theoretical ARMSE curve of the BSLE estimator as it results from (26). This looks

slightly optimistic compared to the Monte-Carlo performance, probably because of the linearization error.

The second test case, reported in the left panel of Fig. 3, compares the accuracy of BSLE and WLS-based state estimators when both active/reactive power pseudo-measurements with $\sigma_L = 0.5$ (50%) and a growing number of PMUs are used. Three different values of PMU accuracy are considered, i.e. $\sigma_{PMU} = (1\%, 0.1\%, 0.01\%)$. In all cases the PMU location results from the greedy placement Algorithm 2. The figure shows that as soon as at least one PMU is included, the theoretical performance computed according to (23) matches almost exactly the results obtained through simulations with both BSLE and WLS for $\sigma_{PMU} < 0.1\%$. The figure also shows that a single PMU with accuracy 0.1% provides better performance than using PMUs with accuracy 1% at all nodes. Similarly, using three PMUs with accuracy 0.01% assures better results than using PMUs with accuracy 0.1% at all nodes, thus indicating a possible trade-off between PMU deployment costs and accuracy.

The third test case, reported in the middle panel of Fig. 3, compares the optimal and greedy-based theoretical ARMSE curves as a function of the number of PMUs, when $\sigma_L = 0.5$ (50%) and $\sigma_{PMU} = 0.001$ (0.1%). The sequence based on the greedy placement algorithm (excluding the feeder at node 1) is $\mathcal{M}^{gr}(N) = \{3, 7, 13, 15, 10, 14, 8, 12, 5, 11, 6, 9, 4, 2\}$. Although the PMU placement is in general different for the two strategies (for up to 4 PMUs the optimal placement sets are $\mathcal{M}^{op}(1) = \{3\}, \mathcal{M}^{op}(2) = \{3, 7\}, \mathcal{M}^{op}(3) = \{7, 12, 15\}, \mathcal{M}^{op}(4) = \{7, 10, 12, 15\}$), performances are very close, thus suggesting that the greedy algorithm might be a viable solution for PMU placement in large networks, where the true optimal positions can be hardly found for computational limits in solving the general combinatorial problem.

The final test case, reported in the right panel of Fig. 3, shows the $\widehat{ARMSE}$ values associated with the WLS-based estimator as a function of the number of instruments when voltage magnitudes only are measured. Also in this case, all pseudo-measurements are affected by 50% uncertainty, while the instrument accuracy is $1\%, 0.1\%$ or 0.01%. The results of this figure confirm the importance of measuring phasor angles. In fact, the magnitude-only measurements cannot improve estimation accuracy beyond a certain limit even when such measurements are extremely accurate (i.e. $\sigma_{PMU} = 0.01\%$).
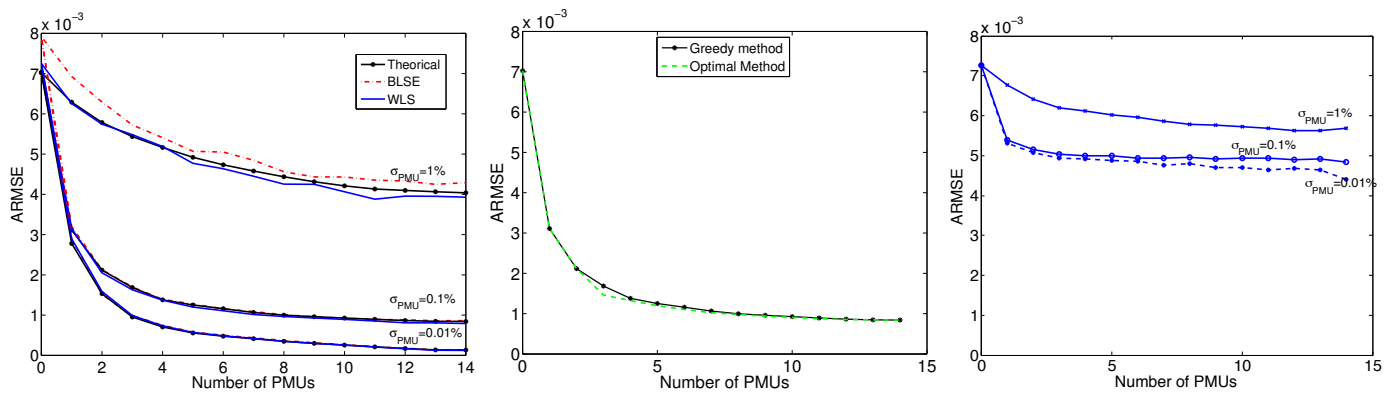
Fig. 3. ARMSE curves as a function of number of PMUs. In all cases, the relative uncertainty of pseudo-measurements is $\sigma_L = 0.5\%$. **Left panel:** BLSE (red dashed line), WLS (blue solid line) and theoretical ARMSE values (black marker line) for $\sigma_{PMU} = (1\%, 0.1\%, 0.01\%)$. **Middle panel:** theoretical BLSE results obtained with both the greedy-based and the optimal PMU placement. **Right panel:** Simulation-based results associated with the WLS estimator assuming to measure nodal voltage magnitudes only with 1%, 0.1% or 0.01% accuracy. Again, instruments are positioned sequentially according to the greedy placement algorithm.

IX. Conclusions

In this work we addressed the problem of state estimation in distribution networks based on pseudo-measurements and PMU measurements. The proposed Bayesian linear state estimator (BLSE) is shown to provide the same performance of the standard non-linear weighted least square estimator (WLS) but with major computational benefits. The main novelty of this approach is that the performance can be computed off-line and can be used to address several problems such as optimal PMU placement or trade-offs between number of PMUs versus their accuracy, without running extensive Monte-Carlo simulations. Moreover, because the estimation error in distribution systems tends to be dominated by uncertainty in loads and scarcity of instrumented nodes, the linearized method along with the use of high-precision PMUs may be a suitable way to facilitate on-line state estimation where it was previously impractical. Future research directions include the extension of the proposed strategy to unbalanced three-phase networks, large scale networks, and dynamic state estimation such as Kalman-filtering.

References

[1] A. Abur and A. Exposito, *Power System State Estimation (Theory and Implementation)*. Marcel Dekker, 2004.

[2] J. T. A. Phadke and K. Karimi, "State estimation with phasor measurements," *IEEE Trans. on Power Systems*, vol. 1, pp. 233–238, 1986.

[3] M. Zhou, V. Centeno, J. Thorp, and A. Phadke, "An alternative for including phasor measurements in state estimators," *IEEE Transactions on Power Systems*, vol. 21, pp. 1930–1937, 2006.

[4] T. Bi, X. Qin, and Q. Yang, "A novel hybrid state estimator for including synchronized phasor measurements," *Electric Power Systems Research*, vol. 78, no. 8, pp. 1343–1352, 2008.

[5] A. von Meier, D. Culler, A. McEachen, and R. Arghandeh, "Micro-synchrophasors for distribution systems," in *Proceedings of IEEE Innovative Smart Grid Technologies Conference (ISGT2014)*, 2014.

[6] R. Sevlian and R. Rajagopal, "Short term electricity load forecasting on varying levels of aggregation," *submitted to IEEE Transactions on Power Systems*.

[7] A. von Meier and R. Arghandeh, *Renewable Energy Integration*. Academic Press, ch. Every Moment Counts: Synchrophasors for Distribution Networks with Variable Resources, pp. 429–438.

[8] K. Purchala, L. Meeus, D. V. Dommelen, and R. Belmans, "Usefulness of DC power flow for active power flow analysis," in *Proceedings of IEEE PES general meeting*, 2005.

[9] C. Lu, J. Teng, and W.-H. Liu, "Distribution system state estimation," *IEEE Transactions on Power Systems*, vol. 10, pp. 229–240, 1995.

[10] H. Wang and N. N. Schulz, "A revised branch current-based distribution system state estimation algorithm and meter placement impact," *IEEE Transactions on Power Systems*, vol. 19, pp. 207–213, 2004.

[11] D. Haughton and G. Heydt, "A linear state estimation formulation for smart distribution systems," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 1187–1195, 2013.

[12] P. Janssen, T. Sezi, and J.-C. Maun, "Distribution system state estimation using unsynchronized phasor measurements," in *Proceedings of IEEE Innovative Smart Grid Technologies Conference Europe (ISGT2012 Europe)*, 2012.

[13] S. Bolognani and S. Zampieri, "A distributed control strategy for reactive power compensation in smart microgrids," *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2818–2833, 2013.

[14] ——, "On the existence and linear approximation of the power flow solution in power distribution networks," University of Padova, Technical report (available on http://arxiv.org, 2014.

[15] U. Eminoglua and M. Hocaoglub, "Distribution systems forward/backward sweep-based power flow algorithms: A review and comparison study," *Electric Power Components and Systems*, vol. 37, no. 1, pp. 91–110, 2008.

[16] R. Arthur and V. Vittal, *Power System Analysis*. 2nd edition, Prentice Hall, Inc., 2000.

[17] W. H. Kersting, "Radial distribution test feeders," in *IEEE Power Engineering Society Winter Meeting*, vol. 2, 2001, p. 908912.

[18] D. Macii, G. Barchi, and L. Schenato, "On the on the role of phasor measurement units for distribution system state estimation," in *Proceedings of IEEE Workshop on Environmental, Energy and Structural Monitoring Systems (EESMS)*, Sep. 2014.

[19] D. Das, D. Kothari, and A. Kalam, "Simple and efficient method for load flow solution of radial distribution networks," *International Journal of Electrical Power & Energy Systems*, vol. 17, no. 5, pp. 335 – 346, 1995.

[20] P. Pegoraro and S. Sulis, "On the robustness in distribution system state estimation," in *Instrumentation and Measurement Technology Conf. (I2MTC), 2012 IEEE International*, May 2012, pp. 776–780.