The Language of Physics Equations

Bruce L. Sherin
4533 Tolman Hall
University of California, Berkeley
Berkeley, CA 94720
bsherin@socrates.berkeley.edu

Abstract

The central hypothesis of this paper is that physics students learn to understand equations in terms of a number of conceptual elements that are referred to as "symbolic forms." Each symbolic form associates a simple conceptual schema with a pattern of symbols in an equation. Taken together, the set of symbolic forms constitutes a vocabulary of elements out of which novel expressions can be constructed, and in terms of which expressions can be understood. The work described here is based on an extensive analysis of a corpus of videotapes of moderately advanced university students solving physics problems.

Introduction

Research on physics problem solving has made significant strides in describing how students and experts solve textbook problems. A few different perspectives have each, working in their own way, helped to fill in different portions of the physics problem solving puzzle. Furthermore, this work has contributed to our understanding of the nature of problem solving, considered more generally.

What have been most successful – or, at least, most strongly predictive – are models that attempt to account for the sequence of steps in a solution. These models trace the equations that are written, the sequence in which they are written, and the steps by which they are manipulated to get to a solution. In the most basic of these models, equations are selected from what is essentially a remembered database, based simply on the quantities that appear in the problem. The solver selects equations which have variables that correspond to the quantities given or the quantities desired (e.g., Bhaskar & Simon, 1977; Larkin et al., 1980). In addition, there have been attempts to build models in which the solution is driven by more sophisticated varieties of understanding. For example, Larkin (1983) describes a model in which expert problem solvers possess schemata, each of which is roughly associated with a fundamental physical principle, that guide the steps in a solution.

However, there is one piece of the puzzle that has not been well described by these existing analyses. In all of these models, there is only a limited sense in which the solver understands the equations; the equations are simply written from memory or constructed according to a limited set of rules. Thus, the issue to be addressed in this paper is whether there is a more fundamental level of equation understanding. Can physicists and physics students read an equation and understand what it "says?" If so, in what terms do they understand it?

The issue of how equations – and symbolic expressions generally – are meaningful to people is an important question with a rich history (e.g., Goodman, 1976; Kaput, 1987; Kieran, 1992). In this paper, I will present one particular viewpoint: I will argue that physics students learn to understand equations in terms of a relatively idiosyncratic vocabulary of overlapping elements that I call "symbolic forms" or just "forms," for short. Each symbolic form associates a simple conceptual schema with a pattern of symbols in an equation.

The Data Corpus and Its Analysis

The work described here is based on an extensive analysis of a corpus of videotapes of 5 pairs of students solving physics problems. The subjects in this study were UC Berkeley students enrolled in "Physics 7C," the third course in the introductory sequence intended primarily for engineering students.

All of the experimental sessions were conducted in a laboratory setting. Students worked with their partner at a blackboard to solve a pre-specified set of problems. Most of these problems were fairly traditional textbook problems, but a few more unusual tasks were also included.

A total of 27 hours of videotape were collected. A subset of this data, corresponding to student work on the tasks shown in Table 1, was selected for more focused analysis. This subset of the corpus, which totaled approximately 11.5 hours of videotape, was carefully transcribed and analyzed.

The analysis of the corpus was primarily qualitative in character. In Sherin (1996) an argument for the view presented here is made using numerous examples of both brief and extended episodes. However, although the argument leans heavily on detailed analysis of cases, steps were also taken to systematize the qualitative analysis and to ensure that the examples selected were representative of the corpus as a whole.

The first phase of the systematic analysis included two components, an *utterance-centered* analysis and an *equation-centered* analysis. The purpose of the utterance-centered analysis was to locate interpretation events; every student utterance was examined and utterances identified in which a student interpreted an equation. Here, "interpretation" was defined very broadly to be any statement in which a student referred to a written equation. For example, a student might have pointed to an expression as they made a statement, or they might have mentioned a portion of an expression in an utterance.

1. A person gives a block a shove so that it slides across a table and then comes to rest. Talk about the forces and what's happening. How does the situation differ if the block is heavier?
2. (a) Suppose a pitcher throws a baseball straight up at 100 mph. Ignoring air resistance, how high does it go? (b) How long does it take to reach that height?
3. Imagine that two objects are dropped from a great height. These two objects are identical in size and shape, but one object has twice the mass of the other object. Because of air resistance, both objects eventually reach terminal velocity. (a) Compare the terminal velocities of the two objects. Are their terminal velocities the same? (b) Suppose that there was a wind blowing straight up when the objects were dropped, how would your answer differ? What if the wind was blowing straight down?
4. A mass hangs from a spring attached to the ceiling. How does the equilibrium position of the mass depend upon the spring constant, k, and the mass, m?
5. Peggy Fleming (a one-time famous figure skater) is stuck on a patch of frictionless ice. Cleverly, she takes off one of her ice skates and throws it as hard as she can. (a) Roughly, how far does she travel? (b) Roughly, how fast does she travel?
6. An ice cube, with edges of length L, is placed in a large container of water. How far below the surface does the cube sink?
7 Suppose that you need to cross the street during a steady downpour and you don't have an umbrella. Is it better to walk or run across the street? Make a simple computation, assuming that you're shaped like a tall rectangular crate. Also, you can assume that the rain is falling straight down. Would it affect your result if the rain was falling at an angle?

Table 1. Tasks included in the focused analysis.

The purpose of the equation-centered analysis was to locate events in which an equation was constructed from some conceptual content to be expressed, rather than from memory or through symbolic manipulation. These events were identified by first coding every equation in the focal corpus as either *written from memory*, *derived through manipulation*, or *constructed*. Constructed expressions were, at least in part, invented by students.

Overall, the goal of this first stage of the systematic analysis was to identify events in which equations were interpreted or constructed by students. The presumption is that these events require the sort of deep equation understanding that is the concern of this work.

The utterance-centered analysis identified a total of 144 interpretation events, and the equation-centered analysis, looking at 547 separate equations, identified 75 construction events. This immediately suggests that the type of phenomena under consideration here are not entirely rare. Of the 547 expressions that students wrote, 14% were constructed in some manner. Furthermore, the 144 interpretive utterances were applied to 93 separate equations. Thus, there were interpretive utterances associated with 17% of all the expressions written.

In the next phase of the analysis, the goal was to identify the set of symbolic forms that could account for the 219 interpretation and construction events. To accomplish this, these events were iteratively coded and recoded in terms of symbolic forms, with the set of symbolic forms refined between each coding. A number of principles were employed to evaluate the adequacy of the coding. These included:

Coverage. The full set of forms must cover all instances of interpretation and construction.

Consistency with statements. Statements made by students must suggest or be consistent with the focus implied in the symbolic form.

Harmony. The full set of forms should fall at approximately the same level of abstraction and should constitute a roughly complete set.

The iterative coding process, the principles employed, as well as the rest of the analysis procedure and more extended examples, are described in detail in Sherin (1996).

Symbolic Forms

The central hypothesis of this work is that physics students learn to express a vocabulary of simple ideas in equations, and to read these same ideas out of equations. Corresponding to the components of this vocabulary are knowledge elements that I call "symbolic forms." A symbolic form has two components:

1. *Conceptual Schema*. Each symbolic form includes a conceptual schema. The particular schemata associated with forms turn out to be relatively simple structures, involving only a few entities and a small number of simple relations among these entities. These schemata are similar to diSessa's (1993) "p-prims" and Johnson's (1987) "image schemata," which are also presumed to have relatively simple structures.
2. *Symbol Template*. Each form is associated with a specific template for a symbolic expressions.

Stated simply, the schema is the "idea" to be expressed in the equation and the symbol template is the specification of how that "idea" is written in symbols.

The nature of symbolic forms is best illustrated with a brief example episode from the data corpus. In this episode, two students were working on the first task listed in Table 1, which concerns blocks sliding along a surface with friction.

The pair of students, Mike and Karl, were unhappy with the solution that they obtained to this task, and they decided that their difficulties might stem from some assumptions that are typically made in physics courses. Usually, the coefficient of friction, μ, is treated as a constant that depends only on the properties of the two materials that are rubbed together. For various reasons, Mike and Karl decided that it might be more accurate to have an expression for the coefficient of friction that depends on the mass of the block:

μ = (some function of mass)

More specifically, these students thought that μ should decrease with increasing mass.

Karl I guess what we're saying is that the larger the weight, the less the coefficient of friction would be.

During a span of approximately 10 minutes, Mike and Karl gradually refined their specification for an expression for μ. The following is an excerpt from their discussion:

Karl Well, yeah, maybe you could consider the frictional force as having two components. One that goes to zero and the other one that's constant. So that one component would be dependent on the weight. And the other component would be independent of the weight.

Mike So, do you mean the sliding friction would be dependent on the weight?

Karl Well I'm talking about the sliding friction would have two components. One component would be fixed based on whatever it's made out of. The other component would be a function of the normal force. The larger the normal force, the smaller that component.

Finally, the students undertook to write an expression. After a few minutes and some false starts, they settled on the following equation:

$$\mu = \mu_1 + C\frac{\mu_2}{m}$$

Here, m is the mass, and μ_1, C, and μ_2 are constants. There are some difficulties with this expression; most notably, μ tends to infinity as the mass becomes small. Nonetheless, this equation captures much of what the students intended.

So, Mike and Karl have constructed a novel expression. Clearly, this expression was not simply written out from memory (you will not find it in any textbook) and it was not derived by manipulating other equations. Thus, the question is: How did Mike and Karl write this equation? For example, how did they know to write a '+' instead of a '×' between the two terms? And how did they know to put the m in the denominator?

I hypothesize that a number of symbolic forms underlie the construction of this expression. The first of these symbolic forms is suggested by Karl's statement that "the sliding friction would have two components." I call this form *parts-of-a-whole.*

PARTS-OF-A-WHOLE

Schema	*Symbol Template*
A whole is composed of two or more parts.	□ + □ +...

The point here is that, because the *parts-of-a-whole* form associates a conceptual schema with a template for an expression, this dictates – at a certain degree of specificity – what the students must write. In this case, the form dictates that the students should write two or more terms separated by plus signs. In fact, the students initially began with marks that played the role of placeholders for each term, gradually filling in details.

The second symbolic form involved here goes with Karl's statement that: "The other component would be a function of the normal force. The larger the normal force, the smaller that component." This is the *proportionality minus* or *prop-* form. The idea of the *prop-* form is that, if you want a term to decrease as some quantity increases, then the quantity must appear in the denominator of the term.

PROP-

One quantity varies inversely with another.	$\frac{?}{\ldots x \ldots}$

Finally, I will just briefly mention two other symbolic forms that played a role in Mike and Karl's construction of their expression for μ. First, the *coefficient* form permitted the writing of the 'C' that multiplies the second term. Lastly, the *identity* form permitted the writing of the "μ=" with which the expression begins.

To be sure, Mike and Karl's construction of a novel expression for the coefficient of friction was a somewhat unusual event; it is rare that the solving of a textbook physics problem requires the construction of novel equations. If symbolic forms were only implicated in these unusual construction events, then their existence might not be very important. However, there are many places that knowledge of this sort is useful in more typical problem solving. As they work, students can, for example:

- Judge the reasonableness of an expression.
- Reconstruct partly remembered expressions.
- Justify expressions in intuitive terms.

Limited space prevents me from illustrating these various roles of symbolic forms.

The Six Clusters of Symbolic Forms

Altogether, the analysis of the data corpus identified 21 symbolic forms. These are listed in Table 2, arranged into 6 groups that are referred to as "clusters." Within a given cluster, the various schemata involve entities of the same or similar ontological type. In addition, the forms in a cluster tend to parse an expression at the same level of detail. While I believe that the list in Table 2 represents a reasonable portrait of the vocabulary of symbolic forms that are involved in understanding physics expressions, it should be kept in mind that the content of this list is at least partly specific to the data corpus on which this work was based. A brief description of each cluster follows.

Competing Terms. One way that a physics equation may be understood is as an arrangement of *terms* that

Competing Terms Cluster		Terms are Amounts Cluster		Dependence Cluster	
competing terms	□ ± □ ± □ ...	*parts-of-a-whole*	□ + □ + ...	*dependence*	$\ldots x \ldots$
opposition	□ − □	*base ± change*	□ ± Δ	*no dependence*	...
balancing	□ = □	*whole - part*	□ - □	*sole dependence*	$\ldots x \ldots$
canceling	0 = □ − □	*same amount*	□ = □		

Multiplication Cluster		Coefficient Cluster		Other	
intensive•extensive	x × y	*coefficient*	*c* □	*identity*	$x = \ldots$
extensive•extensive	x × y	*scaling*	*n* □	*dying away*	$e^{-x} \ldots$

Proportionality Cluster			
prop+	$\frac{\ldots x \ldots}{?}$	*ratio*	$\frac{\ldots x \ldots}{\ldots y \ldots}$
prop-	$\frac{?}{\ldots x \ldots}$	*canceling(b)*	$\frac{\ldots x \ldots}{\ldots x \ldots}$

Table 2. The full set of symbolic forms identified in the analysis.

conflict and support, and that oppose and balance. The Competing Terms Cluster contains the forms related to seeing equations in this manner, as terms associated with influences in competition. An example from this cluster, the *balancing* form, is discussed later in this paper.

Dependence. The forms in this cluster have to do with the simple fact of whether a specific individual symbol appears in an expression does or does not appear in an expression. Most basic of these forms is *no dependence*, whose symbol pattern involves the absence, rather than the presence of symbols. In contrast, the *dependence* form specifies only that a given symbol appears in an expression.

Proportionality. When a physics student looks at an equation, the line that divides the top from the bottom of a ratio – the numerator from the denominator – is a major landmark. Forms in the Proportionality Cluster involve the seeing of individual symbols as either above or below this important landmark. The *prop-* form, discussed above, is an element of this cluster.

Terms are Amounts. Like the forms in the Competing Terms Cluster, these forms address expressions at the level of terms. However, rather than describing a battle between competing influences, these expressions concern the collecting of a generic substance, putting some in and taking some away. Thus, while '+' and '–' signs in Competing Terms expressions are commonly associated with directions in physical space, signs in this cluster generally signal adding on or taking away.

Coefficient. In the *coefficient* forms, a product of factors is seen as broken into two parts. One part is the coefficient itself, which often involves only a single symbol and is usually written on the left.

Multiplication. The forms in this cluster also break down products of factors into parts. In this case, the parts are intensive or extensive quantities. Stated roughly, an intensive quantity specifies an amount of something per unit of something else, while an extensive quantity is a number of units.

Forms and Problem Solving Schemata

In stating that physics equations are understood in terms of symbolic forms, I am making a very particular hypothesis about the way equations are understood; I am claiming that equations are understood in terms of certain type of abstraction with a specific level of generality. In the remaining sections of this paper, I will attempt to simultaneously accomplish two jobs: I will try to clarify exactly what this level of generality is, and I will argue for the importance of abstractions of this sort.

To start, I will contrast my view with one alternative. Note that it is possible that people write and understand physics expressions only at the level of whole equations associated with formal principles. In other words, students and physicists might only know that a given equation is an expression of a particular formal physics principle. In contrast, I am arguing that equations have meaningful sub-structure, and, because of this, that they "say something" to the student that knows the vocabulary. To further examine this contrast, I will refer to a model of physics problem solving proposed by Larkin (1983).

In Larkin's model, physics problem solving is guided by a set of schemata, such as what Larkin calls the "Forces Schema" and the "Work-Energy Schema." These schemata are quite closely related to physical principles, as these principles would be presented in a physics textbook. For example, Larkin says that the Forces Schema "corresponds to the physical principle that the total force on a system (along a particular direction) is equal to the system's mass times its acceleration (along that direction)." This is essentially Newton's Second Law, F=ma. The schema includes rules that correspond to "force laws," which are the laws that allow a physicist to compute the forces on an object given the arrangement of objects in a physical system. Stated simply, this adds up to the following image of problem solving: The construction rules allow the problem solver to find each of the forces acting on an object, then these forces are totaled and substituted into $F_{tot} = ma$.

The Forces Schema could easily be used to solve some of the tasks in Table 1. For example, it is applicable to Task 4 in which a mass hangs at rest from a spring, and Task 6 in which an ice cube floats in a glass of water. In both cases, there is a force acting upward on the object and a force acting downward. Following the Forces Schema, the solution would proceed by first finding each of these forces, then substituting into $F_{tot} = ma$ to write:

$$F_{up} + F_{down} = ma$$

Since in both of these tasks the object is not moving, the acceleration is zero and we can write $F_{up} = -F_{down}$. Once this equation is written, it is possible to solve for the desired quantities. For example, in the floating ice cube problem, the upward force is the buoyant force of water, and the downward force is the force of gravity.

Significantly, none of the five pairs of students solved the ice cube problem in precisely this manner. Instead, all of the pairs jumped directly to equating the upward and downward forces as in the following expression, dealing with sign issues in a "hand-waving" manner, if at all.

$$F_{up} = F_{down}$$

Student justifications for this expression included assertions that this equation must be true "at equilibrium" and that the forces must "balance:"

Alan At equilibrium, they're equal.

Jack Um, so we know the force down is M G and that has to be balanced by the force of the water

I explain this by hypothesizing that the behavior is being driven by a symbolic form, the *balancing* form. In *balancing*, two competing influences are seen as precisely equal and opposite, and this schematization is bound to a symbol template in which two expressions are separated by an equal sign.

BALANCING

Two influence are precisely in balance.	□ = □

How does this account differ from Larkin's? In arguing that a symbolic form is driving the equating of the up and down forces, I am asserting that students are working directly from a more basic and general schematization of the physical situation. They see *balancing* in the situation, and this directly dictates the form of the expression to be written.

To further clarify the distinction, note that the *balancing* form is more general than any schemata that pertain only to forces. Because it is not specific to forces, *balancing* can be applied to explain student work on some tasks that are not usually solved with forces, such as Task 5. In this task, an ice skater who is stranded on frictionless ice throws an ice skate. When this is done, the skater recoils in the opposite direction. In some cases, the students in this study solved this problem by immediately equating the momenta of the skate and the skater just after the collision.

$$M_{skate}V_{skate} = M_{skater}V_{skater}$$

In contrast, a more formal solution would begin with a statement of the conservation of momentum. This would involve equating the momenta before and after the collision, rather than the momenta of the skate and skater. Again, I explain the students' behavior by appeal to the *balancing* form. As in the case of the spring and ice cube problems, *balancing* directly drives the writing of an expression. But, in this case, *balancing* is applied to momenta rather than forces. This property of symbolic forms – that they cut across physical principles – is one of the important characteristics of the abstractions embodied in forms.

I have not tried to provide strong evidence for symbolic forms over models like Larkin's. The simple fact that students appear to skip steps that would appear in a more rigorous solution does not constitute conclusive evidence. For example, the students could just be leaving out steps that they feel are obvious, or they could be applying special purpose schemata for the cases of balanced forces and balanced momenta. For a full comparison and discussion of the relevant evidence, see Sherin (1996).

Forms and Intuitive Understanding

Additional support for the type of abstractions associated with symbolic forms can be found in studies of intuitive physics. It turns out some researchers have argued that, prior to any formal physics instruction, students understand the physical world in terms of abstractions that are very similar to those embodied in forms. Such results add plausibility to the existence of forms, since they suggest possible origins for this knowledge.

First, diSessa (1993) describes a portion of intuitive physics knowledge that he calls the "sense-of-mechanism." According to diSessa, elements of the sense-of-mechanism – which he calls "phenomenological primitives" or "p-prims" – constitute the base level of our intuitive explanations of physical phenomena. P-prims appear to live at a level of abstraction that is very similar to symbolic forms. For example, diSessa lists p-prims that he calls "balancing" and "dying away," both of which have direct correlates in the forms vocabulary.

A second relevant body of research concerns what the researchers involved have called "qualitative reasoning about physical systems" (deKleer & Brown, 1984; Forbus, 1984). This research gives a prominent role to proportionality relations, such as the *prop-* form discussed above, in reasoning about physical systems. If we accept that these relations are a central part of how we understand physical systems, this adds to the plausibility that proportionality relations should be elements in the vocabulary that we can read and write in equations.

Finally, I want to very briefly mention some related research that pertains to how students solve elementary mathematics problems such as the following: John has five apples and Mary gives him three more, how many does he have? The research in question is a collection of papers that identify what Greeno (1987) has called "patterns" in arithmetic word problems. (See, for example, Carpenter & Moser, 1983; Riley, Greeno, & Heller, 1983). To cite an instance, Riley and colleagues list four categories of arithmetic word problems: "change," "equalization," "combine," and "compare." In a *change* problem, for example, some amount is added on to an original quantity, increasing the size of that original quantity. In contrast, in *combine* problems, two quantities are combined to produce a new third quantity.

Clearly, these patterns live at a similar level of abstraction to symbolic forms. For example, the *parts-of-a-whole* form and *combine* pattern seem to involve similar schematizations. Because schematizations of this sort are important to how young students understand the quantitative relations in problem situations, it is plausible that similar schematizations continue to be important in later, more advanced work with equations. This is supported by the work of Izsak (1997), who has attempted to trace the origin and development of some symbolic forms.

Conclusion

In this paper, I have attempted to suggest that a full description of physics problem solving should include some account of a deep level of equation understanding. In particular, I have argued that even moderately advanced students learn a vocabulary of "symbolic forms," in terms of which they can construct and understand expressions. The existence of this knowledge is important for elaborating our models of problem solving and building theories of equation meaning, as well as ultimately for improving instruction.

Generally, the attitude of this work is that symbolic forms and other sorts of knowledge, such as knowledge tied to physical principles, are complementary; we will need both to explain physics problem solving. Thus, I do not want to argue for symbolic forms as a strict alternative to existing models of physics problem solving; I only maintain that there are gaps in what these models can explain, and that symbolic forms can fill some of these gaps.

Most importantly, we would like to be able to model more flexible and generative varieties of knowledge. Traditional models of problem solving involve schemata that contain, within their structure, an outline for the solution of a problem. Alternatively, we would like to be able to describe the way in which a "deep understanding of physics" can drive problem solving in a flexible manner, adapting on-the-fly, and generating new and creative solutions. In hypothesizing symbolic forms, I am filling in part of the story of how intuitive understanding can play a role in problem solving. Because they can understand the content of expressions, students can depart from more formal solutions, judge the reasonableness of expressions, and construct new expressions for new content.

Acknowledgments

This work was funded in part by a Spencer Foundation fellowship and by the Spencer Foundation under grant B-1393, Andrea A. diSessa, principal investigator.

Thanks to Andrea A. diSessa for comments on an earlier draft of this paper.

References

Bhaskar, R., & Simon, H. A. (1977). Problem solving in semantically rich domains: An example from engineering thermodynamics. *Cognitive Science,* 1, 193-215.

Carpenter, T. P., & Moser, J. M. (1983). The development of addition and subtraction problem-solving skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and Subtraction: A Cognitive Perspective* (pp. 9-24). Hillsdale, NJ: Erlbaum.

deKleer, J., & Brown, J. S. (1984). A qualitative physics based on confluences. *Artificial Intelligence,* 24, 7-83.

diSessa, A. (1993). Toward an epistemology of physics. *Cognition and Instruction,* 10(2&3), 165-255.

Forbus, K. D. (1984). Qualitative process theory. *Artificial Intelligence,* 24, 85-168.

Goodman, N. (1976). *Languages of Art* (Rev. ed.). Indianapolis, IN: Hackett.

Greeno, J. G. (1987). Instructional representations based on research about understanding. In A. H. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education* (pp. 61-88). Hillsdale, NJ: Erlbaum.

Izsak, A. G. (1997). *Inscribing the winch: A study of how students develop ways to use representations.* Paper presented at the Annual Meeting of the AERA, Chicago.

Johnson, M. (1987). *The Body in the Mind: The Bodily Basis of Meaning, Imagination, and Reason.* Chicago: University of Chicago Press.

Kaput, J. J. (1987). Towards a theory of symbol use in mathematics. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 159-195). Hillsdale, NJ: Erlbaum.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 390-419). New York: Macmillan.

Larkin, J. (1983). The role of problem representation in physics. In D. Gentner, & A. Stevens (Ed.), *Mental Models* (pp. 75-98). Hillsdale, NJ: Erlbaum.

Larkin, J., McDermott, J., Simon, D. P., & Simon, H. A. (1980). Expert and novice performance in solving physics problems. *Science,* 208, 1335-1342.

Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsberg (Ed.), *The Development of Mathematical Thinking* (pp. 153-196). New York: Academic Press.

Sherin, B. L. (1996). The symbolic basis of physical intuition: A study of two symbol systems in physics instruction. Doctoral dissertation. Berkeley: University of California, Berkeley.