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Graded Representations of Current Algebras

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Kayla Marie Murray

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Dissertation Committee:

Professor Vyjayanthi Chari, Chairperson
Professor Wee Liang Gan
Professor Carl Mautner

The Dissertation of Kayla Marie Murray is approved:

Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

Graded Representations of Current Algebras

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Kayla Marie Murray

Doctor of Philosophy, Graduate Program in Mathematics
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Professor Vyjayanthi Chari, Chairperson

For a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we study the representations of the associated current algebra $\mathfrak{g}[t]$. We focus our attention on $V(\boldsymbol{\xi})$ modules, which are a large family of indecomposable representations that are quotients of local Weyl modules and include Demazure modules when $\mathfrak{g}$ is simply laced. We establish three new presentations of $V(\boldsymbol{\xi})$ modules, which show that they are finitely presented as quotients of local Weyl modules. We establish each presentation for $\mathfrak{sl}_2$ and then for an arbitrary $\mathfrak{g}$. With these presentations, we establish the existence of two short exact sequences of $V(\boldsymbol{\xi})$ modules. These short exact sequences were used to establish a character formula for the tensor product of a level 2 Demazure module and a local Weyl module.

Contents

Introduction

One reason for the interest in graded representations of the current algebra associated to a simple Lie algebra $\mathfrak{g}$ is their connection to representations of quantum affine algebras. It was shown in [3] and [2] that the classical limit of certain irreducible representations of the quantum affine algebra associated to $\mathfrak{g}$ give rise to indecomposable graded representations of the associated current algebra. Hence, understanding graded representations of current algebras could help in the understanding of representations of quantum affine algebras. We focus on $V(\boldsymbol{\xi})$ modules which are a family of indecomposable graded representations of current algebras that were first defined in [5]. Based on their definition, these representations are quotients of local Weyl modules. When $\mathfrak{g}$ is simply laced, this family of modules includes both Demazure modules and local Weyl modules. After their introduction, $V(\boldsymbol{\xi})$ modules were later defined in the case of twisted current algebras in [9].

Since their definition, $V(\boldsymbol{\xi})$ modules have been of interest due to their connection to Demazure modules. Demazure modules associated to a simple Lie algebra were first introduced in [6] as modules for a Borel subalgebra of a Lie algebra. It was shown in [10] that Demazure modules for current algebras have filtrations by higher level Demazure modules.

This result was extended in [4] to show that $V(\boldsymbol{\xi})$ modules also have filtrations by Demazure modules, but this proof was not constructive.

In chapters 2 and 3, we provide a total of three new presentations of $V(\xi)$ modules. These presentations show that these modules are finitely presented as quotients of local Weyl modules, which was previously known only in special cases of $\boldsymbol{\xi}$. In chapters 4 and 5, we establish the existence of two short exact sequences of $V(\boldsymbol{\xi})$ modules. These short exact sequences were used in [1] to establish character formulas for tensor products of Demazure modules.

Chapter 1

Simple Lie Algebras and Current Algebras

In this chapter, we provide the background of the representation theory of simple Lie algebras. Then, we define the current algebra associated to a simple Lie algebra. We also introduce $V(\boldsymbol{\xi})$ modules, which are a family of representations of the current algebra.

1.1 Simple Lie Algebras

We start by setting some notation. Throughout, $\mathbb{C}$ denotes the field of complex numbers. Additionally, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of integers and nonnegative integers, respectively.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathbf{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $n = \dim \mathfrak{h}$, and R be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We set $I = \{1, 2, \ldots, n\}$. Let $\{\alpha_i : i \in I\}$ and $\{\omega_i : i \in I\}$ be a set of simple roots and fundamental weights, respectively. We let R^+ denote the set of postive roots and

P^+ denote the $\mathbb{Z}_+-$span of the fundamental weights. Let $x^{\pm}_{\alpha_i}$ and h_{α_i} with $1 \leq i \leq n$ be a Chevalley basis of $\mathfrak{g}$. Then, we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

where $\mathfrak{n}^{\pm} = \oplus_{i=1}^{n} \mathbb{C} x^{\pm}_{\alpha_i}$ and $\mathfrak{h} = \oplus_{i=1}^{n} \mathbb{C} h_{\alpha_i}$.

For $\lambda \in P^+$, let $V(\lambda)$ be the simple finite-dimensional $\mathfrak{g}-$module generated by v_λ with defining relations

$$\mathfrak{n}^+ v_\lambda = 0$$

$$h_{\alpha_i} v_\lambda = \lambda(h_{\alpha_i}) v_\lambda$$

$$(x^-_{\alpha_i})^{\lambda(h_{\alpha_i})+1} v_\lambda = 0$$

for $1 \leq i \leq n$. It is well known that these modules provide a classification of the finite-dimensional $\mathfrak{g}-$modules. So, any finite-dimensional $\mathfrak{g}-$module is isomorphic to a direct sum of these modules. Thus, for any finite-dimensional $\mathfrak{g}-$module V, we can write $V = \oplus_{\nu \in P} V_\nu$ where $V_\nu = \{v \in V : hv = \nu(h)v, h \in \mathfrak{h}\}$.

1.2 The Lie Algebra $\mathfrak{sl}_2$

Now, we consider the simple Lie algebra $\mathfrak{sl}_2$, which is the algebra of 2×2 traceless matrices. Let

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and } h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then, $\{x, y, h\}$ is the standard basis of $\mathfrak{sl}_2$ with brackets $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$ and Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$. There is one fundamental weight of $\mathfrak{sl}_2$. Hence, we identify P^+ with $\mathbb{Z}_+$. Therefore, the irreducible representations of $\mathfrak{sl}_2$ described in the previous section are the modules $V(n)$ with $n \in \mathbb{Z}_+$ which are generated by v_n where $xv_n = 0$, $hv_n = nv_n$ and $y^{n+1}v_n = 0$. Also, we have $\dim V(n) = n + 1$.

We consider the Lie algebra $\mathfrak{sl}_2$ for two reasons. First, there is a connection between any simple Lie algebra $\mathfrak{g}$ and $\mathfrak{sl}_2$. We have that $\mathfrak{h}$ acts on $\mathfrak{g}$ via the adjoint action. Since $\mathfrak{h}$ has a basis consisting of commuting semisimple elements, the action of $\mathfrak{h}$ is simultaneously diagonalizable. Hence, $\mathfrak{g}$ decomposes into weight spaces. Since the action is the adjoint action, the weight spaces are called root spaces. So, we have the root space decomposition $\mathfrak{g} = \oplus_{\alpha \in R \cup \{0\}} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : h.x = \alpha(h)x \ \forall h \in \mathfrak{h}\}$. Let $\mathfrak{n}^\pm = \oplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$. Then, we can fix non-zero elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ and $h_\alpha \in \mathfrak{h}$ such that $[h_\alpha, x_\alpha^\pm] = \pm 2x_\alpha^\pm$ and $[x_\alpha^+, x_\alpha^-] = h_\alpha$. Therefore, the subalgebra of $\mathfrak{g}$ generated by $\{x_\alpha^+, x_\alpha^-, h_\alpha\}$ is isomorphic to $\mathfrak{sl}_2$. Additionally, in much of our work, we first restrict to the case $\mathfrak{sl}_2$ and then consider arbitrary simple Lie algebras.

1.3 Current Algebra $\mathfrak{g}[t]$

The current algebra associated to $\mathfrak{g}$, denoted $\mathfrak{g}[t]$, is the Lie algebra of polynomial maps $\mathbb{C} \to \mathfrak{g}$. As a vector space, $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$, where t is an indeterminate. The bracket of $\mathfrak{g}[t]$ is given by

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg$$

for all $a, b \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t]$. We note that $\mathbb{C}[t]$ has a grading given by the degree of a polynomial. Hence, $\mathfrak{g}[t]$ inherits this $\mathbb{Z}_+$−grading.

Our focus will be on graded representations of $\mathfrak{g}[t]$. A representation V of $\mathfrak{g}[t]$ is $\mathbb{Z}-$graded if $V = \oplus_{r\in\mathbb{Z}} V[r]$ where $(g \otimes t^s).V[r] \subset V[r+s]$ for all $g \in \mathfrak{g}$ and $s \in \mathbb{Z}_+$. Now, we define the grade shift operator τ_r for $r \in \mathbb{Z}$. If V is a $\mathbb{Z}-$graded $\mathfrak{g}[t]-$module, then $\tau_r V$ is the $\mathbb{Z}-$graded $\mathfrak{g}[t]-$module where the graded pieces (ie. $V[s]$) are shifted up uniformly by r with the action of $\mathfrak{g}[t]$ on V unchanged.

We have the canonical inclusion $i : \mathfrak{g} \to \mathfrak{g}[t]$ given by $g \mapsto g \otimes 1$. Using this inclusion, every representation of $\mathfrak{g}[t]$ is also a representation of $\mathfrak{g}$. Now, for $z \in \mathbb{C}$, we define $ev_z : \mathfrak{g}[t] \to \mathfrak{g}$ by $ev_z(a \otimes f) = f(z)a$. This gives us a way of producing a representation of $\mathfrak{g}[t]$ from a representation of $\mathfrak{g}$. These pullbacks of representations of $\mathfrak{g}$ are called evaluation modules. The irreducible representations of $\mathfrak{g}[t]$ are pullbacks of the irreducible $\mathfrak{g}-$modules, which we denote $ev_z^* V(\lambda)$. We note that the only $\mathbb{Z}-$graded evaluation modules occur when $z = 0$. Hence, all finite-dimensional graded irreducible representations of $\mathfrak{g}[t]$ are given by $\tau_r ev_0^* V(\lambda)$ where $r \in \mathbb{Z}_+$.

1.4 $V(\xi)$ modules

We start by defining local Weyl modules. For $\lambda \in P^+$, the local Weyl module $W_{loc}(\lambda)$ is the $\mathfrak{g}[t]-$module generated by w_λ with defining relations

$$(x_i^+ \otimes \mathbb{C}[t])w_\lambda = 0, \quad (h_i \otimes t^s)w_\lambda = \lambda(h_i)\delta_{s,0}w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0.$$

Local Weyl modules were shown to be finite-dimensional. By declaring the grade of w_λ to be zero, we have that $W_{loc}(\lambda)$ is $\mathbb{Z}_+-$graded and $ev_0 V(\lambda)$ is the unique graded irreducible quotient of $W_{loc}(\lambda)$.

Let $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m > 0)$ be a partition. The length of the partition ξ, denoted $|\xi|$, is the sum of the entries in ξ. Thus, $|\xi| = \xi_1 + \xi_2 + \cdots + \xi_m$. Now, we define $V(\boldsymbol{\xi})$ modules. These modules were first defined by Chari and Venkatesh in [5]. They provided three presentations of these modules, which we give here. Let $\lambda \in P^+$ and $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ be an $|R^+|-$tuple of partitions satisfying $|\xi^\alpha| = \lambda(h_\alpha)$ for all $\alpha \in R^+$. Then, $V(\boldsymbol{\xi})$ is the $\mathfrak{g}[t]-$module generated by $v_{\boldsymbol{\xi}}$ with defining relations

$$\mathfrak{n}^+[t]v_{\boldsymbol{\xi}} = 0$$

$$(h \otimes t^s)v_{\boldsymbol{\xi}} = \delta_{s,0}\lambda(h)v_{\boldsymbol{\xi}}, \quad h \in \mathfrak{h}, s \in \mathbb{Z}_+$$

$$(x_{\alpha_i}^- \otimes 1)^{\lambda(h_i)+1}v_{\boldsymbol{\xi}} = 0, \quad i \in I$$

$$(x_\alpha^+ \otimes t)^s(x_\alpha^- \otimes 1)^{s+r}v_{\boldsymbol{\xi}} = 0, \alpha \in R^+, s, r \in \mathbb{N}, s + r \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha, \tag{1.4.1}$$

for some $k \in \mathbb{N}$.

Now, in order to provide the second presentation, we define

$$S(r,s) = \{(b_p)_{p\geq 0} \ : \ b_p \in \mathbb{Z}_+, \ \sum_{p\geq 0} b_p = r, \ \sum_{p \geq 0} pb_p = s\}$$

for $r, s \in \mathbb{Z}_+$. We note that for a given r and s, $S(r,s)$ is finite. For $x \in \mathfrak{g}$ and $r, s \in \mathbb{Z}_+$, we define the elements $\mathbf{x}(r,s) \in \mathbf{U}(\mathfrak{g}[t])$ by

$$\mathbf{x}(r,s) = \sum_{(b_p)_{p\geq 0} \in S(r,s)} (x \otimes 1)^{(b_0)}(x \otimes t)^{(b_1)} \cdots (x \otimes t^s)^{(b_s)} \tag{1.4.2}$$

where $(x \otimes t^j)^{(p)} := \frac{(x \otimes t_j)^p}{p!}$ for $j, p \in \mathbb{Z}_+$. It was shown that you can replace (1.4.1) by

$$\mathbf{x}_\alpha^-(r,s)v_{\boldsymbol{\xi}} = 0 \text{ if } r + s \geq 1 + rk + \sum_{j \geq k+1} \xi_j^\alpha \tag{1.4.3}$$

for some $k \in \mathbb{N}$. This gives us the second presentation of $V(\boldsymbol{\xi})$.

For the final presentation, we start by defining ${}_kS(r,s)$ for $k \in \mathbb{Z}_+$ to be the subset of $S(r,s)$ consisting of $(b_p)_{p\geq 0}$ such that $b_p = 0$ for $p < k$. Using this, we define

$$ {}_k\mathbf{x}(r,s) = \sum_{(b_p)\in {}_kS(r,s)} (x \otimes t^k)^{(b_k)}(x \otimes t^{k+1})^{(b_{k+1})} \cdots (x \otimes t^s)^{(b_s)}. \qquad (1.4.4) $$

Then, we can replace (1.4.3) by

$$ {}_k\mathbf{x}_\alpha^-(r,s)v_{\boldsymbol{\xi}} = 0, \alpha \in R^+, r+s \geq 1 + rk + \sum_{j\geq k+1} \xi_j^\alpha \qquad (1.4.5) $$

for some $k \in \mathbb{N}$.

We note that $V(\boldsymbol{\xi})$ is a large family of modules for $\mathfrak{g}[t]$ as it includes the local Weyl modules and Demazure modules when $\mathfrak{g}$ is simply laced.

Chapter 2

First New Presentation of $V(\xi)$

In this chapter, we provide the first of three new presentations for the module $V(\boldsymbol{\xi})$. We establish the presentation first for $\mathfrak{sl}_2$. Then, we provide the presentation in the case of an arbitrary simple Lie algebra.

2.1 Motivation

The motivation for this presentation comes from the three presentations of $V(\boldsymbol{\xi})$ modules given in [5]. In all of these presentations, there is one set of relations involving nonnegative integers r and s that depend on an inequality. This inequality involves both r and s and also depends on another nonnegative integer k. The relationship between k and the pair r and s is unknown in general. Additionally, we note that s only appears on the larger side of the inequality. This then implies that we have an infinite number of these relations. Hence, we wanted to find a presentation of $V(\boldsymbol{\xi})$ that did not involve an inequality for these relations.

2.2 Presentation when $\mathfrak{g} = \mathfrak{sl}_2$

Let ξ be a nonempty partition. First, we write ξ as $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m > 0)$. Now, let $\ell := \xi_1$. For $1 \leq i \leq \ell$, define $m_i(\xi)$ to be the number of times i occurs in the partition ξ. Then, we also write ξ as $\xi = \ell^{m_\ell(\xi)}(\ell-1)^{m_{\ell-1}(\xi)} \cdots 1^{m_1(\xi)}$. For $i \geq 1$, let $\nu_i(\xi)$ be the number of parts of ξ greater than or equal to i. Then, we have

$$\sum_{j=1}^{i} \nu_j(\xi) = m_1(\xi) + 2m_2(\xi) + \cdots + (i-1)m_{i-1}(\xi) + i(m_i(\xi) + m_{i+1}(\xi) + \cdots + m_\ell(\xi))$$

$$\xi^{tr} = \nu_1(\xi) \geq \nu_2(\xi) \geq \cdots \geq \nu_\ell(\xi) > 0$$

where ξ^{tr} is the transpose partition of ξ. Given an integer $j \geq 1$, we define a new partition $\xi^{(j)}$ given by $\xi^{(j)} := (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_j \geq 0)$. Then, since $|\xi| = \sum_{j\geq 1} \xi_j$, we have $|\xi^{tr(j)}| = \sum_{i=1}^{j} \nu_i(\xi)$. When ξ is fixed, we write m_i and ν_i for $m_i(\xi)$ and $\nu_i(\xi)$, respectively.

Since $\mathfrak{sl}_2$ has only one positive root α, we write the terms of (1.4.3) as $y(r,s)$. Also, for $k \in \mathbb{Z}_+$ and $g \in \mathfrak{sl}_2$, we write $g_k := g \otimes t^k$. For $r, s \geq 0$, recall

$$y(r,s) = \sum y_0^{(b_0)} y_1^{(b_1)} \cdots y_s^{(b_s)} \in \mathbf{U}(\mathfrak{sl}_2[t]), \tag{2.2.1}$$

where $y_j^{(b_j)} = \dfrac{y_j^{b_j}}{(b_j)!}$ and the sum is over all sequences $(b_j)_{j\geq 0}$, $b_j \in \mathbb{Z}_+$ such that $\sum_{k\geq 0} b_k = r$ and $\sum_{k\geq 0} kb_k = s$. Now, we give the first new presentation of $V(\xi)$.

Theorem 2.2.1. *The module $V(\xi)$ is isomorphic to the quotient of the local Weyl module $W_{\mathrm{loc}}(|\xi|)$ by the $\mathfrak{sl}_2[t]$–submodule generated by the elements*

$$\{y(r, |(\xi^{tr})^{(r)}| - r + 1)w_{|\xi|} : \quad 1 \leq r \leq \ell - 1\}.$$

The proof of this theorem and the corresponding statement for an arbitrary simple Lie algebra will occupy the remainder of this chapter.

2.3 Univeral Enveloping Algebra of $\mathfrak{sl}_2[t]$

Using (2.2.1), we have

$$r!y(r,s) = \sum_{j_1+\cdots+j_r=s} y_{j_1}\cdots y_{j_r}.$$

We have $y(0,0)=1$ and $y(0,s)=0$ if $s>0$. Now, define $Y_0(u) := \sum_{j\geq 0} y_j u^j$, where u is an indeterminate. Then, $y(r,s)$ is the coefficient of u^s in $Y_0(u)^r$. Thus, for all $1\leq m\leq r$, we have

$$\binom{r}{m} y(r,s) = \sum_{0\leq p\leq s} y(m,p)y(r-m,s-p).$$

In particular, if $m=1$, we have

$$y(r,s) = \frac{1}{r!}\sum_{j=0}^{s} y_j y(r-1,s-j). \tag{2.3.1}$$

Now, we prove the following.

Lemma 2.3.1. *In* $\mathbf{U}(\mathfrak{sl}_2[t])$*, we have*

$$[h_1, y(r,s)] = -2ry(r,s+1) + 2y_0 y(r-1,s+1), \quad r>0, \quad s\geq 0.$$

Proof. We proceed by induction on r. If $r=1$, then

$$[h_1, y(1,s)] = [h_1, y_s] = -2y_{s+1} = -2y(1,s+1)$$

for all $s\geq 0$, proving induction starts. Now, suppose

$$[h_1, y(r,s)] = -2ry(r,s+1) + 2y_0 y(r-1,s+1)$$

for some $r \geq 1$ and all $s \geq 0$. Using (2.3.1), we get

$$\begin{aligned}
[h_1, y(r+1,s)] &= \left[h_1, \frac{1}{r+1}\sum_{j=0}^{s} y_j y(r, s-j)\right] \\
&= \frac{1}{r+1}\sum_{j=0}^{s}[h_1, y_j y(r, s-j)] \\
&= \frac{1}{r+1}\sum_{j=0}^{s}(-2y_{j+1}y(r, s-j) + y_j[h_1, y(r, s-j)]). \qquad (2.3.2)
\end{aligned}$$

For the first term on the right hand side of (2.3.2), notice

$$\begin{aligned}
\frac{-2}{r+1}\sum_{j=0}^{s} y_{j+1}y(r, s-j) &= \frac{-2}{r+1}\sum_{j=1}^{s+1} y_j y(r, s+1-j) \\
&= \frac{2}{r+1}y_0 y(r, s+1) + \frac{-2}{r+1}\sum_{j=0}^{s+1} y_j y(r, s+1-j) \\
&= \frac{2}{r+1}y_0 y(r, s+1) + -2y(r+1, s+1).
\end{aligned}$$

For the second term on the right hand side of (2.3.2), the induction hypothesis implies

$$\begin{aligned}
\frac{1}{r+1}\sum_{j=0}^{s} y_j[h_1, y(r, s-j)] &= \frac{1}{r+1}\sum_{j=0}^{s} y_j(-2ry(r, s+1-j) + 2y_0 y(r-1, s+1-j)) \\
&= \sum_{j=0}^{s}\left(\frac{-2r}{r+1}y_j y(r, s+1-j) + \frac{2}{r+1}y_j y_0 y(r-1, s+1-j)\right) \\
&= \frac{2r}{r+1}y_{s+1}y(r, 0) - \frac{2}{r+1}y_{s+1}y_0 y(r-1, 0) \\
&\quad + \sum_{j=0}^{s+1}\left(\frac{-2r}{r+1}y_j y(r, s+1-j) + \frac{2}{r+1}y_j y_0 y(r-1, s+1-j)\right) \\
&= \frac{2ry_{s+1}y_0^r}{(r+1)r!} - \frac{2y_{s+1}y_0^r}{(r+1)(r-1)!} - 2ry(r+1, s+1) + \frac{2r}{r+1}y_0 y(r, s+1) \\
&= -2ry(r+1, s+1) + \frac{2r}{r+1}y_0 y(r, s+1).
\end{aligned}$$

Combining these two results, we have

$$[h_1, y(r+1, s)] = -2(r+1)y(r+1, s+1) + 2y_0 y(r, s+1).$$

□

2.4 Relations in $V(\xi)$

We establish that the terms in Theorem 2.2.1 are relations in $V(\xi)$.

Lemma 2.4.1. *For* $1 \leq r \leq \ell - 1$, $y(r, |(\xi^{tr})^{(r)}| - r + 1)v_\xi = 0$.

Proof. Let $1 \leq r \leq \ell - 1$. To show that $y(r, |(\xi^{tr})^{(r)}| - r + 1)v_\xi = 0$, we need to find $k \in \mathbb{Z}_+$ such that $|(\xi^{tr})^{(r)}| + 1 \geq 1 + rk + \sum_{j \geq k+1} \xi_j$. This is equivalent finding $k \in \mathbb{Z}_+$ such that $|(\xi^{tr})^{(r)}| \geq rk + \sum_{j \geq 0} \xi_j$. Now, let $k = \nu_r$. Then,

$$
\begin{aligned}
rk + \sum_{j \geq k+1} \xi_j &= r(\nu_r) + \sum_{j \geq \nu_r + 1} \xi_j \\
&= r(m_\ell + m_{\ell-1} + \cdots + m_r) + (r-1)m_{r-1} + (r-2)m_{r-2} + \cdots + m_1 \\
&= |(\xi^{tr})^{(r)}|.
\end{aligned}
$$

Therefore, $y(r, |(\xi^{tr})^{(r)}| - r + 1)v_\xi = 0$. □

2.5 Minimal value of s

In this section, we show that the relations from the previous section provide us with a minimal value of s given $1 \leq r \leq \ell - 1$.

Lemma 2.5.1. *For* $1 \leq r \leq \ell - 1$ *and* $s \in \mathbb{Z}_+$, *if there exists* $k \in \mathbb{Z}_+$ *such that* $r + s \geq 1 + rk + \sum_{j \geq k+1} \xi_j$, *then*

$$s \geq |(\xi^{tr})^{(r)}| - r + 1.$$

Proof. Assume $r = 1$ and there exists $k \in \mathbb{Z}_+$ such that $s \geq k + \sum_{j \geq k+1} \xi_j$.

If $k \geq |(\xi^{tr})^{(1)}| = \nu_1$, then $s \geq |(\xi^{tr})^{(1)}|$ and we are done. Otherwise, $k = \nu_1 - k'$ where

$0 < k' \leq \nu_1$. Then,

$$\begin{aligned} s &\geq \nu_1 - k' + \sum_{j \geq \nu_1 - k' + 1} \xi_j \\ &\geq m_1 + m_2 + \cdots + m_\ell - k' + (1)k' \\ &= m_1 + m_2 + \cdots + m_\ell \\ &= |(\xi^{tr})^{(1)}| \end{aligned}$$

completing the base case.

Now, assume $2 \leq r \leq \ell - 1$ and there exists $k \in \mathbb{Z}_+$ such that $r + s \geq 1 + rk + \sum_{j \geq k+1} \xi_j$. Hence, $s \geq (-r+1) + rk + \sum_{j \geq k+1} \xi_j$. As before, we proceed with cases for k.

If $k \geq \nu_1$, then

$$\begin{aligned} s &\geq (-r+1) + r(m_1 + m_2 + \cdots + m_\ell) \\ &= (-r+1) + m_1 + 2m_2 + \cdots + (r-1)m_{r-1} + r(m_r + m_{r+1} + \cdots + m_\ell) + (r-1)m_1 \\ &\quad + (r-2)m_2 + \cdots + m_{r-1}. \end{aligned}$$

Since $r > 1$ and $m_j \geq 0$ for all j,

$$\begin{aligned} s &\geq -r + 1 + m_1 + 2m_2 + \cdots + (r-1)m_{r-1} + r(m_r + m_{r+1} + \cdots + m_\ell) \\ &= |(\xi^{tr})^{(r)}| - r + 1. \end{aligned}$$

If $\nu_{i+1} \le k < \nu_i$ for some i where $1 \le i \le \ell - 1$, then $k = \nu_{i+1} + k'$ where $0 \le k' < m_i$. Thus, we have

$$s \ge -r + 1 + r(\nu_{i+1} + k') + \sum_{j \ge \nu_{i+1}+k'+1} \xi_j$$

$$= -r + 1 + r(m_\ell + m_{\ell-1} + \cdots + m_{i+1} + k') + (m_i - k')i + m_{i-1}(i-1)$$

$$+ m_{i-2}(i-2) + \cdots + m_1$$

$$= -r + 1 + r(m_\ell + m_{\ell-1} + \cdots + m_{i+1}) + rk' + (m_i - k')i + m_{i-1}(i-1)$$

$$+ m_{i-2}(i-2) + \cdots + m_1.$$

If $i \le r - 1$, the desired inequality is obvious. Otherwise, suppose $i \ge r$. So,

$$s \ge -r + 1 + r(m_\ell + m_{\ell-1} + \cdots + m_r) + (m_i - k')(i - r) + m_{i-1}(i - 1 - r)$$

$$+ m_{i-2}(i - 2 - r) + \cdots + m_{r+1} + (r-1)m_{r-1} + (r-2)m_{r-2} + \cdots + m_1$$

$$= \nu_1 + \nu_2 + \cdots + \nu_r - r + 1 + (m_i - k')(i - r) + m_{i-1}(i - 1 - r)$$

$$+ m_{i-2}(i - 2 - r) + \cdots + m_{r+1}.$$

Since $i \ge r, m_i > k'$ and $m_j \ge 0$ for all j, $s \ge |(\xi^{tr})^{(r)}| - r + 1$.

Finally, if $0 \le k < \nu_\ell$, then

$$s \ge -r + 1 + rk + \sum_{j \ge k+1} \xi_j$$

$$= -r + 1 + rk + \ell(m_\ell - k) + (\ell - 1)m_{\ell-1} + (\ell - 2)m_{\ell-2} + \cdots + m_1$$

$$= -r + 1 + m_1 + 2m_2 + \cdots + (r-1)m_{r-1} + r(m_\ell + m_{\ell-1} + \cdots + m_r)$$

$$+ (\ell - r)(m_\ell - k) + (\ell - 1 - r)m_{\ell-1} + \cdots + m_{r+1}.$$

Since $r \le \ell - 1, m_j \ge 0$ for all j, $s \ge |(\xi^{tr})^{(r)}| - r + 1$. $\square$

2.6 Proof of Theorem

Now, we provide the proof of Theorem 2.2.1.

Proof. Let U be the $\mathfrak{sl}_2[t]-$submodule generated by the elements

$$\{y(r, |(\xi^{tr})^{(r)}| - r + 1)w_{|\xi|} : \quad 1 \leq r \leq \ell - 1\}$$

and w be the generator of $W_{\text{loc}}(|\xi|)/U$. We need to show $y(r, s)w = 0$ if there exists $k \in \mathbb{Z}_+$ such that $r + s \geq 1 + rk + \sum_{j \geq k_1} \xi_j$. Now, either $1 \leq r \leq \ell - 1$ or $r \geq \ell$.

Suppose $1 \leq r \leq \ell - 1$. By Lemma 2.5.1, we have $s \geq |(\xi^{tr})^{(r)}| - r + 1$. We proceed by induction on r. Assume $r = 1$ and $s \geq |(\xi^{tr})^{(1)}| = \nu_1$. We have $y(1, \nu_1)w = y_{\nu_1}w = 0$. Since $s \geq \nu_1$, we have $y(1, s)w = y_s w = 0$ by Lemma 2.3.1.

Now, assume $2 \leq r \leq \ell - 1$ and $y(r - 1, s')w = 0$ for all $s' \geq |(\xi^{tr})^{(r-1)}| - (r - 1) + 1$. Assume $s \geq |(\xi^{tr})^{(r)}| - r + 1$. Then, $s = |(\xi^{tr})^{(r)}| - r + 1 + n$ for some $n \in \mathbb{Z}_+$. We now proceed with induction on n. If $n = 0$, then the result is trivial. Suppose $n \geq 1$ and we have $y(r, s - 1)w = 0$. Then, by Lemma 2.3.1,

$$\begin{aligned} 0 &= [h_1, y(r, s - 1)]w \\ &= -2ry(r, s)w + 2y_0 y(r - 1, s)w. \end{aligned}$$

Since $\nu_r \geq 0$ and $n \geq 1$, then

$$\begin{aligned} s &= |(\xi^{tr})^{(r)}| - r + 1 + n \\ &\geq |(\xi^{tr})^{(r-1)}| - r + 2. \end{aligned}$$

Hence, by the inductive hypothesis, we have $y(r - 1, s)w = 0$. Therefore, $-2ry(r, s)w = 0$ and $y(r, s)w = 0$.

Suppose r, s are such that $r \geq \ell$ and $r + s \geq 1 + rk + \sum_{j \geq k+1} \xi_j$. Since $r \geq \ell = \xi_1$ and $\xi_1 \geq \xi_j$ for all j,

$$\begin{array}{rcl} r+s & \geq & 1 + rk + \sum_{j\geq k+1} \xi_j \\ & \geq & 1 + \xi_1 k + \sum_{j \geq k+1} \xi_j \\ & \geq & 1 + \xi_1 + \xi_2 + \cdots + \xi_k + \sum_{j\geq k+1} \xi_j \\ & = & 1 + \sum_{j \geq 1} \xi_j \\ & = & 1 + |\xi|. \end{array}$$

Since $y_0^{m+1} w = 0$ if $m \geq |\xi|$, we have $y_0^{s+r} w = 0$. Therefore, $x_1^r y_0^{s+r} w = 0$. By [5], $y(r, s)w = x_1^{(r)} y_0^{(s+r)} w = 0$ completing the proof. □

Remark 2.6.1. As a consequence of our proof, we have shown in $V(\xi)$ we have the relation

$$y(r, s)v_\xi = 0, \quad s \geq |(\xi^{tr})^{(r)}| - r + 1, \quad r \geq 1.$$

2.7 Presentation for arbitrary simple Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\lambda \in P^+$ and $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ be an $|R^+|$ tuple of partitions such that $\lambda(h_\alpha) = |\xi^\alpha|$. Now, we provide our first presentation of $V(\boldsymbol{\xi})$ for $\mathfrak{g}$.

Corollary 2.7.1. *The module $V(\boldsymbol{\xi})$ is isomorphic to the quotient of the local Weyl module $W_{\rm loc}(\lambda)$ by the $\mathfrak{g}[t]$–submodule generated by the elements*

$$\{x_\alpha^-(r, |((\xi^\alpha)^{tr})^{(r)}| - r + 1)w_\lambda : \quad \alpha \in R^+, \ 1 \leq r \leq \xi_1^\alpha - 1\}.$$

The proof of this is immediate from Theorem 2.2.1 once you fix $\alpha \in R^+$.

Chapter 3

Second and Third New Presentation of $V(\xi)$

In this chapter, we provide two additional new presentations for the module $V(\boldsymbol{\xi})$. As with our previous presentation, we first establish the presentations for $\mathfrak{sl}_2$ and then for an arbitrary simple Lie algebra.

3.1 Second New Presentation

3.1.1 Motivation

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_2$. The motivation for our second presentation comes from two specific partitions. If $\xi = \ell^{m_\ell}$ (ie. ξ is rectangular), it was shown in [5] that $V(\xi)$ has one $y(r,s)$ defining relation as a quotient of $W_{loc}(|\xi|)$. If $\xi = \ell^{m_\ell}(\ell-1)^{m_{\ell-1}}$ (i.e. ξ is a consecutive fat hook), it was show in [11] that $V(\xi)$ also has one $y(r,s)$ defining relation as a quotient of $W_{loc}(|\xi|)$. The presentation we gave in the previous chapter has $\ell - 1$ $y(r,s)$

defining relations for both of these partitions. This led us to conjecture that we should be able to reduce the number of $y(r,s)$ relations in our previous presentation.

3.1.2 Presentation for $\mathfrak{sl}_2$

Let ξ be a nonempty partition. For this presentation, we write ξ in a different way. Define $n(\xi)$ to be the number of distinct integers in ξ and let $\ell_j(\xi)$ for $1 \leq j \leq n(\xi)$ with $0 < \ell_1(\xi) < \ell_2(\xi) < \cdots < \ell_{n(\xi)}(\xi)$ be the distinct numbers appearing in ξ. For $1 \leq i \leq n(\xi)$, let $m_i(\xi)$ be the number of times $\ell_i(\xi)$ occurs in ξ. Then, we write

$$\xi = \ell_{n(\xi)}(\xi)^{m_{n(\xi)}(\xi)} \ell_{n(\xi)-1}(\xi)^{m_{n(\xi)-1}(\xi)} \cdots \ell_1(\xi)^{m_1(\xi)}.$$

When ξ is fixed, we write ℓ_i, m_i, and n for $\ell_i(\xi)$, $m_i(\xi)$, and $n(\xi)$ respectively.

Theorem 3.1.1. *The module $V(\xi)$ is isomorphic to the quotient of the local Weyl module $W_{\text{loc}}(|\xi|)$ by the $\mathfrak{sl}_2[t]-$submodule generated by the elements*

$$\{y_{\nu_1} w_{|\xi|}\} \cup \{y(r, |(\xi^{tr})^{(r)}| - r + 1) w_{|\xi|} \; : \; \ell_1 + 1 \leq r \leq \ell_n - 1\}.$$

Proof. Let U be the $\mathfrak{sl}_2[t]$ submodule generated by the elements

$$\{y_{\nu_1} w_{|\xi|}\} \cup \{y(r, |(\xi^{tr})^{(r)}| - r + 1) w_{|\xi|} \; : \; \ell_1 + 1 \leq r \leq \ell_n - 1\}$$

and w be the generator of $W_{\text{loc}}(|\xi|)/U$. By Theorem 2.2.1, it suffices to show

$$y(r, |(\xi^{tr})^{(r)}| - r + 1) w = 0$$

for $2 \leq r \leq \ell_1$. We begin by induction on r. Assume $r = 2 \leq \ell_1$. Then, by (2.3.1), we have

$$\begin{aligned} y(2, |(\xi^{tr})^{(2)}| - 1)w &= y(2, 2\nu_{\ell_1} - 1)w \\ &= \frac{1}{2} \sum_{j=0}^{\nu_{\ell_1}-1} y_j y(1, 2\nu_{\ell_1} - 1 - j)w \\ &= \frac{1}{2} \sum_{j=0}^{\nu_{\ell_1}-1} y_j y_{2\nu_{\ell_1}-1-j} w. \end{aligned}$$

Now, for $0 \leq j \leq \nu_{\ell_1} - 1$, we have

$$\begin{aligned} 2\nu_{\ell_1} - 1 - j &\geq 2\nu_{\ell_1} - 1 - (\nu_{\ell_1} - 1) \\ &\geq \nu_{\ell_1}. \end{aligned}$$

Hence, $y_{2\nu_{\ell_1}-1-j}w = 0$ for all $0 \leq j \leq \nu_{\ell_1} - 1$. Therefore, $y(2, |(\xi^{tr})^{(2)}| - 1)w = 0$ and induction starts.

Now, assume $3 \leq r \leq \ell_1$ and

$$\begin{aligned} y(r-1, |(\xi^{tr})^{(r-1)}| - (r-1) + 1)w &= y(r-1, (r-1)\nu_{\ell_1} - r + 2)w \\ &= 0. \end{aligned}$$

First, by (2.3.1), we have

$$\begin{aligned} y(r, |(\xi^{tr})^{(r)}| - r + 1)w &= y(r, r\nu_{\ell_1} - r + 1)w \\ &= \frac{1}{r} \sum_{i_1=0}^{\nu_{\ell_1}-1} y_{i_1} y(r-1, r\nu_{\ell_1} - r + 1 - i_1)w. \end{aligned}$$

Now, when $i_1 = \nu_{\ell_1} - 1$, we have

$$\begin{aligned} r\nu_{\ell_1} - r + 1 - i_1 &= r\nu_{\ell_1} - r + 1 - (\nu_{\ell_1} - 1) \\ &= (r-1)\nu_{\ell_1} - r + 2 \end{aligned}$$

and hence $y(r-1, r\nu_{\ell_1} - r + 1 - i_1)w = 0$ by the inductive hypothesis. So, we have

$$\begin{aligned} y(r, |(\xi^{tr})^{(r)}| - r + 1)w &= \frac{1}{r} \sum_{i_1=0}^{\nu_{\ell_1}-2} y_{i_1} y(r-1, r\nu_{\ell_1} - r + 1 - i_1)w \\ &= \frac{1}{r(r-1)} \sum_{i_1=0}^{\nu_{\ell_1}-2} \sum_{i_2=0}^{\nu_{\ell_1}-1} y_{i_1} y_{i_2} y(r-2, r\nu_{\ell_1} - r + 1 - i_1 - i_2)w \\ &= \frac{1}{r!} \sum_{i_1=0}^{\nu_{\ell_1}-2} \sum_{i_2=0}^{\nu_{\ell_1}-1} \cdots \sum_{i_{r-1}=0}^{\nu_{\ell_1}-1} y_{i_1} y_{i_2} \cdots y_{i_{r-1}} y_{r\nu_{\ell_1} - r + 1 - i_1 - i_2 - \cdots - i_{r-1}} w. \end{aligned}$$

Then, we have

$$\begin{aligned} r\nu_{\ell_1} - r + 1 - i_1 - i_2 - \cdots - i_{r-1} &\geq r\nu_{\ell_1} - r + 1 - (\nu_{\ell_1} - 2) - (r-2)(\nu_{\ell_1} - 1) \\ &\geq \nu_{\ell_1} + 1. \end{aligned}$$

Hence, we have $y_{r\nu_{\ell_1} - r + 1 - i_1 - i_2 - \cdots - i_{r-1}} w = 0$. Therefore, $y(r, |(\xi^{tr})^{(r)}| - r + 1)w = 0$ completing the proof. □

3.1.3 Presentation for a simple Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\lambda \in P^+$ and $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ be an $|R^+|$–tuple of partitions where $\lambda(h_\alpha) = |\xi^\alpha|$. We provide our second presentation for $V(\boldsymbol{\xi})$.

Corollary 3.1.2. *The module $V(\boldsymbol{\xi})$ is isomorphic to the quotient of the local Weyl module $W_{\text{loc}}(\lambda)$ by the $\mathfrak{g}[t]$–submodule generated by the elements*

$$\{x_\alpha^-(r, |((\xi^\alpha)^{tr})^{(r)}| - r + 1)w_\lambda \ : \ \alpha \in R^+, \ r = 1 \ \textit{or} \ \ell_1(\xi^\alpha) + 1 \leq r \leq \ell_{n(\xi^\alpha)}(\xi^\alpha) - 1\}.$$

3.2 Third New Presentation

3.2.1 Motivation

We consider the Lie algebra $\mathfrak{sl}_2$ again. Using the notation from the previous section, a partition ξ is called a special fat hook if $\xi = \ell_2^{m_2}\ell_1^{m_1}$. It was shown in [5] that if ξ is a special fat hook then $V(\xi)$ has two $y(r,s)$ defining relations as a quotient of $W_{loc}(|\xi|)$. Our second presentation in the last section has $\ell_2 - \ell_1$ $y(r,s)$ defining relations for this partition. Again, this led us to believe that we should be able to further reduce the number of $y(r,s)$ relations in our previous presentation. We conjecture that this third presentation gives a minimal set of $y(r,s)$ relations for any partition ξ.

3.2.2 Presentation for $\mathfrak{sl}_2$

Recall the elements ${}_k y(r,s)$ in $\mathbf{U}(\mathfrak{sl}_2[t])$ from (1.4.4).

Theorem 3.2.1. *The module $V(\xi)$ is isomorphic to the quotient of the local Weyl module $W_{\mathrm{loc}}(|\xi|)$ by the $\mathfrak{sl}_2[t]$–submodule generated by the elements*

$$\{y_{\nu_1} w_{|\xi|}\} \cup \{{}_{\nu_r} y(r, |(\xi^{tr})^{(r)}| - r + 1) w_{|\xi|} \ : \ 1 \le j \le n-1, \ \ell_j + 1 \le r \le s_j\}$$

where

$$s_j = \begin{cases} \min\{|(\xi^{tr})^{(\ell_j)}| + 1, \ell_{j+1}\}, & 1 \le j \le n-2, \\ \min\{|(\xi^{tr})^{(\ell_{n-1})}| + 1, \ell_n - 1\}, & j = n-1. \end{cases}$$

The proof of this theorem and the corresponding statement for an arbitrary simple Lie algebra occupies the remainder of this section.

3.2.3 Alternate Form of Second New Presentation

Using the results from [5] and our second presentation, we obtain the following alternate form of our second presentation.

Corollary 3.2.2. *The module* $V(\xi)$ *is isomorphic to the quotient of the local Weyl module* $W_{\mathrm{loc}}(|\xi|)$ *by the* $\mathfrak{sl}_2[t]$*–submodule generated by the elements*

$$\{y_{\nu_1}w_{|\xi|}\} \cup \{\ {}_{\nu_r}y(r, |(\xi^{tr})^{(r)}| - r + 1)w_{|\xi|}\ :\ \ell_1 + 1 \le r \le \ell_n - 1\}.$$

3.2.4 Proof of Theorem

Now, we provide the proof of Theorem 3.2.1.

Proof. Let U be the $\mathfrak{sl}_2[t]-$ submodule generated by the elements

$$\{y_{\nu_1}w_{|\xi|}\} \cup \{{}_{\nu_r}y(r, |(\xi^{tr})^{(r)}| - r + 1)w_{|\xi|}\ :\ 1 \le j \le n-1,\ \ell_j + 1 \le r \le s_j\}$$

and w be the generator of $W_{\mathrm{loc}}(|\xi|)/U$. By Corollary 3.2.2, it suffices to show for all $1 \le j \le n-2$ and $s_j + 1 \le r \le \ell_{j+1}$ we have

$${}_{\nu_r}y(r, |(\xi^{tr})^{(r)}| - r + 1)w = 0$$

and for all $s_{n-1} \le r \le \ell_n - 1$ we have

$${}_{\nu_r}y(r, |(\xi^{tr})^{(r)}| - r + 1)w = 0.$$

Fix $1 \le j \le n-1$. If $s_j = \ell_{j+1}$, then there is nothing to do. Now, suppose $s_j \ne \ell_{j+1}$. Then, $s_j = |(\xi^{tr})^{(\ell_j)}| + 1$ and $s_j = \ell_j m_j + \ell_{j-1}m_{j-1} + \cdots + \ell_1 m_1 + 1$. Also, assume $s_j + 1 \le r \le \ell_{j+1}$ (except in the case of $j = n-1$, we assume $s_j + 1 \le r \le \ell_n - 1$). Then,

$${}_{\nu_r}y(r, |(\xi^{tr})^{(r)}| - r + 1) = {}_{m_{j+1}+\cdots+m_n}y(r, r(m_{j+1} + \cdots + m_n) + \ell_j m_j + \cdots + \ell_1 m_1 - r + 1).$$

Now, consider a single element of this term, given by $y_{i_1} \cdots y_{i_r}$. Then, we have

$$r(m_{j+1} + \cdots + m_n) + \ell_j m_j + \cdots + \ell_1 m_1 - r + 1 = i_1 + i_2 + \cdots + i_r$$
$$\geq r(m_{j+1} + \cdots + m_n).$$

But, this would mean that $r \leq \ell_j m_j + \cdots + \ell_1 m_1 + 1$, which is impossible. So, we have ${}_{\nu_r}y(r, |(\xi^{tr})^{(r)}| - r + 1) = 0$, which completes the proof. □

3.2.5 Presentation for a simple Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\lambda \in P^+$ and $\boldsymbol{\xi} = (\xi^\alpha)_{\alpha \in R^+}$ be an $|R^+|-$tuple of partitions where $\lambda(h_\alpha) = |\xi^\alpha|$. We provide our third presentation for $V(\boldsymbol{\xi})$.

Corollary 3.2.3. *The module $V(\boldsymbol{\xi})$ is isomorphic to the quotient of the local Weyl module $W_{\text{loc}}(\lambda)$ by the $\mathfrak{g}[t]$–submodule generated by the elements*

$$\{x_\alpha^-(1, \nu_1(\xi^\alpha))w_\lambda : \ \alpha \in R^+\} \cup \{{}_{\nu_r(\xi^\alpha)}x_\alpha^-(r, |((\xi^\alpha)^{tr})^{(r)}| - r + 1)w_\lambda : \ \alpha \in R^+, \ \ell_j(\xi^\alpha) + 1 \leq r \leq s_j(\xi^\alpha)\}$$

where

$$s_j(\xi^\alpha) = \begin{cases} \min\{|((\xi^\alpha)^{tr})^{(\ell_j(\xi^\alpha))}| + 1, \ell_{j+1}(\xi^\alpha)\}, & 1 \leq j \leq n(\xi^\alpha) - 2, \\ \min\{|((\xi^\alpha)^{tr})^{(\ell_{n(\xi^\alpha)-1}(\xi^\alpha))}| + 1, \ell_{n(\xi^\alpha)}(\xi^\alpha) - 1\}, & j = n(\xi^\alpha) - 1. \end{cases}$$

Chapter 4

First Short Exact Sequence

In this chapter, we restrict ourselves to only the Lie algebra $\mathfrak{sl}_2$. We use our presentations of $V(\xi)$ modules from the previous chapters to establish the existence of a short exact sequence of $V(\xi)$ modules. While this short exact sequence existed in the literature, the proof contained errors.

4.1

Let $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m > 0)$ be a partition. We define two new partitions ξ^+ and ξ^- as follows. If $m = 1$, then $\xi^+ = \xi$ and ξ^- is the empty partition. Otherwise, $\xi^- := (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{m-2} \geq \xi_{m-1} - \xi_m \geq 0)$ and ξ^+ is the unique partition associated to the n-tuple $(\xi_1, \xi_2, \ldots, \xi_{m-2}, \xi_{m-1} + 1, \xi_m - 1)$. Recall for $r \in \mathbb{Z}_+$, τ_r is the grade shift operator. Now, we establish the following result.

Theorem 4.1.1. *For $m > 1$, there exists a short exact sequence of $\mathfrak{sl}_2[t]-$modules*

$$0 \to \tau_{(m-1)\xi_m} V(\xi^-) \overset{\varphi^-}{\to} V(\xi) \overset{\varphi^+}{\to} V(\xi^+) \to 0.$$

The existence of this short exact sequence was shown in [7] and [8]. Additionally, it was shown in [5] with errors. We are able to provide a different proof because of our presentations of $V(\xi)$ modules that we established earlier. In particular, we provide a different construction of φ^-. The proof of this theorem occupies the rest of this chapter.

4.2

We state the following lemma from [5].

Lemma 4.2.1. *Given $a, b, p \in \mathbb{Z}_+$, we have*

$$[x_a, y_b^{(p)}] = y_b^{(p-1)} h_{a+b} - y_{a+2b} y_b^{(p-2)}$$

$$[h_a, y_b^{(p)}] = -2 y_{a+b} y_b^{(p-1)}$$

$$[y_a, x_b^{(p)}] = -x_b^{(p-1)} h_{a+b} - x_b^{(p-2)} x_{a+2b}$$

where $x_b^p = 0$ and $y_b^{(p)} = 0$ if $p < 0$.

4.3

For $s, r \in \mathbb{Z}_+$, we set $X(r, s) := x_1^{(s)} y_0^{(s+r)}$. We know from [5], $X(r, s)v_\xi = y(r, s)v_\xi$.

Lemma 4.3.1. *For $r, s \in \mathbb{Z}_+$,*

$$(s + 1)X(r + 1, s + 1) = X(r + 1, s)h_1 - X(r, s)y_1 + X(r + 2, s)x_1.$$

Proof. Using the first equation of Lemma 4.2.1, we have

$$
\begin{aligned}
(s+1)X(r+1,s+1) &= x_1^{(s)}x_1y_0^{(r+s+2)} \\
&= x_1^{(s)}\left[x_1, y_0^{(r+s+2)}\right] + x_1^{(s)}y_0^{(r+s+2)}x_1 \\
&= x_1^{(s)}y_0^{(r+s+1)}h_1 - x_1^{(s)}y_1y_0^{(r+s)} + x_1^{(s)}y_0^{(r+s+2)}x_1 \\
&= x_1^{(s)}y_0^{(r+s+1)}h_1 - x_1^{(s)}y_0^{(r+s)}y_1 + x_1^{(s)}y_0^{(r+s+2)}x_1 \\
&= X(r+1,s)h_1 - X(r,s)y_1 + X(r+2,s)x_1.
\end{aligned}
$$

□

4.4

Lemma 4.4.1. *For all $i \geq 0$ and $r, s \in \mathbb{Z}_+$, we have we have*

$$
\begin{aligned}
y_iX(r,s) \quad &= \quad X(r,s)y_i - X(r+1,s-1)h_{i+1} - X(r+2,s-2)x_{i+2} \\
&\quad +2X(r,s-1)y_{i+1} - X(r+1,s-2)h_{i+2} + X(r,s-2)y_{i+2}. \qquad (4.4.1)
\end{aligned}
$$

Proof. First, using the third equation of Lemma 4.2.1, we have

$$
\begin{aligned}
y_iX(r,s) &= y_ix_1^{(s)}y_0^{(r+s)} \\
&= [y_i, x_1^{(s)}]y_0^{(r+s)} + x_1^(s)y_iy_0^{(r+s)} \\
&= -x_1^{(s-1)}h_{i+1}y_0^{(r+s)} - x_1^{(s-2)}x_{i+2}y_0^{(r+s)} + x_1^{(s)}y_iy_0^{(r+s)} \\
&= -x_1^{(s-1)}h_{i+1}y_0^{(r+s)} - x_1^{(s-2)}x_{i+2}y_0^{(r+s)} + X(r,s)y_i.
\end{aligned}
$$

Now, using the second equation of Lemma 4.2.1, we have

$$\begin{aligned} y_i X(r,s) &= -x_1^{(s-1)}[h_{i+1}, y_0^{(r+s)}] - x_1^{(s-1)} y_0^{(r+s)} h_{i+1} - x_1^{(s-2)} x_{i+2} y_0^{(r+s)} + X(r,s) y_i \\ &= 2x_1^{(s-1)} y_0^{(r+s-1)} y_{i+1} - X(r+1, s-1) h_{i+1} - x_1^{(s-2)} x_{i+2} y_0^{(r+s)} + X(r,s) y_i \\ &= 2X(r, s-1) y_{i+1} - X(r+1, s-1) h_{i+1} - x_1^{(s-2)} x_{i+2} y_0^{(r+s)} + X(r,s) y_i. \end{aligned}$$

Finally, using the first equation of Lemma 4.2.1, we have

$$\begin{aligned} y_i X(r,s) &= 2X(r, s-1) y_{i+1} - X(r+1, s-1) h_{i+1} - x_1^{(s-2)} [x_{i+2}, y_0^{(r+s)}] \\ &\quad - x_1^{(s-2)} y_0^{(r+s)} x_{i+2} + X(r,s) y_i \\ &= 2X(r, s-1) y_{i+1} - X(r+1, s-1) h_{i+1} - x_1^{(s-2)} y_0^{(r+s-1)} h_{i+2} \\ &\quad + x_1^{(s-2)} y_{i+2} y_0^{(r+s-2)} - X(r+2, s-2) x_{i+2} + X(r,s) y_i \\ &= 2X(r, s-1) y_{i+1} - X(r+1, s-1) h_{i+1} - X(r+1, s-2) h_{i+2} \\ &\quad + X(r, s-2) y_{i+2} - X(r+2, s-2) x_{i+2} + X(r,s) y_i. \end{aligned}$$

□

4.5

Lemma 4.5.1. *For $j \in \mathbb{Z}_+$, we have*

$$X(r,s) y_j v_\xi = 0, \quad s \geq |(\xi^{tr})^{(r+1)}| - r - j.$$

In particular,

$$X(r,s) y_{\nu_1 - 1} v_\xi = 0, \quad s \geq |(\xi^{tr})^{(r+1)}| - \nu_1 - r + 1.$$

Proof. First, we have

$$\begin{aligned}X(r,s)y_0v_\xi &= x_1^{(s)}y_0^{(r+s)}y_0v_\xi \\ &= (r+s+1)x_1^{(s)}y_0^{(r+s+1)}v_\xi \\ &= (r+s+1)X(r+1,s)v_\xi.\end{aligned}$$

We have $X(r+1,s)v_\xi = 0$ if $s \geq |(\xi^{tr})^{(r+1)}| - (r+1) + 1$. Hence, $X(r,s)y_0v_\xi = 0$ if $s \geq |(\xi^{tr})^{(r+1)}| - r$. This finishes the case when $j = 0$. Using Lemma 4.3.1 and the relations of $V(\xi)$, we have

$$(s+1)X(r+1,s+1)v_\xi = -X(r,s)y_1v_\xi. \tag{4.5.1}$$

The left hand side of (4.5.1) is zero if $s+1 \geq |(\xi^{tr})^{(r+1)}| - r$. Hence, we have $X(r,s)y_1v_\xi = 0$ if $s \geq |(\xi^{tr})^{(r+1)}| - r - 1$. This finishes the case when $j = 1$.

Using the $V(\xi)$ relations and Lemma 4.4.1, we have

$$y_iX(r,s+2)v_\xi = X(r,s+2)y_iv_\xi + 2X(r,s+1)y_{i+1}v_\xi + X(r,s)y_{i+2}v_\xi \tag{4.5.2}$$

for all $i \geq 0$. If we let $i = 0$ in (4.5.2), we get

$$y_0X(r,s+2)v_\xi = X(r,s+2)y_0v_\xi + 2X(r,s+1)y_1v_\xi + X(r,s)y_2v_\xi. \tag{4.5.3}$$

Suppose $s \geq |(\xi^{tr})^{(r+1)}| - r - 2$. Then, since $1 \leq r \leq \ell - 1$, $\nu_{r+1} \geq 1$. So, $s \geq |(\xi^{tr})^{(r)}| - r - 1$. Therefore, the left side of (4.5.3) is zero. Also, since $s \geq |(\xi^{tr})^{(r+1)}| - r - 2$, $X(r,s+2)y_0v_\xi = 0$ and $X(r,s+1)y_1v_\xi = 0$. Therefore, we have shown $X(r,s)y_2v_\xi = 0$.

Now, we proceed by induction on j. Suppose $j \geq 2$ and we have the result for all $k \leq j$. If we let $i = j - 1$ in (4.5.2), we get

$$y_{j-1}X(r,s+2)v_\xi = X(r,s+2)y_{j-1}v_\xi + 2X(r,s+1)y_jv_\xi + X(r,s)y_{j+1}v_\xi.$$

Assume $s \geq |(\xi^{tr})^{(r+1)}| - r - (j+1)$. Then, $X(r, s+2)y_{j-1}v_\xi = 0$ and $X(r, s+1)y_j v_\xi = 0$ by the inductive hypothesis. Also, $y_{j-1}X(r, s+2)v_\xi = 0$ since $\nu_{r+1} \geq 1$. Therefore, we have $X(r,s)y_{j+1}v_\xi = 0$.

Taking $j = \nu_1 - 1$, we obtain $X(r,s)y_{\nu_1 - 1}v_\xi = 0, \quad s \geq |(\xi^{tr})^{(r+1)}| - \nu_1 - r + 1$.

□

4.6

Write $\xi = \ell_n^{m_n}\ell_{n-1}^{m_{n-1}} \cdots \ell_1^{m_1}$, where $m_j > 0$ for all $1 \leq j \leq n$. Now, define

$$\tilde{\xi}_1 = \begin{cases} \ell_n^{m_n}\ell_{n-1}^{m_{n-1}} \cdots \ell_2^{m_2}\ell_1^{m_1-2}(\ell_1 - 1)^2, \quad m_1 \geq 2, \\ \\ \ell_n^{m_n}\ell_{n-1}^{m_{n-1}} \cdots \ell_2^{m_2-1}(\ell_2 - 1)\ell_1^{m_1-1}(\ell_1 - 1), \quad m_1 = 1, \quad \ell_2 > \ell_1 + 1, \\ \\ \ell_n^{m_n}\ell_{n-1}^{m_{n-1}} \cdots \ell_2^{m_2-1}\ell_1(\ell_1 - 1), \quad m_1 = 1, \ell_2 = \ell_1 + 1. \end{cases}$$

Then, we have

$$m_1 \geq 2 \implies \nu_r(\tilde{\xi}_1) = \begin{cases} \nu_r(\xi), \quad r \neq \ell_1, \\ \\ \nu_{\ell_1}(\xi) - 2, r = \ell_1. \end{cases}$$

$$m_1 = 1 \implies \nu_r(\tilde{\xi}_1) = \begin{cases} \nu_r(\xi), \quad r \neq \ell_1, \ell_2, \\ \\ \nu_r(\xi) - 1, \quad r = \ell_1, \ell_2. \end{cases}$$

Lemma 4.6.1. *There exists a well defined map of $\mathfrak{sl}_2[t]-$modules $V(\tilde{\xi}_1) \to V(\xi)$ extending the assignment $v_{\tilde{\xi}_1} \to y_{\nu_1(\xi)-1}v_\xi$.*

Proof. We write $\tilde{\xi} = \tilde{\xi}_1$. We start by checking that the $W_{loc}(|\tilde{\xi}|)$ relations hold on $y_{\nu_1(\xi)-1}v_\xi$.

For $s \in \mathbb{Z}_+$, we have

$$\begin{aligned} x_s y_{\nu_1(\xi)-1} &= [x_s, y_{\nu_1(\xi)-1}]v_\xi \\ &= h_{s+\nu_1(\xi)-1}v_\xi \\ &= 0 \end{aligned}$$

since $s + \nu_1(\xi) - 1 \geq 1$.

We have

$$\begin{aligned} h_s y_{\nu_1(\xi)-1} v_\xi &= [h_s, y_{\nu_1(\xi)-1}]v_\xi + y_{\nu_1(\xi)-1}h_s v_\xi \\ &= -2y_{\nu_1(\xi)-1+s}v_\xi + \delta_{s,0}|\xi|y_{\nu_1(\xi)-1}v_\xi. \end{aligned}$$

If $s > 0$, then $h_s y_{\nu_1(\xi)-1}v_\xi = 0$ since $\nu_1(\xi) - 1 + s \geq \nu_1(\xi)$. If $s = 0$, we have

$$\begin{aligned} h_0 y_{\nu_1(\xi)-1} &= (|\xi| - 2)y_{\nu_1(\xi)-1}v_\xi \\ &= |\tilde{\xi}|y_{\nu_1(\xi)-1}v_\xi. \end{aligned}$$

For the final local Weyl module relation, we need to verify $y_0^{|\xi|-1}y_{\nu_1(\xi)-1}v_\xi = 0$. We know $y_0^{|\xi|+1}v_\xi = 0$. Then, by Lemma 4.2.1, we have

$$\begin{aligned} 0 &= h_{\nu_1(\xi)-1}y_0^{|\xi|+1}v_\xi \\ &= [h_{\nu_1(\xi)-1}, y_0^{|\xi|+1}]v_\xi \\ &= -2y_{\nu_1(\xi)-1}y_0^{(|\xi|)}v_\xi. \end{aligned}$$

Therefore, $y_0^{|\xi|} y_{\nu_1(\xi)-1} v_\xi = 0$. Again, by Lemma 4.2.1,

$$
\begin{aligned}
0 &= x_0 y_0^{|\xi|} y_{\nu_1(\xi)-1} v_\xi \\
&= [x_0, y_0^{|\xi|}] y_{\nu_1(\xi)-1} v_\xi + y_0^{|\xi|} x_0 y_{\nu_1(\xi)-1} v_\xi \\
&= y_0^{(|\xi|-1)} h_0 y_{\nu_1(\xi)-1} v_\xi - y_0 y_0^{(|\xi|-2)} y_{\nu_1(\xi)-1} v_\xi + y_0^{|\xi|} x_0 y_{\nu_1(\xi)-1} v_\xi \\
&= y_0^{(|\xi|-1)} [h_0, y_{\nu_1(\xi)-1}] v_\xi + y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} h_0 v_\xi - (|\xi|-1) y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi + y_0^{|\xi|} x_0 y_{\nu_1(\xi)-1} v_\xi \\
&= -2 y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi + |\xi| y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi - (|\xi|-1) y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi + y_0^{|\xi|} x_0 y_{\nu_1(\xi)-1} v_\xi \\
&= -1 y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi + y_0^{|\xi|} [x_0, y_{\nu_1(\xi)-1}] v_\xi \\
&= -1 y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi + y_0^{|\xi|} h_{\nu_1(\xi)-1} v_\xi \\
&= -1 y_0^{(|\xi|-1)} y_{\nu_1(\xi)-1} v_\xi
\end{aligned}
$$

since $\nu_1(\xi) - 1 > 0$. Therefore, we have shown $y_0^{|\xi|-1} y_{\nu_1(\xi)-1} v_\xi = 0$.

Now, we know that

$$X(r,s) v_{\tilde{\xi}} = 0, \quad s \geq |(\tilde{\xi}^{tr})^{(r)}| - r + 1,$$

and also that

$$X(r,s) y_{\nu_1(\xi)-1} v_\xi = 0, \quad s \geq |(\xi^{tr})^{(r+1)}| - \nu_1(\xi) - r + 1.$$

Hence, the map will exist if we prove that

$$|(\tilde{\xi}^{tr})^{(r)}| \geq |(\xi^{tr})^{(r+1)}| - \nu_1(\xi).$$

Suppose $m_1 \geq 2$ and $r < \ell_1$; then the inequality is just $\nu_1(\xi) \geq \nu_{r+1}(\xi)$, which is trivially true. If $r \geq \ell_1$, then the inequality becomes $\nu_1(\xi) - 2 \geq \nu_{\ell_1+1}(\xi)$ which is also clearly true since $m_1 \geq 2$.

Suppose that $m_1 = 1$. Again, if $r < \ell_1$, the inequality is just $\nu_1(\xi) \geq \nu_{r+1}(\xi)$. If $\ell_1 \leq r < \ell_2$, then the inequality becomes $\nu_1(\xi) - 1 \geq \nu_{\ell_1+1}(\xi)$ which is true since $m_1 = 1$ and if $r \geq \ell_2$ then we must prove $\nu_1(\xi) - 2 \geq \nu_{r+1}(\xi)$ which is again true since $m_1 + m_2 \geq 2$. $\square$

4.7

Now, we provide the proof of the Theorem 4.1.1.

Proof. First, we define $V(\xi) \xrightarrow{\varphi^+} V(\xi^+)$ by $v_\xi \mapsto v_{\xi^+}$. Since $|\xi| = |\xi^+|$, $V(\xi)$ and $V(\xi^+)$ are both quotients of the same local Weyl module. Hence, to show that φ^+ is well defined it suffices to show that $y(r, |(\xi^{tr})^{(r)}| - r + 1)v_{\xi^+} = 0$ for $1 \leq r \leq \xi_1 - 1$. But, this is immediate since $\sum_{i=1}^{j} \nu_i(\xi^+) \leq \sum_{i=1}^{j} \nu_i(\xi)$ for $1 \leq j \leq \xi_1 - 1$.

Now, we establish the existence of φ^-. Recall that we are using the notation $\ell_1 = \xi_m$. Applying Lemma 4.6.1 to the partition ξ, we obtain a map $\psi_1 : V(\tilde{\xi}) \to V(\xi)$ where $\psi_1(v_{\tilde{\xi}}) = y_{\nu_1(\xi)-1}v_\xi$. Now, we write $\xi^1 := \tilde{\xi}$. Then, ξ^1 is the partition

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{\nu_1(\xi)-1} - 1 \geq \ell_1 - 1 \geq 0.$$

If $\ell_1 - 1 = 0$, we stop. Otherwise, we apply Lemma 4.6.1 to the partition ξ^1 to obtain a map $\psi_2 : V(\tilde{\xi^1}) \to V(\xi^1)$ where $\psi_2(v_{\tilde{\xi^1}}) = y_{\nu_1(\xi)-1}v_{\xi^1}$ since $\nu_1(\xi^1) = \nu_1(\xi)$. Now, we write $\xi^2 := \tilde{\xi^1}$. Then, ξ^2 is the parition

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{\nu_1(\xi)-1} - 2 \geq \ell_1 - 2 \geq 0.$$

If $\ell_1 - 2 = 0$, we stop. Otherwise, we continue this process to obtain maps $\psi_j : V(\xi^j) \mapsto V(\xi^{j-1})$ where $\psi_j(v_{\xi^j}) = y_{\nu_1(\xi)-1}v_{\xi^{j-1}}$ for $1 \leq j \leq \ell_1$. Now, if $m_1 = 1$, then

$$\xi^{\ell_1} = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{\nu_1(\xi)-2} \geq \xi_{\nu_1(\xi)-1} - \ell_1 > 0).$$

If $m_1 \geq 2$, then

$$\xi^{\ell_1} = \xi_1 \geq \xi_2 \geq \cdots \geq \xi_{\nu_1(\xi)-2} > 0.$$

In either case, $\xi^{\ell_1} = \xi^-$. If we take the composition $\psi_1 \circ \psi_2 \circ \cdots \circ \psi_{\ell_1}$, we obtain a map from $V(\xi^-)$ to $V(\xi)$ where $v_{\xi^-} \mapsto (y_{\nu_1(\xi)-1})^{\ell_1} v_\xi$. Therefore, we set $\varphi^- := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_{\ell_1}$.

Finally, we need to show that our definition of φ^+ and φ^- give us a short exact sequence. We start by showing $\operatorname{im}\varphi^- \subset \ker\varphi^+$. Hence, we must show that $y_{m-1}^{\xi_m} v_\xi^+ = 0$ since $m = \nu_1(\xi)$. But, this is equivalent to showing $y(\xi_m, \xi_m(m-1))v_{\xi^+} = 0$. We know that $y(\xi_m, |((\xi^+)^{tr})^{(\xi_m)}| - \xi_m + 1)v_{\xi^+} = 0$. Then, we have

$$|((\xi^+)^{tr})^{(\xi_m)}| = \begin{cases} \xi_m(m-1), & \xi_m = 1 \\ m(\xi_m - 1) + (m-1), & \xi_m \neq 1. \end{cases}$$

In either case, $|((\xi^+)^{tr})^{(\xi_m)}| - \xi_m + 1 \leq \xi_m(m-1)$. Therefore, by Remark 2.6.1,

$$y(\xi_m, \xi_m(m-1))v_{\xi^+} = 0.$$

Now, we need to show that $\ker\varphi^+ \subset \operatorname{im}\varphi^-$. Since $\varphi^+(v_\xi) = v_{\xi^+}$ and $V(\xi)$ and $V(\xi^+)$ are quotients of the same local Weyl module, we have

$$\ker\varphi^+ = \{y(r, \nu_1(\xi^+) + \nu_2(\xi^+) + \cdots + \nu_r(\xi^+) - r + 1)v_\xi : \quad 1 \leq r \leq \xi_1^+ - 1\}.$$

We note that

$$\nu_1(\xi) + \nu_2(\xi) + \cdots + \nu_r(\xi) = \nu_1(\xi^+) + \nu_2(\xi^+) + \cdots + \nu_r(\xi^+)$$

for $1 \leq r \leq \xi_m - 1$ and $r \geq \xi_{m-1} + 1$. Therefore, we have

$$\ker\varphi^+ = \{y(r, \nu_1(\xi^+) + \nu_2(\xi^+) + \cdots + \nu_r(\xi^+) - r + 1)v_\xi : \quad \xi_m \leq r \leq \xi_{m-1}\}.$$

Let $\xi_m \le r \le \xi_{m-1}$. Then, we have

$$\begin{aligned}
\nu_1(\xi^+) + \nu_2(\xi^+) + \cdots + \nu_r(\xi^+) - r + 1 &= m(\xi_m - 1) + (m-1)(r - (\xi_m - 1)) - r + 1 \\
&= m(\xi_m - 1) + mr - m(\xi_m - 1) - r + (\xi_m - 1) - r + 1 \\
&= (m-2)r + \xi_m.
\end{aligned}$$

Now, consider a single term of $y(r, (m-2)r + \xi_m)v_\xi$ given by $y_{i_1} y_{i_2} \cdots y_{i_r} v_\xi$. We must have $i_1 + i_2 + \cdots + i_r = (m-2)r + \xi_m$ and $0 \le i_j \le m-1$ for all $1 \le j \le r$ since $y_m v_\xi = 0$. If $i_j \le m-2$ for all $1 \le j \le r$, then

$$\begin{aligned}
i_1 + i_2 + \cdots + i_r &\le (m-2)r \\
&< (m-2)r + \xi_m
\end{aligned}$$

which is a contradiction. Hence, there exists j such that $i_j = m-1$. Therefore, $y_{i_1} y_{i_2} \cdots y_{i_r} v_\xi$ is in the submodule generated by $y_{m-1} v_\xi$. We conclude that $y(r, (m-2)r + \xi_m)v_\xi \subset \operatorname{im} \varphi^-$ completing the proof. $\square$

Chapter 5

Second Short Exact Sequence

In this chapter, we again restrict ourselves to $\mathfrak{sl}_2$. We use our presentations of $V(\xi)$ modules from the previous chapters to establish a sequence of lemmas that were used to provide a new short exact sequence of $V(\xi)$ modules.

5.1

Let $\xi = \ell^{m_\ell}(\ell-1)^{m_{\ell-1}} \cdots 1^{m_1}$. Assume there exists $1 \leq i \leq \ell$ such that $m_i \geq 2$. Then, we define the partitions $\xi^+(i)$ and $\xi^-(i)$ as

$$\xi^+(i) := \ell^{m_\ell}(\ell-1)^{m_{\ell-1}} \cdots (i+2)^{m_{i+2}}(i+1)^{m_{i+1}+1} i^{m_i-2}(i-1)^{m_{i-1}+1}(i-2)^{m_{i-2}} \cdots 1^{m_1}$$

and

$$\xi^-(i) := \ell^{m_\ell}(\ell-1)^{m_{\ell-1}} \cdots (i+1)^{m_{i+1}} i^{m_i-2}(i-1)^{m_{i-1}}(i-2)^{m_{i-2}} \cdots 1^{m_1}.$$

Now, we have the following result.

Theorem 5.1.1. *If $i \leq 3$, the following is a short exact sequence of graded $\mathfrak{sl}_2[t]$-modules*

$$0 \to \tau_{\nu_1(\xi)+\cdots+\nu_i(\xi)-i} V(\xi^-(i)) \to V(\xi) \to V(\xi^+(i)) \to 0.$$

Remark 5.1.2. If $i = 1$, this short exact sequence is the same as the short exact sequence from the previous chapter given by ξ^- and ξ^+.

The proof of this theorem can be found in [1]. But, the proof depends on our following sequence of lemmas and propositions. In particular, we help to establish the existence of the map $\tau_{\nu_1(\xi)+\cdots+\nu_i(\xi)-i} V(\xi^-(i)) \to V(\xi)$. The statements and proofs of the lemmas and propositions needed to establish this short exact sequence will take up the remainder of this chapter.

Remark 5.1.3. In [1], this theorem was used to provide a character formulas for the tensor product of level 1 Demazure modules (ie. local Weyl modules) and the tensor product of any level 2 Demazure module with a level 1 Demazure module. This character formula in turn provides evidence that a tensor product of Demazure modules for $\mathfrak{sl}_2[t]$ has a Demazure flag of a certain level.

5.2

Lemma 5.2.1. *In the $\mathfrak{sl}_2$-module $V(r_1) \otimes V(r_2) \otimes \cdots \otimes V(r_k)$, the $r_1 + r_2 + \cdots + r_k - 2$ weight space has dimension k.*

Proof. For $1 \leq j \leq k$, let v_j be the generator of $V(r_j)$. We claim that

$$\{v_1 \otimes v_2 \otimes \cdots \otimes v_{j-1} \otimes y v_j \otimes v_{j+1} \otimes \cdots \otimes v_k : \quad 1 \leq j \leq k\}$$

is a basis for the $r_1 + r_2 + \cdots + r_k - 2$ weight space.

These elements are in the $r_1 + r_2 + \cdots + r_k - 2$ weight space since

$$h(v_1 \otimes v_2 \otimes \cdots \otimes v_{j-1} \otimes yv_j \otimes v_{j+1} \otimes \cdots \otimes v_k)$$
$$= (r_1 + r_2 + \cdots + r_k - 2)v_1 \otimes v_2 \otimes \cdots \otimes v_{j-1} \otimes yv_j \otimes v_{j+1} \otimes \cdots \otimes v_k.$$

Also, it is clear that these elements are linearly independent. Now, we need to show that these elements span the $r_1 + r_2 + \cdots + r_k - 2$ weight space. Suppose v is a vector in the $r_1 + r_2 + \cdots + r_k - 2$ weight space. It suffices to assume v is a simple tensor. So,

$$v = g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_k v_k$$

where $g_j \in \mathfrak{sl}_2$ for $1 \leq j \leq k$. By assumption, we have

$$hv = (r_1 + r_2 + \cdots + r_k - 2)v.$$

On the other hand,

$$\begin{aligned} hv &= h(g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_k v_k) \\ &= \sum_{j=1}^{k} g_1 v_1 \otimes g_2 v_2 \otimes \cdots \otimes g_{j-1} v_{j-1} \otimes h(g_j v_j) \otimes g_{j+1} v_{j+1} \otimes \cdots \otimes g_k v_k. \end{aligned}$$

For all $1 \leq j \leq k$, we know that $h(g_j v_j) = r_j - 2k_j$ for some $k_j \in \mathbb{Z}_+$. Since $hv = r_1 + r_2 + \cdots + r_k - 2$, there exists n such that

$$hg_n v_n = r_n - 2$$

$$hg_j v_j = r_j$$

for $j \neq n$. But, then in $V(r_n)$, the $r_n - 2$ weight space is one dimensional. Hence, $g_n v_n$ is a scalar multiple of yv_n. Also, for $j \neq n$, $g_j v_j$ is a highest weight vector in $V(r_j)$ and hence a scalar multiple of v_j. Thus, v is a scalar multiple of

$$v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1} \otimes yv_n \otimes v_{n+1} \otimes \cdots \otimes v_k.$$

This proves that our set spans the $r_1 + r_2 + \cdots + r_k - 2$ weight space. Therefore, we have a basis for the $r_1 + r_2 + \cdots + r_k - 2$ weight space and the dimension of this weight space is k. □

Proposition 5.2.2. *We have*

$$y(1, \nu_1 - 2)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0.$$

Proof. As $\mathfrak{sl}_2-$modules, we know that the short exact sequence

$$0 \to V(\xi(i)^-) \to V(\xi(i)) \to V(\xi(i)^+) \to 0$$

where

$$\xi^-(i) = \ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_1} \cdots i^{\nu_i-\nu_{i+1}-2}(i-1)^{\nu_{i-1}-\nu_i} \cdots 1^{\nu_1-\nu_2}$$

$$\xi^+(i) = \ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_1} \cdots (i+1)^{\nu_{i+1}-\nu_{i+2}+1} i^{\nu_i-\nu_{i+1}-2}(i-1)^{\nu_{i-1}-\nu_i+1} \cdots 1^{\nu_1-\nu_2}$$

exists. As a $\mathfrak{sl}_2-$module,

$$V(\ell^{\nu_\ell}(\ell-1)^{\nu_{\ell-1}-\nu_\ell} \cdots (i+1)^{\nu_{i+1}-\nu_{i+2}} i^{\nu_i-\nu_{i+1}-2}(i-1)^{\nu_{i-1}-\nu_i} \cdots 1^{\nu_1-\nu_2})$$
$$= V(\ell)^{\otimes \nu_\ell} \otimes V(\ell-1)^{\otimes \nu_{\ell-1}-\nu_\ell} \otimes \cdots \otimes V(i+1)^{\otimes \nu_{i+1}-\nu_{i+2}} \otimes V(i)^{\otimes \nu_i-\nu_{i+1}-2}$$
$$\otimes V(i-1)^{\otimes \nu_{i-1}-\nu_i} \otimes \cdots \otimes V(1)^{\otimes \nu_1-\nu_2}$$

In this module, the dimension of the $|\xi| - 2i - 2$ weight space is $\nu_1 - 2$ by Lemma 5.2.1. Hence, in $V(\xi)$, the dimension of the $|\xi| - 2i - 2$ weight space is $\nu_1 - 2$.

Now, consider the element

$$y(i, |(\xi^{tr})^{(i)}| - i)v_\xi,$$

which has weight $|\xi| - 2i$. Then, the element

$$y(1, v_1 - 2)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi$$

has weight $|\xi| - 2i - 2$.

Since $V(\xi)$ is finite dimensional, there exists m such that

$$y(1, m)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0.$$

Let N be minimal with respect to this property. Now, consider the vectors

$$\{y(1, j)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi : 0 \le j \le N - 1\}$$

These vectors have different grades and hence are linearly independent. But, since the dimension of the $|\xi| - 2i - 2$ weight space is $\nu_1 - 2$, $N \le \nu_1 - 2$. Hence,

$$y(1, v_1 - 2)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0.$$

□

5.3

Proposition 5.3.1. *We have*

$$y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = \ _1y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

Proof. First, we write

$$y(i, |(\xi^{tr})^{(i)}| - i) = \ _1y(1, |(\xi^{tr})^{(i)}| - i) + \sum_{j=1}^{i-1} y_0^{(j)} \ _1y(i - j, |(\xi^{tr})^{(i)}| - i). \tag{5.3.1}$$

Now, we claim

$$ {}_1y(j, |(\xi^{tr})^{(i)}| - i)v_\xi = 0 $$

for all $1 \leq j \leq i-1$. We proceed by induction on j. For $j = 1$, we have

$$ {}_1y(1, |(\xi^{tr})^{(i)}| - i)v_\xi = y_{|(\xi^{tr})^{(i)}|-i}v_\xi. $$

Since $i \geq 2$ and $\nu_k \geq \nu_i \geq 2$ for all $2 \leq k \leq i$, we have

$$ \begin{aligned} |(\xi^{tr})^{(i)}| - i &\geq \nu_1 + 2(i-1) - i \\ &\geq \nu_1 + i - 2 \\ &\geq \nu_1. \end{aligned} $$

Hence,

$$ {}_1y(1, |(\xi^{tr})^{(i)}| - i)v_\xi = 0 $$

by Remark 2.6.1. For the inductive step, assume $2 \leq j \leq i-1$ and ${}_1y(k, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$ for all $1 \leq k < j$. First, we note

$$ |(\xi^{tr})^{(i)}| - i \geq |(\xi^{tr})^{(j)}| - j + 1 $$

since $i \geq j+1$ and $\nu_n \geq 2$ for all $1 \leq n \leq i$. Hence,

$$ y(j, |(\xi^{tr})^{(i)}| - i)v_\xi = 0. $$

Now, we write

$$ y(j, |(\xi^{tr})^{(i)}| - i) = {}_1y(j, |(\xi^{tr})^{(i)}| - i) + \sum_{k=1}^{j-1} y_0^{(k)} \, {}_1y(j-k, |(\xi^{tr})^{(i)}| - i). $$

Hence, by the inductive hypothesis,

$$ y(j, |(\xi^{tr})^{(i)}| - i)v_\xi = {}_1y(j, |(\xi^{tr})^{(i)}| - i)v_\xi = 0, $$

which completes the claim. By the claim and (5.3.1), we now have

$$y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = \ {}_1y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

□

5.4

Lemma 5.4.1. *In* $\mathbf{U}(\mathfrak{sl}_2[t])$, *we have*

$$_1y(r,s) = \frac{1}{r}\sum_{j=1}^{s-r+1} y_j \ {}_1y(r-1, s-j), \quad r \geq 2, \quad s \geq r.$$

Proof. First, we recall that for $r \geq 2$ and $s' \geq 0$, we have

$$y(r,s') = \frac{1}{r}\sum_{j=0}^{s'} y_j y(r-1, s'-j)$$

by (2.3.1). Now, we consider the algebra homomorphism

$$\gamma : \mathbb{C}[y_0, y_1, y_2, \ldots] \to \mathbb{C}[y_1, y_2, y_3, \ldots]$$

given by $\gamma(y_j) = y_{j+1}$ for all $j \geq 0$. Then,

$$\gamma(y(r,s')) = \ {}_1y(r, s'+r).$$

On the other hand, we have

$$
\begin{aligned}
{}_1y(r, s'+r) &= \gamma(y(r,s')) \\
&= \gamma\left(\frac{1}{r}\sum_{j=0}^{s'} y_j y(r-1, s'-j)\right) \\
&= \frac{1}{r}\sum_{j=0}^{s'} \gamma(y_j y(r-1, s'-j)) \\
&= \frac{1}{r}\sum_{j=0}^{s'} y_{j+1}\ {}_1y(r-1, s'-j+r-1) \\
&= \frac{1}{r}\sum_{j=1}^{s'+1} y_j\ {}_1y(r-1, s'-j+r)
\end{aligned}
$$

Now, we let $s = s' + r$. Then, $s \geq r$ and

$$
{}_1y(r,s) = \frac{1}{r}\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-1, s-j),
$$

which completes the proof. □

5.5

Lemma 5.5.1. *In* $\mathbf{U}(\mathfrak{sl}_2[t])$, *we have*

$$
[h_1,\ {}_1y(r,s)] = -2r\ {}_1y(r, s+1) + 2y_1\ {}_1y(r-1, s), \quad r > 0, \quad s \geq r.
$$

Proof. We proceed by induction on r. Assume $r = 1$ and $s \geq 1$. Then, we have

$$
\begin{aligned}
[h_1,\ {}_1y(1,s)] &= [h_1, y_s] \\
&= -2y_{s+1} \\
&= -2\ {}_1y(1, s+1).
\end{aligned}
$$

Now, assume $r = 2$ and $s \geq 2$. Then, by Lemma 5.4.1, we have

$$\begin{aligned}
[h_1,\ {}_1y(2,s)] &= \left[h_1, \frac{1}{2}\sum_{j=1}^{s-1} y_j\ {}_1y(1,s-j)\right] \\
&= \frac{1}{2}\sum_{j=1}^{s-1}[h_1, y_j y_{s-j}] \\
&= \frac{1}{2}\sum_{j=1}^{s-1}[h_1, y_j]y_{s-j} + y_j[h_1, y_{s-j}] \\
&= \frac{1}{2}\sum_{j=1}^{s-1} -2y_{j+1}y_{s-j} + \frac{1}{2}\sum_{j=1}^{s-1} -2y_j y_{s+1-j} \\
&= \frac{1}{2}\sum_{j=2}^{s} -2y_j y_{s+1-j} + \frac{1}{2}\sum_{j=1}^{s-1} -2y_j y_{s+1-j} \\
&= \frac{1}{2}\sum_{j=1}^{s}(-2y_j y_{s+1-j}) + y_1 y_s + \frac{1}{2}\sum_{j=1}^{s}(-2y_j y_{s+1-j}) + y_s y_1 \\
&= -2\ {}_1y(2,s+1) + -2\ {}_1y(2,s+1) + 2y_1 y_s \\
&= -4\ {}_1y(2,s+1) + 2y_1\ {}_1y(1,s).
\end{aligned}$$

Now assume $r \geq 3, s \geq r$. Also, assume that for all $s' \geq r-1$, we have

$$[h_1,\ {}_1y(r-1,s')] = -2(r-1)y(r-1,s'+1) + 2y_1\ {}_1y(r-2,s').$$

Then, by Lemma 5.4.1,

$$
\begin{aligned}
[h_1,\ {}_1y(r,s)] &= \left[h_1, \frac{1}{r}\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-1,s-j)\right] \\
&= \frac{1}{r}\sum_{j=1}^{s-r+1}[h_1, y_j\ {}_1y(r-1,s-j)] \\
&= \frac{1}{r}\sum_{j=1}^{s-r+1} y_j[h_1,\ {}_1y(r-1,s-j)] + \frac{1}{r}\sum_{j=1}^{s-r+1}[h_1,y_j]\ {}_1y(r-1,s-j)] \\
&= \frac{1}{r}\sum_{j=1}^{s-r+1} y_j(-2(r-1)\ {}_1y(r-1,s+1-j) + 2y_1\ {}_1y(r-2,s-j)) \\
&\quad + \frac{1}{r}\sum_{j=1}^{s-r+1} -2y_{j+1}\ {}_1y(r-1,s-j) \\
&= \frac{-2(r-1)}{r}\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-1,s+1-j) + \frac{2}{r}y_1\sum_{j=1}^{s-r+1} y_j\ {}_1y(r-2,s-j) \\
&\quad + \frac{-2}{r}\sum_{j=1}^{s-r+1} y_{j+1}\ {}_1y(r-1,s-j). \qquad (5.5.1)
\end{aligned}
$$

The first term on the right hand side of (5.5.1) is equal to

$$
\frac{2(r-1)}{r}y_{s-r+2}\ {}_1y(r-1,r-1) + \frac{-2(r-1)}{r}\sum_{j=1}^{s+1-r+1} y_j\ {}_1y(r-1,s+1-j).
$$

The second term on the right hand side of (5.5.1) is equal to

$$
\frac{-2}{r}y_{s-r+2}y_1\ {}_1y(r-2,r-2) + \frac{2}{r}y_1\sum_{j=1}^{s-r+2} y_j\ {}_1y(r-2,s-j).
$$

The third term on the right hand side of (5.5.1) is equal to

$$
\frac{2}{r}y_1\ {}_1y(r-1,s) + \frac{-2}{r}\sum_{j=1}^{s+1-r+1} y_j\ {}_1y(r-1,s+1-j).
$$

Now, if we add all these terms up, we get

$$\begin{aligned}[h_1,\ {}_1y(r,s)] &= \frac{2(r-1)}{r} y_{s-r+2} y_1^{(r-1)} - 2(r-1)\ {}_1y(r,s+1) \\ &+ \frac{-2}{r} y_{s-r+2} y_1^{(r-2)} y_1 + \frac{2(r-1)}{r} y_1\ {}_1y(r-1,s) \\ &+ \frac{2}{r} y_1\ {}_1y(r-1,s) - 2\ {}_1y(r,s+1) \\ &= -2r\ {}_1y(r,s+1) + 2y_1\ {}_1y(r-1,s).\end{aligned}$$

□

5.6

Lemma 5.6.1. *For $1 \leq r \leq i$, if $\nu_r \geq 3$ and*

$$ {}_1y(j, |(\xi^{tr})^{(j)}| - 3j + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0,$$

for all $1 \leq j \leq r-1$, then

$$y(r, |(\xi^{tr})^{(r)}| - 3r + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = \ {}_1y(r, |(\xi^{tr})^{(r)}| - 3r + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi.$$

Proof. We proceed by induction on r.

Assume $r = 1$ and $\nu_1 \geq 3$. Then,

$$\begin{aligned} y(1, \nu_1 - 2) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi &= y_{\nu_1 - 2} y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\ &= \ {}_1y(1, \nu_1 - 2) y(i, |(\xi^{tr})^{(i)}| - i) v_x i.\end{aligned}$$

This completes the base case. Let $2 \leq r \leq i$. Assume $\nu_r \geq 3$ and

$$ {}_1y(j, |(\xi^{tr})^{(j)}| - 3j + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0$$

for all $1 \le j \le r-1$.

Now, we claim that

$$ {}_1y(m, s')y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0 $$

for all $1 \le m \le r-1$ and $s' > |(\xi^{tr})^{(m)}| - 3m + 1$.

To prove this claim, we begin by induction on m. Assume $m = 1$ and $s' > \nu_1 - 2$. Then, $s' = \nu_1 - 2 + n$, where $n \in \mathbb{Z}_+$. Now, we continue by induction on n. Assume $n = 1$. By assumption, we have

$$ {}_1y(1, \nu_1 - 2)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0. $$

Hence, by Lemma 5.5.1,

$$ \begin{aligned} 0 =& h_1\ {}_1y(1, \nu_1 - 2)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ =& [h_1,\ {}_1y(1, \nu_1 - 2)]y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &+\ {}_1y(1, \nu_1 - 2)h_1 y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ =& -2\ {}_1y(1, \nu_1 - 2 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &+\ {}_1y(1, \nu_1 - 2)[h_1, y(i, |(\xi^{tr})^{(i)}| - i)]v_\xi \\ =& -2\ {}_1y(1, \nu_1 - 2 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &- 2i\ {}_1y(1, \nu_1 - 2)y(i, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\ &+ 2y_0\ {}_1y(1, \nu_1 - 2)y(i-1, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\ =& -2\ {}_1y(1, \nu_1 - 2 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \end{aligned} $$

since $\nu_i \ge 1$. Therefore,

$$ {}_1y(1, \nu_1 - 2 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0 $$

and this completes the base case $n = 1$.

Now, assume for $n \geq 1$, we have

$$ {}_1y(1, \nu_1 - 2 + n) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0. $$

Hence, by the inductive hypothesis and Lemma 5.5.1,

$$ \begin{aligned} 0 =& h_1 \ {}_1y(1, \nu_1 - 2 + n) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\ =& [h_1, \ {}_1y(1, \nu_1 - 2 + n)] y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\ & + \ {}_1y(1, \nu_1 - 2 + n) h_1 y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\ =& -2 \ {}_1y(1, \nu_1 - 2 + n + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\ & + \ {}_1y(1, \nu_1 - 2 + n) [h_1, y(i, |(\xi^{tr})^{(i)}| - i)] v_\xi \\ =& -2 \ {}_1y(1, \nu_1 - 2 + n + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\ & -2i \ {}_1y(1, \nu_1 - 2 + n) y(i, |(\xi^{tr})^{(i)}| - i + 1) v_\xi \\ & + 2y_0 \ {}_1y(1, \nu_1 - 2 + n) y(i - 1, |(\xi^{tr})^{(i)}| - i + 1) v_\xi \\ =& -2 \ {}_1y(1, \nu_1 - 2 + n + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \end{aligned} $$

since $\nu_i \geq 1$. Therefore,

$$ {}_1y(1, \nu_1 - 2 + n + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0, $$

which completes the induction on n and the base case $m = 1$.

Assume $2 \leq m \leq r - 1$. For the inductive hypothesis, assume

$$ {}_1y(m - 1, \tilde{s}) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0 $$

for all $\tilde{s} > |(\xi^{tr})^{(m-1)}| - 3(m-1) + 1$. Assume $s' > |(\xi^{tr})^{(m)}| - 3m + 1$. Then, $s = |(\xi^{tr})^{(m)}| - 3m + 1 + n$ where $n \in \mathbb{Z}_+$. We proceed by induction on n. Since $m \leq r - 1$, we have

$$ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0. $$

Hence, by Lemma 5.5.1 and the inductive hypothesis,

$$ \begin{aligned} 0 =& h_1\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ =& [h_1,\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)]y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &+\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)h_1 y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ =& -2m\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &+ 2y_1\ {}_1y(m-1, |(\xi^{tr})^{(m)}| - 3m + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &+\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)[h_1, y(i, |(\xi^{tr})^{(i)}| - i)]v_\xi \\ =& -2m\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\ &+ -2i\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)y(i, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\ &+ 2y_0\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1)y(i-1, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\ =& -2m\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \end{aligned} $$

since $\nu_m \geq \nu_r \geq 3$ and $\nu_i \geq 2$. Therefore,

$$ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0 $$

and this completes the base case $n = 1$. Now, assume that for $n \geq 1$, we have

$$ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0. $$

Hence, by the inductive hypothesis and Lemma 5.5.1, we have

$$\begin{aligned}
0 =& h_1\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
=& [h_1,\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)]y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&+\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)h_1 y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
=& -2m\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&+ 2y_1\ {}_1y(m-1, |(\xi^{tr})^{(m)}| - 3m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&+\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)[h_1, y(i, |(\xi^{tr})^{(i)}| - i)]v_\xi \\
=& -2m\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&+ -2i\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\
&+ 2y_0\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n)y(i-1, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\
=& -2m\ {}_1y(m, |(\xi^{tr})^{(m)}| - 3m + 1 + n + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi
\end{aligned}$$

since $\nu_m \geq 3$ and $\nu_i \geq 2$. This completes the induction on n and the claim.

Now, we write

$$\begin{aligned}
y(r, |(\xi^{tr})^{(r)}| - 3r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \quad =& \quad \sum_{j=1}^{r-1} y_0^{(j)}\ {}_1y(r-j, |(\xi^{tr})^{(r)}| - 3r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&+ {}_1y(r, |(\xi^{tr})^{(r)}| - 3r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi. \quad (5.6.1)
\end{aligned}$$

Also, for $1 \leq r - j \leq r - 1$, we have

$$\begin{aligned}
|(\xi^{tr})^{(r)}| - 3r + 1 &\geq |(\xi^{tr})^{(r-j)}| + 3(r - (r - j)) - 3r + 1 \\
&= |(\xi^{tr})^{(r-j)}| - 3(r - j) + 1
\end{aligned}$$

since $\nu_r \geq 3$. Therefore, by the claim

$${}_1y(r - j, |(\xi^{tr})^{(r)}| - 3r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0.$$

Hence, using (5.6.1), we have established

$$y(r, |(\xi^{tr})^{(r)}| - 3r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = {}_1y(r, |(\xi^{tr})^{(r)}| - 3r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

This completes the induction on r. □

5.7

Lemma 5.7.1. *For $i+1 \leq r \leq \ell - 1$, if $\nu_r \geq 1$,*

$${}_1y(j, |(\xi^{tr})^{(j)}| - 3j + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $1 \leq j \leq i$, and

$${}_1y(\tilde{j}, |(\xi^{tr})^{(\tilde{j})}| - 2i - \tilde{j} + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $i+1 \leq \tilde{j} \leq r-1$, then

$$y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = {}_1y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

Proof. We proceed by induction on r.

Let $r = i+1$. Assume $\nu_{i+1} \geq 1, \nu_i - \nu_{i+1} \geq 2$, and

$${}_1y(j, |(\xi^{tr})^{(j)}| - 3j + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $1 \leq j \leq i$. Then $\nu_i \geq 3$ and $\nu_j \geq 3$ for all $1 \leq j \leq i$. By the claim from that was proved in the Lemma 5.6.1,

$${}_1y(j, s)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $1 \leq j \leq i$ and $s \geq |(\xi^{tr})^{(j)}| - 3j + 1$. Now, we write

$$y(i+1, |(\xi^{tr})^{(i+1)}| - 3i)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = \sum_{k=1}^{i} y_0^{(k)} \ {}_1y(i+1-k, |(\xi^{tr})^{(i+1)}| - 3i)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi$$
$$+ \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

Now, for all $1 \leq j \leq i$, we have

$$\begin{aligned}|(\xi^{tr})^{(i+1)}| - 3i &\geq |(\xi^{tr})^{(j)}| + \nu_{j+1} + \cdots + \nu_{i+1} - 3i \\ &\geq |(\xi^{tr})^{(j)}| + 3(i-j) + 1 - 3i \\ &= |(\xi^{tr})^{(j)}| - 3j + 1\end{aligned}$$

since $\nu_i \geq 3$ and $\nu_{i+1} \geq 1$. Thus, by Lemma 5.6.1,

$${}_1y(i+1-k, |(\xi^{tr})^{(i+1)}| - 3i)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $1 \leq k \leq i$. (Note: The claim in Lemma 5.6.1 is only for $1 \leq k < i$, but a symmetric proof holds for $k = i$.) Hence,

$$y(i+1, |(\xi^{tr})^{(i+1)}| - 3i)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

This completes the base case.

Let $i+2 \leq r \leq \ell - 1$. Assume $\nu_r \geq 1$, $\nu_i - \nu_{i+1} \geq 2$,

$${}_1y(j, |(\xi^{tr})^{(j)}| - 3j + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $1 \leq j \leq i$, and

$${}_1y(\tilde{j}, |(\xi^{tr})^{(\tilde{j})}| - 2i - \tilde{j} + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $i + 1 \leq \tilde{j} \leq r - 1$.

Now, write

$$
\begin{aligned}
y(r,&|(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&= \sum_{k=1}^{r-1} y_0^{(k)} \ {}_1y(r-k, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&\quad + \ {}_1y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.
\end{aligned}
$$

Since $r \geq i+2, \nu_r \geq 1$, and $\nu_i - \nu_{i+1} \geq 2$, $\nu_i \geq 3$. Hence, $\nu_j \geq 3$ for all $1 \leq j \leq i$. Also, for all $1 \leq j \leq i$, we have

$$
\begin{aligned}
|(\xi^{tr})^{(r)}| - 2i - r + 1 &= |(\xi^{tr})^{(j)}| + \nu_{j+1} + \cdots + \nu_r - 2i - r + 1 \\
&\geq |(\xi^{tr})^{(j)}| + 3(i-j) + 1(r-i) - 2i - r + 1 \\
&= |(\xi^{tr})^{(j)}| - 3j + 1.
\end{aligned}
$$

Hence, for all $1 \leq j \leq i$,

$$
{}_1y(j, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0
$$

by the claim from Lemma 5.6.1. (Note: Again, the claim in the Lemma 5.6.1 is only for $1 \leq j < i$, but a symmetric proof holds for $j = i$.)

Hence,

$$
\begin{aligned}
y(r,&|(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&= \sum_{k=1}^{r-i-1} y_0^{(k)} \ {}_1y(r-k, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&\quad + \ {}_1y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.
\end{aligned}
$$

Now, we claim that

$$
{}_1y(m, s')y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0
$$

for all $i+1 \leq m \leq r-1$ and $s' > |(\xi^{tr})^{(m)}| - 2i - m + 1$. To prove this claim, we begin by induction on m. Assume $m = i+1$ and $s' > |(\xi^{tr})^{(i+1)}| - 3i$. Then, $s' = |(\xi^{tr})^{(i+1)}| - 3i + n$, where $n \in \mathbb{Z}_+$. Now, we continue by induction on n. Assume $n = 1$. By assumption, we have

$$ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0. $$

Hence, by Lemma 5.5.1,

$$
\begin{aligned}
0 =& h_1 \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\
=& [h_1, \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i)] y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\
&+ \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i) h_1 y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\
=& -2(i+1) \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\
&+ 2y_1 \ {}_1y(i, |(\xi^{tr})^{(i+1)}| - 3i) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\
&+ \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i) [h_1, y(i, |(\xi^{tr})^{(i)}| - i)] v_\xi \\
=& -2(i+1) \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi \\
&+ -2i \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i) y(i, |(\xi^{tr})^{(i)}| - i + 1) v_\xi \\
&+ 2y_0 \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i) y(i-1, |(\xi^{tr})^{(i)}| - i + 1) v_\xi \\
=& -2(i+1) \ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi
\end{aligned}
$$

since $\nu_i \geq 1$ and $\nu_{i+1} \geq 1$. Therefore,

$$ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i + 1) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0 $$

and this completes the base case $n = 1$. Now, assume for $n \geq 1$, we have

$$ {}_1y(i+1, |(\xi^{tr})^{(i+1)}| - 3i + n) y(i, |(\xi^{tr})^{(i)}| - i) v_\xi = 0. $$

Hence, by the inductive hypothesis and Lemma 5.5.1,

$$\begin{aligned}
0 =& h_1\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi \\
=& [h_1,\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n)]y(i,|(\xi^{tr})^{(i)}|-i)v_\xi \\
&+\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n)h_1y(i,|(\xi^{tr})^{(i)}|-i)v_\xi \\
=& -2(i+1)\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n+1)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi \\
&+2y_1\ {}_1y(i,|(\xi^{tr})^{(i+1)}|-3i+n)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi \\
&+\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n)[h_1,y(i,|(\xi^{tr})^{(i)}|-i)]v_\xi \\
=& -2(i+1)\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n+1)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi \\
&+-2i\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n)y(i,|(\xi^{tr})^{(i)}|-i+1)v_\xi \\
&+2y_0\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n)y(i-1,|(\xi^{tr})^{(i)}|-i+1)v_\xi \\
=& -2(i+1)\ {}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n+1)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi
\end{aligned}$$

since $\nu_i \geq 1$ and $\nu_{i+1} \geq 1$. Therefore,

$${}_1y(i+1,|(\xi^{tr})^{(i+1)}|-3i+n+1)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi = 0,$$

which completes the induction on n and the base case $m = 1$. Assume $i+2 \leq m \leq r-1$. For the inductive hypothesis, assume

$${}_1y(m-1,\tilde{s})y(i,|(\xi^{tr})^{(i)}|-i)v_\xi = 0$$

for all $\tilde{s} > |(\xi^{tr})^{(m-1)}| - 2i - (m-1) + 1$. Assume $s' > |(\xi^{tr})^{(m)}| - 2i - m + 1$. Then, $s = |(\xi^{tr})^{(m)}| - 2i - m + 1 + n$ where $n \in \mathbb{Z}_+$. We proceed by induction on n. Since $i+1 \leq m \leq r-1$, we have

$${}_1y(m,|(\xi^{tr})^{(m)}|-2i-m+1)y(i,|(\xi^{tr})^{(i)}|-i)v_\xi = 0.$$

Hence, by Lemma 5.5.1 and the inductive hypothesis,

$$\begin{aligned}
0 =& h_1 \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
=& [h_1, \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1)]y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1)h_1 y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
=& -2m \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + 2y_1 \ {}_1y(m-1, |(\xi^{tr})^{(m)}| - 2i - m + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1)[h_1, y(i, |(\xi^{tr})^{(i)}| - i)]v_\xi \\
=& -2m \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + -2i \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1)y(i, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\
& + 2y_0 \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1)y(i-1, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\
=& -2m \ {}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi
\end{aligned}$$

since $\nu_m \geq \nu_r \geq 1$ and $\nu_i \geq 2$. Therefore,

$${}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

and this completes the base case $n = 1$. Now, assume that for $n \geq 1$, we have

$${}_1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0.$$

Hence, by the inductive hypothesis and Lemma 5.5.1, we have

$$
\begin{aligned}
0 =& h_1 \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
=& [h_1, \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)]y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)h_1 y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
=& -2m \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + 2y_1 \; _1y(m-1, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)[h_1, y(i, |(\xi^{tr})^{(i)}| - i)]v_\xi \\
=& -2m \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
& + -2i \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)y(i, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\
& + 2y_0 \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n)y(i-1, |(\xi^{tr})^{(i)}| - i + 1)v_\xi \\
=& -2m \; _1y(m, |(\xi^{tr})^{(m)}| - 2i - m + 1 + n + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi
\end{aligned}
$$

since $\nu_m \geq 1$ and $\nu_i \geq 2$. This completes the induction on n and the claim.

Now, recall

$$
\begin{aligned}
y(r, & |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&= \sum_{k=1}^{r-i-1} y_0^{(k)} \; _1y(r-k, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi \\
&+ \; _1y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.
\end{aligned}
$$

Also, for $i + 1 \leq j \leq r - 1$, we have

$$
\begin{aligned}
|(\xi^{tr})^{(r)}| - 2i - r + 1 &\geq |(\xi^{tr})^{(j)}| + (r - j) - 2i - r + 1 \\
&= |(\xi^{tr})^{(j)}| - 2i - j + 1
\end{aligned}
$$

since $\nu_r \geq 1$. Therefore, by the claim

$$_1y(j, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = 0$$

for all $i + 1 \leq j \leq r - 1$.

Hence, we have established

$$y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi = {}_1y(r, |(\xi^{tr})^{(r)}| - 2i - r + 1)y(i, |(\xi^{tr})^{(i)}| - i)v_\xi.$$

This completes the induction on r. $\square$

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